

**EFFICIENT METHODS FOR SOLVING FUZZY LINEAR  
SYSTEMS**

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**Thesis**

**Submitted in fulfilment of the requirements for the degree of**

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**by**

**NEETU**  
(Regn. No. 950911005)

**to the**



**SCHOOL OF MATHEMATICS**  
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# CERTIFICATE

This is to certify that the thesis entitled, “**Efficient Methods for Solving Fuzzy Linear Systems**” submitted by **Neetu** in the fulfilment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar University, Patiala, is a record of candidate’s own work carried out by her under my supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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# DECLARATION

It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.



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***DEDICATED***

***TO***

***MY PARENTS***

***&***

***MY HUSBAND***



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Gurur-Brahma Gurur-Vishnur-Gururdevo Maheshvarah |

Gurure va Param Brahma Tasmai Shrii-Gurave Namah||

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# Abstract

In this thesis the limitations of existing methods for solving fully fuzzy linear system, dual fully fuzzy linear system and dual fully fuzzy matrix equation are pointed out. Also, to overcome the limitations of the existing methods, new methods are proposed to find the solution of fully fuzzy linear system, dual fully fuzzy linear system and dual fully fuzzy matrix equation.

The thesis comprises seven chapters. A brief outline of the chapters is as follows:

## Chapter 1 Introduction

Chapter 1 is introductory in nature. In this chapter, a brief review of the work done in the literature for solving fully fuzzy linear system, dual fully fuzzy linear system and dual fully fuzzy matrix equations is presented.

## Chapter 2 Mehar method-I for solving fully fuzzy linear system with unrestricted fuzzy coefficients

Dehgan et al. [49] pointed that there is no method in the literature to find the solution of fully fuzzy linear system and proposed some methods to find the solution of such fully fuzzy linear system in which all elements of the coefficient matrix and unknown vector are represented by non-negative triangular fuzzy numbers. After that a lot of researchers have proposed several direct and iterative methods for solving the same type of fully fuzzy linear system. In this chapter, limitations of these existing methods are pointed out. Also, to overcome these limitations, a new method (named as Mehar method-I) is proposed. The proposed Mehar method-I is illustrated by numerical examples.

### **Chapter 3 Mehar method-II for solving fully fuzzy linear system with unrestricted fuzzy variables**

The methods, proposed in Chapter 2, can be used to find the solution of such fully fuzzy linear system in which the elements of the coefficient matrix and right hand side vector are represented by unrestricted triangular fuzzy numbers whereas all the elements of the unknown vector are represented by non-negative triangular fuzzy numbers. However, the method, proposed in Chapter 2, cannot be used to find the solution of such fully fuzzy linear system in which elements of coefficient matrix are represented by non-negative triangular fuzzy numbers whereas all the elements of unknown vector and right hand side vector are represented by unrestricted triangular fuzzy numbers. In this chapter, a new method (named as Mehar method-II) is proposed to find the solution of same type of fully fuzzy linear system. The proposed Mehar method-II is illustrated with the help of some numerical examples.

### **Chapter 4 Mehar method-III for solving fully fuzzy linear system with unrestricted fuzzy coefficients and unrestricted fuzzy variables**

In the methods, proposed in previous chapters, either the elements of coefficient matrix and elements of right hand side vector or elements of unknown vector are assumed as non-negative triangular fuzzy numbers. However, in the real life problems there is need to find the solution of fully fuzzy linear system without any restriction on these elements. To overcome the limitations of the existing methods and the methods proposed in previous chapters, in this chapter, a new method (named as Mehar method-III) is proposed to find the solution of such fully fuzzy linear system in which elements of coefficient matrix, right hand side vector as well as the elements of unknown vector are represented by unrestricted

triangular fuzzy numbers. The proposed Mehar method-III is illustrated by numerical examples.

## **Chapter 5 Mehar method-IV for solving dual fully fuzzy linear system with unrestricted fuzzy coefficients and unrestricted fuzzy variables**

In this chapter, limitations of the existing methods for solving dual fully fuzzy linear system are pointed out. Also, to overcome these limitations, a new method (named as Mehar method-IV) is proposed to find the solution of such dual fully fuzzy linear system in which all the elements of all the matrices and elements of unknown vector are represented by unrestricted triangular fuzzy numbers. The proposed Mehar method-IV is illustrated by numerical examples.

## **Chapter 6 Mehar method-V for solving dual fully fuzzy matrix equation with unrestricted fuzzy coefficient and unrestricted fuzzy variables**

Otadi and Mosleh [113] extended the concept of fully fuzzy linear system into fully fuzzy matrix equations and proposed a method to find the solution of such fully fuzzy matrix equations in which all the elements of unknown matrix are represented by non-negative triangular fuzzy numbers whereas the elements of coefficient matrix and right hand matrix are represented by unrestricted triangular fuzzy numbers. On the same direction, in this chapter, the concept of dual fully fuzzy linear system is extended into dual fully fuzzy matrix equations and a new method (named as Mehar method-V) is proposed to find the solution of such dual fully fuzzy matrix equation in which all the elements of all the matrices are represented by unrestricted trapezoidal fuzzy numbers. The proposed Mehar method-V is illustrated by numerical examples.

## **Chapter 7 Future Scope**

Finally, in this Chapter, based on the presented study future work has been suggested

## List of Publications

1. **Neetu**, Amit Kumar, Abhinav Bansal: Solving fully fuzzy linear system with arbitrary triangular fuzzy numbers  $(m, \alpha, \beta)$ . *Soft Computing* **17**, 691-702 (2013). (SCI)
2. **Neetu**, Amit Kumar, Abhinav Bansal: Linear programming approach to find the solution of fully fuzzy linear systems with arbitrary fuzzy coefficients. *Journal of Intelligent & Fuzzy Systems* **25**, 747-753 (2013). (SCI)
3. Amit Kumar, **Neetu**, Abhinav Bansal: A new method to solve fully fuzzy linear system with trapezoidal fuzzy numbers. *Canadian Journal on Science and Engineering Mathematics* **1**, 45-56 (2010).
4. Amit Kumar, **Neetu**, Abhinav Bansal: A method for solving fully fuzzy linear system with trapezoidal fuzzy numbers. *Iranian Journal of Optimization* **2**, 359-374 (2010)
5. Amit Kumar, **Neetu**, Abhinav Bansal: A new approach for solving fully fuzzy linear systems. *Advances in Fuzzy Systems* **2011**, 1-8 (2011).
6. Amit Kumar, Abhinav Bansal, **Neetu**: Solution of fully fuzzy linear system with arbitrary coefficients. *International Journal of Applied Mathematics and Computation* **3**, 232-237 (2011).
7. Amit Kumar, **Neetu**, Abhinav Bansal: A new computational method to solve fully fuzzy linear systems for negative coefficient matrix. *International Journal of Manufacturing Technology and Management* **25**, 19-31 (2012).
8. Amit Kumar, **Neetu**, Abhinav Bansal: A new computational method for solving fully fuzzy linear systems of triangular fuzzy numbers. *Fuzzy Information and Engineering* **4**, 63-73 (2012).

9. Amit Kumar, **Neetu**, Abhinav Bansal: Some new computational methods to solve dual fully fuzzy linear system of arbitrary triangular fuzzy numbers. *New Mathematics and Natural Computation* **9**, 13-26 (2013).
10. Amit Kumar, Abhinav Bansal, **Neetu**: Fully fuzzy linear systems of triangular fuzzy numbers (a,b,c). *International Journal of Intelligent Computing and Cybernetics* **6**, 21-44 (2013).

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# Chapter 1

## Introduction

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System of simultaneous linear equations is one of the important field of applied mathematics that has many applications in various area of science and engineering. Some of the areas in which system of simultaneous linear equations play a major role are operational research, physics, statistics, economics, biological sciences, chemical engineering processes, power system networks etc.

In general a system of simultaneous linear equations is represented as  $AX = b$ , where  $A = [a_{ij}]_{m \times n}$  is called as coefficient matrix,  $X = [x_j]_{n \times 1}$  is called as unknown vector and  $b = [b_i]_{m \times 1}$  is called as right hand side vector. The same system may also be written as  $\sum_{i=1}^m a_{ij}x_j = b_i, \forall i = 1, 2, \dots, m$ .

To find the solution  $\{x_j\}$  of the system of linear equations  $\sum_{i=1}^m a_{ij}x_j = b_i, \forall i = 1, 2, \dots, m$ , the values of  $a_{ij}$  and  $b_i$  are either collected from experts or measured through experiments. If the values of  $a_{ij}$  and / or  $b_i$  are imprecise and only some vague knowledge about these values is available then it is better to represent  $a_{ij}$  and / or  $b_i$  as fuzzy numbers [141] instead of real numbers.

A system of linear equation in which  $a_{ij}$  and  $b_i$  are represented by fuzzy numbers whereas  $x_j$  is represented by real number or  $x_j$  and  $b_i$  are represented by fuzzy numbers whereas  $a_{ij}$  is represented by a real number is known as fuzzy linear system. While, a system of linear equations in which all  $a_{ij}, b_i$  and  $x_j$  are represented by fuzzy numbers is known as fully fuzzy linear system.

Further, a system of linear equation  $\sum_{i=1}^m a_{ij}x_j + b_i = \sum_{i=1}^m c_{ij}x_j + d_i, \forall i = 1, 2, \dots, m$ , in which  $a_{ij}, b_i, c_{ij}, d_i$  and  $x_j$  are real numbers, can be easily transformed into  $\sum_{i=1}^m (a_{ij} - c_{ij})x_j = d_i - b_i, \forall i = 1, 2, \dots, m$ . However it is not possible for the same type of fuzzy linear system and fully fuzzy linear system as on subtracting a fuzzy number from itself the obtained fuzzy number is not zero fuzzy number.

A system of linear equation  $AX + b = Cx + d$  i.e.,  $\sum_{i=1}^m a_{ij}x_j + b_i = \sum_{i=1}^m c_{ij}x_j + d_i, \forall i = 1, 2, \dots, m$ , in which  $a_{ij}, b_i, c_{ij}, d_i$  and  $x_j$  are all represented by fuzzy numbers are known as dual fully fuzzy linear system.

To find the solution of a fully fuzzy linear system and dual fully fuzzy linear systems, firstly it is transformed into crisp system of linear equations and then using the crisp solution of transformed crisp system of linear equations, the fuzzy solution of fully fuzzy linear system and dual fully fuzzy linear system is obtained.

The existing methods for solving fully fuzzy linear system and dual fully fuzzy linear system on the basis of types of fuzzy numbers and the methods, used for solving the crisp system of linear equations obtained from fully fuzzy linear system / dual fully fuzzy linear systems, can be classified as follows:

- (1) Existing methods for solving such fully fuzzy linear systems in which all the elements of coefficients matrix, unknown vector and right hand vectors are represented by non-negative triangular fuzzy numbers / trapezoidal fuzzy numbers and direct methods are used to find the solution of transformed crisp system of linear equations.
- (2) Existing methods for solving such fully fuzzy linear systems in which all the elements of coefficients matrix, unknown vector and right hand vectors are represented by non-negative triangular fuzzy numbers / trapezoidal fuzzy numbers

and iterative methods are used to find the solution of transformed crisp system of linear equations.

- (3) Existing methods for solving such dual fully fuzzy linear systems in which all the elements of coefficients matrix, unknown vector and right hand vectors are represented by non-negative triangular / trapezoidal fuzzy numbers and direct / iterative methods are used to find the solution of transformed crisp system of linear equations.

A good brief review of all these existing methods is recently published by Daud et al. [46]. Since, the paper, published by Daud et al. [46] is easily available. So, the same is not presented in this thesis.

After a deep study of the existing methods for solving fully fuzzy linear system and dual fully fuzzy linear system, it was found that there was no method in the literature to find the solution of following fully fuzzy linear system / dual fully fuzzy linear system.

- (i) Fully fuzzy linear system / dual fully fuzzy linear system in which elements of coefficient matrix and right hand side vector are represented by unrestricted triangular fuzzy numbers whereas elements of unknown vector are represented by non-negative triangular fuzzy numbers and vice-versa.
- (ii) Fully fuzzy linear system / dual fully fuzzy linear system in which elements of coefficient matrix, right hand side vector and unknown vector are represented by unrestricted triangular fuzzy numbers.

To fill this gap, the research work was started to find the solution of such fully fuzzy linear system in which elements of coefficient matrix and right hand side vector are represented by unrestricted triangular fuzzy numbers whereas the elements of unknown vector are represented by non-negative triangular fuzzy numbers. After successful completion

of the same, a new method is proposed to find the solution of such fully fuzzy linear system in which elements of the coefficient matrix are represented by non-negative triangular fuzzy number where as elements of unknown vector and right hand side vector is represented by unrestricted triangular fuzzy number. Finally, new methods are proposed to find the solutions of such fully fuzzy linear system / dual fully fuzzy linear system / dual fully fuzzy matrix equation in which all the elements of coefficients matrix, right hand side vector and unknown vector are represented by unrestricted triangular / trapezoidal fuzzy numbers.

The Chapter wise summary of the thesis is as follows

## **Chapter 2 Mehar method-I for solving fully fuzzy linear system with unrestricted fuzzy coefficients**

Dehgan et al. [49] pointed that there is no method in the literature to find the solution of fully fuzzy linear system and proposed some methods to find the solution of such fully fuzzy linear system in which all elements of the coefficient matrix and unknown vector are represented by non-negative triangular fuzzy numbers. After that a lot of researchers have proposed several direct and iterative methods for solving the same type of fully fuzzy linear system. In this chapter, limitations of these existing methods are pointed out. Also, to overcome these limitations, a new method (named as Mehar method-I) is proposed. The proposed Mehar method-I is illustrated by numerical examples.

## **Chapter 3 Mehar method-II for solving fully fuzzy linear system with unrestricted fuzzy variables**

The methods, proposed in Chapter 2, can be used to find the solution of such fully fuzzy linear system in which the elements of the coefficient matrix and right hand side vector are represented by unrestricted triangular fuzzy numbers whereas all the elements of the

unknown vector are represented by non-negative triangular fuzzy numbers. However, the method, proposed in Chapter 2, cannot be used to find the solution of such fully fuzzy linear system in which elements of coefficient matrix are represented by non-negative triangular fuzzy numbers whereas all the elements of unknown vector and right hand side vector are represented by unrestricted triangular fuzzy numbers. In this chapter, a new method (named as Mehar method-II) is proposed to find the solution of same type of fully fuzzy linear system. The proposed Mehar method-II is illustrated with the help of some numerical examples.

#### **Chapter 4 Mehar method-III for solving fully fuzzy linear system with unrestricted fuzzy coefficients and unrestricted fuzzy variables**

In the methods, proposed in previous chapters, either the elements of coefficient matrix and elements of right hand side vector or elements of unknown vector are assumed as non-negative triangular fuzzy numbers. However, in the real life problems there is need to find the solution of fully fuzzy linear system without any restriction on these elements. To overcome the limitations of the existing methods and the methods proposed in previous chapters, in this chapter, a new method (named as Mehar method-III) is proposed to find the solution of such fully fuzzy linear system in which elements of coefficient matrix, right hand side vector as well as the elements of unknown vector are represented by unrestricted triangular fuzzy numbers. The proposed Mehar method-III is illustrated by numerical examples.

#### **Chapter 5 Mehar method-IV for solving dual fully fuzzy linear system with unrestricted fuzzy coefficients and unrestricted fuzzy variables**

In this chapter, limitations of the existing methods for solving dual fully fuzzy linear system are pointed out. Also, to overcome these limitations, a new method (named as Mehar method-IV) is proposed to find the solution of such dual fully fuzzy linear system in which all the elements of all the matrices and elements of unknown vector are represented by unrestricted triangular fuzzy numbers. The proposed Mehar method-IV is illustrated by numerical examples.

## **Chapter 6 Mehar method-V for solving dual fully fuzzy matrix equation with unrestricted fuzzy coefficient and unrestricted fuzzy variables**

Otadi and Mosleh [113] extended the concept of fully fuzzy linear system into fully fuzzy matrix equations and proposed a method to find the solution of such fully fuzzy matrix equations in which all the elements of unknown matrix are represented by non-negative triangular fuzzy numbers whereas the elements of coefficient matrix and right hand matrix are represented by unrestricted triangular fuzzy numbers. On the same direction, in this chapter, the concept of dual fully fuzzy linear system is extended into dual fully fuzzy matrix equations and a new method (named as Mehar method-V) is proposed to find the solution of such dual fully fuzzy matrix equation in which all the elements of all the matrices are represented by unrestricted trapezoidal fuzzy numbers. The proposed Mehar method-V is illustrated by numerical examples.

## **Chapter 7 Future Scope**

Finally, in this Chapter, based on the presented study future work has been suggested

## Chapter 2

# Mehar method-I for solving fully fuzzy linear system with unrestricted coefficients<sup>1</sup>

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Dehgan et al. [49] pointed that there is no method in the literature to find the solution of fully fuzzy linear system and proposed some methods to find the solution of such fully fuzzy linear system in which all elements of the coefficient matrix and unknown vector are represented by non-negative triangular fuzzy numbers. After that a lot of researchers have proposed several direct and iterative methods for solving the same type of fully fuzzy linear system. In this chapter, limitations of these existing methods are pointed out. Also, to overcome these limitations, a new method (named as Mehar method-I) is proposed. The proposed Mehar method-I is illustrated by numerical examples.

### 2.1. Preliminaries

In this section, some basic definitions related to fuzzy numbers as well as arithmetic operations of triangular fuzzy numbers are presented [53].

#### 2.1.1. Basic definitions

In this section, some basic definitions related to fuzzy numbers are presented.

**Definition 2.1.** Let  $X$  denote a universal set. Then the fuzzy subset  $\tilde{A}$  of  $X$  is defined by its membership function  $\mu_{\tilde{A}} : X \rightarrow [0,1]$  which assigns a real number  $\mu_{\tilde{A}}(x)$  in the interval  $[0,1]$ ,

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<sup>1</sup> Some part of this chapter is published in *Soft Computing* 17 (2013) 69-702 and remaining part is published in *International Journal of Applied Mathematics and Computation* 3 (2011) 232-237.

to each element  $x \in X$ , where the value of  $\mu_{\tilde{A}}(x)$  at  $x$  shows the grade of membership of  $x$  in  $\tilde{A}$ .

**Definition 2.2.** A fuzzy set  $\tilde{A}$ , defined on the universal set of real number  $R$ , is said to be a fuzzy number if its membership function has the following characteristics:

(i)  $\tilde{A}$  is convex i.e.,  $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \text{minimum}(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \forall x_1, x_2 \in R, \forall \lambda \in [0,1]$ .

(ii)  $\tilde{A}$  is normal i.e.,  $\exists x_0 \in R$  such that  $\mu_{\tilde{A}}(x_0) = 1$ .

(iii)  $\mu_{\tilde{A}}$  is piecewise continuous.

**Definition 2.3.** A fuzzy number  $\tilde{A} = (m, \alpha, \beta)$  is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha} & m - \alpha \leq x < m, \alpha > 0 \\ 1 - \frac{x-m}{\beta} & m \leq x < m + \beta, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

**Definition 2.4.** A triangular fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  is said to be non-negative (positive) triangular fuzzy number i.e.,  $\tilde{M} \geq 0$  ( $\tilde{M} > 0$ ) if and only if  $m - \alpha \geq 0$  ( $m - \alpha > 0$ ).

**Definition 2.5.** A triangular fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  is said to be non-positive (negative) triangular fuzzy number i.e.,  $\tilde{M} \leq 0$  ( $\tilde{M} < 0$ ) if and only if  $m + \beta \leq 0$  ( $m + \beta < 0$ ).

**Definition 2.6.** A triangular fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  is said to be zero triangular fuzzy number i.e.,  $\tilde{M} = 0$  if  $m = 0, \alpha = 0$  and  $\beta = 0$ . A zero triangular fuzzy number is denoted by  $(0,0,0)$ .

**Definition 2.7.** A fuzzy number  $\tilde{M}$  called a “Near Zero” fuzzy number if it is neither positive, nor negative and not even zero i.e., a triangular fuzzy number  $\tilde{M} = (m, \alpha, \beta) \equiv 0$  is Near zero if and only if  $m - \alpha < 0 < m + \beta$ . It is denoted by  $\tilde{M} \equiv 0$ .

For example (1,2,1) is a “near zero” fuzzy number as it is neither positive, nor negative and not even zero.

**Definition 2.8.** A triangular fuzzy number  $\tilde{M} = (m, \alpha, \beta)$  is said to be unrestricted triangular fuzzy number if and only if  $m - \alpha$  is a real number.

**Definition 2.9.** Two fuzzy numbers  $\tilde{M} = (m, \alpha, \beta)$  and  $\tilde{N} = (n, \gamma, \delta)$  are said to be equal if and only if  $m = n, \alpha = \gamma$  and  $\beta = \delta$ .

**Definition 2.10** A matrix  $\tilde{A} = (\tilde{a}_{ij})$  is called a fully fuzzy matrix, if each element of  $\tilde{A}$  is a fuzzy number.  $\tilde{A}$  will be positive (negative) and denoted by  $\tilde{A} > 0$  ( $\tilde{A} < 0$ ) if each element of  $\tilde{A}$  be positive (negative).  $\tilde{A}$  will be non-positive (non-negative) and denoted by  $\tilde{A} \leq 0$  ( $\tilde{A} \geq 0$ ) if each element of  $\tilde{A}$  is non-positive (non-negative). We may represent  $n \times m$  fuzzy matrix  $\tilde{A} = (\tilde{a}_{ij})_{n \times m}$ , where  $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij})$  and  $a_{ij}, b_{ij}, c_{ij} \in R$ .

**Definition 2.11.** Let  $\tilde{A} = (\tilde{a}_{ij})$  and  $\tilde{B} = (\tilde{b}_{ij})$  be two  $m \times n$  and  $n \times p$  fuzzy matrices. We define  $\tilde{A} \otimes \tilde{B}$  which is the  $m \times p$  matrix  $\tilde{C} = (\tilde{c}_{ij})$  where

$$\tilde{c}_{ij} = \sum_{k=1}^n \tilde{a}_{ik} \otimes \tilde{b}_{kj} \quad \forall i = 1, 2, \dots, m; j = 1, 2, \dots, p.$$

### 2.1.2. Arithmetic operations

In this section, some arithmetic operations between triangular fuzzy numbers are presented. For two triangular fuzzy numbers  $\tilde{M} = (m, \alpha, \beta)$  and  $\tilde{N} = (n, \gamma, \delta)$  the formula

for extended addition, extended subtraction and extended multiplication are summarized as follows:

(i) **Addition**

$$(m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta)$$

also

$$-\tilde{M} = -(m, \alpha, \beta) = (-m, \beta, \alpha)$$

(ii) **Subtraction**

$$\tilde{M} \ominus \tilde{N} = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m - n, \alpha + \delta, \beta + \gamma)$$

(iii) **Multiplication** The approximate formulas for the extended multiplication of two triangular fuzzy numbers can be summarized as follows:

If  $\tilde{M} > 0$  and  $\tilde{N} > 0$  and when spreads are negligible then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta)$$

If  $\tilde{M} < 0$  and  $\tilde{N} > 0$  and when spreads are negligible then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha - m\delta, n\beta - m\gamma)$$

If  $\tilde{M} > 0$  and  $\tilde{N} > 0$  and when spreads are not negligible then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha - \alpha\gamma, m\delta + n\beta + \beta\delta)$$

If  $\tilde{M} < 0$  and  $\tilde{N} > 0$  and when spreads are not negligible then

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha - m\delta + \alpha\delta, n\beta - m\gamma - \beta\gamma)$$

In general for two triangular fuzzy  $\tilde{M}, \tilde{N}$  the extended multiplication operation is defined as follows:

$\tilde{M} = (m, \alpha, \beta)$  and  $\tilde{N} = (n, \gamma, \delta)$  as:

$$\tilde{M} \otimes \tilde{N} = \begin{cases} (mn, f_1, f_2) & \text{if } (m, \alpha, \beta) \geq 0 \\ (mn, f_3, f_4) & \text{if } (m, \alpha, \beta) \leq 0 \\ (mn, f_5, f_6) & (m, \alpha, \beta) \equiv 0 \end{cases}$$

$$f_1 = mn - \min((m - \alpha)(n - \gamma), (m + \beta)(n - \gamma)),$$

$$f_2 = \max((m - \alpha)(n + \delta), (m + \beta)(n + \delta)) - mn$$

$$f_3 = mn - \min((m - \alpha)(n + \delta), (m + \beta)(n + \delta))$$

$$f_4 = \max((m - \alpha)(n - \gamma), (m + \beta)(n - \gamma)) - mn$$

$$f_5 = mn - \min((m - \alpha)(n + \delta), (m + \beta)(n - \gamma))$$

$$f_6 = \max((m - \alpha)(n - \gamma), (m + \beta)(n + \delta)) - mn$$

$$\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|, \max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|.$$

#### (iv) *Scalar multiplication*

$$\lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda \alpha, \lambda \beta), & \lambda \geq 0 \\ (\lambda m, -\lambda \beta, -\lambda \alpha), & \lambda < 0 \end{cases}$$

## 2.2. Limitations of existing methods

The existing methods [3, 5, 9, 10, 25-27, 31, 32, 46, 48-50, 52, 54, 57, 61, 63, 64, 72, 76,78,79, 82-85, 89, 100, 101,106, 107, 109, 110, 111, 114, 116-118, 126, 127, 134] can be used only to find the solution of fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \quad \forall i = 1, 2, \dots, n$  where  $\tilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij})$ ,  $\tilde{x}_j = (x_j, y_j, z_j)$ ,  $\tilde{b}_i = (b_i, g_i, h_i)$  are non-negative triangular fuzzy numbers.

However, none of these existing methods can be used to find the solution of the following fully fuzzy linear system.

Fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \quad \forall i = 1, 2, \dots, n$ , where  $\tilde{x}_j = (x_j, y_j, z_j)$  is non-negative triangular fuzzy number and  $\tilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij}), \tilde{b}_i = (b_i, g_i, h_i)$  are unrestricted triangular fuzzy numbers. For example the following fully fuzzy linear systems, chosen in Example 2.1, Example 2.2 cannot be solved by any of the above mentioned existing methods.

**Example 2.1.**

$$(4,6,1) \otimes (x_1, y_1, z_1) \oplus (4,2,4) \otimes (x_2, y_2, z_2) = (24,26,21)$$

$$(3,2,1) \otimes (x_1, y_1, z_1) \oplus (2,3,1) \otimes (x_2, y_2, z_2) = (14,18,13)$$

$$(x_1, y_1, z_1) \geq 0, (x_2, y_2, z_2) \geq 0.$$

**Example 2.2.**

$$(-1,2,3) \otimes (x_1, y_1, z_1) \oplus (2,3,2) \otimes (x_2, y_2, z_2) = (3,18,17).$$

$$(-4,1,10) \otimes (x_1, y_1, z_1) \oplus (-1,1,2) \otimes (x_2, y_2, z_2) = (-6,20,33)$$

$$(x_1, y_1, z_1) \geq 0, (x_2, y_2, z_2) \geq 0.$$

**2.3. Proposed Mehar method-I**

In this section, to overcome the limitations of the existing methods discussed in Section 2.2., a new method (named as Mehar method-I) is proposed to find the solution of fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \quad \forall i = 1, 2, \dots, n$ , where  $\tilde{x}_j$  is non-negative triangular fuzzy numbers and  $\tilde{a}_{ij}, \tilde{b}_i$  are unrestricted triangular fuzzy numbers.

The steps of the proposed Mehar method-I are as follows:

**Step 1.** Assuming  $\tilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij})$ ,  $\tilde{x}_j = (x_j, y_j, z_j)$  and  $\tilde{b}_i = (b_i, g_i, h_i)$ , the fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \quad \forall i = 1, 2, \dots, n$ , can be transformed into the following fully fuzzy linear system.

$$\sum_{j=1}^n (m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, n.$$

**Step 2.** Assuming  $(m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (f_{ij}, p_{ij}, q_{ij})$ , the fully fuzzy linear system, obtained in Step 1, can be transformed into the following fully fuzzy linear system.

$$\sum_{j=1}^n (f_{ij}, p_{ij}, q_{ij}) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, n.$$

**Step 3.** Using arithmetic operations

$$\sum_{j=1}^n (m_j, \alpha_j, \beta_j) = \left( \sum_{j=1}^n m_j, \sum_{j=1}^n \alpha_j, \sum_{j=1}^n \beta_j \right)$$

the fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$\left( \sum_{j=1}^n f_{ij}, \sum_{j=1}^n p_{ij}, \sum_{j=1}^n q_{ij} \right) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, n.$$

**Step 4.** Using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \implies m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ , the fully fuzzy linear system, obtained in Step 3, can be transformed into crisp system of linear equations.

$$\sum_{j=1}^n f_{ij} = b_i \quad \forall i = 1, 2, \dots, n$$

$$\sum_{j=1}^n p_{ij} = g_i \quad \forall i = 1, 2, \dots, n$$

$$\sum_{j=i}^n q_{ij} = h_i \quad \forall i = 1, 2, \dots, n.$$

**Step 5.** Use any appropriate existing method to find the solution  $\{x_j, y_j, z_j\}$  of the crisp system of linear equations obtained in Step 4.

**Step 6.** Using the solution  $\{x_j, y_j, z_j\}$ , obtained in Step 5, the fuzzy solution of fully fuzzy linear system is  $(x_j, y_j, z_j)$ .

## 2.4. Solution of chosen fully fuzzy linear systems

In Section 2.2, some fully fuzzy linear systems are chosen which cannot be solved by the existing methods. In this section, the solution of these chosen fully fuzzy linear system are obtained by using the Mehar method-I as proposed in Section 2.3.

### 2.4.1. Solution of fully fuzzy linear system chosen in Example 2.1

Using the Mehar method-I, proposed in Section 2.3., the solution of fully fuzzy linear system, chosen in Example 2.1., can be obtained as follows.

**Step 1.** Using the multiplication of an unrestricted triangular fuzzy number with a non-negative triangular fuzzy number, defined in Section 2.1.2, the fully fuzzy linear system, chosen in Example 2.1, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & (4x_1, 4x_1 - \min(-2(x_1 - y_1), -2(x_1 + z_1)), \max(5(x_1 - y_1), 5(x_1 + z_1)) - 4x_1) \oplus \\ & (4x_2, 4x_2 - \min(2(x_2 - y_2), 2(x_2 + z_2)), \max(8(x_2 - y_2), 8(x_2 + z_2)) - 4x_2) \\ & = (24, 26, 21) \end{aligned}$$

$$(3x_1, 3x_1 - \min((x_1 - y_1), (x_1 + z_1)), \max(4(x_1 - y_1), 4(x_1 + z_1)) - 3x_1) \oplus$$

$$(2x_2, 2x_2 - \min(-1(x_2 - y_2), -1(x_2 + z_2)), \max(4(x_2 - y_2), 4(x_2 + z_2)) - 2x_2)$$

$$= (14, 18, 13)$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$  the fully fuzzy linear system, obtained in Step 1, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & \left(4x_1, 4x_1 - \left(\left(\frac{-2x_1 + 2y_1 - 2x_1 - 2z_1}{2}\right) - \left|\frac{-2x_1 + 2y_1 + 2x_1 + 2z_1}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{5x_1 - 5y_1 + 5x_1 + 5z_1}{2}\right) + \left|\frac{5x_1 - 5y_1 - 5x_1 - 5z_1}{2}\right|\right) - 4x_1\right) \\ & \oplus \left(4x_2, 4x_2 - \left(\left(\frac{2x_2 - 2y_2 + 2x_2 + 2z_2}{2}\right) - \left|\frac{2x_2 - 2y_2 - 2x_2 - 2z_2}{2}\right|\right), \right. \\ & \quad \left.-4x_2\right) = (24, 26, 21) \end{aligned}$$

$$\begin{aligned} & \left(3x_1, 3x_1 - \left(\left(\frac{x_1 - y_1 + x_1 + z_1}{2}\right) - \left|\frac{x_1 - y_1 - x_1 - z_1}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{4x_1 - 4y_1 + 4x_1 + 4z_1}{2}\right) + \left|\frac{4x_1 - 4y_1 - 4x_1 - 4z_1}{2}\right|\right) - 3x_1\right) \\ & \oplus \left(2x_2, 2x_2 - \left(\left(\frac{-x_2 + y_2 - x_2 - z_2}{2}\right) - \left|\frac{-x_2 + y_2 + x_2 + z_2}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{4x_2 - 4y_2 + 4x_2 + 4z_2}{2}\right) + \left|\frac{4x_2 - 4y_2 - 4x_2 - 4z_2}{2}\right|\right) - 2x_2\right) \\ & = (14, 18, 13) \end{aligned}$$

**Step 3.** Since  $x_1, y_1, z_1, x_2, y_2, z_2 \geq 0$  so fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$(4x_1, 4x_1 + 2(x_1 + z_1), 5(x_1 + z_1) - 4x_1) \oplus (4x_2, 4x_2 - 2(x_2 - y_2), 8(x_2 + z_2) - 4x_2) \\ = (24, 26, 21)$$

$$(3x_1, 3x_1 - (x_1 - y_1), 4(x_1 + z_1) - 3x_1) \oplus (2x_2, 2x_2 + 2(x_2 + z_2), 3(x_2 + z_2) - 2x_2) \\ = (14, 18, 13)$$

which can be further written as

$$(4x_1, 6x_1 + 2z_1, x_1 + 5z_1) \oplus (4x_2, 2x_2 + 2y_2, 4x_2 + 8z_2) = (24, 26, 21)$$

$$(3x_1, 2x_1 + y_1, x_1 + 4z_1) \oplus (2x_2, 4x_2 + 2z_2, x_2 + 3z_2) = (14, 18, 13)$$

**Step 4.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the fully fuzzy linear system, obtained in Step 3, can be transformed into the following fully fuzzy linear system.

$$(4x_1 + 4x_2, 6x_1 + 2z_1 + 2x_2 + 2y_2, x_1 + 5z_1 + 4x_2 + 8z_2) = (24, 26, 21)$$

$$(3x_1 + 2x_2, 2x_1 + y_1 + 4x_2 + 2z_2, x_1 + 4z_1 + x_2 + 3z_2) = (14, 18, 13)$$

**Step 5.** Using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \Rightarrow m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ ,

the fully fuzzy linear system can be transformed into the following crisp system of linear equations.

$$4x_1 + 4x_2 = 24$$

$$3x_1 + 2x_2 = 14$$

$$6x_1 + 2z_1 + 2x_2 + 2y_2 = 26$$

$$2x_1 + y_1 + 4x_2 + 2z_2 = 18$$

$$x_1 + 5z_1 + 4x_2 + 8z_2 = 21$$

$$x_1 + 4z_1 + x_2 + 3z_2 = 13$$

**Step 6.** The solution of crisp system linear system of equations, obtained in Step 5, is as follows.

$$x_1 = 2, y_1 = 1, z_1 = 1, x_2 = 4, y_2 = 2, z_2 = 1$$

**Step 7.** Putting the values of  $x_1, y_1, z_1, x_2, y_2, z_2$  in  $\tilde{x}_1 = (x_1, y_1, z_1)$  and  $\tilde{x}_2 = (x_2, y_2, z_2)$ , the fuzzy solution of chosen fully fuzzy linear system is

$$\tilde{x}_1 = (2,1,1) \text{ and } \tilde{x}_2 = (4,2,1).$$

#### 2.4.2. Solution of fully fuzzy linear system chosen in Example 2.2.

Using the Mehar method-I, proposed in Section 2.3., the solution of fully fuzzy linear system, chosen in Example 2.2., can be obtained as follows.

**Step 1.** Using the multiplication of an unrestricted triangular fuzzy number with a non-negative triangular fuzzy number, defined in Section 2.1.2, the fully fuzzy linear system, chosen in Example 2.3 can be transformed into the following fully fuzzy linear system.

$$(-x_1, -x_1 - \min(-3(x_1 - y_1), -3(x_1 + z_1)), \max(2(x_1 - y_1), 2(x_1 + z_1)) + x_1) \oplus$$

$$(2x_2, 2x_2 - \min(-1(x_2 - y_2), -1(x_2 + z_2)), \max(4(x_2 - y_2), 4(x_2 + z_2)) - 2x_2)$$

$$= (3,18,17)$$

$$(-4x_1, -4x_1 - \min(-5(x_1 - y_1), -5(x_1 + z_1)), \max(6(x_1 - y_1), 6(x_1 + z_1)) + 4x_1)$$

$$\oplus$$

$$(-x_2, -x_2 - \min(-2(x_2 - y_2), -2(x_2 + z_2)), \max((x_2 - y_2), (x_2 + z_2)) + x_2)$$

$$= (-6,20,33)$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$  the fully fuzzy linear system, obtained in Step 1, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & \left( -x_1, -x_1 - \left( \left( \frac{-3x_1 + 3y_1 - 3x_1 - 3z_1}{2} \right) - \left| \frac{-3x_1 + 3y_1 + 3x_1 + 3z_1}{2} \right| \right) \right), \\ & \left( \left( \frac{2x_1 - 2y_1 + 2x_1 + 2z_1}{2} \right) + \left| \frac{2x_1 - 2y_1 - 2x_1 - 2z_1}{2} \right| \right) + x_1 \Big) \\ & \oplus \left( 2x_2, 2x_2 - \left( \left( \frac{-x_2 + y_2 - x_2 - z_2}{2} \right) - \left| \frac{-x_2 + y_2 + x_2 + z_2}{2} \right| \right) \right), \\ & \left( \left( \frac{4x_2 - 4y_2 + 4x_2 + 4z_2}{2} \right) + \left| \frac{4x_2 - 4y_2 - 4x_2 - 4z_2}{2} \right| \right) - 2x_2 \Big) \\ & = (3, 18, 17) \end{aligned}$$

$$\begin{aligned} & \left( -4x_1, -4x_1 - \left( \left( \frac{-5x_1 + 5y_1 - 5x_1 - 5z_1}{2} \right) - \left| \frac{-5x_1 + 5y_1 + 5x_1 + 5z_1}{2} \right| \right) \right), \\ & \left( \left( \frac{6x_1 - 6y_1 + 6x_1 + 6z_1}{2} \right) + \left| \frac{6x_1 - 6y_1 - 6x_1 - 6z_1}{2} \right| \right) + 4x_1 \Big) \\ & \oplus \left( -x_2, -x_2 \right. \\ & \left. - \left( \left( \frac{-2x_2 + 2y_2 - 2x_2 - 2z_2}{2} \right) - \left| \frac{-2x_2 + 2y_2 + 2x_2 + 2z_2}{2} \right| \right) \right), \\ & \left( \left( \frac{x_2 - y_2 + x_2 + z_2}{2} \right) + \left| \frac{x_2 - y_2 - x_2 - z_2}{2} \right| \right) + x_2 \Big) = (-6, 20, 33) \end{aligned}$$

**Step 3.** Since  $x_1, y_1, z_1, x_2, y_2, z_2 \geq 0$  so fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & ((-x_1, -x_1 + 3(x_1 + z_1), 2(x_1 + z_1) + x_1)) \oplus (2x_2, 2x_2 + (x_2 + z_2), 4(x_2 + z_2) - 2x_2) \\ & = (3, 18, 17) \end{aligned}$$

$$\begin{aligned} &((-4x_1, -4x_1 + 5(x_1 + z_1), 6(x_1 + z_1) + 4x_1)) \\ &\oplus (-x_2, -x_2 + 2(x_2 + z_2), (x_2 + z_2) + x_2) = (-6, 20, 33) \end{aligned}$$

which can be further written as

$$(-x_1, 2x_1 + 3z_1, 3x_1 + 2z_1) \oplus (2x_2, 3x_2 + z_2, 2x_2 + 4z_2) = (3, 18, 17)$$

$$(-4x_1, x_1 + 5z_1, 10x_1 + 6z_1) \oplus (-x_2, x_2 + 2z_2, 2x_2 + z_2) = (-6, 20, 33)$$

**Step 4.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the fully fuzzy linear system, obtained in Step 3, can be transformed into the following fully fuzzy linear system.

$$(-x_1 + 2x_2, 2x_1 + 3z_1 + 3x_2 + z_2, 3x_1 + 2z_1 + 2x_2 + 4z_2) = (3, 18, 17)$$

$$(-4x_1 - x_2, x_1 + 5z_1 + x_2 + 2z_2, 10x_1 + 6z_1 + 2x_2 + z_2) = (-6, 20, 33)$$

**Step 5.** Using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \Rightarrow m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ , the fully fuzzy linear system can be transformed into the following crisp system of linear equations.

$$-x_1 + 2x_2 = 3$$

$$-4x_1 - x_2 = -6$$

$$2x_1 + 3x_2 + 3z_1 + z_2 = 18$$

$$x_1 + x_2 + 5z_1 + 2z_2 = 20$$

$$3x_1 + 2x_2 + 2z_1 + 4z_2 = 17$$

$$10x_1 + 2x_2 + 6z_1 + z_2 = 33$$

**Step 6.** The solution of crisp system of linear system of equations obtained in Step 5, is

$$x_1 = 1, x_2 = 2, \text{ and } z_1 = 3, z_2 = 1.$$

**Step 7.** Further, this fully fuzzy linear system has infinite feasible solutions which may be represented as follows

$$\tilde{x} = \{\tilde{x}_1 = (1, \alpha, 3); \tilde{x}_2 = (2, \beta, 1) | \alpha \in [0,1], \beta \in [0,2]\}.$$

## 2.5. Need of linear programming method to solve the crisp system of linear equations obtained from fully fuzzy linear system

It obvious from Step 6, of the method proposed in Section 2.3., that the fuzzy solution  $(x_j, y_j, z_j)$  of the fully fuzzy linear system will be obtained with the help of crisp solution  $\{x_j, y_j, z_j\}$  of the crisp system of linear equations obtained in Step 5.

Since, in the fully fuzzy linear system  $(x_j, y_j, z_j)$  is representing a non-negative triangular fuzzy number. So, for the solution  $\{x_j, y_j, z_j\}$  of crisp system of linear equations obtained in Step 4, the conditions  $x_j - y_j \geq 0$  and  $y_j \geq 0, z_j \geq 0$  should always be satisfied. i.e., there is need to solve the crisp system of linear equations, obtained in Step 4 of the proposed method with the restrictions  $x_j - y_j \geq 0$  and  $y_j \geq 0, z_j \geq 0$ . The only way to solve the crisp system of linear equations, obtained in Step 4 of the proposed method with the restrictions  $x_j - y_j \geq 0$  and  $y_j \geq 0, z_j \geq 0$ , is linear programming method.

The solution of the crisp system of linear equations, obtained in Step 4, of the proposed method with the restrictions, can be obtained by solving following linear programming problem:

$$\text{Minimize } \bar{Z} = \sum_{i=1}^m Z_i + \sum_{i=1}^m Z'_i + \sum_{i=1}^m Z''_i$$

subject to

$$\sum_{j=1}^n f_{ij} + Z_i = b_i \quad \forall i = 1, 2, \dots, m$$

$$\sum_{j=1}^n p_{ij} + Z'_i = g_i \quad \forall i = 1, 2, \dots, m$$

$$\sum_{j=i}^n q_{ij} + Z''_i = h_i \quad \forall i = 1, 2, \dots, m$$

$$x_j - y_j \geq 0 \quad \forall j = 1, 2, \dots, m$$

$$Z_j, Z'_j, Z''_j \geq 0 \quad \forall j = 1, 2, \dots, m.$$

**Remark 2.1.** The fully fuzzy linear system will be consistent i.e., solution of fully fuzzy linear system will exist if and only if the linear programming problem is consistent. Further, the necessary condition for the above linear programming problem to generate a feasible solution (if it is consistent) is that the value of the objective function  $\bar{Z} = 0$  otherwise the solution will be termed as an infeasible solution.

**Remark 2.2.** The nature of the solutions of the fully fuzzy linear system depends upon the nature of the solutions of the linear programming which may be unique, trivial or infinitely many i.e., the fully fuzzy linear system may yield no solution, unique solution or infinitely many solutions depending upon the nature of the linear programming problem.

**Example 2.3.** Solve the following fully fuzzy linear system by the proposed method

$$(3,5,1)\tilde{x}_1 \oplus (5,1,1)\tilde{x}_2 \oplus (-4,1,2)\tilde{x}_3 = (10,20,26)$$

$$(2,1,1)\tilde{x}_1 \oplus (3,1,1)\tilde{x}_2 \oplus (2,0,2)\tilde{x}_3 = (24,9,21)$$

$$(-2,1,1)\tilde{x}_1 \oplus (-4,0,3)\tilde{x}_2 \oplus (4,2,1)\tilde{x}_3 = (-4,19,19)$$

where  $\tilde{x}_1 = (x_1, y_1, z_1), \tilde{x}_2 = (x_2, y_2, z_2), \tilde{x}_3 = (x_3, y_3, z_3) \geq 0$

**Solution:** Using the proposed method, the solution of fully fuzzy linear system, chosen in Example 2.3, can be obtained as follows:

**Step 1.** Using the multiplication of an unrestricted triangular fuzzy number with a non-negative triangular fuzzy number, defined in Section 2.1.2, the fully fuzzy linear system, chosen in Example 2.3 can be transformed into the following fully fuzzy linear system:

$$(3x_1, 3x_1 - \min(-2(x_1 - y_1), -2(x_1 + z_1)), \max(4(x_1 - y_1), 4(x_1 + z_1)) - 3x_1) \oplus$$

$$(5x_2, 5x_2 - \min(4(x_2 - y_2), 4(x_2 + z_2)), \max(6(x_2 - y_2), 6(x_2 + z_2)) - 5x_2) \oplus$$

$$(-4x_3, -4x_3 - \min(-5(x_3 - y_3), -5(x_3 + z_3)), \max(-2(x_3 - y_3), -2(x_3 + z_3)) - 4x_3) = (10, 20, 26)$$

$$(2x_1, 2x_1 - \min((x_1 - y_1), (x_1 + z_1)), \max(3(x_1 - y_1), 3(x_1 + z_1)) - 2x_1) \oplus$$

$$(3x_2, 3x_2 - \min(2(x_2 - y_2), 2(x_2 + z_2)), \max(4(x_2 - y_2), 4(x_2 + z_2)) - 3x_2) \oplus$$

$$(2x_3, 2x_3 - \min(2(x_3 - y_3), 2(x_3 + z_3)), \max(4(x_3 - y_3), 4(x_3 + z_3)) - 2x_3) = (24, 9, 21)$$

$$(-2x_1, -2x_1 - \min(-3(x_1 - y_1), -3(x_1 + z_1)), \max(-1(x_1 - y_1), -1(x_1 + z_1)) + 2x_1) \oplus$$

$$(-4x_2, -4x_2 - \min(-4(x_2 - y_2), -4(x_2 + z_2)), \max(-1(x_2 - y_2), -1(x_2 + z_2)) + 4x_2) \oplus$$

$$(4x_3, 4x_3 - \min(2(x_3 - y_3), 2(x_3 + z_3)), \max(5(x_3 - y_3), 5(x_3 + z_3)) - 4x_3) = (-4, 19, 19)$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$  the fully fuzzy linear system, obtained in Step 1, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned}
& \left( 3x_1, 3x_1 - \left( \left( \frac{-2x_1 + 2y_1 - 2x_1 - 2z_1}{2} \right) - \left| \frac{-2x_1 + 2y_1 + 2x_1 + 2z_1}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{4x_1 - 4y_1 + 4x_1 + 4z_1}{2} \right) + \left| \frac{4x_1 - 4y_1 - 4x_1 - 4z_1}{2} \right| \right) - 3x_1 \right) \oplus \\
& \left( 5x_2, 5x_2 - \left( \left( \frac{4x_2 - 4y_2 + 4x_2 + 4z_2}{2} \right) - \left| \frac{4x_2 - 4y_2 - 4x_2 - 4z_2}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{6x_2 - 6y_2 + 6x_2 + 6z_2}{2} \right) - \left| \frac{6x_2 - 6y_2 - 6x_2 - 6z_2}{2} \right| \right) - 5x_2 \right) \\
& \oplus \left( -4x_3, -4x_3 - \left( \left( \frac{-5x_3 + 5y_3 - 5x_3 - 5z_3}{2} \right) - \left| \frac{-5x_3 + 5y_3 + 5x_3 + 5z_3}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{-2x_3 + 2y_3 - 2x_3 - 2z_3}{2} \right) - \left| \frac{-2x_3 + 2y_3 + 2x_3 + 2z_3}{2} \right| \right) - 4x_3 \right) \\
& = (10, 20, 26) \\
& \left( 2x_1, 2x_1 - \left( \left( \frac{x_1 - y_1 + x_1 + z_1}{2} \right) - \left| \frac{x_1 - y_1 + x_1 + z_1}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{3x_1 - 3y_1 + 3x_1 + 3z_1}{2} \right) + \left| \frac{3x_1 - 3y_1 - 3x_1 - 3z_1}{2} \right| \right) - 2x_1 \right) \oplus \\
& \left( 3x_2, 3x_2 - \left( \left( \frac{2x_2 - 2y_2 + 2x_2 + 2z_2}{2} \right) - \left| \frac{2x_2 - 2y_2 - 2x_2 - 2z_2}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{4x_2 - 4y_2 + 4x_2 + 4z_2}{2} \right) - \left| \frac{4x_2 - 4y_2 - 4x_2 - 4z_2}{2} \right| \right) - 3x_2 \right)
\end{aligned}$$

$$\begin{aligned} & \oplus \left( 2x_3, 2x_3 - \left( \left( \frac{2x_3 + 2y_3 + 2x_3 + 2z_3}{2} \right) - \left| \left( \frac{2x_3 + 2y_3 - 2x_3 - 2z_3}{2} \right) \right| \right), \right. \\ & \quad \left. \left( \left( \frac{4x_3 - 4y_3 + 4x_3 + 4z_3}{2} \right) - \left| \left( \frac{4x_3 - 4y_3 - 4x_3 - 4z_3}{2} \right) \right| \right) - 2x_3 \right) \\ & = (24,9,21) \end{aligned}$$

$$\begin{aligned} & \left( -2x_1, -2x_1 - \left( \left( \frac{-3x_1 + 3y_1 - 3x_1 - 3z_1}{2} \right) - \left| \frac{-3x_1 + 3y_1 + 3x_1 + 3z_1}{2} \right| \right), \right. \\ & \quad \left. \left( \left( \frac{-x_1 + y_1 - x_1 - z_1}{2} \right) + \left| \frac{x_1 - y_1 + x_1 + z_1}{2} \right| \right) + 2x_1 \right) \oplus \end{aligned}$$

$$\begin{aligned} & \left( -4x_2, -4x_2 - \left( \left( \frac{-4x_2 + 4y_2 - 4x_2 - 4z_2}{2} \right) - \left| \frac{-4x_2 + 4y_2 + 4x_2 + 4z_2}{2} \right| \right), \right. \\ & \quad \left. \left( \left( \frac{-x_2 + y_2 - x_2 - z_2}{2} \right) - \left| \frac{x_2 - y_2 + x_2 + z_2}{2} \right| \right) + 4x_2 \right) \end{aligned}$$

$$\begin{aligned} & \oplus \left( 4x_3, 4x_3 - \left( \left( \frac{2x_3 - 2y_3 + 2x_3 + 2z_3}{2} \right) - \left| \left( \frac{2x_3 - 2y_3 - 2x_3 - 2z_3}{2} \right) \right| \right), \right. \\ & \quad \left. \left( \left( \frac{5x_3 - 5y_3 + 5x_3 + 5z_3}{2} \right) - \left| \left( \frac{5x_3 - 5y_3 - 5x_3 - 5z_3}{2} \right) \right| \right) - 4x_3 \right) \\ & = (-4,19,19) \end{aligned}$$

**Step 3.** Since  $x_1, y_1, z_1, x_2, y_2, z_2 \geq 0$  so fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & (3x_1, 3x_1 + 2(x_1 + z_1), 4(x_1 + z_1) - 3x_1) \oplus (5x_2, 5x_2 - 4(x_2 - y_2), 6(x_2 + z_2) - 5x_2) \\ & \oplus (-4x_3, -4x_3 + 5(x_3 + z_3), -2(x_3 - y_3) - 4x_3) = (10,20,26) \end{aligned}$$

$$\begin{aligned} & (2x_1, 2x_1 - (x_1 - y_1), 3(x_1 + z_1) - 2x_1) \oplus (3x_2, 3x_2 - 2(x_2 - y_2), 4(x_2 + z_2) - 3x_2) \oplus \\ & (2x_3, 2x_3 - 2(x_3 - y_3), 4(x_3 + z_3) - 2x_3) = (24,9,21) \end{aligned}$$

$$(-2x_1, -2x_1 + 3(x_1 + z_1), 2x_1 - (x_1 - y_1))$$

$$\oplus (-4x_2, -4x_2 + 4(x_2 + z_2), 4x_2 - (x_2 - y_2))$$

$$\oplus (4x_3, 4x_3 - 2(x_3 - y_3), 5(x_3 + z_3) - 4x_3) = (-4, 19, 19)$$

**Step 4.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the fully fuzzy linear system, obtained in Step 3, can be transformed into the following fully fuzzy linear system.

$$3x_1 + 5x_2 - 4x_3 = 10$$

$$3x_1 + 2(x_1 + z_1) + 5x_2 - 4(x_2 - y_2) - 4x_3 + 5(x_3 + z_3) = 20$$

$$4(x_1 + z_1) - 3x_1 + 6(x_2 + z_2) - 5x_2 - 2(x_3 - y_3) - 4x_3 = 26$$

$$2x_1 + 3x_2 + 2x_3 = 24$$

$$2x_1 - (x_1 - y_1) + 3x_2 - 2(x_2 - y_2) + 2x_3 - 2(x_3 - y_3) = 9$$

$$3(x_1 + z_1) - 2x_1 + 4(x_2 + z_2) - 3x_2 + 4(x_3 + z_3) - 2x_3 = 21$$

$$-2x_1 - 4x_2 + 4x_3 = -4$$

$$-2x_1 + 3(x_1 + z_1) - 4x_2 + 4(x_2 + z_2) + 4x_3 - 2(x_3 - y_3) = 19$$

$$2x_1 - (x_1 - y_1) + 4x_2 - (x_2 - y_2) + 5(x_3 + z_3) - 4x_3 = 19$$

$$y_i - x_i \geq 0 \quad \forall i = 1, 2, 3.$$

which can be further written as:

$$3x_1 + 5x_2 - 4x_3 = 10$$

$$5x_1 + 2z_1 + x_2 + y_2 + x_3 + 5z_3 = 20$$

$$x_1 + 4z_1 + x_2 + 6z_2 + 2y_3 - 2x_3 = 26$$

$$2x_1 + 3x_2 + 2x_3 = 24$$

$$x_1 + y_1 + x_2 + 2y_2 + 2y_3 = 9$$

$$x_1 + 3z_1 + x_2 + 4z_2 + 2x_3 + 2z_3 = 21$$

$$-2x_1 - 4x_2 + 4x_3 = -4$$

$$x_1 + 3z_1 + 4z_2 + 2x_3 + 2y_3 = 19$$

$$x_1 + y_1 + 3x_2 + y_2 + x_3 + 5z_3 = 19$$

$$y_i - x_i \geq 0 \quad \forall i = 1, 2, 3.$$

**Step 5.** The solution of the system, obtained in Step 4, can be obtained by solving the following linear programming problem.

$$\text{Minimize } \bar{Z} = \sum_{i=1}^m Z_i + \sum_{i=1}^m Z'_i + \sum_{i=1}^m Z''_i$$

$$3x_1 + 5x_2 - 4x_3 + Z_1 = 10$$

$$5x_1 + 2z_1 + x_2 + y_2 + x_3 + 5z_3 + Z_2 = 20$$

$$x_1 + 4z_1 + x_2 + 6z_2 + 2y_3 - 2x_3 + Z_3 = 26$$

$$2x_1 + 3x_2 + 2x_3 + Z'_1 = 24$$

$$x_1 + y_1 + x_2 + 2y_2 + 2y_3 + Z'_2 = 9$$

$$x_1 + 3z_1 + x_2 + 4z_2 + 2x_3 + 2z_3 + Z'_3 = 21$$

$$-2x_1 - 4x_2 + 4x_3 + Z''_1 = -4$$

$$x_1 + 3z_1 + 4z_2 + 2x_3 + 2y_3 + Z_2'' = 19$$

$$x_1 + y_1 + 3x_2 + y_2 + x_3 + 5z_3 + Z_3'' = 19$$

$$y_i - x_i \geq 0 \quad \forall i = 1, 2, 3.$$

On solving the above linear programming problem, the following solution is obtained.

$$x_1 = 2, y_1 = 1, z_1 = 1, x_2 = 4, y_2 = 0, z_2 = 1, x_3 = 4, y_3 = 1, z_3 = 0 \text{ and}$$

$$Z_1, Z_2, Z_3, Z_1', Z_2', Z_3', Z_1'', Z_2'', Z_3'' = 0$$

**Step 6.** Putting the values of  $x_1, y_1, z_1, x_2, y_2, z_2$  in  $\tilde{x}_1 = (x_1, y_1, z_1)$  and  $\tilde{x}_2 = (x_2, y_2, z_2)$ , the fuzzy solution of chosen fully fuzzy linear system is  $\tilde{x}_1 = (2, 1, 1)$ ,  $\tilde{x}_2 = (4, 0, 1)$  and  $\tilde{x}_3 = (4, 1, 0)$ .

## 2.6. Results and discussion

The fuzzy solution of fully fuzzy linear systems, chosen in Example 2.1, Example 2.2 and Example 2.3, are shown in Table 2.1. It is obvious from the results, shown in Table 2.1 that the fully fuzzy linear system, chosen in Example 2.1-2.3, cannot be solved by the existing methods but can be solved by the proposed Mehar method-I.

**Table 2.1:** Comparison of results by using proposed and existing methods

Examples	Solution using proposed method	Solution using existing methods
Example 2.1	$\tilde{x}_1 = (2, 1, 1)$ $\tilde{x}_2 = (4, 2, 1)$	Not applicable
Example 2.2	$\tilde{x}_1 = (1, \alpha, 3)$ $\tilde{x}_2 = (2, \beta, 1)$ where $\alpha \in [0, 1], \beta \in [0, 2]$	Not applicable
Example 2.3	$\tilde{x}_1 = (2, 2, 1)$ $\tilde{x}_2 = (4, 0, 1)$ $\tilde{x}_3 = (4, 1, 0)$	Not applicable

## **2.7. Conclusion**

On the basis of present study, it can be concluded that the fully fuzzy linear system which can be solved by existing methods can also be solved by the proposed Mehar method-I. However, there exist several fully fuzzy linear system whose solution can be obtained by the proposed Mehar method-I but cannot be obtained by the existing methods Hence, it is better to use the proposed Mehar method-I as compared to the existing method.

## Chapter 3

# Mehar method-II for solving fully fuzzy linear system with unrestricted fuzzy variables

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The methods, proposed in Chapter 2, can be used to find the solution of such fully fuzzy linear system in which the elements of the coefficient matrix and right hand side vector are represented by unrestricted triangular fuzzy numbers whereas all the elements of the unknown vector are represented by non-negative triangular fuzzy numbers. However, the methods, proposed in Chapter 2, cannot be used to find the solution of such fully fuzzy linear system in which elements of coefficient matrix are represented by non-negative triangular fuzzy numbers whereas all the elements of unknown vector and right hand side vector are represented by unrestricted triangular fuzzy numbers. In this chapter, a new method (named as Mehar method-II) is proposed to find the solution of same type of fully fuzzy linear system. The proposed Mehar method-II is illustrated with the help of some numerical examples.

### 3.1. Limitation of existing methods and the methods proposed in previous chapter

Neither the existing methods [3, 5, 9, 10, 25-27, 31, 32, 46, 48-50, 52, 54, 57, 61, 63, 64, 72, 76,78,79, 82-85, 89, 100, 101,106, 107, 109, 110, 111, 114, 116-118, 126, 127, 134] nor the Mehar method-II, proposed in previous chapter, can be used to find the solution of fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i$  where  $\tilde{x}_j, \tilde{b}_i$  are unrestricted triangular fuzzy numbers and  $\tilde{a}_{ij}$  is a non-negative triangular fuzzy number. For example it is not possible to find the solution of following fully fuzzy linear system, chosen in Example 3.1, by using existing methods or methods proposed in Chapter 2.

### Example 3.1

$$(5,1,1) \otimes (x_1, y_1, z_1) \oplus (6,1,2) \otimes (x_2, y_2, z_2) = (8,22,12)$$

$$(3,1,4) \otimes (x_1, y_1, z_1) \oplus (5,2,1) \otimes (x_2, y_2, z_2) = (9,31,7)$$

where  $(x_1, y_1, z_1), (x_2, y_2, z_2)$  are unrestricted triangular fuzzy numbers.

### 3.2. Proposed Mehar method-II

In this section, to overcome the limitations of the existing methods and the Mehar method-I as discussed in Section 3.1., a new method (named as Mehar method-II) is proposed to find the solution of fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i$  where  $\tilde{x}_j, \tilde{b}_i$  are unrestricted triangular fuzzy numbers and  $\tilde{a}_{ij}$  is non-negative triangular fuzzy numbers.

The steps of the proposed Mehar method-II are as follows:

**Step 1.** Assuming  $\tilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij}), \tilde{x}_j = (x_j, y_j, z_j)$  and  $\tilde{b}_i = (b_i, g_i, h_i)$ , the fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i$  can be transformed into the following fully fuzzy linear system.

$$\sum_{j=1}^n (m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, m$$

**Step 2.** Assuming  $(m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (f_{ij}, p_{ij}, q_{ij})$ , the fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$\sum_{j=1}^n (f_{ij}, p_{ij}, q_{ij}) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, m$$

where

$$(f_{ij}, p_{ij}, q_{ij}) = (m_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j)$$

$$f_{ij} = m_{ij}x_j$$

$$p_{ij} = f_{ij} - \min\left((m_{ij} - \alpha_{ij})(x_j - y_j), (m_{ij} + \beta_{ij})(x_j - y_j)\right)$$

$$= f_{ij} - \frac{2m_{ij} - \alpha_{ij} + \beta_{ij}}{2}(x_j - y_j) + \left|\frac{\alpha_{ij} + \beta_{ij}}{2}\right| |x_j - y_j|$$

$$q_{ij} = \max\left((m_{ij} - \alpha_{ij})(x_j + z_j), (m_{ij} + \beta_{ij})(x_j + z_j)\right) - p_{ij}$$

$$= -p_{ij} + \frac{2m_{ij} - \alpha_{ij} + \beta_{ij}}{2}(x_j + z_j) + \left|\frac{\alpha_{ij} + \beta_{ij}}{2}\right| |x_j + z_j|$$

$$\forall i = 1, 2, \dots, m, \forall j = 1, 2, \dots, n$$

**Step 3.** Using the arithmetic operation

$$\sum_{j=1}^n (m_j, \alpha_j, \beta_j) = \left( \sum_{j=1}^n m_j, \sum_{j=1}^n \alpha_j, \sum_{j=1}^n \beta_j \right)$$

the fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$\left( \sum_{j=1}^n f_{ij}, \sum_{j=1}^n p_{ij}, \sum_{j=1}^n q_{ij} \right) = (b_i, g_i, h_i) \forall i = 1, 2, \dots, m$$

**Step 4.** Using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \Rightarrow m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ , the fully fuzzy linear system, obtained in Step 3, can be transformed into the following crisp system of non-linear equations.

$$\sum_{j=1}^n f_{ij} = b_i$$

$$\sum_{j=1}^n p_{ij} = g_i$$

$$\sum_{j=1}^n q_{ij} = h_i$$

**Step 5.** Substituting values of  $f_{ij}, p_{ij}, q_{ij}$  and assuming  $x_j - y_j = L_j$  and  $x_j + z_j = R_j \forall j = 1, 2, \dots, n$ . (where  $x_j, y_j, z_j, w_j, L_j, R_j \in R$ ), the crisp system of non-linear equations, obtained in Step 4, can be transformed into the following crisp system of non-linear equations.

For  $i = 1, 2, \dots, m$ .

$$\sum_{j=1}^n f_{ij} = b_i \Rightarrow \sum_{j=1}^n m_{ij} x_j = b_i$$

similarly,

$$\sum_{j=1}^n p_{ij} = g_i \Rightarrow \sum_{j=1}^n \frac{2m_{ij} - \alpha_{ij} + \beta_{ij}}{2} L_j - \left( \frac{\alpha_{ij} + \beta_{ij}}{2} \right) |L_j| = b_i - g_i$$

$$\sum_{j=1}^n q_{ij} = h_i \Rightarrow \sum_{j=1}^n \frac{2m_{ij} - \alpha_{ij} + \beta_{ij}}{2} R_j + \left( \frac{\alpha_{ij} + \beta_{ij}}{2} \right) |R_j| = b_i + h_i$$

**Step 6.** Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$ , where  $L'_j = \begin{cases} L_j & \text{if } L_j > 0 \\ 0 & \text{otherwise} \end{cases}$  and

$$L''_j = \begin{cases} -L_j & \text{if } L_j < 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly,  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$ , the crisp system of non-linear equations, obtained in Step 5, can be transformed into the following crisp system of linear equations

$$\sum_{j=1}^n m_{ij}x_j = b_i$$

$$\sum_{j=1}^n \frac{2m_{ij} - \alpha_{ij} + \beta_{ij}}{2} (L'_j - L''_j) - \left( \frac{\alpha_{ij} + \beta_{ij}}{2} \right) (L'_j + L''_j) = b_i - g_i$$

$$\sum_{j=1}^n \frac{2\alpha_{ij} - \alpha_{ij} + \beta_{ij}}{2} (R'_j - R''_j) + \left( \frac{\alpha_{ij} + \beta_{ij}}{2} \right) (R'_j + R''_j) = b_i + h_i$$

$$L'_j, L''_j, R'_j, R''_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

**Step 7.** The solution of system obtained in Step 5, can be obtained by solving the following linear programming problem, which is obtained by adding the artificial variable  $Z_i, \forall i = 1, 2, \dots, m$ .

$$\text{Minimize } \bar{Z} = \sum_{i=1}^m Z_i + \sum_{i=1}^m Z'_i + \sum_{i=1}^m Z''_i$$

subject to

$$\sum_{j=1}^n m_{ij}x_j + Z_i = b_i \quad \forall i = 1, 2, \dots, m$$

$$\sum_{j=1}^n \left( (m_{ij} - \alpha_{ij})L'_j - (m_{ij} + \beta_{ij})L''_j \right) + Z'_i = b_i - g_i \quad \forall i = 1, 2, \dots, m$$

$$\sum_{j=1}^n \left( (m_{ij} + \beta_{ij})R'_j - (m_{ij} - \alpha_{ij})R''_j \right) + Z''_i = b_i + h_i \quad \forall i = 1, 2, \dots, m$$

$$x_j - L'_j + L''_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$R'_j - R''_j - x_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

$$L'_j, L''_j, R'_j, R''_j, Z_j, Z'_j, Z''_j \geq 0 \quad \forall j = 1, 2, \dots, n$$

**Step 8.** Solve the linear programming problem, obtained in Step 7, and substitute the values of  $x_j, L', L'', R', R''$  obtained in Step 7 in  $\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \forall j = 1, 2, \dots, n$  to find the solution of the fully fuzzy linear system.

**Remark 3.1.** The fully fuzzy linear system will be consistent i.e., it will generate a solution if and only if the linear programming problem is consistent. Further, the necessary condition for the above linear programming problem to generate a feasible solution (if it is consistent) is that the value of the objective function  $\bar{Z} = 0$  otherwise the solution will be termed as an infeasible solution. The sufficient condition(s) for the fully fuzzy linear system to have a feasible (strong) fuzzy solution are  $L'.L''^T = R'.R''^T = \bar{Z} = 0$ .

### 3.3. Solution of chosen fully fuzzy linear system

In Section 3.1, a fully fuzzy linear system is chosen which cannot be solved by the existing methods as well as Mehar method-I. In this section, the solution of the chosen fully fuzzy linear system is obtained by the Mehar method-II proposed in Section 3.2.

#### 3.3.1. Solution of fully fuzzy linear system chosen in Example 3.1.

Using the method proposed in Section 3.2, the solution of fully fuzzy linear system chosen in Example 3.1, can be obtained as follow:

**Step 1.** Using the multiplication of an unrestricted triangular fuzzy number with a non-negative triangular fuzzy number, defined in Section 2.1.2, the fully fuzzy linear system, chosen in Example 3.1 can be transformed into the following fully fuzzy linear system:

$$(5x_1, 5x_1 - \min(4(x_1 - y_1), 6(x_1 - y_1)), \max(4(x_1 + z_1), 6(x_1 + z_1)) - 5x_1) \oplus$$

$$\begin{aligned} & (6x_2, 6x_2 - \min(5(x_2 - y_2), 8(x_2 - y_2)), \max(5(x_2 + z_2), 8(x_2 + z_2)) - 6x_2) \\ & = (8, 22, 12) \end{aligned}$$

$$\begin{aligned} & (3x_1, 3x_1 - \min(2(x_1 - y_1), 7(x_1 - y_1)), \max(2(x_1 + z_1), 7(x_1 + z_1)) - 3x_1) \oplus \\ & (5x_2, 5x_2 - \min(3(x_2 - y_2), 6(x_2 - y_2)), \max(3(x_2 + z_2), 6(x_2 + z_2)) - 5x_2) \\ & = (9, 31, 7) \end{aligned}$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$ , the fully fuzzy linear system, obtained in Step 1, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & (5x_1, 5x_1 - 5(x_1 - y_1) + |(x_1 - y_1)|, -5x_1 + 5(x_1 + z_1) + |(x_1 + z_1)|) \oplus \\ & \left(6x_2, 6x_2 - \frac{13}{2}(x_2 - y_2) + \frac{3}{2}|(x_2 - y_2)|, -6x_2 + \frac{13}{2}(x_2 + z_2) + \frac{3}{2}|(x_2 + z_2)|\right) \\ & = (8, 22, 12) \\ & \left(3x_1, 3x_1 - \frac{9}{2}(x_1 - y_1) + \frac{5}{2}|(x_1 - y_1)|, -3x_1 + \frac{9}{2}(x_1 + z_1) + \frac{5}{2}|(x_1 + z_1)|\right) \oplus \\ & \left(5x_2, 5x_2 - \frac{9}{2}(x_2 - y_2) + \frac{3}{2}|(x_2 - y_2)|, -5x_2 + \frac{9}{2}(x_2 + z_2) + \frac{3}{2}|(x_2 + z_2)|\right) \\ & = (9, 31, 7) \end{aligned}$$

**Step 3.** Substituting  $x_1 - y_1 = L_1$ ,  $x_2 - y_2 = L_2$  and  $x_1 + z_1 = R_1$ ,  $x_2 + z_2 = R_2$ , in the fully fuzzy linear system obtained in Step 2, can be transformed into the following crisp system of non-linear equations.

$$\begin{aligned} & (5x_1, 5x_1 - 5L_1 + |L_1|, -5x_1 + 5R_1 + |R_1|) \\ & \oplus \left(6x_2, 6x_2 - \frac{13}{2}L_2 + \frac{3}{2}|L_2|, -6x_2 + \frac{13}{2}R_2 + \frac{3}{2}|R_2|\right) = (8, 22, 12) \end{aligned}$$

$$\left(3x_1, 3x_1 - \frac{9}{2}L_1 + \frac{5}{2}|L_1|, -3x_1 + \frac{9}{2}R_1 + \frac{5}{2}|R_1|\right) \\ \oplus \left(5x_2, 5x_2 - \frac{9}{2}L_2 + \frac{3}{2}|L_2|, -5x_2 + \frac{9}{2}(R_2) + \frac{3}{2}|R_2|\right) = (9,31,7)$$

**Step 4.** Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$ , where  $L'_j = \begin{cases} L_j & \text{if } L_j > 0 \\ 0 & \text{otherwise} \end{cases}$  and

$L''_j = \begin{cases} -L_j & \text{if } L_j < 0 \\ 0 & \text{otherwise} \end{cases}$ . Similarly,  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$ , the crisp system of

non-linear equations, obtained in Step 3, can be transformed into the following crisp system of linear equations.

$$5x_1 + 6x_2 = 8$$

$$3x_1 + 5x_2 = 9$$

$$4L'_1 - 6L''_1 + 5L'_2 - 8L''_2 = -14$$

$$2L'_1 - 7L''_1 + 3L'_2 - 6L''_2 = -22$$

$$6R'_1 - 4R''_1 + 8R'_2 - 5R''_2 = 20$$

$$7R'_1 - 2R''_1 + 6R'_2 - 3R''_2 = 16$$

$$x_1 - L'_1 + L''_1 \geq 0$$

$$x_2 - L'_2 + L''_2 \geq 0$$

$$R'_1 - R''_1 - x_1 \geq 0$$

$$R'_2 - R''_2 - x_2 \geq 0$$

**Step 5.** The solution of system obtained in Step 4, can be obtained by solving the following linear programming problem, which is obtained by adding the artificial variable  $Z_i, \forall i = 1, 2, \dots, m$ .

$$\text{Minimize } Z = (Z_1 + Z_2 + Z'_1 + Z'_2 + Z''_1 + Z''_2)$$

subject to

$$5x_1 + 6x_2 + Z_1 = 8$$

$$3x_1 + 5x_2 + Z_2 = 9$$

$$4L'_1 - 6L''_1 + 5L'_2 - 8L''_2 + Z'_1 = -14$$

$$2L'_1 - 7L''_1 + 3L'_2 - 6L''_2 + Z'_2 = -22$$

$$6R'_1 - 4R''_1 + 8R'_2 - 5R''_2 + Z''_1 = 20$$

$$7R'_1 - 2R''_1 + 6R'_2 - 3R''_2 + Z''_2 = 16$$

$$x_1 - L'_1 + L''_1 \geq 0$$

$$x_2 - L'_2 + L''_2 \geq 0$$

$$R'_1 - R''_1 - x_1 \geq 0$$

$$R'_2 - R''_2 - x_2 \geq 0$$

$$Z_1, Z_2, Z'_1, Z'_2, Z''_1, Z''_2, L'_1, L'_2, L''_1, L''_2, R'_1, R'_2, R''_1, R''_2 \geq 0$$

**Step 5.** On solving the above linear programming problem, the following solution is obtained:

$$x_1 = -2, x_2 = 3, L'_1 = 0, L''_1 = 4, L'_2 = 2, L''_2 = 0, R'_1 = 0, R''_1 = 1, R'_2 = 3, R''_2 = 0,$$

$$Z_1 = Z_2 = Z'_1 = Z'_2 = Z''_1 = Z''_2 = 0$$

**Step 6.** Substituting the values, obtained in Step 5, in  $\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \quad \forall j = 1, 2$ . Thus,

$$\tilde{x}_1 = (-2,2,1) \text{ and } \tilde{x}_2 = (3,1,0)$$

### 3.4. Results and discussion

The solution of fully fuzzy linear system, chosen in Example 3.1, is shown in Table 3.1. It is obvious from the results, shown in Table 3.1, that the fully fuzzy linear system, chosen in Example 3.1, can neither be solved by the existing methods nor by the Mehar method-I, but can be solved by the Mehar method-II proposed in this chapter.

**Table 3.1:** Comparison of results by using proposed and existing methods

Examples	Solution using Mehar method-II	Solution using existing methods	Solution using Mehar method-I
Example3.1	$\tilde{x}_1 = (-2,2,1)$ $\tilde{x}_2 = (3,1,0)$	Not applicable	Not applicable

### 3.5. Conclusion

On the basis of present study, it can be concluded that the fully fuzzy linear system which can be solved by existing methods can also be solved by the proposed Mehar method-II. However, there exist several fully fuzzy linear system whose solution can be obtained by the proposed Mehar method-II but cannot be obtained by the existing methods. Hence, it is better to use the proposed Mehar method-II as compared to the existing methods.

## Chapter 4

# Mehar method-III for solving fully fuzzy linear system with unrestricted fuzzy coefficients and unrestricted fuzzy variables

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In the methods, proposed in previous chapters, either the elements of coefficient matrix and elements of right hand side vector or elements of unknown vector are assumed as non-negative triangular fuzzy numbers. However, in the real life problems there is need to find the solution of fully fuzzy linear system without any restriction on these elements. To overcome the limitations of the existing methods and the methods proposed in previous chapters, in this chapter, a new method (named as Mehar method-III) is proposed to find the solution of such fully fuzzy linear system in which elements of coefficient matrix, right hand side vector as well as the elements of unknown vector are represented by unrestricted triangular fuzzy numbers. The proposed Mehar method-III is illustrated by numerical examples.

### 4.1. Limitations of the existing methods and methods proposed in previous chapters

Neither the existing methods [3, 5, 9, 10, 25-27, 31, 32, 46, 48-50, 52, 54, 57, 61, 63, 64, 72, 76,78,79, 82-85, 89, 100, 101,106, 107, 109, 110, 111, 114, 116-118, 126, 127, 134] nor the Mehar method-I and Mehar method -II, proposed in previous chapters, can be used to find the solution of fully fuzzy linear system when elements of both the coefficient matrix as well as the elements of solution vector are unrestricted triangular fuzzy numbers. For example, the fully fuzzy linear system chosen in Example 4.1 and Example 4.2 can neither be solved by using the existing method nor by the Mehar method-I and Mehar method-II, proposed in previous chapters.

**Example 4.1.**

$$(3,5,1) \otimes (x_1, y_1, z_1) \oplus (2,4,1) \otimes (x_2, y_2, z_2) = (8,21,6)$$

$$(2,1,0) \otimes (x_1, y_1, z_1) \oplus (4,0,1) \otimes (x_2, y_2, z_2) = (8,22,6)$$

where  $(x_1, y_1, z_1), (x_2, y_2, z_2)$  are unrestricted triangular fuzzy numbers.

**Example 4.2.**

$$\begin{bmatrix} (2,1,1) & (-1,1,0) & (3,1,0) \\ (-2,1,1) & (4,2,1) & (4,1,0) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{bmatrix} = \begin{bmatrix} (0,5,8) \\ (-10,14,6) \end{bmatrix}$$

where  $(x_1, y_1, z_1), (x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$  are unrestricted triangular fuzzy numbers.

**4.2. Proposed Mehar method-III**

To overcome the limitations of the existing methods and the Mehar method-I and Mehar method-II, proposed in previous chapters , in this section, a new method (named as Mehar method-III) is proposed to solve the fully fuzzy linear system  $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i$  where  $\tilde{a}_{ij}, \tilde{b}_i$  and  $\tilde{x}_j$  are unrestricted triangular fuzzy numbers.

The steps of the proposed Mehar method-III are as follows:

**Step 1.** Assuming  $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}), \tilde{b}_i = (b_i, g_i, h_i)$  and  $\tilde{x}_j = (x_j, y_j, z_j)$  the fully fuzzy linear system can be transformed into the following fully fuzzy linear system.

$$\sum_{j=1}^n (a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, m$$

**Step 2.** Substituting  $(a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j) = (p_{ij}, q_{ij}, r_{ij})$ , the fully fuzzy linear system obtained in Step 1, may be written as

$$\sum_{j=1}^n (p_{ij}, q_{ij}, r_{ij}) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, m$$

where

$$(p_{ij}, q_{ij}, r_{ij}) = (a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j)$$

**Case 1:** If  $(a_{ij}, b_{ij}, c_{ij}) \geq 0$  i. e.,  $a_{ij} - b_{ij} \geq 0$

$$p_{ij} = a_{ij}x_j$$

$$q_{ij} = a_{ij}x_j - \min\left((a_{ij} - b_{ij})(x_j - y_j), (a_{ij} + c_{ij})(x_j - y_j)\right)$$

$$r_{ij} = \max\left((a_{ij} - b_{ij})(x_j + z_j), (a_{ij} + c_{ij})(x_j + z_j)\right) - a_{ij}x_j$$

This set of equation can be further reduced as follows.

$$p_{ij} = a_{ij}x_j$$

$$q_{ij} = p_{ij} - w_{ij}$$

$$w_{ij} = \min\left((a_{ij} - b_{ij})(x_j - y_j), (a_{ij} + c_{ij})(x_j - y_j)\right)$$

$$= \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(x_j - y_j) - \left|\frac{c_{ij} + b_{ij}}{2}\right| |x_j - y_j|$$

$$r_{ij} = s_{ij} - p_{ij}$$

$$s_{ij} = \max\left((a_{ij} - b_{ij})(x_j + z_j), (a_{ij} + c_{ij})(x_j + z_j)\right)$$

$$= \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(x_j + z_j) + \left|\frac{c_{ij} + b_{ij}}{2}\right| |x_j + z_j|$$

To solve this system we substitute  $L_j = x_j - y_j$  as the left coefficient and  $R_j = x_j + z_j$  as the right coefficient. This can be further reduced as follows. Define variables  $L'_j$  and  $L''_j$  as

$$L'_j = \begin{cases} L_j & \text{if } L_j > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } L''_j = \begin{cases} -L_j & \text{if } L_j < 0 \\ 0 & \text{otherwise} \end{cases}$$

Putting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$p_{ij} = a_{ij}x_j$$

$$w_{ij} = \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(L'_j - L''_j) - \left(\frac{c_{ij} + b_{ij}}{2}\right)(L'_j + L''_j)$$

$$s_{ij} = \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(R'_j - R''_j) + \left(\frac{c_{ij} + b_{ij}}{2}\right)(R'_j + R''_j)$$

**Case 2.** if  $(a_{ij}, b_{ij}, c_{ij}) < 0$  i. e.,  $a_{ij} + c_{ij} < 0$

$$p_{ij} = a_{ij}x_j$$

$$q_{ij} = a_{ij}x_j - \min\left((a_{ij} - b_{ij})(x_j + z_j), (a_{ij} + c_{ij})(x_j + z_j)\right)$$

$$r_{ij} = \max\left((a_{ij} - b_{ij})(x_j - y_j), (a_{ij} + c_{ij})(x_j - y_j)\right) - a_{ij}x_j$$

This set of equation can be similarly reduced as follows.

$$p_{ij} = a_{ij}x_j$$

$$q_{ij} = p_{ij} - w_{ij}$$

$$w_{ij} = \min\left((a_{ij} - b_{ij})(x_j + z_j), (a_{ij} + c_{ij})(x_j + z_j)\right)$$

$$= \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(x_j + z_j) - \left|\frac{c_{ij} + b_{ij}}{2}\right||x_j + z_j|$$

$$r_{ij} = s_{ij} - p_{ij}$$

$$\begin{aligned} s_{ij} &= \max \left( (a_{ij} - b_{ij})(x_j - y_j), (a_{ij} + c_{ij})(x_j - y_j) \right) \\ &= \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(x_j - y_j) + \left| \frac{c_{ij} + b_{ij}}{2} \right| |x_j - y_j| \end{aligned}$$

Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$p_{ij} = a_{ij}x_j$$

$$s_{ij} = \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(L'_j - L''_j) + \left( \frac{c_{ij} + b_{ij}}{2} \right)(L'_j + L''_j)$$

$$w_{ij} = \frac{2a_{ij} + c_{ij} - b_{ij}}{2}(R'_j - R''_j) - \left( \frac{c_{ij} + b_{ij}}{2} \right)(R'_j + R''_j)$$

**Case 3.** If  $(a_{ij}, b_{ij}, c_{ij}) \equiv 0$  i. e.,  $a_{ij} - b_{ij} < 0 < a_{ij} + c_{ij}$

$$p_{ij} = a_{ij}x_j$$

$$q_{ij} = a_{ij}x_j - \min \left( (a_{ij} - b_{ij})(x_j + z_j), (a_{ij} + c_{ij})(x_j - y_j) \right)$$

$$r_{ij} = \max \left( (a_{ij} - b_{ij})(x_j - y_j), (a_{ij} + c_{ij})(x_j + z_j) \right) - a_{ij}x_j$$

This set of equation can be similarly reduced as follows.

$$p_{ij} = a_{ij}x_j$$

$$q_{ij} = p_{ij} - w_{ij}$$

$$w_{ij} = \min \left( (a_{ij} - b_{ij})(x_j + z_j), (a_{ij} + c_{ij})(x_j - y_j) \right)$$

$$r_{ij} = s_{ij} - p_{ij}$$

$$s_{ij} = \max \left( (a_{ij} - b_{ij})(x_j - y_j), (a_{ij} + c_{ij})(x_j + z_j) \right)$$

Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j, R_j = R'_j - R''_j, |R_j| = R'_j + R''_j$ . Also let

$$(a_{ij} - b_{ij})R_j - (a_{ij} + c_{ij})L_j = u'_{ij} - u''_{ij}$$

This implies

$$|(a_{ij} - b_{ij})R_j - (a_{ij} + c_{ij})L_j| = u'_{ij} + u''_{ij}.$$

Similarly

$$(a_{ij} - b_{ij})L_j - (a_{ij} + c_{ij})R_j = v'_{ij} - v''_{ij}$$

and therefore

$$|(a_{ij} - b_{ij})L_j - (a_{ij} + c_{ij})R_j| = v'_{ij} + v''_{ij}$$

where  $L'_j, L''_j, R'_j, R''_j, u'_{ij}, u''_{ij}, v'_{ij}, v''_{ij} \geq 0$

Substituting we get:

$$p_{ij} = a_{ij}x_j$$

$$w_{ij} = \frac{(a_{ij} - b_{ij})(R'_j - R''_j) + (a_{ij} + c_{ij})(L'_j - L''_j)}{2} - \left( \frac{u'_{ij} + u''_{ij}}{2} \right)$$

$$s_{ij} = \frac{(a_{ij} + c_{ij})(R'_j - R''_j) + (a_{ij} - b_{ij})(L'_j - L''_j)}{2} + \left( \frac{v'_{ij} + v''_{ij}}{2} \right)$$

with two additional constraints:

$$(a_{ij} - b_{ij})R_j - (a_{ij} + c_{ij})L_j = u'_{ij} - u''_{ij}$$

$$(a_{ij} - b_{ij})L_j - (a_{ij} + c_{ij})R_j = v'_{ij} - v''_{ij}$$

**Step 3.** Using arithmetic operations

$$\sum_{j=1}^n (m_j, \alpha_j, \beta_j) = \left( \sum_{j=1}^n m_j, \sum_{j=1}^n \alpha_j, \sum_{j=1}^n \beta_j \right)$$

the fully fuzzy linear system, obtained in Step 2, can be transformed into the following fully fuzzy linear system.

$$\left( \sum_{j=1}^n p_{ij}, \sum_{j=1}^n q_{ij}, \sum_{j=1}^n r_{ij} \right) = (b_i, g_i, h_i) \quad \forall i = 1, 2, \dots, m$$

**Step 4.** Using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \Rightarrow m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ , the fully fuzzy linear system, obtained in Step 3, can be transformed into the following crisp system of linear equations.

$$\begin{cases} \sum_{j=1}^n p_{ij} = b_i & i = 1, 2, \dots, m \\ \sum_{j=1}^n w_{ij} = b_i - g_i & i = 1, 2, \dots, m \\ \sum_{j=i}^n s_{ij} = h_i + b_i & i = 1, 2, \dots, m \end{cases}$$

with appropriate constraints obtained from Step 2.

**Step 5.** The solution of system obtained in Step 4, can be obtained by solving the following linear programming problem, which is obtained by adding the artificial variable  $Z_i, \forall i = 1, 2, \dots, m$ .

$$\text{Minimize } \bar{Z} = \sum_{i=1}^m Z_i + \sum_{i=1}^m Z'_i + \sum_{i=1}^m Z''_i$$

subject to

$$\sum_{j=1}^n p_{ij} + Z_i = b_i \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n w_{ij} + Z'_i = b_i - g_i \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n s_{ij} + Z''_i = h_i - b_i \quad i = 1, 2, \dots, m$$

$$Z_i, Z'_i, Z''_i \geq 0 \quad \forall i = 1, 2, \dots, m$$

**Step 6.** Solve the above linear programming problem and put the values of  $L'_j, L''_j, R'_j, R''_j, x_j$  obtained in

$$\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \quad \forall j = 1, 2, \dots, n$$

to find the solution of fully fuzzy linear system. The solution would be termed as (feasible) strong fuzzy solution if  $x_j - L'_j + L''_j \geq 0$  and  $R'_j - R''_j - x_j \geq 0$ . Otherwise the solution would be termed as (infeasible) weak fuzzy solution

**Remark 4.1.** The sufficient conditions for the solution to be termed as a strong (feasible) solution for  $i = 1, 2, \dots, m, j = 1, 2, \dots, n$  are:

$$x_j - L'_j + L''_j \geq 0 \text{ and } R'_j - R''_j - x_j \geq 0$$

$$L'_j \cdot L''_j = u'_{ij} \cdot u''_{ij} = v'_{ij} \cdot v''_{ij} = R'_j \cdot R''_j = \bar{Z} = 0$$

### 4.3. Solution of chosen fully fuzzy linear systems

In Section 4.1, some fully fuzzy linear systems are chosen which cannot be solved by the existing methods. In this section, the solution of these chosen fully fuzzy linear systems are obtained by the Mehar method-III as proposed in Section 4.2.

#### 4.3.1. Solution of fully fuzzy linear system chosen in Example 4.1

Using the Mehar method-III, proposed in Section 4.2, the solution of fully fuzzy linear system chosen in Example 4.1, be obtained as follows:

**Step 1.** Using the multiplication of an unrestricted triangular fuzzy number, defined in Section 2.1.2, the fully fuzzy linear system, chosen in Example 4.1, can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & (3x_1, 3x_1 - \min(-2(x_1 + z_1), 4(x_1 - y_1)), \max(-2(x_1 - y_1), 4(x_1 + z_1)) - 3x_1) \oplus \\ & (2x_2, 2x_2 - \min(-2(x_2 + z_2), 3(x_2 - y_2)), \max(-2(x_2 - y_2), 3(x_2 + z_2)) - 2x_2) \\ & = (8, 21, 6) \end{aligned}$$

$$(2x_1, 2x_1 - \min((x_1 - y_1), 2(x_1 - y_1)), \max((x_1 + z_1), 2(x_1 + z_1)) - 2x_1) \oplus$$

$$\begin{aligned} & (4x_2, 4x_2 - \min(4(x_2 - y_2), 5(x_2 - y_2)), \max(4(x_2 + z_2), 5(x_2 + z_2)) - 4x_2) \\ & = (8, 22, 6) \end{aligned}$$

which can be further written as

$$(3x_1, 3x_1 - w_{11}, s_{11} - 3x_1) \oplus (2x_2, 2x_2 - w_{12}, s_{12} - 2x_2) = (8, 21, 6)$$

$$(2x_1, 2x_1 - w_{21}, s_{21} - 2x_1) \oplus (4x_2, 4x_2 - w_{22}, s_{22} - 4x_2) = (8, 22, 6)$$

where  $w_{11} = \min(-2(x_1 + z_1), 4(x_1 - y_1))$ ,  $s_{11} = \max(-2(x_1 - y_1), 4(x_1 + z_1))$

$$w_{12} = \min(-2(x_2 + z_2), 3(x_2 - y_2)), s_{12} = \max(-2(x_2 - y_2), 3(x_2 + z_2))$$

$$w_{21} = \min(x_1 - y_1, 2(x_1 - y_1)), s_{21} = \max(x_1 + z_1, 2(x_1 + z_1))$$

$$w_{22} = \min(4(x_2 - y_2), 5(x_2 - y_2)), s_{22} = \max(4(x_2 + z_2), 5(x_2 + z_2))$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$ ,  $x_i - y_i = L_i$  and  $x_i + z_i = R_i$  as well as  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$ ,  $R_j = R'_j - R''_j$ ,  $|R_j| = R'_j + R''_j$ , the fully fuzzy linear system, obtained in Step 1, can be written as:

$$(3x_1, 3x_1 - w_{11}, s_{11} - 3x_1) \oplus (2x_2, 2x_2 - w_{12}, s_{12} - 2x_2) = (8, 21, 6)$$

$$(2x_1, 2x_1 - w_{21}, s_{21} - 2x_1) \oplus (4x_2, 4x_2 - w_{22}, s_{22} - 4x_2) = (8, 22, 6)$$

where

$$w_{11} = \frac{-2(R'_1 - R''_1) + 4(L'_1 - L''_1)}{2} - \left(\frac{u'_{11} + u''_{11}}{2}\right)$$

$$s_{11} = \frac{4(R'_1 - R''_1) - 2(L'_1 - L''_1)}{2} + \left(\frac{v'_{11} + v''_{11}}{2}\right)$$

with two additional equations:

$$-2(R'_1 - R''_1) - 4(L'_1 - L''_1) = u'_{11} - u''_{11}$$

$$-2(L'_1 - L''_1) - 4(R'_1 - R''_1) = v'_{11} - v''_{11}$$

$$w_{12} = \frac{-2(R'_2 - R''_2) + 3(L'_2 - L''_2)}{2} - \left(\frac{u'_{12} + u''_{12}}{2}\right)$$

$$s_{12} = \frac{3(R'_2 - R''_2) - 2(L'_2 - L''_2)}{2} + \left(\frac{v'_{12} + v''_{12}}{2}\right)$$

with two additional equations:

$$-2(R'_2 - R''_2) - 3(L'_2 - L''_2) = u'_{12} - u''_{12}$$

$$-2(L'_2 - L''_2) - 3(R'_2 - R''_2) = v'_{12} - v''_{12}$$

$$w_{21} = 1.5(L'_1 - L''_1) - 0.5(L'_1 + L''_1) = L'_1 - 2L''_1$$

$$s_{21} = 1.5(R'_1 - R''_1) + 0.5(R'_1 + R''_1) = 2R'_1 - R''_1$$

$$w_{22} = 4.5(L'_2 - L''_2) - 0.5(L'_2 + L''_2) = 4L'_2 - 5L''_2$$

$$s_{22} = 4.5(R'_2 - R''_2) + 0.5(R'_2 + R''_2) = 5R'_2 - 4R''_2$$

**Step 3.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the fully fuzzy linear system, obtained in Step 2, and using the relation  $(a, b, c) = (d, e, f) \Rightarrow a = d, b = e, c = f$ , the crisp system of linear equations can be transformed into the following crisp system of linear equations.

$$3x_1 + 2x_2 = 8$$

$$2x_1 + 4x_2 = 8$$

$$w_{11} + w_{12} = 8 - 21 = -13$$

$$w_{21} + w_{22} = 8 - 22 = -14$$

$$s_{11} + s_{12} = 8 + 6 = 14$$

$$s_{21} + s_{22} = 8 + 6 = 14$$

**Step 4.** Using Phase 1 of the two-phase method, the following linear programming problem can be used to find the solution of crisp system of linear equations, obtained in Step 3.

$$\text{Minimize } \bar{Z} = Z_1 + Z_2 + Z'_1 + Z'_2 + Z''_1 + Z''_2$$

subject to

$$3x_1 + 2x_2 + Z_1 = 8$$

$$2x_1 + 4x_2 + Z_2 = 8$$

$$-R'_1 + R''_1 + 2L'_1 - 2L''_1 - 0.5u'_{11} + 0.5u''_{11} - R'_2 + R''_2$$

$$+ 1.5L'_2 - 1.5L''_2 - 0.5u'_{12} + 0.5u''_{12} + Z'_1 = -13$$

$$L'_1 - 2L''_1 + 4L'_2 - 5L''_2 + Z'_2 = -14$$

$$2R'_1 - 2R''_1 - L'_1 + L''_1 + 0.5v'_{11} + 0.5v''_{11} + 1.5R'_2 - 1.5R''_2$$

$$-L'_2 + L''_2 + 0.5v'_{12} + 0.5v''_{12} + Z''_1 = 14$$

$$2R'_1 - R''_1 + 5R'_2 - 4R''_2 + Z''_2 = 14$$

$$-2(R'_1 - R''_1) - 4(L'_1 - L''_1) - u'_{11} + u''_{11} = 0$$

$$-2(L'_1 - L''_1) - 4(R'_1 - R''_1) - v'_{11} + v''_{11} = 0$$

$$-2(R'_2 - R''_2) - 4(L'_2 - L''_2) - u'_{12} + u''_{12} = 0$$

$$-2(L'_2 - L''_2) - 4(R'_2 - R''_2) - v'_{12} + v''_{12} = 0$$

$$L'_1, L''_1, L'_2, L''_2, u'_{11}, u''_{11}, v'_{11}, v''_{11}, R'_1, R''_1, R'_2, R''_2, u'_{12}, u''_{12}, v'_{12}, v''_{12},$$

$$Z_1, Z_2, Z'_1, Z'_2, Z''_1, Z''_2 \geq 0$$

**Step 5.** On solving the above linear programming problem, we get  $x_1 = 2, x_2 = 1, L'_1 = 1, L''_1 = 0, u'_{11} = 0, u''_{11} = 8, v'_{11} = 0, v''_{11} = 10, L'_2 = 0, L''_2 = 3, R'_1 = 2, R''_1 = 0, R'_2 = 2, R''_2 = 0, u'_{12} = 5, u''_{12} = 0, v'_{12} = 0, v''_{12} = 0, Z_1 = 0, Z_2 = 0, Z'_1 = 0, Z'_2 = 0, Z''_1 = 0$  and  $Z''_2 = 0$ .

**Step 6.** Putting the values of  $L'_j, L''_j, R'_j, R''_j, x_j$  obtained from Step 6, in  $\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \forall j = 1, 2, \dots, n$

the solution of fully fuzzy linear system is  $\tilde{x}_1 = (2,1,0)$  and  $\tilde{x}_2 = (1,4,1)$ .

#### 4.3.2. Solution of fully fuzzy linear system chosen in Example 4.2.

Using the Mehar method-III, proposed in Section 4.2, the solution of fully fuzzy linear system chosen in Example 4.2, be obtained as follows:

**Step 1.** Using the multiplication of triangular fuzzy numbers as proposed in Step 2, of the proposed method the fully fuzzy linear system, chosen in Example 4.2 can be transformed into the following fully fuzzy linear system.

$$\begin{aligned} & (2x_1, 2x_1 - \min((x_1 - y_1), 3(x_1 - y_1)), \max((x_1 + z_1), 3(x_1 + z_1)) - 2x_1) \oplus \\ & (-x_2, -x_2 - \min(-2(x_2 + z_2), -1(x_2 + z_2)), \max(-2(x_2 - y_2), -1(x_2 - y_2)) + x_2) \\ & \oplus (3x_3, 3x_3 - \min(2(x_3 - y_3), 3(x_3 - y_3)), \max(2(x_3 + z_3), 3(x_3 + z_3)) - 3x_3) \\ & = (0,5,8) \end{aligned}$$

$$\begin{aligned} & (-2x_1, -2x_1 - \min(-3(x_1 + z_1), -1(x_1 + z_1)), \max(-3(x_1 - y_1), -1(x_1 - y_1)) \\ & + 2x_1) \oplus \end{aligned}$$

$$(4x_2, 4x_2 - \min(2(x_2 - y_2), 3(x_2 - y_2)), \max(2(x_2 + z_2), 3(x_2 + z_2)) - 2x_2) \oplus$$

$$\begin{aligned} & (4x_3, 4x_3 - \min(3(x_3 - y_3), 4(x_3 - y_3)), \max(3(x_3 + z_3), 4(x_3 + z_3)) - 4x_3) \\ & = (-10,14,6) \end{aligned}$$

which can be further written as

$$\begin{aligned} & (2x_1, 2x_1 - w_{11}, s_{11} - 2x_1) \oplus (-x_2, -x_2 - w_{12}, s_{12} + x_2) \oplus (3x_3, 3x_3 - w_{13}, s_{13} - 3x_3) \\ & = (0,5,8) \end{aligned}$$

$$\begin{aligned} &(-2x_1, -2x_1 - w_{21}, s_{21} + 2x_1) \oplus (4x_2, 4x_2 - w_{22}, s_{22} - 2x_2) \\ &\oplus (4x_3, 4x_3 - w_{23}, s_{23} - 4x_3) = (-10, 14, 6) \end{aligned}$$

where  $w_{11} = \min((x_1 - y_1), 3(x_1 - y_1))$ ,  $s_{11} = \max((x_1 + z_1), 3(x_1 + z_1))$

$$w_{12} = \min(-2(x_2 + z_2), -1(x_2 + z_2)), s_{12} = \max(-2(x_2 - y_2), -1(x_2 - y_2))$$

$$w_{13} = \min(2(x_3 - y_3), 3(x_3 - y_3)), s_{13} = \max(3(x_3 + z_3), 4(x_3 + z_3))$$

$$w_{21} = \min(-3(x_1 + z_1), -1(x_1 + z_1)), s_{21} = \max(-3(x_1 - y_1), -1(x_1 - y_1))$$

$$w_{22} = \min(2(x_2 - y_2), 3(x_2 - y_2)), s_{22} = \max(2(x_2 + z_2), 3(x_2 + z_2))$$

$$w_{23} = \min(3(x_3 - y_3), 4(x_3 - y_3)), s_{23} = \max(3(x_3 + z_3), 4(x_3 + z_3))$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$ ,

$x_i - y_i = L_i$  and  $x_i + z_i = R_i$  as well as  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$ ,  $R_j = R'_j -$

$R''_j$ ,  $|R_j| = R'_j + R''_j$ , the fully fuzzy linear system obtained in Step 1, can be written as:

$$\begin{aligned} &(2x_1, 2x_1 - w_{11}, s_{11} - 2x_1) \oplus (-x_2, -x_2 - w_{12}, s_{12} + x_2) \oplus (3x_3, 3x_3 - w_{13}, s_{13} - 3x_3) \\ &= (0, 5, 8) \end{aligned}$$

$$\begin{aligned} &(-2x_1, -2x_1 - w_{21}, s_{21} + 2x_1) \oplus (4x_2, 4x_2 - w_{22}, s_{22} - 2x_2) \\ &\oplus (4x_3, 4x_3 - w_{23}, s_{23} - 4x_3) = (-10, 14, 6) \end{aligned}$$

where

$$w_{11} = L'_1 - 3L''_1, w_{12} = -2R'_2 - R''_2$$

$$w_{13} = 2L'_3 - 3L''_3, s_{11} = 3R'_1 - R''_1$$

$$s_{12} = -L'_2 + 2L''_2, s_{13} = 3R'_3 - 2R''_3$$

$$w_{21} = -3R'_1 + R''_1, w_{22} = 2L'_2 - 5L''_2$$

$$w_{23} = 3L'_3 - 4L''_3, s_{21} = -L'_1 + 3L''_1$$

$$s_{22} = 5R'_2 - 2R''_2, s_{23} = 4R'_3 - 3R''_3$$

**Step 3.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the fully fuzzy linear system, obtained in Step 3, can be transformed into the following fully fuzzy linear system.

$$2x_1 - x_2 + 3x_3 = 0$$

$$-2x_1 + 4x_2 + 4x_3 = -10$$

$$L'_1 - 3L''_1 - 2R'_2 - R''_2 + 2L'_3 - 3L''_3 = -5$$

$$3R'_1 - R''_1 - L'_2 + 2L''_2 + 3R'_3 - 2R''_3 = 8$$

$$-3R'_1 + R''_1 + 2L'_2 - 5L''_2 + 3L'_3 - 4L''_3 = -24$$

$$-L'_1 + 3L''_1 + 5R'_2 - 2R''_2 + 4R'_3 - 3R''_3 = -4$$

$$L'_1, L''_1, L'_2, L''_2, R'_1, R''_1, R'_2, R''_2 \geq 0$$

**Step 4.** Using Phase 1 of the two-phase method, the following linear programming problem can be used to find the solution of crisp system of linear equations, obtained in Step 3

$$\text{Minimize } \bar{Z} = Z_1 + Z_2 + Z'_1 + Z'_2 + Z''_1 + Z''_2$$

subject to

$$2x_1 - x_2 + 3x_3 + Z_1 = 0$$

$$-2x_1 + 4x_2 + 4x_3 + Z_2 = -10$$

$$L'_1 - 3L''_1 - 2R'_2 - R''_2 + 2L'_3 - 3L''_3 + Z'_1 = -5$$

$$3R'_1 - R''_1 - L'_2 + 2L''_2 + 3R'_3 - 2R''_3 + Z''_1 = 8$$

$$-3R'_1 + R''_1 + 2L'_2 - 5L''_2 + 3L'_3 - 4L''_3 + Z'_2 = -24$$

$$-L'_1 + 3L''_1 + 5R'_2 - 2R''_2 + 4R'_3 - 3R''_3 + Z''_2 = -4$$

$$L'_1, L''_1, L'_2, L''_2, R'_1, R''_1, R'_2, R''_2, Z_1, Z_2, Z'_1, Z''_1, Z'_2, Z''_2 \geq 0$$

On solving the above linear programming problem ,we get

$$x_1 = 1, x_2 = -1, x_3 = -1, L'_1 = 1, L''_1 = 0, L'_2 = 0, L''_2 = 2, L'_3 = 0, L''_3 = 2, R'_1 = 2, R''_1 = 0, R'_2 = 0, R''_2 = 0, R'_3 = 0, R''_3 = 1, Z_1 = 0, Z_2 = 0, Z'_1 = 0, Z''_1 = 0, Z'_2 = 0, Z''_2 = 0.$$

**Step 5.** Putting the values of  $L'_j, L''_j, R'_j, R''_j, x_j$ , obtained from Step 5, in  $\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \forall j = 1,2,3$ , the solution of fully fuzzy linear system is  $\tilde{x}_1 = (1,0,1)$ ,  $\tilde{x}_2 = (-1,1,1)$  and  $\tilde{x}_3 = (-1,1,0)$

#### 4.4. Results and discussion

The solution of fully fuzzy linear system, chosen in Example 4.1 and Example 4.2, are shown in Table 4.1. It is obvious from the results, shown in Table 4.1, that the fully fuzzy linear systems, chosen in Example 4.1 and Example 4.2 can neither be solved by the existing methods nor by the Mehar method-I and Mehar method-II , proposed in previous chapters but can be solved by the Mehar method-III proposed in this chapter.

**Table 4.1** : Comparison of results by using proposed and existing methods

Example	Solution using Mehar method-III	Solution using methods discussed in previous chapter as well as existing methods
Example 4 .1	$\tilde{x}_1 = (2,1,0)$ $\tilde{x}_2 = (1,4,1)$	Not applicable
Example 4.2	$\tilde{x}_1 = (1,0,1)$ $\tilde{x}_2 = (-1,1,1)$ $\tilde{x}_3 = (-1,1,0)$	Not applicable

#### 4.5. Conclusion

On the basis of present study, it can be concluded that the fully fuzzy linear system which can be solved by existing methods and by using the Mehar method-I, Mehar method-II proposed in previous chapters, can also be solved by the Mehar method-III, proposed in this chapter. However, there exist several fully fuzzy linear systems whose solution can be obtained by the Mehar method-III but can neither be obtained by the existing methods nor by the Mehar method-I and Mehar method-II. Hence, it is better to use the Mehar method-III, as compared to the existing methods and the Mehar method-I and Mehar method-II.



## Chapter 5

# Mehar method-IV for solving dual fully fuzzy linear system with unrestricted coefficients and unrestricted fuzzy variables<sup>2</sup>

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In this chapter, limitations of the existing methods for solving dual fully fuzzy linear system are pointed out. Also, to overcome these limitations, a new method (named as Mehar method-IV) is proposed to find the solution of such dual fully fuzzy linear system in which all the elements of all the matrices and elements of unknown vector are represented by unrestricted triangular fuzzy numbers. The proposed Mehar method-IV is illustrated by numerical examples.

### 5.1 Limitations of existing methods

The existing methods [30, 86, 98, 112, 123, 125] can be used only to find the solution of such dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  where the elements of triangular fuzzy matrices  $\tilde{A}$  and  $\tilde{C}$  as well as the elements of triangular fuzzy vectors  $\tilde{x}$ ,  $\tilde{b}$  and  $\tilde{d}$  all are non-negative triangular fuzzy numbers. However, none of these existing methods, can be used to find the solution of following types of dual fully fuzzy linear system.

- (1) Dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  where the elements of fuzzy matrices  $\tilde{A}$  and  $\tilde{C}$  as well as the elements of fuzzy vectors  $\tilde{b}$  and  $\tilde{d}$  and the unknown vector  $\tilde{x}$  all are unrestricted triangular fuzzy numbers. For example

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<sup>2</sup> A part of this chapter is published in *New Mathematics and Natural Computation* **9** (2013) 13-26.

**Example 5.1.**

$$(3,5,1) \otimes (x_1, y_1, z_1) \oplus (5,1,1) \otimes (x_2, y_2, z_2) \oplus (-4,1,2) \otimes (x_3, y_3, z_3) \oplus (18,2,1) =$$

$$(5,2,1) \otimes (x_1, y_1, z_1) \oplus (2,0,1) \otimes (x_2, y_2, z_2) \oplus (2,1,0) \otimes (x_3, y_3, z_3) \oplus (2,10,12),$$

$$(2,1,1) \otimes (x_1, y_1, z_1) \oplus (3,1,1) \otimes (x_2, y_2, z_2) \oplus (2,0,2) \otimes (x_3, y_3, z_3) \oplus (10,20,10) =$$

$$(3,1,1) \otimes (x_1, y_1, z_1) \oplus (2,4,1) \otimes (x_2, y_2, z_2) \oplus (3,1,1) \otimes (x_3, y_3, z_3) \oplus (8,3,14),$$

$$(-2,1,1) \otimes (x_1, y_1, z_1) \oplus (-4,0,3) \otimes (x_2, y_2, z_2) \oplus (4,2,1) \otimes (x_3, y_3, z_3) \oplus (24,24,1) =$$

$$(-4,0,3) \otimes (x_1, y_1, z_1) \oplus (2,4,1) \otimes (x_2, y_2, z_2) \oplus (3,5,1) \otimes (x_3, y_3, z_3) \oplus (8,1,6)$$

where,  $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$  are all unrestricted triangular fuzzy numbers.

(2) Dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  where the elements of fuzzy matrices  $\tilde{A}$  and  $\tilde{C}$  as well as the elements of fuzzy vectors  $\tilde{b}$  and  $\tilde{d}$  are unrestricted triangular fuzzy numbers but elements of unknown vector  $\tilde{x}$ , are non-negative triangular fuzzy numbers.

$$(2,3,1) \otimes (x_1, y_1, z_1) \oplus (3,1,1) \otimes (x_2, y_2, z_2) \oplus (3,55,0) = (-4,1,1) \otimes (x_1, y_1, z_1) \oplus$$

$$(-3,3,1) \otimes (x_2, y_2, z_2) \oplus (45,47,3),$$

$$(-3,1,1) \otimes (x_1, y_1, z_1) \oplus (5,1,1) \otimes (x_2, y_2, z_2) \oplus (2,17,0)$$

$$= (2,1,1) \otimes (x_1, y_1, z_1) \oplus (-2,3,1) \otimes (x_2, y_2, z_2) \oplus (3,4,13)$$

where,  $(x_1, y_1, z_1), (x_2, y_2, z_2) \geq 0$ .

**5.2. Proposed Mehar method-IV**

In this section, a new method (named as Mehar method-IV) is proposed to solve a dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  with no restrictions on the parameters.

The steps of the proposed Mehar method-IV to find the solution of dual fully fuzzy linear system are as follows:

**Step 1** Substituting  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ ,  $\tilde{x} = (\tilde{x}_i)_{n \times 1}$ ,  $\tilde{C} = (\tilde{c}_{ij})_{m \times n}$ ,  $\tilde{d} = (\tilde{d}_i)_{m \times 1}$  and  $\tilde{b} = (\tilde{b}_i)_{m \times 1}$  the dual fully fuzzy linear system can be transformed into the following dual fully fuzzy linear system.

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \oplus \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \otimes \tilde{x}_j \oplus \tilde{d}_i \quad \forall i = 1, 2, \dots, m$$

**Step 2** If all the parameters  $\tilde{a}_{ij}$ ,  $\tilde{x}_j$ ,  $\tilde{c}_{ij}$ ,  $\tilde{b}_i$  and  $\tilde{d}_i$  are represented by triangular fuzzy numbers  $(a_{ij}^1, a_{ij}^2, a_{ij}^3)$ ,  $(x_j, y_j, z_j)$ ,  $(c_{ij}^1, c_{ij}^2, c_{ij}^3)$ ,  $(b_i^1, b_i^2, b_i^3)$  and  $(d_i^1, d_i^2, d_i^3)$  respectively, then the dual fully fuzzy linear system, obtained in Step 1, can be transformed into the following dual fully fuzzy linear system.

$$\sum_{j=1}^n (a_{ij}^1, a_{ij}^2, a_{ij}^3) \otimes (x_j, y_j, z_j) \oplus (b_i^1, b_i^2, b_i^3) = \sum_{j=1}^n (c_{ij}^1, c_{ij}^2, c_{ij}^3) \otimes (x_j, y_j, z_j) \oplus (d_i^1, d_i^2, d_i^3)$$

$$\forall i = 1, 2, \dots, m.$$

**Step 3** Assuming  $(a_{ij}^1, a_{ij}^2, a_{ij}^3) \otimes (x_j, y_j, z_j) = (A_{ij}^1, A_{ij}^2, A_{ij}^3)$ ,

and  $(c_{ij}^1, c_{ij}^2, c_{ij}^3) \otimes (x_j, y_j, z_j) = (C_{ij}^1, C_{ij}^2, C_{ij}^3)$  the dual fully fuzzy linear system, obtained in Step 2, can be transformed into the following dual fully fuzzy linear system.

$$\sum_{j=1}^n (A_{ij}^1, A_{ij}^2, A_{ij}^3) \oplus (b_i^1, b_i^2, b_i^3) = \sum_{j=1}^n (C_{ij}^1, C_{ij}^2, C_{ij}^3) \oplus (d_i^1, d_i^2, d_i^3)$$

$$\forall i = 1, 2, \dots, m$$

$$(A_{ij}^1, A_{ij}^2, A_{ij}^3) = (a_{ij}^1, a_{ij}^2, a_{ij}^3) \otimes (x_j, y_j, z_j)$$

Using the extended multiplication operation, defined in Section 2.1.2, the following cases can be obtained:

**Case I.** if  $(a_{ij}^1, a_{ij}^2, a_{ij}^3) \geq 0$  i. e.,  $a_{ij}^1 - a_{ij}^2 \geq 0$

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$A_{ij}^2 = a_{ij}^1 x_j - \min\left((a_{ij}^1 - a_{ij}^2)(x_j - y_j), (a_{ij}^1 + a_{ij}^3)(x_j - y_j)\right)$$

$$A_{ij}^3 = \max\left((a_{ij}^1 - a_{ij}^2)(x_j + z_j), (a_{ij}^1 + a_{ij}^3)(x_j + z_j)\right) - a_{ij}^1 x_j$$

This set of equation can be further reduced as follows

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$A_{ij}^2 = A_{ij}^1 - W_{ij}$$

$$W_{ij} = \min\left((a_{ij}^1 - a_{ij}^2)(x_j - y_j), (a_{ij}^1 + a_{ij}^3)(x_j - y_j)\right)$$

$$= \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(x_j - y_j) - \left| \frac{a_{ij}^3 + a_{ij}^2}{2} \right| |x_j - y_j|$$

$$A_{ij}^3 = S_{ij} - A_{ij}^1$$

$$S_{ij} = \max\left((a_{ij}^1 - a_{ij}^2)(x_j + z_j), (a_{ij}^1 + a_{ij}^3)(x_j + z_j)\right)$$

$$= \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(x_j + z_j) + \left| \frac{a_{ij}^3 + a_{ij}^2}{2} \right| |x_j + z_j|$$

To solve this system we substitute  $L_j = x_j - y_j$  as the left coefficient and  $R_j = x_j + z_j$  as the right coefficient. This can be further reduced as follows. Define variables  $L'_j$  and  $L''_j$

$$\text{as } L'_j = \begin{cases} L_j & \text{if } L_j > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } L''_j = \begin{cases} -L_j & \text{if } L_j < 0 \\ 0 & \text{otherwise} \end{cases}$$

Putting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$W_{ij} = \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(L'_j - L''_j) - \left( \frac{a_{ij}^3 + a_{ij}^2}{2} \right)(L'_j + L''_j)$$

$$S_{ij} = \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(R'_j - R''_j) + \left( \frac{a_{ij}^3 + a_{ij}^2}{2} \right)(R'_j + R''_j)$$

**Case2.** If  $(a_{ij}^1, a_{ij}^2, a_{ij}^3) \leq 0$  i.e.,  $a_{ij}^1 + a_{ij}^3 \leq 0$

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$A_{ij}^2 = a_{ij}^1 x_j - \min \left( (a_{ij}^1 - a_{ij}^2)(x_j + z_j), (a_{ij}^1 + a_{ij}^3)(x_j + z_j) \right)$$

$$A_{ij}^3 = \max \left( (a_{ij}^1 - a_{ij}^2)(x_j - y_j), (a_{ij}^1 + a_{ij}^3)(x_j - y_j) \right) - a_{ij}^1 x_j$$

This set of equation can be similarly reduced as follows.

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$A_{ij}^2 = A_{ij}^1 - W_{ij}$$

$$\begin{aligned}
W_{ij} &= \min \left( (a_{ij}^1 - a_{ij}^2)(x_j + z_j), (a_{ij}^1 + a_{ij}^3)(x_j + z_j) \right) \\
&= \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(x_j + z_j) - \left| \frac{a_{ij}^3 + a_{ij}^2}{2} \right| |x_j + z_j|
\end{aligned}$$

$$A_{ij}^3 = S_{ij} - A_{ij}^1$$

$$\begin{aligned}
S_{ij} &= \max \left( (a_{ij}^1 - a_{ij}^2)(x_j - y_j), (a_{ij}^1 + a_{ij}^3)(x_j - y_j) \right) \\
&= \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(x_j - y_j) + \left| \frac{a_{ij}^3 + a_{ij}^2}{2} \right| |x_j - y_j|
\end{aligned}$$

Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly

$R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$S_{ij} = \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(L'_j - L''_j) + \left( \frac{a_{ij}^3 + a_{ij}^2}{2} \right)(L'_j + L''_j)$$

$$W_{ij} = \frac{2a_{ij}^1 + a_{ij}^3 - a_{ij}^2}{2}(R'_j - R''_j) - \left( \frac{a_{ij}^3 + a_{ij}^2}{2} \right)(R'_j + R''_j)$$

**Case 3.** If  $(a_{ij}^1, a_{ij}^2, a_{ij}^3) \equiv 0$  i. e.,  $a_{ij}^1 - a_{ij}^2 < 0 < a_{ij}^1 + a_{ij}^3$

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$A_{ij}^2 = a_{ij}^1 x_j - \min \left( (a_{ij}^1 - a_{ij}^2)(x_j + z_j), (a_{ij}^1 + a_{ij}^3)(x_j - y_j) \right)$$

$$A_{ij}^3 = \max \left( (a_{ij}^1 - a_{ij}^2)(x_j - y_j), (a_{ij}^1 + a_{ij}^3)(x_j + z_j) \right) - a_{ij}^1 x_j$$

This set of equation can be similarly reduced as follows.

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$A_{ij}^2 = A_{ij}^1 - W_{ij}$$

$$W_{ij} = \min \left( (a_{ij}^1 - a_{ij}^2)(x_j + z_j), (a_{ij}^1 + a_{ij}^3)(x_j - y_j) \right)$$

$$A_{ij}^3 = S_{ij} - A_{ij}^1$$

$$S_{ij} = \max \left( (a_{ij}^1 - a_{ij}^2)(x_j - y_j), (a_{ij}^1 + a_{ij}^3)(x_j + z_j) \right)$$

Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j, R_j = R'_j - R''_j, |R_j| = R'_j + R''_j$ .

Also let

$$(a_{ij}^1 - a_{ij}^2)R_j - (a_{ij}^1 + a_{ij}^3)L_j = u'_{ij} - u''_{ij}$$

this implies

$$|(a_{ij}^1 - a_{ij}^2)R_j - (a_{ij}^1 + a_{ij}^3)L_j| = u'_{ij} + u''_{ij}.$$

Similarly,

$$(a_{ij}^1 - a_{ij}^2)L_j - (a_{ij}^1 + a_{ij}^3)R_j = v'_{ij} - v''_{ij}$$

and therefore

$$|(a_{ij}^1 - a_{ij}^2)L_j - (a_{ij}^1 + a_{ij}^3)R_j| = v'_{ij} + v''_{ij}$$

where  $L'_j, L''_j, R'_j, R''_j, u'_{ij}, u''_{ij}, v'_{ij}, v''_{ij} \geq 0$

thus

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$W_{ij} = \frac{(a_{ij}^1 - a_{ij}^2)(R'_j - R''_j) + (a_{ij}^1 + a_{ij}^3)(L'_j - L''_j)}{2} - \left( \frac{u'_{ij} + u''_{ij}}{2} \right)$$

$$S_{ij} = \frac{(a_{ij}^1 + a_{ij}^3)(R_j' - R_j'') + (a_{ij}^1 - a_{ij}^2)(L_j' - L_j'')}{2} + \left( \frac{v_{ij}' + v_{ij}''}{2} \right)$$

with two additional constraints:

$$(a_{ij}^1 - a_{ij}^2)R_j - (a_{ij}^1 + a_{ij}^3)L_j = u_{ij}' - u_{ij}''$$

$$(a_{ij}^1 - a_{ij}^2)L_j - (a_{ij}^1 + a_{ij}^3)R_j = v_{ij}' - v_{ij}''$$

**Step 4.** In the same way, we may formulate  $C_{ij}^1, C_{ij}^2, C_{ij}^3$  as follows.

**Case 1.** If  $(c_{ij}^1, c_{ij}^2, c_{ij}^3) \geq 0$  i. e.,  $c_{ij}^1 - c_{ij}^2 \geq 0$

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$C_{ij}^2 = c_{ij}^1 x_j - \min \left( (c_{ij}^1 - c_{ij}^2)(x_j - y_j), (c_{ij}^1 + c_{ij}^3)(x_j - y_j) \right)$$

$$C_{ij}^3 = \max \left( (c_{ij}^1 - c_{ij}^2)(x_j + z_j), (c_{ij}^1 + c_{ij}^3)(x_j + z_j) \right) - c_{ij}^1 x_j$$

This set of equation can be further reduced as follows.

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$C_{ij}^2 = C_{ij}^1 - W_{ij}'$$

$$W_{ij}' = \min \left( (c_{ij}^1 - c_{ij}^2)(x_j - y_j), (c_{ij}^1 + c_{ij}^3)(x_j - y_j) \right)$$

$$= \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2} (x_j - y_j) - \left| \frac{c_{ij}^3 + c_{ij}^2}{2} \right| |x_j - y_j|$$

$$C_{ij}^3 = S_{ij}' - C_{ij}^1$$

$$S'_{ij} = \max \left( (c_{ij}^1 - c_{ij}^2)(x_j + z_j), (c_{ij}^1 + c_{ij}^3)(x_j + z_j) \right)$$

$$= \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(x_j + z_j) + \left| \frac{c_{ij}^3 + c_{ij}^2}{2} \right| |x_j + z_j|$$

To solve this system we substitute  $L_j = x_j - y_j$  as the left coefficient and  $R_j = x_j + z_j$  as the right coefficient. This can be further reduced as follows. Define variables  $L'_j$  and  $L''_j$

$$\text{as } L'_j = \begin{cases} L_j & \text{if } P_j > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } L''_j = \begin{cases} -L_j & \text{if } P_j < 0 \\ 0 & \text{otherwise} \end{cases}$$

Putting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$W'_{ij} = \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(L'_j - L''_j) - \left( \frac{c_{ij}^3 + c_{ij}^2}{2} \right)(L'_j + L''_j)$$

$$S'_{ij} = \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(R'_j - R''_j) + \left( \frac{c_{ij}^3 + c_{ij}^2}{2} \right)(R'_j + R''_j)$$

**Case 2.** if  $(c_{ij}^1, c_{ij}^2, c_{ij}^3) \leq 0$  i. e.,  $c_{ij}^1 + c_{ij}^3 \leq 0$

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$C_{ij}^2 = c_{ij}^1 x_j - \min \left( (c_{ij}^1 - c_{ij}^2)(x_j + z_j), (c_{ij}^1 + c_{ij}^3)(x_j + z_j) \right)$$

$$C_{ij}^3 = \max \left( (c_{ij}^1 - c_{ij}^2)(x_j - y_j), (c_{ij}^1 + c_{ij}^3)(x_j - y_j) \right) - c_{ij}^1 x_j$$

This set of equation can be similarly reduced as follows

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$C_{ij}^2 = C_{ij}^1 - W'_{ij}$$

$$\begin{aligned}
W'_{ij} &= \min\left((c_{ij}^1 - c_{ij}^2)(x_j + z_j), (c_{ij}^1 + c_{ij}^3)(x_j + z_j)\right) \\
&= \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(x_j + z_j) - \left|\frac{c_{ij}^3 + c_{ij}^2}{2}\right| |x_j + z_j|
\end{aligned}$$

$$C_{ij}^3 = S'_{ij} - C_{ij}^1$$

$$\begin{aligned}
S'_{ij} &= \max\left((c_{ij}^1 - c_{ij}^2)(x_j - y_j), (c_{ij}^1 + c_{ij}^3)(x_j - y_j)\right) \\
&= \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(x_j - y_j) + \left|\frac{c_{ij}^3 + c_{ij}^2}{2}\right| |x_j - y_j|
\end{aligned}$$

Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$A_{ij}^1 = a_{ij}^1 x_j$$

$$S'_{ij} = \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(L'_j - L''_j) + \left(\frac{c_{ij}^3 + c_{ij}^2}{2}\right)(L'_j + L''_j)$$

$$W'_{ij} = \frac{2c_{ij}^1 + c_{ij}^3 - c_{ij}^2}{2}(R'_j - R''_j) - \left(\frac{a_{ij}^3 + a_{ij}^2}{2}\right)(R'_j + R''_j)$$

**Case 3.** If  $(c_{ij}^1, c_{ij}^2, c_{ij}^3) \equiv 0$  i. e.,  $c_{ij}^1 - c_{ij}^2 < 0 < c_{ij}^1 + c_{ij}^3$

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$C_{ij}^2 = c_{ij}^1 x_j - \min\left((c_{ij}^1 - c_{ij}^2)(x_j + z_j), (c_{ij}^1 + c_{ij}^3)(x_j - y_j)\right)$$

$$C_{ij}^3 = \max\left((c_{ij}^1 - c_{ij}^2)(x_j - y_j), (c_{ij}^1 + c_{ij}^3)(x_j + z_j)\right) - c_{ij} x_j$$

This set of equation can be similarly reduced as follows.

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$C_{ij}^2 = C_{ij}^1 - W'_{ij}$$

$$W'_{ij} = \min\left((c_{ij}^1 - c_{ij}^2)(x_j + z_j), (c_{ij}^1 + c_{ij}^3)(x_j - y_j)\right)$$

$$C_{ij}^3 = S'_{ij} - C_{ij}^1$$

$$S'_{ij} = \max\left((c_{ij}^1 - c_{ij}^2)(x_j - y_j), (c_{ij}^1 + c_{ij}^3)(x_j + z_j)\right)$$

Substituting  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$  and similarly  $R_j = R'_j - R''_j$  and  $|R_j| = R'_j + R''_j$  where  $L'_j, L''_j, R'_j, R''_j \geq 0$ , we get:

$$(c_{ij}^1 - c_{ij}^2)R_j - (c_{ij}^1 + c_{ij}^3)L_j = u'_{ij} - u''_{ij}$$

This implies

$$|(c_{ij}^1 - c_{ij}^2)R_j - (c_{ij}^1 + c_{ij}^3)L_j| = u'_{ij} + u''_{ij}$$

Similarly,

$$(c_{ij}^1 - c_{ij}^2)L_j - (c_{ij}^1 + c_{ij}^3)R_j = v'_{ij} - v''_{ij}$$

$$|(c_{ij}^1 - c_{ij}^2)L_j - (c_{ij}^1 + c_{ij}^3)R_j| = v'_{ij} + v''_{ij}$$

where  $L'_j, L''_j, R'_j, R''_j, u'_{ij}, u''_{ij}, v'_{ij}, v''_{ij} \geq 0$

Substituting we get:

$$C_{ij}^1 = c_{ij}^1 x_j$$

$$W'_{ij} = \frac{(c_{ij}^1 - c_{ij}^2)(R'_j - R''_j) + (c_{ij}^1 + c_{ij}^3)(L'_j - L''_j)}{2} - \left(\frac{u'_{ij} + u''_{ij}}{2}\right)$$

$$S'_{ij} = \frac{(c_{ij}^1 + c_{ij}^3)(R_j - R_j'') + (c_{ij}^1 - c_{ij}^2)(L_j - L_j'')}{2} + \left( \frac{v'_{ij} + v''_{ij}}{2} \right)$$

with two additional constraints:

$$(c_{ij}^1 - c_{ij}^2)R_j - (c_{ij}^1 + c_{ij}^3)L_j = u'_{ij} - u''_{ij}$$

$$(c_{ij}^1 - c_{ij}^2)L_j - (c_{ij}^1 + c_{ij}^3)R_j = v'_{ij} - v''_{ij}$$

**Step 5.** Thus the dual fully fuzzy linear system , obtained from Step 3 and Step 4, can be transformed into the following crisp system of linear equations:

$$\begin{cases} \sum_{j=1}^n A_{ij}^1 + b_i^1 = \sum_{j=1}^n C_{ij}^1 + d_i^1 & ; & i = 1, 2, \dots, m \\ \sum_{j=1}^n A_{ij}^2 + b_i^2 = \sum_{j=1}^n C_{ij}^2 + d_i^2 & ; & i = 1, 2, \dots, m \\ \sum_{j=1}^n A_{ij}^3 + b_i^3 = \sum_{j=1}^n C_{ij}^3 + d_i^3 & ; & i = 1, 2, \dots, m \end{cases}$$

with appropriate constraints obtained from Step 3 and Step 4.

**Step 6.** The solution of system obtained in Step 5, can be obtained by solving the following linear programming problem, which is obtained by adding the artificial variable  $Z_i, \forall i = 1, 2, \dots, m$ .

$$\text{Minimize } \bar{Z} = \sum_{i=1}^n Z_i + \sum_{i=1}^n Z'_i + \sum_{i=1}^n Z''_i$$

subject to

$$\sum_{j=1}^n A_{ij}^1 - \sum_{j=1}^n C_{ij}^1 + Z_i = d_i^1 - b_i^1 \quad ; \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n A_{ij}^2 - \sum_{j=1}^n C_{ij}^2 + Z_i' = d_i^2 - b_i^2 \quad ; \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n A_{ij}^3 - \sum_{j=1}^n C_{ij}^3 + Z_i'' = d_i^3 - b_i^3 \quad ; \quad i = 1, 2, \dots, m$$

**Step 7.** Solve the linear programming problem, obtained from Step 6, and put the values in

$$\tilde{x}_j = (x_j, x_j - L_j' + L_j'', R_j' - R_j'' - x_j) \quad \forall j = 1, 2, \dots, n$$

to find the solution of fully fuzzy linear system. If the linear programming problem is inconsistent and does not yield any solution then the given dual fully fuzzy linear system is inconsistent and has no solution.

### 5.3. Solution of chosen fully fuzzy linear systems

In Section 5.1, some dual fully fuzzy linear systems are chosen which cannot be solved by the existing methods [30, 86, 98, 112, 123, 125]. In this section, the solution of the chosen dual fully fuzzy linear system are obtained by the Mehar method-IV, proposed in Section 5.2.

#### 5.3.1. Solution of dual fully fuzzy linear system chosen in Example 5.1

Using the Mehar method-IV, proposed in Section 5.2, the solution of dual fully fuzzy linear system, chosen in Example 5.1, can be obtained as follows:

**Step 1.** Using the multiplication of triangular fuzzy numbers as proposed in Step 2, of the proposed method the dual fully fuzzy linear system, chosen in Example 5.1 can be transformed into the following dual fully fuzzy linear system.

$$(3x_1, 3x_1 - \min(-2(x_1 + z_1), 4(x_1 - y_1)), \max(-2(x_1 - y_1), 4(x_1 + z_1)) - 3x_1) \oplus$$

$$\begin{aligned}
& (5x_2, 5x_2 - \min(4(x_2 - y_2), 6(x_2 - y_2)), \max(4(x_2 + z_2), 6(x_2 + z_2)) - 5x_2) \\
& \oplus (-4x_3, -4x_3 - \min(-5(x_3 + z_3), -2(x_3 + z_3)), \max(-5(x_3 - y_3), -2(x_3 - y_3)) + \\
& 4x_3) \oplus (18, 2, 1) =
\end{aligned}$$

$$\begin{aligned}
& (5x_1, 5x_1 - \min(3(x_1 - y_1), 6(x_1 - y_1)), \max(3(x_1 + z_1), 6(x_1 + z_1)) - 5x_1) \oplus \\
& (2x_2, 2x_2 - \min(2(x_2 - y_2), 3(x_2 - y_2)), \max(2(x_2 + z_2), 3(x_2 + z_2)) - 2x_2) \\
& \oplus (2x_3, 2x_3 - \min((x_3 - z_3), 3(x_3 - y_3)), \max((x_3 + z_3), 3(x_3 + z_3)) - 2x_3) \\
& \oplus (2, 10, 12),
\end{aligned}$$

$$\begin{aligned}
& (2x_1, 2x_1 - \min((x_1 - y_1), 3(x_1 - y_1)), \max((x_1 + z_1), 3(x_1 + z_1)) - 2x_1) \oplus \\
& (3x_2, 3x_2 - \min(2(x_2 - y_2), 4(x_2 - y_2)), \max(2(x_2 + z_2), 4(x_2 + z_2)) - 3x_2) \\
& \oplus (2x_3, 2x_3 - \min(2(x_3 - y_3), 4(x_3 - y_3)), \max(2(x_3 + z_3), 4(x_3 + z_3)) - 2x_3) \\
& \oplus (10, 20, 10) =
\end{aligned}$$

$$\begin{aligned}
& (3x_1, 3x_1 - \min(2(x_1 - y_1), 4(x_1 - y_1)), \max(2(x_1 + z_1), 4(x_1 + z_1)) - 3x_1) \oplus \\
& (2x_2, 2x_2 - \min(-2(x_2 + z_2), 3(x_2 - y_2)), \max(-2(x_2 - y_2), 3(x_2 + z_2)) - 2x_2) \\
& \oplus (3x_3, 3x_3 - \min(2(x_3 - y_3), 4(x_3 - y_3)), \max(2(x_3 + z_3), 4(x_3 + z_3)) - 3x_3) \\
& \oplus (8, 3, 14)
\end{aligned}$$

and

$$\begin{aligned}
& (-2x_1, -2x_1 - \min(-3(x_1 + z_1), -1(x_1 + z_1)), \max(-3(x_1 - y_1), -1(x_1 - y_1)) \\
& + 2x_1) \oplus
\end{aligned}$$

$$(-4x_2, -4x_2 - \min(-4(x_2 + z_2), -1(x_2 + z_2)), \max(-4(x_2 - y_2), -1(x_2 - y_2)) \\ + 4x_2)$$

$$\oplus (4x_3, 4x_3 - \min(2(x_3 - y_3), 5(x_3 - y_3)), \max(2(x_3 + z_3), 5(x_3 + z_3)) - 4x_3)$$

$$\oplus (24, 24, 1) =$$

$$(-4x_1, -4x_1 - \min(-3(x_1 + z_1), -1(x_1 + z_1)), \max(-4(x_1 - y_1), -1(x_1 - y_1)) \\ + 4x_1) \oplus$$

$$(2x_2, 2x_2 - \min(-2(x_2 + z_2), 3(x_2 - y_2)), \max(-2(x_2 - y_2), 3(x_2 + z_2)) - 2x_2)$$

$$\oplus (3x_3, 3x_3 - \min(-2(x_3 + z_3), 4(x_3 - y_3)), \max(-2(x_3 - y_3), 4(x_3 + z_3)) - 3x_3) \\ \oplus (8, 1, 6)$$

which can be further written as

$$(3x_1, 3x_1 - W_{11}, S_{11} - 3x_1) \oplus (5x_2, 5x_2 - W_{12}, S_{12} - 5x_2) \\ \oplus (-4x_3, -4x_3 - W_{13}, S_{13} + 4x_3) \oplus (18, 2, 1) \\ = (5x_1, 5x_1 - W'_{11}, S'_{11} - 5x_1) \oplus (2x_2, 2x_2 - W'_{12}, S'_{12} - 2x_2) \\ \oplus (2x_3, 2x_3 - W'_{13}, S'_{13} - 2x_3) \oplus (2, 10, 12)$$

$$(2x_1, 2x_1 - W_{21}, S_{21} - 2x_1) \oplus (3x_2, 3x_2 - W_{22}, S_{22} - 3x_2) \oplus (2x_3, 2x_3 - W_{23}, S_{23} - 2x_3) \\ \oplus (10, 20, 10) \\ = (3x_1, 3x_1 - W'_{21}, S'_{21} - 3x_1) \oplus (2x_2, 2x_2 - W'_{22}, S'_{22} - 2x_2) \\ \oplus (3x_3, 3x_3 - W'_{23}, S'_{23} - 3x_3) \oplus (8, 3, 14)$$

and

$$\begin{aligned}
& (-2x_1, -2x_1 - W_{31}, S_{31} + 2x_1) \oplus (-4x_2, -4x_2 - W_{32}, S_{32} + 4x_2) \\
& \oplus (4x_3, 4x_3 - W_{33}, S_{33} - 4x_3) \oplus (24, 24, 1) \\
& = (-4x_1, -4x_1 - W'_{31}, S'_{31} + 4x_1) \oplus (2x_2, 2x_2 - W'_{32}, S'_{32} - 2x_2) \\
& \oplus (3x_3, 3x_3 - W'_{33}, S'_{33} - 3x_3) \oplus (8, 1, 6)
\end{aligned}$$

where  $W_{11} = \min(-2(x_1 + z_1), 4(x_1 - y_1))$ ,  $S_{11} = \max(-2(x_1 - y_1), 4(x_1 + z_1))$

$$W_{12} = \min(4(x_2 - y_2), 6(x_2 - y_2)), S_{12} = \max(4(x_2 + z_2), 6(x_2 + z_2))$$

$$W_{13} = \min(-5(x_3 + z_3), -2(x_3 + z_3)), S_{13} = \max(-5(x_3 - y_3), -2(x_3 - y_3))$$

$$W'_{11} = \min(3(x_1 - y_1), 6(x_1 - y_1)), S'_{11} = \max(3(x_1 + z_1), 6(x_1 + z_1))$$

$$W'_{12} = \min(2(x_2 - y_2), 3(x_2 - y_2)), S'_{12} = \max(2(x_2 + z_2), 3(x_2 + z_2))$$

$$W'_{13} = \min((x_3 - y_3), 3(x_3 - y_3)), S'_{13} = \max((x_3 + z_3), 3(x_3 + z_3))$$

$$W_{21} = \min((x_1 - y_1), 3(x_1 - y_1)), S_{21} = \max((x_1 + z_1), 3(x_1 + z_1))$$

$$W_{22} = \min(2(x_2 - y_2), 4(x_2 - y_2)), S_{22} = \max(2(x_2 + z_2), 4(x_2 + z_2))$$

$$W_{23} = \min(2(x_3 - y_3), 4(x_3 - y_3)), S_{23} = \max(2(x_3 + z_3), 4(x_3 + z_3))$$

$$W'_{21} = \min(2(x_1 - y_1), 4(x_1 - y_1)), S'_{21} = \max(2(x_1 + z_1), 4(x_1 + z_1))$$

$$W'_{22} = \min(-2(x_2 + z_2), 3(x_2 - y_2)), S'_{22} = \max(-2(x_2 - y_2), 3(x_2 + z_2))$$

$$W'_{23} = \min(2(x_3 - y_3), 4(x_3 - y_3)), S'_{23} = \max(2(x_3 + z_3), 4(x_3 + z_3))$$

$$W_{31} = \min(-3(x_1 + z_1), -1(x_1 + z_1)), S_{31} = \max(-3(x_1 - y_1), -1(x_1 - y_1))$$

$$W_{32} = \min(-4(x_2 + z_2), -1(x_2 + z_2)), S_{32} = \max(-4(x_2 - y_2), -1(x_2 - y_2))$$

$$W_{33} = \min(2(x_3 - y_3), 5(x_3 - y_3)), S_{33} = \max(2(x_3 + z_3), 5(x_3 + z_3))$$

$$W'_{31} = \min(-3(x_1 + z_1), -1(x_1 + z_1)), S'_{31} = \max(-4(x_1 - y_1), -1(x_1 - y_1))$$

$$W'_{32} = \min(-2(x_3 + z_3), 4(x_3 - y_3)), S'_{32} = \max(-2(x_3 - y_3), 4(x_3 + z_3))$$

$$W'_{33} = \min(-2(x_3 + z_3), 4(x_3 - y_3)), S'_{33} = \max(-2(x_3 - y_3), 4(x_3 + z_3))$$

**Step 3.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$ ,

$x_i - y_i = L_i$  and  $x_i + z_i = R_i$  as well as  $L_j = L'_j - L''_j$  and  $|L_j| = L'_j + L''_j$ ,  $R_j = R'_j -$

$R''_j$ ,  $|R_j| = R'_j + R''_j$ , the fully fuzzy linear system, obtained in Step 2, can be written as:

$$\begin{aligned} & (3x_1, 3x_1 - W_{11}, S_{11} - 3x_1) \oplus (5x_2, 5x_2 - W_{12}, S_{12} - 5x_2) \\ & \oplus (-4x_3, -4x_3 - W_{13}, S_{13} + 4x_3) \oplus (18, 2, 1) \\ & = (5x_1, 5x_1 - W'_{11}, S'_{11} - 5x_1) \oplus (2x_2, 2x_2 - W'_{12}, S'_{12} - 2x_2) \\ & \oplus (2x_3, 2x_3 - W'_{13}, S'_{13} - 2x_3) \oplus (2, 10, 12) \end{aligned}$$

$$\begin{aligned} & (2x_1, 2x_1 - W_{21}, S_{21} - 2x_1) \oplus (3x_2, 3x_2 - W_{22}, S_{22} - 3x_2) \oplus (2x_3, 2x_3 - W_{23}, S_{23} - 2x_3) \\ & \oplus (10, 20, 10) \\ & = (3x_1, 3x_1 - W'_{21}, S'_{21} - 3x_1) \oplus (2x_2, 2x_2 - W'_{22}, S'_{22} - 2x_2) \\ & \oplus (3x_3, 3x_3 - W'_{23}, S'_{23} - 3x_3) \oplus (8, 3, 14) \end{aligned}$$

and

$$\begin{aligned} & (-2x_1, -2x_1 - W_{31}, S_{31} + 2x_1) \oplus (-4x_2, -4x_2 - W_{32}, S_{32} + 4x_2) \\ & \oplus (4x_3, 4x_3 - W_{33}, S_{33} - 4x_3) \oplus (24, 24, 1) \end{aligned}$$

$$= (-4x_1, -4x_1 - W'_{31}, S'_{31} + 4x_1) \oplus (2x_2, 2x_2 - W'_{32}, S'_{32} - 2x_2) \\ \oplus (3x_3, 3x_3 - W'_{33}, S'_{33} - 3x_3) \oplus (8, 1, 6)$$

Where

$$W_{11} = \min(-2(x_1 + z_1), 4(x_1 - y_1)) \Rightarrow W_{11} = \frac{-2(R'_1 - R''_1) + 4(L'_1 - L''_1)}{2} - \left(\frac{u'_{11} + u''_{11}}{2}\right)$$

$$S_{11} = \max(-2(x_1 - y_1), 4(x_1 + z_1)) \Rightarrow S_{11} = \frac{4(R'_1 - R''_1) - 2(L'_1 - L''_1)}{2} + \left(\frac{v'_{11} + v''_{11}}{2}\right)$$

with two additional equations:

$$-2(R'_1 - R''_1) - 4(L'_1 - L''_1) = u'_{11} - u''_{11}$$

$$-2(L'_1 - L''_1) - 4(R'_1 - R''_1) = v'_{11} - v''_{11}$$

$$W_{12} = \min(4(x_2 - y_2), 6(x_2 - y_2)) \Rightarrow W_{12} = 5(L'_2 - L''_2) - (L'_2 + L''_2)$$

$$S_{12} = \max(4(x_2 + z_2), 6(x_2 + z_2)) \Rightarrow S_{12} = 5(R'_2 - R''_2) + (R'_2 + R''_2)$$

$$W_{13} = \min(-5(x_3 + z_3), -2(x_3 + z_3)) \Rightarrow \frac{-7}{2}(R'_3 - R''_3) - \frac{3}{2}(R'_3 + R''_3)$$

$$S_{13} = \max(-5(x_3 - y_3), -2(x_3 - y_3)) \Rightarrow \frac{-7}{2}(L'_3 - L''_3) + \frac{3}{2}(L'_3 + L''_3)$$

$$W'_{11} = \min(3(x_1 - y_1), 6(x_1 - y_1)) \Rightarrow \frac{9}{2}(L'_1 - L''_1) - \frac{3}{2}(L'_1 + L''_1)$$

$$S'_{11} = \max(3(x_1 + z_1), 6(x_1 + z_1)) \Rightarrow \frac{9}{2}(R'_1 - R''_1) + \frac{3}{2}(R'_1 + R''_1)$$

$$W'_{12} = \min(2(x_2 - y_2), 3(x_2 - y_2)) \Rightarrow \frac{5}{2}(L'_2 - L''_2) - \frac{1}{2}(L'_2 + L''_2)$$

$$S'_{12} = \max(2(x_2 + z_2), 3(x_2 + z_2)) \Rightarrow \frac{5}{2}(R'_2 - R''_2) + \frac{1}{2}(R'_2 + R''_2)$$

$$W'_{13} = \min((x_3 - y_3), 3(x_3 - y_3)) \Rightarrow 2(L'_3 - L''_3) - (L'_3 + L''_3)$$

$$S'_{13} = \max((x_3 + z_3), 3(x_3 + z_3)) \Rightarrow 2(R'_3 - R''_3) + (R'_3 + R''_3)$$

$$W_{21} = \min((x_1 - y_1), 3(x_1 - y_1)) \Rightarrow 2(L'_1 - L''_1) - (L'_1 + L''_1)$$

$$S_{21} = \max((x_1 + z_1), 3(x_1 + z_1)) \Rightarrow 2(R'_1 - R''_1) + (R'_1 + R''_1)$$

Similarly, this can be obtained  $W'_{22}, S'_{22}, W'_{23}, S'_{23}, W'_{31}, S'_{31}, W'_{32}, S'_{32}, W'_{33}, S'_{33}$ .

**Step 4.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the dual fully fuzzy linear system, obtained in Step 3, can be transformed into the following dual fully fuzzy linear system.

$$\begin{aligned} & (3x_1 + 5x_2 - 4x_3, 3x_1 - W_{11} + 5x_2 - W_{12} - 4x_3 - W_{13}, S_{11} - 3x_1 + S_{12} - 5x_2 + S_{13} \\ & \quad + 4x_3) \oplus (18, 2, 1) \\ = & (5x_1 + 2x_2 + 2x_3, 5x_1 - W'_{11} + 2x_2 - W'_{12} + 2x_3 - W'_{13}, S'_{11} - 5x_1 + S'_{12} - 2x_2 + S'_{13} \\ & \quad - 2x_3) \oplus (2, 10, 12) \end{aligned}$$

$$\begin{aligned} & (2x_1 + 3x_2 + 2x_3, 2x_1 - W_{21} + 3x_2 - W_{22} + 2x_3 - W_{23}, S_{21} - 2x_1 + S_{22} - 3x_2 + S_{23} \\ & \quad - 2x_3) \oplus (10, 20, 10) \\ = & (3x_1 + 2x_2 + 3x_3, 3x_1 - W'_{21} + 2x_2 - W'_{22} + 3x_3 - W'_{23}, S'_{21} - 3x_1 + S'_{22} - 2x_2 + S'_{23} \\ & \quad - 3x_3) \oplus (8, 3, 14) \end{aligned}$$

and

$$\begin{aligned} & (-2x_1 - 4x_2 + 4x_3, -2x_1 - W_{31} - 4x_2 - W_{32} + 4x_3 - W_{33}, S_{31} + 2x_1 + S_{32} + 4x_2 + S_{33} \\ & \quad - 4x_3) \oplus (24, 24, 1) \end{aligned}$$

$$= (-4x_1 + 2x_2 + 3x_3, -4x_1 - W'_{31} + 2x_2 - W'_{32} + 3x_3 - W'_{33}, S'_{31} + 4x_1 + S'_{32} - 2x_2 + S'_{33} - 3x_3) \oplus (8,1,6)$$

**Step 5.** Using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \Rightarrow m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ , the dual fully fuzzy linear system can be transformed into the following crisp system of linear equations.

$$-2x_1 + 3x_2 - 6x_3 + 16 = 0$$

$$-2x_1 + 3x_2 - 6x_3 + L'_1 - 4L''_1 - 2L'_2 + 3L''_2 + L'_3 - 2L''_3 + R'_1 - R''_1 + 5R'_3 - 2R''_3 + 0.5U'_{11} + 0.5U''_{11} - 8 = 0$$

$$2x_1 - 3x_2 + 6x_3 - L'_1 + L''_1 - 4R'_1 + 2R''_1 + 3R'_2 - 2R''_2 - 3L'_3 - 2R'_3 + R''_3 + 0.5V'_{11} + 0.5V''_{11} - 11 = 0$$

$$-x_1 + x_2 - x_3 + 2 = 0$$

$$-x_1 + x_2 + x_3 + L'_1 - L''_1 - 0.5L'_2 + 2.5L''_2 - R'_2 + R''_2 - 0.5U'_{12} - 0.5U''_{12} + 17 = 0$$

$$-x_1 + x_2 + x_3 - 3R'_1 - R''_1 + 2.5R'_2 - 0.5R''_2 + L'_2 - L''_2 - 0.5V'_{21} - 0.5V''_{21} - 4 = 0$$

$$2x_1 - 2x_2 + x_3 + 16 = 0$$

$$2x_1 - 6x_2 + x_3 - 3R'_1 + 3R'_2 - R'_3 - 7L''_3 + 1.5L'_2 - 1.5L''_2 - 0.5U'_{21} - 0.5U''_{21} - 0.5U'_{33} - 0.5U''_{33} + 16 = 0$$

$$-2x_1 + 6x_2 - x_3 - L'_1 + 3L''_1 + 2L''_2 + 2L'_3 - 4L''_3 + 3R'_3 - 1.5R'_2 + 1.5R''_2 - 0.5V'_{21} - 0.5V''_{21} - 0.5V'_{33} - 0.5V''_{33} - 6 = 0$$

$$-2R'_1 + 2R''_1 - 4L'_1 + 4L''_1 - U'_{11} + U''_{11} = 0$$

$$-2L'_1 + 2L''_1 - 4R'_1 + R''_1 - V'_{11} + V''_{11} = 0$$

$$-2R'_2 + 2R''_2 - 3L'_2 + 3L''_2 - U'_{21} + U''_{21} = 0$$

$$-2L'_2 + 2L''_2 - 3R'_2 + 3R''_2 - V'_{21} + V''_{21} = 0$$

$$-2R'_3 + 2R''_3 - 4L'_3 + 4L''_3 - U'_{33} + U''_{33} = 0$$

$$-2L'_3 + 2L''_3 - 4R'_3 + 4R''_3 - V'_{33} + V''_{33} = 0$$

$$x_1, x_2, x_3, R'_1, R'_2, R''_1, R''_2, R'_3, R''_3, L'_1, L''_1, L'_2, L''_2, L'_3, L''_3, U'_{11}, U''_{11}, V'_{11}, V''_{11}, U'_{21}, U''_{21}, V'_{21}, V''_{21}, U'_{33}, U''_{33}, V'_{33}, V''_{33} \geq 0$$

**Step 6.** Using Phase 1 of the two-phase method, the following linear programming problem can be used to find the solution of crisp system of linear equations, obtained in Step 5.

$$\text{Minimize } Z = Z_1 + Z_2 + Z_3 + Z'_1 + Z'_2 + Z'_3 + Z''_1 + Z''_2 + Z''_3$$

subject to

$$-2x_1 + 3x_2 - 6x_3 + 16 + Z_1 = 0$$

$$\begin{aligned} -2x_1 + 3x_2 - 6x_3 + L'_1 - 4L''_1 - 2L'_2 + 3L''_2 + L'_3 - 2L''_3 + R'_1 - R''_1 + 5R'_3 - 2R''_3 + 0.5U'_{11} \\ + 0.5U''_{11} - 8 + Z_2 = 0 \end{aligned}$$

$$\begin{aligned} 2x_1 - 3x_2 + 6x_3 - L'_1 + L''_1 - 4R'_1 + 2R''_1 + 3R'_2 - 2R''_2 - 3L'_3 - 2R'_3 + R''_3 + 0.5V'_{11} \\ + 0.5V''_{11} - 11 + Z_3 = 0 \end{aligned}$$

$$-x_1 + x_2 - x_3 + 2 + Z'_1 = 0$$

$$-x_1 + x_2 + x_3 + L'_1 - L''_1 - 0.5L'_2 + 2.5L''_2 - R'_2 + R''_2 - 0.5U'_{12} - 0.5U''_{12} + 17 + Z'_2 = 0$$

$$-x_1 + x_2 + x_3 - 3R'_1 - R''_1 + 2.5R'_2 - 0.5R''_2 + L'_2 - L''_2 - 0.5V'_{21} - 0.5V''_{21} - 4 + Z'_3 = 0$$

$$2x_1 - 2x_2 + x_3 + 16 + Z''_1 = 0$$

$$2x_1 - 6x_2 + x_3 - 3R'_1 + 3R'_2 - R'_3 - 7L''_3 + 1.5L'_2 - 1.5L''_2 - 0.5U'_{21} - 0.5U''_{21} - 0.5U'_{33} - 0.5U''_{33} + 16 + Z''_2 = 0$$

$$-2x_1 + 6x_2 - x_3 - L'_1 + 3L''_1 + 2L''_2 + 2L'_3 - 4L''_3 + 3R'_3 - 1.5R'_2 + 1.5R''_2 - 0.5V'_{21} - 0.5V''_{21} - 0.5V'_{33} - 0.5V''_{33} - 6 + Z''_3 = 0$$

$$-2R'_1 + 2R''_1 - 4L'_1 + 4L''_1 - U'_{11} + U''_{11} = 0$$

$$-2L'_1 + 2L''_1 - 4R'_1 + R''_1 - V'_{11} + V''_{11} = 0$$

$$-2R'_2 + 2R''_2 - 3L'_2 + 3L''_2 - U'_{21} + U''_{21} = 0$$

$$-2L'_2 + 2L''_2 - 3R'_2 + 3R''_2 - V'_{21} + V''_{21} = 0$$

$$-2R'_3 + 2R''_3 - 4L'_3 + 4L''_3 - U'_{33} + U''_{33} = 0$$

$$-2L'_3 + 2L''_3 - 4R'_3 + 4R''_3 - V'_{33} + V''_{33} = 0$$

$$x_1, x_2, x_3, R'_1, R'_2, R''_1, R''_2, R'_3, R''_3, L'_1, L''_1, L'_2, L''_2, L'_3, L''_3, U'_{11}, U''_{11}, V'_{11}, V''_{11}, U'_{21}, U''_{21}, V'_{21}, V''_{21}, U'_{33}, U''_{33}, V'_{33}, V''_{33} \geq 0$$

On solving the above LPP,  $x_1 = 2, x_2 = 4, x_3 = 4, L'_1 = 1, L''_1 = 0, L'_2 = 4, L''_2 = 0, L'_3 = 3, L''_3 = 0, R'_1 = 3, R''_1 = 0, R'_2 = 5, R''_2 = 0, R'_3 = 4, R''_3 = 0, U'_{11} = -1, U''_{11} = 9, V'_{11} = -11, V''_{11} = 3, U'_{21} = 6, U''_{21} = 28, V'_{21} = -6, V''_{21} = 17, V'_{33} = -8, V''_{33} = 18, Z_2 = 0, Z_3 = 0, Z'_1 = 0, Z'_2 = 0, Z'_3 = 0, Z''_1 = 0, Z''_2 = 0, Z''_3 = 0,$

**Step 7.** Putting the values of  $L'_j, L''_j, R'_j, R''_j, x_j$  obtained from Step 6, in  $\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \forall j = 1, 2, \dots, n,$

$$\tilde{x}_1 = (2, 1, 1), \tilde{x}_2 = (4, 0, 1) \text{ and } \tilde{x}_3 = (4, 1, 0)$$

#### 5.4. Solution of dual fully fuzzy linear system with non-negative restriction on elements of unknown vector

In this section, special case of  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  has been discussed in which the elements of coefficient matrix are unrestricted triangular fuzzy numbers whereas the elements of unknown vector are non-negative triangular fuzzy numbers.

The steps of the proposed method are as follows:

**Step 1** Substituting  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ ,  $\tilde{x} = (\tilde{x}_i)_{n \times 1}$ ,  $\tilde{C} = (\tilde{c}_{ij})_{m \times n}$ ,  $\tilde{d} = (\tilde{d}_i)_{m \times 1}$  and  $\tilde{b} = (\tilde{b}_i)_{m \times 1}$  the  $m \times n$  dual fully fuzzy linear system may be written as

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \oplus \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \otimes \tilde{x}_j \oplus \tilde{d}_i \quad \forall i = 1, 2, \dots, m$$

**Step 2** If all the parameters  $\tilde{a}_{ij}$ ,  $\tilde{x}_j$ ,  $\tilde{c}_{ij}$ ,  $\tilde{d}_i$  and  $\tilde{b}_i$  are represented by triangular fuzzy numbers  $(a_{ij}, m_{ij}, n_{ij})$ ,  $(x_j, y_j, z_j) \geq 0$ ,  $(c_{ij}, p_{ij}, q_{ij})$ ,  $(d_i, e_i, f_i)$  and  $(b_i, g_i, h_i)$  respectively, then the dual fully fuzzy linear system, obtained in Step 1, may be written as:

$$\sum_{j=1}^n (a_{ij}, m_{ij}, n_{ij}) \otimes (x_j, y_j, z_j) \oplus (b_i, g_i, h_i) = \sum_{j=1}^n (c_{ij}, p_{ij}, q_{ij}) \otimes (x_j, y_j, z_j) \oplus (d_i, e_i, f_i)$$

$$\forall i = 1, 2, \dots, m.$$

**Step 3** Assuming  $(a_{ij}, m_{ij}, n_{ij}) \otimes (x_j, y_j, z_j) = (A_{ij}, M_{ij}, N_{ij})$ , and  $(c_{ij}, p_{ij}, q_{ij}) \otimes (x_j, y_j, z_j) = (C_{ij}, P_{ij}, Q_{ij})$  the dual fully fuzzy linear system, obtained in Step 2, may be written as:

$$\sum_{j=1}^n (A_{ij}, M_{ij}, N_{ij}) \oplus (b_i, g_i, h_i) = \sum_{j=1}^n (C_{ij}, P_{ij}, Q_{ij}) \oplus (d_i, e_i, f_i) \quad \forall i = 1, 2, \dots, m$$

Using arithmetic operations, defined in Section 3.1,

$$(A_{ij}, M_{ij}, N_{ij}) = (a_{ij}, m_{ij}, n_{ij}) \otimes (x_j, y_j, z_j)$$

$$A_{ij} = a_{ij}x_j$$

$$M_{ij} = A_{ij} - R_{ij}$$

$$N_{ij} = S_{ij} - A_{ij}$$

$$R_{ij} = \min \left( (a_{ij} - m_{ij})(x_j - y_j), (a_{ij} - m_{ij})(x_j + z_j) \right)$$

$$S_{ij} = \max \left( (a_{ij} + n_{ij})(x_j - y_j), (a_{ij} + n_{ij})(x_j + z_j) \right)$$

Similarly we obtain  $C_{ij}, P_{ij}, Q_{ij}$  as follows:

$$C_{ij} = c_{ij}x_j, P_{ij} = C_{ij} - T_{ij}, Q_{ij} = V_{ij} - C_{ij}$$

$$T_{ij} = \min \left( (c_{ij} - p_{ij})(x_j - y_j), (c_{ij} - p_{ij})(x_j + z_j) \right)$$

$$V_{ij} = \max \left( (c_{ij} + q_{ij})(x_j - y_j), (c_{ij} + q_{ij})(x_j + z_j) \right)$$

**Step 4** The dual fully fuzzy linear system obtained in Step 3, is transformed into crisp system of linear equations

$$\begin{cases} \sum_{j=1}^n A_{ij} + b_i = \sum_{j=1}^n C_{ij} + d_i \\ \sum_{j=1}^n M_{ij} + g_i = \sum_{j=1}^n P_{ij} + e_i \\ \sum_{j=1}^n N_{ij} + h_i = \sum_{j=1}^n Q_{ij} + f_i \end{cases}$$

This can be further reduced as

$$\left\{ \begin{array}{l} \sum_{j=1}^n A_{ij} + b_i = \sum_{j=1}^n C_{ij} + d_i \\ \sum_{j=1}^n (A_{ij} - R_{ij}) + g_i = \sum_{j=1}^n (C_{ij} - T_{ij}) + e_i \\ \sum_{j=1}^n (S_{ij} - A_{ij}) + h_i = \sum_{j=1}^n (V_{ij} - C_{ij}) + f_i \end{array} \right.$$

which can be further reduced as

$$\left\{ \begin{array}{l} \sum_{j=1}^n a_{ij}x_j + b_i = \sum_{j=1}^n c_{ij}x_j + d_i \\ \sum_{j=1}^n R_{ij} - \sum_{j=1}^n T_{ij} = (d_i - e_i) - (b_i - g_i) \\ \sum_{j=1}^n S_{ij} - \sum_{j=1}^n V_{ij} = (d_i + f_i) - (b_i + h_i) \end{array} \right.$$

where

$$R_{ij} = \begin{cases} (a_{ij} - m_{ij})(x_j - y_j) & \text{if } a_{ij} - m_{ij} \geq 0 \\ (a_{ij} - m_{ij})(x_j + z_j) & \text{if } a_{ij} - m_{ij} < 0 \end{cases}$$

$$T_{ij} = \begin{cases} (c_{ij} - p_{ij})(x_j - y_j) & \text{if } c_{ij} - p_{ij} \geq 0 \\ (c_{ij} - p_{ij})(x_j + z_j) & \text{if } c_{ij} - p_{ij} < 0 \end{cases}$$

$$S_{ij} = \begin{cases} (a_{ij} + n_{ij})(x_j - y_j) & \text{if } a_{ij} + n_{ij} < 0 \\ (a_{ij} + n_{ij})(x_j + z_j) & \text{if } a_{ij} + n_{ij} \geq 0 \end{cases}$$

$$V_{ij} = \begin{cases} (c_{ij} + q_{ij})(x_j - y_j) & \text{if } c_{ij} + q_{ij} < 0 \\ (c_{ij} + q_{ij})(x_j + z_j) & \text{if } c_{ij} + q_{ij} \geq 0 \end{cases}$$

**Step 5.** The solution of system obtained in Step 4, can be obtained by solving the following linear programming problem, which is obtained by adding the artificial variable  $Z_i, \forall i = 1, 2, \dots, m$ .

Minimize

$$\bar{Z} = \sum_{i=1}^n Z_i + \sum_{i=1}^n Z'_i + \sum_{i=1}^n Z''_i$$

subject to

$$\sum_{j=1}^n a_{ij}^1 - \sum_{j=1}^n c_{ij}^1 + Z_i = d_i - b_i \quad ; \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n R_{ij} - \sum_{j=1}^n T_{ij} + Z'_i = (d_i - e_i) - (b_i - g_i); \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n S_{ij} - \sum_{j=1}^n V_{ij} + Z''_i = (d_i + f_i) \quad ; \quad i = 1, 2, \dots, m$$

$$y_i - x_i \geq 0 \quad \forall i = 1, 2, \dots, m$$

**Step 6.** Solve the above linear programming problem and put the above values in

$$\tilde{x}_j = (x_j, x_j - L'_j + L''_j, R'_j - R''_j - x_j) \quad \forall j = 1, 2, \dots, n$$

to find the solution of dual fully fuzzy linear system. If the linear programming problem is inconsistent and does not yield any solution then the given dual fully fuzzy linear system is inconsistent and has no solution.

**Remark 5.1.** The  $m \times n$  dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  is not equivalent to the fully fuzzy linear system  $\tilde{M} \otimes \tilde{x} = \tilde{n}$  where  $\tilde{M} = \tilde{A} \ominus \tilde{C}$  and  $\tilde{n} = \tilde{d} \ominus \tilde{b}$  because for a fuzzy number say  $\tilde{A}$ , the statement  $\tilde{A} \ominus \tilde{B} = \tilde{C}$  does not implies  $\tilde{A} = \tilde{C} \oplus \tilde{B}$

**Remark 5.2.** The  $m \times n$  dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  can be reduced to standard fully fuzzy linear system  $\tilde{M} \otimes \tilde{x} = \tilde{n}$  if we take  $\tilde{A} = \tilde{M}, \tilde{b} = \tilde{0}, \tilde{C} = \tilde{0}$  and  $\tilde{d} = \tilde{n}$ .

#### 5.4.1. Solution of dual fully fuzzy linear system chosen in Example 5.2

In this section, the solution of the dual fully fuzzy linear system chosen in Example 5.2, is obtained by the method proposed in Section 5.4.

Using the method proposed in Section 5.4, the solution of dual fully fuzzy linear system, chosen in Example 5.2, can be obtained as follows:

**Step 1.** Using the multiplication of unrestricted triangular fuzzy numbers with non-negative triangular fuzzy numbers, defined in Section 2.1.2, the dual fully fuzzy linear system, chosen in Example 5.2, can be transformed into the following dual fully fuzzy linear system.

$$\begin{aligned} & (2x_1, 2x_1 - \min(-1(x_1 - y_1), -1(x_1 + z_1)), \max(3(x_1 - y_1), 3(x_1 + z_1)) - 2x_1) \oplus \\ & (3x_2, 3x_2 - \min(2(x_2 - y_2), 2(x_2 + z_2)), \max(4(x_2 - y_2), 4(x_2 + z_2)) - 3x_2) \oplus (3, 55, 0) \\ & = (-4x_1, -4x_1 - \min(-5(x_1 - y_1), -5(x_1 + z_1)), \max(-3(x_1 - y_1), -3(x_1 + z_1)) + \\ & 4x_1) \oplus (-3x_2, -3x_2 - \min(-6(x_2 - y_2), -6(x_2 + z_2)), \max(-2(x_2 - y_2), -2(x_2 + \\ & z_2) + 3x_2) \oplus (45, 47, 3) \end{aligned}$$

and

$$\begin{aligned} & (-3x_1, -3x_1 - \min(-4(x_1 - y_1), -4(x_1 + z_1)), \max(-2(x_1 - y_1), -2(x_1 + z_1)) + 3x_1) \\ & \oplus (5x_2, 5x_2 - \min(4(x_2 - y_2), 4(x_2 + z_2)), \max(6(x_2 - y_2), 6(x_2 + z_2)) - 5x_2) \\ & \oplus (2, 17, 0) = \\ & (2x_1, 2x_1 - \min((x_1 - y_1), (x_1 + z_1)), \max(3(x_1 - y_1), 3(x_1 + z_1)) - 2x_1) \oplus \end{aligned}$$

$$(-2x_2, -2x_2 - \min(-5(x_2 - y_2), -5(x_2 + z_2)), \max(-1(x_2 - y_2), -1(x_2 + z_2) + 2x_2) \oplus (3,4,13).$$

**Step 2.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$  the dual fully fuzzy linear system, obtained in Step 1, can be transformed into the following dual fully fuzzy linear system.

$$\begin{aligned} & \left(2x_1, 2x_1 - \left(\left(\frac{-x_1 + y_1 - x_1 - z_1}{2}\right) - \left|\frac{-x_1 + y_1 + x_1 + z_1}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{3x_1 - 3y_1 + 3x_1 + 3z_1}{2}\right) + \left|\frac{3x_1 - 3y_1 - 3x_1 - 3z_1}{2}\right|\right) - 2x_1\right) \\ & \oplus \left(3x_2, 3x_2 - \left(\left(\frac{2x_2 - 2y_2 + 2x_2 + 2z_2}{2}\right) - \left|\frac{2x_2 - 2y_2 - 2x_2 - 2z_2}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{4x_2 - 4y_2 + 4x_2 + 4z_2}{2}\right) + \left|\frac{4x_2 - 4y_2 - 4x_2 - 4z_2}{2}\right|\right) - 3x_2\right) \oplus (3,55,0) \\ & = \left(-4x_1, -4x_1 - \left(\left(\frac{-5x_1 + 5y_1 - 5x_1 - 5z_1}{2}\right) - \left|\frac{-5x_1 + 5y_1 + 5x_1 + 5z_1}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{-3x_1 + 3y_1 - 3x_1 - 3z_1}{2}\right) + \left|\frac{-3x_1 + 3y_1 + 3x_1 + 3z_1}{2}\right|\right) + 4x_1\right) \\ & \oplus \left(-3x_2, -3x_2 \right. \\ & \quad \left. - \left(\left(\frac{-6x_2 + 6y_2 - 6x_2 - 6z_2}{2}\right) - \left|\frac{-6x_2 + 6y_2 + 6x_2 + 6z_2}{2}\right|\right), \right. \\ & \quad \left.\left(\left(\frac{-2x_2 + 2y_2 - 2x_2 - 2z_2}{2}\right) + \left|\frac{-2x_2 + 2y_2 + 2x_2 + 2z_2}{2}\right|\right) + 3x_2\right) \\ & \oplus (45,47,3) \end{aligned}$$

$$\begin{aligned}
& \left( -3x_1, -3x_1 - \left( \left( \frac{-4x_1 + 4y_1 - 4x_1 - 4z_1}{2} \right) - \left| \frac{-4x_1 + 4y_1 + 4x_1 + 4z_1}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{-2x_1 + 2y_1 - 2x_1 - 2z_1}{2} \right) + \left| \frac{-2x_1 + 2y_1 + 2x_1 + 2z_1}{2} \right| \right) + 3x_1 \right) \\
& \oplus \left( 5x_2, 5x_2 - \left( \left( \frac{4x_2 - 4y_2 + 4x_2 + 4z_2}{2} \right) - \left| \frac{4x_2 - 4y_2 - 4x_2 - 4z_2}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{6x_2 - 6y_2 + 6x_2 + 6z_2}{2} \right) + \left| \frac{6x_2 - 6y_2 - 6x_2 - 6z_2}{2} \right| \right) - 5x_2 \right) \oplus (2,17,0) \\
& = \left( 2x_1, 2x_1 - \left( \left( \frac{x_1 - y_1 + x_1 + z_1}{2} \right) - \left| \frac{x_1 - y_1 - x_1 - z_1}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{3x_1 - 3y_1 + 3x_1 + 3z_1}{2} \right) + \left| \frac{3x_1 - 3y_1 - 3x_1 - 3z_1}{2} \right| \right) - 2x_1 \right) \oplus \\
& \left( -2x_2, -2x_2 - \left( \left( \frac{-5x_2 + 5y_2 - 5x_2 - 5z_2}{2} \right) - \left| \frac{-5x_2 + 5y_2 + 5x_2 + 5z_2}{2} \right| \right), \right. \\
& \quad \left. \left( \left( \frac{-x_2 + y_2 - x_2 - z_2}{2} \right) + \left| \frac{-x_2 + y_2 + x_2 + z_2}{2} \right| \right) + 2x_2 \right) \oplus (3,4,13)
\end{aligned}$$

**Step 3.** Since  $x_1, y_1, z_1, x_2, y_2, z_2, \geq 0$  so dual fully fuzzy linear system, obtained in Step 2, can be transformed into following dual fully fuzzy linear system.

$$\begin{aligned}
& (2x_1, 2x_1 + (x_1 + z_1), 3(x_1 + z_1) - 2x_1) \oplus (3x_2, 3x_2 - 2(x_2 - y_2), 4(x_2 + z_2) - 3x_2) \oplus \\
& (3,55,0) = (-4x_1, -4x_1 + 5(x_1 + z_1), -3(x_1 - y_1) + 4x_1) \oplus (-3x_2, -3x_2 + \\
& 6(x_2 + z_2), -2(x_2 - y_2) + 3x_2) \oplus (45,47,3),
\end{aligned}$$

$$\begin{aligned}
& (-3x_1, -3x_1 + 4(x_1 + z_1), -2(x_1 - y_1) + 3x_1) \oplus (5x_2, 5x_2 - 4(x_2 - y_2), 6(x_2 + z_2) - \\
& 5x_2) \oplus (2,17,0) = (2x_1, 2x_1 - (x_1 - y_1), 3(x_1 + z_1) - 2x_1) \oplus (-2x_2, -2x_2 + \\
& 5(x_2 + z_2), -1(x_2 - y_2) + 2x_2) \oplus (3,4,13).
\end{aligned}$$

**Step 4.** Using the addition of two triangular fuzzy numbers, defined in Section 2.1.2., the dual fully fuzzy linear system, obtained in Step 3, and using the relation  $(m_1, \alpha_1, \beta_1) = (m_2, \alpha_2, \beta_2) \Rightarrow m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$ , the fully fuzzy linear system can be transformed into the following crisp system of linear equations.

$$6x_1 + 6x_2 = 42$$

$$-5x_1 + 7x_2 = 1$$

$$2x_1 - 4z_1 - 2x_2 - 2y_2 - 6z_2 = -8$$

$$-3y_1 - 2y_2 + 3z_1 + 4z_2 = 3$$

$$y_1 + 2x_2 - 4y_2 - 4z_1 + 5z_2 = 13$$

$$2y_1 - y_2 - 3z_1 + 6z_2 = 13$$

$$y_i - x_i \geq 0 \quad \forall i = 1, 2.$$

**Step 5.** Using Phase 1 of the two-phase method, the following linear programming problem can be used to find the solution of crisp system of linear equations, obtained in Step 4,

$$\text{Minimize } \bar{Z} = Z_1 + Z_2 + Z'_1 + Z'_2 + Z''_1 + Z''_2$$

subject to

$$6x_1 + 6x_2 + Z_1 = 42$$

$$-5x_1 + 7x_2 + Z_2 = 1$$

$$2x_1 - 4z_1 - 2x_2 - 2y_2 - 6z_2 + Z'_1 = -8$$

$$-3y_1 - 2y_2 + 3z_1 + 4z_2 + Z'_2 = 3$$

$$y_1 + 2x_2 - 4y_2 - 4z_1 + 5z_2 + Z''_1 = 13$$

$$2y_1 - y_2 - 3z_1 + 6z_2 + Z_2'' = 13$$

$$y_i - x_i \geq 0 \quad \forall i = 1, 2.$$

$$Z_1, Z_2, Z_1', Z_2', Z_1'', Z_2'' \geq 0$$

On solving the above linear programming problem,

$$x_1 = 4, y_1 = 1, z_1 = 0, x_2 = 3, y_2 = 1, z_2 = 2 \text{ and } Z_1, Z_2, Z_1', Z_2', Z_1'', Z_2'' = 0$$

Clearly  $x_i - y_i \geq 0, y_i \geq 0$  and  $z_i \geq 0$ . Hence the solution is feasible and can be written as

$$\tilde{x}_1 = (4, 1, 0) \text{ and } \tilde{x}_2 = (3, 1, 2).$$

### 5.5. Results and discussion

The solution of dual fully fuzzy linear system chosen in Example 5.1 and Example 5.2, are obtained by using the existing methods and the method proposed are shown in Table 5.1. It is obvious from the result shown in Table 5.1 that the dual fully fuzzy linear systems, chosen in Example 5.1 and Example 5.2. cannot be solved by any of the existing methods, but can be solved by the Mehar method-IV proposed in this chapter.

**Table 5.1:** Comparison of results by using proposed and existing method

Examples	Solution using proposed method	Solution using existing methods
Example 5.1	$\tilde{x}_1 = (2, 1, 1)$ $\tilde{x}_2 = (4, 0, 1)$ $\tilde{x}_3 = (4, 1, 0)$	Not applicable
Example 5.2	$\tilde{x}_1 = (4, 1, 0)$ $\tilde{x}_2 = (3, 1, 2)$	Not applicable

## 5.6. Conclusion

In this chapter, the solution of dual fully fuzzy linear system  $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$  is proposed, which is a generalised form of  $\tilde{A} \otimes \tilde{x} = \tilde{b}$ . The main advantage of this method is that the solution of dual fully fuzzy linear system can be obtained without any restriction on the coefficient matrix as well as on solution vector. Moreover the methods proposed in Chapter 2, 3 and 4 are the special cases of the method proposed in this chapter. The method is illustrated with the help of numerical examples.

## Chapter 6

# Mehar method-V for solving dual fully fuzzy matrix equation with unrestricted fuzzy coefficient and unrestricted fuzzy variables <sup>3</sup>

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Otadi and Mosleh [113] extended the concept of fully fuzzy linear system into fully fuzzy matrix equations and proposed a method to find the solution of such fully fuzzy matrix equations in which all the elements of unknown matrix are represented by non-negative triangular fuzzy numbers whereas the elements of coefficient matrix and right hand matrix are represented by unrestricted triangular fuzzy numbers. On the same direction, in this chapter, the concept of dual fully fuzzy linear system is extended into dual fully fuzzy matrix equations and a new method (named as Mehar method-V) is proposed to find the solution of such dual fully fuzzy matrix equation in which all the elements of all the matrices are represented by unrestricted trapezoidal fuzzy numbers. The proposed method is illustrated by numerical examples.

### 6.1 Preliminaries

In the previous chapters, triangular fuzzy numbers are used to represent the elements of matrices and unknown vector. Further  $(m, \alpha, \beta)$  representation of triangular fuzzy numbers have been used in the previous chapters. In this chapter, instead of  $(m, \alpha, \beta)$  representation of triangular fuzzy number  $(m - \alpha, m, m + \beta)$  is used to represent the triangular fuzzy number and  $(m - \alpha, m, n, m + \beta)$  is used to represent the trapezoidal fuzzy number. In this representation the definition, presented in Section 2.1.1 of Chapter 2, and arithmetic operations, presented in Section 2.1.2 of Chapter 2, may be defined as follows.

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<sup>3</sup> A part of this chapter is published in *Journal of Intelligent & Fuzzy Systems* **25** (2013) 747-753.

### 6.1.1. Basic definitions

In this section, some basic definitions related to fuzzy numbers as well as arithmetic operations on trapezoidal fuzzy numbers are represented.

**Definition 6.1.** A fuzzy number  $\tilde{M} = (a, b, c, d)$  is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{M}}(x) = \begin{cases} \frac{x-a}{b-a}, & a < x \leq b, \\ 1, & b \leq x < c, \\ \frac{d-x}{d-c}, & c \leq x < d \\ 0 & \text{otherwise} \end{cases}$$

where  $a, b, c, d \in R$

**Definition 6.2.** A trapezoidal fuzzy number  $\tilde{M} = (a, b, c, d)$  may be non-negative, negative, or near zero.

(1)  $\tilde{M}$  is a non-negative trapezoidal fuzzy number denoted by  $\tilde{M}_p = (a, b, c, d) \geq 0$  if  $a \geq 0$

(2)  $\tilde{M}$  is a negative trapezoidal fuzzy number denoted by  $\tilde{M}_n = (a, b, c, d) < 0$  if  $d < 0$

(3)  $\tilde{M}$  is a near zero trapezoidal fuzzy number denoted by  $\tilde{M}_o = (a, b, c, d) \equiv 0$  if  $a < 0 < d$ . Further there may be three categories

$$3.1. \tilde{M}_\alpha = (a, b, c, d) \equiv 0_1 \text{ if } a \leq b \leq c < 0 < d$$

$$3.2. \tilde{M}_\beta = (a, b, c, d) \equiv 0_2 \text{ if } a \leq b < 0 < c \leq d$$

$$3.3. \tilde{M}_\gamma = (a, b, c, d) \equiv 0_3 \text{ if } a < 0 \leq b \leq c \leq d$$

**Definition 6.3.** A trapezoidal fuzzy number  $\tilde{M} = (a, b, c, d)$  is said to be zero trapezoidal fuzzy number if and only if  $a = 0, b = 0, c = 0, d = 0$ .

**Definition 6.4.** Two trapezoidal fuzzy number  $\tilde{M} = (a, b, c, d)$  and  $\tilde{N} = (e, f, g, h)$  are said to be equal i.e.,  $\tilde{M} = \tilde{N}$  if and only if  $a = e, b = f, c = g, d = h$ .

### 6.1.2. Arithmetic operations on trapezoidal fuzzy numbers

Let  $\tilde{M} = (a, b, c, d)$  and  $\tilde{N} = (e, f, g, h)$  be two trapezoidal fuzzy numbers. Then,

$$(i) \quad \tilde{M} \oplus \tilde{N} = (a, b, c, d) \oplus (e, f, g, h) = (a + e, b + f, c + g, d + h)$$

$$(ii) \quad \tilde{M} \ominus \tilde{N} = (a, b, c, d) \ominus (e, f, g, h) = (a - h, b - g, c - f, d - e)$$

$$(iii) \quad \text{If } \tilde{N} = (e, f, g, h) \geq 0 \text{ then,}$$

$$(a, b, c, d) \otimes (e, f, g, h) = (\min(ae, ah), \min(bf, bg), \max(cf, cg), \max(ae, ah))$$

Similarly if  $\tilde{M} = (a, b, c, d) \geq 0$ , then

$$(a, b, c, d) \otimes (e, f, g, h) = (\min(ae, de), \min(bf, cf), \max(bg, cg), \max(ah, dh))$$

where,

$$\min(ae, de) = \left(\frac{a+d}{2}\right)e - \left|\frac{a-d}{2}\right||e|$$

$$\min(bf, cf) = \left(\frac{b+c}{2}\right)f - \left|\frac{b-c}{2}\right||f|$$

$$\max(bg, cg) = \left(\frac{b+c}{2}\right)g + \left|\frac{b-c}{2}\right||g|$$

$$\max(ah, dh) = \left(\frac{a+d}{2}\right)h + \left|\frac{a-d}{2}\right||h|$$

Since,  $d \geq a$  and  $c \geq b$  therefore  $\left|\frac{a-d}{2}\right|$ ,  $\left|\frac{b-c}{2}\right|$  can be replaced by  $\left(\frac{d-a}{2}\right)$  and  $\left(\frac{c-b}{2}\right)$

respectively in above set of equations.

For two trapezoidal fuzzy numbers  $\tilde{M} = (a, b, c, d)$  and  $\tilde{N} = (e, f, g, h)$  the extended multiplication operation between two fuzzy numbers may be defined as follows

$$\tilde{M} \otimes \tilde{N} = \begin{cases} (\min(ae, de), \min(bf, cf), \max(bg, cg), \max(ah, dh)) & \text{if } \tilde{M} = \tilde{M}_p \\ (\min(ah, dh), \min(bg, cg), \max(bf, cf), \max(ae, de)) & \text{if } \tilde{M} = \tilde{M}_n \\ (\min(ah, de), \min(bg, cg), \max(bf, cf), \max(ae, dh)) & \text{if } \tilde{M} = \tilde{M}_\alpha \\ (\min(ah, de), \min(bf, cg), \max(bg, cf), \max(ae, dh)) & \text{if } \tilde{M} = \tilde{M}_\beta \\ (\min(ah, de), \min(bf, cf), \max(bg, cg), \max(ae, dh)) & \text{if } \tilde{M} = \tilde{M}_\gamma \end{cases}$$

## 6.2. Limitations of the existing methods

The existing method [113] cannot be used to find the solution of dual fully fuzzy matrix equations  $\tilde{A} \otimes \tilde{X} \oplus \tilde{B} = \tilde{C} \otimes \tilde{X} \oplus \tilde{D}$  where  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$  and  $\tilde{X}$  are fuzzy number matrices.

For example, it is not possible to solve the dual fully fuzzy matrix equations chosen in given Example 6.1 and Example 6.2, by using the existing methods due to unrestricted trapezoidal fuzzy numbers.

### Example 6.1.

$$\begin{bmatrix} (1,2,3,4) & (-5, -2, -2, -1) \\ (-5,2,3,4) & (2,3,4,4) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} \oplus \begin{bmatrix} (-15, -6, -6, 5) & (0,0,0,0) \\ (0,0,0,0) & (-4, -2, -2, 1) \end{bmatrix} = \\ \begin{bmatrix} (0,0,0,0) & (1,3,4,4) \\ (-4, -2, -1, -1) & (0,0,0,0) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} \oplus \begin{bmatrix} (-1,8,10,33) & (-32, -16, -16, -7) \\ (-15, -2,1,1) & (-3,24,33,44) \end{bmatrix}$$

where  $\tilde{x}_{11} = (x_{11}, y_{11}, z_{11}, w_{11})$ ,  $\tilde{x}_{12} = (x_{12}, y_{12}, z_{12}, w_{12})$ ,  $\tilde{x}_{21} = (x_{21}, y_{21}, z_{21}, w_{21})$ ,  $\tilde{x}_{22} = (x_{22}, y_{22}, z_{22}, w_{22})$  are unrestricted trapezoidal fuzzy numbers.

### Example 6.2.

$$\begin{bmatrix} (-2,2,4,5) & (1,2,3,6) \\ (2,3,4,6) & (3,4,5,7) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} \oplus \begin{bmatrix} (2,6,8,9) & (4,5,10,11) \\ (5,9,12,13) & (4,9,13,15) \end{bmatrix} = \\ \begin{bmatrix} (1,2,3,6) & (-1,3,4,5) \\ (3,4,5,6) & (4,5,6,8) \end{bmatrix} \otimes \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} \oplus \begin{bmatrix} (2,4,6,11) & (2,4,9,13) \\ (1,3,4,7) & (1,2,4,8) \end{bmatrix}$$

where,

$$\tilde{x}_{11} = (x_{11}, y_{11}, z_{11}, w_{11}) \geq 0, \tilde{x}_{12} = (x_{12}, y_{12}, z_{12}, w_{12}) \geq 0, \tilde{x}_{21} = (x_{21}, y_{21}, z_{21}, w_{21}) \geq 0, \tilde{x}_{22} = (x_{22}, y_{22}, z_{22}, w_{22}) \geq 0$$

### 6.3. Proposed Mehar method-V

In this section, to overcome the limitations of the existing methods, pointed out in Section 6.2, a new method (named as Mehar method-V) is proposed to find solution of dual fully fuzzy matrix equation  $\tilde{A} \otimes \tilde{X} \oplus \tilde{B} = \tilde{C} \otimes \tilde{X} \oplus \tilde{D}$ .

The steps of the proposed Mehar method-V are as follows:

**Step 1.** Substituting  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ ,  $\tilde{x} = (\tilde{x}_{ij})_{n \times k}$  and  $\tilde{b} = (\tilde{b}_{ij})_{m \times k}$ ,  $\tilde{C} = (\tilde{c}_{ij})_{m \times n}$ ,  $\tilde{d} = (\tilde{d}_{ij})_{m \times k}$  the  $m \times k$  dual fully fuzzy matrix equation may be written as:

$$\sum_{j=1, \dots, n}^{\oplus} \tilde{a}_{ij} \otimes \tilde{x}_{ij} \oplus \tilde{b}_{ij} = \sum_{j=1, \dots, n}^{\oplus} \tilde{c}_{ij} \otimes \tilde{x}_{ij} \oplus \tilde{d}_{ij} \quad \forall i = 1, 2, \dots, m$$

**Step 2.** If all the parameters  $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij}, \tilde{d}_{ij}$  are represented by trapezoidal fuzzy numbers  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}), (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}), (\alpha''_{ik}, \beta''_{ik}, \gamma''_{ik}, \delta''_{ik}), (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij})$  and  $\tilde{x}_j = (x_{kj}, y_{kj}, z_{kj}, w_{kj})$  respectively, then the dual fully fuzzy matrix equation obtained in Step 1, can be transformed into the following dual fully fuzzy matrix equation

$$\begin{aligned} & \sum_{j=1}^n (\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) \oplus (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}) \\ & = \sum_{j=1}^n (\alpha''_{ik}, \beta''_{ik}, \gamma''_{ik}, \delta''_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) \oplus (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij}) \quad \forall i \\ & = 1, 2, \dots, m \end{aligned}$$

**Step 3.** Assuming  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) = (f_{ik}^j, p_{ik}^j, q_{ik}^j, r_{ik}^j)$ , and  $(\alpha''_{ik}, \beta''_{ik}, \gamma''_{ik}, \delta''_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) = (F_{ik}^j, P_{ik}^j, Q_{ik}^j, R_{ik}^j)$ ,  $1 \leq i, j, k \leq n$  where each  $(x_{kj}, y_{kj}, z_{kj}, w_{kj})$  is an unrestricted trapezoidal fuzzy number, the dual fully fuzzy matrix equation, obtained in Step 2, can be transformed into the following dual fully fuzzy matrix equation:

$$\begin{aligned} & \sum_{j=1, \dots, n}^{\oplus} (f_{ik}^j, p_{ik}^j, q_{ik}^j, r_{ik}^j) \oplus (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}) \\ &= \sum_{j=1, \dots, n}^{\oplus} (F_{ik}^j, P_{ik}^j, Q_{ik}^j, R_{ik}^j) \oplus (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij}) \quad \forall i = 1, 2, \dots, m \end{aligned}$$

**Case 1.** If  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \geq 0$  i.e.,  $\alpha'_{ik} \geq 0$

$$f_{ik}^j = \min(\alpha'_{ik} x_{kj}, \delta'_{ik} x_{kj}) = \frac{\alpha'_{ik} + \delta'_{ik}}{2} x_{kj} - \frac{\delta'_{ik} - \alpha'_{ik}}{2} |x_{kj}|$$

$$p_{ik}^j = \min(\beta'_{ik} y_{kj}, \gamma'_{ik} y_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} y_{kj} - \frac{\gamma'_{ik} - \beta'_{ik}}{2} |y_{kj}|$$

$$q_{ik}^j = \max(\beta'_{ik} z_{kj}, \gamma'_{ik} z_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} z_{kj} + \frac{\gamma'_{ik} - \beta'_{ik}}{2} |z_{kj}|$$

$$r_{ik}^j = \max(\alpha'_{ik} w_{kj}, \delta'_{ik} w_{kj}) = \frac{\alpha'_{ik} + \delta'_{ik}}{2} w_{kj} + \frac{\delta'_{ik} - \alpha'_{ik}}{2} |w_{kj}|$$

**Case 2.** If  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) < 0$  i.e.,  $\delta'_{ik} \geq 0$

$$f_{ik}^j = \min(\alpha'_{ik} w_{kj}, \delta'_{ik} w_{kj}) = \frac{\alpha'_{ik} + \delta'_{ik}}{2} w_{kj} - \frac{\delta'_{ik} - \alpha'_{ik}}{2} |w_{kj}|$$

$$p_{ik}^j = \min(\beta'_{ik} z_{kj}, \gamma'_{ik} z_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} z_{kj} - \frac{\gamma'_{ik} - \beta'_{ik}}{2} |z_{kj}|$$

$$q_{ik}^j = \max(\beta'_{ik} y_{kj}, \gamma'_{ik} y_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} y_{kj} + \frac{\gamma'_{ik} - \beta'_{ik}}{2} |y_{kj}|$$

$$r_{ik}^j = \max(\alpha'_{ik} x_{kj}, \delta'_{ik} x_{kj}) = \frac{\alpha'_{ik} + \delta'_{ik}}{2} x_{kj} + \frac{\delta'_{ik} - \alpha'_{ik}}{2} |x_{kj}|$$

**Case 3.** If  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \equiv 0_1$  i. e.,  $\alpha'_{ik} \leq \beta'_{ik} \leq \gamma'_{ik} < 0 < \delta'_{ik}$

$$f_{ik}^j = \min(\alpha'_{ik} w_{kj}, \delta'_{ik} x_{kj}) = \frac{\alpha'_{ik} w_{kj} + \delta'_{ik} x_{kj}}{2} - \left| \frac{\alpha'_{ik} w_{kj} - \delta'_{ik} x_{kj}}{2} \right|$$

$$p_{ik}^j = \min(\beta'_{ik} z_{kj}, \gamma'_{ik} z_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} z_{kj} - \frac{\gamma'_{ik} - \beta'_{ik}}{2} |z_{kj}|$$

$$q_{ik}^j = \max(\beta'_{ik} y_{kj}, \gamma'_{ik} y_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} y_{kj} + \frac{\gamma'_{ik} - \beta'_{ik}}{2} |y_{kj}|$$

$$r_{ik}^j = \max(\alpha'_{ik} x_{kj}, \delta'_{ik} w_{kj}) = \frac{\alpha'_{ik} x_{kj} + \delta'_{ik} w_{kj}}{2} + \left| \frac{\delta'_{ik} w_{kj} - \alpha'_{ik} x_{kj}}{2} \right|$$

**Case 4.** If  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \equiv 0_2$  i. e.,  $\alpha'_{ik} \leq \beta'_{ik} < 0 < \gamma'_{ik} \leq \delta'_{ik}$

$$f_{ik}^j = \min(\alpha'_{ik} w_{kj}, \delta'_{ik} x_{kj}) = \frac{\alpha'_{ik} w_{kj} + \delta'_{ik} x_{kj}}{2} - \left| \frac{\alpha'_{ik} w_{kj} - \delta'_{ik} x_{kj}}{2} \right|$$

$$p_{ik}^j = \min(\beta'_{ik} y_{kj}, \gamma'_{ik} z_{kj}) = \frac{\beta'_{ik} y_{kj} + \gamma'_{ik} z_{kj}}{2} - \left| \frac{\beta'_{ik} y_{kj} - \gamma'_{ik} z_{kj}}{2} \right|$$

$$q_{ik}^j = \max(\beta'_{ik} z_{kj}, \gamma'_{ik} y_{kj}) = \frac{\beta'_{ik} z_{kj} + \gamma'_{ik} y_{kj}}{2} + \left| \frac{\beta'_{ik} z_{kj} - \gamma'_{ik} y_{kj}}{2} \right|$$

$$r_{ik}^j = \max(\alpha'_{ik} x_{kj}, \delta'_{ik} w_{kj}) = \frac{\alpha'_{ik} x_{kj} + \delta'_{ik} w_{kj}}{2} + \left| \frac{\delta'_{ik} w_{kj} - \alpha'_{ik} x_{kj}}{2} \right|$$

**Case 5.** If  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \equiv 0_2$  i. e.,  $\alpha'_{ik} < 0 \leq \beta'_{ik} \leq \gamma'_{ik} \leq \delta'_{ik}$

$$f_{ik}^j = \min(\alpha'_{ik} w_{kj}, \delta'_{ik} x_{kj}) = \frac{\alpha'_{ik} w_{kj} + \delta'_{ik} x_{kj}}{2} - \left| \frac{\alpha'_{ik} w_{kj} - \delta'_{ik} x_{kj}}{2} \right|$$

$$p_{ik}^j = \min(\beta'_{ik} y_{kj}, \gamma'_{ik} z_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} y_{kj} - \frac{\gamma'_{ik} - \beta'_{ik}}{2} |y_{kj}|$$

$$q_{ik}^j = \max(\beta'_{ik} z_{kj}, \gamma'_{ik} x_{kj}) = \frac{\beta'_{ik} + \gamma'_{ik}}{2} z_{kj} - \frac{\gamma'_{ik} - \beta'_{ik}}{2} |z_{kj}|$$

$$r_{ik}^j = \max(\alpha'_{ik} x_{kj}, \delta'_{ik} w_{kj}) = \frac{\alpha'_{ik} x_{kj} + \delta'_{ik} w_{kj}}{2} + \left| \frac{\delta'_{ik} w_{kj} - \alpha'_{ik} x_{kj}}{2} \right|$$

In the same way we can formulate  $(F_{ik}^j, P_{ik}^j, Q_{ik}^j, R_{ik}^j)$ .

**Step 4** Using the arithmetic operations, defined in Section 6.1.2, the dual fully fuzzy matrix equation, obtained in Step 3, can be transformed into the following dual fully fuzzy matrix equation.

$$\begin{aligned} & \left( \sum_{j=1}^n f_{ik}^j, \sum_{j=1}^n p_{ik}^j, \sum_{j=1}^n q_{ik}^j, \sum_{j=1}^n r_{ik}^j \right) \oplus (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}) \\ &= \left( \sum_{j=1}^n F_{ik}^j, \sum_{j=1}^n P_{ik}^j, \sum_{j=1}^n Q_{ik}^j, \sum_{j=1}^n R_{ik}^j \right) \oplus (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij}) \forall i = 1, 2, \dots, m \end{aligned}$$

**Step 5.** The dual fully fuzzy matrix equation, obtained in Step 1, can be transformed into the following crisp linear system of non-linear equations.

$$\left\{ \begin{array}{l} \sum_{j=1}^n f_{ik}^j + b'_{ij} = \sum_{j=1}^n F_{ik}^j + b''_{ij} \quad ; \quad i = 1, 2, \dots, m \\ \sum_{j=1}^n p_{ik}^j + g'_{ij} = \sum_{j=1}^n P_{ik}^j + g''_{ij} \quad ; \quad i = 1, 2, \dots, m \\ \sum_{j=i}^n q_{ik}^j + h'_{ij} = \sum_{j=i}^n Q_{ik}^j + h''_{ij} \quad ; \quad i = 1, 2, \dots, m \\ \sum_{j=i}^n r_{ik}^j + l'_{ij} = \sum_{j=i}^n R_{ik}^j + l''_{ij} \quad ; \quad i = 1, 2, \dots, m \end{array} \right.$$

$$y_{kj} - x_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$z_{kj} - y_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$w_{kj} - z_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$x_{kj}, y_{kj}, z_{kj}, w_{kj} \geq 0$$

**Step 6.** The crisp system of non-linear equations obtained in Step 5, can be directly solved by any existing methods or by converting the non-linear system into linear system as proposed in Chapter 5.

**Remark 6.1.** The  $m \times n$  dual fully fuzzy matrix equation  $\tilde{A} \otimes \tilde{X} \oplus \tilde{B} = \tilde{C} \otimes \tilde{X} \oplus \tilde{D}$  can be reduced to standard FFME  $\tilde{A} \otimes \tilde{X} = \tilde{N}$  if we take  $\tilde{B} = \tilde{0}$ ,  $\tilde{C} = \tilde{0}$  and  $\tilde{D} = \tilde{N}$ .

#### 6.4. Solution of chosen dual fully fuzzy matrix equations

In Section 6.2, some dual fully fuzzy matrix equation are chosen which cannot be solved by the existing methods [64]. In this section, the solution of the chosen dual fully fuzzy matrix equation are obtained by the Mehar method-V proposed in Section 6.3.

##### 6.4.1. Solution of dual fully fuzzy matrix equation chosen in Example 6.1

Using the methods, proposed in Section 6.3, the solution of dual fully fuzzy matrix equation chosen in Example 6.1, can be obtained as follows:

$$\begin{aligned} \text{Step 1. } & (1,2,3,4) \otimes (x_{11}, y_{11}, z_{11}, w_{11}) \oplus (-5, -2, -2, -1) \otimes (x_{21}, y_{21}, z_{21}, w_{21}) \oplus \\ & (-15, -6, -6, 5) = (1,3,4,4) \otimes (x_{21}, y_{21}, z_{21}, w_{21}) \oplus (-1,8,10,33) \end{aligned}$$

and

$$\begin{aligned} & (1,2,3,4) \otimes (x_{12}, y_{12}, z_{12}, w_{12}) \oplus (-5, -2, -2, -1) \otimes (x_{22}, y_{22}, z_{22}, w_{22}) \\ & = (1,3,4,4) \otimes (x_{22}, y_{22}, z_{22}, w_{22}) \oplus (-32, -16, -16, -7), \end{aligned}$$

and

$$\begin{aligned} & (-5,2,3,4) \otimes (x_{11}, y_{11}, z_{11}, w_{11}) \oplus (2,3,4,4) \otimes (x_{21}, y_{21}, z_{21}, w_{21}) \\ & = (-4, -2, -1, -1) \otimes (x_{11}, y_{11}, z_{11}, w_{11}) \oplus (-15, -2, 1, 11), \end{aligned}$$

and

$$\begin{aligned} & (-5,2,3,4) \otimes (x_{12}, y_{12}, z_{12}, w_{12}) \oplus (2,3,4,4) \otimes (x_{22}, y_{22}, z_{22}, w_{22}) \oplus (-4, -2, -2, -1) \\ & = (-4, -2, -1, -1) \otimes (x_{12}, y_{12}, z_{12}, w_{12}) \oplus (-3,24,33,44). \end{aligned}$$

$$y_{11} - x_{11} \geq 0, z_{11} - y_{11} \geq 0, w_{11} - z_{11} \geq 0$$

$$y_{12} - x_{12} \geq 0, z_{12} - y_{12} \geq 0, w_{12} - z_{12} \geq 0$$

$$y_{21} - x_{21} \geq 0, z_{21} - y_{21} \geq 0, w_{21} - z_{21} \geq 0$$

$$y_{22} - x_{22} \geq 0, z_{22} - y_{22} \geq 0, w_{22} - z_{22} \geq 0$$

**Step 2.** Using Step 3, of the proposed method the dual fully fuzzy matrix equation, chosen in Example 6.1, can be transformed into the following dual fully fuzzy linear system.

$$\begin{aligned}
& ((\min(x_{11}, 4x_{11}), \min(2y_{11}, 3y_{11}), \max(2z_{11}, 3z_{11}), \max(w_{11}, 4w_{11})) \oplus \\
& (\min(-5w_{21}, -w_{21}), \min(-2z_{21}, -2z_{21}), \max(-2y_{21}, -2y_{21}), \max(-5x_{21}, x_{21})) \\
& \oplus (-15, -6, -6, 5) \\
& = (\min(x_{21}, 4x_{21}), \min(3y_{21}, 4y_{21}), \max(3z_{21}, 4z_{21}), \max(w_{21}, 4w_{21})) \\
& \oplus (-1, 8, 10, 33), \\
& ((\min(x_{12}, 4x_{12}), \min(2y_{12}, 3y_{12}), \max(2z_{12}, 3z_{12}), \max(w_{12}, 4w_{12})) \oplus \\
& (\min(-5w_{22}, -w_{22}), \min(-2z_{22}, -2z_{22}), \max(-2y_{22}, -2y_{22}), \max(-5x_{22}, x_{22})) \\
& = (\min(x_{22}, 4x_{22}), \min(3y_{22}, 4y_{22}), \max(3z_{22}, 4z_{22}), \max(w_{22}, 4w_{22})) \\
& \oplus (-32, -16, -16, -7), \\
& ((\min(-5w_{11}, 4x_{11}), \min(2y_{11}, 3y_{11}), \max(2z_{11}, 3z_{11}), \max(-5x_{11}, 4w_{11})) \oplus \\
& (\min(2x_{21}, 4x_{21}), \min(3y_{21}, 4y_{21}), \max(3z_{21}, 4z_{21}), \max(2w_{21}, 4w_{21})) \\
& = (\min(-4w_{11}, -w_{11}), \min(-2z_{11}, -z_{11}), \max(-2y_{11}, -y_{11}), \max(-4x_{11}, -x_{11})) \\
& \oplus (-15, -2, 1, 11), \\
& ((\min(-5w_{12}, 4x_{12}), \min(2y_{12}, 3y_{12}), \max(2z_{12}, 3z_{12}), \max(-5x_{12}, 4w_{12})) \oplus \\
& (\min(2x_{22}, 4x_{22}), \min(3y_{22}, 4y_{22}), \max(3z_{22}, 4z_{22}), \max(2w_{22}, 4w_{22})) \\
& \oplus (-4, -2, -2, -1)
\end{aligned}$$

$$= (\min(-4w_{12}, -w_{12}), \min(-2z_{12}, -z_{12}), \max(-2y_{12}, -y_{12}), \max(-4x_{12}, -x_{12}))$$

$$\oplus (-3, 24, 33, 44).$$

$$y_{11} - x_{11} \geq 0, z_{11} - y_{11} \geq 0, w_{11} - z_{11} \geq 0$$

$$y_{12} - x_{12} \geq 0, z_{12} - y_{12} \geq 0, w_{12} - z_{12} \geq 0$$

$$y_{21} - x_{21} \geq 0, z_{21} - y_{21} \geq 0, w_{21} - z_{21} \geq 0$$

$$y_{22} - x_{22} \geq 0, z_{22} - y_{22} \geq 0, w_{22} - z_{22} \geq 0$$

**Step 3.** Using the relation  $\min(x, y) = \left(\frac{x+y}{2}\right) - \left|\frac{x-y}{2}\right|$ ,  $\max(x, y) = \left(\frac{x+y}{2}\right) + \left|\frac{x-y}{2}\right|$  and definition 7.4, we get the following system of equations:

$$\frac{5}{2}x_{11} - \frac{3}{2}|x_{11}| - 3w_{21} - 2|w_{21}| - \frac{5}{2}x_{21} + \frac{3}{2}|x_{21}| - 14 = 0$$

$$\frac{5}{2}y_{11} - \frac{1}{2}|y_{11}| - 2z_{21} - \frac{7}{2}y_{21} + \frac{1}{2}|y_{21}| - 14 = 0$$

$$\frac{5}{2}z_{11} + \frac{1}{2}|z_{11}| - 2y_{21} - \frac{7}{2}z_{21} - \frac{1}{2}|z_{21}| - 16 = 0$$

$$\frac{5}{2}w_{11} + \frac{3}{2}|x_{11}| - 2x_{21} + 3|x_{21}| - \frac{5}{2}w_{21} - \frac{3}{2}|w_{21}| - 28 = 0$$

$$\frac{5}{2}x_{12} - \frac{3}{2}|x_{12}| - 3w_{22} - 2|w_{22}| - \frac{5}{2}x_{22} + \frac{3}{2}|x_{22}| + 32 = 0$$

$$\frac{5}{2}y_{12} - \frac{1}{2}|y_{12}| - 2z_{22} - \frac{7}{2}y_{22} + \frac{1}{2}|y_{22}| + 16 = 0$$

$$\frac{5}{2}z_{12} + \frac{1}{2}|z_{12}| - 2y_{22} - \frac{7}{2}z_{22} - \frac{1}{2}|z_{22}| + 16 = 0$$

$$\frac{5}{2}w_{12} + \frac{3}{2}|w_{12}| - 3x_{22} + 2|x_{22}| - \frac{5}{2}w_{22} - \frac{3}{2}|w_{22}| + 7 = 0$$

$$2x_{11} - \frac{1}{2}|5w_{11} + 4x_{11}| + 3x_{21} - |x_{21}| + \frac{3}{2}|w_{11}| + 15 = 0$$

$$\frac{5}{2}y_{11} - \frac{1}{2}|y_{11}| + \frac{7}{2}y_{21} - \frac{1}{2}|y_{21}| + \frac{3}{2}z_{11} + \frac{1}{2}|z_{11}| + 2 = 0$$

$$\frac{5}{2}z_{11} + \frac{1}{2}|z_{11}| + \frac{7}{2}z_{21} + \frac{1}{2}|z_{21}| + \frac{3}{2}y_{11} - \frac{1}{2}|y_{11}| - 1 = 0$$

$$\frac{5}{2}x_{11} + 2w_{11} + \frac{1}{2}|5x_{11} + 4w_{11}| + 3w_{21} + |w_{21}| + \frac{5}{2}x_{11} - \frac{3}{2}|x_{11}| - 11 = 0$$

$$2x_{12} - \frac{1}{2}|5w_{12} + 4x_{12}| + 3x_{22} - |x_{22}| + \frac{3}{2}|w_{12}| - 1 = 0$$

$$\frac{5}{2}y_{12} - \frac{1}{2}|y_{12}| + \frac{7}{2}y_{22} - \frac{1}{2}|y_{22}| + \frac{3}{2}z_{12} + \frac{1}{2}|z_{12}| - 26 = 0$$

$$\frac{5}{2}z_{12} + \frac{1}{2}|z_{12}| + \frac{7}{2}z_{22} + \frac{1}{2}|z_{22}| + \frac{3}{2}y_{12} - \frac{1}{2}|y_{12}| - 35 = 0$$

$$2w_{12} + \frac{1}{2}|5x_{12} + 4w_{12}| + 3w_{22} + |w_{22}| - \frac{3}{2}|x_{12}| - 45 = 0$$

$$y_{11} - x_{11} \geq 0, z_{11} - y_{11} \geq 0, w_{11} - z_{11} \geq 0$$

$$y_{12} - x_{12} \geq 0, z_{12} - y_{12} \geq 0, w_{12} - z_{12} \geq 0$$

$$y_{21} - x_{21} \geq 0, z_{21} - y_{21} \geq 0, w_{21} - z_{21} \geq 0$$

$$y_{22} - x_{22} \geq 0, z_{22} - y_{22} \geq 0, w_{22} - z_{22} \geq 0$$

On solving the above system, the obtained optimal solution is  $x_{11} = 1, y_{11} = 1, z_{11} = 2, w_{11} = 3, x_{12} = 1, y_{12} = 3, z_{12} = 4, w_{12} = 5,$   $x_{21} = -3, y_{21} = -2, z_{21} = -2, w_{21} = -1, x_{22} = 3, y_{22} = 4, z_{22} = 5, w_{22} = 6.$

**Step 4.** Putting the values of  $x_{11}, y_{11}, z_{11}, w_{11}, x_{12}, y_{12}, z_{12}, w_{12}, x_{21}, y_{21}, z_{21}, w_{21}, x_{22}, y_{22}, z_{22}, w_{22}$

$z_{21}, w_{21}, x_{22}, y_{22}, z_{22}, w_{22}$  in  $(x_{11}, y_{11}, z_{11}, w_{11}), (x_{12}, y_{12}, z_{12}, w_{12}), (x_{21}, y_{21}, z_{21}, w_{21}),$  and  $(x_{22}, y_{22}, z_{22}, w_{22}),$  the solution of given dual fully fuzzy matrix equation is

$$\tilde{x}_{11} = (x_{11}, y_{11}, z_{11}, w_{11}) = (1, 1, 2, 3)$$

$$\tilde{x}_{12} = (x_{12}, y_{12}, z_{12}, w_{12}) = (1, 3, 4, 5)$$

$$\tilde{x}_{21} = (x_{21}, y_{21}, z_{21}, w_{21}) = (-3, -2, -2, -1)$$

$$\tilde{x}_{22} = (x_{22}, y_{22}, z_{22}, w_{22}) = (3, 4, 5, 6).$$

### 6.5. Solution of dual fully fuzzy matrix equation with non-negative restriction on elements of unknown fuzzy matrix

In this section, special case of dual fully fuzzy matrix equation  $\tilde{A} \otimes \tilde{X} \oplus \tilde{B} = \tilde{C} \otimes \tilde{X} \oplus \tilde{D}$  has been discussed in which non-negative restriction on variable fuzzy matrix has been proposed i.e.,  $\tilde{X} \geq 0$ .

The steps of the proposed method are as follows:

**Step 1** Substituting  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ ,  $\tilde{x} = (\tilde{x}_{ij})_{n \times k}$  and  $\tilde{b} = (\tilde{b}_{ij})_{m \times k}$ ,  $\tilde{C} = (\tilde{c}_{ij})_{m \times n}$ ,  $\tilde{d} = (\tilde{d}_{ij})_{m \times k}$  the  $m \times k$  dual fully fuzzy matrix equation may be written as:

$$\sum_{j=1, \dots, n}^{\oplus} \tilde{a}_{ij} \otimes \tilde{x}_{ij} \oplus \tilde{b}_{ij} = \sum_{j=1, \dots, n}^{\oplus} \tilde{c}_{ij} \otimes \tilde{x}_{ij} \oplus \tilde{d}_{ij} \quad \forall i = 1, 2, \dots, m$$

**Step 2** If all the parameters  $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_{ij}, \tilde{d}_i$  and  $\tilde{x}_j$  are represented by trapezoidal fuzzy numbers  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}), (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}), (\alpha''_{ik}, \beta''_{ik}, \gamma''_{ik}, \delta''_{ik}), (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij})$  and  $(x_{kj}, y_{kj}, z_{kj}, w_{kj})$  respectively then the dual obtained in Step 1, may be written as:

$$\begin{aligned}
& \sum_{j=1, \dots, n}^{\oplus} (\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) \oplus (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}) \\
&= \sum_{j=1, \dots, n}^{\oplus} (\alpha''_{ij}, \beta''_{ij}, \gamma''_{ij}, \delta''_{ij}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) \oplus (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij}) \quad \forall i \\
&= 1, 2, \dots, m
\end{aligned}$$

**Step 3** Assuming  $(\alpha'_{ik}, \beta'_{ik}, \gamma'_{ik}, \delta'_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) = (f'_{ik}, p'_{ik}, q'_{ik}, r'_{ik})$ , and

$(\alpha''_{ik}, \beta''_{ik}, \gamma''_{ik}, \delta''_{ik}) \otimes (x_{kj}, y_{kj}, z_{kj}, w_{kj}) = (F'_{ik}, P'_{ik}, Q'_{ik}, R'_{ik})$ ,  $1 \leq i, j, k \leq n$  where each

$(x_{kj}, y_{kj}, z_{kj}, w_{kj})$  is a non-negative trapezoidal number the dual fully fuzzy matrix

equation, obtained in Step 2, may be written as:

$$\begin{aligned}
& \sum_{j=1, \dots, n}^{\oplus} (f'_{ik}, p'_{ik}, q'_{ik}, r'_{ik}) \oplus (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}) \\
&= \sum_{j=1, \dots, n}^{\oplus} (F'_{ik}, P'_{ik}, Q'_{ik}, R'_{ik}) \oplus (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij}) \quad \forall i = 1, 2, \dots, m
\end{aligned}$$

where

$$f'_{ik} = \min(\alpha'_{ik} x_{kj}, \alpha'_{ik} w_{kj}) = \begin{cases} \alpha'_{ik} x_{kj} & \text{if } \alpha'_{ik} \geq 0 \\ \alpha'_{ik} w_{kj} & \text{if } \alpha'_{ik} < 0 \end{cases}$$

$$p'_{ik} = \min(\beta'_{ik} y_{kj}, \beta'_{ik} z_{kj}) = \begin{cases} \beta'_{ik} y_{kj} & \text{if } \beta'_{ik} \geq 0 \\ \beta'_{ik} z_{kj} & \text{if } \beta'_{ik} < 0 \end{cases}$$

$$q'_{ik} = \max(\gamma'_{ik} y_{kj}, \gamma'_{ik} z_{kj}) = \begin{cases} \gamma'_{ik} y_{kj} & \text{if } \gamma'_{ik} \geq 0 \\ \gamma'_{ik} z_{kj} & \text{if } \gamma'_{ik} < 0 \end{cases}$$

$$r'_{ik} = \max(\delta'_{ik} w_{kj}, \delta'_{ik} x_{kj}) = \begin{cases} \delta'_{ik} w_{kj} & \text{if } \delta'_{ik} \geq 0 \\ \delta'_{ik} x_{kj} & \text{if } \delta'_{ik} < 0 \end{cases}$$

$$F_{ik}^j = \min(\alpha''_{ik}x_{kj}, \alpha''_{ik}w_{kj}) = \begin{cases} \alpha''_{ik}x_{kj} & \text{if } \alpha''_{ik} \geq 0 \\ \alpha''_{ik}w_{kj} & \text{if } \alpha''_{ik} < 0 \end{cases}$$

$$P_{ik}^j = \min(\beta''_{ik}y_{kj}, \beta''_{ik}z_{kj}) = \begin{cases} \beta''_{ik}y_{kj} & \text{if } \beta''_{ik} \geq 0 \\ \beta''_{ik}z_{kj} & \text{if } \beta''_{ik} < 0 \end{cases}$$

$$Q_{ik}^j = \max(\gamma''_{ik}y_{kj}, \gamma''_{ik}z_{kj}) = \begin{cases} \gamma''_{ik}y_{kj} & \text{if } \gamma''_{ik} \geq 0 \\ \gamma''_{ik}z_{kj} & \text{if } \gamma''_{ik} < 0 \end{cases}$$

$$R_{ik}^j = \max(\delta''_{ik}w_{kj}, \delta''_{ik}x_{kj}) = \begin{cases} \delta''_{ik}w_{kj} & \text{if } \delta''_{ik} \geq 0 \\ \delta''_{ik}x_{kj} & \text{if } \delta''_{ik} < 0 \end{cases}$$

**Step 4** Using the arithmetic operations, defined in Section 7.1.2, the dual fully fuzzy matrix equation, obtained in Step 3, may be written as :

$$\begin{aligned} & \left( \sum_{j=1}^n f_{ik}^j, \sum_{j=1}^n p_{ik}^j, \sum_{j=1}^n q_{ik}^j, \sum_{j=1}^n r_{ik}^j \right) \oplus (b'_{ij}, g'_{ij}, h'_{ij}, l'_{ij}) \\ & = \left( \sum_{j=1}^n F_{ik}^j, \sum_{j=1}^n P_{ik}^j, \sum_{j=1}^n Q_{ik}^j, \sum_{j=1}^n R_{ik}^j \right) \oplus (b''_{ij}, g''_{ij}, h''_{ij}, l''_{ij}) \forall i = 1, 2, \dots, m \end{aligned}$$

**Step 5** The dual fully fuzzy matrix equation, obtained in Step 4, may be transformed into the following crisp system of matrix equations.

$$\left\{ \begin{array}{l} \sum_{j=1}^n f_{ik}^j + b'_{ij} = \sum_{j=1}^n F_{ik}^j + b''_{ij} \quad ; \quad i = 1, 2, \dots, m \\ \sum_{j=1}^n p_{ik}^j + g'_{ij} = \sum_{j=1}^n P_{ik}^j + g''_{ij} \quad ; \quad i = 1, 2, \dots, m \\ \sum_{j=i}^n q_{ik}^j + h'_{ij} = \sum_{j=i}^n Q_{ik}^j + h''_{ij} \quad ; \quad i = 1, 2, \dots, m \\ \sum_{j=i}^n r_{ik}^j + l'_{ij} = \sum_{j=i}^n R_{ik}^j + l''_{ij} \quad ; \quad i = 1, 2, \dots, m \end{array} \right.$$

$$y_{kj} - x_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$z_{kj} - y_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$w_{kj} - z_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$x_{kj}, y_{kj}, z_{kj}, w_{kj} \geq 0$$

**Step 6** The solution of system obtained in Step 5, can be obtained by solving the following linear programming problem, which is obtained by adding the artificial variable  $Z_i, \forall i = 1, 2, \dots, m$ .

$$\text{Minimize } R_1 + R_2 + \dots + R_{4n^2}$$

subject to

$$\sum_{k=1}^n f_{1k}^{(1)} - \sum_{j=1}^n F_{1k}^{(1)} + R_1 = b''_{11} - b'_{11}$$

$$\sum_{k=1}^n f_{2k}^{(1)} - \sum_{j=1}^n F_{1k}^{(1)} + R_2 = b''_{21} - b'_{21}$$

⋮

$$\sum_{j=1}^n f_{nk}^{(1)} - \sum_{j=1}^n F_{1k}^{(1)} + R_n = b''_{n1} - b'_{n1}$$

$$\sum_{j=1}^n f_{1k}^{(2)} - \sum_{j=1}^n F_{1k}^{(2)} + R_{n+1} = b''_{12} - b'_{12}$$

⋮

$$\sum_{k=1}^n r_{nk}^n - \sum_{j=1}^n R_{nk}^{(n)} + R_{4n^2} = l''_{11} - l'_{11}$$

$$y_{kj} - x_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$z_{kj} - y_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$w_{kj} - z_{kj} \geq 0 \quad \forall j, k = 1, 2, \dots, n$$

$$x_{kj}, y_{kj}, z_{kj}, w_{kj} \geq 0$$

**Step 7.** Using the solution, obtained in Step 6, find the fuzzy solution of dual fully fuzzy matrix equations.

### 6.5.1 Solution of dual fully fuzzy linear matrix equation chosen in Example 6.2

In Section 6.2, some dual fully fuzzy matrix equation are chosen which cannot be solved by the existing methods [123]. In this section, the solution of the dual fully fuzzy matrix equation, chosen in Example 6.2, is obtained by the Mehar method-V, proposed in Section 6.5.

Using the Mehar method-V, proposed in Section 6.5, the solution of dual fully fuzzy matrix equation chosen in Example 6.2, can be obtained as follows:

**Step 1** Using the arithmetic operations, defined in Section 6.1, and using Step 3, 4 of the proposed method, the chosen dual fully fuzzy matrix equation may be written as:

$$\begin{aligned} &(-2w_{11} + x_{21} + 2, 3y_{11} + 2y_{21} + 6, 4z_{11} + 3z_{21} + 8, 5w_{11} + 6w_{21} + 9) \\ &= (x_{11} - w_{21} + 2, 2y_{11} + 3y_{21} + 4, 3z_{11} + 4z_{21} + 6, 6w_{11} + 5w_{21} + 11) \end{aligned}$$

$$\begin{aligned} &(-2w_{12} + x_{22} + 4, 3y_{12} + 2y_{22} + 5, 4z_{12} + 3z_{22} + 10, 5w_{12} + 6w_{22} + 11) \\ &= (x_{12} - w_{22} + 2, 2y_{12} + 3y_{22} + 4, 3z_{12} + 4z_{22} + 9, 6w_{12} + 5w_{22} + 13) \end{aligned}$$

$$\begin{aligned} &(-2x_{11} + 3x_{21} + 5, 3y_{11} + 4y_{21} + 9, 4z_{11} + 5z_{21} + 12, 6w_{11} + 7w_{21} + 13) \\ &= (3x_{11} + 4x_{21} + 1, 4y_{11} + 5y_{21} + 3, 5z_{11} + 6z_{21} + 4, 6w_{11} + 8w_{21} + 7) \end{aligned}$$

$$\begin{aligned} &(2x_{12} + 3x_{22} + 4, 3y_{12} + 4y_{22} + 9, 4z_{12} + 5z_{22} + 13, 6w_{12} + 7w_{22} + 15) \\ &= (3x_{12} + 4x_{22} + 1, 4y_{12} + 5y_{22} + 2, 5z_{12} + 6z_{22} + 4, 6w_{12} + 8w_{22} + 8) \end{aligned}$$

$$y_{11} - x_{11} \geq 0, z_{11} - y_{11} \geq 0, w_{11} - z_{11} \geq 0$$

$$y_{12} - x_{12} \geq 0, z_{12} - y_{12} \geq 0, w_{12} - z_{12} \geq 0$$

$$y_{21} - x_{21} \geq 0, z_{21} - y_{21} \geq 0, w_{21} - z_{21} \geq 0$$

$$y_{22} - x_{22} \geq 0, z_{22} - y_{22} \geq 0, w_{22} - z_{22} \geq 0$$

Using Definition 6.4, the dual fully fuzzy matrix equation, obtained in Step 1, is transformed into the following crisp linear system of linear equations

$$-2w_{11} + x_{21} + 2 = x_{11} - w_{21} + 2$$

$$3y_{11} + 2y_{21} + 6 = 2y_{11} + 3y_{21} + 4$$

$$4z_{11} + 3z_{21} + 8 = 3z_{11} + 4z_{21} + 6$$

$$5w_{11} + 6w_{21} + 9 = 6w_{11} + 5w_{21} + 11$$

$$-2w_{12} + x_{22} + 4 = x_{12} - w_{22} + 2$$

$$3y_{12} + 2y_{22} + 5 = 2y_{12} + 3y_{22} + 4$$

$$4z_{12} + 3z_{22} + 10 = 3z_{12} + 4z_{22} + 9$$

$$5w_{12} + 6w_{22} + 11 = 6w_{12} + 5w_{22} + 13$$

$$-2x_{11} + 3x_{21} + 5 = 3x_{11} + 4x_{21} + 1$$

$$3y_{11} + 4y_{21} + 9 = 4y_{11} + 5y_{21} + 3$$

$$4z_{11} + 5z_{21} + 12 = 5z_{11} + 6z_{21} + 4$$

$$6w_{11} + 7w_{21} + 13 = 6w_{11} + 8w_{21} + 7$$

$$2x_{12} + 3x_{22} + 4 = 3x_{12} + 4x_{22} + 1$$

$$3y_{12} + 4y_{22} + 9 = 4y_{12} + 5y_{22} + 2$$

$$4z_{12} + 5z_{22} + 13 = 5z_{12} + 6z_{22} + 4$$

$$6w_{12} + 7w_{22} + 15 = 6w_{12} + 8w_{22} + 8$$

i.e.,

$$x_{21} - x_{11} - 2w_{11} + w_{21} = 0$$

$$x_{12} + 2w_{12} - w_{22} - x_{22} = 2$$

$$y_{21} - y_{11} = 2, z_{21} - z_{11} = 2, w_{21} - w_{11} = 2$$

$$y_{22} - y_{12} = 1, z_{22} - z_{12} = 1, w_{22} - w_{12} = 2$$

$$x_{21} + x_{11} = 4, y_{11} - y_{21} = 6, z_{21} + z_{11} = 8$$

$$w_{21} = 6$$

$$x_{12} + x_{22} = 3, y_{12} + y_{22} = 7, z_{12} + z_{22} = 9$$

$$w_{22} = 7$$

$$y_{11} - x_{11} \geq 0, z_{11} - y_{11} \geq 0, w_{11} - z_{11} \geq 0$$

$$y_{12} - x_{12} \geq 0, z_{12} - y_{12} \geq 0, w_{12} - z_{12} \geq 0$$

$$y_{21} - x_{21} \geq 0, z_{21} - y_{21} \geq 0, w_{21} - z_{21} \geq 0$$

$$y_{22} - x_{22} \geq 0, z_{22} - y_{22} \geq 0, w_{22} - z_{22} \geq 0$$

$$x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2, x_3, y_3, z_3, w_3 \geq 0$$

**Step 2.** The solution of linear system, obtained in Step 1, may be obtained by solving the following crisp linear programming problem:

$$\begin{aligned} \text{Minimize } (R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + R_{13} \\ + R_{14} + R_{15} + R_{16}) \end{aligned}$$

subject to

$$x_{21} - x_{11} - 2w_{11} + w_{21} + R_1 = 0$$

$$y_{21} - y_{11} + R_2 = 2, z_{21} - z_{11} + R_3 = 2$$

$$w_{21} - w_{11} + R_4 = 2, x_{12} + 2w_{12} - w_{22} - x_{22} + R_5 = 2$$

$$y_{22} - y_{12} + R_6 = 1, z_{22} - z_{12} + R_7 = 1,$$

$$w_{22} - w_{12} + R_8 = 2, x_{21} + x_{11} + R_9 = 4$$

$$y_{11} - y_{21} + R_{10} = 6, z_{21} + z_{11} + R_{11} = 8$$

$$w_{21} + R_{12} = 6, x_{12} + x_{22} + R_{13} = 3$$

$$y_{12} + y_{22} + R_{14} = 7, z_{12} + z_{22} + R_{15} = 9$$

$$w_{22} + R_{16} = 7,$$

$$y_{11} - x_{11} \geq 0, z_{11} - y_{11} \geq 0, w_{11} - z_{11} \geq 0$$

$$y_{12} - x_{12} \geq 0, z_{12} - y_{12} \geq 0, w_{12} - z_{12} \geq 0$$

$$y_{21} - x_{21} \geq 0, z_{21} - y_{21} \geq 0, w_{21} - z_{21} \geq 0$$

$$y_{22} - x_{22} \geq 0, z_{22} - y_{22} \geq 0, w_{22} - z_{22} \geq 0$$

$$x_{11}, y_{11}, z_{11}, w_{11}, x_{12}, y_{12}, z_{12}, w_{12}, x_{21}, y_{21}, z_{21}, w_{21}, x_{22}, y_{22}, z_{22}, w_{22} \geq 0$$

$$R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}, R_{11}, R_{12}, R_{13},$$

$$R_{14}, R_{15}, R_{16} \geq 0$$

On solving the above linear programming problem , the optimal solution is  $x_{11} = 1, y_{11} = 2, z_{11} = 3, w_{11} = 4, x_{12} = 1, y_{12} = 3, z_{12} = 4, w_{12} = 5, x_{21} = 3, y_{21} = 4, z_{21} = 5, w_{21} = 6, x_{22} = 2, y_{22} = 4, z_{22} = 5, w_{22} = 7$ .

**Step 3** Putting the values of  $x_{11}, y_{11}, z_{11}, w_{11}, x_{12}, y_{12}, z_{12}, w_{12}, x_{21}, y_{21}, z_{21}, w_{21}, x_{22}, y_{22}, z_{22}, w_{22}$

in  $(x_{11}, y_{11}, z_{11}, w_{11}), (x_{12}, y_{12}, z_{12}, w_{12}), (x_{21}, y_{21}, z_{21}, w_{21}),$  and  $(x_{22}, y_{22}, z_{22}, w_{22}),$  the exact solution of given fully fuzzy matrix equation is

$$\tilde{x}_{11} = (x_{11}, y_{11}, z_{11}, w_{11}) = (1, 2, 3, 4)$$

$$\tilde{x}_{12} = (x_{12}, y_{12}, z_{12}, w_{12}) = (1, 3, 4, 5)$$

$$\tilde{x}_{21} = (x_{21}, y_{21}, z_{21}, w_{21}) = (3, 4, 5, 6)$$

$$\tilde{x}_{22} = (x_{22}, y_{22}, z_{22}, w_{22}) = (2, 4, 5, 7)$$

## 6.6. Results and discussions

The solution of fully fuzzy matrix equation chosen in Example 6.1 and 6.2 are shown in Table 6.1. It is obvious from the results, shown in Table 6.1, that the dual fully fuzzy matrix equation, chosen in Example 6.1 and Example 6.2, cannot be solved by the existing method [113] but can be solved by the Mehar method-V proposed in this chapter.

**Table 6.1:** Comparison of results by using proposed and existing methods

Examples	Solution using proposed method	Solution using existing method [123]
Example 6.1	$\tilde{x}_{11} = (1, 1, 2, 3)$ $\tilde{x}_{12} = (1, 3, 4, 5)$ $\tilde{x}_{21} = (-3, -2, -2, -1)$ $\tilde{x}_{22} = (3, 4, 5, 6).$	Not Applicable
Example 6.2	$\tilde{x}_{11} = (1, 2, 3, 4)$ $\tilde{x}_{12} = (1, 3, 4, 5)$ $\tilde{x}_{21} = (3, 4, 5, 6)$ $\tilde{x}_{22} = (2, 4, 5, 7)$	Not Applicable

## 6.7. Conclusion

On the basis of present study, it can be concluded that the dual fully fuzzy matrix equations which can be solved by existing methods can also be solved by the proposed Mehar method-V. However, there exist several dual fully fuzzy matrix equations whose solution can be obtained by the proposed method but cannot be obtained by the existing methods. Hence, it is better to use the proposed Mehar method-V as compared to the existing methods.



# Chapter 7

## Future Scope

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The following work may be considered as future research work.

1. In future the real life applications of Mehar method-I, Mehar method-II, Mehar method-III, Mehar method-IV and Mehar method-V may be proposed.
2. In all the methods, proposed in this thesis the solution of fully fuzzy linear system / dual fully fuzzy linear system / fully fuzzy matrix equations / dual fully fuzzy matrix equations obtained by transforming into the equivalent crisp system. In future it may be tried to develop the methods to find the solution of this system without transforming into crisp system.
3. To the best of my knowledge till now there is no method in the literature to find the solution of such linear system / matrix equation in which elements of the matrices are represented by complex fuzzy numbers. In future the methods may be proposed to find the solution of such systems.
4. Since most of real life systems are nonlinear in nature. Therefore, in future there is a need to explore methods to find the solution of nonlinear fuzzy systems.



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