Mathematical Modeling of Discs at the Centre of Galaxies

Thesis submitted in partial fulfillment of the requirement for the award of the degree of

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By

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Dedicated to my parents and my supervisor.

Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **"Mathematical Modeling of Discs at the Centre of Galaxies"** in partial fulfillment of the requirements for the award of degree of **Masters of Science** in **Mathematics and Computing** and submitted to the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Mamta Gulati,** Assistant Professor and other research work which is duly listed in the reference section.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from any institution.

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Abstract

The galactic system can be mathematically modeled as Ellipsoidals (elliptical galaxies) or thin (zero thickness) axisymmetric discs. To study the dynamical evolution of galactic discs we can model the system in different ways depending upon the physical process we wish to study. In some cases disc is assumed to be continum and the fluid equations are used to study the evolution of such systems. In some other cases, the continuum approximation doesnot hold and we need to go to particle description, for such cases we use collisionless Boltzmann equation, as the galactic systems can be safely assumed to be collisionless. In this thesis, we first discuss the approximations under which continuum behaviour of fluid is valid. Next we move on to discuss various types of galactic system and their components. Some observational features that we aim to explain by the means of studying the dynamical evolution of the systems using perturbation analysis are also discussed. The centres of some galaxies exhibit a counter-intuitive lopsided distribution of stars. In the last part of this thesis we give a dynamical model to explain such features as a part of time evolution of gas and star discs at the centres of galaxies. The gravitationally coupled gas and stellar discs turns out to be stable to non-axisymmetric perturbations. The observed lopsided distribution of stars will only be long lived if there is a continuous source of trigger for such perturbations.

Contents

Chapter 1

INTRODUCTION

1.1 Fluids

Any substance that under an applied shear stress flows continually is termed as a fluid. It includes both liquid and gases. Permanent deformation and capability to flow are characteristic features of fluids. Fluids are categorised into two types:

- Ideal Fluid - An ideal fluid is non-viscous in nature and cannot be compressed. These fluids doesnot exist in reality since all fluids possess atleast some viscosity. However studying ideal fluid is important to understand some of basic phenomenas of fluid motion, for instance in aerodynamics.
- Real Fluid - Real fluids are those that possess viscosity. All the fluids are real fluids. Real fluids are subdivided into two categories:
  1. Newtonian Fluid - A real fluid that follows Newton's law of viscosity ie., sheer stress $\propto$ rate of strain, is termed as Newtonian fluids. Water and honey are examples of newtonian fluids.
  2. Non-Newtonian Fluid - A real fluid that doesnot follow Newton's law of viscosity is called Non-Newtonian fluid. Most of the fluids are Non-Newtonian in nature. Blood, paint, ketchup are some of the examples.

In this thesis, we will mainly concentrate on the mathematical modeling of astrophysical systems at the centres of galaxies. Extraterrestrial objects comprise of gases, dust, stars, clouds in interstellar medium, gas between the galaxies or cluster of galaxies and all these components behave like fluid under certain approximations. Fluid under consideration can be real or ideal depending upon the physical phenomena we are interested in studying. Gas component of galaxies are considered as ideal fluids for all practical purposes. Gases in discs around stars behave like real fluids.

1.2 Plan of the thesis

This thesis is divided in three parts. In first part, we review the basic properties of fluid ie., under which conditions the continuum behaviour of fluid is applicable. We thus derive equations of fluid for continuous system. The second part of thesis ie, chapter 3 consist of a review of galaxies and its components. We also review the literature survey done on mathematical modeling of centres of galaxies in the past few years. In third part, we discuss the proposed model for the centre of galaxies to explain certain observations. The observations will be hence explained in details in chapter 4. Lastly, in chapter 5, we offer some conclusions of the results obtained in this thesis along with some future directions.

Chapter 2

FLUID APPROXIMATION

This chapter is a review of standard fluid equations. We shall also discuss the approximations under which these equations are valid. The review is based on the book "The Physics of Fluids and Plasmas by" Choudhuri [1998].

2.1 From N-body to continum

Fluids, in general, is a collection of particles. We study these by modeling them as a continuum governed by a set of macroscopic equations. To develop the dynamical theory of fluid systems from N-particles there are four levels. The first level is called level 0. Here, the system is assumed to be a collection of N-microscopic particles and they obey quantum mechanics. An N-particle wave function explains the motion of particles. The evolution of wave function is governed by Schrodinger's equation. At the next level 1, the system is modeled as a collection of N classical particles. Each particle is prescribed by its position and velocity (momentum) coordinates. The whole system collectively will be described by 2N position and momentum coordinates namely $(\mathbf{q_s}, \mathbf{p_s}; \mathbf{s} = \mathbf{1}, ..., \mathbf{n})$. The time evolution of the system is studied by Newton's laws of motion.

For large value of N it is not possible to solve the equations of motion for all the position and velocity coordinates. Thus, we move to next level, ie., level 2, in which a distribution

LEVEL	DESCRIPTION	DYNAMICAL EQUATIONS
0	N quantum particles	Schrodinger's equation
1	N classical particles	Newton's laws of motion
2	Distribution function	Boltzmann equation
3	Continuum model	Hydrodynamic equations

Table 2.1: Different levels for neutral fluids

function $\mathbf{f}(\mathbf{p}, \mathbf{q}, \mathbf{t})$ is introduced. It gives particle number density in six-dimensional $(\mathbf{p}, \mathbf{q})$ space at time t. Boltzmann equation is used to depict the evolution of $(\mathbf{p}, \mathbf{q})$ with time. Finally we move from level 2 to level 3, where the system is modeled as continuum and hydrodynamic equations are used. We summarize these levels in table 2.1 and below we explain the transitions from lower level to next higher level.

2.2 Passing from level 0 to level 1

In this section we shall discuss under what approximations we can pass from level 0 to level 1. The measure of the sizes of wave packets for individual particles is given by

$$\lambda = \frac{h}{p} \approx \frac{h}{\sqrt{m\kappa_B T}} \tag{2.1}$$

where λ is de-Broglie wavelength, p is momentum of the particles, mass is given by m, κ_B is the Boltzmann constant and T denotes the temperature. For non-overlapping wave packets we compare it with the distance between particles which is given by $n^{-1/3}$ for n particles per unit volume. Particles will be non-overlapping if $\lambda \ll n^{-1/3}$ which gives

$$\frac{hn^{1/3}}{\sqrt{m\kappa_B T}} \ll 1. \tag{2.2}$$

Hence, if this inequality is satisfied, an individual wave packet evolving according to Schrodinger's equation moves like a classical particle. This gives us the condition by which we can go from level 0 to level 1.

It is not always necessary that we can pass on from level 0 to level 1 because certain collections of particles are inherently quantum and a classical description is not adequate. The material inside a white dwarf star is an example of such a system. Description at level 1 is possible only when the wave packets for the different particles are widely separated so that quantum interference is not important. Thus eq (2.2) is necessary condition to pass from level 0 to level 1.

2.3 Passing from level 1 to level 2

In level 1, we have a system of N classical particles and the system of the particles is described by generalised position and momentum coordinates $(q_s, p_s); s = 1, ..., N$ of each particle. These evolves according to Hamilton's equations :

$$\dot{p_s} = -\frac{\partial H}{\partial q_s}, \tag{2.3}$$

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \tag{2.4}$$

where the Hamiltonian $H(q_s, p_s, t)$ can be function of all the coordinates and time.
Let $\rho_{ens}(q_s, p_s, t)$ be the density of ensemble points at a location in the phase space. Consider one member of the ensemble and its trajectory $(q_s(t), p_s(t))$ in the phase space. Then the rate of change of density, $\rho_{ens}(q_s(t), p_s(t), t)$, is zero while moving along the trajectory, i.e.

$$\frac{d\rho_{ens}}{dt} = 0 \tag{2.5}$$

where d/dt denotes the time derivative taken along the particle trajectory. This is called Liouville's theorem.
If (q_s, p_s) and $(q_s + \delta q_s, p_s + \delta p_s)$ denote the system at times t and $t + \delta t$ on the trajectory then

$$\frac{d\rho_{ens}}{dt} = \lim_{\delta t \to 0} \frac{\rho_{ens}(q_s + \delta q_s, p_s + \delta p_s, t + \delta t) - \rho_{ens}(q_s, p_s, t)}{\delta t} \tag{2.6}$$

Keeping only first order terms in Taylor series we get,

$$\frac{d\rho_{ens}}{dt} = \frac{\partial \rho_{ens}}{\partial t} + \sum_s \dot{q}_s \frac{\partial \rho_{ens}}{\partial q_s} + \sum_s \dot{p}_s \frac{\partial \rho_{ens}}{\partial p_s} \tag{2.7}$$

In the system of N classical particles, it is assumed that all the particles are similar. In such a case the system will be completely described by $6N$ position and velocity coordinates. The corresponding $6N$-dimensional phase space is referred as the Γ-space and the system is represented by one point in this space. We can also consider a 6-dimensional space with each dimension corresponding to coordinates of position and velocity of the particles and is called μ-space. The system at a given instant of time is represented by N points in μ-space. In Γ-space the time evolution is a single trajectory which will be mapped to N trajectories in μ-space correspond to N particles.
The distribution function $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{t})$ in μ-space defined as

$$\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{t}) = \lim_{\delta V \to 0^+} \frac{\delta N}{\delta V} \tag{2.8}$$

where δN be the number of points in a small volume δV of the μ-space. The limit of δV is taken in special way. δV is small compared to the size of the system, but large to accomodate sufficiently large number of particles inside it. Introducing the distribution function $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{t})$ is possible only if this special limit exists, and then only we can move from level 1 to level 2.
In level 2, trajectories of the points in the μ-space can be obtained from Hamiltonian $H(x, u, t)$ using the equations

$$\dot{u} = -\nabla H \tag{2.9}$$

$$\dot{x} = \nabla_u H \tag{2.10}$$

and so for collisionless system we have

$$\frac{df}{dt} = 0. \tag{2.11}$$

Just as the total derivative in the γ-space could be put in form of eqn (2.7) , the total derivative in the μ-space can be similarly put in form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x} \cdot \nabla f + \dot{u} \cdot \nabla_u f \tag{2.12}$$

where ∇ is gradient w.r.t. position and ∇_u is the gradient w.r.t. velocity. Thus for collisionless systems we will have,

$$\frac{\partial f}{\partial t} + \dot{x} \cdot \nabla f + \dot{u} \cdot \nabla_u f = 0 \tag{2.13}$$

which is the collisionless Boltzmann equation (CBE).

2.4 Passing from level 2 to level 3

We now focus on the evolution of the system by taking collisions into account for a dilute neutral fluid. The system is said to be dilute if the total volume of fluid particles is negligible in comparision to the volume of the system. Mathematically this translates to

$$na^3 \ll 1 \tag{2.14}$$

where a is radius of the particles. The average distance travelled by a particle between two collisions is known as the mean free path and is given by

$$\lambda = \frac{1}{\sqrt{2} n \pi a^2}. \tag{2.15}$$

Since the fluid is dilute, we have $\lambda \gg a$ which is coming from the fact that each particle will have to travel a large distance before colliding with another particle.
The probability of multi-particle collisions is much smaller than the probability of binary collisions. Hence it is sufficient to consider only binary collisions. Collisions produce changes in the value of $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{t})$, thus eqn (2.11) is modified to the form

$$\frac{df}{dt} d^3x d^3u = -C_{out} + C_{in} \tag{2.16}$$

where C_{out} and C_{in} are the rates at which particles leave and enter the elementary volume d^3xd^3u of the μ-space due to collisions. Since C_{out} must be equal to the number of collisions per unit time within the volume d^3xd^3u, thus

$$C_{out} = d^3x d^3u \int d^3u_1 \int d\Omega \sigma(u, u_1|u^{'}, u_1^{'})|u - u_1| f(x, u, t) f(x, u_1, t) \tag{2.17}$$

where u, u_1 are initial velocities of particles, $u^{'}$, $u_1^{'}$ are velocities after collision, $d\Omega$ is angle of deflection, constant $\sigma(u, u_1|u^{'}, u_1^{'})$ is the differential scattering cross-section. Similarly, the term C_{in} is

$$C_{in} = d^3xd^3u \int d^3u_1 \int d\Omega\sigma(u, u_1|u^{'}, u_1^{'})|u - u_1|f(x, u^{'}, t)f(x, u_1^{'}, t). \tag{2.18}$$

Putting the values in eqn (2.16), we get

$$\frac{\partial f}{\partial t} + u \cdot \nabla f + \frac{F}{m} \cdot \nabla_u f = \int d^3u_1 \int d\Omega|u - u_1|\sigma(\Omega)(f^{'}f_1^{'} - ff_1) \tag{2.19}$$

where $f = f(x, u, t)$, $f_1 = f(x, u_1, t)$, $f^{'} = f(x, u^{'}, t)$, $f_1^{'} = f(x, u_1^{'}, t)$, $F = m\dot{u}$. The above equation is a nonlinear integro-differential equation followed by the distribution function.
If χ is a conserved quantity in our dynamical system, we multiply both sides of eqn (2.19) by χ and take the integral over u to get the conservation equation given by,

$$\frac{\partial}{\partial t}(n\langle\chi\rangle) + \frac{\partial}{\partial x_i}(n\langle u_i\chi\rangle) - n\left\langle u_i\frac{\partial\chi}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i\frac{\partial\chi}{\partial u_i}\right\rangle - \frac{n}{m}\left\langle\frac{\partial F_i}{\partial u_i}\chi\right\rangle = 0 \tag{2.20}$$

where χ stands for any quantity conserved in binary collisions and $n\langle\chi\rangle$ is the amount of χ per unit volume. Since mass, momentum and energy are conserved in binary collisions between particles, substitute these for χ in above equation.

- For $\chi = m$

$$\frac{\partial}{\partial t}(nm) + \frac{\partial}{\partial x_i}(nm\langle u_i\rangle) = 0 \tag{2.21}$$

  The term $mn = \rho$ (say) is the density and $v = \langle u\rangle$ is average velocity of the particles. The above equation becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0. \tag{2.22}$$

- For $\chi = mu_j$

$$\frac{\partial}{\partial t}(nm\langle u_j\rangle) + \frac{\partial}{\partial x_i}(nm\langle u_iu_j\rangle) - nF_j = 0 \tag{2.23}$$

  To simplify above equation, let us define a tensor

$$P_{ij} = nm\langle(u_i - v_i)(u_j - v_j)\rangle\,, \tag{2.24}$$
$$nm\langle u_iu_j\rangle = P_{ij} + nmv_iv_j. \tag{2.25}$$

  Substituting in eqn (2.23) we get

$$\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_iv_j) = -\frac{\partial P_{ji}}{\partial x_i} + \frac{\rho}{m}F_j. \tag{2.26}$$

Simplifying it further gives

$$\rho(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i}) = -\frac{\partial P_{ji}}{\partial x_i} + \frac{\rho}{m} F_j. \tag{2.27}$$

- For $\chi = \frac{1}{2}m|u-v|^2$
  If the gas is monatomic and translational kinectic energy is conserved in binary collisions, then $\chi = \frac{1}{2}m|u-v|^2$. Substituting this in eqn (2.20) and simplifying, we get

$$\frac{\partial}{\partial t}(\rho\epsilon) + \frac{\partial}{\partial x_i}(\rho\epsilon v_i) + \frac{\partial q_i}{\partial x_i} + P_{ij}\Lambda_{ij} = 0 \tag{2.28}$$

  where

$$\epsilon = \frac{1}{2}\langle|u-v|^2\rangle \tag{2.29}$$

$$q = \frac{1}{2}\rho\langle(u-v)|u-v|^2\rangle \tag{2.30}$$

  are the internal energy per unit mass and energy flux respectively. Also

$$\Lambda_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{2.31}$$

  Thus we get

$$\rho\left(\frac{\partial \epsilon}{\partial t} + v_i \frac{\partial \epsilon}{\partial x_i}\right) + \frac{\partial q_i}{\partial x_i} + P_{ij}\Lambda_{ij} = 0 \tag{2.32}$$

Equations (2.22), (2.27), and (2.32) are called the moment equations of the CBE.

Collisions are important in a system of particles to produce Maxwellian distribution and also they are important in establishing fluidlike behaviour. Thus there is a connection between a Maxwellian distribution at microscopic level and fluidlike bahaviour at macroscopic level. Assume the distribution function at each point within a system of particles to be

$$f^{(0)}(x,u,t) = n(x,t)\left[\frac{m}{2\pi\kappa_B T(x,t)}\right]^{3/2} exp\left[-\frac{m(u-v(x,t))^2}{2\kappa_B T(x,t)}\right] \tag{2.33}$$

where n is number density, T temperature, v the mean flow velocity.
We calculate P_{ij} from eqn (2.24)

$$P_{ij} = mn\left(-\frac{m}{2\pi\kappa_B T}\right)^{3/2} \int d^3U U_i U_j exp\left(-\frac{mU^2}{2\kappa_B T}\right) \tag{2.34}$$

where $U = u - v$. Since integration is over whole of velocity space, the integral vanishes when integrand is odd, and we have

$$P_{ij} = n\kappa_B T\delta_{ij} = p\delta_{ij}. \tag{2.35}$$

The energy flux gives rise to odd integrand and thus, we have $q = 0$. Internal energy is

$$\epsilon = \frac{3}{2}\frac{\kappa_B T}{m}. \tag{2.36}$$

Further we have

$$P_{ij}\Lambda_{ij} = \frac{1}{2}p\delta_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = p\nabla \cdot v. \tag{2.37}$$

Substituting values for P_{ij}, q, $P_{ij}\Lambda_{ij}$ in the moment equations we get

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho}\nabla p + \frac{F}{m}, \tag{2.38}$$

$$\rho\left(\frac{\partial \epsilon}{\partial t} + v \cdot \nabla\epsilon\right) + p\nabla \cdot v = 0, \tag{2.39}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0. \tag{2.40}$$

These make up a system of five independent scalar equations with five independent variables, thus, leading to a dynamical theory of macroscopic nature. But these equations for fluid does not exhibit transport phenomena as $q = 0$ implies that heat energy is not transported from one part of the system to another. Tensor P_{ij} is diagonal means the absence of viscosity that allows for transport of momentum from one layer of fluid to other so the relative motions inside fluid are also damped out.
To handle transport phenomena we write distribution function as

$$f(x, u, t) = f^{(0)}(x, u, t) + g(x, u, t) \tag{2.41}$$

where $\mathbf{f^{(0)}}(\mathbf{x}, \mathbf{u}, \mathbf{t})$ is the Maxwellian distribution given by eqn (2.33) and $\mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{t})$ is departure from Maxwellian which is assumed to be small. Substituting this value in RHS of Boltzmann equation given in eqn (2.19)

$$RHS \approx \int d^3u_1 \int d\Omega|u - u_1|\sigma(\Omega)(f^{(0)'}g_1^{'} + f_1^{(0)'}g^{'} - f^{(0)}g_1 - f_1^{(0)}g) \tag{2.42}$$

where the quadratic terms in small quantity g are neglected. Estimation of order of magnitude of the collision integral can be obtained from one of the terms, say last term

$$-\int d^3u_1 \int d\Omega|u - u_1|\sigma(\Omega)f^{(0)}(x, u_1, t)g(x, u, t) \approx -g(x, u, t).n\sigma_{tot}\bar{u}_{rel} \tag{2.43}$$

where σ_{tot} is total collision cross-section, $\bar{u}_{rel}$ is average relative velocity between particles, $n\sigma_{tot}\bar{u}_{rel}$ is the inverse of collision time τ.
Thus Boltzmann equation becomes

$$\left(\frac{\partial}{\partial t} + u.\nabla + \frac{F}{m}.\nabla_u\right) f = -\frac{f - f^{(0)}}{\tau}. \tag{2.44}$$

Approximating f by $f^{(0)}$ gives

$$g = -\tau \left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} + \frac{F_i}{m} \frac{\partial}{\partial u_i} \right) f^{(0)}. \tag{2.45}$$

After evaluating $\partial f^{(0)}/\partial t$ and $\partial f^{(0)}/\partial x_i$, equation becomes

$$g = -\tau \left[\frac{1}{T} \frac{\partial T}{\partial x_i} U_i \left(\frac{m}{2\kappa_B T} U^2 - \frac{5}{2} \right) + \frac{m}{\kappa_B T} \Lambda_{ij} \left(U_i U_j - \frac{1}{3} \delta_{ij} U^2 \right) \right] f^{(0)}. \tag{2.46}$$

The above equation gives departure from Maxwellian distribution. For modified distribution $f^{(0)} + g$, values of q and P_{ij} is given by

$$q = -K \nabla T \tag{2.47}$$

$$P_{ij} = p\delta_{ij} + \pi_{ij} \tag{2.48}$$

where $K = \frac{5}{2} \tau n \frac{\kappa_B^2 T}{m}$ and $\pi_{ij} = -2\mu(\Lambda_{ij} - \frac{1}{3}\delta_{ij} \nabla \cdot v)$. Thus, we have

$$\frac{\partial P_{ij}}{\partial x_i} = \frac{\partial p}{\partial x_j} - \mu \left[\nabla^2 v_j + \frac{1}{3} \frac{\partial}{\partial j} (\nabla \cdot v) \right] \tag{2.49}$$

and

$$P_{ij}\Lambda_{ij} = p\nabla \cdot v - 2\mu \left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3} (\nabla \cdot v)^2 \right]. \tag{2.50}$$

Substituting values in moment equations (2.27), and (2.32) gives

$$\rho \left(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} \right) = -\frac{\partial p}{\partial x_j} + \mu \left[\nabla^2 v_j + \frac{1}{3} \frac{\partial}{\partial j} (\nabla . \cdot v) \right] + \frac{\rho}{m} F_j \tag{2.51}$$

and

$$\rho \left(\frac{\partial \epsilon}{\partial t} + v \cdot \nabla \epsilon \right) - \nabla \cdot (K \nabla T) + p \nabla \cdot v - 2\mu \left[\Lambda_{ij}\Lambda_{ij} - \frac{1}{3} (\nabla \cdot v)^2 \right] = 0. \tag{2.52}$$

Further simplifying these equations and also adding equation of continuity gives the set of hydrodynamic equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{2.53}$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + F + \frac{\mu}{\rho} \nabla^2 v, \tag{2.54}$$

$$\rho \left(\frac{\partial \epsilon}{\partial t} + v \cdot \nabla \epsilon \right) - \nabla \cdot (K \nabla T) + p \nabla \cdot v = 0 \tag{2.55}$$

where small terms have been neglected and F/m is replaced by F. These form the set of fluid equations including viscosity and transport phenomena.

2.5 Conclusion

The approximation under which we pass from a lower level to higher one are as follows-

- **Level 0 to level 1:** The size of the particles should be very small compared to the distance between them, ie., $\lambda \ll n^{-1/3}$.
- **Level 1 to level 2:** To construct the distribution function for the particles the infinitesimal volume of the δV taken should be smaller than the size of the system but should be large enough to accomodate large number of particles in it.
- **Level 2 to level 3:**
  The fluid is assumed to be dilute so we ignore multi-particle collision and consider only binary collisions, and we get the continum behaviour of the fluid.

Chapter 3

GALAXIES AND THEIR REVIEW

This chapter is an introduction to extra-terrestrial objects called galaxy and a review of their properties. The review is done from "Galactic Dynamics" by Binney and Tremaine [2008].

3.1 The galaxies and its components

Galaxies are defined as a massive system bounded by the gravitation. It can be as massive as 10^{13} times the mass of sun. It consists of stars, stellar remnants, dark matter and interstellar medium composed of gas and dust. The size of galaxies vary from dwarfs with ten million stars to giant galaxies with a hundred trillion stars where each of these stars are orbiting the center of the mass of the galaxy. Sun and all stars in the sky visible to us belong to the Milky Way galaxy. Its diameter ranges from 100,000 to 180,000 light years and contain approx 100 to 400 billion stars. From Earth we see a hazy band of stars, as shown in figure 3.1, which is the star disk of our galaxy, the Milky way.

Stars in the galaxy lie in galactic disk which is a flattened, roughly axisymmetric structure or an elliptical structure. The midplane of this disk is known as the galactic plane. Flat disk galaxies also contain a small, unstructured, centrally located stellar system known as bulge. It is thicker than the disk and comprises approx 15 percent of the total light coming from galaxy. The stars in the bulge or elliptical galaxies are believed to form at the same time as that of the formation of galaxies themselves, whereas stars in the disk are forming continuously. Thus, disk stars widely vary in ages. The distribution of bulge stars is symmetric about the galactic midplane and the bulge is little brighter and thicker on one side of the galactic center compared to the other side.

Figure 3.1: Image of Milky way in night sky. (Credit: Besancon Arnaud)

Another component of galaxies is the Stellar Halo. It is the part of the galaxy that contains old stars with low metallicity. It contains about 1 percent stellar mass of the galaxy. The stars in the halo have low metallicity implying that these are among the first components to form in the Galaxy. A large part of the halo comprises of the debris of disrupted stellar systems, such as globular clusters and small satellite galaxies.
The galaxy also contains many small stellar systems known as star clusters. They vary from 10^2 to 10^6 stars. These clusters can be of two types - Open Clusters and Globular Clusters. Open Clusters are irregular stellar systems containing approx of 10^2 to 10^4 stars. New open clusters are continuously formed in the Galactic disk and most of them are younger than 1 Gyr. Globular Clusters are old and much more massive stellar systems. They contain 10^4 to 10^6 stars in nearly spherical distribution. Stellar systems can be mathematically modelled using collisionless Boltzmann equation or fluid equation depending upon the physical phenomena we are interested to explore.

3.2 Types of Galaxies

According to Hubble classification system, galaxies can be categorised into mainly following four types

- Elliptical Galaxies
  They are elliptical in appearance as the name suggests. They range from tens of millions to over hundred of trillion stars and these are mostly older stars. They are dim and has little gas and dust. The star formation activity is little as the interstellar medium ie., the gas component is sparse. A number of globular clusters surround these galaxies. Their shapes vary from nearly circular, named as E0 to flattened systems

in which the major axis of ellipse is much larger than the minor axis, named as E7. An example of elliptical galaxy is shown in figure 3.2 . Mathematically, we can model elliptical galaxies as spherically symmetric systems comprising of particles (stars) and fluid (gas).

Figure 3.2: Image of Elliptical galaxy IC 1101 (Credit: NASA/Chandra)

- Spiral Galaxies
  These galaxies contain a central bulge which is orbited by a flat disk. The disk is composed of stars, gas and dust. The disk is separated into arms where continuously stars are being formed. These galaxies can be easily spotted as they are brighter because of the young stars inhabiting them.
  Some spirals also have a bar at the center instead of a bulge where the spiral arms unwind from the ends of a bar-shaped concentration. Our own galaxy, the Milky way, is also a barred spiral galaxy which is shown in figure 3.3. Spirals can be mathematically modeled as two-dimensional nearly axisymmetric disk. The spiral arms are actually perturbations on an initially axisymmetric disk.
- Lenticular Galaxies
  Lenticular galaxies are transition objects between elliptical and spiral galaxies. They contain a central bulge or bar and are surrounded by a rotating disk as seen in spiral galaxies. Similar to elliptical galaxies, they have little cool gas, almost no recent star formation and hence contains mostly aging stars. Example of lenticular galaxy is NGC 4866 shown in figure 3.4
- Irregular Galaxies
  They have no regular geometric shape and thus have a chaotic appearance. They contain large amount of gas and dust. They are small in size and are thus prone to environmental effects like colliding with bigger galaxies. IC 3583 is an example of

Figure 3.3: The Milkyway galaxy (Image Credit: Katyanna Quach)

irregular galaxy as shown in figure 3.5

3.3 Observations of Centers of galaxies

In this section, we shall explain the observed features at the center of galaxies. These features are counter-intuitive and are the ones which we shall attempt to explain mathematically in this thesis.

The centers of galaxies are also called galactic nuclie. Observations made about galactic nuclei are limited largely by the resolutions of present telescopes. Out of a few galaxies for which such observations are known, two galaxies: NGC4486B (in the Virgo cluster) and M31 (closest neighbour to our galaxy) show an unusual double peak distribution of stars in their nuclei as seen in figure 3.6 and 3.7 .

There are subtle differences in the double nucleus of the two galaxies. Morphologically both NGC4486B and M31 are different. NGC4486B is an elliptical galaxy whereas M31 is a spiral galaxy. Physical separation of peak in NGC4486B is $5-7$ times larger than that of M31. Both the peaks in NGC4486B are at similar distances from center while in M31 the fainter peak is within 0.05" of the center. Central surface brightness of peaks are more closely matched in NGC4486B as compared to M31.

Observation of double peak nucleus in two morphologically different galaxies needs a theoretical modeling for the nucleus in which this feature comes out naturally. Tremaine [1995] proposed an eccentric disk model to explain such stellar distribution which is further explored in Bacon et al. [2001]; Salow and Statler [2001]; Sambhus and Sridhar [2002]. It was suggested by Touma [2002] that a possible origin of such high eccentricity orbits is instability of

Figure 3.4: Image of Lenticular Galaxy NGC 4866. (Image Credit: European Space Agency)

counter-rotating streams of stars. It could occur when a globular cluster gets disrupted and is added to the stellar system. Sridhar and Saini [2010] studied these instabilities analytically in the WKB limit for softened gravity disc. However their analysis had certain limitations like they could not calculate eigenfunctions and only eigenvalues for equal counter-rotation could be calculated.

This work was further taken forward by Gulati et al. [2012], who formulated an eigenvalue problem for a softened gravity disk. They also included a counter-rotating component in the disk which gave rise to instabilities, supporting the formation of double peak nucleus. Gulati et al. [2012], however, had a drawback that softened gravity disk only supports m=1 modes. These could not explain the symmetric double peak nucleus in elliptical galaxies, which could possibly be due to m=2 or m=4 (i.e. even m) modes.

This work was then further extended in Gulati and Saini [2016] and Gulati and Saini [2017]. Authors modelled the stellar disk using collisionless Boltzmann equation which supports all m-modes and could explain the observations in elliptical galaxies.

Eccentric modes for galactic system till now are studied for either gas disc or softened gravity disc or collisionless stellar discs. However, gas and stars coexist in the galaxies and should be studied as a system of two coupled co-planar discs, one comprising of gas and other one being star disc. Similar study was done by Jalali [2013] who combined gas disc and particle disc. But his study was applicable only under the approximation that gas surface density is dominant as compared to particle surface density. He showed that in particle phase two unstable modes exist that give rise to planetesimals. These are helpful in planet formation. However this is not applicable to galactic systems. In the next chapter we propose a mathematical model for a coupled system of gas and particles (stars in our case) applicable to galactic systems.

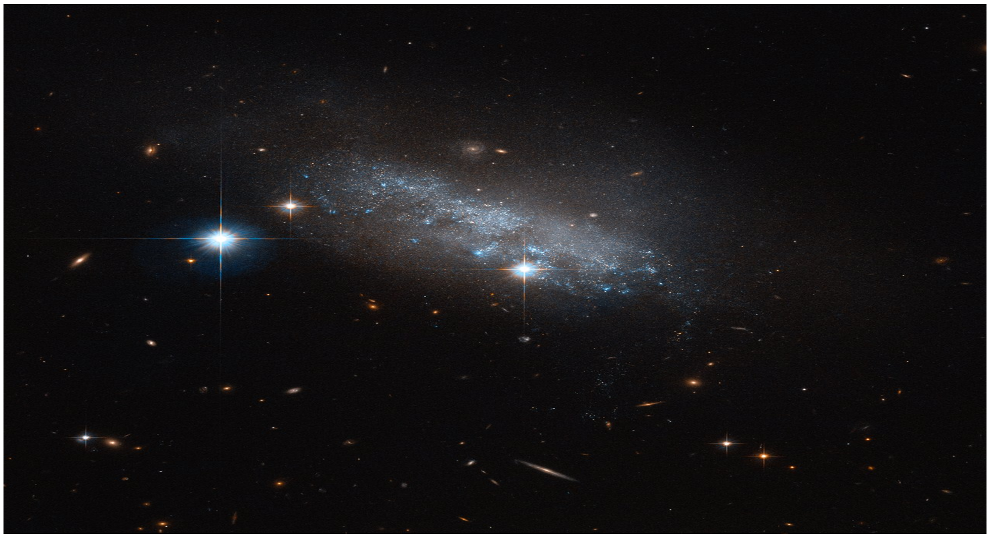

Figure 3.5: Image of galaxy IC 3583, an Irregular galaxy (Credit: ESA/Hubble and NASA)

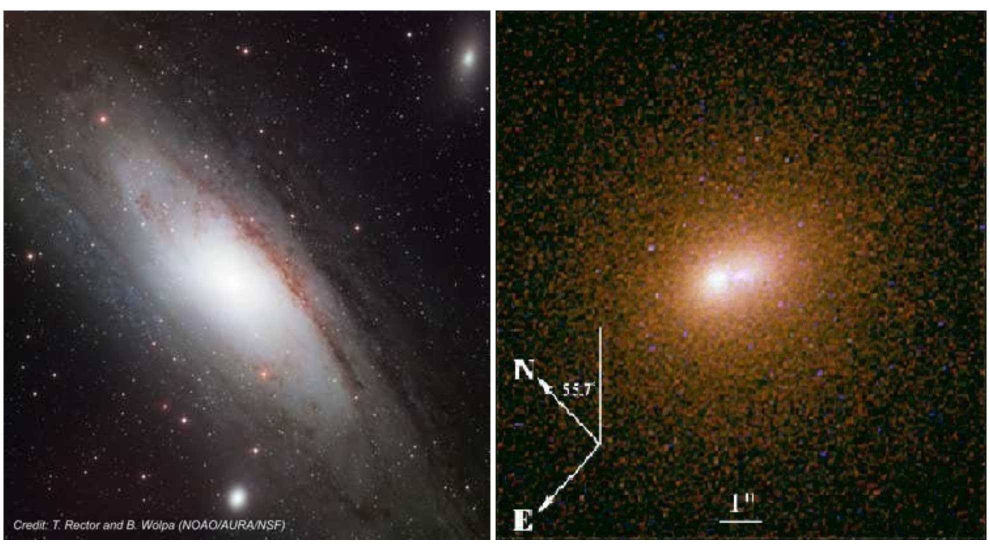

Figure 3.6: Two images of M31; left panel is full image of the spiral M31 (Credit: T.Rector and B.Wolpa (NOAO/AURO/NSF)) and the right one is image of central few arcsec with scale and directions mentioned on the image (Lauer et al., 1998). Double peak structure is clearly revealed in second image.

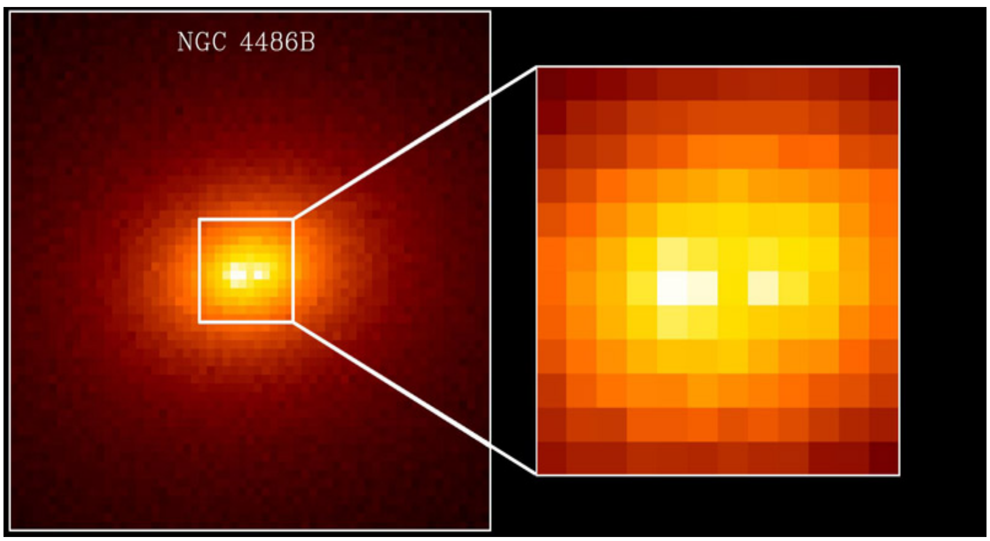

Figure 3.7: Image of NGC4486B.(Credit: Karl Gebhardt, Tod Lauer and NASA). Dimensions of left panel is 2.7 arsec in sky and right panel is a blow-up of central 0.5 arsec. Two brightness peaks come out distinctively in the images.

Chapter 4

PERTURBATIVE ANALYSIS FOR NEARLY KEPLERIAN GAS AND PARTICLE DISC

4.1 Introduction

In this chapter, to explain the observations of double peak nucleus using a more realistic model for galactic disc composed of particles and gases, we propose a model. The gas and dust component of the galactic disk composes the gaseous disc while stars form particle disc. The motion of particles is governed by gravitational forces due to both gases and particles. Jalali [2013] studied a system of coupled gas and particle disc applicable only on planetary discs, ie., authors confined their study to disks with $\Sigma_g/\Sigma_p \gg 1$ where Σ_g is surface density of gas and Σ_p is surface density of particle phases. However this approximation is not valid for galactic discs. Hence we propose a model for system with $\Sigma_g/\Sigma_p \lesssim 1$.

4.2 Unperturbed Disc

Unperturbed disc is divided into two parts: gas disc and particle disc. Below we explain the equilibrium configuration for both of them.

4.2.1 Gas Disc

Gases extend over vast regions of space with only few particles per cm^3. As already pointed in chapter 1, gases behave like fluids and can be modelled as continuum. Continuum fluid equations are thus applicable.

We start by considering thin disc of ideal fluid orbiting a central mass M. Although viscous forces are responsible for the accretion flows but we ignore them assuming they have little effect on the perturbed flow [Saini et al., 2009]. We consider continuity and Euler equations in cylinderical polar coordinates with the disc being in z=0 plane;

$$\frac{\partial \Sigma_{dg}}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma_{dg}v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma_{dg}v_\phi) \;=\; 0, \tag{4.1}$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} \;=\; -\frac{GM}{R^2} - \frac{\partial}{\partial R}(\Phi + h), \tag{4.2}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_R v_\phi}{R} \;=\; -\frac{1}{R}\frac{\partial}{\partial \phi}(\Phi + h) \tag{4.3}$$

where Σ_{dg} is surface density of the disc, v_R and v_ϕ are radial and azimuthal components of the fluid velocity, h is enthalpy per unit mass, Φ is the total gravitational potential due to disc and will be discussed later. The equation of state assumed to be $p = D\Sigma_{dg}^{\Gamma}$,ie., we assume a barotropic fluid where $D > 0$ and Γ is a barotropic index. Thus, the isentropic sound speed and enthalpy are given by

$$c_s^2 = \Gamma D \Sigma_{dg}^{\Gamma-1}, \tag{4.4}$$

$$h = \frac{\Gamma D}{\Gamma - 1}\Sigma_{dg}^{\Gamma-1} = \frac{c_s^2}{\Gamma - 1}. \tag{4.5}$$

4.2.2 Particle disc

Stellar disk can be tackled in two ways using softened gravity disc or collisionless particle disc.

- Softened gravity disc
  The concept of softened gravity was introduced by Miller [1971] to avoid computational glitches due to potential blow up in situations of close encounters of two gravitating bodies. Thus, he modified prescription for gravitational potential to $(R^2 + b^2)^{-1/2}$, where b is the softening length, rather than $1/|R|$.
  It is similar to fluid disc and thus fluid equations are applicable here which are given as following

$$\frac{\partial \Sigma_{ds}}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma_{ds}v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma_{ds}v_\phi) = 0, \tag{4.6}$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{GM}{R^2} - \frac{\partial \Phi}{\partial R}, \tag{4.7}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_R v_\phi}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} \tag{4.8}$$

  where Σ_{ds} is surface density of the disc, v_R and v_ϕ are radial and azimuthal components of the fluid velocity, Φ is gravitational potential due to disc.

- Collisionless particle disc
  We have discussed the collisionless Boltzmann equation already in chapter 2. We assume that the orbits in the disc can be described by the epicycle approximation. By epicycle approximation we mean that the orbits are nearly circular. The solution of CBE for such system can be written in form of Schwarzschild distribution function given as
  $$f_0(v_R, \tilde{v_\phi}, L_Z) = \frac{\gamma\Sigma(R)}{2\pi\sigma_R^2(R)} exp\left[-\frac{v_R^2 + \gamma^2\tilde{V_\phi^2}}{2\sigma_R^2(R)}\right] \tag{4.9}$$
  where $\Sigma(R)$ is surface density, $\tilde{v_\phi} = v_\phi - v_c(R)$, $v_c(R) = R\Omega(R)$ is the circular speed, $\sigma_R(R)$ is the radial velocity dispersion, and $\gamma(R) = 2\Omega(R)/\kappa(R)$ [Binney and Tremaine, 2008].

4.2.3 Potential Theory

Being a scalar potential at any point in the disk is the addition of the potential due to centeral mass, given by $-GM/R$, potential of gas disc and stellar disc. Potential of gas disc is given by
$$\Phi_{dg}(R) = -G\int \frac{\Sigma_{dg}(R')}{|R - R'|} d^2R' \,. \tag{4.10}$$
And the potential of the stellar disc is
$$\Phi_{ds}(R) = -G\int \frac{\Sigma_{ds}(R')}{\sqrt{|R - R'|^2 + b^2}} d^2R' \tag{4.11}$$
where b is Miler softening length. Thus total potential is
$$\Phi(R) = -\frac{GM}{R} + \Phi_{dg}(R) + \Phi_{ds}(R). \tag{4.12}$$
The azimuthal frequency Ω is given by
$$\Omega^2(R) = \frac{GM}{R^3} + \frac{1}{R}\frac{d}{dR}(\Phi_{ds} + \Phi_{dg}). \tag{4.13}$$

4.3 WKB Dispersion Relation

We shall use the perturbative analysis to study the evolution of the unperturbed system explained in the previous section. To begin with, we assume that only the particle disc is perturbed and the gas disc only contributes to the unperturbed potential. Also we shall use the softened gravity disc to mimic the particle disc for the derivation of dispersion relation. However the complete analysis would require the use of collisionless Boltzmann equation.

Let $X = X_0 + X_1$ be the perturbation of any physical quantity X. Substituting this and keeping only first order perturbed quantities in equations (4.6), (4.7), (4.8), we get

$$\frac{\partial \Sigma_1}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma_0 v_{R1}) + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} + \Omega\frac{\partial \Sigma_1}{\partial \phi} = 0, \tag{4.14}$$

$$\frac{\partial v_{R1}}{\partial t} + \Omega\frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial \Phi_1}{\partial R}, \tag{4.15}$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \Omega\frac{\partial v_{\phi 1}}{\partial \phi} + \left[\frac{d\Omega R}{dR} + \Omega\right] v_{R1} = -\frac{1}{R}\frac{\partial \Phi_1}{\partial \phi}. \tag{4.16}$$

Any solution of the above equations is of the form $X_1 = X_a(R)e^{i(m\phi - \omega t)}$, where $m \geq 0$ is an integer. Substituting this definition in equations (4.14), (4.15), (4.16) and then solving for v_{Ra} and $v_{\phi a}$, we get

$$-i(m\Omega - \omega)\Sigma_a = \frac{1}{R}\frac{d}{dR}(R\Sigma_0 v_{Ra}) + \frac{im\Sigma_0}{R} v_{\phi a}, \tag{4.17}$$

$$v_{Ra} = -\frac{i}{D}\left[(m\Omega - \omega)\frac{d\Phi_a}{dR} + \frac{2m\Omega}{R}\Phi_a\right], \tag{4.18}$$

$$v_{\phi a} = \frac{1}{D}\left[\frac{\kappa^2}{2\Omega}\frac{d\Phi_a}{dR} + \frac{m(m\Omega - \omega)}{R}\Phi_a\right] \tag{4.19}$$

where

$$D = \kappa^2 - (m\Omega - \omega)^2.$$

The above equations determine v_{Ra}, $v_{\phi a}$, Σ_a in terms of perturbing potential Φ_a. Manipulating the Poisson integral for the stellar component, as the perturbed potential will only have contribution from particle disc, we obtain

$$\Phi_a(R) = \int_0^\infty R' dR' P_m(R, R')\Sigma_a(R') \tag{4.20}$$

where P_m is the polarization function that relates Φ_a to Σ_a [Gulati et al., 2012].
Under WKB approximation, ie., if radial wave number k is larger ($|kR| \gg m$), then perturbed density is of the form $\Sigma_a(R) = \Sigma_a(R) exp[i\int^R dRk(R)]$ which gives

$$\Phi_a = -2\pi G\frac{exp(-|k|b)}{|k|}\Sigma_a. \tag{4.21}$$

The term with d/dR in RHS of eqs (4.17), (4.18), (4.19) is $O(|kR|/m)$ larger than the others and thus, neglecting the latter, we get

$$\Sigma_a = -k^2\frac{\Sigma_0}{D}\Phi_a, \tag{4.22}$$

$$v_{Ra} = \frac{(m\Omega - w)}{D} k\Phi_a, \tag{4.23}$$

$$v_{\phi a} = i\frac{\kappa^2}{2\Omega D} k\Phi_a. \tag{4.24}$$

Substituting value of Σ_a from eqn (4.22) in eqn (4.21) gives the WKB dispersion relation

$$D = 2\pi G|k|exp(-|k|b)\Sigma_0. \tag{4.25}$$

4.4 Results and Discussion

The dispersion relationship obtained in above section is same as dispersion relationship derived by Sridhar and Saini [2010] and Gulati et al. [2012] for single disc. The only difference it has is that $\Omega(R)$ has contribution from both star disc and gas disc. This doesnot have any direct implication on the stability of the disc and hence their conclusion that a nearly Keplerian single disc is stable to the perturbations holds for our analysis as well. The gravitationally coupled gas and stellar discs turns out to be stable to non-axisymmetric perturbations. The observed lopsided distribution of stars will only be long lived if there is a continuous source of trigger for such perturbations.

Our results serve as the first step for a detailed numerical simulation without the WKB approximation. Further investigation is required for giving the details of eigenvalues and eigenvectors and the nature of perturbation which will be carried over in due course.

Chapter 5

CONCLUSION

In this thesis, we discussed the mathematical modeling of galactic systems, largely the centre of galaxies. The basic motivation behind this was the counter-intuitive observations that we see at the centre of galaxies. In past two decades many works have been done on explaining these structures but each theory proposed has its own drawback which we have discussed in detail at various steps in the thesis. In the present work we aim to extend the theory further to model the actual systems of galaxies which consist of both gas and stars.

In the first chapter of this thesis we have given an introduction to fluids and the plan of thesis. In the second chapter, we discussed the basic approximations under which the fluid behave as continuum. Conditions such as the particles should be non-overlapping, ie., $\lambda << n^{-1/3}$ and each particle have to travel large distance before colliding with another forms the basic conditions under which continuum actually holds. In chapter 3, we discussed various observations of the centres of galaxies and the mathematical modeling of the galactic systems that is done till date. Similar work was done by Jalali [2013] combining gas and star disc but under approximation that gas component dominates.

Next in chapter 4, we give our model to explain eccentric modes, applicable to the system which have both gas and particle disc coupled together. We have assumed that both gas and particle component in the disc are comparable. We formulated a dispersion relationship in which the gas disc and the star disc are considered as two coplanar discs. The perturbation occurs only in the star disc while the gas disc add to the unperturbed potential. We concluded that the discs are largely stable. We cannot comment on the full nature of eigenvalues and eigenvectors. The nature of eigenvalues and eigenvectors can only be described after the formulation of complete eigenvalue problem which is the topic of future investigation.

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