SOLVING DIFFERENT TYPES OF TRANSPORTATION PROBLEMS IN FUZZY ENVIRONMENT

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Submitted by

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DEDICATED

TO

MY PARENTS, GOD AND MEHAR

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"Solving different types of transportation problems in fuzzy environment"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

In todays highly competitive market, the pressure on organizations to find better ways to create and deliver value to customers become stronger. How and when to send the products to the customers in the quantities, they want in a cost-effective manner, become more challenging. Transportation models provide a powerful framework to meet this challenge. Transportation models ensure the efficient movement and timely availability of raw materials and finished goods.

The solid transportation problem can be considered as extension of transportation problem. In solid transportation problem, different modes of transport called conveyances, such as truck, cargo flights etc. are used to transport a homogeneous product from different sources to various destinations.

This thesis is devoted to solving different types of transportation problems in fuzzy environment. The main topics are transportation problem and solid transportation problem in fuzzy environment.

The chapter-wise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes basic definitions, arithmetic operations and concepts used throughout the work.

Chapter 2 presents brief review of the work done in the area of fuzzy transportation problem and fuzzy solid transportation problem.

In **Chapter 3,** fuzzy transportation problem is studied. Fuzzy vogel's approximation method is used to find the fuzzy initial basic feasible solution and fuzzy modified distribution method is used to find fuzzy optimal solution of the said problem. To illustrate the presented method, a numerical example is solved.

In **Chapter 4,** two types of fuzzy transportation problems are studied. The

problems are solved using extension principle. To illustrate the presented methods, two numerical examples are solved.

In **Chapter 5,** the solution procedure of fuzzy solid transportation problem using extension principle is presented. To illustrate the presented method, a numerical example is solved.

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Chapter 1

Introduction

1.1 Introduction

The transportation problem is one of the earliest application of the linear programming problems. The basic transportation problem was originally developed by Hitchcock [12]. In classical form, the transportation problem minimizes the cost of transporting a product which is available at some sources and is required at various destinations. In general transportation problem is solved with the assumptions that the coefficients or cost parameters, availability and demand are specified in a precise way i.e. in crisp environment. In real life, there are many diverse situations due to uncertainty in judgement, lack of evidence etc., sometimes it is not possible to get relevant precise data for all parameters. This type of imprecise data is not always well represented by random variable selected from a probability distribution. Fuzzy numbers introduced by Zadeh [29], may represents this data. So fuzzy decision making method is needed here. The basic definitions [30] used throughout the work are as follows:

Definition 1.1 A crisp set or a classical set A is defined as a collection of distinct and distinguishable objects. The objects are called elements of A. A crisp set A, defined on the universal set X, can also be represented by $A = \{(x, \mu_A(x)) : x \in X\}$ where $\mu_A : X \to \{0, 1\}$ is called characteristic function defined by

$$\mu_A(x)=\begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

Definition 1.2 The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\widetilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\widetilde{A}} : X \rightarrow [0,1]$. The assigned value indicates the membership grade of the element in the set A.

Definition 1.3 The function $\mu_{\widetilde{A}}$ is called the membership function and the set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$ defined by $\mu_{\widetilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 1.4 Let $\widetilde{A}$ be a fuzzy set then $(\widetilde{A})_\alpha = \{x \in X : \mu_{\widetilde{A}}(x) \geq \alpha, 0 \leq \alpha \leq 1\}$ is said to be an α-cut of $\widetilde{A}$.

Definition 1.5 The support of a fuzzy set $\widetilde{A}$ is the crisp subset of X and is defined as:

$$supp(\widetilde{A}) = \{x \in X : \mu_{\widetilde{A}}(x) > 0\}$$

Definition 1.6 A fuzzy set $\widetilde{A}$, defined on the universal set X, is said to be convex if $\mu_{\widetilde{A}}(\alpha_1 x_1 + \alpha_2 x_2) \geq min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}\ \forall\, x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \geq 0,\ \ \alpha_1 + \alpha_2 = 1$ and $\widetilde{A}$ is said to be normal if there exists at least one $x \in X$ such that $\mu_{\widetilde{A}}(x) = 1$.

Definition 1.7 A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

1. $\mu_{\widetilde{A}} : R \longrightarrow [0,1]$ is continuous.
2. $\mu_{\widetilde{A}}(x) = 0$, for all $x \in (-\infty, a] \cup [d, \infty)$.
3. $\mu_{\widetilde{A}}(x)$ is strictly increasing on $[a,b]$ and is strictly decreasing on $[c,d]$.
4. $\mu_{\widetilde{A}}(x) = 1$, for all $x \in [b,c]$.

Definition 1.8 A fuzzy number $\widetilde{A}$ defined on the universal set of real numbers R, denoted as $\widetilde{A} = (a,b,c,d)$, is said to be a trapezoidal fuzzy number if its membership function, $\mu_{\widetilde{A}}(x)$, is given by

$$\mu_{\widetilde{A}}(x)=\begin{cases} \frac{(x-a)}{(b-a)}, & a \le x < b \\ 1, & b \le x \le c \\ \frac{(x-d)}{(c-d)}, & c < x \le d \\ 0, & \text{otherwise} \end{cases}$$

1.2 Arithmetic operations

In this section, some arithmetic operations between trapezoidal fuzzy numbers, defined on universal set of real numbers R, are presented.

Let $\widetilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers

(i) $\widetilde{A}_1 \oplus \widetilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

(ii) $\widetilde{A}_1 \ominus \widetilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$

(iii) $k\widetilde{A}_1 = \begin{cases} (ka_1, kb_1, kc_1, kd_1) & \text{if } k \ge 0 \\ (kd_1, kc_1, kb_1, ka_1) & \text{if } k \le 0 \end{cases}$

(iv)$\widetilde{A}_1 \otimes \widetilde{A}_2 = (a, b, c, d)$

where, $a = min(a_1a_2, a_1d_2, a_2d_1, d_1d_2)$, $b = min(b_1b_2, b_1c_2, c_1b_2, c_1c_2)$

$c = max(b_1b_2, b_1c_2, c_1b_2, c_1c_2)$, $d = max(a_1a_2, a_1d_2, a_2d_1, d_1d_2)$

1.3 Ranking function

A convenient method for comparing of fuzzy numbers is by use of ranking function [16]. A ranking function $\Re : F(R) \rightarrow R$, where $F(R)$ represents a set of all trapezoidal fuzzy numbers defined on a set of real numbers, which maps each fuzzy number into a real number of $F(R)$.

Let $\widetilde{A} = (a, b, c, d)$ then $\Re(\widetilde{A}) = \dfrac{a + b + c + d}{4}$

Let $\widetilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\widetilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers

(i) $\widetilde{A}_1 \succeq \widetilde{A}_2$ if and only if $\Re(\widetilde{A}_1) \ge \Re(\widetilde{A}_2)$

(ii) $\widetilde{A}_1 \succ \widetilde{A}_2$ if and only if $\Re(\widetilde{A}_1) > \Re(\widetilde{A}_2)$

(iii) $\widetilde{A}_1 \approx \widetilde{A}_2$ if and only if $\Re(\widetilde{A}_1) = \Re(\widetilde{A}_2)$

1.4 Transportation problem

Consider a transportation problem with m sources and n destination. Let a_i be the availability at i^{th} source, b_j be the demand at j^{th} destination, c_{ij} be the cost of transporting one unit of the product from i^{th} source to j^{th} destination and x_{ij} be the quantity transported from i^{th} source to j^{th} destination. The crisp transportation problem is:

$$
\begin{aligned}
Z = min \; & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
& subject \; to \\
& \quad \sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, ..., m, && (1.1) \\
& \quad \sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, ..., n, && (1.2) \\
& \quad x_{ij} \geq 0, \quad \forall \;\; i, j.
\end{aligned}
$$

The necessary and sufficient condition for above transportation problem to have initial basic feasible solution is $\sum\limits_{i=1}^{m} a_i = \sum\limits_{j=1}^{n} b_j$.

Definition 1.4.1 A set of non-negative allocations x_{ij} is said to be feasible solution of above transportation problem if it satisfies the constraints (1.1) and (1.2).

Definition 1.4.2 A feasible solution to above transportation problem is said to be basic feasible solution if the number of allocations are $m + n - 1$.

Definition 1.4.3 If the number of allocations in a basic feasible solutions are less than $m + n - 1$, it is called degenerate basic feasible solution (otherwise non-degenerate).

Definition 1.4.4 A basic feasible solution is said to be optimal solution if it minimizes the total transportation cost.

Chapter 2

Literature review

The transportation problem is one of the earliest applications of linear programming problems. The basic transportation problem was originally developed by Hitchcock [12]. Efficient methods of solution derived from the simplex algorithm were developed in 1947, primarily by Dantzig [8] and then by Charnes and Cooper [7]. The transportation problem can be modeled as a standard linear programming problem, which can then be solved by the simplex method. However, because of its very special mathematical structure, it was recognized early that the simplex method applied to the transportation problem can be made quite efficient in terms of how to evaluate the necessary simplex-method information (variable to enter the basis, variable to leave the basis and optimality conditions).

In 1963, Dantzig used the simplex method to the transportation problem as the primal simplex transportation method. An initial basic feasible solution for the transportation problem can be obtained by using the north west corner rule, row minima, column minima, matrix minima or the vogel's approximation method. The modified distribution method is useful for finding the optimal solution for the transportation problems. Charnes and Cooper [7] developed the stepping stone method which provides an alternative way of determining the simplex-method information. The Linear Interactive and Discrete Optimization (LINDO) [20] and many other commercial and academic packages are useful to find the solution of the transportation problems.

In general, transportation problems are solved with the assumptions that the coefficients or cost parameters are specified in a precise way i.e. in crisp environment. In real life, there are many diverse situations due to uncertainty in judgments, lack of evidence etc. Sometimes it is not possible to get relevant precise data for the cost parameter. This type of imprecise data is not always well represented by random variable selected from a probability distribution. Fuzzy number may represent this data. So, the concept of fuzzy set theory is introduced by Zadeh [29]. fuzzy decision making method is needed here.

Zimmerman [31] showed that solutions obtained by fuzzy linear programming are always efficient. Subsequently, Zimmermann's fuzzy linear programming has developed into several fuzzy optimization methods for solving the transportation problems. Chanas et al. [4] presented a fuzzy linear programming model for solving transportation problems with crisp cost coefficients and fuzzy availability and demand values. Moreover, Chanas and Kuchta [5] proposed the concept of the optimal solution for the transportation problem with fuzzy coefficients expressed as $L-R$ fuzzy numbers and developed an algorithm for obtaining the optimal solution. Chanas and Kuchta [6] designed an algorithm for solving the integer fuzzy transportation problem with fuzzy availability and demand volumes in the sense of maximizing the joint satisfaction of the fuzzy goal and constraints.

Kikuchi [17] suggested that in many problems of transportation engineering and planning, the observed or derived values of the variables are approximate, yet the variables themselves must satisfy a set of rigid relationships dictated by physical principle. When the observed values do not satisfy the relationships, each value is adjusted until they satisfy the relationship. They proposed a simple adjustment method that finds the most appropriate set of crisp numbers. The method assumes that each observed value is an approximate number (or a fuzzy number) and the true value is found in the support of the membership function. For each of many possible sets of values that satisfy the relationships, the lowest membership grade is checked and the set whose lowest membership grade is the highest, is chosen as the best set of values for the problem. This process is performed using the fuzzy linear programming method. Their paper presents the model, the computational process and applications. Chanas and Kuchta [5] and Saad and Samir [25] discussed the solution algorithm for solving the transportation problems in fuzzy environment. Grzegorzewski [10] and Chanas [3] approximated the fuzzy number to its nearest interval. Liu and Kao [21] described a method for solving fuzzy transportation problems based on extension principle [30].

The solid transportation problem was proposed by Shell [26]. Haley [11] introduced the solution procedure of solid transportation problem which is an extension of the modified distribution method. For finding an optimal solution, the solid transportation problem requires $m + n + l - 2$ nonzero values of the decision variables to start with a basic feasible solution. Patel and Tripathy [23] developed a computationally superior method for a solid transportation problem with mixed constraints. Bit et al. [2] developed the fuzzy programming model for a multi-objective solid transportation problem. Vajda [27] proposed an algorithm for a multi-index transportation problem which is an extension of the modified distribution method. Gen et al. [9] gave a genetic algorithm for solving a bicriteria solid transportation problem with fuzzy numbers. Li et al. [19] designed a neural network approach for a multicriteria solid transportation problem. Li et al. [18] improved the genetic algorithm given by [19] to solve the fuzzy multi-objective solid transportation problem with fuzzy numbers. Jimenez and Verdegay [15] investigated multi-objective solid transportation problem via genetic algorithm. Jimenez and Verdegay [13] obtained a solution procedure for uncertain solid transportation problem. Jimenez and Verdegay [14] developed a parametric approach for solving fuzzy solid transportation problems by an evolutionary algorithm.

Chapter 3

Solution of fully fuzzy transportation problems using ranking function

In this chapter, a method is presented to solve such fuzzy transportation problems (FFTP) in which all the parameters are represented by trapezoidal fuzzy numbers. Fuzzy vogel's approximation method is used to find fuzzy initial basic feasible solution and fuzzy modified distribution method is used to find fuzzy optimal solution. To illustrate the presented method a numerical example is solved.

3.1 Fully fuzzy transportation problems

In crisp transportation problems, it is assumed that decision maker is sure about the precise values of transportation cost, availability and demand of the product. In real world applications, all these parameters of the transportation problems may not be known precisely due to controllable factors. To deal with such situations, fuzzy set theory is applied in literature to solve the transportation problems.

The balanced fuzzy transportation problems, in which a decision maker is uncertain about the precise values of transportation cost, availability and demand can be formulated as follows:

$$\widetilde{Z} = min \sum_{i=1}^{m} \sum_{j=1}^{n} \widetilde{C}_{ij} \otimes \widetilde{X}_{ij}$$

$$subject\ to$$

$$\begin{aligned}
&\sum_{j=1}^{n} \widetilde{X}_{ij} \approx \widetilde{a}_i, && i = 1, 2, ..., m \\
&\sum_{i=1}^{m} \widetilde{X}_{ij} \approx \widetilde{b}_j, && j = 1, 2, ..., n \\
&\sum_{i=1}^{m} \widetilde{a}_i \approx \sum_{j=1}^{n} \widetilde{b}_j \\
&\widetilde{X}_{ij} \succeq 0
\end{aligned}$$

where, m : total number of sources ; n : total number of destinations

$\widetilde{a}_i$: the fuzzy availability of the product at i^{th} source

$\widetilde{b}_j$: the fuzzy demand of the product at j^{th} destination

$\widetilde{C}_{ij}$: the fuzzy transportation cost for one unit quantity of the product from i^{th} source to j^{th} destination

$\widetilde{X}_{ij}$: the fuzzy quantity of the product that should be transported from i^{th} source to j^{th} destination

$\sum_{i=1}^{m} \widetilde{a}_i$: total fuzzy availability of the product

$\sum_{j=1}^{n} \widetilde{b}_j$: total fuzzy demand of the product

$\sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{C}_{ij} \otimes \widetilde{X}_{ij}$: total fuzzy transportation cost

Table 3.1 represents the tabular form of FFTP

Table 3.1 Tabular representation of FFTP

Destination (j)→ Source (i) ↓	1	2	...	n	Fuzzy availability ($\widetilde{a}_i$)
1	$\widetilde{C}_{11}$ $\widetilde{X}_{11}$	$\widetilde{C}_{12}$ $\widetilde{X}_{12}$	... ...	$\widetilde{C}_{1n}$ $\widetilde{X}_{1n}$	$\widetilde{a}_1$
2	$\widetilde{C}_{21}$ $\widetilde{X}_{21}$	$\widetilde{C}_{22}$ $\widetilde{X}_{22}$	... ...	$\widetilde{C}_{2n}$ $\widetilde{X}_{2n}$	$\widetilde{a}_2$
⋮	⋮	⋮	⋮	⋮	⋮
m	$\widetilde{C}_{m1}$ $\widetilde{X}_{m1}$	$\widetilde{C}_{m2}$ $\widetilde{X}_{m2}$	... ...	$\widetilde{C}_{mn}$ $\widetilde{X}_{mn}$	$\widetilde{a}_m$
Fuzzy demand ($\widetilde{b}_j$)	$\widetilde{b}_1$	$\widetilde{b}_2$	...	$\widetilde{b}_n$	$\sum_{i=1}^{m} \widetilde{a}_i \approx \sum_{j=1}^{n} \widetilde{b}_j$

Remark 3.1 If $\sum_{i=1}^{m} \widetilde{a}_i \approx \sum_{j=1}^{n} \widetilde{b}_j$ then the FFTP is said to be balanced FFTP, otherwise it is called unbalanced FFTP.

3.2 Solution of fully fuzzy transportation problems

The solution of FFTP can be obtained in two stages, namely fuzzy initial basic feasible solution and fuzzy optimal solution. For finding initial basic feasible solution of a fuzzy transportation problem there are numerous methods but fuzzy vogel's approximation method is preferred over the other methods. Since the initial fuzzy basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

3.2.1 Fuzzy vogel's approximation method

The various steps of fuzzy vogel's approximation method are:

Step 1 Find the fuzzy penalties, namely the fuzzy difference between the smallest and next smallest fuzzy costs in each row and column.

Step 2 Among the fuzzy penalties as found in Step 1, choose the fuzzy maximum penalty, by ranking method. If the maximum penalties are more than one, choose any one arbitrarily.

Step 3 In the selected row or column as by Step 2, find out the cell having the least fuzzy cost. Allocate to this cell as much as possible depending on the fuzzy availability and fuzzy demands.

Step 4 Delete the row or column which is fully exhausted. Again compute column and row fuzzy penalties for the reduced fuzzy transportation table and then go to Step 2, repeat the procedure until all the demands are satisfied.

Once the fuzzy initial basic feasible solution is computed, the next step in the problem is to determine whether the solution obtained is optimal or not.

Fuzzy optimality test can be conducted to any fuzzy initial basic feasible solution of a fuzzy transportation provided such allocations has exactly $(m+n-1)$ non-negative allocations where m is the number of sources and n is the number of destinations.

3.2.2 Fuzzy modified distribution method

The method is used to find the fuzzy optimal solution. The various steps of the method are:

Step 1 Find out a set of numbers $\widetilde{U}_i$ and $\widetilde{V}_j$ for each row and column satisfying $\widetilde{U}_i \oplus \widetilde{V}_j \approx \widetilde{C}_{ij}$ for each basic cell.

Step 2 To start with we assign a zero trapezoidal fuzzy number to any row or column having maximum number of allocations. If the maximum number of allocation is more than one, then choose any one arbitrary.

Step 3 Find out for each nonbasic cell the net evaluation value $\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j$ this step gives the optimality conclusion.

Case (i) If $\Re(\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j) > 0 \;\; \forall\, i, j$ then the solution is optimal and a unique solution exists.

Case (ii) If $\Re(\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j) \geq 0 \;\; \forall\, i, j$ then the solution is fuzzy optimal, but an alternate solution exists.

Case (iii) If $\Re(\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j) < 0$ for at least one i, j then the solution is not fuzzy optimal.

In this case we go to next step, to improve the total fuzzy transportation cost.

Step 4 Select the nonbasic cell having the most negative value of $\Re(\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j)$ from this cell draw a closed path horizontally and vertically to the nearest basic cell with the restriction that the corner of the closed path must not lie in any nonbasic cell. Assign sign $+$ and $-$ alternately and find the fuzzy minimum allocation from the cell having negative sign. This allocation should be added to the allocation having negative sign.

Step 5 The above step yields a better solution by making one (or more) basic cell as nonbasic cell and one nonbasic cell as basic cell. For this new set of fuzzy basic feasible solution repeat from Step 1, until a fuzzy optimal solution is obtained.

where, $\widetilde{U}_i = (u_i^{(1)}, u_i^{(2)}, u_i^{(3)}, u_i^{(4)})$ and $\widetilde{V}_j = (v_j^{(1)}, v_j^{(2)}, v_j^{(3)}, v_j^{(4)})$

3.3 Numerical example

In this section, a FFTP is solved by using the methods discussed in Section 3.2.

Example 3.1 Suppose there are three sources and four destination. Let $\widetilde{C}_{ij}$ be the fuzzy transportation cost for unit quantity of the product from i^{th} source to j^{th} destination, $\widetilde{a}_i$ be the fuzzy availability at i^{th} source and $\widetilde{b}_j$ be the fuzzy demand at j^{th} destination are shown in Table 3.2. Find the fuzzy quantity of the product transported from each source to various destinations so that the total fuzzy transportation cost is minimum.

Table 3.2 Numerical example

Destination (j)→ Source (i) ↓	1	2	3	4	Fuzzy availability
1	$(-2,0,2,8)$	$(-2,0,2,8)$	$(-2,0,2,8)$	$(-1,0,1,4)$	$(0,2,4,6)$
2	$(4,8,12,16)$	$(4,7,9,12)$	$(2,4,6,8)$	$(1,3,5,7)$	$(2,4,9,13)$
3	$(2,4,9,13)$	$(0,6,8,10)$	$(0,6,8,10)$	$(4,7,9,12)$	$(2,4,6,8)$
Fuzzy demand	$(1,3,5,7)$	$(0,2,4,6)$	$(1,3,5,7)$	$(1,3,5,7)$	

Solution: $\sum_{i=1}^{m} \widetilde{a}_i = (4,10,19,27)$ and $\sum_{j=1}^{n} \widetilde{b}_j = (3,11,19,27)$

Since $\Re(\sum_{i=1}^{m} \widetilde{a}_i) = 15 = \Re(\sum_{j=1}^{n} \widetilde{b}_j)$

So $\sum_{i=1}^{m} \widetilde{a}_i \approx \sum_{j=1}^{n} \widetilde{b}_j$, the problem is balanced fuzzy transportation problem.

Using the steps, discussed in Section 3.2.1, the obtained fuzzy initial basic feasible solution is shown in Table 3.3:

Table 3.3 Fuzzy initial basic feasible solution

Destination $(j)\rightarrow$ Source $(i)\downarrow$	1	2	3	4	Fuzzy availability
1	$(-2,0,2,8)$ $(0,2,4,6)$	$(-2,0,2,8)$	$(-2,0,2,8)$	$(-1,0,1,4)$	$(0,2,4,6)$
2	$(4,8,12,16)$	$(4,7,9,12)$	$(2,4,6,8)$ $(-5,-1,6,12)$	$(1,3,5,7)$ $(1,3,5,7)$	$(2,4,9,13)$
3	$(2,4,9,13)$ $(-5,-1,3,7)$	$(0,6,8,10)$ $(0,2,4,6)$	$(0,6,8,10)$ $(-11,-3,6,12)$	$(4,7,9,12)$	$(2,4,6,8)$
Fuzzy demand	$(1,3,5,7)$	$(0,2,4,6)$	$(1,3,5,7)$	$(1,3,5,7)$	

Since the number of basic cells are $m+n-1=6$, so the solution is non degenerate fuzzy basic feasible solution.

The initial total fuzzy transportation cost is:

$$\widetilde{C}_{11}\otimes\widetilde{X}_{11}\oplus\widetilde{C}_{23}\otimes\widetilde{X}_{23}\oplus\widetilde{C}_{24}\otimes\widetilde{X}_{24}\oplus\widetilde{C}_{31}\otimes\widetilde{X}_{31}\oplus\widetilde{C}_{32}\otimes\widetilde{X}_{32}\oplus\widetilde{C}_{33}\otimes\widetilde{X}_{33}=$$

$$(-2,0,2,8)\otimes(0,2,4,6)\oplus(2,4,6,8)\otimes(-5,-1,6,12)\oplus(1,3,5,7)\otimes(1,3,5,7)\oplus$$

$$(2,4,9,13)\otimes(-5,-1,3,7)\oplus(0,6,8,10)\otimes(0,2,4,6)\oplus(0,6,8,10)\otimes(-11,-3,6,12)$$

$$\approx(-226,-18,176,464)$$

where, $\widetilde{C}_{11},\widetilde{C}_{23},\widetilde{C}_{24},\widetilde{C}_{31},\widetilde{C}_{32},\widetilde{C}_{33}$ are fuzzy cost coefficients and $\widetilde{X}_{11},\widetilde{X}_{23},\widetilde{X}_{24},\widetilde{X}_{31},\widetilde{X}_{32},\widetilde{X}_{33}$ are fuzzy allocations.

Applying the fuzzy modified distribution method, determine a set of numbers $\widetilde{U}_i$ and $\widetilde{V}_j$ for each row and column such that $\widetilde{C}_{ij}\approx\widetilde{U}_i\oplus\widetilde{V}_j$ for each basic cell. Since 3^{rd} row has maximum numbers of allocations, take $\widetilde{U}_3=(-2,-1,1,2)$. The remaining $\widetilde{U}_i$ and $\widetilde{V}_j$ can be obtained as given below.

$$\widetilde{C}_{31}\approx\widetilde{U}_3\oplus\widetilde{V}_1 \qquad \therefore\widetilde{V}_1\approx(0,3,10,15)$$

$$\widetilde{C}_{32}\approx\widetilde{U}_3\oplus\widetilde{V}_2 \qquad \therefore\widetilde{V}_2\approx(-2,5,9,12)$$

$$\widetilde{C}_{33}\approx\widetilde{V}_3\oplus\widetilde{V}_3 \qquad \therefore\widetilde{V}_3\approx(-2,5,9,12)$$

$$\widetilde{C}_{11}\approx\widetilde{U}_1\oplus\widetilde{V}_1 \qquad \therefore\widetilde{U}_1\approx(-17,-10,-1,8)$$

$$\widetilde{C}_{23}\approx\widetilde{U}_2\oplus\widetilde{V}_3 \qquad \therefore\widetilde{U}_2\approx(-10,-5,1,10)$$

$$\widetilde{C}_{24} \approx \widetilde{U}_2 \oplus \widetilde{V}_4 \qquad \therefore \widetilde{V}_4 \approx (-9, 2, 10, 17)$$

For each nonbasic cell, the net evaluation $\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j$ is calculated and shown in Table 3.4:

Table 3.4 Fuzzy optimal solution

	1	2	3	4	$\widetilde{a}_i$
1	$(-2, 0, 2, 8)$ (0,2,4,6)	$(-2, 0, 2, 8)$ $*(-22, -8, 7, 27)$	$(-2, 0, 2, 8)$ $*(-22, -8, 7, 27)$	$(-1, 0, 1, 4)$ $*(-26, -9, 9, 30)$	(0,2,4,6)
2	(4,8,12,16) $*(-21, -3, 14, 26)$	(4,7,9,12) $*(-18, -3, 9, 24)$	(2,4,6,8) $(-5, -1, 6, 12)$	(1,3,5,7) (1,3,5,7)	(2,4,9,13)
3	(2,4,9,13) $(-5, -1, 3, 7)$	(0,6,8,10) (0,2,4,6)	(0,6,8,10) $(-11, -3, 6, 12)$	(4,7,9,12) $*(-15, -4, 8, 23)$	(2,4,6,8)
$(\widetilde{b}_j)$	(1,3,5,7)	(0,2,4,6)	(1,3,5,7)	(1,3,5,7)	

where, $\widetilde{U}_i = (u_i^{(1)}, u_i^{(2)}, u_i^{(3)}, u_i^{(4)})$ and $\widetilde{V}_j = (v_j^{(1)}, v_j^{(2)}, v_j^{(3)}, v_j^{(4)})$,

Since $\Re(\widetilde{C}_{ij} \ominus \widetilde{U}_i \ominus \widetilde{V}_j) > 0$ for each nonbasic cell so the solution is fuzzy optimal and unique.

$\therefore$ The fuzzy optimal solution in terms of trapezoidal fuzzy numbers is:

$$\widetilde{X}_{11} = (0, 2, 4, 6), \quad \widetilde{X}_{23} = (-5, -1, 6, 12), \quad \widetilde{X}_{24} = (1, 3, 5, 7)$$

$$\widetilde{X}_{31} = (-5, -1, 3, 7), \quad \widetilde{X}_{32} = (0, 2, 4, 6), \quad \widetilde{X}_{33} = (-11, -3, 6, 12)$$

Hence, the minimum total fuzzy transportation cost is:

$\widetilde{C}_{11} \otimes \widetilde{X}_{11} \oplus \widetilde{C}_{23} \otimes \widetilde{X}_{23} \oplus \widetilde{C}_{24} \otimes \widetilde{X}_{24} \oplus \widetilde{C}_{31} \otimes \widetilde{X}_{31} \oplus \widetilde{C}_{32} \otimes \widetilde{X}_{32} \oplus \widetilde{C}_{33} \otimes \widetilde{X}_{33} =$

$(-2, 0, 2, 8) \otimes (0, 2, 4, 6) \oplus (2, 4, 6, 8) \otimes (-5, -1, 6, 12) \oplus (1, 3, 5, 7) \otimes (1, 3, 5, 7) \oplus$

$(2, 4, 9, 13) \otimes (-5, -1, 3, 7) \oplus (0, 6, 8, 10) \otimes (0, 2, 4, 6) \oplus (0, 6, 8, 10) \otimes (-11, -3, 6, 12)$

$\approx (-226, -18, 176, 464)$

Conclusion

In this chapter, an existing method for solving a FFTP in which all the parameters are represented by trapezoidal fuzzy numbers, is presented. To illustrate the existing method, a numerical example is solved and obtained results are discussed.

Chapter 4

Solution of fully fuzzy transportation problems based on extension principle

In this chapter, two methods, based on extension principle [30], is presented to solve such two types of FFTP in which all the parameters are represented by trapezoidal fuzzy numbers. To illustrate the presented methods, two numerical examples are solved and obtained results are discussed.

4.1 Fully fuzzy transportation problems

In this section, the formulation of two types of FFTP are presented:

4.1.1 Fully fuzzy transportation problems with inequality constraints

The formulation of FFTP with inequality constraints is:

$$
\begin{aligned}
&\widetilde{Z} = min \ \sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{C}_{ij}x_{ij} \\
&\qquad subject\ to \\
&\qquad\qquad \sum_{j=1}^{n} x_{ij} \leq \widetilde{S}_i, \qquad i = 1,2,...,m, \\
&\qquad\qquad \sum_{i=1}^{m} x_{ij} \geq \widetilde{D}_j, \qquad j = 1,2,...,n, \\
&\qquad\qquad x_{ij} \geq 0, \qquad \forall \ \ i,j.
\end{aligned}
\tag{$P_{4.1}$}
$$

where, m : total number of sources ; n : total number of destinations

$\widetilde{S}_i$: the fuzzy availability of the product at i^{th} source

$\widetilde{D}_j$: the fuzzy demand of the product at j^{th} destination

$\widetilde{C}_{ij}$: the fuzzy transportation cost for unit quantity of the product from i^{th} source to j^{th} destination

x_{ij} : the quantity of the product that should be transported from i^{th} source to j^{th} destination

$\sum\limits_{i=1}^{m} \widetilde{S}_i$: total fuzzy availability of the product
$\sum\limits_{j=1}^{n} \widetilde{D}_j$: total fuzzy demand of the product
$\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} \widetilde{C}_{ij}x_{ij}$: total fuzzy transportation cost

4.1.2 Fully fuzzy transportation problems with equality constraints

The formulation of FFTP with equality constraints is:

$$\widetilde{Z} = min \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{C}_{ij}x_{ij}$$

$$subject\ to$$

$$\begin{aligned} &\sum_{j=1}^{n} x_{ij} = \widetilde{S}_i, && i = 1,2,...,m,\\ &\sum_{i=1}^{m} x_{ij} = \widetilde{D}_j, && j = 1,2,...,n, \\ &x_{ij} \geq 0, && \forall \ \ i,j. \end{aligned} \tag{$P_{4.2}$}$$

4.2 Solution of fully fuzzy transportation problems

4.2.1 Solution of fully fuzzy transportation problems with inequality constraint

The α-cuts of $\widetilde{C}_{ij}$, $\widetilde{S}_i$ and $\widetilde{D}_j$ are:

$$(\widetilde{C}_{ij})_\alpha = [(C_{ij})^L_\alpha, (C_{ij})^U_\alpha] \tag{4.1}$$

where, $(C_{ij})^L_\alpha = min\{c_{ij} \in S(\widetilde{C}_{ij}) | \mu_{\widetilde{C}_{ij}}(c_{ij}) \geq \alpha\}$

$(C_{ij})^U_\alpha = max\{c_{ij} \in S(\widetilde{C}_{ij}) | \mu_{\widetilde{C}_{ij}}(c_{ij}) \geq \alpha\}$

$$(\widetilde{S}_i)_\alpha = [(S_i)^L_\alpha, (S_i)^U_\alpha] \tag{4.2}$$

where, $(S_i)^L_\alpha = min\{s_i \in S(\widetilde{S}_i) | \mu_{\widetilde{S}_i}(s_i) \geq \alpha\}$

$(S_i)^U_\alpha = max\{s_i \in S(\widetilde{S}_i) | \mu_{\widetilde{S}_i}(s_i) \geq \alpha\}$

$$(\widetilde{D}_j)_\alpha = [(D_j)^L_\alpha, (D_j)^U_\alpha] \tag{4.3}$$

where, $(D_j)^L_\alpha = min\{d_j \in S(\widetilde{D}_j) | \mu_{\widetilde{D}_j}(d_j) \geq \alpha\}$

$(D_j)^U_\alpha = max\{d_j \in S(\widetilde{D}_j) | \mu_{\widetilde{D}_j}(d_j) \geq \alpha\}$

These intervals indicate where the unit transporting cost, availability and demand lie at possibility level α. In deriving the membership function of the total transportation cost $\widetilde{Z}$. The major difficulty lies on how to deal with the varying ranges of the unit transporting costs, the availabilities and the demand quantities. One idea is to apply Zadeh's extension principle [28, 29, 30].

Based on the extension principle, the membership function $\mu_{\widetilde{Z}}$ can be defined as:

$$\mu_{\widetilde{Z}}(z) = \sup_{c,s,d} min\{\mu_{\widetilde{C}_{ij}}(c_{ij}), \mu_{\widetilde{S}_i}(s_i), \mu_{\widetilde{D}_j}(d_j) \forall i, j | z = Z(c, s, d)\} \tag{4.4}$$

where $Z(c, s, d)$, is defined in model $(P_{4.3})$.

$$\begin{aligned} Z = min \ & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ & subject\ to \\ & \quad \sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i = 1, 2, ..., m, \\ & \quad \sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1, 2, ..., n, \\ & \quad x_{ij} \geq 0, \quad \forall \ i, j. \end{aligned} \tag{$P_{4.3}$}$$

where, m : total number of sources ; n : total number of destinations

s_i : the availability of the product at i^{th} source

d_j : the demand of the product at j^{th} destination

c_{ij} : the transportation cost for unit quantity of the product from i^{th} source to j^{th} destination

x_{ij} : the quantity of the product that should be transported from i^{th} source to j^{th} destination

$\sum\limits_{i=1}^{m} s_i$: total availability of the product

$\sum\limits_{j=1}^{n} d_j$: total demand of the product

$\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} c_{ij}\ x_{ij}$: total transportation cost

If the α-cuts of $\widetilde{Z}$ at all α values degenerate to the same point, then the total transportation cost is a crisp number. Otherwise, it is a fuzzy number. In Eq. (4.4), several membership functions are involved. To derive $\mu_{\widetilde{Z}}$ in closed form is hardly possible. According to (4.4), $\mu_{\widetilde{Z}}$ is the minimum of $\mu_{\widetilde{C}_{ij}}$, $\mu_{\widetilde{S}_i}$ and $\mu_{\widetilde{D}_j}$, $\forall\ i,j$. So $\mu_{\widetilde{C}_{ij}}(c_{ij}) \geq \alpha$, $\mu_{\widetilde{S}_i}(s_i) \geq \alpha$, $\mu_{\widetilde{D}_j}(d_j) \geq \alpha$ and at least one $\mu_{\widetilde{C}_{ij}}(c_{ij})$, $\mu_{\widetilde{S}_i}(s_i)$ or $\mu_{\widetilde{D}_j}(d_j)$, $\forall\ i,j$ equal to α such that $z = Z(c,s,d)$ to satisfy $\mu_{\widetilde{Z}}(z) = \alpha$. To find the membership function $\mu_{\widetilde{Z}}$, it suffices to find the left shape function and right shape function of $\mu_{\widetilde{Z}}$, which is equivalent to finding the lower bound Z^L_α and upper bound Z^U_α of the α-cuts of $\widetilde{Z}$. Since Z^L_α is the minimum of $Z(c,s,d)$ and Z^U_α is the maximum of $Z(c,s,d)$, they can be expressed as:

$Z^L_\alpha = min\{Z(c,s,d) | (C_{ij})^L_\alpha \leq c_{ij} \leq (C_{ij})^U_\alpha, (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha, (D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha, \forall\ i,j\}$

$Z^U_\alpha = max\{Z(c,s,d) | (C_{ij})^L_\alpha \leq c_{ij} \leq (C_{ij})^U_\alpha, (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha, (D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha, \forall\ i,j\}$

which can be reformulated as the following pair of two-level mathematical programs:

$$Z^L_\alpha = min(p) \begin{cases} min \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ subject\ to \\ \qquad \displaystyle\sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i = 1,2,...,m, \\ \qquad \displaystyle\sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1,2,...,n, \\ \qquad x_{ij} \geq 0, \qquad \forall\ i,j. \end{cases} \tag{$P_{4.4}$}$$

where,

$$p = (C_{ij})^L_\alpha \leq c_{ij} \leq (C_{ij})^U_\alpha, (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha, (D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha, \forall\ i,j$$

$$Z_\alpha^U = max(p)\begin{cases} min \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ subject\ to \\ \qquad \displaystyle\sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i = 1, 2, ..., m, \\ \qquad \displaystyle\sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1, 2, ..., n, \\ \qquad x_{ij} \geq 0, \quad \forall \ \ i, j. \end{cases} \tag{$P_{4.5}$}$$

where,

$$p = (C_{ij})_\alpha^L \leq c_{ij} \leq (C_{ij})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, \forall \ \ i, j$$

At least one c_{ij}, s_i or d_j must hit the boundary of their α-cuts to satisfy $\mu_{\tilde{Z}}(z) = \alpha$.

A necessary and sufficient condition for Model $(P_{4.4})$ and $(P_{4.5})$ to have feasible solutions is $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$. In the first level of Model $(P_{4.4})$ and $(P_{4.5})$, s_i and d_j are allowed to vary in the range of $[(S_i)_\alpha^L, (S_i)_\alpha^U]$ and $[(D_j)_\alpha^L, (D_j)_\alpha^U]$ respectively. However, to ensure the transportation problem to be feasible at second level, it is necessary to impose the constraints $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$ at the first level. Hence, Model $(P_{4.4})$ and $(P_{4.5})$ becomes:

$$Z_\alpha^L = min(p^*)\begin{cases} min \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ subject\ to \\ \qquad \displaystyle\sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i = 1, 2, ..., m, \\ \qquad \displaystyle\sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1, 2, ..., n, \\ \qquad x_{ij} \geq 0, \quad \forall \ \ i, j. \end{cases} \tag{$P_{4.6}$}$$

where,

$$p^* = (C_{ij})_\alpha^L \leq c_{ij} \leq (C_{ij})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, \sum_{i=1}^{m} s_i \geq$$

$\sum_{j=1}^{n} d_j, \forall \ \ i,j$

$$Z_\alpha^U = max(p^*) \begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ subject \ to \\ \qquad \sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i=1,2,...,m, \\ \qquad \sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j=1,2,...,n, \\ \qquad x_{ij} \geq 0, \quad \forall \ \ i,j. \end{cases} \qquad (P_{4.7})$$

where,

$p^* = (C_{ij})_\alpha^L \leq c_{ij} \leq (C_{ij})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \forall \ \ i,j$

Model $(P_{4.6})$ and $(P_{4.7})$ will be infeasible for any α level if $\sum_{i=1}^{m}(s_i)_{\alpha=0}^U \leq \sum_{j=1}^{n}(d_j)_{\alpha=0}^L$. In other words, a fuzzy transportation problem is feasible if the upper bound of the total fuzzy availability is greater than or equal to the lower bound of the total fuzzy demand.

To find the minimum objective value set c_{ij} to its lower bound $(C_{ij})_\alpha^L \ \forall \ \ i,j$ in Model $(P_{4.6})$. Hence, Model $(P_{4.6})$ can be reformulated as:

$$Z_\alpha^L = min(p^{**}) \begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n} (C_{ij})_\alpha^L x_{ij} \\ subject \ to \\ \qquad \sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i=1,2,...,m, \\ \qquad \sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j=1,2,...,n, \\ \qquad x_{ij} \geq 0, \quad \forall \ \ i,j. \end{cases} \qquad (P_{4.8})$$

where,

$$p^{**} = (S_i)^L_\alpha \le s_i \le (S_i)^U_\alpha, (D_j)^L_\alpha \le d_j \le (D_j)^U_\alpha, \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j, \forall \ \ i, j$$

Since Model ($P_{4.8}$) is to find the minimum of all the minimum objective values, one can insert the constraints of level 1 into level 2 and simplify the two-level mathematical program to the one-level program as follows:

$$\begin{aligned}
Z^L_\alpha = min \ \ & \sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij})^L_\alpha x_{ij} \\
& subject \ to \\
& \sum_{j=1}^{n} x_{ij} \le s_i, \qquad i = 1, 2, ..., m, \\
& \sum_{i=1}^{m} x_{ij} \ge d_j, \qquad j = 1, 2, ..., n, \\
& \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j \\
& (S_i)^L_\alpha \le s_i \le (S_i)^U_\alpha, \qquad i = 1, 2, ..., m \\
& (D_j)^L_\alpha \le d_j \le (D_j)^U_\alpha, \quad j = 1, 2, ..., n \qquad (P_{4.9}) \\
& x_{ij} \ge 0, \quad \forall \ \ i, j.
\end{aligned}$$

This model is a linear program which can be solved easily. In this model, since all c_{ij} have been set to the lower bounds of their α-cuts, that is, $\mu_{\widetilde{C}_{ij}}(c_{ij}) = \alpha$, this assures $\mu_{\widetilde{Z}}(z) = \alpha$ as required by (4.4).

To solve Model ($P_{4.7}$), the dual of the level 2 problem is formulated to become a maximization problem to be consistent with the maximization operation of level 1. It is well-known from the duality theorem of linear programming that the primal model and the dual model have the same objective value. Thus Model ($P_{4.7}$) becomes:

$$Z^U_\alpha = max(p^*) \left\{ \begin{aligned}
& max - \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
& subject \ to \\
& \qquad -u_i + v_j \le c_{ij}, \qquad i = 1, 2, ..., m, \ j = 1, 2, ..., n, \\
& \qquad u_i, v_j \ge 0, \quad \forall \ \ i, j.
\end{aligned} \right. \qquad (P_{4.10})$$

where,

$$p^* = (C_{ij})^L_\alpha \leq c_{ij} \leq (C_{ij})^U_\alpha, (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha, (D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \forall \ \ i,j$$

Since $(C_{ij})^L_\alpha \leq c_{ij} \leq (C_{ij})^U_\alpha, \forall \ \ i,j$ in Model $(P_{4.10})$, one can derive the upper bound of the objective value by setting c_{ij} to its upper bound $(C_{ij})^U_\alpha \ \forall \ \ i,j$, because this gives the largest feasible region. Thus, we can reformulate Model $(P_{4.10})$ as:

$$Z^U_\alpha = max(p^{**}) \begin{cases} max \ -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\ subject \ to \\ \qquad -u_i + v_j \leq (C_{ij})^U_\alpha, \quad i = 1,2,...,m, \ j = 1,2,...,n, \\ \qquad u_i, v_j \geq 0, \quad \forall \ \ i,j. \end{cases} \qquad (P_{4.11})$$

where,

$$p^{**} = (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha, (D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \forall \ \ i,j$$

Now, since both level 1 and level 2 perform the same maximization operation, their constraints can be combined to form the following one-level mathematical program:

$$\begin{aligned} Z^U_\alpha = max \quad & -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\ subject \ to \quad & \\ & -u_i + v_j \leq (C_{ij})^U_\alpha, \quad i = 1,2,...,m, \ \ j = 1,2,...,n, \\ & \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j \\ & (S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha, \quad i = 1,2,...,m, \\ & (D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha, \quad j = 1,2,...,n, \qquad (P_{4.12}) \\ & u_i, v_j \geq 0, \ \ \forall \ \ i,j \end{aligned}$$

This model is a linearly constrained nonlinear program. There are several effective and efficient methods for solving this problem [14]. Similar to Model $(P_{4.9})$ since all

c_{ij} have been set to the upper bounds of their α-cuts, that is, $\mu_{\widetilde{C}_{ij}}(c_{ij}) = \alpha$, this assures $\mu_{\widetilde{Z}}(z) = \alpha$ as required by (4.4).

Problems $(P_{4.6})$ and $(P_{4.7})$ are assured to be feasible if the lower bound of the total fuzzy demand is smaller than the upper bound of the total fuzzy availability, i.e., $\sum_{j=1}^{n}(D_j)^L_{\alpha=0} \leq \sum_{i=1}^{m}(S_i)^U_{\alpha=0}$. If this condition is not satisfied, then the problem will be infeasible. In this case, a fictitious availability point $m+1$ with an amount of $s_{m+1} \geq \sum_{j=1}^{n}(D_j)^L_{\alpha=0} - \sum_{i=1}^{m}(S_i)^U_{\alpha=0}$ just like the crisp transportation problem can be assumed to make the problem feasible. The amount to be transported from the fictitious availability point is the shortage of that demand point.

For two possibility levels α_1 and α_2 such that $0 < \alpha_2 < \alpha_1 \leq 1$, the feasible regions defined by α_1 in Models $(P_{4.9})$ and $(P_{4.12})$ are smaller than those defined by α_2. Consequently, $(Z)^L_{\alpha_1} \geq (Z)^L_{\alpha_2}$ and $(Z)^U_{\alpha_1} \leq (Z)^U_{\alpha_2}$; in other words, the left shape function is nondecreasing and the right shape function is non increasing. This property, based on the definition of "convex fuzzy set", assures the convexity of $\widetilde{Z}$. If both $(Z)^L_\alpha$ and $(Z)^U_\alpha$ are invertible with respect to α, then a left shape function $L(z) = (Z^L_\alpha)^{-1}$ and a right shape function $R(z) = (Z^U_\alpha)^{-1}$ can be obtained. From $L(z)$ and $R(z)$, the membership function $\mu_{\widetilde{Z}}$ is constructed as:

$$\mu_{\widetilde{Z}} = \begin{cases} \text{L(z)}, & (Z)^L_{\alpha=0} \leq z \leq (Z)^L_{\alpha=1} \\ 1, & (Z)^L_{\alpha=1} \leq z \leq (Z)^U_{\alpha=1} \\ \text{R(z)}, & (Z)^U_{\alpha=1} \leq z \leq (Z)^U_{\alpha=0} \end{cases} \tag{4.5}$$

In most cases, the values of $(Z)^L_\alpha$ and $(Z)^U_\alpha$ may not be solved analytically. However, the numerical solutions for $(Z)^L_\alpha$ and $(Z)^U_\alpha$ at different possibility level α can be collected to approximate the shapes of $L(z)$ and $R(z)$.

4.2.2 Solution of fully fuzzy transportation problems with equality constraints

In this section, we discuss the transportation model with equality constraints:

Similar to the discussion of the inequality-constraints case, the lower and upper bounds of $\widetilde{Z}$ at possibility level α can be solved from the following pair of two-level mathematical programs:

$$Z_\alpha^L = min(p^*)\begin{cases} min\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ subject\ to \\ \qquad \displaystyle\sum_{j=1}^{n} x_{ij} = s_i, \qquad i = 1,2,...,m, \\ \qquad \displaystyle\sum_{i=1}^{m} x_{ij} = d_j, \qquad j = 1,2,...,n, \\ \qquad x_{ij} \geq 0, \quad \forall\ \ i,j. \end{cases} \tag{$P_{4.13}$}$$

$$Z_\alpha^U = max(p^*)\begin{cases} min\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ subject\ to \\ \qquad \displaystyle\sum_{j=1}^{n} x_{ij} = s_i, \qquad i = 1,2,...,m, \\ \qquad \displaystyle\sum_{i=1}^{m} x_{ij} = d_j, \qquad j = 1,2,...,n, \\ \qquad x_{ij} \geq 0, \quad \forall\ \ i,j. \end{cases} \tag{$P_{4.14}$}$$

where,

$$p^* = (C_{ij})_\alpha^L \leq c_{ij} \leq (C_{ij})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j, \forall\ \ i,j$$

The corresponding pair of one-level mathematical program is:

$$Z_\alpha^L = min \quad \sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij})_\alpha^L x_{ij}$$

$$subject\ to$$

$$
\begin{aligned}
&\sum_{j=1}^{n} x_{ij} = s_i, \qquad i = 1,2,...,m,\\
&\sum_{i=1}^{m} x_{ij} = d_j, \qquad j = 1,2,...,n,\\
&\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.\\
&(S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, \quad i = 1,2,...,m,\\
&(D_j)_\alpha^L \le d_j \le (D_j)_\alpha^U, \quad j = 1,2,...,n,\\
&x_{ij} \ge 0, \quad \forall\ i,j.
\end{aligned}
\qquad (P_{4.15})
$$

$$
\begin{aligned}
Z_\alpha^U = max \quad & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j\\
subject\ to \quad &\\
& u_i + v_j \le (C_{ij})_\alpha^U, \quad i = 1,2,...,m,\ \ j = 1,2,...,n,\\
& \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j\\
& (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, \quad i = 1,2,...,m,\\
& (D_j)_\alpha^L \le d_j \le (D_j)_\alpha^U, \ \ j = 1,2,...,n,\\
& u_i, v_j \text{ unrestricted in sign}, \quad \forall\ \ i,j
\end{aligned}
\qquad (P_{4.16})
$$

The lower and upper bounds of the total transportation cost at α level can be obtained by solving Model ($P_{4.15}$) and ($P_{4.16}$). The α-level sets $[Z_\alpha^L, Z_\alpha^U]$ of $\widetilde{Z}$ at different possibility levels constitute the membership function $\mu_{\widetilde{Z}}$.

4.3 Numerical example

4.3.1 Numerical example with inequality constraints

In this section, solution of FFTP with inequality constraint is solved by using the method discussed in Section 4.2.1.

Example 4.1 Consider a transportation problem with one fuzzy transporting cost, two fuzzy supplies, and three fuzzy demands. Supply 1 and demand 3 are triangular fuzzy numbers and the remainders are trapezoidal fuzzy numbers. The problem has the following form:

$$\widetilde{Z} = min \quad 10x_{11}{+}50x_{12}{+}80x_{13}{+}(60{,}70{,}80{,}90)x_{21}{+}60x_{22}{+}20x_{23}$$

$$subject\ to$$

$$x_{11}{+}x_{12}{+}x_{13}{\leq}\ (70, 90, 100)$$

$$x_{21}{+}x_{22}{+}x_{23}{\leq}\ (40, 60, 70, 80)$$

$$x_{11}{+}x_{21}{\geq}\ (30, 40, 50, 70)$$

$$x_{12}{+}x_{22}{\geq}\ (20, 30, 40, 50)$$

$$x_{13}{+}x_{23}{\geq}\ (40, 50, 80)$$

$$x_{11},\ x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{23}\ \geq 0$$

The total supply is $\widetilde{S}_1 + \widetilde{S}_2 = (110, 150, 160, 180)$ and the total demand is $\widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3 = (90, 120, 140, 200)$ Since $\widetilde{S} \cap \widetilde{D} \neq \phi$, in other words, the upper bound of total fuzzy supply is greater than the lower bound of total fuzzy demand. So the problem has feasible solution. According to Models $(P_{4.9})$ and $(P_{4.12})$, the lower and upper bounds of $\widetilde{Z}$ at possibility level α can be solved as:

$$Z_\alpha^L = min \quad 10x_{11}{+}50x_{12}{+}80x_{13}{+}(60{+}10\alpha)x_{21}{+}60x_{22}{+}20x_{23}$$

$$subject\ to$$

$$x_{11}{+}x_{12}{+}x_{13}{\leq}\ s_1$$

$$x_{21}{+}x_{22}{+}x_{23}{\leq}\ s_2$$

$$x_{11}{+}x_{21}{\geq}\ d_1$$

$$x_{12}{+}x_{22}{\geq}\ d_2$$

$$x_{13}{+}x_{23}{\geq}\ d_3$$

$$s_1 + s_2 \geq d_1 + d_2 + d_3$$

$$70 + 20\alpha \leq s_1 \leq 100 - 10\alpha, \quad 40 + 20\alpha \leq s_2 \leq 80 - 10\alpha,$$

$$30 + 10\alpha \leq d_1 \leq 70 - 20\alpha, \quad 20 + 10\alpha \leq d_2 \leq 50 - 10\alpha,$$

$$40 + 10\alpha \le d_3 \le 80 - 30\alpha,$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \ge 0 \qquad (P_{4.17})$$

$$Z_\alpha^U = max \;\; -s_1u_1 - s_2u_2 + d_1v_1 + d_2v_2 + d_3v_3$$

subject to

$$-u_1 + v_1 \le 10$$

$$-u_1 + v_2 \le 50$$

$$-u_1 + v_3 \le 80$$

$$-u_2 + v_1 \le (90 - 10\alpha)$$

$$-u_2 + v_2 \le 60$$

$$-u_2 + v_3 \le 20$$

$$s_1 + s_2 \ge d_1 + d_2 + d_3$$

$$70 + 20\alpha \le s_1 \le 100 - 10\alpha, \quad 40 + 20\alpha \le s_2 \le 80 - 10\alpha,$$

$$30 + 10\alpha \le d_1 \le 70 - 20\alpha, \quad 20 + 10\alpha \le d_2 \le 50 - 10\alpha,$$

$$40 + 10\alpha \le d_3 \le 80 - 30\alpha,$$

$$u_1, u_2, v_1, v_2, v_3 \ge 0 \qquad (P_{4.18})$$

Table 4.1 Total transportation cost at different values of α

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z_α^L	2100	2180	2260	2340	2420	2500	2580	2660	2740	2820	2900
Z_α^U	5800	5600	5400	5200	5000	4800	4440	4080	3860	3680	3500

A mathematical programming solver Lingo [20] is used to solve Model ($P_{4.17}$). Table 4.1 lists the α-cuts of the total transportation cost at different α values: $0, 0.1, 0.2, ..., 1.0$. The $\alpha = 0$ cut shows the total transportation cost lies between 2100 and 5800. The $\alpha = 1.0$ cut shows the total transportation cost lies between 2900 and 3500

For the $\alpha = 0$ cut of $\widetilde{Z}$, the lower bound of $Z^* = 2100$ occurs at $x_{11}^* = 30$, $x_{12}^* = 20$, $x_{13}^* = 0$, $x_{21}^* = 0$, $x_{22}^* = 0$, $x_{23}^* = 40$ with $s_1 = 70$, $s_2 = 40$, $d_1 = 30$, $d_2 = 20$, $d_3 = 40$.

For the $\alpha = 1.0$ cut of $\widetilde{Z}$, the lower bound of $Z^* = 2900$ occurs at $x_{11}^* = 40$, $x_{12}^* = 30$, $x_{13}^* = 0$, $x_{21}^* = 0$, $x_{22}^* = 0$, $x_{23}^* = 50$ with $s_1 = 90$, $s_2 = 60$, $d_1 = 40$, $d_2 = 30$, $d_3 = 50$.

For the $\alpha = 0$ the upper bound of $Z^* = 5800$ occurs at $x_{11}^* = 30$, $x_{12}^* = 30$, $x_{13}^* = 40$, $x_{21}^* = 0$, $x_{22}^* = 0$, $x_{23}^* = 40$ with $s_1 = 100$, $s_2 = 40$, $d_1 = 30$, $d_2 = 30$, $d_3 = 80$.

For the $\alpha = 1.0$ the upper bound of $Z^* = 3500$ occurs at $x_{11}^* = 50$, $x_{12}^* = 40$, $x_{13}^* = 0$, $x_{21}^* = 0$, $x_{22}^* = 0$, $x_{23}^* = 50$ with $s_1 = 90$, $s_2 = 50$, $d_1 = 50$, $d_2 = 40$, $d_3 = 50$.

4.3.2 Numerical example with equality constraints

In this section, solution of FFTP with equality constraint is solved by using the methods discussed in Section 4.2.2.

Example 4.2 Consider Example 4.1 Suppose the inequality constraints are replaced by equality constraints:

$$\widetilde{Z} = min \quad 10x_{11}+50x_{12}+80x_{13}+(60,70,80,90)x_{21}+60x_{22}+20x_{23}$$

$$subject\ to$$

$$x_{11}+x_{12}+x_{13}= (70, 90, 100)$$

$$x_{21}+x_{22}+x_{23}= (40, 60, 70, 80)$$

$$x_{11}+x_{21}= (30, 40, 50, 70)$$

$$x_{12}+x_{22}= (20, 30, 40, 50)$$

$$x_{13}+x_{23}= (40, 50, 80)$$

$$x_{11},\ x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{23} \geq 0$$

Based on Model ($P_{4.15}$) and ($P_{4.16}$), the lower and upper bounds of the α-cut of $\widetilde{Z}$ can be derived by solving the following pair of mathematical programs:

$$Z_\alpha^L = min \quad 10x_{11}+50x_{12}+80x_{13}+(60+10\alpha)x_{21}+60x_{22}+20x_{23}$$

subject to

$$x_{11}+x_{12}+x_{13}= s_1$$

$$x_{21}+x_{22}+x_{23}= s_2$$

$$x_{11}+x_{21}= d_1$$

$$x_{12}+x_{22}= d_2$$

$$x_{13}+x_{23}= d_3$$

$$s_1 + s_2 = d_1 + d_2 + d_3$$

$$70 + 20\alpha \leq s_1 \leq 100 - 10\alpha, \quad 40 + 20\alpha \leq s_2 \leq 80 - 10\alpha,$$

$$30 + 10\alpha \leq d_1 \leq 70 - 20\alpha, \quad 20 + 10\alpha \leq d_2 \leq 50 - 10\alpha,$$

$$40 + 10\alpha \leq d_3 \leq 80 - 30\alpha,$$

$$x_{11},\, x_{12},\, x_{13},\, x_{21},\, x_{22},\, x_{23} \geq 0$$

$$Z_\alpha^U = max \;\; s_1u_1 + s_2u_2 + d_1v_1 + d_2v_2 + d_3v_3$$

subject to

$$u_1 + v_1 \leq 10$$

$$u_1 + v_2 \leq 50$$

$$u_1 + v_3 \leq 80$$

$$u_2 + v_1 \leq (90 - 10\alpha)$$

$$u_2 + v_2 \leq 60$$

$$u_2 + v_3 \leq 20$$

$$s_1 + s_2 = d_1 + d_2 + d_3$$

$$70 + 20\alpha \leq s_1 \leq 100 - 10\alpha, \quad 40 + 20\alpha \leq s_2 \leq 80 - 10\alpha,$$

$$30+10\alpha \le d_1 \le 70-20\alpha, \quad 20+10\alpha \le d_2 \le 50-10\alpha,$$

$$40+10\alpha \le d_3 \le 80-30\alpha,$$

u_1, u_2, v_1, v_2, v_3 unrestricted in sign.

For $\alpha = 0$, the lower bound of the objective value is 2300, occurring at $x^*_{11} = 50$, $x^*_{12} = 20$, $x^*_{13} = 0$, $x^*_{21} = 0$, $x^*_{22} = 0$, $x^*_{23} = 40$ with $s_1 = 70$, $s_2 = 40$, $d_1 = 50$, $d_2 = 20$, $d_3 = 40$.

For the $\alpha = 0.9$, the α-cut is a single point 3680. The associated optimal solution is $x^*_{11} = 52$, $x^*_{12} = 36$, $x^*_{13} = 0$, $x^*_{21} = 0$, $x^*_{22} = 5$, $x^*_{23} = 53$ with $s_1 = 88$, $s_2 = 58$, $d_1 = 52$, $d_2 = 41$, $d_3 = 53$.

For the $\alpha = 1$, the lower bound is infeasible.

For the $\alpha = 0$, the upper bound of $Z^* = 5800$ occurs at $x^*_{11} = 30$, $x^*_{12} = 30$, $x^*_{13} = 40$, $x^*_{21} = 0$, $x^*_{22} = 0$, $x^*_{23} = 40$ with $s_1 = 100$, $s_2 = 40$, $d_1 = 30$, $d_2 = 30$, $d_3 = 80$.

For the $\alpha = 1$, the upper bound is infeasible.

Table 4.2 Total transportation cost at different values of α

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z^L_α	2300	2400	2500	2600	2700	2800	2900	3040	3260	3680	Infeasible
Z^U_α	5800	5600	5400	5200	5000	4800	4440	4080	3860	3680	Infeasible

Conclusion

In this chapter, an existing method for solving an exact solution of FFTP based on extension principle, is presented. Two different types of the fuzzy transportation problems are discussed: one with inequality constraints and the other with equality constraints. To illustrate the existing method, numerical examples are solved and obtained results are discussed.

Chapter5

Solution of fuzzy solid transportation problem based on extension principle

In this chapter, a method, based on extension principle, is presented to solve such fuzzy solid transportation problem in which all the parameters are represented by trapezoidal fuzzy numbers. To illustrate the presented method a numerical example is solved and obtained results are discussed.

5.1 Solid transportation problem

$$
\begin{aligned}
Z = min \ & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\
& subject\ to \\
& \quad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq s_i, \qquad i = 1, 2, ..., m, \\
& \quad \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \geq d_j, \qquad j = 1, 2, ..., n, \\
& \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1, 2, ..., l, \\
& \quad x_{ijk} \geq 0, \quad \forall \ \ i, j, k.
\end{aligned}
\qquad (P_{5.1})
$$

where, m : total number of sources ; n : total number of destinations

l : total number of modes of transportation

s_i : the availability of the product at i^{th} source

d_j : the demand of the product at j^{th} destination

e_k : the conveyance capacity of the product at k^{th} mode of transportation

c_{ijk} : the transportation cost for unit quantity of the product from i^{th} source to j^{th} destination through k^{th} conveyance

x_{ijk} : the quantity of the product that should be transported from

i^{th} source to j^{th} destination through k^{th} conveyance

$\sum\limits_{i=1}^{m} s_i$: total availability of the product

$\sum\limits_{j=1}^{n} d_j$: total demand of the product

$\sum\limits_{k=1}^{l} e_k$: total conveyance capacity of the product

$\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{l} c_{ijk}\ x_{ijk}$: total solid transportation cost

Intuitively, if any of the parameters c_{ijk}, s_i, d_j or e_k is fuzzy, the total transportation cost becomes fuzzy as well. Model $(P_{5.1})$ the crisp solid transportation problem turns into the fuzzy solid transportation problem.

Suppose the unit transporting cost c_{ijk}, availability s_i, demand d_j and conveyance capacity e_k are approximately known. They can be represented by the convex fuzzy numbers $\widetilde{C}_{ijk}$, $\widetilde{S}_i$, $\widetilde{D}_j$ and $\widetilde{E}_k$, respectively, with membership functions $\mu_{\widetilde{C}_{ijk}}$, $\mu_{\widetilde{S}_i}$, $\mu_{\widetilde{D}_j}$ and $\mu_{\widetilde{E}_k}$:

$$\begin{aligned}
\widetilde{C}_{ijk} &= \{(c_{ijk},\mu_{\widetilde{C}_{ijk}}(c_{ijk}))|c_{ijk}\in \mathrm{S}(\widetilde{C}_{ijk})\},\\
\tilde{S}_i &= \{(s_i,\mu_{\widetilde{S}_i}(s_i))|s_i\in \mathrm{S}(\widetilde{S}_i)\}, \qquad (5.1)\\
\tilde{D}_j &= \{(d_j,\mu_{\widetilde{D}_j}(d_j))|d_j\in \mathrm{S}(\widetilde{D}_j)\},\\
\tilde{E}_k &= \{(e_k,\mu_{\widetilde{E}_k}(e_k))|e_k\in \mathrm{S}(\widetilde{E}_k)\},
\end{aligned}$$

where, $\mathrm{S}(\widetilde{C}_{ijk})$, $\mathrm{S}(\widetilde{S}_i)$, $\mathrm{S}(\widetilde{D}_j)$ and $\mathrm{S}(\widetilde{E}_k)$ are the supports of $\widetilde{C}_{ij}$, $\widetilde{S}_i$, $\widetilde{D}_j$ and $\widetilde{E}_k$, which denote the universe sets of the unit transporting cost, the availability of the product at i^{th} source and the demand of the product at j^{th} destination and the conveyance capacity of the product at k^{th} mode of transportation respectively.

5.2 Fuzzy solid transportation problem

In this section, the fuzzy solid transportation problem is of the following mathematical form:

$$\begin{aligned}
&\widetilde{Z} = min \ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\widetilde{C}_{ijk}x_{ijk}\\
&\qquad subject\ to\\
&\qquad\quad \sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} \leq \widetilde{S}_i, \qquad i = 1,2,...,m,\\
&\qquad\quad \sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} \geq \widetilde{D}_j, \qquad j = 1,2,...,n,\\
&\qquad\quad \sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} \leq \widetilde{E}_k, \qquad k = 1,2,...,l,\\
&\qquad\quad x_{ijk} \geq 0, \qquad \forall\ i,j,k. \qquad\qquad (P_{5.2})
\end{aligned}$$

where, m : total number of sources ; n : total number of destinations

l : total number of modes of transportation ;

$\widetilde{S}_i$: the fuzzy availability of the product at i^{th} source

$\widetilde{D}_j$: the fuzzy demand of the product at j^{th} destination

$\widetilde{E}_k$: the fuzzy conveyance capacity of the product at k^{th} mode of transportation

$\widetilde{C}_{ijk}$: the fuzzy transportation cost for unit quantity of the product from i^{th} source to j^{th} destination through k^{th} conveyance

x_{ijk} : the quantity of the product that should be transported from i^{th} source to j^{th} destination through k^{th} conveyance

$\sum_{i=1}^{m}\widetilde{S}_i$: total fuzzy availability of the product

$\sum_{j=1}^{n}\widetilde{D}_j$: total fuzzy demand of the product

$\sum_{k=1}^{l}\widetilde{E}_k$: total fuzzy conveyance capacity of the product

$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\widetilde{C}_{ijk}x_{ijk}$: total fuzzy solid transportation cost

5.3 Solution of fuzzy solid transportation problem

The α-cuts of $\widetilde{C}_{ijk}$, $\widetilde{S}_i$, $\widetilde{D}_j$ and $\widetilde{E}_k$ as

$$(\widetilde{C}_{ijk})_\alpha = [(C_{ijk})^L_\alpha, (C_{ijk})^U_\alpha] \tag{5.2}$$

where, $(C_{ijk})^L_\alpha = min\{c_{ijk} \in S(\widetilde{C}_{ijk}) | \mu_{\widetilde{C}_{ijk}}(c_{ijk}) \geq \alpha\}$

$$(C_{ijk})^U_\alpha = max\{c_{ijk} \in S(\widetilde{C}_{ijk}) | \mu_{\widetilde{C}_{ijk}}(c_{ijk}) \geq \alpha\}$$

$$(\widetilde{S}_i)_\alpha = [(S_i)^L_\alpha, (S_i)^U_\alpha] \tag{5.3}$$

where, $(S_i)^L_\alpha = min\{s_i \in S(\widetilde{S}_i) | \mu_{\widetilde{S}_i}(s_i) \geq \alpha\}$

$$(S_i)^U_\alpha = max\{s_i \in S(\widetilde{S}_i) | \mu_{\widetilde{S}_i}(s_i) \geq \alpha\}$$

$$(\widetilde{D}_j)_\alpha = [(D_j)^L_\alpha, (D_j)^U_\alpha] \tag{5.4}$$

where, $(D_j)^L_\alpha = min\{d_j \in S(\widetilde{D}_j) | \mu_{\widetilde{D}_j}(d_j) \geq \alpha\}$

$$(D_j)^U_\alpha = max\{d_j \in S(\widetilde{D}_j) | \mu_{\widetilde{D}_j}(d_j) \geq \alpha\}$$

$$(\widetilde{E}_k)_\alpha = [(E_k)^L_\alpha, (E_k)^U_\alpha] \tag{5.5}$$

where, $(E_k)^L_\alpha = min\{e_k \in S(\widetilde{E}_k) | \mu_{\widetilde{E}_k}(e_k) \geq \alpha\}$

$$(E_k)^U_\alpha = max\{e_k \in S(\widetilde{E}_k) | \mu_{\widetilde{E}_k}(e_k) \geq \alpha\}$$

These intervals indicate where the unit transporting cost, availability, demand and conveyance lie at possibility level α. In deriving the membership function of the total transportation cost $\widetilde{Z}$. Since $\widetilde{Z}$ is a fuzzy number, instead of a crisp number, it cannot be minimized directly. To tackle this problem, one can transform the fuzzy solid transportation problems, which is based on Zadeh's extension principle [28, 29, 30], to a family of mathematical programs to be solved.

Based on the extension principle, the membership function $\mu_{\widetilde{Z}}$ can be defined as:

$$\mu_{\widetilde{Z}}(z) = \sup_{c,s,d,e} min\{\mu_{\widetilde{C}_{ijk}}(c_{ijk}), \mu_{\widetilde{S}_i}(s_i), \mu_{\widetilde{D}_j}(d_j), \mu_{\widetilde{E}_k}(e_k) \forall\ i,j,k | z = Z(c,s,d,e)\} \tag{5.6}$$

where $Z(c,s,d,e)$, is defined in Model $(P_{5.1})$.

where as Eq. (5.6), several membership functions are involved. To derive $\mu_{\widetilde{Z}}$ in closed form is hardly possible. According to (5.6), $\mu_{\widetilde{Z}}$ is the minimum of $\mu_{\widetilde{C}_{ijk}}$, $\mu_{\widetilde{S}_i}$, $\mu_{\widetilde{D}_j}$ and $\mu_{\widetilde{E}_k}$, $\forall\ \ i,j,k$. So $\mu_{\widetilde{C}_{ijk}}(c_{ijk}) \geq \alpha$, $\mu_{\widetilde{S}_i}(s_i) \geq \alpha$, $\mu_{\widetilde{D}_j}(d_j) \geq \alpha$, $\mu_{\widetilde{E}_k}(e_k) \geq \alpha$ and at least one $\mu_{\widetilde{C}_{ijk}}(c_{ijk})$, $\mu_{\widetilde{S}_i}(s_i)$, $\mu_{\widetilde{D}_j}(d_j)$ or $\mu_{\widetilde{E}_k}(e_k)$, $\forall\ \ i,j,k$ equal to α such that $z = Z(c,s,d,e)$ to satisfy $\mu_{\widetilde{Z}}(z) = \alpha$. To find the membership function $\mu_{\widetilde{Z}}$, it suffices to find the left shape function and right shape function of $\mu_{\widetilde{Z}}$, which is equivalent to finding the lower bound Z_α^L and upper bound Z_α^U of the α-cuts of $\widetilde{Z}$. Since Z_α^L is the minimum of $Z(c,s,d,e)$ and Z_α^U is the maximum of $Z(c,s,d,e)$, they can be expressed as:

$Z_\alpha^L = min\{Z(c,s,d,e) | (C_{ijk})_\alpha^L \leq c_{ijk} \leq (C_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U \ \ \forall\ \ i,j,k\}$

$Z_\alpha^U = max\{Z(c,s,d,e) | (C_{ijk})_\alpha^L \leq c_{ijk} \leq (C_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U \ \ \forall\ \ i,j,k\}$

which can be reformulated as the following pair of two-level mathematical programs:

$$Z_\alpha^L = min(p) \left\{ \begin{array}{l} min \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ subject\ to \\ \qquad \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq s_i, \qquad i = 1,2,...,m, \\ \qquad \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \geq d_j, \qquad j = 1,2,...,n, \\ \qquad \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1,2,...,l, \\ \qquad x_{ijk} \geq 0, \quad \forall\ \ i,j,k. \end{array} \right. \qquad (P_{5.3})$$

where, $p = (C_{ijk})_\alpha^L \leq c_{ijk} \leq (C_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U, \forall\ \ i,j,k$

$$Z_\alpha^U = max(p)\begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ subject\ to \\ \qquad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \qquad i=1,2,...,m, \\ \qquad \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge d_j, \qquad j=1,2,...,n, \\ \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \qquad k=1,2,...,l, \\ \qquad x_{ijk} \ge 0, \quad \forall\ \ i,j,k. \end{cases} \qquad (P_{5.4})$$

where, $p = (C_{ijk})_\alpha^L \le c_{ijk} \le (C_{ijk})_\alpha^U, (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, (D_j)_\alpha^L \le d_j \le (D_j)_\alpha^U, (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \forall\ \ i,j,k$

A necessary and sufficient condition for Model ($P_{5.3}$) and ($P_{5.4}$) to have feasible solutions is $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} d_j$. In the first level of Model ($P_{5.3}$) and ($P_{5.4}$), s_i, d_j and e_k are allowed to vary in the range of $[(S_i)_\alpha^L, (S_i)_\alpha^U]$, $[(D_j)_\alpha^L, (D_j)_\alpha^U]$ and $[(E_k)_\alpha^L, (E_k)_\alpha^U]$, respectively. However, to be feasible, it is necessary to impose the constraints $\sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} d_j$ at first level. Hence, Model ($P_{5.3}$) and ($P_{5.4}$) becomes:

$$Z_\alpha^L = min(p^*)\begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ subject\ to \\ \qquad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \qquad i=1,2,...,m, \\ \qquad \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge d_j, \qquad j=1,2,...,n, \\ \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \qquad k=1,2,...,l, \\ \qquad x_{ijk} \ge 0, \quad \forall\ \ i,j,k. \end{cases} \qquad (P_{5.5})$$

$$Z_\alpha^U = max(p^*)\left\{\begin{array}{l} min\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ subject\ to \\ \qquad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq s_i, \qquad i = 1,2,...,m, \\ \qquad \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \geq d_j, \qquad j = 1,2,...,n, \\ \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1,2,...,l, \\ \qquad x_{ijk} \geq 0, \quad \forall\ \ i,j,k. \end{array}\right. \qquad (P_{5.6})$$

where, $p^* = (C_{ijk})_\alpha^L \leq c_{ijk} \leq (C_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j, \forall\ \ i,j,k$

To find the minimum objective value set c_{ijk} to its lower bound $(C_{ijk})_\alpha^L\ \ \forall\ \ i,j,k$ in Model $(P_{5.5})$. Hence, Model $(P_{5.5})$ can be reformulated as:

$$Z_\alpha^L = min(p^{**})\left\{\begin{array}{l} min\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} (C_{ijk})_\alpha^L x_{ijk} \\ subject\ to \\ \qquad \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq s_i, \qquad i = 1,2,...,m, \\ \qquad \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \geq d_j, \qquad j = 1,2,...,n, \\ \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \qquad k = 1,2,...,l, \\ \qquad x_{ijk} \geq 0, \quad \forall\ \ i,j,k. \end{array}\right. \qquad (P_{5.7})$$

where, $p^{**} = (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j, \forall\ \ i,j,k$

Since Model $(P_{5.7})$ is to find the minimum of all the minimum objective values, one can insert the constraints of level 1 into level 2 and simplify the two-level mathematical program to the one-level program as follows:

$$
\begin{aligned}
Z_\alpha^L = min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(C_{ijk})_\alpha^L x_{ijk} \\
\quad subject\ to \quad & \\
& \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le s_i, \qquad i = 1,2,...,m, \\
& \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge d_j, \qquad j = 1,2,...,n, \\
& \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k, \qquad k = 1,2,...,l, \\
& \sum_{i=1}^{m} s_i \ge \sum_{j=1}^{n} d_j \\
& \sum_{k=1}^{l} e_k \ge \sum_{j=1}^{n} d_j \\
& (S_i)_\alpha^L \le s_i \le (S_i)_\alpha^U, \qquad i = 1,2,...,m \\
& (D_j)_\alpha^L \le d_j \le (D_j)_\alpha^U, \quad j = 1,2,...,n \\
& (E_k)_\alpha^L \le e_k \le (E_k)_\alpha^U, \quad k = 1,2,...,l \qquad (P_{5.8}) \\
& x_{ijk} \ge 0, \quad \forall \ \ i,j,k.
\end{aligned}
$$

This is a linear program which can be solved easily. In this model, since all c_{ijk} have been set to the lower bounds of their α-cuts, that is, $\mu_{\widetilde{C}_{ijk}}(c_{ijk}) = \alpha$, this assures $\mu_{\widetilde{Z}}(z) = \alpha$ as required by (5.6).

To solve Model ($P_{5.6}$), the dual of the level 2 problem is formulated to become a maximization problem to be consistent with the maximization operation of level 1. It is well known from the duality theorem of linear programming that the primal model and the dual model have the same objective value. Thus Model ($P_{5.6}$) becomes:

$$Z_\alpha^U = max(p^*)\begin{cases} max \ -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{l} e_k w_k \\ subject\ to \\ \qquad -u_i + v_j - w_k \leq c_{ijk}, \\ \qquad i = 1, 2, ..., m,\ j = 1, 2, ..., n,\ k = 1, 2, ..., l, \\ \qquad u_i, v_j, w_k \geq 0, \quad \forall \ \ i, j, k. \end{cases} \qquad (P_{5.9})$$

where, $p^* = (C_{ijk})_\alpha^L \leq c_{ijk} \leq (C_{ijk})_\alpha^U, (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j, \forall \ \ i, j, k$

Since $(C_{ijk})_\alpha^L \leq c_{ijk} \leq (C_{ijk})_\alpha^U, \forall \ \ i, j, k$ in Model $(P_{5.9})$, one can derive the upper bound of the objective value by setting c_{ijk} to its upper bound $(C_{ijk})_\alpha^U \ \forall \ \ i, j, k$, because this gives the largest feasible region. Thus, we can reformulate Model $(P_{5.9})$ as:

$$Z_\alpha^U = max(p^{**})\begin{cases} max \ -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{l} e_k w_k, \\ subject\ to \\ \qquad -u_i + v_j - w_k \leq (C_{ijk})_\alpha^U, \\ \qquad i = 1, 2, ..., m,\ j = 1, 2, ..., n,\ k = 1, 2, ..., l, \\ \qquad u_i, v_j, w_k \geq 0, \quad \forall \ \ i, j, k. \end{cases} \qquad (P_{5.10})$$

where, $p^{**} = (S_i)_\alpha^L \leq s_i \leq (S_i)_\alpha^U, (D_j)_\alpha^L \leq d_j \leq (D_j)_\alpha^U, (E_k)_\alpha^L \leq e_k \leq (E_k)_\alpha^U, \sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j, \sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j, \forall \ \ i, j, k$

Now, since both level 1 and level 2 perform the same maximization operation, their constraints can be combined to form the following one-level mathematical program:

$$Z_\alpha^U = max \qquad -\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j - \sum_{k=1}^{l} e_k w_k$$

subject to

$$-u_i + v_j - w_k \leq (C_{ijk})^U_\alpha,$$

$$i = 1, 2, ..., m,\ j = 1, 2, ..., n,\ k = 1, 2, ..., l,$$

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$$

$$\sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j$$

$$(S_i)^L_\alpha \leq s_i \leq (S_i)^U_\alpha,\ \ i = 1, 2, ..., m, \tag{5.6}$$

$$(D_j)^L_\alpha \leq d_j \leq (D_j)^U_\alpha,\ j = 1, 2, ..., n, \tag{5.7}$$

$$(E_k)^L_\alpha \leq e_k \leq (E_k)^U_\alpha,\ \ \ k = 1, 2, ..., l \tag{5.8}$$

$$x_{ijk} \geq 0\ \forall\ i, j, k \tag{$P_{5.11}$}$$

This model is a linearly constrained nonlinear program. There are several effective and efficient methods for solving this problem [1, 24]. Similar to Model ($P_{5.8}$), since all c_{ijk} have been set to the upper bounds of their α-cuts, that is, $\mu_{\widetilde{C}_{ijk}}(c_{ijk}) = \alpha$, this assures $\mu_{\widetilde{Z}}(z) = \alpha$ as required by (5.6).

If the total availability and total conveyance capacity are greater than the total demand at all α values, respectively, i.e., $\sum_{i=1}^{m}(S_i)^L_{\alpha=0} \geq \sum_{j=1}^{n}(D_j)^U_{\alpha=0}$ and $\sum_{k=1}^{l}(E_k)^L_{\alpha=0} \geq \sum_{j=1}^{n}(D_j)^U_{\alpha=0}$ then the constraints $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$ and $\sum_{k=1}^{l} e_k \geq \sum_{j=1}^{n} d_j$ can be deleted from Model ($P_{5.11}$). Multiplying constraints (5.6) to (5.8) by u_i, v_j and w_k, respectively, and substituting $s_i u_i$ by p_i, $d_j v_j$ by q_j and $e_k w_k$ by r_k. Model ($P_{5.11}$) is transformed into the following linear program:

$$Z^U_\alpha = max \quad -\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j - \sum_{k=1}^{l} r_k$$

subject to

$$-u_i + v_j - w_k \leq (C_{ijk})^U_\alpha,$$

$$i = 1, 2, ..., m,\ j = 1, 2, ..., n,\ k = 1, 2, ..., l,$$

$$(S_i)^L_\alpha \le p_i \le (S_i)^U_\alpha, \quad i = 1, 2, ..., m,$$

$$(D_j)^L_\alpha \le q_j \le (D_j)^U_\alpha, \;\; j = 1, 2, ..., n,$$

$$(E_k)^L_\alpha \le r_k \le (E_k)^U_\alpha, \quad k = 1, 2, ..., l \qquad (P_{5.12})$$

$$p_i, q_j, r_k \ge 0 \;\forall\;\; i, j, k$$

In this case, the upper bound of the total transportation cost Z^U_α at α level can be found more easily.

Model $(P_{5.3})$ and $(P_{5.4})$ are assured to be feasible if the lower bound of the total fuzzy demand is smaller than both of the upper bound of the total fuzzy availability and the upper bound of the total conveyance capacity, i.e., $\sum_{j=1}^{n}(D_j)^L_{\alpha=0} \le \sum_{i=1}^{m}(S_i)^U_{\alpha=0}$ and $\sum_{j=1}^{n}(D_j)^L_{\alpha=0} \le \sum_{k=1}^{l}(E_k)^U_{\alpha=0}$

5.4 Numerical example

Example 5.1 Consider a solid transportation problem with two fuzzy availabilities, three fuzzy demands and two fuzzy conveyance capacities. The notation used in this example is (a, b, c, d) for a trapezoidal fuzzy number with a, b, c and d as the coordinates of the four vertices of the trapezoid and (x, y, z) for a triangular fuzzy number with x, y and z as the coordinates of the three vertices of the triangle. The problem has the following mathematical form:

$$\widetilde{Z} = min \quad (20, 30, 40)x_{111}+70x_{112}+60x_{121}+60x_{122}+50x_{131}+30x_{132}$$

$$+(10, 20, 30)x_{211}+40x_{212}+30x_{221}+50x_{222}+40x_{231}+50x_{232}$$

subject to

$$x_{111}+x_{112}+x_{121}+x_{122}+x_{131}+x_{132} \le (70, 80, 100, 120),$$

$$x_{211}+x_{212}+x_{221}+x_{222}+x_{231}+x_{232} \le (60, 70, 90),$$

$$x_{111}+x_{112}+x_{211}+x_{212} \ge (10, 30, 40, 50),$$

$$x_{121}+x_{122}+x_{221}+x_{222} \ge (40, 50, 60),$$

$$x_{131}+x_{132}+x_{231}+x_{232}\geq (30,40,60,70),$$

$$x_{111}+x_{121}+x_{131}+x_{211}+x_{221}+x_{231}\leq (70,80,100),$$

$$x_{112}+x_{122}+x_{132}+x_{212}+x_{222}+x_{232}\leq (60,70,90), \qquad (P_{5.13})$$

$$x_{ijk}\geq 0,\ \ i=1,2,\ \ j=1,2,3,\ \ k=1,2.$$

The total availability $\widetilde{S}=\widetilde{S}_1+\widetilde{S}_2=(130,150,170,210)$, the total demand $\widetilde{D}=\widetilde{D}_1+\widetilde{D}_2+\widetilde{D}_3=(80,120,150,180)$ and the total conveyance capacity $\widetilde{E}=\widetilde{E}_1+\widetilde{E}_2=(130,150,190)$. Since $\widetilde{S}\cap\widetilde{D}\cap\widetilde{E}\neq\phi$, in other words, the upper bound of total fuzzy availability and total fuzzy conveyance capacity is greater than the lower bound of total fuzzy demand. So, it has feasible solution. According to Model $(P_{5.8})$ and $(P_{5.11})$, the lower and upper bounds of $\widetilde{Z}$ at possibility level α can be solved as:

$$Z_\alpha^L=min \quad 20x_{111}+70x_{112}+60x_{121}+60x_{122}+50x_{131}+30x_{132}$$

$$+10x_{211}+40x_{212}+30x_{221}+50x_{222}+40x_{231}+50x_{232}$$

subject to

$$x_{111}+x_{112}+x_{121}+x_{122}+x_{131}+x_{132}\leq s_1,$$

$$x_{211}+x_{212}+x_{221}+x_{222}+x_{231}+x_{232}\leq s_2,$$

$$x_{111}+x_{112}+x_{211}+x_{212}\geq d_1,$$

$$x_{121}+x_{122}+x_{221}+x_{222}\geq d_2,$$

$$x_{131}+x_{132}+x_{231}+x_{232}\geq d_3,$$

$$x_{111}+x_{121}+x_{131}+x_{211}+x_{221}+x_{231}\leq e_1,$$

$$x_{112}+x_{122}+x_{132}+x_{212}+x_{222}+x_{232}\leq e_2,$$

$$s_1+s_2\geq d_1+d_2+d_3, \qquad e_1+e_2\geq d_1+d_2+d_3,$$

$$(70+10\alpha)\leq s_1\leq(120-20\alpha), \quad (60+10\alpha)\leq s_2\leq(90-20\alpha),$$

$$(10+20\alpha)\leq d_1\leq(50-10\alpha), \quad (40+10\alpha)\leq d_2\leq(60-10\alpha),$$

$$(30+10\alpha)\leq d_3\leq(70-10\alpha), \quad (70+10\alpha)\leq e_1\leq(100-20\alpha),$$

$$(60+10\alpha) \leq e_2 \leq (90-20\alpha),$$

$$x_{ijk} \geq 0,\ i=1,2, j=1,2,3, k=1,2.$$

$$Z_\alpha^U = max \quad -s_1u_1 - s_2u_2 + d_1v_1 + d_2v_2 + d_3v_3 - e_1w_1 - e_2w_2$$

subject to

$$-u_1 + v_1 - w_1 \leq (40-10\alpha), \quad -u_1 + v_1 - w_2 \leq 70,$$

$$-u_1 + v_2 - w_1 \leq 60, \quad -u_1 + v_2 - w_2 \leq 20,$$

$$-u_1 + v_3 - w_1 \leq 50, \quad -u_1 + v_3 - w_2 \leq 30,$$

$$-u_2 + v_1 - w_1 \leq (30-10\alpha), \quad -u_2 + v_1 - w_2 \leq 40$$

$$-u_2 + v_2 - w_1 \leq 30, \quad -u_2 + v_2 - w_2 \leq 50,$$

$$-u_2 + v_3 - w_1 \leq 40, \quad -u_2 + v_3 - w_2 \leq 50,$$

$$s_1 + s_2 \geq d_1 + d_2 + d_3, \quad e_1 + e_2 \geq d_1 + d_2 + d_3,$$

$$(70+10\alpha) \leq s_1 \leq (120-20\alpha), \quad (60+10\alpha) \leq s_2 \leq (90-20\alpha),$$

$$(10+20\alpha) \leq d_1 \leq (50-10\alpha), \quad (40+10\alpha) \leq d_2 \leq (60-10\alpha),$$

$$(30+10\alpha) \leq d_3 \leq (70-10\alpha), \quad (70+10\alpha) \leq e_1 \leq (100-20\alpha),$$

$$(60+10\alpha) \leq e_2 \leq (90-20\alpha),$$

$$u_1,\ u_2,\ v_1,\ v_2,\ v_3, w_1,\ w_2 \geq 0.$$

Table 5.1 Total transportation cost at different values of α

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z_α^L	1800	1882	1968	2058	2152	2250	2392	2538	2688	2842	3000
Z_α^U	5700	5531	5364	5199	5036	4875	4716	4559	4404	4251	4100

A mathematical programming solver Lingo [20] is used to solve Model ($P_{5.17}$). Table 5.1 lists the α-cuts of the total transportation cost at different α values: $0, 0.1, 0.2, ..., 1.0$. The $\alpha = 0$ cut shows the total transportation cost lies between 1800 and 5700. The $\alpha = 1.0$ cut shows the total transportation cost lies between

3000 and 4100

For the $\alpha = 0$ cut of $\widetilde{Z}$, the lower bound of $Z^* = 1800$ occurs at $x^*_{122} = 40$, $x^*_{132} = 30$, $x^*_{211} = 10$ with $s_1 = 70$, $s_2 = 60$, $d_1 = 10$, $d_2 = 40$, $d_3 = 30$, $e_1 = 100$, $e_2 = 70$ and the other decision variables are 0.

For $\alpha = 1$, the lower bound of $Z^* = 3000$ occurs at $x^*_{122} = 50$, $x^*_{132} = 20$, $x^*_{211} = 30$, $x^*_{231} = 20$ with $s_1 = 80$, $s_2 = 70$, $d_1 = 30$, $d_2 = 50$, $d_3 = 40$, $e_1 = 80$, $e_2 = 70$ and the other decision variables are 0.

For $\alpha = 0$, the upper bound of $Z^* = 5700$ occurs at $x^*_{111} = 40$, $x^*_{122} = 10$, $x^*_{132} = 70$, $x^*_{211} = 10$, $x^*_{221} = 50$ with $s_1 = 70$, $s_2 = 60$, $d_1 = 10$, $d_2 = 40$, $d_3 = 30$, $e_1 = 100$, $e_2 = 70$ and the other decision variables are 0.

For $\alpha = 1$, the upper bound of $Z^* = 4100$ occurs at $x^*_{111} = 10$, $x^*_{132} = 60$, $x^*_{211} = 30$, $x^*_{221} = 40$ with $s_1 = 80$, $s_2 = 70$, $d_1 = 40$, $d_2 = 50$, $d_3 = 60$, $e_1 = 80$, $e_2 = 70$ and the other decision variable s are 0.

Notably, the values of the decision variables derived in this example are also fuzzy.

Conclusion

In this chapter, an existing method for finding an exact solution of fuzzy solid transportation problems, is presented. To illustrate the existing method, numerical example is solved and obtained results are discussed.

References

[1] Bazaraa, M.S., Sherali, H.D. and Shetty, C.M., Nonlinear Programming: Theory and Algorithms, 2nd ed., John Wiley & Sons, New York, 1993.

[2] Bit, A.K., Biswal, M.P. and Alam, S.S., *Fuzzy programming approach to multiobjective solid transportation problem*, Fuzzy Sets and Systems, Vol. 57, 1993, pp. 183–194.

[3] Chanas, S., *On the interval approximation of a fuzzy number*, Fuzzy Sets and Systems, Vol. 122, 2001, pp. 353–356.

[4] Chanas, S., Kolodziejczyk, W. and Machaj, A., *A fuzzy approach to the transportation problem* , Fuzzy Sets and Systems, Vol. 13, 1984, pp. 211–221.

[5] Chanas, S. and Kuchta, D., *A concept of the optimal solution of the transportation problem with fuzzy cost coefficients*, Fuzzy Sets and Systems, Vol. 82, 1996, pp. 299–305.

[6] Chanas, S. and Kuchta, D., *Fuzzy integer transportation problem*, Fuzzy Sets and Systems, Vol. 98, 1998, pp. 291–298.

[7] Charnes, A. and Cooper, W.W., *The stepping-stone method for explaining linear programming calculation in transportation problem*, Management Science, Vol. l, 1954, pp. 49–69.

[8] Dantzig, G.B., Application of the simplex method to a transportation problem, Chap 23, in Koopmans, Activity Analysis of production and allocation, Cowls Commission Monograph 13, John Wiley & Sons, New york, 1951.

[9] Gen, M., Ida, K., Li, Y. and Kubota, E., *Solving bicriteria solid transportation problem with fuzzy numbers by a genetic algorithm*, Computers and Industrial Engineering, Vol. 29, 1995, pp. 537–541

[10] Grzegorzewski, P., *Nearest interval approximation of a fuzzy number*, Fuzzy Sets and Systems, Vol. 130, 2002, pp. 321–330.

[11] Haley, K.B., *The multi-index problem*, Operations Research, Vol. 11, 1962, pp. 446–448.

[12] Hitchcock, *Distribution of a product from several sources to numerous localities*, Journal of Mathematical Physics, Vol. 12, 1978,pp. 224–230

[13] Jimenez, F. and Verdegay J.L., *Uncertain solid transportation problems*, Fuzzy Sets and Systems, Vol. 100, 1998, pp. 45–57.

[14] Jimenez, F. and Verdegay J.L., *Solving fuzzy solid transportation problems by an evolutionary algorithm based parametric approach*, European Journal of Operational Research, Vol. 117, 1999, pp. 485–510.

[15] Jimenez, F. and Verdegay J.L., *Interval multiobjective solid transportation problems via genetic algorithms*, Management of Uncertainity in Knowledge Based Systems, Vol. 2, 1996, pp. 787–792.

[16] Kaufmann, A. and Gupta, M.M., Introduction to Fuzzy Arithmetics: Theory and Applications, Van Nostrand Reinhold, 1991, New York.

[17] Kikuchi, S., *A method to defuzzify the number: transportation problem application*, Fuzzy Sets Systems, Vol. 116, 2000, pp. 3–9.

[18] Li, Y., Ida, K. and Gen, M., *Improved genetic algorithm for solving multiobjective solid transportation problem with fuzzy numbers*, Computers and Industrial Engineering, Vol. 33, 1997, pp. 589–592.

[19] Li, Y., Ida, K., Gen, M. and Kobuchi, R., *Neural network approach for multicriteria solid transportation problem*, Computers and Industrial Engineering, Vol. 33, 1997, pp. 465–468.

[20] Lingo, *User's Guide*, LINDO Systems Inc., Chicago, 1999.

[21] Liu, S.T. and Kao, C., *Solving fuzzy transportation problems based on extension principle*, European Journal of Operational Research, Vol. 153, 2004, pp. 661–674.

[22] Liu, S.T., *Fuzzy total transportation cost measures for solid transportation problem*, Applied Mathematics and Computation, Vol. 174, 2006, pp. 927–941

[23] Patel, G. and Tripathy, J., *The solid transportation problem and its variants*, International Journal of Management and Systems , Vol. 5, 1989, pp. 17–36.

[24] Reklatis, G.V., Ravindran, A. and Ragsdell, K.M., Engineering Optimization, John Wiley & Sons, New York, 1983.

[25] Saad, M.O. and Abass A.S., *A parametric study on transportation problem under fuzzy environment*, The Journal of Fuzzy Mathematics, Vol. 11, 2003, pp. 115–124.

[26] Shell, E., *Distribution of a product by several properties*, Directorate of Management Analysis, Proc. 2nd Symp. on Linear Programming, Vol. 2, 1955, pp. 615–642, DCS/Comptroller H.Q.U.S.A.F., Washington, DC.

[27] Vajda, S., Readings in Linear Programming, Pitman, London, 1988.

[28] Yager, R.R., *A characterization of the extension principle*, Fuzzy Sets and Systems, Vol. 18, 1986, pp. 205–217.

[29] Zadeh, L.A., *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems, Vol. 1, 1978, pp. 3–28.

[30] Zimmermann, H.J., Fuzzy Set Theory and Its Applications, 3rd ed., Kluwer-Nijhoff, Boston, 1996.

[31] Zimmermann, H.J., *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems, Vol. 1, 1978, pp. 45–55