ALGORITHMS FOR SOLVING SOME SINGLE AND MULTICRITERIA DECISION MAKING PROBLEMS IN FUZZY ENVIRONMENT

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CERTIFICATE

This is to certify that the thesis entitled, **"Algorithms for Solving Some Single and Multicriteria Decision Making Problems in Fuzzy Environment"**, submitted by Mrs. Anila Gupta in the fulfilment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidates's own work carried out by her under our supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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TO

THE ALMIGHTY

&

MY FAMILY

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Abstract

To the best of our knowledge, only the existing method [56] is proposed in the literature for solving time minimizing transportation problems in fuzzy environment and no method is proposed in the literature for solving bi-objective warehouse selection problems in fuzzy environment as well as fully fuzzy multi-objective transportation problems (multi-objective transportation problems in which each parameter as well as each decision variable is represented by a fuzzy number). In this thesis, to resolve the shortcomings of existing method [56], a new method is proposed for solving time minimizing transportation problems in fuzzy environment and new methods are proposed for solving bi-objective warehouse selection problems in fuzzy environment and fully fuzzy multi-objective transportation problems.

The chapter wise summary of the thesis is as follows:

Chapter 1

It is introductory in nature. It briefly reviews the earlier work done in the areas of warehouse selection problems, time minimizing transportation problems and multi-objective transportation problems in fuzzy environment.

Chapter 2

Dinagar and Palanivel [56] claimed that there is no method in the literature for solving time minimizing transportation problems in fuzzy environment and proposed

a method to solve fuzzy time minimizing transportation problems by modifying an existing method for solving time minimizing transportation problems in crisp environment. Keeping this in view, in this chapter, bi-objective warehouse selection problems in fuzzy environment have been considered. The two objectives are to minimize the total fuzzy cost and fuzzy time of meeting the requirements of all the ration shops from their assigned warehouses at the selected sites. An iterative algorithm using ranking function is developed to find the set of efficient solutions of these problems in fuzzy environment.

Chapter 3

In this chapter, the shortcomings of the existing method [56] are pointed out and a new method is proposed to resolve these shortcomings. Also, an alternative method is proposed for solving warehouse selection problems in fuzzy environment and it is shown that it is much easy to apply the proposed method in this chapter as compared to the method proposed in the previous chapter.

Chapter 4

In this chapter, shortcomings of existing formulation of time minimizing transportation problems in fuzzy environment are pointed out and to resolve these shortcomings, a new formulation is proposed. Also, with the help of proposed formulation and the existing method [99], a new method is proposed for solving time minimizing transportation problems in fuzzy environment.

Chapter 5

Gupta et al. [68] pointed out that there is no method in the literature for solving fully fuzzy multi-objective transportation problems and proposed a method for its solution. In this chapter, the limitations of this method are pointed out and to overcome these limitations, a new method is proposed by modifying the existing method. The advantages of the proposed method over the existing method are discussed. To illustrate the proposed method some numerical examples are solved.

Chapter 6

There are several methods in the literature for solving transportation problems where the parameters are fuzzy numbers. Chiang [44] pointed out that it is better to represent the parameters as level (λ, ρ) interval-valued triangular fuzzy numbers instead of fuzzy numbers and proposed a method to find the optimal solution of single objective transportation problems by representing the availability and demand as level (λ, ρ) interval-valued triangular fuzzy numbers. In this chapter, the limitations of this existing method are pointed out and to overcome these limitations a new method is proposed. To illustrate the proposed method numerical examples are solved. The advantages of the proposed method over the existing method are also discussed.

Chapter 7

This chapter concludes the work herein. Utility and possible future extensions of the work are indicated.

List of Research Papers

1. **A. Gupta**, A. Kumar and M.K. Sharma, Application of tabu search for solving the bi-objective warehouse problem in a fuzzy environment, Iranian Journal of Fuzzy Systems, 9 (2012) 1–19.

2. **A. Gupta** and A. Kumar, A new method for solving linear multi-objective transportation problems with fuzzy parameters, Applied Mathematical Modelling, 36 (2012) 1421–1430.

3. A. Kumar, **A. Gupta** and M.K. Sharma, Applications of fuzzy linear programming problem with generalized LR flat fuzzy parameters, Journal of Applied Mathematics and Informatics (Communicated).

4. **A. Gupta**, A. Kumar and M.K. Sharma, Bi-objective warehouse problem in a fuzzy environment: A new approach, Information Sciences (Communicated).

5. **A. Gupta**, A. Kumar and M.K. Sharma, Fully fuzzy multi-objective transportation problems with non-linear membership - A new insight, The Asian Journal of Mathematics (Communicated).

6. **A. Gupta**, A. Kumar and M.K. Sharma, Fully fuzzy multi-objective transportation problems with non-linear interval-valued LR membership, Optimization Letters (Communicated).

7. A. Kumar, **A. Gupta** and M.K. Sharma, Time minimizing transportation problems in fuzzy environment - A novel formulation, Journal of Intelligent and Fuzzy Systems (Communicated).

8. A. Kumar, **A. Gupta** and M.K. Sharma, Algorithm on bi-objective warehouse problem in fuzzy environment, Eighth Triennial Conference of Association of Asian Pacific Operational Research Societies (APORS), Jaipur, Dec. 6-9, 2009.

9. A. Kumar and **A. Gupta**, A new method for solving generalized assignment problem, International Congress on Productivity, Quality, Reliability, Optimization and Modelling, Delhi, Feb. 7-8, 2011.

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Chapter 1

Introduction

Decision making is the process of identifying and choosing alternatives based on the values and preferences of the decision maker. Decision making based solely on a single criterion appears insufficient as soon as the decision-making process deals with complex organizational environment. It is difficult, if not impossible, to summarize the complexity of opinions, the motivations and the goals found in organizations in a single objective. Thus, the need to satisfy several criteria which often are non-commensurable and are at least partially contradictory led to the development of multi-criteria decision making. Optimization is a kind of the decision making, or more specifically, as one of the major quantitative tools in the machinery of decision making, in which decisions have to be taken to optimize one or more objectives under some prescribed set of circumstances [145].

In multi-objective optimization problems, objectives are usually non commensurable and cannot be combined into single-objective. Moreover, the objectives often conflict with each other in that any improvement in one objective function can be achieved only at the expense of other. Accordingly, the aim is to find the compromise or satisficing solution of the decision maker which is also Pareto optimal based on his/her subjective value judgement [42, 147, 163]. However, when formulating the single and multi-objective programming problem which closely describes and represents the real decision situation, various factors of the real system should be reflected in the description of the objective functions and constraints. Naturally, these objective functions and constraints involve many parameters whose possible values may be assigned by the experts. In the conventional approach, such parameters are fixed at some values in an experimental and/or subjective manner through the expert's understanding of the nature of the parameters.

In most real-world situations, it may be reasonable to assume that the possible values of these parameters are often only imprecisely or ambiguously known to the experts. In this case, it would certainly be more appropriate to interpret the expert's understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy subsets of the real line known as fuzzy numbers [58, 161]. The resulting single and multi-objective programming problems involving fuzzy parameters would be viewed as the more realistic version of the conventional one [140].

1.1 Literature review

Bellman and Zadeh [17] were the first to consider the application of the fuzzy set theory in solving optimization problems. Zimmermann [165, 166] applied the fuzzy set theory to the linear programming and multi-criteria decision making problems. Leberling [102] presented an application of fuzzy approaches to the linear vector maximum problems. He used hyperbolic membership function for solution of multi-objective linear programming problems and deduced that the solutions obtained by using the fuzzy min-operator together with linear as well as special non-linear membership functions (hyperbolic) are always compromise solutions of the original multi-criteria problem. Oheigeartaigh [121] proposed an algorithm for solving transportation problems where the availabilities and demands are represented by triangular fuzzy numbers. Chanas et al. [39] pointed out that there are grievous errors in the study of Oheigeartaigh [121] and observed that the final solution admitted to be optimal is inferior to the intial solution. They presented a fuzzy linear programming model for fuzzy transportation problems with crisp cost coefficients and fuzzy supply and demand values and used parametric programming for its solution.

Ringuest and Rinks [133] developed two interactive algorithms for multi-objective transportation problems and solved two numerical examples to illustrate these algorithms and to demonstrate their viability. The first algorithm proceeds from one efficient extreme point to the next along the edges of the feasible decision variable space. The search is continued until a satisfactory solution is obtained. However second algorithm proceeds by optimizing a function which passes through the l currently most preferred nondominated solutions. This procedure is repeated iteratively until a nondominated solution is repeated or an inferior solution results. Thus, the algorithm is allowed to move back and forth across the objective function space.

Bit et al. [30] applied fuzzy linear programming technique to solve linear multi-objective transportation problems with K objectives. The fuzzy programming method gives K non dominated solutions and an optimal compromise solution in contrast to more than K non dominated and dominated solutions provided by the interactive algorithms [133]. Bit et al. [31] extended fuzzy linear programming technique to solve linear multi-objective solid transportation problems. Bit et al. [32] proposed an additive fuzzy programming model for multi-objective transportation problems in which weights and priorities for nonequivalent objectives were also incorporated. This model gives a non dominated solution which is nearer to the best compromise solution. They pointed out that the algorithms proposed by Ringuest and Rinks [133] and Bit et al. [30] are not applicable to those multi-objective transportation problems where the relative importance or priority of the objectives are given.

Gen et al. [66] proposed a genetic algorithm for the solution of bi-criteria solid transportation problems with fuzzy numbers and pointed out that this algorithm can be adapted to solve non-linear multi-objective solid transportation problems. Chanas and Kuchta [40] proposed the concept of the optimal solution of the transportation problem with fuzzy coefficients expressed as interval fuzzy numbers and developed an algorithm for the same. Tzeng et al. [153] studied the real life problem of coal allocation planning of Taipower, the official electricity authorithy of Taiwan and formulated the problem as a fuzzy bi-criteria multi-index transportation problem. Li et al. [106] improved the genetic algorithm to solve the multi-objective solid transportation problems with fuzzy numbers. Verma et al. [156] used fuzzy programming technique to solve multi-objective transportation problems with hyberbolic and exponential membership functions and obtained optimal compromise solution. An algorithm is designed by Chanas and Kuchta [41] for solving the integer fuzzy transportation problem with fuzzy supply and demand values in the sense of maximizing the joint satisfaction of the fuzzy goal and constraints and it is pointed out that there is no need to solve any parametric problem. Moreover, fuzzy numbers defining the problem need not be trapezoidal fuzzy numbers and they can differ from each other and be of any type.

Hussien [73] studied the complete set of α-possibly efficient solutions of multi-objective transportation problems with possibilistic coefficients of the objective functions. Jimenez and Verdegay [82] used the technique of parametric programming to obtain fuzzy solutions of the fuzzy problem of uncertain solid transportation problems. Teng and Tzeng [152] proposed the fuzzy multi-objective programming for the problem of transportation investment project selection. Das et al. [47] proposed a solution procedure of the interval multi-objective transportation problems, where the co-efficients of the objective functions, source and destination parameters have been considered as interval. Three different type of transportation problems were solved to illustrate the solution methodology. The real life problem of cement transportation planning of Taiwan has been considered by Shih [144] and three types of fuzzy linear programming models are used to determine the optimal transportation amount and the capacity of new facilities.

Avineri et al. [11] provided a methodology for the selection and ranking of transportation projects using fuzzy sets theory. A comparative study of the results of the proposed method, obtained by a fuzzy expert system and the results obtained by an ordinary crisp process was carried out. Kikuchi [91] represented all the parameters of transportation problem by triangular fuzzy numbers and used the fuzzy linear programming approach to find the set of values such that the smallest membership grade among them is maximized. Li and Lai [107] presented a fuzzy compromise programming approach to multi-objective transportation problems. A characteristic feature of the proposed approach is that the various objectives are synthetically considered with the marginal evaluation for individual objectives and the global evaluation for all objectives and a compromise programming model is formulated.

Sakawa et al. [137] dealt with actual problems on production and work force assignment of a housing material manufacturer. Abd El-Wahed [1] presented a fuzzy programming approach to determine the optimal compromise solution of multi-objective transportation problems. This study showed that the fuzzy approach outperforms the interactive procedure as the number of objectives and constraints increases. Ahlatcioglu et al. [3] proposed a method which solves the transportation problem where the demands and supplies are bounded lower and upper triangular fuzzy numbers. Sakawa et al. [139] described the decentralized two-level transportation problem of a housing material manufacturer with Interactive fuzzy programming approach. They took into account not only the degree of satisfaction of housing material manufacturer but also of two of its forwarding agents. Liu [110] developed a method to calculate the lower and upper bounds of total transportation cost when the supply and demand quantities are varying. Liu and Kao [112] developed a procedure to derive the fuzzy objective value of the fuzzy transportation problem where the cost coefficients, supply and demand quantities were fuzzy numbers. Their idea was based on the extension principle. A pair of mathematical programs is formulated to calculate the lower and upper bounds of the fuzzy total transportation cost at possibility level α.

Ammar and Youness [7] investigated the efficient solutions and stability of multi-objective transportation problem with fuzzy cost coefficient, fuzzy supply and fuzzy demand. They introduced the concept of α-fuzzy efficient and α-parametric efficient solutions. Chiang [44] pointed out that it is better to use interval-valued fuzzy numbers instead of normal fuzzy numbers and proposed a method to find the optimal solution of single objective transportation problems with demand and availability as interval-valued fuzzy numbers. Jana and Roy [79] developed a solution procedure of multi-objective fuzzy linear programming problems with mixed constraints and the method is further applied to a linear multi-objective solid transportation problems. Gani and Razak [61] used a parametric approach to obtain fuzzy solution of two stage cost minimizing fuzzy transportation problems in which supplies and demands are trapezoidal fuzzy numbers. Gani and Razak [62] presented fuzziness in preemptive goal programming formulation of multi-objective unbalanced transportation problems with budgetary constraints in which demand and budget are specified imprecisely. Abd El-Wahed and Lee [2] proposed an interactive fuzzy goal programming approach to determine the preferred compromise solution for the multi-objective transportation problem. The proposed approach considers the imprecise nature of the input data by implementing the minimum operator and also assumes that each objective function has a fuzzy goal. The approach focuses on minimizing the worst upper bound to obtain an efficient solution which is close to the best lower bound of each objective function. The solution procedure controls the search direction via updating both the membership values and the aspiration levels.

Gupta and Mehlawat [69] studied a real life problem of selecting a new type of coal for a steel manufacturing company in India. They used level (λ, ρ) interval-valued fuzzy numbers and developed an algorithm to find the non-dominated solutions. Zangiabadi and Maleki [162] presented a fuzzy goal programming approach to determine an optimal compromise solution for the multi-objective transportation problem by assuming that each objective function has a fuzzy goal. They showed that their proposed method and the fuzzy programming method are equivalent. Li et al. [105] described a new method based on fuzzy goal programming for solving transportation problems with fuzzy costs. Maleki and Khodaparasti [119] applied fuzzy goal programming approach to solve non-linear multi-objective transportation problems. Pramanik and Roy [128] presented a priority based fuzzy goal programming approach for solving a multi-objective transportation problem with fuzzy coefficients. Jana and Roy [81] presented the solution procedure of fuzzy linear programming problems with fuzzy variables and further applied it to fuzzy capacitated transportation problems. Chakraborty and Chakraborty [36] considered bi-criteria transportation problem with fuzzy parameters. Fuzzy parametric programming was used to handle impreciseness and the resulting bi-criteria problem was solved by prioritized goal programming approach. To illustrate the methodology, a real life case study was made with inputs from Jharia coalfield in India.

Lohgaonkar and Bajaj [113] used fuzzy programming technique to find the optimal compromise solution of a multi-objective capacitated transportation problems with linear and non-linear membership function (hyperbolic, exponential). Lohgaonkar et al. [114] used fuzzy goal programming approach to unbalanced transportation problems with additive multiple fuzzy goals where the goals are considered to be of equal importance. Sudhakar and Kumar [148] proposed a zero suffix method to solve multi-objective two stage fuzzy transportation problems. Ammar and Kozae [6] used parametric programming in vector fuzzy transportation problems with fuzzy data. Dinagar and Palanivel [56] pointed out that there is no method in literature to solve fuzzy time minimizing transportation problems and proposed a method for the same. Kocken and Ahlatcioglu [96] presented a compensatory approach to solve multi-objective linear transportation problems with fuzzy cost coefficients by using Werner's μ_{and} operator which is useful in computational efficiency and always generates Pareto optimal solutions. Jaikumar et al. [78] proposed a new algorithm for solving transportation problems and called it Modified Next to Next Minimum Penalty Method and suggested that their new algorithm is more efficient than Modified Vogel's Approximation Method [142].

Although several authors have tried to develop new methods for solving warehouse selection problems in crisp environment; but to the best of our knowledge, there is no method in the literature to solve warehouse selection problems in fuzzy environment. Ignizio and Cavalier [75] formulated and solved the problem of selecting upto a fixed number of sites from among a given number of potential sites for locating warehouses at them and clustering customers to the selected sites in such a way that each customer is assigned to unique selected site. The problem has a single objective to minimize the sum of distances from each customer to his/her assigned site. Praveena et al. [129] extended this problem to bi-objective warehouse selection problem. The two objectives are to minimize the total cost and the duration of meeting requirements of all the ration shops from their assigned warehouses to the selected sites. The problem is solved using a heuristic algorithm consisting of a combination of add and drop rules. Prakash et al. [124] solved the same problem using heuristic algorithm incorporating tabu search and obtained efficient solutions. Also, it is shown that incorporation of tabu search results into increasing the number of efficient solutions than those obtained earlier without its incorporation, thus leading to better results. A discussion about efficient solutions can be found in [75, 124, 147].

After reviewing the literature, it can be concluded that there is no method in the literature for solving warehouse selection problems in fuzzy environment and there are some shortcomings and limitations in the existing methods for solving transportation problems in fuzzy environment. In this thesis, new methods for solving warehouse selection problems in fuzzy environment, time minimizing transportation problems in fuzzy environment and multi-objective transportation problems in fuzzy environment are proposed to overcome the limitations and shortcomings of the existing methods.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Dinagar and Palanivel [56] claimed that there is no method in the literature for solving time minimizing transportation problems in fuzzy environment and proposed a method to solve fuzzy time minimizing transportation problems by modifying an existing method for solving time minimizing transportation problems in crisp environment. Keeping this in view, in this chapter, bi-objective warehouse selection problems in fuzzy environment have been considered. The two objectives are to minimize the total fuzzy cost and fuzzy time of meeting the requirements of all the ration shops from their assigned warehouses at the selected sites. An iterative algorithm using ranking function is developed to find the set of efficient solutions of these problems in fuzzy environment.

Chapter 3

In this chapter, the shortcomings of the existing method [56] are pointed out and a new method is proposed to resolve these shortcomings. Also, an alternative method is proposed for solving warehouse selection problems in fuzzy environment and it is shown that it is much easy to apply the proposed method in this chapter as compared to the method proposed in previous chapter.

Chapter 4

In this chapter, shortcomings of existing formulation of time minimizing transportation problems in fuzzy environment are pointed out and to resolve these shortcomings, a new formulation is proposed. Also, with the help of proposed formulation and the existing method [99], a new method is proposed for solving time minimizing transportation problems in fuzzy environment.

Chapter 5

Gupta et al. [68] pointed out that there is no method in the literature for solving fully fuzzy multi-objective transportation problems and proposed a method for its solution. In this chapter, the limitations of this method are pointed out and to overcome these limitations, a new method is proposed by modifying the existing method. The advantages of the proposed method over the existing method are discussed. To illustrate the proposed method some numerical examples are solved.

Chapter 6

There are several methods in the literature for solving transportation problems where the parameters are fuzzy numbers. Chiang [44] pointed out that it is better to represent the parameters as level (λ, ρ) interval-valued triangular fuzzy numbers instead of fuzzy numbers and proposed a method to find the optimal solution of single objective transportation problems by representing the availability and demand as level (λ, ρ) interval-valued triangular fuzzy numbers. In this chapter, the limitations of this existing method are pointed out and to overcome these limitations a new method is proposed. To illustrate the proposed method numerical examples are solved. The advantages of the proposed method over the existing method are also discussed.

Chapter 7

This chapter concludes the work herein. Utility and possible future extensions of the work are indicated.

Chapter 2

Optimal Way To Select Warehouse Sites For Clustering Ration Shops In Fuzzy Environment

Dinagar and Palanivel [56] claimed that there is no method in the literature for solving time minimizing transportation (TMT) problems in fuzzy environment and proposed a method to solve fuzzy TMT problems by modifying an existing method for solving TMT problems in crisp environment. Keeping this in view, in this chapter, bi-objective warehouse selection problems with two objectives in fuzzy environment have been considered. The two objectives are to minimize the total fuzzy cost and fuzzy time of meeting the requirements of all the ration shops from their assigned warehouses at the selected sites. An iterative algorithm using ranking function is developed to find the set of efficient solutions of these problems in fuzzy environment.

The work presented in this chapter is published in the *Iranian Journal of Fuzzy Systems* 9 (2012) 1–19 (Impact factor: 0.592).

2.1 Preliminaries

In this section, some basic definitions and arithmetic operations of trapezoidal fuzzy numbers are presented.

2.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 2.1 [86] Let X be a classical set of objects. Then, the set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$ is called a fuzzy set in X where $\mu_{\widetilde{A}} : X \to [0;1]$ is called the membership function.

Definition 2.2 [86] Let $\widetilde{A}$ be a fuzzy set in X and $\lambda \in [0,1]$ be a real number. Then, the classical set $A^{\lambda} = \{x \in X : \mu_{\widetilde{A}}(x) \geq \lambda\}$ is called λ-level set or λ-cut of $\widetilde{A}$.

Definition 2.3 [86] A fuzzy set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$ is said to be normalized fuzzy set if and only if $Supremum\{\mu_{\widetilde{A}}(x) : \forall\ x \in X\} = 1$.

Definition 2.4 [86] A fuzzy set $\widetilde{A}$ is called a convex fuzzy set if and only if $\mu_{\widetilde{A}}(\alpha x_1 + (1-\alpha)x_2) \geq Minimum\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}$, $\forall\ x_1, x_2 \in X$ and $\alpha \in [0,1]$.

Definition 2.5 [86] A convex normalized fuzzy set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in \mathbb{R}\}$ on real line $\mathbb{R}$ is called a fuzzy number if and only if $\mu_{\widetilde{A}}(x)$ is piecewise continuous in $\mathbb{R}$.

Definition 2.6 [86] A fuzzy number $\widetilde{A}$ defined on the universal set of real numbers $\mathbb{R}$ denoted as $\widetilde{A} = (a, b, c, d)$ is said to be a trapezoidal fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} (\frac{x-a}{b-a}), & a \leq x < b \\ 1, & b \leq x \leq c \\ (\frac{x-d}{c-d}), & c < x \leq d \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.7 [86] Let $\widetilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then, its λ-cut A^λ is defined as follows:

$$A^\lambda = [a + (b - a)\lambda, d - (d - c)\lambda],\ 0 \leq \lambda \leq 1.$$

Definition 2.8 [109] A ranking function $\Re : F(\mathbb{R}) \to \mathbb{R}$, where $F(\mathbb{R})$ set of fuzzy numbers defined on set of real numbers, maps each fuzzy number into a real number, where a natural order exists.

Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then,

(i) $\widetilde{A} \prec \widetilde{B}$ if and only if $\Re(\widetilde{A}) < \Re(\widetilde{B})$

(ii) $\widetilde{A} \preceq \widetilde{B}$ if and only if $\Re(\widetilde{A}) \leq \Re(\widetilde{B})$

(iii) $\widetilde{A} \approx \widetilde{B}$ if and only if $\Re(\widetilde{A}) = \Re(\widetilde{B})$

where, $\widetilde{A} = \frac{(a_1+b_1+c_1+d_1)}{4}$ and $\widetilde{B} = \frac{(a_2+b_2+c_2+d_2)}{4}$.

Definition 2.9 [56] A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be zero trapezoidal fuzzy number if and only if $\Re(\widetilde{A}) = 0$.

Definition 2.10 [56] A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be non-negative trapezoidal fuzzy number if and only if $\Re(\widetilde{A}) \geq 0$.

Definition 2.11 [56] A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be positive trapezoidal fuzzy number if and only if $\Re(\widetilde{A}) > 0$.

2.1.2 Arithmetic operations

In this section, some arithmetic operations between two trapezoidal fuzzy numbers defined on the universal set of real numbers $\mathbb{R}$ are presented.

Let $\widetilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\widetilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then,

(i) $\widetilde{A}_1 \oplus \widetilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

(ii) $\widetilde{A}_1 \ominus \widetilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$

(iii) $\widetilde{A}_1 \otimes \widetilde{A}_2 \simeq (a, b, c, d)$

where

$$a = \ Minimum\ \{a_1a_2, a_1d_2, a_2d_1, d_1d_2\},\ b = \ Minimum\ \{b_1b_2, b_1c_2, c_1b_2, c_1c_2\},$$

$$c = \ Maximum\ \{b_1b_2, b_1c_2, c_1b_2, c_1c_2\},\ d = \ Maximum\ \{a_1a_2, a_1d_2, a_2d_1, d_1d_2\}$$

(iv) $\gamma A_1 = \begin{cases} (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1), & \gamma \geq 0 \\ (\gamma d_1, \gamma c_1, \gamma b_1, \gamma a_1), & \gamma < 0 \end{cases}$

Remark 2.1 Let $\{\widetilde{a}_i : i = 1, 2, \ldots, n\}$ be a set of trapezoidal fuzzy numbers. If $\Re(\widetilde{a}_k) \leq \Re(\widetilde{a}_i), \forall\ i$ then, the fuzzy number $\widetilde{a}_k$ is the $Minimum\{\widetilde{a}_i : i = 1, 2, \ldots, n\}$ and if $\Re(\widetilde{a}_k) \geq \Re(\widetilde{a}_i), \forall\ i$ then, the fuzzy number $\widetilde{a}_k$ is the $Maximum\{\widetilde{a}_i : i = 1, 2, \ldots, n\}$.

2.2 Mathematical formulation and tabular representation of balanced time minimizing transportation problem in crisp environment

In TMT problems, the time of transporting a homogenous product from certain sources to various destinations is minimized. For example, during wars, it is required to transport armaments in the shortest possible time. Time minimization is desirable during transportation of perishable goods. The basic difference between cost minimizing transportation problems and TMT problems is that the cost of transportation changes with variation in the quantity of product, but the time is independent of quantity of the product transported. This was first studied by Hammer [71, 72]. The objective of TMT problems is to minimize the maximum time of transporting the product from certain sources to various destinations.

Suppose that there are p sources say $S_1, S_2, \ldots, S_p$ and q destinations say D_1, $D_2, \ldots, D_q$. Let a_i be the availability at i^{th} source S_i, b_j be the demand at j^{th} destination D_j, x_{ij} be the units of the product transported from i^{th} source to j^{th} destination and t_{ij} be the time of transporting the product from i^{th} source to j^{th} destination irrespective of the units of the product to be transported, $\sum\limits_{i=1}^{p} a_i$ be the total availability and $\sum\limits_{j=1}^{q} b_j$ be the total demand of the product. Then, a balanced TMT problem can be formulated into mathematical programming problem $(P_{2.1})$ and can be represented by Table 2.1.

The objective function which is sought to be minimized is

$$T = \ Maximum\{t_{ij} : x_{ij} > 0; \ i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q\}$$

subject to

$$\begin{aligned} &\sum_{j=1}^{q} x_{ij} = a_i, \qquad i = 1, 2, \ldots, p \\ &\sum_{i=1}^{p} x_{ij} = b_j, \qquad j = 1, 2, \ldots, q \\ &\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j \\ &x_{ij} \geq 0. \end{aligned} \tag{$P_{2.1}$}$$

Table 2.1: Tabular representation of crisp TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	t_{11}	$\cdots$	t_{1j}	$\cdots$	t_{1q}	$= a_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	t_{i1}	$\cdots$	t_{ij}	$\cdots$	t_{iq}	$= a_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	t_{p1}	$\cdots$	t_{pj}	$\cdots$	t_{pq}	$= a_p$
Demand	$= b_1$	$\cdots$	$= b_j$	$\cdots$	$= b_q$	$\sum\limits_{i=1}^{p} a_i = \sum\limits_{j=1}^{q} b_j$

Remark 2.2 If $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j$, then $(P_{2.1})$ is said to be balanced. Otherwise, it is said to be unbalanced.

2.2.1 Feasible and optimal solutions of time minimizing transportation problem in crisp environment

In this section, the definitions of feasible and optimal solutions of TMT problem in crisp environment are presented.

Definition 2.12 [143] Any set of non-negative allocations $\{x_{ij}\}$ which satisfies the constraints of the problem $(P_{2.1})$ is said to be a feasible solution.

Definition 2.13 [143] A feasible solution of the problem $(P_{2.1})$ is called a basic feasible solution if the number of non negative allocations are at the most $p+q-1$.

Definition 2.14 [143] A basic feasible solution of the problem $(P_{2.1})$ which optimizes the objective function is called optimal basic feasible solution.

2.2.2 Existing method for solving time minimizing transportation problem in crisp environment

Using the existing method [143], based on tabular representation, the optimal solution of TMT problem in crisp environment can be obtained by using the following steps:

Step 1 Examine that the chosen problem is balanced or not.

Case (i) If the problem is balanced, then Go to Step 2.

Case (ii) If the problem is not balanced, then check that $\sum_{i=1}^{p} a_i < \sum_{j=1}^{q} b_j$ or $\sum_{i=1}^{p} a_i > \sum_{j=1}^{q} b_j$.

Case (a) If $\sum_{i=1}^{p} a_i < \sum_{j=1}^{q} b_j$, then add a dummy source S_{p+1} with availability $a_{p+1} = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i$ and assume that $t_{(p+1)j} = 0,\ \forall\ j = 1, 2, \ldots, q$.

Case (b) If $\sum_{i=1}^{p} a_i > \sum_{j=1}^{q} b_j$, then add a dummy destination D_{q+1} with demand $b_{q+1} = \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j$ and assume that $t_{i(q+1)} = 0, \ \forall \ i = 1, 2, \ldots, p$.

Step 2 Find an initial basic feasible solution by using any of classical transportation methods, say North-West corner rule whose detailed procedure is given below. Start with the North West corner of Table 2.1 and consider the cell $(1, 1)$. In this cell a_1 and b_1 are the availability and demand respectively. Check that $a_1 > b_1$ or $a_1 < b_1$ or $a_1 = b_1$

Case (i) If $a_1 > b_1$, then, assign $x_{11} = b_1$ in the cell $(1, 1)$. Move horizontally to cell $(1, 2)$ in next column. Decrease a_1 by b_1 until the availability of the first source is exhausted. Cross out the first column.

Case (ii) If $a_1 < b_1$, then, assign $x_{11} = a_1$ in the cell $(1, 1)$. Move vertically below to cell $(2, 1)$ in next row. Decrease b_1 by a_1 until the demand of this destination is satisfied. Cross out the first row.

Case (iii) If $a_1 = b_1$, then, assign a_1 or b_1 to x_{11} in the cell $(1, 1)$ and move diagonally to the cell $(2, 2)$ determined by the next column of the next row.

Proceed the process until the South-East corner of the table is reached.

Step 3 Compute $T = \ Maximum\{t_{ij} : x_{ij} > 0; \ i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q\}$ for the current basic feasible solution and cross out all the non-basic cells for which $t_{ij} \geq T$.

Step 4 Construct a closed loop for the basic cells corresponding to T in such a way that when the values at cells are shifted around, the value at the cell containing T tends towards (not-necessarily) zero and no variable becomes negative. If no such closed path is possible, the solution under test is optimal basic feasible solution, otherwise Go to Step 2.

2.3 Mathematical formulation and tabular representation of balanced time minimizing transportation problem in fuzzy environment

Dinagar and Palanivel [56] obtained the mathematical formulation ($P_{2.2}$) and tabular representation (Table 2.2) of TMT problems in fuzzy environment by replacing all the parameters t_{ij}, x_{ij}, a_i and b_j of problem ($P_{2.1}$) and Table 2.1 by fuzzy parameters $\widetilde{t}_{ij}, \widetilde{x}_{ij}, \widetilde{a}_i$ and $\widetilde{b}_j$ respectively.

The objective function which is sought to be minimized is:

$\widetilde{T} = \ Maximum\{\widetilde{t}_{ij} : \widetilde{x}_{ij}$ is a positive trapezoidal fuzzy number; $i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q\}$

subject to

$$\begin{aligned} &\sum_{j=1}^{q} \widetilde{x}_{ij} \approx \widetilde{a}_i, \qquad i = 1, 2, \ldots, p \\ &\sum_{i=1}^{p} \widetilde{x}_{ij} \approx \widetilde{b}_j, \qquad j = 1, 2, \ldots, q \\ &\sum_{i=1}^{p} \widetilde{a}_i \approx \sum_{j=1}^{q} \widetilde{b}_j \\ &\widetilde{x}_{ij} \text{ is a non-negative trapezoidal fuzzy number.} \end{aligned} \qquad (P_{2.2})$$

where,

$\widetilde{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$: the fuzzy availability of the product at i^{th} source (S_i).

$\widetilde{b}_j = (b_{j1}, b_{j2}, b_{j3}, b_{j4})$: the fuzzy demand of the product at j^{th} destination (D_j).

$\widetilde{t}_{ij} = (t_{ij1}, t_{ij2}, t_{ij3}, t_{ij4})$: the fuzzy time of transporting the product from i^{th} source (S_i) to j^{th} destination (D_j).

$\widetilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4})$: fuzzy decision variable representing the quantity of the product transported from i^{th} source (S_i) to j^{th} destination (D_j).

$\sum_{i=1}^{p} \widetilde{a}_i$: total fuzzy availability of the product.

$\sum_{j=1}^{q} \widetilde{b}_j$: total fuzzy demand of the product.

Table 2.2: Tabular representation of fuzzy TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	$\widetilde{t}_{11}$	$\cdots$	$\widetilde{t}_{1j}$	$\cdots$	$\widetilde{t}_{1q}$	$\approx \widetilde{a}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	$\widetilde{t}_{i1}$	$\cdots$	$\widetilde{t}_{ij}$	$\cdots$	$\widetilde{t}_{iq}$	$\approx \widetilde{a}_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	$\widetilde{t}_{p1}$	$\cdots$	$\widetilde{t}_{pj}$	$\cdots$	$\widetilde{t}_{pq}$	$\approx \widetilde{a}_p$
Demand	$\approx \widetilde{b}_1$	$\cdots$	$\approx \widetilde{b}_j$	$\cdots$	$\approx \widetilde{b}_q$	$\sum_{i=1}^{p} \widetilde{a}_i \approx \sum_{j=1}^{q} \widetilde{b}_j$

Remark 2.3 If $\sum_{i=1}^{p} \widetilde{a}_i \approx \sum_{j=1}^{q} \widetilde{b}_j$, then $(P_{2.2})$ is said to be balanced. Otherwise, it is said to be unbalanced.

2.3.1 Feasible and optimal solutions of time minimizing transportation problems in fuzzy environment

In this section, the existing definitions of feasible and optimal solutions of TMT problems in fuzzy environment are presented.

Definition 2.15 [56] Any set of non negative fuzzy allocations $\{\widetilde{x}_{ij}\}$ which satisfies the constraints of the problem $(P_{2.2})$ is a fuzzy feasible solution.

Definition 2.16 [56] A fuzzy feasible solution of the problem $(P_{2.2})$ is a fuzzy basic feasible solution if the number of non negative allocations are at the most $p+q-1$.

Definition 2.17 [56] A fuzzy basic feasible solution of the problem $(P_{2.2})$ which optimizes the objective function is called optimal fuzzy basic feasible solution.

2.3.2 Existing method for solving time minimizing transportation problems in fuzzy environment

Dinagar and Palanivel [56] claimed that there is no method in the literature for solving TMT problems in fuzzy environment and proposed a method for the same with the help of the existing method, presented in Section 2.2.2 for solving TMT problems in crisp environment.

The steps of the existing method are as follows [56]:

Step 1 Examine that the chosen problem is balanced or not.

Case (i) If the problem is balanced, then Go to Step 2.

Case (ii) If the problem is not balanced, then check that $\sum_{i=1}^{p} \widetilde{a}_i \prec \sum_{j=1}^{q} \widetilde{b}_j$ or $\sum_{i=1}^{p} \widetilde{a}_i \succ \sum_{j=1}^{q} \widetilde{b}_j$.

Case (a) If $\sum_{i=1}^{p} \widetilde{a}_i \prec \sum_{j=1}^{q} \widetilde{b}_j$, then add a dummy source S_{p+1} with fuzzy availability $\widetilde{a}_{p+1} = \sum_{j=1}^{q} \widetilde{b}_j \ominus \sum_{i=1}^{p} \widetilde{a}_i$ and assume that $\widetilde{t}_{(p+1)j} = (0,0,0,0),\ \forall\ j = 1,2,\ldots,q$, then Go to Step 2.

Case (b) If $\sum_{i=1}^{p} \widetilde{a}_i \succ \sum_{j=1}^{q} \widetilde{b}_j$, then add a dummy destination D_{q+1} with demand $\widetilde{b}_{q+1} = \sum_{i=1}^{p} \widetilde{a}_i \ominus \sum_{j=1}^{q} \widetilde{b}_j$ and assume that $\widetilde{t}_{i(q+1)} = (0,0,0,0),\ \forall\ i = 1,2,\ldots,p$, then Go to Step 2.

Step 2 Find an initial fuzzy basic feasible solution using any transportation method in fuzzy environment, say fuzzy North-West corner rule whose detailed procedure is given below.

Start with the North West corner of Table 2.2 and consider the cell $(1,1)$. In this cell $(a_{11}, a_{12}, a_{13}, a_{14})$ and $(b_{11}, b_{12}, b_{13}, b_{14})$ are the fuzzy availability and fuzzy demand respectively. Check that $(a_{11}, a_{12}, a_{13}, a_{14}) \succ (b_{11}, b_{12}, b_{13}, b_{14})$ or $(a_{11}, a_{12}, a_{13}, a_{14}) \prec (b_{11}, b_{12}, b_{13}, b_{14})$ or $(a_{11}, a_{12}, a_{13}, a_{14}) \approx (b_{11}, b_{12}, b_{13}, b_{14})$.

Case (i) If $(a_{11}, a_{12}, a_{13}, a_{14}) \succ (b_{11}, b_{12}, b_{13}, b_{14})$, then assign $\widetilde{x}_{11} = (b_{11}, b_{12}, b_{13}, b_{14})$ in the cell $(1, 1)$. Move horizontally to cell $(1, 2)$ in next column. Decrease $(a_{11}, a_{12}, a_{13}, a_{14})$ by $(b_{11}, b_{12}, b_{13}, b_{14})$, until the fuzzy availability of the first source is satisfied. Cross out the first column.

Case (ii) If $(a_{11}, a_{12}, a_{13}, a_{14}) \prec (b_{11}, b_{12}, b_{13}, b_{14})$, then assign $\widetilde{x}_{11} = (a_{11}, a_{12}, a_{13}, a_{14})$ in the cell $(1, 1)$ and move vertically below to cell $(2, 1)$ in the next row. Decrease $(b_{11}, b_{12}, b_{13}, b_{14})$ by $(a_{11}, a_{12}, a_{13}, a_{14})$, until the fuzzy demand of this destination is satisfied. Cross out the first row.

Case (iii) If $(a_{11}, a_{12}, a_{13}, a_{14}) \approx (b_{11}, b_{12}, b_{13}, b_{14})$, then assign $(a_{11}, a_{12}, a_{13}, a_{14})$ or $(b_{11}, b_{12}, b_{13}, b_{14})$ to $\widetilde{x}_{11}$ in the cell $(1, 1)$ and move diagonally to the cell $(2, 2)$ determined by the next column of the next row.

Proceed the process until the South East corner of the table is reached.

Step 3 Find $\widetilde{T} = \; Maximum\{\widetilde{t}_{ij} : \widetilde{x}_{ij} \succ 0;\; i = 1, 2, \ldots, p;\; j = 1, 2, \ldots, q\}$ corresponding to the current fuzzy basic feasible solution and cross out all such non basic cells for which $(t_{k1}, t_{k2}, t_{k3}, t_{k4}) \succeq \widetilde{T}$

Step 4 Construct a closed loop for the basic cells corresponding to $\widetilde{T}$ in such a way that when the values at the cells are shifted around, the value of the cell containing $\widetilde{T}$ tends towards (not-necessarily) zero and no variable becomes negative. If no such closed path is possible, the solution under test is optimal fuzzy basic feasible solution, otherwise Go to Step 2.

2.3.3 Illustrative example

Dinagar and Palanivel [56] solved fuzzy TMT problem, chosen in Example 2.1, to illustrate their proposed method.

Example 2.1 [56] Solve the fuzzy TMT problem represented by Table 2.3 to find the optimal fuzzy transportation time.

Table 2.3: Tabular representation of numerical problem

Destinations→ Sources↓	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	$(7, 9, 11, 13)$	$(-5, -1, 1, 5)$	$(14, 18, 22, 26)$	$(8, 11, 12, 13)$	$(12, 14, 16, 18)$
S_2	$(-2, 0, 2, 4)$	$(5, 6, 7, 10)$	$(7, 8, 9, 12)$	$(14, 18, 22, 26)$	$(22, 24, 26, 28)$
S_3	$(9, 11, 13, 15)$	$(11, 13, 15, 17)$	$(13, 15, 17, 19)$	$(15, 17, 19, 21)$	$(3, 4, 5, 8)$
$\widetilde{b}_j$	$(9, 11, 13, 15)$	$(6, 7, 9, 10)$	$(12, 14, 16, 18)$	$(7, 9, 11, 13)$	$(37, 42, 47, 54)$ $\approx$ $(34, 41, 49, 56)$

Solution: Using the existing method [56], the fuzzy optimal solution of fuzzy TMT problem, chosen in Example 2.1, can be obtained as follows:

Step 1 $\sum_{i=1}^{3} \widetilde{a}_i = (37, 42, 47, 54)$ and $\sum_{j=1}^{4} \widetilde{b}_j = (34, 41, 49, 56)$, $\sum_{i=1}^{3} \widetilde{a}_i \approx \sum_{j=1}^{4} \widetilde{b}_j$. So, the problem is balanced.

Step 2 The initial fuzzy basic feasible solution obtained by using fuzzy North-West corner rule is shown in Table 2.4.

Table 2.4: Initial fuzzy basic feasible solution

Destinations→ Sources↓	D_1	D_2	D_3	D_4
S_1	$(7, 9, 11, 13)$ $(9, 11, 13, 15)$	$(-5, -1, 1, 5)$ $(-2, 2, 4, 8)$	$(14, 18, 22, 26)$	$(8, 11, 12, 13)$
S_2	$(-2, 0, 2, 4)$	$(5, 6, 7, 10)$ $(3, 4, 5, 8)$	$(7, 8, 9, 12)$ $(12, 14, 16, 18)$	$(14, 18, 22, 26)$ $(3, 4, 5, 8)$
S_3	$(9, 11, 13, 15)$	$(11, 13, 15, 17)$	$(13, 15, 17, 19)$	$(15, 17, 19, 21)$ $(3, 4, 5, 8)$

Step 3 For the fuzzy basic feasible solution, shown in Table 2.4, $\widetilde{T}$ is $(14, 18, 22, 26)$. So, according to Step 3, the cell $(1, 3)$ is crossed out. This is shown in Table 2.5.

Table 2.5: Illustrating Step 3

Destinations→ Sources↓	D_1	D_2	D_3	D_4
S_1	$(7, 9, 11, 13)$ $(9, 11, 13, 15)$	$(-5, -1, 1, 5)$ $(-2, 2, 4, 8)$	~~(14,18,22,26)~~	$(8, 11, 12, 13)$
S_2	$(-2, 0, 2, 4)$	$(5, 6, 7, 10)$ $(3, 4, 5, 8)$	$(7, 8, 9, 12)$ $(12, 14, 16, 18)$	$(14, 18, 22, 26)$ $(3, 4, 5, 8)$
S_3	$(9, 11, 13, 15)$	$(11, 13, 15, 17)$	$(13, 15, 17, 19)$	$(15, 17, 19, 21)$ $(3, 4, 5, 8)$

Step 4 According to Step 4, a closed loop is formed starting with basic cell (2,4) containing $\widetilde{T}$. It is clear from closed loop that $(x_{241}, x_{242}, x_{243}, x_{244})$ can be decreased only by $(-2, 2, 4, 8)$, for otherwise $(x_{121}, x_{122}, x_{123}, x_{124})$ will become negative.

Table 2.6: Closed loop starting from cell (2,4)

Destinations→ Sources↓	D_1	D_2	D_3	D_4
S_1	$(7, 9, 11, 13)$ $(9, 11, 13, 15)$	$(-5, -1, 1, 5)$ $(-2, 2, 4, 8)$ →	~~(14,18,22,26)~~	$(8, 11, 12, 13)$ ↓
S_2	$(-2, 0, 2, 4)$	$(5, 6, 7, 10)$ $(3, 4, 5, 8)$ ↑	$(7, 8, 9, 12)$ $(12, 14, 16, 18)$	$(14, 18, 22, 26)$ $(3, 4, 5, 8)$ ←
S_3	$(9, 11, 13, 15)$	$(11, 13, 15, 17)$	$(13, 15, 17, 19)$	$(15, 17, 19, 21)$ $(3, 4, 5, 8)$

Step 5 From Table 2.6, the new fuzzy basic feasible solution is obtained and is shown in Table 2.7.

Table 2.7: New fuzzy basic feasible solution

Destinations→ Sources↓	D_1	D_2	D_3	D_4
S_1	$(7,9,11,13)$ $(9,11,13,15)$	$(-5,-1,1,5)$	~~(14,18,22,26)~~	$(8,11,12,13)$ $(-2,2,4,8)$
S_2	$(-2,0,2,4)$	$(5,6,7,10)$ $(1,6,9,16)$	$(7,8,9,12)$ $(12,14,16,18)$	$(14,18,22,26)$ $(-5,0,3,10)$
S_3	$(9,11,13,15)$	$(11,13,15,17)$	$(13,15,17,19)$	$(15,17,19,21)$ $(3,4,5,8)$

Step 6 Similarly, proceeding in the same manner, the optimal fuzzy basic feasible solution is obtained. Table 2.8 provides the optimal fuzzy basic feasible solution which is $x_{11} = (-19, 1, 11, 27)$, $x_{14} = (-4, 6, 12, 26)$, $x_{21} = (-5, 0, 3, 10)$, $x_{22} = (1, 6, 9, 16)$, $x_{23} = (12, 14, 16, 18)$, $x_{31} = (3, 4, 5, 8)$ and optimal fuzzy transportation time is $\widetilde{T} = (9, 11, 13, 15)$.

Table 2.8: Optimal fuzzy basic feasible solution

Destinations→ Sources↓	D_1	D_2	D_3	D_4
S_1	$(7,9,11,13)$ $(-19,1,11,27)$	$(-5,-1,1,5)$	~~(14,18,22,26)~~	$(8,11,12,13)$ $(-4,6,12,26)$
S_2	$(-2,0,2,4)$ $(-5,0,3,10)$	$(5,6,7,10)$ $(1,6,9,16)$	$(7,8,9,12)$ $(12,14,16,18)$	~~(14,18,22,26)~~
S_3	$(9,11,13,15)$ $(3,4,5,8)$	~~(11,13,15,17)~~	~~(13,15,17,19)~~	~~(15,17,19,21)~~

2.4 Mathematical formulation and tabular representation of warehouse selection problem in crisp environment

The problem of selecting upto a fixed number of sites from among a given number of potential warehouse sites for clustering a given number of ration shops to them in optimal manner is an important real life problem. Prakash et al. [124] proposed a new method for solving warehouse selection problems in crisp environment. In this section, the mathematical formulation as well as tabular representation of the same problem is presented.

Suppose that there are M ration shops, N potential warehouse sites and k is maximum number of sites which can be selected from among the N potential sites for locating warehouses at them. The M ration shops are to be clustered upto k sites in such a way that each shop is assigned to unique site which is selected for the task. There is no restriction on number of ration shops to be clustered to a selected site. Let c_{ij} and t_{ij} $(i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N)$ units be the cost and time respectively of meeting requirements of i^{th} ration shop from j^{th} site. Let b_j units be the setup cost of warehouse at j^{th} site and B units be the budgetary amount allocated for set up of warehouses. Let x_{ij} be the variable assuming value 0 or 1 according as i^{th} ration shop is not assigned or assigned to j^{th} site and y_j be the variable assuming the value 0 or 1 according as j^{th} site is not selected or selected for locating a warehouse at it. Let C and T denote the total cost and maximum time respectively of meeting requirements of all the ration shops from their assigned warehouses. The mathematical formulation is as follows:

The two objective functions which are sought to be minimized are

$$C = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} x_{ij}$$

$$T = \ Maximum\{t_{ij} : x_{ij} = 1;\ i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N\}$$

subject to

$$\sum_{j=1}^{N} y_j \le k$$

$$\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, 2, \ldots, M$$

$$x_{ij} - y_j \leq 0, \quad i = 1, 2, \ldots, M; \ j = 1, 2, \ldots, N$$
$$\sum_{j=1}^{N} b_j y_j \leq B \qquad (P_{2.3})$$
$$\sum_{i=1}^{M} x_{ij} \geq y_j, \quad j = 1, 2, \ldots, N$$
$$x_{ij}, y_j = 0 \text{ or } 1, \quad i = 1, 2, \ldots, M; \ j = 1, 2, \ldots, N$$

The tabular representation of which is provided in Table 2.9

Table 2.9: Tabular representation of warehouse selection problem

Warehouse sites (j) $\rightarrow$ Ration shops (i) $\downarrow$	1	$\cdots$	j	$\cdots$	N
1	c_{11} t_{11}	$\cdots$	c_{1j} t_{1j}	$\cdots$	c_{1N} t_{1N}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	c_{i1} t_{i1}	$\cdots$	c_{ij} t_{ij}	$\cdots$	c_{iN} t_{iN}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	c_{M1} t_{M1}	$\cdots$	c_{Mj} t_{Mj}	$\cdots$	c_{MN} t_{MN}
Setup cost (b_j)	b_1	$\cdots$	b_j	$\cdots$	b_N
k	B				

2.4.1 Efficient solutions of warehouse selection problems in crisp environment

In this section, the existing definitions of 1^{st}, 2^{nd} and subsequent efficient solutions of warehouse selection problems in crisp environment are presented [124].

(i) A solution (X^1, Y^1) is called the 1^{st} efficient solution if there exists no solution (X, Y) of the problem given by $(P_{2.3})$ satisfying the conditions (a) $C(X) \leq C(X^1)$ and (b) $T(X) \leq T(X^1)$ with strict inequality holding in at least one of the conditions out of (a) and (b).

where,

X^l: the combination of x_{ij}'s during l^{th} efficient solution.

Y^l: the combination of y_j's during l^{th} efficient solution.

(ii) A solution (X^2, Y^2) is called the 2^{nd} efficient solution if no efficient solution (X, Y) of the problem exist satisfying the conditions (a) $C(X^1) < C(X) < C(X^2)$ and (b) $T(X^1) > T(X) > T(X^2)$.

(iii) 3^{rd} and the subsequent efficient solutions are defined in the same way as is defined for the 2^{nd} efficient solution.

2.4.2 Existing method for warehouse selection problems in crisp environment

Prakash et al. [124] proposed the following method for solving warehouse selection problem in crisp environment.

Step 1 Find $a = Minimum\{\sum_{i=1}^{M} c_{ij} : j = 1, 2, \ldots, N\}$ and $T_j = Maximum\{t_{ij} : i = 1, 2, \ldots, M\}$ for $j = 1, 2, \ldots, N$.

Case (i) If 'a' occurs corresponding to unique value of j, i.e., $j = t$, where $t \in \{1, 2, \ldots, N\}$, then t is the first potential site for the first warehouse.

Case (ii) If 'a' occurs corresponding to more than one values of j, i.e., $j = p_1, p_2, \ldots, p_m$, then find $Minimum\{T_{p_1}, T_{p_2}, \ldots, T_{p_m}\}$.

Case (a) If minimum is unique and it is corresponding to p_l, then p_l is the first potential site for the first warehouse.

Case (b) If minimum occurs corresponding to more than one values of j i.e., $j = p_{s_1}, p_{s_2}, \ldots, p_{s_q}$, then choose any j from $p_{s_1}, p_{s_2}, \ldots, p_{s_q}$ for the first potential site for the first warehouse.

Step 2 Let the first potential site selected for the first warehouse be q_1 where $q_1 \in \{1, 2, \ldots, N\}$. Now for selecting the next warehouse site q_2 where $q_2 \neq q_1$, construct Table 2.10 as:

Table 2.10: Tabular representation of crisp warehouse selection problem

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	$(q_1, 1)$	$\cdots$	(q_1, j)	$\cdots$	(q_1, N)
1	c^1_{11} t^1_{11}	$\cdots$	c^1_{1j} t^1_{1j}	$\cdots$	c^1_{1N} t^1_{1N}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	c^1_{i1} t^1_{i1}	$\cdots$	c^1_{ij} t^1_{ij}	$\cdots$	c^1_{iN} t^1_{iN}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	c^1_{M1} t^1_{M1}	$\cdots$	c^1_{Mj} t^1_{Mj}	$\cdots$	c^1_{MN} t^1_{MN}
Setup cost (b_j)	$b_{q_1} + b_1$	$\cdots$	$b_{q_1} + b_j$	$\cdots$	$b_{q_1} + b_N$
k	B				

Let the cost and time of $(i, j)^{th}$ cell, $j \neq q_1$ in new table be c^1_{ij} and t^1_{ij} where

$$c^1_{ij} = Minimum\{c_{iq_1}, c_{ij}\} \text{ and } t^1_{ij} = \begin{cases} t_{ij} & \text{if } Minimum\{c_{iq_1}, c_{ij}\} = c_{ij} \\ t_{iq_1} & \text{if } Minimum\{c_{iq_1}, c_{ij}\} = c_{iq_1} \end{cases}$$

The newly constructed Table 2.10 is same as Table 2.9 except it has $N - 1$ columns. Now apply Step 1 and find new combination of warehouses say (q_1, q_2) where $q_2 \in \{1, 2, \ldots, N\}$ and $q_2 \neq q_1$.

Step 3 Let the pair (q_1, q_2) represent the two potential sites for the two warehouses. Similarly, find k-potential sites for k-warehouses by applying Step 2. Let the combination of k-potential sites for k-warehouses be $(q_1, q_2, \ldots, q_k)$. Selection of sites using add rule, in the way explained above, yields the 0^{th} iterative solution (X^1_0, Y^1_0) for the 1^{st} efficient solution (X^1, Y^1) and is also the incumbent solution for it. The incumbent solution for the 1^{st} efficient solution (X^1, Y^1) is the best solution obtained so far. $C(X^1_0)$ and $T(X^1_0)$ provides the total cost and maximum time of 0^{th} iterative solution (X_0^1, Y_0^1).

Step 4 Invoke drop and add rules with tabu search for further improving the solution. Drop the site which was first selected i.e., q_1. The site dropped recently is included in the tabu list for add which will be prohibited from being selected in the next 1^{st} iterative solution (X_1^1, Y_1^1) and $k-1$, the most recently added warehouses i.e., $(q_2, q_3, \ldots, q_k)$ appearing in the order of their selection in the 0^{th} iterative solution are included in the tabu list for drop. Find all the entries of the column corresponding to combination $(q_2, q_3, \ldots, q_k)$ using Steps 1 and 2. Then, add the new site by replacing q_1 by $(q_2, q_3, \ldots, q_k)$ in Step 2. Suppose the new combination of k-potential sites for the k-warehouses are $(q_2, q_3, \ldots, q_k, q_h)$ where $q_h \neq q_1, q_2, \ldots, q_k$. Selection of sites $(q_2, q_3, \ldots, q_k, q_h)$ in the way explained above yields the 1^{st} iterative solution (X_1^1, Y_1^1) for the 1^{st} efficient solution (X^1, Y^1). Find the cost $C(X_1^1)$ and time $T(X_1^1)$ for $(q_2, q_3, \ldots, q_k, q_h)$. If $C(X_1^1) < C(X_0^1)$, then the current iterative solution will be treated as the incumbent solution for the next iteration, otherwise there will be no change in incumbent solution.

Step 5 The 2^{nd} iterative solution (X_2^1, Y_2^1) is obtained in similar manner as explained in Step 4 by changing the both tabu lists. The tabu list for the drop rule used for obtaining 2^{nd} iterative solution (X_2^1, Y_2^1) will now include the $k-1$ most recent selected sites included in the 1^{st} iterative solution (X_1^1, Y_1^1) i.e., $(q_3, \ldots, q_k, q_h)$ appearing in the order of their selection. Similarly, the tabu list for the add rule used for obtaining the 2^{nd} iterative solution (X_2^1, Y_2^1) will now include the 1^{st} site selected in the 1^{st} iterative solution (X_1^1, Y_1^1) i.e., q_2. If the

2^{nd} iterative solution (X_2^1, Y_2^1) is better than the incumbent solution, then the current iterative solution will be treated as incumbent solution for the next iteration otherwise the incumbent solution does not change. The 3^{rd} iterative solution (X_3^1, Y_3^1) and subsequent ones will be obtained in the same way as done in obtaining the 1^{st} and 2^{nd} iterative solutions. Change the incumbent solution to the current iterative solution if it is better than the incumbent solution. This process of obtaining newer iterative solutions is terminated when either an already obtained iterative solution is revisited or there are no sites left that will serve to improve the solution. When this happens, then the most recent incumbent solution provides the 1^{st} efficient solution (X^1, Y^1) of the formulated problem.

Step 6 To find the 2^{nd} efficient solution (X^2, Y^2) of the formulated problem, replace t_{ij}'s $\geq T(X^1)$ by an arbitrary large number L and repeat Steps 1 to 5.

Step 7 The incumbent solution at the end of iterative process yields 2^{nd} efficient solution (X^2, Y^2) of formulated problem. 3^{rd} and the subsequent solutions are obtained in the same way as is done to obtain the 2^{nd} efficient solution. The process of obtaining the efficient solutions terminates when it is no longer possible to obtain a new efficient solution with lesser duration.

2.5 Proposed mathematical formulation and tabular representation of warehouse selection problem in fuzzy environment

As discussed in Section 2.3, Dinagar and Palanivel [56] obtained the mathematical formulation and tabular representation of TMT problems in fuzzy environment by replacing the crisp parameters t_{ij}, x_{ij}, a_i and b_j of crisp TMT problems with fuzzy parameters $\widetilde{t}_{ij}, \widetilde{x}_{ij}, \widetilde{a}_i$ and $\widetilde{b}_j$. In the same direction, in this section, the mathematical formulation and tabular representation of warehouse selection problems in fuzzy environment is obtained by replacing the crisp parameters c_{ij}, t_{ij}, b_j and B of warehouse selection problems in crisp environment by fuzzy parameters $\widetilde{c}_{ij}, \widetilde{t}_{ij}, \widetilde{b}_j$ and $\widetilde{B}$ respectively.

The resultant mathematical formulation in fuzzy environment is as follows:
The objective functions which are sought to be minimized are

$$\widetilde{C} = \sum_{i=1}^{M} \sum_{j=1}^{N} \widetilde{c}_{ij} x_{ij}$$

$$\widetilde{T} = \ Maximum\{\widetilde{t}_{ij} : x_{ij} = 1;\ i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N\}$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{N} y_j \leq k \\
&\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, 2, \ldots, M \\
&x_{ij} - y_j \leq 0, \quad i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N \\
&\sum_{j=1}^{N} \widetilde{b}_j y_j \preceq \widetilde{B} \qquad\qquad (P_{2.4}) \\
&\sum_{i=1}^{M} x_{ij} \geq y_j, \quad j = 1, 2, \ldots, N \\
&x_{ij}, y_j = 0 \text{ or } 1, \quad i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N
\end{aligned}$$

where,

$\widetilde{c}_{ij} = (c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4})$: fuzzy cost of meeting requirement from j^{th} site to i^{th} ration shop.

$\widetilde{t}_{ij} = (t_{ij1}, t_{ij2}, t_{ij3}, t_{ij4})$: fuzzy time of meeting requirement from j^{th} site to i^{th} ration shop.

$\widetilde{b}_j = (b_{j1}, b_{j2}, b_{j3}, b_{j4})$: fuzzy setup cost of setting up of a warehouse at the j^{th} site.

$\widetilde{B} = (B_1, B_2, B_3, B_4)$: total fuzzy budgetary amount allocated for setting up of warehouses.

$\widetilde{C}$: total fuzzy cost of meeting requirements.

$\widetilde{T}$: maximum fuzzy time of meeting requirements.

Table 2.11: Tabular representation of fuzzy warehouse selection problem

Warehouse sites (j) → Ration shops (i) ↓	1	$\cdots$	j	$\cdots$	N
1	$\widetilde{c}_{11}$ $\widetilde{t}_{11}$	$\cdots$	$\widetilde{c}_{1j}$ $\widetilde{t}_{1j}$	$\cdots$	$\widetilde{c}_{1N}$ $\widetilde{t}_{1N}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\widetilde{c}_{i1}$ $\widetilde{t}_{i1}$	$\cdots$	$\widetilde{c}_{ij}$ $\widetilde{t}_{ij}$	$\cdots$	$\widetilde{c}_{iN}$ $\widetilde{t}_{iN}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	$\widetilde{c}_{M1}$ $\widetilde{t}_{M1}$	$\cdots$	$\widetilde{c}_{Mj}$ $\widetilde{t}_{Mj}$	$\cdots$	$\widetilde{c}_{MN}$ $\widetilde{t}_{MN}$
Setup cost ($\widetilde{b}_j$)	$\widetilde{b}_1$	$\cdots$	$\widetilde{b}_j$	$\cdots$	$\widetilde{b}_N$
k			$\widetilde{B}$		

2.5.1 Proposed efficient solutions of warehouse selection problems in fuzzy environment

In this section, the definitions of 1^{st}, 2^{nd} and subsequent efficient solutions of warehouse selection problems in fuzzy environment are proposed [124].

(i) A solution (X^1, Y^1) is called the 1^{st} efficient solution in fuzzy environment if there exists no solution (X, Y) of the problem $(P_{2.4})$ satisfying the conditions (a) $\widetilde{C}(X) \preceq \widetilde{C}(X^1)$ and (b) $\widetilde{T}(X) \preceq \widetilde{T}(X^1)$ with strict inequality holding in at least one of the conditions out of (a) and (b).

(ii) A solution (X^2, Y^2) is called the 2^{nd} efficient solution in fuzzy environment if no efficient solution (X, Y) of the problem exists satisfying the conditions (a) $\widetilde{C}(X^1) \prec \widetilde{C}(X) \prec \widetilde{C}(X^2)$ and (b) $\widetilde{T}(X^1) \succ \widetilde{T}(X) \succ \widetilde{T}(X^2)$.

(iii) 3^{rd} and the subsequent efficient solutions in fuzzy environment are found in the same way as is found for the 2^{nd} efficient solution.

2.5.2 Proposed method for solving warehouse selection problems in fuzzy environment

Dinagar and Palanivel [56] claimed that there is no method in the literature for solving TMT problems in fuzzy environment and proposed a method, with the help of an existing method for solving TMT problems in crisp environment for the same. Similarly, to the best of our knowledge, there is no method in the literature for solving warehouse selection problems in fuzzy environment; So, in the same direction, in this chapter with the help of an existing method for solving warehouse selection problems in crisp environment, a new method is proposed for solving warehouse selection problems in fuzzy environment.

The steps of proposed method are as follows:

Step 1 Find $\widetilde{a} = Minimum\{\sum_{i=1}^{M} \widetilde{c}_{ij} : j = 1, 2, \ldots, N\}$ and $\widetilde{T}_j = Maximum\{\widetilde{t}_{ij} : i = 1, 2, \ldots, M\}$ for $j = 1, 2, \ldots, N$.

Case (i) If $\widetilde{a}$ occurs corresponding to unique value of j, i.e., $j = t$ where $t \in \{1, 2, \ldots, N\}$, then t is the first potential site for the first warehouse.

Case (ii) If $\widetilde{a}$ occurs corresponding to more than one values of j, i.e., $j = p_1, p_2, \ldots, p_m$, then find $Minimum\{\widetilde{T}_{p_1}, \widetilde{T}_{p_2}, \ldots, \widetilde{T}_{p_m}\}$.

Case (a) If minimum is unique and it is corresponding to p_l, then p_l is first potential site for first warehouse.

Case (b) If minimum occurs corresponding to more than one values of j i.e., $j = p_{s_1}, p_{s_2}, \ldots, p_{s_q}$, then choose any j from $p_{s_1}, p_{s_2}, \ldots, p_{s_q}$ for the first potential site for the first warehouse.

Step 2 Let the first potential site selected for the first warehouse be q_1 where $q_1 \in \{1, 2, \ldots, N\}$. Now for selecting the next warehouse site q_2 where $q_2 \neq q_1$, construct Table 2.12 as:

Table 2.12: Tabular representation of warehouse selection problem

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	$(q_1, 1)$	$\cdots$	(q_1, j)	$\cdots$	(q_1, N)
1	$\widetilde{c}^1_{11}$ $\widetilde{t}^1_{11}$	$\cdots$	$\widetilde{c}^1_{1j}$ $\widetilde{t}^1_{1j}$	$\cdots$	$\widetilde{c}^1_{1N}$ $\widetilde{t}^1_{1N}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\widetilde{c}^1_{i1}$ $\widetilde{t}^1_{i1}$	$\cdots$	$\widetilde{c}^1_{ij}$ $\widetilde{t}^1_{ij}$	$\cdots$	$\widetilde{c}^1_{iN}$ $\widetilde{t}^1_{iN}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	$\widetilde{c}^1_{M1}$ $\widetilde{t}^1_{M1}$	$\cdots$	$\widetilde{c}^1_{Mj}$ $\widetilde{t}^1_{Mj}$	$\cdots$	$\widetilde{c}^1_{MN}$ $\widetilde{t}^1_{MN}$
k	$\widetilde{B}$				

Let the fuzzy cost and fuzzy time of $(i, j)^{th}$ cell, $j \neq q_1$ in new table be $\widetilde{c}^1_{ij}$ and $\widetilde{t}^1_{ij}$ where $\widetilde{c}^1_{ij} = Minimum\{\widetilde{c}_{iq_1}, \widetilde{c}_{ij}\}$ and $\widetilde{t}^1_{ij} = \begin{cases} \widetilde{t}_{ij}, & \text{if } Minimum\{\widetilde{c}_{iq_1}, \widetilde{c}_{ij}\} = \widetilde{c}_{ij} \\ \widetilde{t}_{iq_1}, & \text{if } Minimum\{\widetilde{c}_{iq_1}, \widetilde{c}_{ij}\} = \widetilde{c}_{iq_1} \end{cases}$
The newly constructed Table 2.12 is same as Table 2.11 except it has $N - 1$ columns. Now apply Step 1 and find new combination of warehouses say (q_1, q_2) where $q_2 \in \{1, 2, \ldots, N\}$ and $q_2 \neq q_1$.

Step 3 Let the pair (q_1, q_2) represents the two potential sites for the two warehouses. Similarly, find k-potential sites for k-warehouses by applying Step 2. Let the combination of k-potential sites for k-warehouses be $(q_1, q_2, \ldots, q_k)$. Selection of sites using add rule, in the way explained above, yields the 0^{th} iterative solution (X^1_0, Y^1_0) for the 1^{st} efficient solution (X^1, Y^1) and is also the incumbent solution for it. The incumbent solution for the 1^{st} efficient solution (X^1, Y^1) is the best solution obtained so far. $\widetilde{C}(X_0^1)$ and $\widetilde{T}(X_0^1)$ provides the total fuzzy cost and maximum fuzzy time of 0^{th} iterative solution.

Step 4 Invoke drop and add rules with tabu search for further improving the solution. Drop the site which was first selected i.e., q_1. The site dropped recently is included in the tabu list for add which will be prohibited from being selected in the next 1^{st} iterative solution (X_1^1, Y_1^1) and $k-1$, the most recently added warehouses i.e., $(q_2, q_3, \ldots, q_k)$ appearing in the order of their selection in the 0^{th} iterative solution are included in the tabu list for drop. Find all the entries of the column corresponding to combination $(q_2, q_3, \ldots, q_k)$ using Steps 1 and 2. Then, add the new site by replacing q_1 by $(q_2, q_3, \ldots, q_k)$ in Step 2. Suppose the new combination of k-potential sites for the k-warehouses are $(q_2, q_3, \ldots, q_k, q_h)$ where $q_h \neq q_1, q_2, \ldots, q_k$. Selection of sites $(q_2, q_3, \ldots, q_k, q_h)$ in the way explained above yields the 1^{st} iterative solution (X_1^1, Y_1^1) for the 1^{st} efficient solution (X^1, Y^1). Find the total fuzzy cost $\widetilde{C}(X_1^1)$ and fuzzy time $\widetilde{T}(X_1^1)$ for $(q_2, q_3, \ldots, q_k, q_h)$. If $\widetilde{C}(X_1^1) \prec \widetilde{C}(X_0^1)$, then the current iterative solution will be treated as the incumbent solution for the next iteration, otherwise there will be no change in incumbent solution.

Step 5 The 2^{nd} iterative solution (X_2^1, Y_2^1) is obtained in similar manner as explained in Step 4 by changing the both tabu lists. The tabu list for the drop rule used for obtaining 2^{nd} iterative solution (X_2^1, Y_2^1) will now include the $k-1$ most recent selected sites included in the 1^{st} iterative solution (X_1^1, Y_1^1) i.e., $(q_3, \ldots, q_k, q_h)$ appearing in the order of their selection. Similarly, the tabu list for the add rule used for obtaining the 2^{nd} iterative solution (X_2^1, Y_2^1) will now include the 1^{st} site selected in the 1^{st} iterative solution (X_1^1, Y_1^1) i.e., q_2. If the 2^{nd} iterative solution (X_2^1, Y_2^1) is better than the incumbent solution, then the current iterative solution will be treated as incumbent solution for the next iteration otherwise the incumbent solution does not change. The 3^{rd} iterative solution (X_3^1, Y_3^1) and subsequent ones will be obtained in the same way as done in obtaining the 1^{st} and 2^{nd} iterative solutions. Change the incumbent solution to the current iterative solution if it is better than the incumbent solution. This process of obtaining newer iterative solutions is terminated when either an already obtained iterative solution is revisited or there are no sites left that will serve to improve the solution. When this happens, then the most recent incumbent solution provides the 1^{st} efficient solution (X^1, Y^1) of the formulated problem.

Step 6 To find the 2^{nd} efficient solution (X^2, Y^2) of the formulated problem, replace $\widetilde{t}_{ij} \succeq \widetilde{T}(X^1)$ by an arbitrary large number fuzzy number (L, L, L, L) and repeat Steps 1 to 5.

Step 7 The incumbent solution at the end of iterative process yields 2^{nd} efficient solution (X^2, Y^2) of formulated problem. 3^{rd} and the subsequent solutions are obtained in the same way as is done to obtain the 2^{nd} efficient solution. The process of obtaining the efficient solutions terminates when it is no longer possible to obtain a new efficient solution with lesser duration.

2.5.3 Illustrative example

In this section, fuzzy warehouse selection problem, chosen in Example 2.2, is solved to illustrate the proposed method.

Example 2.2 Solve the fuzzy warehouse selection problem, represented by Table 2.13, to find all the fuzzy efficient solutions.

Table 2.13: Tabular representation of numerical problem

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	1	2	3	4	5	6	7
1	$(36, 38, 42, 44)$ $(5, 8, 9, 10)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(44, 49, 53, 54)$ $(9, 10, 11, 14)$	$(117, 118, 121, 124)$ $(3, 6, 7, 8)$	$(175, 177, 183, 185)$ $(8, 9, 11, 12)$	$(165, 167, 169, 179)$ $(6, 8, 9, 13)$	$(143, 145, 155, 157)$ $(5, 6, 8, 9)$
2	$(68, 69, 70, 73)$ $(3, 6, 7, 8)$	$(73, 80, 82, 85)$ $(3, 6, 7, 8)$	$(128, 129, 130, 133)$ $(6, 8, 9, 13)$	$(128, 129, 130, 133)$ $(5, 6, 8, 9)$	$(165, 167, 169, 179)$ $(5, 8, 9, 10)$	$(133, 136, 144, 147)$ $(8, 9, 11, 12)$	$(11, 15, 25, 29)$ $(9, 10, 11, 14)$
3	$(73, 80, 82, 85)$ $(5, 8, 9, 10)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(186, 190, 205, 209)$ $(3, 4, 5, 8)$	$(68, 69, 70, 73)$ $(5, 8, 9, 10)$	$(133, 136, 144, 147)$ $(9, 10, 11, 14)$	$(128, 129, 130, 133)$ $(9, 10, 14, 15)$	$(56, 58, 62, 64)$ $(8, 9, 11, 12)$
4	$(56, 58, 62, 64)$ $(9, 10, 11, 14)$	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(68, 69, 70, 73)$ $(8, 12, 14, 18)$	$(73, 80, 82, 85)$ $(9, 10, 11, 14)$	$(28, 29, 31, 32)$ $(6, 8, 9, 13)$	$(156, 159, 160, 165)$ $(5, 8, 9, 10)$	$(199, 203, 217, 221)$ $(3, 6, 7, 8)$
5	$(81, 89, 93, 97)$ $(8, 9, 11, 12)$	$(89, 98, 104, 109)$ $(10, 12, 13, 17)$	$(16, 18, 22, 24)$ $(5, 8, 9, 10)$	$(56, 58, 62, 64)$ $(9, 10, 14, 15)$	$(36, 38, 42, 44)$ $(3, 6\ 7, 8)$	$(44, 49, 53, 54)$ $(2, 6, 7, 9)$	$(197, 205, 230, 248)$ $(8, 14, 16, 18)$
Setup cost $(\widetilde{b}_j)$	$(89, 98, 104, 109)$	$(275, 288, 309, 328)$	$(680, 690, 700, 730)$	$(712, 784, 832, 872)$	$(380, 392, 410, 418)$	$(178, 196, 208, 218)$	$(445, 490, 520, 545)$
$k = 3$	$\widetilde{B} = (1380, 1390, 1410, 1420)$						

Solution: The total fuzzy cost $(\sum_{i=1}^{5} \widetilde{c}_{ij})$ and fuzzy time $\widetilde{T}_j$, obtained by Step 1, is shown in Table 2.14.

Table 2.14: Total fuzzy cost and fuzzy time for 1^{st} warehouse site

Warehouse sites $(j) \rightarrow$	1	2	3	4	5	6	7
Total fuzzy cost $(\sum_{i=1}^{5} \widetilde{c}_{ij})$	$(314, 334, 349, 363)$	$(209, 231, 253, 267)$	$(442, 455, 480, 503)$	$(442, 454, 465, 479)$	$(537, 547, 569, 587)$	$(626, 640, 656, 678)$	$(606, 626, 689, 719)$
$\widetilde{T}_j$	$(9, 10, 11, 14)$	$(10, 12, 13, 17)$	$(8, 12, 14, 18)$	$(9, 10, 14, 15)$	$(9, 10, 11, 14)$	$(9, 10, 14, 15)$	$(8, 14, 16, 18)$

Since, the minimum value of total fuzzy cost $(\widetilde{a})$ is corresponding to $j = 2$; So, $q_1 = 2$ is first potential site for first warehouse site. Use Step 2 for selecting second warehouse site. The results are depicted in Table 2.15 below.

Table 2.15: Total fuzzy cost and fuzzy time for two warehouse sites

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	$(2, 1)$	$(2, 3)$	$(2, 4)$	$(2, 5)$	$(2, 6)$	$(2, 7)$
1	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$
2	$(68, 69, 70, 73)$ $(3, 6, 7, 8)$	$(73, 80, 82, 85)$ $(3, 6, 7, 8)$	$(73, 80, 82, 85)$ $(3, 6, 7, 8)$	$(73, 80, 82, 85)$ $(3, 6, 7, 8)$	$(73, 80, 82, 85)$ $(3, 6, 7, 8)$	$(11, 15, 25, 29)$ $(9, 10, 11, 14)$
3	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$
4	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$
5	$(81, 89, 93, 97)$ $(8, 9, 11, 12)$	$(16, 18, 22, 24)$ $(5, 8, 9, 10)$	$(56, 58, 62, 64)$ $(9, 10, 14, 15)$	$(36, 38, 42, 44)$ $(3, 6\ 7, 8)$	$(44, 49, 53, 54)$ $(2, 6, 7, 9)$	$(89, 98\ 104, 109)$ $(10, 12, 13, 17)$
Total fuzzy cost $(\sum_{i=1}^{5} \widetilde{c}_{ij})$	$(196, 211, 230, 243)$	$(136, 151, 171, 182)$	$(176, 191, 211, 222)$	$(156, 171, 191, 202)$	$(164, 182, 202, 212)$	$(147, 166, 196, 211)$
$\widetilde{T}_j$	$(8, 9, 11, 12)$	$(6, 8, 9, 13)$	$(9, 10, 14, 15)$	$(6, 8, 9, 13)$	$(6, 8, 9, 13)$	$(10, 12, 13, 17)$
$k = 3$	$\widetilde{B} = (1380, 1390, 1410, 1420)$					

Since, the minimum value of total fuzzy cost ($\widetilde{a}$) is corresponding to $(2,3)$; So, the pair $(2,3)$ represents the two potential sites for two warehouse sites. For next warehouse site, again, apply Step 2 and thus, Table 2.16 is obtained.

Table 2.16: Total fuzzy cost and fuzzy time for three warehouse sites

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	$(2,3,1)$	$(2,3,5)$	$(2,3,6)$
1	$(28,29,31,32)$ $(1,2,2,3)$	$(28,29,31,32)$ $(1,2,2,3)$	$(28,29,31,32)$ $(1,2,2,3)$
2	$(68,69,70,73)$ $(3,6,7,8)$	$(73,80,82,85)$ $(3,6,7,8)$	$(73,80,82,85)$ $(3,6,7,8)$
3	$(11,15,25,29)$ $(6,8,9,13)$	$(11,15,25,29)$ $(6,8,9,13)$	$(11,15,25,29)$ $(6,8,9,13)$
4	$(8,9,11,12)$ $(3,6,7,8)$	$(8,9,11,12)$ $(3,6,7,8)$	$(8,9,11,12)$ $(3,6,7,8)$
5	$(16,18,22,24)$ $(5,8,9,10)$	$(16,18,22,24)$ $(5,8,9,10)$	$(16,18,22,24)$ $(5,8,9,10)$
Total fuzzy cost $(\sum_{i=1}^{5} \widetilde{c}_{ij})$	$(131,140,159,170)$	$(136,151,171,182)$	$(136,151,171,182)$
$\widetilde{T}_j$	$(6,8,9,13)$	$(6,8,9,13)$	$(6,8,9,13)$
$k=3$	$\widetilde{B}=(1380,1390,1410,1420)$		

Combination of sites $(2,3,4)$ and $(2,3,7)$ are not chosen in Table 2.16 as they do not satisfy the budgetary constraint $\sum_{j=1}^{N} \widetilde{b}_j y_j \preceq \widetilde{B}$. Since the minimum value of total fuzzy cost $(\widetilde{a})$ is corresponding to $(2,3,1)$, so the combination $(2,3,1)$ represent the three potential sites for three warehouse sites. Hence the current and incumbent solutions of 0^{th} iteration are shown in Table 2.17.

Table 2.17: Current and incumbent solutions of 0^{th} iteration for 1^{st} efficient solution

Current Solution (X_0^1, Y_0^1)			Incumbent Solution (X_0^1, Y_0^1)		
Variables at level 1	$\widetilde{C}(X_0^1)$	$\widetilde{T}(X_0^1)$	Variables at level 1	$\widetilde{C}(X_0^1)$	$\widetilde{T}(X_0^1)$
$y_2, y_3, y_1, x_{12}, x_{21}, x_{32}, x_{42}, x_{53}$	$(131, 140, 159, 170)$	$(6, 8, 9, 13)$	$y_2, y_3, y_1, x_{12}, x_{21}, x_{32}, x_{42}, x_{53}$	$(131, 140, 159, 170)$	$(6, 8, 9, 13)$

Hereafter, use tabu search method to improve the solution with respect to first objective function.

Iteration No. 1. Since, the combination of sites selected in 0^{th} iterative solution for setting up of warehouses is $(2,3,1)$; Thus, the resultant two tabu lists derived are given in Table 2.18.

Table 2.18: Two tabu lists in 1^{st} iteration for 1^{st} efficient solution

Tabu list for drop	Tabu list for add
$\{y_3, y_1\}$	$\{y_2\}$

Using Step 2, add the new site to the combination of sites $(3,1)$. Hence the solutions at this stage are:

Table 2.19: Total fuzzy cost and fuzzy time for 1^{st} iteration

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	$(3,1,5)$	$(3,1,6)$	$(3,1,7)$
1	$(36,38, 42,44)$ $(5,8, 9,10)$	$(36,38, 42,44)$ $(5,8, 9,10)$	$(36,38, 42,44)$ $(5,8, 9,10)$
2	$(68,69, 70,73)$ $(3,6, 7,8)$	$(68,69, 70,73)$ $(3,6, 7,8)$	$(11,15, 25,29)$ $(9,10, 11,14)$
3	$(73,80, 82,85)$ $(5,8, 9,10)$	$(73,80, 82,85)$ $(5,8, 9,10)$	$(56,58, 62,64)$ $(8,9, 11,12)$
4	$(28,29, 31,32)$ $(6,8, 9,13)$	$(56,58, 62,64)$ $(9,10, 11,14)$	$(56,58, 62,64)$ $(9,10, 11,14)$
5	$(16,18, 22,24)$ $(5,8, 9,10)$	$(16,18, 22,24)$ $(5,8, 9,10)$	$(16,18, 22,24)$ $(5,8, 9,10)$
Total fuzzy cost $(\sum_{i=1}^{5} \widetilde{c}_{ij})$	$(221,234, 247,258)$	$(249,263, 278,290)$	$(175,187, 213,225)$
$\widetilde{T}_j$	$(6,8, 9,13)$	$(9,10, 11,14)$	$(9,10, 11,14)$
$k=3$	$\widetilde{B}=(1380,1390,1410,1420)$		

Combination of sites $(3,1,4)$ is not chosen in Table 2.19 as it does not satisfy budgetary constraint $\sum_{j=1}^{N} \widetilde{b}_j y_j \preceq \widetilde{B}$. The value of $\widetilde{a}$ is corresponding to $(3,1,7)$. The total fuzzy cost of 1^{st} iteration $\widetilde{C}(X_1^1) = (175,187,213,225)$ is greater than total fuzzy cost $\widetilde{C}(X_0^1)$ in incumbent solution. So, the incumbent solution will remain the same. The current and incumbent solutions are given in Table 2.20.

Table 2.20: Current and incumbent solutions during 1^{st} iteration for 1^{st} efficient solution

Current Solution (X_1^1, Y_1^1)			Incumbent Solution (X_0^1, Y_0^1)		
Variables at level 1	$\widetilde{C}(X_1^1)$	$\widetilde{T}(X_1^1)$	Variables at level 1	$\widetilde{C}(X_0^1)$	$\widetilde{T}(X_0^1)$
$y_3, y_1, y_7,$ $x_{11}, x_{27}, x_{37},$ x_{41}, x_{53}	$(175,187$ $213,225)$	$(9,10,$ $11,14)$	$y_2, y_3, y_1,$ $x_{12}, x_{21}, x_{32},$ x_{42}, x_{53}	$(131,140$ $159,170)$	$(6,8,$ $9,13)$

Iteration No. 2. Since, the combination of sites selected in 1^{st} iterative solution for setting up of warehouses is $(3,1,7)$; Thus, the resultant two tabu lists derived are given in Table 2.21.

Table 2.21: Two tabu lists in 2^{nd} iteration of 1^{st} efficient solution

Tabu list for drop	Tabu list for add
$\{y_1, y_7\}$	$\{y_3\}$

Similarly proceeding in the same manner as explained above, we have Table 2.22 showing all the iterative solutions for 1^{st} efficient solution. The iterative solutions (X_6^1, Y_6^1) and (X_2^1, Y_2^1) are identical i.e., the 6^{th} iterative solution revisits the 2^{nd} iterative solution, indicating that the computation for obtaining the 1^{st} efficient solution should be stopped. Note that (X_3^1, Y_3^1), being the most recent incumbent solution and also, is the best solution among all iterative solutions, provides the 1^{st} efficient solution (X^1, Y^1) of the numerical problem.

Table 2.22: The iterative solutions $(X_r^1, Y_r^1)(r = 0, 1, \ldots, 6)$ for 1^{st} non-dominated solution

Iterative solutions	Variables x_{ij}'s and y_j's in the order of appearance	Total fuzzy cost $\widetilde{C}(X_r^1)$	Fuzzy time $\widetilde{T}(X_r^1)$	Tabu list for drop rule	Tabu list for add rule	Incumbent solution
(X_0^1, Y_0^1) 0^{th} iterative solution	$x_{12}, x_{21}, x_{32},$ x_{42}, x_{53} y_2, y_3, y_1	$(131, 140,$ $159, 170)$	$(6, 8,$ $9, 13)$	$\{y_3, y_1\}$	$\{y_2\}$	(X_0^1, Y_0^1)
(X_1^1, Y_1^1) 1^{st} iterative solution	$x_{11}, x_{27}, x_{37},$ x_{41}, x_{53} y_3, y_1, y_7	$(175, 187,$ $213, 225)$	$(9, 10,$ $11, 14)$	$\{y_1, y_7\}$	$\{y_3\}$	(X_0^1, Y_0^1)
(X_2^1, Y_2^1) 2^{nd} iterative solution	$x_{12}, x_{27}, x_{32},$ x_{42}, x_{51} y_1, y_7, y_2	$(139, 157,$ $185, 199)$	$(9, 10,$ $11, 14)$	$\{y_7, y_2\}$	$\{y_1\}$	(X_0^1, Y_0^1)
(X_3^1, Y_3^1) 3^{rd} iterative solution	$x_{12}, x_{27}, x_{32},$ x_{42}, x_{55} y_7, y_2, y_5	$(94, 106,$ $134, 146)$	$(9, 10,$ $11, 14)$	$\{y_2, y_5\}$	$\{y_7\}$	(X_3^1, Y_3^1)
(X_4^1, Y_4^1) 4^{th} iterative solution	$x_{12}, x_{21}, x_{32},$ x_{42}, x_{55} y_2, y_5, y_1	$(151, 160,$ $179, 190)$	$(6, 8,$ $9, 13)$	$\{y_5, y_1\}$	$\{y_2\}$	(X_3^1, Y_3^1)
(X_5^1, Y_5^1) 5^{th} iterative solution	$x_{11}, x_{27}, x_{37},$ x_{45}, x_{55} y_5, y_1, y_7	$(167, 178,$ $202, 213)$	$(9, 10,$ $11, 14)$	$\{y_1, y_7\}$	$\{y_5\}$	(X_3^1, Y_3^1)
(X_6^1, Y_6^1) 6^{th} iterative solution	$x_{12}, x_{27}, x_{32},$ x_{42}, x_{51} y_1, y_7, y_2	$(139, 157,$ $185, 199)$	$(9, 10,$ $11, 14)$	$\{y_7, y_2\}$	$\{y_1\}$	(X_3^1, Y_3^1)

2^{nd} **efficient solution** Since, rank of $\widetilde{T}(X^1)$ is 11; So, replace all $\widetilde{t}_{ij}$'s by (L, L, L, L) (large positive fuzzy number) whose rank is ≥ 11 in Table 2.13.

Table 2.23: Tabular representation of the numerical problem for 2^{nd} non-dominated solution

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	1	2	3	4	5	6	7
1	$(36, 38, 42, 44)$ $(5, 8, 9, 10)$	$(28, 29, 31, 32)$ $(1, 2, 2, 3)$	$(44, 49, 53, 54)$ (L, L, L, L)	$(117, 118, 121, 124)$ $(3, 6, 7, 8)$	$(175, 177, 183, 185)$ $(8, 9, 11, 12)$	$(165, 167, 169, 179)$ $(6, 8, 9, 13)$	$(143, 145, 153, 157)$ $(5, 6, 8, 9)$
2	$(68, 69, 70, 73)$ $(3, 6, 7, 8)$	$(73, 80, 82, 85)$ $(3, 6, 7, 8)$	$(128, 129, 130, 133)$ $(6, 8, 9, 13)$	$(128, 129, 130, 133)$ $(5, 6, 8, 9)$	$(165, 167, 169, 179)$ $(5, 8, 9, 10)$	$(133, 136, 144, 147)$ $(8, 9, 11, 12)$	$(11, 15, 25, 29)$ (L, L, L, L)
3	$(73, 80, 82, 85)$ $(5, 8, 9, 10)$	$(11, 15, 25, 29)$ $(6, 8, 9, 13)$	$(186, 190, 205, 209)$ $(3, 4, 5, 8)$	$(68, 69, 70, 73)$ $(5, 8, 9, 10)$	$(133, 136, 144, 147)$ (L, L, L, L)	$(128, 129, 130, 133)$ (L, L, L, L)	$(56, 58, 62, 64)$ $(8, 9, 11, 12)$
4	$(56, 58, 62, 64)$ (L, L, L, L)	$(8, 9, 11, 12)$ $(3, 6, 7, 8)$	$(68, 69, 70, 73)$ (L, L, L, L)	$(73, 80, 82, 85)$ (L, L, L, L)	$(28, 29, 31, 32)$ $(6, 8, 9, 13)$	$(156, 159, 160, 165)$ $(5, 8, 9, 10)$	$(199, 203, 217, 221)$ $(3, 6, 7, 8)$
5	$(81, 89, 93, 97)$ $(8, 9, 11, 12)$	$(89, 98, 104, 109)$ (L, L, L, L)	$(16, 18, 22, 24)$ $(5, 8, 9, 10)$	$(56, 58, 62, 64)$ (L, L, L, L)	$(36, 38, 42, 44)$ $(3, 6\ 7, 8)$	$(44, 49, 53, 54)$ $(2, 6, 7, 9)$	$(197, 205, 230, 248)$ (L, L, L, L)
Setup cost $(\widetilde{b}_j)$	$(89, 98, 104, 109)$	$(275, 288, 309, 328)$	$(680, 690, 700, 730)$	$(712, 784, 832, 872)$	$(380, 392, 410, 418)$	$(178, 196, 208, 218)$	$(445, 490, 520, 545)$

Proceeding in the same manner as in for 1^{st} efficient solution, we will get 2^{nd} efficient solution which is shown in Table 2.24.

Table 2.24: 2^{nd} efficient solution of the numerical problem

2^{nd} Efficient solution	Variables at level 1	Total fuzzy cost of meeting requirement $(\widetilde{C}(X^2))$	Fuzzy time of meeting requirement $(\widetilde{T}(X^2))$
(X^2, Y^2)	$y_2, y_3, y_1,$ $x_{12}, x_{21}, x_{32},$ x_{42}, x_{53}	$(131, 140, 159, 170)$	$(6, 8, 9, 13)$

Similarly, solve 3^{rd} and subsequent efficient solutions, as solved for 2^{nd} efficient solution. It is found that there is no improvement in fuzzy time with the 5^{th} efficient solution. So, the procedure of finding efficient solutions stops here. The four fuzzy efficient solutions of the numerical example obtained are given in following Table 2.25.

Table 2.25: Efficient solutions of numerical problem

Efficient solutions	Variables at level 1	Total fuzzy cost of meeting requirement	Fuzzy time of meeting requirement
(X^1, Y^1)	$y_7, y_2, y_5,$ $x_{12}, x_{27}, x_{32},$ x_{42}, x_{55}	$(94, 106,$ $134, 146)$	$(9, 10,$ $11, 14)$
(X^2, Y^2)	$y_2, y_3, y_1,$ $x_{12}, x_{21}, x_{32},$ x_{42}, x_{53}	$(131, 140,$ $159, 170)$	$(6, 8,$ $9, 13)$
(X^3, Y^3)	$y_2, y_3, y_1,$ $x_{12}, x_{21}, x_{31},$ x_{42}, x_{53}	$(193, 205,$ $216, 226)$	$(5, 8,$ $9, 10)$
(X^4, Y^4)	$y_2, y_5, y_3,$ $x_{12}, x_{22}, x_{33},$ x_{42}, x_{55}	$(331, 346,$ $371, 392)$	$(3, 6,$ $7, 8)$

2.6 Conclusion

On the basis of presented study, it can be concluded that there was no method in the literature for solving warehouse selection problem in fuzzy environment and in this chapter, a method is proposed for the same.

Chapter 3

Efficient Method For Time Minimizing Transportation Problems And Alternative Method For Warehouse Selection Problems In Fuzzy Environment

In this chapter, the shortcomings of the existing method [56] are pointed out and a new method is proposed to resolve these shortcomings. Also, an alternative method is proposed for solving warehouse selection problems in fuzzy environment and it is shown that it is much easy to apply the proposed method in this chapter as compared to the method proposed in previous chapter.

3.1 Shortcomings of the existing method

On solving the existing fuzzy TMT problem [56], presented in Example 2.1, only the unique fuzzy optimal solution $\widetilde{x}_{11} = (-19, 1, 11, 27)$, $\widetilde{x}_{14} = (-4, 6, 12, 26)$, $\widetilde{x}_{21} = (-5, 0, 3, 10)$, $\widetilde{x}_{22} = (1, 6, 9, 16)$, $\widetilde{x}_{23} = (12, 14, 16, 18)$, $\widetilde{x}_{31} = (3, 4, 5, 8)$ is obtained. However, it can be easily verified that all the trapezoidal fuzzy numbers $\widetilde{x}_{11} = (x_{111}, x_{112}, x_{113}, x_{114})$, $\widetilde{x}_{14} = (x_{141}, x_{142}, x_{143}, x_{144})$, $\widetilde{x}_{21} = (x_{211}, x_{212}, x_{213}, x_{214})$,

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$\widetilde{x}_{22} = (x_{221}, x_{222}, x_{223}, x_{224})$, $\widetilde{x}_{23} = (x_{231}, x_{232}, x_{233}, x_{234})$ and $\widetilde{x}_{31} = (x_{311}, x_{312}, x_{313}, x_{314})$ which will satisfy the conditions $\frac{(x_{111}+x_{112}+x_{113}+x_{114})}{4} = 5$, $\frac{(x_{141}+x_{142}+x_{143}+x_{144})}{4} = 10$, $\frac{(x_{211}+x_{212}+x_{213}+x_{214})}{4} = 2$, $\frac{(x_{221}+x_{222}+x_{223}+x_{224})}{4} = 8$, $\frac{(x_{231}+x_{232}+x_{233}+x_{234})}{4} = 15$ and $\frac{(x_{311}+x_{312}+x_{313}+x_{314})}{4} = 5$ are the fuzzy optimal solutions of the same problem. Thus, it is obvious that it is not possible to find all the fuzzy optimal solutions of fuzzy TMT problems by using the existing method [56].

3.2 Proposed method for solving time minimizing transportation problems in fuzzy environment

In this section, to overcome the shortcomings of the existing method [56] pointed out in Section 3.1, a new method is proposed for solving TMT problems in fuzzy environment.

The steps of proposed method are as follows:

Step 1 Represent the chosen TMT problem in fuzzy environment into tabular form as shown in Table 2.2.

Step 2 Find the values of $\Re(\widetilde{t}_{ij})$, $\Re(\widetilde{a}_i)$ and $\Re(\widetilde{b}_j)$ by using the ranking formula discussed in Section 2.1.1.

Step 3 Construct a Table 3.1 by replacing the fuzzy parameters $\widetilde{t}_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ by $\Re(\widetilde{t}_{ij})$, $\Re(\widetilde{a}_i)$ and $\Re(\widetilde{b}_j)$ respectively.

Table 3.1: Crisp TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	$\Re(\widetilde{t}_{11})$	$\cdots$	$\Re(\widetilde{t}_{1j})$	$\cdots$	$\Re(\widetilde{t}_{1q})$	$\Re(\widetilde{a}_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	$\Re(\widetilde{t}_{i1})$	$\cdots$	$\Re(\widetilde{t}_{ij})$	$\cdots$	$\Re(\widetilde{t}_{iq})$	$\Re(\widetilde{a}_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	$\Re(\widetilde{t}_{p1})$	$\cdots$	$\Re(\widetilde{t}_{pj})$	$\cdots$	$\Re(\widetilde{t}_{pq})$	$\Re(\widetilde{a}_p)$
Demand	$\Re(\widetilde{b}_1)$	$\cdots$	$\Re(\widetilde{b}_j)$	$\cdots$	$\Re(\widetilde{b}_q)$	$\sum_{i=1}^{p}\Re(\widetilde{a}_i)=\sum_{j=1}^{q}\Re(\widetilde{b}_j)$

Step 4 Use the existing method, discussed in Section 2.2.2, to find the optimal solution and optimal time of transportation of TMT problem in crisp environment.

Step 5 Let on solving the TMT problem in crisp environment, obtained in Step 4, the value of $(i,j)^{th}$ decision variable x_{ij} in the optimal solution be A. Then, all the trapezoidal fuzzy number (x,y,z,w) which satisfy the condition $\frac{(x+y+z+w)}{4} = A$, can be treated as $(i,j)^{th}$ fuzzy decision variable $\widetilde{x}_{ij}$ of fuzzy optimal solution of the chosen TMT problem in fuzzy environment.

Step 6 Let on solving the TMT problem in crisp environment, obtained in Step 4, the optimal time of transportation be $t_{\alpha\beta}$. Then, optimal fuzzy time of transportation of chosen TMT problem in fuzzy environment will be $\widetilde{t}_{\alpha\beta}$.

3.3 Proposed alternative method for solving warehouse selection problem in fuzzy environment

In this section, an alternative method is proposed for solving warehouse selection problems in fuzzy environment.

The steps of proposed method are as follows:

Step 1 Represent the chosen warehouse selection problem in fuzzy environment into tabular form as shown in Table 2.11.

Step 2 Find the values of $\Re(\widetilde{c}_{ij})$, $\Re(\widetilde{t}_{ij})$, $\Re(\widetilde{b}_j)$ and $\Re(\widetilde{B})$ by using the ranking formula discussed in Section 2.1.1.

Step 3 Construct Table 3.2 by replacing the fuzzy parameters $\widetilde{c}_{ij}$, $\widetilde{t}_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ by $\Re(\widetilde{c}_{ij})$, $\Re(\widetilde{t}_{ij})$, $\Re(\widetilde{a}_i)$ and $\Re(\widetilde{b}_j)$ respectively.

Table 3.2: Crisp warehouse selection problem

Warehouse sites $(j) \rightarrow$ Ration shops $(i) \downarrow$	1	$\cdots$	j	$\cdots$	N
1	$\Re(\widetilde{c}_{11})$ $\Re(\widetilde{t}_{11})$	$\cdots$	$\Re(\widetilde{c}_{1j})$ $\Re(\widetilde{t}_{1j})$	$\cdots$	$\Re(\widetilde{c}_{1N})$ $\Re(\widetilde{t}_{1N})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\Re(\widetilde{c}_{i1})$ $\Re(\widetilde{t}_{i1})$	$\cdots$	$\Re(\widetilde{c}_{ij})$ $\Re(\widetilde{t}_{ij})$	$\cdots$	$\Re(\widetilde{c}_{iN})$ $\Re(\widetilde{t}_{iN})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	$\Re(\widetilde{c}_{M1})$ $\Re(\widetilde{t}_{M1})$	$\cdots$	$\Re(\widetilde{c}_{Mj})$ $\Re(\widetilde{t}_{Mj})$	$\cdots$	$\Re(\widetilde{c}_{MN})$ $\Re(\widetilde{t}_{MN})$
Setup cost $(\Re(\widetilde{b}_j))$	$\Re(\widetilde{b}_1)$	$\cdots$	$\Re(\widetilde{b}_j)$	$\cdots$	$\Re(\widetilde{b}_N)$
k			$\Re(\widetilde{B})$		

Step 4 Use the existing method, discussed in Section 2.4.2, to find all the efficient solutions of warehouse selection problem in crisp environment.

Step 5 Let N be the total number of efficient solutions of warehouse selection problem in crisp environment. Consider the k^{th} efficient solution (X^k, Y^k). Let $C(X^k) = \sum_{x_{ij} \in X^k} c_{ij}x_{ij}$ and $T(X^k) = Maximum\{t_{ij} : x_{ij} \in X^k\} = t_{\alpha\beta}$ (say) be the total cost and time of meeting requirements of all the ration shops from their assigned warehouse sites. Then $\widetilde{t}_{\alpha\beta}$ will be fuzzy time of meeting requirements of all the ration shops from their assigned warehouse sites for k^{th}

efficient solution of chosen warehouse selection problem in fuzzy environment and $\widetilde{C}(X^k) = \sum_{x_{ij} \in X^k} \widetilde{c}_{ij} x_{ij}$ will be the total fuzzy cost of meeting requirements of all the ration shops from their assigned warehouses for k^{th} efficient solution of chosen warehouse selection problem in fuzzy environment.

3.4 Advantages of the proposed methods

The main advantage of the proposed methods over the existing method [56] and the method proposed in the previous chapter is that these are much easier to apply as compared to the existing method [56] and the method proposed in the previous chapter. Moreover, on using the method proposed in this chapter for solving TMT problems in fuzzy environment, all the shortcomings of the existing method [56], pointed out in Section 3.1, are resolved.

3.5 Illustrative examples

In this section, to illustrate the proposed methods, the fuzzy TMT problem and the fuzzy warehouse selection problem, chosen in Example 2.1 and Example 2.2, are solved.

3.5.1 Optimal solution of chosen fuzzy time minimizing transportation problem

Using the method, proposed in Section 3.2, the fuzzy optimal solution of TMT problem in fuzzy environment, chosen in Example 2.1, can be obtained as follows:

Step 1 Using Section 2.3.3 of Chapter 2, the chosen TMT problem in fuzzy environment can be represented by Table 2.3.

Step 2 Using the ranking formula, discussed in Section 2.1.1, the values of $\Re(\widetilde{t}_{11})$, $\Re(\widetilde{t}_{12})$, $\Re(\widetilde{t}_{13})$, $\Re(\widetilde{t}_{14})$, $\Re(\widetilde{t}_{21})$, $\Re(\widetilde{t}_{22})$, $\Re(\widetilde{t}_{23})$, $\Re(\widetilde{t}_{24})$, $\Re(\widetilde{t}_{31})$, $\Re(\widetilde{t}_{32})$, $\Re(\widetilde{t}_{33})$, $\Re(\widetilde{t}_{34})$, $\Re(\widetilde{a}_1)$, $\Re(\widetilde{a}_2)$, $\Re(\widetilde{a}_3)$, $\Re(\widetilde{b}_1)$, $\Re(\widetilde{b}_2)$, $\Re(\widetilde{b}_3)$ and $\Re(\widetilde{b}_4)$ are 10, 0, 20, 11, 1, 7, 9, 20, 12, 14, 16, 18, 15, 25, 5, 12, 8, 15, 10 respectively.

Step 3 Using Step 3 of the proposed method, Table 2.3 can be converted into Table 3.3.

Table 3.3: Crisp TMT problem

Destinations→ Sources↓	D_1	D_2	D_3	D_4	a_i
S_1	10	0	20	11	15
S_2	1	7	9	20	25
S_3	12	14	16	18	5
b_j	12	8	15	10	

Step 4 On solving the crisp TMT problem, obtained in Step 3, by using the existing method, discussed in Section 2.2.2, the optimal solution is $x_{11} = 5$, $x_{14} = 10$, $x_{21} = 2$, $x_{22} = 8$, $x_{23} = 15$, $x_{31} = 5$ and the obtained optimal time is $t_{31} = 12$.

Step 5 Since the values of x_{11}, x_{14}, x_{21}, x_{22}, x_{23}, x_{31} are 5, 10, 2, 8, 15 and 5 respectively; So, all the trapezoidal fuzzy numbers $\widetilde{x}_{11} = (x_{111}, x_{112}, x_{113}, x_{114})$, $\widetilde{x}_{14} = (x_{141}, x_{142}, x_{143}, x_{144})$, $\widetilde{x}_{21} = (x_{211}, x_{212}, x_{213}, x_{214})$, $\widetilde{x}_{22} = (x_{221}, x_{222}, x_{223}, x_{224})$, $\widetilde{x}_{23} = (x_{231}, x_{232}, x_{233}, x_{234})$ and $\widetilde{x}_{31} = (x_{311}, x_{312}, x_{313}, x_{314})$ which will satisfy the conditions $\frac{(x_{111}+x_{112}+x_{113}+x_{114})}{4} = 5$, $\frac{(x_{141}+x_{142}+x_{143}+x_{144})}{4} = 10$, $\frac{(x_{211}+x_{212}+x_{213}+x_{214})}{4} = 2$, $\frac{(x_{221}+x_{222}+x_{223}+x_{224})}{4} = 8$, $\frac{(x_{231}+x_{232}+x_{233}+x_{234})}{4} = 15$ and $\frac{(x_{311}+x_{312}+x_{313}+x_{314})}{4} = 5$ can be treated as fuzzy optimal solution of the chosen problem.

Step 6 Since the optimal time of transportation is $t_{31} = 12$; So, $\widetilde{t}_{31} = (9, 11, 13, 15)$ is the optimal fuzzy time of transportation.

3.5.2 Optimal solution of chosen fuzzy warehouse selection problem

Using the method, proposed in Section 3.3 , the optimal solution of fuzzy warehouse selection problem, chosen in Example 2.2 can be obtained as follows:

Step 1 Using Section 2.5.3 of Chapter 2, the chosen fuzzy warehouse selection problem can be represented by Table 2.13.

Step 2 Using the ranking formula, discussed in Section 2.1.1, Table 2.13 can be converted into Table 3.4.

Table 3.4: Crisp warehouse selection problem

Warehouse sites (j) → Ration shops (i) ↓	1	2	3	4	5	6	7
1	40 8	30 2	50 11	120 6	180 10	170 9	150 7
2	70 6	80 6	130 9	130 7	170 8	140 10	20 11
3	80 8	20 9	200 5	70 8	140 11	130 12	60 10
4	60 11	10 6	70 13	80 11	30 9	160 8	210 6
5	90 10	100 13	20 8	60 12	40 6	50 6	220 14
Setup cost (b_j)	100	300	700	800	400	200	500
$k = 3$	$B = 1400$						

Step 3 On solving the crisp warehouse selection problem, obtained in Step 2, by using the existing method, discussed in Section 2.4.2, the obtained efficient solutions are shown in Table 3.5.

Table 3.5: Efficient solutions

Efficient solutions	Variables y_j's and x_{ij}'s	Total cost of meeting requirements (C)	Time of meeting requirements (T)
(X^1, Y^1)	$y_7, y_2, y_5, x_{12}, x_{27}, x_{32}, x_{42}, x_{55}$	$c_{12} + c_{27} + c_{32} + c_{42} + c_{55} = 120$	$t_{27} = 11$
(X^2, Y^2)	$y_2, y_3, y_1, x_{12}, x_{21}, x_{32}, x_{42}, x_{53}$	$c_{12} + c_{21} + c_{32} + c_{42} + c_{53} = 150$	$t_{32} = 9$
(X^3, Y^3)	$y_2, y_3, y_1, x_{12}, x_{21}, x_{31}, x_{42}, x_{53}$	$c_{12} + c_{21} + c_{31} + c_{42} + c_{53} = 210$	$t_{53} = 8$
(X^4, Y^4)	$y_2, y_5, y_3, x_{12}, x_{22}, x_{33}, x_{42}, x_{55}$	$c_{12} + c_{22} + c_{33} + c_{42} + c_{55} = 360$	$t_{55} = 6$

Consider the first efficient solution (X^1, Y^1). Since $C = c_{12}+c_{27}+c_{32}+c_{42}+c_{55}$ and $T = t_{27}$ are the total cost and time of meeting requirements of all the ration shops from their assigned warehouses; So, $\widetilde{C} = \widetilde{c}_{12} \oplus \widetilde{c}_{27} \oplus \widetilde{c}_{32} \oplus \widetilde{c}_{42} \oplus \widetilde{c}_{55} = (94, 106, 134, 146)$ and $\widetilde{t}_{27} = (9, 10, 11, 14)$ are total fuzzy cost and fuzzy time of meeting all the requirements for 1^{st} efficient solution of chosen fuzzy warehouse selection problem. The remaining fuzzy efficient solutions can be obtained in the same manner. All the fuzzy efficient solutions of the chosen problem are shown in Table 3.6.

Table 3.6: Efficient solutions of chosen fuzzy warehouse selection problem

Efficient solutions	Variables y_j's and x_{ij}'s	Total fuzzy cost of meeting requirements ($\widetilde{C}$)	Fuzzy time of meeting requirements ($\widetilde{T}$)
(X^1, Y^1)	$y_7, y_2, y_5, x_{12}, x_{27}, x_{32}, x_{42}, x_{55}$	$\widetilde{c}_{12} \oplus \widetilde{c}_{27} \oplus \widetilde{c}_{32} \oplus \widetilde{c}_{42} \oplus \widetilde{c}_{55} = (94, 106, 134, 146)$	$\widetilde{t}_{27} = (9, 10, 11, 14)$
(X^2, Y^2)	$y_2, y_3, y_1, x_{12}, x_{21}, x_{32}, x_{42}, x_{53}$	$\widetilde{c}_{12} \oplus \widetilde{c}_{21} \oplus \widetilde{c}_{32} \oplus \widetilde{c}_{42} \oplus \widetilde{c}_{53} = (131, 140, 159, 170)$	$\widetilde{t}_{32} = (6, 8, 9, 13)$
(X^3, Y^3)	$y_2, y_3, y_1, x_{12}, x_{21}, x_{31}, x_{42}, x_{53}$	$\widetilde{c}_{12} \oplus \widetilde{c}_{21} \oplus \widetilde{c}_{31} \oplus \widetilde{c}_{42} \oplus \widetilde{c}_{53} = (193, 205, 216, 226)$	$\widetilde{t}_{53} = (5, 8, 9, 10)$
(X^4, Y^4)	$y_2, y_5, y_3, x_{12}, x_{22}, x_{33}, x_{42}, x_{55}$	$\widetilde{c}_{12} \oplus \widetilde{c}_{22} \oplus \widetilde{c}_{33} \oplus \widetilde{c}_{42} \oplus \widetilde{c}_{55} = (331, 346, 371, 392)$	$\widetilde{t}_{55} = (3, 6, 7, 8)$

3.6 Comparative study

The optimal solution of the problems, chosen in Example 2.1 and Example 2.2, obtained by using the existing method [56] and the method proposed in previous chapter, as well as the methods proposed in this chapter are shown in Table 3.7 and Table 3.8.

Table 3.7: Comparative study for fuzzy TMT problem

Example	Existing method [56]	Proposed method
2.1	$\widetilde{x}_{11} = (-19, 1, 11, 27)$, $\widetilde{x}_{14} = (-4, 6, 12, 26)$, $\widetilde{x}_{21} = (-5, 0, 3, 10)$, $\widetilde{x}_{22} = (1, 6, 9, 16)$, $\widetilde{x}_{23} = (12, 14, 16, 18)$, $\widetilde{x}_{31} = (3, 4, 5, 8)$, $\widetilde{T} = (9, 11, 13, 15)$	$\widetilde{x}_{11} = (x_{111}, x_{112}, x_{113}, x_{114})$, $\widetilde{x}_{14} = (x_{141}, x_{142}, x_{143}, x_{144})$, $\widetilde{x}_{21} = (x_{211}, x_{212}, x_{213}, x_{214})$, $\widetilde{x}_{22} = (x_{221}, x_{222}, x_{223}, x_{224})$, $\widetilde{x}_{23} = (x_{231}, x_{232}, x_{233}, x_{234})$, $\widetilde{x}_{31} = (x_{311}, x_{312}, x_{313}, x_{314})$ which will satisfy the conditions $\frac{(x_{111}+x_{112}+x_{113}+x_{114})}{4} = 5$, $\frac{(x_{141}+x_{142}+x_{143}+x_{144})}{4} = 10$, $\frac{(x_{211}+x_{212}+x_{213}+x_{214})}{4} = 2$, $\frac{(x_{221}+x_{222}+x_{223}+x_{224})}{4} = 8$, $\frac{(x_{231}+x_{232}+x_{233}+x_{234})}{4} = 15$, $\frac{(x_{311}+x_{312}+x_{313}+x_{314})}{4} = 5$ and $\widetilde{T} = (9, 11, 13, 15)$

Table 3.8: Comparative study for fuzzy warehouse selection problem

Example 2.2	Method proposed in Chapter 2	Method proposed in this chapter
(X^1, Y^1)	$y_7, y_2, y_5,\ x_{12},\ x_{27},\ x_{32},\ x_{42}, x_{55}$, $\widetilde{C} = (94, 106, 134, 146)$, $\widetilde{T} = (9, 10, 11, 14)$	$y_7, y_2, y_5,\ x_{12},\ x_{27},\ x_{32},\ x_{42}, x_{55}$, $\widetilde{C} = (94, 106, 134, 146)$, $\widetilde{T} = (9, 10, 11, 14)$
(X^2, Y^2)	$y_2, y_3, y_1,\ x_{12},\ x_{21},\ x_{32},\ x_{42}, x_{53}$, $\widetilde{C} = (131, 140, 159, 170)$, $\widetilde{T} = (6, 8, 9, 13)$	$y_2, y_3, y_1,\ x_{12},\ x_{21},\ x_{32},\ x_{42}, x_{53}$, $\widetilde{C} = (131, 140, 159, 170)$, $\widetilde{T} = (6, 8, 9, 13)$
(X^3, Y^3)	$y_2, y_3, y_1,\ x_{12},\ x_{21},\ x_{31},\ x_{42}, x_{53}$, $\widetilde{C} = (193, 205, 216, 226)$, $\widetilde{T} = (5, 8, 9, 10)$	$y_2, y_3, y_1,\ x_{12},\ x_{21},\ x_{31},\ x_{42}, x_{53}$, $\widetilde{C} = (193, 205, 216, 226)$, $\widetilde{T} = (5, 8, 9, 10)$
(X^4, Y^4)	$y_2, y_5, y_3,\ x_{12},\ x_{22},\ x_{33},\ x_{42}, x_{55}$, $\widetilde{C} = (331, 346, 371, 392)$, $\widetilde{T} = (3, 6, 7, 8)$	$y_2, y_5, y_3,\ x_{12},\ x_{22},\ x_{33},\ x_{42}, x_{55}$, $\widetilde{C} = (331, 346, 371, 392)$, $\widetilde{T} = (3, 6, 7, 8)$

It is obvious from the results, shown in Table 3.7, that on solving the fuzzy TMT problem, chosen in Example 2.1, by using the existing method [56] only a unique fuzzy optimal solution is obtained. While, on solving the same problem by using the method, proposed in Section 3.2, infinite fuzzy optimal solutions are obtained and moreover, it is much easy to apply the proposed method as compared to the existing method [56].

Also, it is obvious from the results, shown in Table 3.8, that on solving the fuzzy warehouse selection problem, chosen in Example 2.2, by using the method proposed in Section 2.5.2 as well as the method proposed in Section 3.3, the same results are obtained. However, it is much easy to apply the method proposed in Section 3.3, as compared to the method proposed in previous chapter.

3.7 Conclusion

On the basis of presented study, it can be concluded that it is better to use methods proposed in this chapter as compared to the existing method [56] and the method proposed in previous chapter.

Chapter 4

Time Minimizing Transportation Problems In Fuzzy Environment - A Novel Formulation

In this chapter, shortcomings of existing formulation of TMT problems in fuzzy environment are pointed out and to resolve these shortcomings, a new formulation is proposed. Also, with the help of proposed formulation and the existing method [99], a new method is proposed for solving TMT problems in fuzzy environment.

4.1 Preliminaries

The definitions of non-negative trapezoidal fuzzy number, zero trapezoidal fuzzy number, positive trapezoidal fuzzy number and equality of trapezoidal fuzzy numbers as presented in foregoing chapter depend on ranking function. However, the same numbers can be defined without the help of any ranking function as described in the literature [86].

In this section, the latter definitions are presented [86].

Definition 4.1 A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be non-negative trapezoidal fuzzy number if and only if $a \geq 0$.

The work presented in this chapter is communicated in the *Journal of Intelligent and Fuzzy Systems.*

Definition 4.2 A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be zero trapezoidal fuzzy number if and only if $a = 0,\ b = 0,\ c = 0$ and $d = 0$.

Definition 4.3 A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be positive trapezoidal fuzzy number if and only if $a \geq 0,\ b \geq 0,\ c \geq 0$ and $d > 0$.

Definition 4.4 Two trapezoidal fuzzy numbers $\widetilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\widetilde{A}_2 = (a_2, b_2, c_2, d_2)$ are said to be equal i.e., $\widetilde{A}_1 = \widetilde{A}_2$ if and only if $a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2$ and $d_1 = d_2$.

4.2 Shortcomings of existing mathematical formulation of time minimizing transportation problem in fuzzy environment

The existing fuzzy mathematical formulation $(P_{2.2})$ of TMT problem in fuzzy environment is obtained by replacing the crisp parameters $t_{ij},\ x_{ij},\ a_i$ and b_j of mathematical formulation $(P_{2.1})$ by fuzzy parameters $\widetilde{t}_{ij},\ \widetilde{x}_{ij},\ \widetilde{a}_i$ and $\widetilde{b}_j$ respectively and the crisp equality by the fuzzy equality. However, it is not genuine to use this conversion method due to the following reasons:

Since, there is not a unique way for finding minimum and maximum of fuzzy numbers; So, it is appropriate to use one of the existing method for the same. Nevertheless, there is a unique way, discussed in Definition 4.4, for conversion of fuzzy equality into crisp equality. So, it is not genuine to use any other method for the same. However, in the existing formulation, $(P_{2.2})$, the existing method [109] is used for converting fuzzy equality into crisp equality as well as for checking that the problem is balanced or not. Hence, it is not appropriate to use the existing formulation $(P_{2.2})$ for finding the fuzzy optimal solution of TMT problem in fuzzy environment.

4.3 Proposed mathematical formulation and tabular representation of balanced time minimizing transportation problem in fuzzy environment

In this section, to resolve the shortcomings of existing formulation ($P_{2.2}$), a new mathematical formulation and tabular representation of TMT problem in fuzzy environment is obtained by replacing $\approx$ by $=$ in ($P_{2.2}$).

The objective function which is sought to be minimized is:

$\widetilde{T} = Maximum\{\widetilde{t}_{ij} : \widetilde{x}_{ij}$ is a positive trapezoidal fuzzy number, $i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q\}$

subject to

$$\begin{aligned} &\sum_{j=1}^{q} \widetilde{x}_{ij} = \widetilde{a}_i, \qquad i = 1, 2, \ldots, p \\ &\sum_{i=1}^{p} \widetilde{x}_{ij} = \widetilde{b}_j, \qquad j = 1, 2, \ldots, q \\ &\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j \\ &\widetilde{x}_{ij} \text{ is a non-negative trapezoidal fuzzy number.} \end{aligned} \qquad (P_{4.1})$$

Table 4.1: Tabular representation of fuzzy TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	$\widetilde{t}_{11}$	$\cdots$	$\widetilde{t}_{1j}$	$\cdots$	$\widetilde{t}_{1q}$	$= \widetilde{a}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	$\widetilde{t}_{i1}$	$\cdots$	$\widetilde{t}_{ij}$	$\cdots$	$\widetilde{t}_{iq}$	$= \widetilde{a}_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	$\widetilde{t}_{p1}$	$\cdots$	$\widetilde{t}_{pj}$	$\cdots$	$\widetilde{t}_{pq}$	$= \widetilde{a}_p$
Demand	$= \widetilde{b}_1$	$\cdots$	$= \widetilde{b}_j$	$\cdots$	$= \widetilde{b}_q$	$\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j$

Remark 4.1: If $\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j$, then ($P_{4.1}$) is said to be balanced; Otherwise, it is said to be unbalanced.

4.4 Proposed method

Kumar and Kaur [99] proposed a method to solve fuzzy cost minimizing transportation problems. In the same direction, in this section, a new method is proposed for solving TMT problems in fuzzy environment.

The steps of the proposed method are as follows:

Step 1 Find the total fuzzy availability $\sum_{i=1}^{p} \widetilde{a}_i$ and the total fuzzy demand $\sum_{j=1}^{q} \widetilde{b}_j$. Let $\sum_{i=1}^{p} \widetilde{a}_i = (a^1, a^2, a^3, a^4)$ and $\sum_{j=1}^{q} \widetilde{b}_j = (b^1, b^2, b^3, b^4)$. Examine whether the problem is balanced or not.

Case (i) If the problem is balanced, then Go to Step 2.

Case (ii) If the problem is unbalanced, then convert the unbalanced problem into balanced problem by using the existing method [99] as follows:

Case (a) If $a^1 \leq b^1$, $a^2 - a^1 \leq b^2 - b^1$, $a^3 - a^2 \leq b^3 - b^2$ and $a^4 - a^3 \leq b^4 - b^3$, then introduce a dummy source with fuzzy availability $\Big(b^1 - a^1, b^2 - a^2, b^3 - a^3, b^4 - a^4\Big)$. Assume fuzzy time of transportation from the introduced dummy source to all destinations as zero trapezoidal fuzzy number. Then, Go to Step 2.

Case (b) If $a^1 \geq b^1$, $a^2 - a^1 \geq b^2 - b^1$, $a^3 - a^2 \geq b^3 - b^2$ and $a^4 - a^3 \geq b^4 - b^3$, then introduce a dummy destination with fuzzy demand $\Big(a^1 - b^1, a^2 - b^2, a^3 - b^3, a^4 - b^4\Big)$. Assume fuzzy time of transportation from the introduced dummy destination to all sources as zero trapezoidal fuzzy number. Then, Go to Step 2.

Case (c) If neither Case (a) nor Case (b) is satisfied, then introduce a dummy source with fuzzy availability $\Big(Maximum\{0, b^1 - a^1\}$, $Maximum$

$\{0, b^1 - a^1\}$+ $Maximum$ $\{0, (b^2 - b^1) - (a^2 - a^1)\}$, $Maximum$ $\{0, b^1 - a^1\}$+ $Maximum$ $\{0, (b^2 - b^1) - (a^2 - a^1)\}$+ $Maximum$ $\{0, (b^3 - b^2) - (a^3 - a^2)\}$, $Maximum$ $\{0, b^1 - a^1\}$+ $Maximum$ $\{0, (b^2 - b^1) - (a^2 - a^1)\}$+ $Maximum$ $\{0, (b^3 - b^2) - (a^3 - a^2)\}$+ $Maximum$ $\{0, (b^4 - b^3) - (a^4 - a^3)\}\Big)$ and a dummy destination with fuzzy demand $\Big(Maximum$ $\{0, a^1 - b^1\}$, $Maximum$ $\{0, a^1 - b^1\}$+ $Maximum$ $\{0, (a^2 - a^1) - (b^2 - b^1)\}$, $Maximum$ $\{0, a^1 - b^1\}$+ $Maximum$ $\{0, (a^2 - a^1) - (b^2 - b^1)\}$+ $Maximum$ $\{0, (a^3 - a^2) - (b^3 - b^2)\}$, $Maximum$ $\{0, a^1 - b^1\}$+ $Maximum$ $\{0, (a^2 - a^1) - (b^2 - b^1)\}$+ $Maximum$ $\{0, (a^3 - a^2) - (b^3 - b^2)\}$+ $Maximum$ $\{0, (a^4 - a^3) - (b^4 - b^3)\}\Big)$. Assume fuzzy time of transportation from the introduced dummy source to all destinations as zero trapezoidal fuzzy number and from all sources to the introduced dummy destination as zero trapezoidal fuzzy number. Then, Go to Step 2.

Step 2 Represent the balanced TMT problem in fuzzy environment, obtained from Step 1, into tabular form as depicted in Table 4.1.

Step 3 Convert the balanced TMT problem in fuzzy environment, represented by Table 4.1, into four crisp TMT problems as depicted in Table 4.2, Table 4.3, Table 4.4 and Table 4.5.

Table 4.2: First crisp TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	η_{11}	$\cdots$	η_{1j}	$\cdots$	η_{1q}	a_{11}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	η_{i1}	$\cdots$	η_{ij}	$\cdots$	η_{iq}	a_{i1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	η_{p1}	$\cdots$	η_{pj}	$\cdots$	η_{pq}	a_{p1}
Demand	b_{11}	$\cdots$	b_{j1}	$\cdots$	b_{q1}	$\sum_{i=1}^{p}(a_{i1}) = \sum_{j=1}^{q}(b_{j1})$

where, $\eta_{ij} = \frac{t_{ij1}+t_{ij2}+t_{ij3}+t_{ij4}}{4}$, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$

Table 4.3: Second crisp TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	ρ_{11}	$\cdots$	ρ_{1j}	$\cdots$	ρ_{1q}	$a_{12} - a_{11}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	ρ_{i1}	$\cdots$	ρ_{ij}	$\cdots$	ρ_{iq}	$a_{i2} - a_{i1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	ρ_{p1}	$\cdots$	ρ_{pj}	$\cdots$	ρ_{pq}	$a_{p2} - a_{p1}$
Demand	$b_{12} - b_{11}$	$\cdots$	$b_{j2} - b_{j1}$	$\cdots$	$b_{q2} - b_{q1}$	$\sum_{i=1}^{p}(a_{i2} - a_{i1}) = \sum_{j=1}^{q}(b_{j2} - b_{j1})$

where, $\rho_{ij} = \frac{t_{ij2}+t_{ij3}+t_{ij4}}{4}$, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$

Table 4.4: Third crisp TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	δ_{11}	$\cdots$	δ_{1j}	$\cdots$	δ_{1q}	$a_{13} - a_{12}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	δ_{i1}	$\cdots$	δ_{ij}	$\cdots$	δ_{iq}	$a_{i3} - a_{i2}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	δ_{p1}	$\cdots$	δ_{pj}	$\cdots$	δ_{pq}	$a_{p3} - a_{p2}$
Demand	$b_{13} - b_{12}$	$\cdots$	$b_{j3} - b_{j2}$	$\cdots$	$b_{q3} - b_{q2}$	$\sum_{i=1}^{p}(a_{i3} - a_{i2}) = \sum_{j=1}^{q}(b_{j3} - b_{j2})$

where, $\delta_{ij} = \frac{t_{ij3}+t_{ij4}}{4}$, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$

Table 4.5: Fourth crisp TMT problem

Destinations→ Sources↓	D_1	$\cdots$	D_j	$\cdots$	D_q	Availability
S_1	ξ_{11}	$\cdots$	ξ_{1j}	$\cdots$	ξ_{1q}	$a_{14}-a_{13}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_i	ξ_{i1}	$\cdots$	ξ_{ij}	$\cdots$	ξ_{iq}	$a_{i4}-a_{i3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_p	ξ_{p1}	$\cdots$	ξ_{pj}	$\cdots$	ξ_{pq}	$a_{p4}-a_{p3}$
Demand	$b_{14}-b_{13}$	$\cdots$	$b_{j4}-b_{j3}$	$\cdots$	$b_{q4}-b_{q3}$	$\sum_{i=1}^{p}(a_{i4}-a_{i3})=\sum_{j=1}^{q}(b_{j4}-b_{j3})$

where, $\xi_{ij}=\frac{t_{ij4}}{4}$, $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$

Step 4 Use any existing method to find the optimal solutions $\{x_{ij1}\}$, $\{x_{ij2}-x_{ij1}\}$, $\{x_{ij3}-x_{ij2}\}$ and $\{x_{ij4}-x_{ij3}\}$ of crisp TMT problems represented by Table 4.2, Table 4.3, Table 4.4 and Table 4.5 respectively.

Step 5 Find the values of $\{x_{ij1}\}$, $\{x_{ij2}\}$, $\{x_{ij3}\}$ and $\{x_{ij4}\}$ by solving the equations obtained in Step 4.

Step 6 Find the fuzzy optimal solution $\{\widetilde{x}_{ij}\}$ by putting the values of x_{ij1}, x_{ij2}, x_{ij3} and x_{ij4} in $\widetilde{x}_{ij}=(x_{ij1},x_{ij2},x_{ij3},x_{ij4})$.

Step 7 Find the optimal fuzzy transportation time by calculating $Maximum\{\widetilde{t}_{ij}:\widetilde{x}_{ij}$ is a positive trapezoidal fuzzy number; $i=1,2,\ldots,p$; $j=1,2,\ldots,q\}$.

4.5 Advantage of the proposed method

The main advantage of the proposed method over the existing method [56] is that the fuzzy optimal solution obtained by using the method proposed in this chapter will satisfy both the existing formulation $(P_{2.2})$ as well as the proposed formulation $(P_{4.1})$. While, the fuzzy optimal solution obtained by using the existing method [56] and the method proposed in previous chapter can satisfy the existing formulation ($P_{2.2}$), but will not satisfy the proposed formulation ($P_{4.1}$). To show the advantage of the proposed method over the method proposed in previous chapter, the fuzzy TMT problem chosen in Example 2.1, is solved and obtained results are discussed.

Step 1 Total fuzzy availability $\sum_{i=1}^{3} \widetilde{a}_i = (37, 42, 47, 54)$ and total fuzzy demand $\sum_{j=1}^{4} \widetilde{b}_j = (34, 41, 49, 56)$. Since $\sum_{i=1}^{3} \widetilde{a}_i \neq \sum_{j=1}^{4} \widetilde{b}_j$, so it is an unbalanced fuzzy TMT problem. Now, using Step 1, the unbalanced fuzzy TMT problem can be converted into a balanced fuzzy TMT problem by introducing a dummy source S_4 with fuzzy availability $(0, 2, 5, 5)$ and a dummy destination D_5 with fuzzy demand $(3, 3, 3, 3)$ so that total fuzzy availability = total fuzzy demand i.e., $(37, 42, 47, 54) \oplus (0, 2, 5, 5) = (34, 41, 49, 56) \oplus (3, 3, 3, 3)$.

Step 2 Assuming that the fuzzy time of transportation of the product from dummy source S_4 to all destinations and from all sources to dummy destination D_5 as zero trapezoidal fuzzy number, i.e., $\widetilde{t}_{41} = \widetilde{t}_{42} = \widetilde{t}_{43} = \widetilde{t}_{44} = \widetilde{t}_{15} = \widetilde{t}_{25} = \widetilde{t}_{35} = \widetilde{t}_{45} = (0, 0, 0, 0)$, the tabular representation of the balanced fuzzy TMT problem, obtained from Step 1, is shown in Table 4.6 given overleaf.

Step 3 Using Step 3 of the proposed method, the balanced fuzzy TMT problem represented by Table 4.6 can be split into four crisp TMT problems as depicted in Table 4.7, Table 4.8, Table 4.9 and Table 4.10.

Table 4.7: First crisp TMT problem

	D_1	D_2	D_3	D_4	D_5	a_{i1}
S_1	10	0	20	11	0	12
S_2	1	7	9	20	0	22
S_3	12	14	16	18	0	3
S_4	0	0	0	0	0	0
b_{j1}	9	6	12	7	3	

Table 4.6: Balanced fuzzy TMT problem

Destinations→ Sources↓	D_1	D_2	D_3	D_4	D_5	$\widetilde{a}_i$
S_1	$(7, 9, 11, 13)$	$(-5, -1, 1, 5)$	$(14, 18, 22, 26)$	$(8, 11, 12, 13)$	$(0, 0, 0, 0)$	$(12, 14, 16, 18)$
S_2	$(-2, 0, 2, 4)$	$(5, 6, 7, 10)$	$(7, 8, 9, 12)$	$(14, 18, 22, 26)$	$(0, 0, 0, 0)$	$(22, 24, 26, 28)$
S_3	$(9, 11, 13, 15)$	$(11, 13, 15, 17)$	$(13, 15, 17, 19)$	$(15, 17, 19, 21)$	$(0, 0, 0, 0)$	$(3, 4, 5, 8)$
S_4	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$(0, 2, 5, 5)$
$\widetilde{b}_j$	$(9, 11, 13, 15)$	$(6, 7, 9, 10)$	$(12, 14, 16, 18)$	$(7, 9, 11, 13)$	$(3, 3, 3, 3)$	$(37, 44, 52, 59)$

Table 4.8: Second crisp TMT problem

	D_1	D_2	D_3	D_4	D_5	$a_{i2}-a_{i1}$
S_1	8.25	1.25	16.50	9	0	2
S_2	1.50	5.75	7.25	16.50	0	2
S_3	9.75	11.25	12.75	14.25	0	1
S_4	0	0	0	0	0	2
$b_{j2}-b_{j1}$	2	1	2	2	0	

Table 4.9: Third crisp TMT problem

	D_1	D_2	D_3	D_4	D_5	$a_{i3}-a_{i2}$
S_1	6	1.50	12	6.25	0	2
S_2	1.50	4.25	5.25	12	0	2
S_3	7	8	9	10	0	1
S_4	0	0	0	0	0	3
$b_{j3}-b_{j2}$	2	2	2	2	0	

Table 4.10: Fourth crisp TMT problem

	D_1	D_2	D_3	D_4	D_5	$a_{i4}-a_{i3}$
S_1	3.25	1.25	6.50	3.25	0	2
S_2	1	2.50	3	6.50	0	2
S_3	3.75	4.25	4.75	5.25	0	3
S_4	0	0	0	0	0	0
$b_{j4}-b_{j3}$	2	1	2	2	0	

Step 4 The optimal solution of crisp TMT problems, shown in Tables 4.7 to 4.10, are $x_{111} = 5$, $x_{121} = 0$, $x_{131} = 0$, $x_{141} = 7$, $x_{151} = 0$, $x_{211} = 4$, $x_{221} = 6$, $x_{231} = 12$, $x_{241} = 0$, $x_{251} = 0$, $x_{311} = 0$, $x_{321} = 0$, $x_{331} = 0$, $x_{341} = 0$, $x_{351} = 3$, $x_{411} = 0$, $x_{421} = 0$, $x_{431} = 0$, $x_{441} = 0$, $x_{451} = 0$, $x_{112}-x_{111} = 1$, $x_{122}-x_{121} = 1$, $x_{132}-x_{131} = 0$, $x_{142}-x_{141} = 0$, $x_{152}-x_{151} = 0$, $x_{212}-x_{211} = 0$, $x_{222}-x_{221} = 0$, $x_{232}-x_{231} = 2$, $x_{242}-x_{241} = 0$, $x_{252}-x_{251} = 0$, $x_{312}-x_{311} = 1$, $x_{322}-x_{321} = 0$, $x_{332}-x_{331} = 0$, $x_{342}-x_{341} = 0$, $x_{352}-x_{351} = 0$, $x_{412}-x_{411} = 0$, $x_{422}-x_{421} = 0$, $x_{432}-x_{431} = 0$, $x_{442}-x_{441} = 2$, $x_{452}-x_{451} = 0$, $x_{113}-x_{112} = 1$, $x_{123}-x_{122} = 1$, $x_{133}-x_{132} = 0$, $x_{143}-x_{142} = 0$, $x_{153}-x_{152} = 0$, $x_{213}-x_{212} = 0$, $x_{223}-x_{222} = 1$, $x_{233}-x_{232} = 1$, $x_{243}-x_{242} = 0$, $x_{253}-x_{252} = 0$, $x_{313}-x_{312} = 1$, $x_{323}-x_{322} = 0$,

$x_{333}-x_{332}=0$, $x_{343}-x_{342}=0$, $x_{353}-x_{352}=0$, $x_{413}-x_{412}=0$, $x_{423}-x_{422}=0$, $x_{433}-x_{432}=1$, $x_{443}-x_{442}=2$, $x_{453}-x_{452}=0$, $x_{114}-x_{113}=2$, $x_{124}-x_{123}=0$, $x_{134}-x_{133}=0$, $x_{144}-x_{143}=0$, $x_{154}-x_{153}=0$, $x_{214}-x_{213}=0$, $x_{224}-x_{223}=1$, $x_{234}-x_{233}=1$, $x_{244}-x_{243}=0$, $x_{254}-x_{253}=0$, $x_{314}-x_{313}=0$, $x_{324}-x_{323}=0$, $x_{334}-x_{333}=1$, $x_{344}-x_{343}=2$, $x_{354}-x_{353}=0$, $x_{414}-x_{413}=0$, $x_{424}-x_{423}=0$, $x_{434}-x_{433}=0$, $x_{444}-x_{443}=0$ and $x_{454}-x_{453}=0$ respectively.

Step 5 On solving the equations, obtained in Step 4, the values of x_{ij1}, x_{ij2}, x_{ij3} and x_{ij4}, $\forall\ i,j$ are $x_{111}=5$, $x_{121}=0$, $x_{131}=0$, $x_{141}=7$, $x_{151}=0$, $x_{211}=4$, $x_{221}=6$, $x_{231}=12$, $x_{241}=0$, $x_{251}=0$, $x_{311}=0$, $x_{321}=0$, $x_{331}=0$, $x_{341}=0$, $x_{351}=3$, $x_{411}=0$, $x_{421}=0$, $x_{431}=0$, $x_{441}=0$, $x_{451}=0$, $x_{112}=6$, $x_{122}=1$, $x_{132}=0$, $x_{142}=7$, $x_{152}=0$, $x_{212}=4$, $x_{222}=6$, $x_{232}=14$, $x_{242}=0$, $x_{252}=0$, $x_{312}=1$, $x_{322}=0$, $x_{332}=0$, $x_{342}=0$, $x_{352}=3$, $x_{412}=0$, $x_{422}=0$, $x_{432}=0$, $x_{442}=2$, $x_{452}=0$, $x_{113}=7$, $x_{123}=2$, $x_{133}=0$, $x_{143}=7$, $x_{153}=0$, $x_{213}=4$, $x_{223}=7$, $x_{233}=15$, $x_{243}=0$, $x_{253}=0$, $x_{313}=2$, $x_{323}=0$, $x_{333}=0$, $x_{343}=0$, $x_{353}=3$, $x_{413}=0$, $x_{423}=0$, $x_{433}=1$, $x_{443}=4$, $x_{453}=0$, $x_{114}=9$, $x_{124}=2$, $x_{134}=0$, $x_{144}=7$, $x_{154}=0$, $x_{214}=4$, $x_{224}=8$, $x_{234}=16$, $x_{244}=0$, $x_{254}=0$, $x_{314}=2$, $x_{324}=0$, $x_{334}=1$, $x_{344}=2$, $x_{355}=3$, $x_{414}=0$, $x_{424}=0$, $x_{434}=1$, $x_{444}=4$ and $x_{454}=0$

Step 6 Putting the values of $x_{ij1}, x_{ij2}, x_{ij3}$ and x_{ij4} in $\widetilde{x}_{ij}=(x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4})$, the fuzzy optimal solution is $\widetilde{x}_{11}=(5,6,7,9)$, $\widetilde{x}_{12}=(0,1,2,2)$, $\widetilde{x}_{13}=(0,0,0,0)$, $\widetilde{x}_{14}=(7,7,7,7)$, $\widetilde{x}_{15}=(0,0,0,0)$, $\widetilde{x}_{21}=(4,4,4,4)$, $\widetilde{x}_{22}=(6,6,7,8)$, $\widetilde{x}_{23}=(12,14,15,16)$, $\widetilde{x}_{24}=(0,0,0,0)$, $\widetilde{x}_{25}=(0,0,0,0)$, $\widetilde{x}_{31}=(0,1,2,2)$, $\widetilde{x}_{32}=(0,0,0,0)$, $\widetilde{x}_{33}=(0,0,0,1)$, $\widetilde{x}_{34}=(0,0,0,2)$, $\widetilde{x}_{35}=(3,3,3,3)$, $\widetilde{x}_{41}=(0,0,0,0)$,

$\widetilde{x}_{42} = (0,0,0,0)$, $\widetilde{x}_{43} = (0,0,1,1)$, $\widetilde{x}_{44} = (0,2,4,4)$ and $\widetilde{x}_{45} = (0,0,0,0)$.

Step 7 The optimal fuzzy transportation time obtained by solving $Maximum\{\widetilde{t}_{11}$, $\widetilde{t}_{12}$, $\widetilde{t}_{14}$, $\widetilde{t}_{21}$, $\widetilde{t}_{22}$, $\widetilde{t}_{23}$, $\widetilde{t}_{31}$, $\widetilde{t}_{33}$, $\widetilde{t}_{34}$, $\widetilde{t}_{35}$, $\widetilde{t}_{43}$, $\widetilde{t}_{44}\}$ is $\widetilde{T} = (15,17,19,21)$.

4.6 Comparative study

The fuzzy optimal solution of the problem, chosen in Example 2.1, obtained by using the method proposed in previous chapter as well as the method proposed in this chapter are shown in Table 4.11.

Table 4.11: Comparative study for fuzzy TMT problem

Example	Method proposed in Chapter 3	Method proposed in this chapter
2.1	$\widetilde{x}_{11} = (x_{111}, x_{112}, x_{113}, x_{114})$, $\widetilde{x}_{14} = (x_{141}, x_{142}, x_{143}, x_{144})$, $\widetilde{x}_{21} = (x_{211}, x_{212}, x_{213}, x_{214})$, $\widetilde{x}_{22} = (x_{221}, x_{222}, x_{223}, x_{224})$, $\widetilde{x}_{23} = (x_{231}, x_{232}, x_{233}, x_{234})$, $\widetilde{x}_{31} = (x_{311}, x_{312}, x_{313}, x_{314})$ which will satisfy the conditions $\frac{(x_{111}+x_{112}+x_{113}+x_{114})}{4} = 5$, $\frac{(x_{141}+x_{142}+x_{143}+x_{144})}{4} = 10$, $\frac{(x_{211}+x_{212}+x_{213}+x_{214})}{4} = 2$, $\frac{(x_{221}+x_{222}+x_{223}+x_{224})}{4} = 8$, $\frac{(x_{231}+x_{232}+x_{233}+x_{234})}{4} = 15$, $\frac{(x_{311}+x_{312}+x_{313}+x_{314})}{4} = 5$ and $\widetilde{T} = (9,11,13,15)$	$\widetilde{x}_{11} = (5,6,7,9)$, $\widetilde{x}_{12} = (0,1,2,2)$, $\widetilde{x}_{14} = (7,7,7,7)$, $\widetilde{x}_{21} = (4,4,4,4)$, $\widetilde{x}_{22} = (6,6,7,8)$, $\widetilde{x}_{23} = (12,14,15,16)$, $\widetilde{x}_{31} = (3,4,5,8)$ and $\widetilde{T} = (15,17,19,21)$

It can be easily verified that the fuzzy optimal solution obtained by using the method proposed in previous chapter is satisfying the existing formulation ($P_{2.2}$) but not the proposed formulation ($P_{4.1}$). While, the fuzzy optimal solution obtained by using the method proposed in this chapter is satisfying both the existing formulation ($P_{2.2}$) and the proposed formulation ($P_{4.1}$).

4.7 Conclusion

On the basis of presented study, it can be concluded that it is better to use method proposed in this chapter as compared to the method proposed in previous chapter for solving TMT problems in fuzzy environment.

Chapter 5

Fuzzy Optimal Solution Of Fully Fuzzy Multi-objective Transportation Problems With *LR* Flat Fuzzy Numbers

Gupta et al. [68] pointed out that there is no method in the literature for solving fully fuzzy multi-objective transportation problems and proposed a method for its solution. In this chapter, the limitations of this method are pointed out and to overcome these limitations, a generalized method is proposed by modifying the existing method. The advantages of the proposed method over the existing method are discussed. To illustrate the proposed method some numerical examples have been considered.

5.1 Preliminaries

In the literature [58, 168], it is pointed out that the computational efforts required to solve a fuzzy linear programming problem can be reduced if the decision maker expresses his data using LR flat fuzzy numbers. All kinds of crisp numbers, triangular fuzzy numbers and trapezoidal fuzzy numbers are particular types of LR

The work presented in this chapter is communicated in *The Asian Journal of Mathematics.*

flat fuzzy numbers. Thus, LR flat fuzzy numbers are frequently used to increase the computational efficiency without limiting the generality beyond the acceptable limits and facilitates the ease of acquisition of data to solve real life problems. The present study comes up with a solution of fully fuzzy multi-objective transportation problems with LR flat fuzzy numbers.

This section presents some basic definitions and arithmetic operations of LR flat fuzzy numbers.

5.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 5.1 [58] A function $L : [0, \infty) \to [0, 1]$ or $(R : [0, \infty) \to [0, 1])$ is said to be reference function of fuzzy number if and only if

(i) $L(0) = 1$ and $L(1) = 0$ (or $R(0) = 1$ and $R(1) = 0$).

(ii) L (or R) is non-increasing on $[0, \infty)$.

Definition 5.2 [58] A fuzzy number $\widetilde{A}$, defined on universal set of real numbers $\mathbb{R}$, denoted as $(m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & x \leq m \\ 1, & m \leq x \leq n \\ R(\frac{x-n}{\beta}), & x \geq n \end{cases}$$

where, α and β are non-negative real numbers.

The cases $\alpha = 0$ and/or $\beta = 0$ are admissible. It is assumed that $L(\frac{m-x}{0}) = 0$ and/or $R(\frac{x-n}{0}) = 0$. Thus, any interval $[m, n]$ and any real number m are also LR flat fuzzy numbers and can be written as $(m, n, 0, 0)_{LR}$ and $(m, m, 0, 0)_{LR}$ respectively.

Definition 5.3 [58] Let $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ be an LR flat fuzzy number and λ be a real number in the interval $[0,1]$. Then, the classical set $A_\lambda = \{x \in X : \mu_{\widetilde{A}}(x) \geq \lambda\} = [m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)]$ is said to be $\lambda-$cut of $\widetilde{A}$.

Definition 5.4 [58] An LR flat fuzzy number $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be zero LR flat fuzzy number if and only if $m = 0$, $n = 0$, $\alpha = 0$ and $\beta = 0$.

Definition 5.5 [58] Two LR flat fuzzy numbers $\widetilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are said to be equal, i.e., $\widetilde{A}_1 = \widetilde{A}_2$ if and only if $m_1 = m_2$, $n_1 = n_2$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Definition 5.6 [49] An LR flat fuzzy number $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be non-negative LR flat fuzzy number if and only if $m - \alpha \geq 0$.

Remark 5.1 If $m = n$ then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be an LR fuzzy number and is denoted as $(m, m, \alpha, \beta)_{LR}$ or $(m, \alpha, \beta)_{LR}$ or $(n, n, \alpha, \beta)_{LR}$ or $(n, \alpha, \beta)_{LR}$.

Remark 5.2 If $m = n$ and $L(x) = R(x) = Maximum\{0, 1 - x\}$, then an LR flat fuzzy number $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be triangular fuzzy number and is denoted as (a, b, c) where, $a = m - \alpha$, $b = m$ (or n), $c = m + \beta$ (or $n + \beta$).

Remark 5.3 If $m \neq n$ and $L(x) = R(x) = Maximum\{0, 1 - x\}$, then an LR flat fuzzy number $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be trapezoidal fuzzy number and is denoted as (a, b, c, d) where, $a = m - \alpha$, $b = m$, $c = n$ and $d = n + \beta$.

5.1.2 Arithmetic operations

In this section, addition and multiplication operations of LR flat fuzzy numbers are presented [58].

(i) Let $\widetilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers. Then, $\widetilde{A}_1 \oplus \widetilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$.

(ii) Let $\widetilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two non-negative LR flat fuzzy numbers. Then,

$\widetilde{A}_1 \otimes \widetilde{A}_2 \simeq (m_1 m_2, n_1 n_2, m_1 m_2 - (m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2) - n_1 n_2)_{LR}$.

5.2 Mathematical formulation of balanced fully fuzzy multi-objective transportation problem

The mathematical formulation of balanced fully fuzzy multi-objective transportation problem is as follows [68]:

$$Minimize \sum_{i=1}^{p} \sum_{j=1}^{q} \widetilde{c}_{ij}^{k} \otimes \widetilde{x}_{ij}, \quad k = 1, 2, \ldots, K$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{q} \widetilde{x}_{ij} = \widetilde{a}_i, \quad i = 1, 2, \ldots, p \\
&\sum_{i=1}^{p} \widetilde{x}_{ij} = \widetilde{b}_j, \quad j = 1, 2, \ldots, q \\
&\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j \\
&\widetilde{x}_{ij} \text{ is a non-negative fuzzy number.}
\end{aligned} \qquad (P_{5.1})$$

where,

p: total number of sources.

q: total number of destinations.

$\widetilde{a}_i$: the fuzzy availability of the product at i^{th} source (S_i).

$\widetilde{b}_j$: the fuzzy demand of the product at j^{th} destination (D_j).

$\widetilde{c}_{ij}^{k}$: the penalty criteria for k^{th} objective function.

$\widetilde{x}_{ij}$: the fuzzy quantity of the product that should be transported from i^{th} source to j^{th} destination (or fuzzy decision variable) in order to minimize K objective functions.

$\sum_{i=1}^{p} \widetilde{a}_i$: total fuzzy availability of the product.

$\sum_{j=1}^{q} \widetilde{b}_j$: total fuzzy demand of the product.

$\sum_{i=1}^{p} \sum_{j=1}^{q} \widetilde{c}_{ij}^k \otimes \widetilde{x}_{ij}$: fuzzy value of k^{th} objective function.

Remark 5.4 If $\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j$, then $(P_{5.1})$ is said to be balanced; Otherwise, it is said to be unbalanced.

5.3 Limitations of the existing method

The existing method [68] can be used for solving such balanced and unbalanced fully fuzzy multi-objective transportation problems in which all the parameters are represented by trapezoidal fuzzy numbers. However, this method [68] cannot be used for solving those balanced and unbalanced fully fuzzy multi-objective transportation problems in which all the parameters are represented by LR flat fuzzy numbers. The fully fuzzy multi-objective balanced transportation problem, chosen in Example 5.1 and the fully fuzzy multi-objective unbalanced transportation problem, chosen in Example 5.2, in which all the parameters are represented by trapezoidal fuzzy numbers can be solved by using the existing method. However, the fully fuzzy multi-objective balanced and unbalanced transportation problems chosen in Example 5.3 and Example 5.4 respectively in which all the parameters are represented by LR flat fuzzy numbers cannot be solved by using the existing method [68].

Example 5.1 A company has three sources S_1, S_2, S_3 and four destinations D_1, D_2, D_3, D_4. The fuzzy penalties for supplying a unit quantity of the product from i^{th} source to j^{th} destination for 1^{st} and 2^{nd} objectives are given in Table 5.1 and

Table 5.2 respectively. The fuzzy availability of the product at sources S_1, S_2, S_3 is $(6,7,9,11)$, $(16,17,19,23)$ and $(14,16,17,21)$ respectively and the fuzzy demand of the product at destinations D_1, D_2, D_3, D_4 is $(9,10,11,14)$, $(1,2,3,6)$, $(12,13,14,17)$ and $(14,15,17,18)$ respectively. The company wants to determine the fuzzy quantities of the product to be transported from each source to various destinations in order to minimize each objective function.

Table 5.1: Fuzzy penalties for 1^{st} objective function

Destination→ Source↓	D_1	D_2	D_3	D_4
S_1	$(0,0.5,1,2.5)$	$(0,1,2,5)$	$(4,7,8,9)$	$(5,6,8,9)$
S_2	$(0,0.5,1,2.5)$	$(7,8,9,12)$	$(1,2,4,5)$	$(2,3,5,6)$
S_3	$(6,7,8,11)$	$(7,8,9,12)$	$(2,3,5,6)$	$(4,5,6,9)$

Table 5.2: Fuzzy penalties for 2^{nd} objective function

Destination→ Source↓	D_1	D_2	D_3	D_4
S_1	$(2,3,5,6)$	$(1,2,5,8)$	$(1,2,4,5)$	$(1,2,3,6)$
S_2	$(3,4,5,8)$	$(6,7,8,11)$	$(7,8,9,12)$	$(8,9,10,13)$
S_3	$(4,5,6,9)$	$(0,1,2,5)$	$(3,4,5,8)$	$(0,1,1.25,1.75)$

Example 5.2 A company has three sources S_1, S_2, S_3 and four destinations D_1, D_2, D_3, D_4. The fuzzy penalties for supplying a unit quantity of the product from i^{th} source to j^{th} destination for 1^{st} and 2^{nd} objectives are given in Table 5.3 and Table 5.4 respectively. The fuzzy availability of the product at sources S_1, S_2, S_3 is $(6,7,8,11)$, $(17,18,19,22)$ and $(14,16,17,21)$ respectively and the fuzzy demand of the product at destinations D_1, D_2, D_3, D_4 is $(9,10,11,14)$, $(1,2,3,6)$, $(12,13,14,17)$ and $(14,15,16,19)$ respectively. The company wants to determine

the fuzzy quantities of the product to be transported from each source to various destinations in order to minimize each objective function.

Table 5.3: Fuzzy penalties for 1^{st} objective function

Destination→ Source↓	D_1	D_2	D_3	D_4
S_1	$(0, 0.5, 1, 2.5)$	$(0, 1, 2, 5)$	$(4, 7, 8, 9)$	$(5, 6, 8, 9)$
S_2	$(0, 0.5, 1, 2.5)$	$(7, 8, 9, 12)$	$(1, 2, 4, 5)$	$(2, 3, 5, 6)$
S_3	$(6, 7, 8, 11)$	$(7, 8, 9, 12)$	$(2, 3, 5, 6)$	$(4, 5, 6, 9)$

Table 5.4: Fuzzy penalties for 2^{nd} objective function

Destination→ Source↓	D_1	D_2	D_3	D_4
S_1	$(2, 3, 5, 6)$	$(1, 2, 5, 8)$	$(1, 2, 4, 5)$	$(1, 2, 3, 6)$
S_2	$(3, 4, 5, 8)$	$(6, 7, 8, 11)$	$(7, 8, 9, 12)$	$(8, 9, 10, 13)$
S_3	$(4, 5, 6, 9)$	$(0, 1, 2, 5)$	$(3, 4, 5, 8)$	$(0, 1, 1.25, 1.75)$

Example 5.3 A company has two sources S_1, S_2 and three destinations D_1, D_2, D_3. The fuzzy penalties for supplying a unit quantity of the product from i^{th} source to j^{th} destination for 1^{st} and 2^{nd} objectives are given in Table 5.5 and Table 5.6 respectively. The fuzzy availability of the product at sources S_1, S_2 is $(90, 92, 10, 3)_{LR}$ and $(60, 88, 10, 28)_{LR}$ respectively. The fuzzy demand of the product at destinations D_1, D_2, D_3 is $(40, 50, 5, 15)_{LR}$, $(30, 40, 6, 6)_{LR}$ and $(80, 90, 9, 10)_{LR}$ respectively where $L(x) = R(x) = Maximum\{0, 1 - x^2\}$. The company wants to determine the fuzzy quantities of the product to be transported from each source to various destinations in order to minimize each objective function.

Table 5.5: Fuzzy penalties for 1^{st} objective function

Destination→ Source↓	D_1	D_2	D_3
S_1	$(14,16,1,3)_{LR}$	$(29,31,2,2)_{LR}$	$(18,19,1,4)_{LR}$
S_2	$(19,21,3,2)_{LR}$	$(14,16,2,2)_{LR}$	$(15,17,4,3)_{LR}$

Table 5.6: Fuzzy penalties for 2^{nd} objective function

Destination→ Source↓	D_1	D_2	D_3
S_1	$(15,16,2,4)_{LR}$	$(5,6,2,2)_{LR}$	$(8,9,3,4)_{LR}$
S_2	$(7,8,1,1)_{LR}$	$(13,14,2,2)_{LR}$	$(11,12,3,6)_{LR}$

Example 5.4 A company has two sources S_1, S_2 and three destinations D_1, D_2, D_3. The fuzzy penalties for supplying a unit quantity of the product from i^{th} source to j^{th} destination for 1^{st} and 2^{nd} objectives are given in Table 5.7 and Table 5.8 respectively. The fuzzy availability of the product at sources S_1, S_2 is $(90,92,10,3)_{LR}$ and $(60,70,10,5)_{LR}$ respectively. The fuzzy demand of the product at destinations D_1, D_2, D_3 is $(40,50,5,15)_{LR}$, $(30,40,6,6)_{LR}$ and $(50,60,3,10)_{LR}$ respectively where $L(x) = R(x) = Maximum\{0, 1-x^2\}$. The company wants to determine the fuzzy quantities of the product to be transported from each source to various destinations in order to minimize each objective function.

Table 5.7: Fuzzy penalties for 1^{st} objective function

Destination→ Source↓	D_1	D_2	D_3
S_1	$(14,16,1,3)_{LR}$	$(29,31,2,2)_{LR}$	$(18,19,1,4)_{LR}$
S_2	$(19,21,3,2)_{LR}$	$(14,16,2,2)_{LR}$	$(15,17,4,3)_{LR}$

Table 5.8: Fuzzy penalties for 2^{nd} objective function

Destination→ Source↓	D_1	D_2	D_3
S_1	$(15,16,2,4)_{LR}$	$(5,6,2,2)_{LR}$	$(8,9,3,4)_{LR}$
S_2	$(7,8,1,1)_{LR}$	$(13,14,2,2)_{LR}$	$(11,12,3,6)_{LR}$

5.4 Proposed method

In this section, to overcome the limitations of existing method [68], discussed in Section 5.3, a new method is proposed to find the exact fuzzy optimal solution of fully fuzzy multi-objective transportation problems occurring in real life situations by representing all the parameters as LR flat fuzzy numbers. The advantages of the proposed method over the existing method are discussed.

The steps of the proposed method are as follows:

Step 1 Find the total fuzzy availability $\sum_{i=1}^{p}\widetilde{a}_i$ and the total fuzzy demand $\sum_{j=1}^{q}\widetilde{b}_j$. Let $\sum_{i=1}^{p}\widetilde{a}_i = (m,n,\alpha,\beta)_{LR}$ and $\sum_{j=1}^{q}\widetilde{b}_j = (m',n',\alpha',\beta')_{LR}$. Examine if the problem is balanced or not.

Case (i) If the problem is balanced, then Go to Step 2.

Case (ii) If the problem is unbalanced, then convert the unbalanced problem into balanced problem as follows.

Case (a) If $m-\alpha \le m'-\alpha'$, $\alpha \le \alpha'$, $n-m \le n'-m'$ and $\beta \le \beta'$, then introduce a dummy source with fuzzy availability $\Big(m'-m, n'-n, \alpha'-\alpha, \beta'-\beta\Big)_{LR}$. Assume the fuzzy penalties $(\widetilde{c}_{ij}^{k})$ from the introduced dummy source to all destinations as zero LR flat fuzzy numbers for all the objectives. Then, Go to Step 2.

Case (b) If $m-\alpha \geq m'-\alpha'$, $\alpha \geq \alpha'$, $n-m \geq n'-m'$ and $\beta \geq \beta'$, then introduce a dummy destination with fuzzy demand $\Big(m-m', n-n', \alpha-\alpha', \beta-\beta'\Big)_{LR}$. Assume the fuzzy penalties ($\widetilde{c}_{ij}^{k}$) from all sources to the introduced dummy destination as zero LR flat fuzzy numbers for all the objectives. Then, Go to Step 2.

Case (c) If neither Case (a) nor Case (b) is satisfied, then introduce a dummy source with fuzzy availability $\Big(Maximum\{0,(m'-\alpha')-(m-\alpha)\}+ Maximum\{0,(\alpha'-\alpha)\}, Maximum\{0,(m'-\alpha')-(m-\alpha)\}+Maximum \{0,(\alpha'-\alpha)\}+ Maximum\{0,(n'-m')-(n-m)\}, Maximum\{0,(\alpha'-\alpha)\}, Maximum\{0,(\beta'-\beta)\}\Big)_{LR}$ and dummy destination with fuzzy demand $\Big(Maximum\{0,(m-\alpha)-(m'-\alpha')\}+ Maximum\{0,(\alpha-\alpha')\}, Maximum\{0,(m-\alpha)-(m'-\alpha')\}+ Maximum\ \{0,(\alpha-\alpha')\}+ Maximum\{0,(n-m)-(n'-m')\}, Maximum\{0,(\alpha-\alpha')\}, Maximum\{0,(\beta-\beta')\}\Big)_{LR}$. Assume the fuzzy penalties ($\widetilde{c}_{ij}^{k}$) from dummy source to all destinations and from all sources to dummy destination as zero LR flat fuzzy numbers for all the objectives. Then, Go to Step 2.

Step 2 Formulate the balanced fully fuzzy multi-objective transportation problem, obtained from Step 1, into fully fuzzy multi-objective linear programming problem ($P_{5.1}$).

Step 3 Assuming $\widetilde{c}_{ij}^{k} = (m_{ij}^{k}, n_{ij}^{k}, \alpha_{ij}^{k}, \beta_{ij}^{k})_{LR}$, $\widetilde{x}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\widetilde{a}_i = (m_i, n_i, \alpha_i, \beta_i)_{LR}$, $\widetilde{b}_j = (m'_j, n'_j, \alpha'_j, \beta'_j)_{LR}$, the fully fuzzy multi-objective linear programming problem, obtained in Step 2, can be written as:

$$Minimize\ \sum_{i=1}^{p}\sum_{j=1}^{q}\Big((m_{ij}^k, n_{ij}^k, \alpha_{ij}^k, \beta_{ij}^k)_{LR}\otimes(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}\Big),\quad k=1,2,\ldots,K$$

subject to

$$\sum_{j=1}^{q}((m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}) = (m_i, n_i, \alpha_i, \beta_i)_{LR},\quad i=1,2,\ldots,p$$
$$\sum_{i=1}^{p}((m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}) = (m'_j, n'_j, \alpha'_j, \beta'_j)_{LR},\quad j=1,2,\ldots,q$$

$(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ is a non-negative LR flat fuzzy number.

Step 4 Using the arithmetic operations, defined in Section 5.1.2, the fully fuzzy multi-objective linear programming problem, obtained in Step 3, can be written as:

$$Minimize\ \Big(\Big(\sum_{i=1}^{p}\sum_{j=1}^{q} m_{ij}^k m_{ij}, \sum_{i=1}^{p}\sum_{j=1}^{q} n_{ij}^k n_{ij}, \sum_{i=1}^{p}\sum_{j=1}^{q}(m_{ij}^k\alpha_{ij} + m_{ij}\alpha_{ij}^k - \alpha_{ij}^k\alpha_{ij}),$$
$$\sum_{i=1}^{p}\sum_{j=1}^{q}(n_{ij}^k\beta_{ij} + n_{ij}\beta_{ij}^k + \beta_{ij}^k\beta_{ij})\Big)_{LR}\Big),\quad k=1,2,\ldots,K$$

subject to

$$\Big(\sum_{j=1}^{q} m_{ij}, \sum_{j=1}^{q} n_{ij}, \sum_{j=1}^{q}\alpha_{ij}, \sum_{j=1}^{q}\beta_{ij}\Big)_{LR} = (m_i, n_i, \alpha_i, \beta_i)_{LR},\quad i=1,2,\ldots,p$$
$$\Big(\sum_{i=1}^{p} m_{ij}, \sum_{i=1}^{p} n_{ij}, \sum_{i=1}^{p}\alpha_{ij}, \sum_{i=1}^{p}\beta_{ij}\Big)_{LR} = (m'_j, n'_j, \alpha'_j, \beta'_j)_{LR},\quad j=1,2,\ldots,q$$

$(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ is a non-negative LR flat fuzzy number.

Step 5 Assuming $\Big(\Big(\sum_{i=1}^{p}\sum_{j=1}^{q} m_{ij}^k m_{ij}, \sum_{i=1}^{p}\sum_{j=1}^{q} n_{ij}^k n_{ij}, \sum_{i=1}^{p}\sum_{j=1}^{q}(m_{ij}^k\alpha_{ij} + m_{ij}\alpha_{ij}^k - \alpha_{ij}^k\alpha_{ij}),$ $\sum_{i=1}^{p}\sum_{j=1}^{q}(n_{ij}^k\beta_{ij} + n_{ij}\beta_{ij}^k + \beta_{ij}^k\beta_{ij})\Big)_{LR}\Big) = (m_0^k, n_0^k, \alpha_0^k, \beta_0^k)_{LR}$ and using Definition 5.5 and Definition 5.6, the fully fuzzy multi-objective linear programming problem, obtained in Step 4, can be converted into the fuzzy multi-objective linear programming problem ($P_{5.2}$):

$$Minimize\ (m_0^k, n_0^k, \alpha_0^k, \beta_0^k)_{LR},\quad k=1,2,\ldots,K$$

subject to

$$\sum_{j=1}^{q} m_{ij} = m_i,\quad i=1,2,\ldots,p$$
$$\sum_{j=1}^{q} n_{ij} = n_i,\quad i=1,2,\ldots,p$$
$$\sum_{j=1}^{q} \alpha_{ij} = \alpha_i,\quad i=1,2,\ldots,p$$

$$\sum_{j=1}^{q} \beta_{ij} = \beta_i, \quad i = 1, 2, \ldots, p \qquad (P_{5.2})$$
$$\sum_{i=1}^{p} m_{ij} = m'_j, \quad j = 1, 2, \ldots, q$$
$$\sum_{i=1}^{p} n_{ij} = n'_j, \quad j = 1, 2, \ldots, q$$
$$\sum_{i=1}^{p} \alpha_{ij} = \alpha'_j, \quad j = 1, 2, \ldots, q$$
$$\sum_{i=1}^{p} \beta_{ij} = \beta'_j, \quad j = 1, 2, \ldots, q$$
$$m_{ij} - \alpha_{ij},\, n_{ij} - m_{ij}\; \alpha_{ij},\, \beta_{ij} \geq 0, \quad \forall\;\; i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q.$$

Step 6 The fuzzy multi-objective linear programming problem, obtained in Step 5, can be converted into the following crisp multi-objective linear programming problem:

$$Minimize\; \Re\Big((m_0^k, n_0^k, \alpha_0^k, \beta_0^k)_{LR}\Big), \qquad k = 1, 2, \ldots, K$$

subject to same restrictions given in $(P_{5.2})$.

Step 7 Using the existing formula [109], $\Re(m_0^k, n_0^k, \alpha_0^k, \beta_0^k)_{LR} = \frac{1}{2}\Big(\int_0^1 (m_0^k - \alpha_0^k L^{-1}(\lambda))d\lambda + \int_0^1 (n_0^k + \beta_0^k R^{-1}(\lambda))d\lambda\Big)$, the crisp multi-objective linear programming problem, obtained in Step 6 can be converted into following crisp multi-objective linear programming problem:

$$Minimize\; \tfrac{1}{2}\Big(\int_0^1 (m_0^k - \alpha_0^k L^{-1}(\lambda))d\lambda + \int_0^1 (n_0^k + \beta_0^k R^{-1}(\lambda))d\lambda\Big), \quad k = 1, 2, \ldots, K$$

subject to same restrictions given in $(P_{5.2})$.

Step 8 Solve the crisp multi-objective linear programming problem, obtained in Step 7, by using any classical multi-objective linear programming approach to find the optimal solution $\{m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\}$.

Step 9 Find the fuzzy optimal solution $\{\widetilde{x}_{ij}\}$ by putting the values of $m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}$ in $\widetilde{x}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$.

Step 10 Find the minimum value of each objective function by putting the values of $\widetilde{x}_{ij}$ in $\sum_{i=1}^{p}\sum_{j=1}^{q}(\widetilde{c}_{ij}^{k} \otimes \widetilde{x}_{ij})$.

Remark 5.5 Let $\widetilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij},)_{LR}$ with $L(x) = R(x) = Maximum\{0, 1 - x^2\}$. Then, $\Re(\widetilde{A}) = \frac{1}{6}(3m_{ij} + 3n_{ij} + 2\beta_{ij} - 2\alpha_{ij})$.

5.4.1 Advantages of the proposed method

The existing method [68] can be used to find the fuzzy optimal solution of only those fully fuzzy multi-objective transportation problems wherein all the parameters are represented by trapezoidal fuzzy numbers; But, it cannot be used for solving those fully fuzzy multi-objective transportation problems in which the parameters are either represented by LR fuzzy numbers or LR flat fuzzy numbers. Since, triangular fuzzy numbers, trapezoidal fuzzy numbers and LR fuzzy numbers are particular types of LR flat fuzzy numbers; So, the methods, proposed in this chapter, can be used for solving all such fully fuzzy multi-objective transportation problems in which all the parameters are represented by triangular fuzzy numbers or trapezoidal fuzzy numbers or LR fuzzy numbers.

5.5 Illustrative examples

In this section, to illustrate the proposed method, the fully fuzzy multi-objective transportation problems, chosen in Example 5.3 and Example 5.4, are solved by using the proposed method.

5.5.1 Fuzzy optimal solution of fully fuzzy multi-objective transportation problem chosen in Example 5.3

Step 1 Total fuzzy availability = $(150, 180, 20, 31)_{LR}$ and total fuzzy demand = $(150, 180, 20, 31)_{LR}$. Since, total fuzzy availability = total fuzzy demand; So, it is a balanced fully fuzzy multi-objective transportation problem.

Step 2 The balanced fully fuzzy multi-objective transportation problem, chosen in Example 5.3, may be formulated into the following fully fuzzy multi-objective linear programming problem:

$Minimize\ \Big((14,16,1,3)_{LR}\otimes\widetilde{x}_{11}\oplus(29,31,2,2)_{LR}\otimes\widetilde{x}_{12}\oplus(18,19,1,4)_{LR}\otimes\widetilde{x}_{13}\oplus$ $(19,21,3,2)_{LR}\otimes\widetilde{x}_{21}\oplus(14,16,2,2)_{LR}\otimes\widetilde{x}_{22}\oplus(15,17,4,3)_{LR}\otimes\widetilde{x}_{23}\Big)$

$Minimize\ \Big((15,16,2,4)_{LR}\otimes\widetilde{x}_{11}\oplus(5,6,2,2)_{LR}\otimes\widetilde{x}_{12}\oplus(8,9,3,4)_{LR}\otimes\widetilde{x}_{13}\oplus$ $(7,8,1,1)_{LR}\otimes\widetilde{x}_{21}\oplus(13,14,2,2)_{LR}\otimes\widetilde{x}_{22}\oplus(11,12,3,6)_{LR}\otimes\widetilde{x}_{23}\Big)$

subject to

$$\widetilde{x}_{11}\oplus\widetilde{x}_{12}\oplus\widetilde{x}_{13}=(90,92,10,3)_{LR}$$

$$\widetilde{x}_{21}\oplus\widetilde{x}_{22}\oplus\widetilde{x}_{23}=(60,88,10,28)_{LR}$$

$$\widetilde{x}_{11}\oplus\widetilde{x}_{21}=(40,50,5,15)_{LR}$$

$$\widetilde{x}_{12}\oplus\widetilde{x}_{22}=(30,40,6,6)_{LR}$$

$$\widetilde{x}_{13}\oplus\widetilde{x}_{23}=(80,90,9,10)_{LR}$$

$\widetilde{x}_{11},\widetilde{x}_{12},\widetilde{x}_{13},\widetilde{x}_{21},\widetilde{x}_{22},\widetilde{x}_{23}$ are non-negative LR flat fuzzy numbers.

Step 3 Assuming $\widetilde{x}_{ij}=(m_{ij},n_{ij},\alpha_{ij},\beta_{ij})_{LR}$ and using arithmetic operations, defined in Section 5.1.2, the fully fuzzy multi-objective linear programming problem, obtained in Step 2, can be converted into the following fully fuzzy linear programming problem:

$Minimize\ (14m_{11}+29m_{12}+18m_{13}+19m_{21}+14m_{22}+15m_{23},16n_{11}+31n_{12}+$ $19n_{13}+21n_{21}+16n_{22}+17n_{23},m_{11}+13\alpha_{11}+2m_{12}+27\alpha_{12}+m_{13}+17\alpha_{13}+$ $3m_{21}+16\alpha_{21}+2m_{22}+12\alpha_{22}+4m_{23}+11\alpha_{23},3n_{11}+19\beta_{11}+2n_{12}+33\beta_{12}+$ $4n_{13}+23\beta_{13}+2n_{21}+23\beta_{21}+2n_{22}+18\beta_{22}+3n_{23}+20\beta_{23})_{LR}$

$Minimize\ (15m_{11}+5m_{12}+8m_{13}+7m_{21}+13m_{22}+11m_{23}, 16n_{11}+6n_{12}+9n_{13}+8n_{21}+14n_{22}+12n_{23}, 2m_{11}+13\alpha_{11}+2m_{12}+3\alpha_{12}+3m_{13}+5\alpha_{13}+m_{21}+6\alpha_{21}+2m_{22}+11\alpha_{22}+3m_{23}+8\alpha_{23}, 4n_{11}+20\beta_{11}+2n_{12}+8\beta_{12}+4n_{13}+13\beta_{13}+n_{21}+9\beta_{21}+2n_{22}+16\beta_{22}+6n_{23}+18\beta_{23})_{LR}$

subject to

$$\Big(\sum_{j=1}^{3} m_{1j}, \sum_{j=1}^{3} n_{1j}, \sum_{j=1}^{3} \alpha_{1j}, \sum_{j=1}^{3} \beta_{1j}\Big)_{LR} = (90, 92, 10, 3)_{LR}$$
$$\Big(\sum_{j=1}^{3} m_{2j}, \sum_{j=1}^{3} n_{2j}, \sum_{j=1}^{3} \alpha_{2j}, \sum_{j=1}^{3} \beta_{2j}\Big)_{LR} = (60, 88, 10, 28)_{LR}$$
$$\Big(\sum_{i=1}^{2} m_{i1}, \sum_{i=1}^{2} n_{i1}, \sum_{i=1}^{2} \alpha_{i1}, \sum_{i=1}^{2} \beta_{i1}\Big)_{LR} = (40, 50, 5, 15)_{LR}$$
$$\Big(\sum_{i=1}^{2} m_{i2}, \sum_{i=1}^{2} n_{i2}, \sum_{i=1}^{2} \alpha_{i2}, \sum_{i=1}^{2} \beta_{i2}\Big)_{LR} = (30, 40, 6, 6)_{LR}$$
$$\Big(\sum_{i=1}^{2} m_{i3}, \sum_{i=1}^{2} n_{i3}, \sum_{i=1}^{2} \alpha_{i3}, \sum_{i=1}^{2} \beta_{i3}\Big)_{LR} = (80, 90, 9, 10)_{LR}$$

$(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ is a non-negative LR flat fuzzy number.

Step 4 Using Definitions 5.5 and 5.6, the fully fuzzy multi-objective linear programming problem, obtained in Step 3, can be converted into the fully fuzzy multi-objective linear programming problem ($P_{5.3}$):

$Minimize\ (14m_{11}+29m_{12}+18m_{13}+19m_{21}+14m_{22}+15m_{23}, 16n_{11}+31n_{12}+19n_{13}+21n_{21}+16n_{22}+17n_{23}, m_{11}+13\alpha_{11}+2m_{12}+27\alpha_{12}+m_{13}+17\alpha_{13}+3m_{21}+16\alpha_{21}+2m_{22}+12\alpha_{22}+4m_{23}+11\alpha_{23}, 3n_{11}+19\beta_{11}+2n_{12}+33\beta_{12}+4n_{13}+23\beta_{13}+2n_{21}+23\beta_{21}+2n_{22}+18\beta_{22}+3n_{23}+20\beta_{23})_{LR}$

$Minimize\ (15m_{11}+5m_{12}+8m_{13}+7m_{21}+13m_{22}+11m_{23}, 16n_{11}+6n_{12}+9n_{13}+8n_{21}+14n_{22}+12n_{23}, 2m_{11}+13\alpha_{11}+2m_{12}+3\alpha_{12}+3m_{13}+5\alpha_{13}+m_{21}+6\alpha_{21}+2m_{22}+11\alpha_{22}+3m_{23}+8\alpha_{23}, 4n_{11}+20\beta_{11}+2n_{12}+8\beta_{12}+4n_{13}+13\beta_{13}+n_{21}+9\beta_{21}+2n_{22}+16\beta_{22}+6n_{23}+18\beta_{23})_{LR}$

subject to

$$\begin{array}{llllr}
\sum_{j=1}^{3} m_{1j} = 90, & \sum_{j=1}^{3} n_{1j} = 92, & \sum_{j=1}^{3} \alpha_{1j} = 10, & \sum_{j=1}^{3} \beta_{1j} = 3 & \\
\sum_{j=1}^{3} m_{2j} = 60, & \sum_{j=1}^{3} n_{2j} = 88, & \sum_{j=1}^{3} \alpha_{2j} = 10, & \sum_{j=1}^{3} \beta_{2j} = 28 & \\
\sum_{i=1}^{2} m_{i1} = 40, & \sum_{i=1}^{2} n_{i1} = 50, & \sum_{i=1}^{2} \alpha_{i1} = 5, & \sum_{i=1}^{2} \beta_{i1} = 15 & (P_{5.3}) \\
\sum_{i=1}^{2} m_{i2} = 30, & \sum_{i=1}^{2} n_{i2} = 40, & \sum_{i=1}^{2} \alpha_{i2} = 6, & \sum_{i=1}^{2} \beta_{i2} = 6 & \\
\sum_{i=1}^{2} m_{i3} = 80, & \sum_{i=1}^{2} n_{i3} = 90, & \sum_{i=1}^{2} \alpha_{i3} = 9, & \sum_{i=1}^{2} \beta_{i3} = 10 & \\
\multicolumn{4}{l}{m_{ij} - \alpha_{ij},\ n_{ij} - m_{ij},\ \alpha_{ij},\ \beta_{ij} \geq 0,\ \forall\ i = 1, 2;\ j = 1, 2, 3.} &
\end{array}$$

Step 5 Using Step 6 of the method, proposed in Section 5.4 and Remark 5.5, the fuzzy optimal solution of the fuzzy multi-objective linear programming problem $(P_{5.3})$, can be obtained by solving the following crisp multi-objective linear programming problem:

$Minimize \left(\frac{1}{6}(40m_{11} + 54n_{11} - 26\alpha_{11} + 38\beta_{11} + 83m_{12} + 97n_{12} - 54\alpha_{12} + 66\beta_{12} + 52m_{13} + 65n_{13} - 34\alpha_{13} + 46\beta_{13} + 51m_{21} + 67n_{21} - 32\alpha_{21} + 46\beta_{21} + 38m_{22} + 52n_{22} - 24\alpha_{22} + 36\beta_{22} + 37m_{23} + 57n_{23} - 22\alpha_{23} + 40\beta_{23})\right)$

$Minimize \left(\frac{1}{6}(41m_{11} + 56n_{11} - 26\alpha_{11} + 40\beta_{11} + 11m_{12} + 22n_{12} - 6\alpha_{12} + 16\beta_{12} + 18m_{13} + 35n_{13} - 10\alpha_{13} + 26\beta_{13} + 19m_{21} + 26n_{21} - 12\alpha_{21} + 18\beta_{21} + 35m_{22} + 46n_{22} - 22\alpha_{22} + 32\beta_{22} + 27m_{23} + 48n_{23} - 16\alpha_{23} + 36\beta_{23})\right)$

subject to same restrictions given in $(P_{5.3})$.

Step 6 Using fuzzy programming technique [165], the optimal solution of the crisp multi-objective linear programming problem, obtained in Step 5, is

$m_{11} = 9.30$, $n_{11} = 9.30$, $\alpha_{11} = 0.30$, $\beta_{11} = 0$, $m_{12} = 0.70$, $n_{12} = 0.70$, $\alpha_{12} = 0.70$, $\beta_{12} = 0$, $m_{13} = 80$, $n_{13} = 82$, $\alpha_{13} = 9$, $\beta_{13} = 3$, $m_{21} = 30.70$, $n_{21} = 40.70$, $\alpha_{21} = 4.70$, $\beta_{21} = 15$, $m_{22} = 29.30$, $n_{22} = 39.30$, $\alpha_{22} = 5.30$, $\beta_{22} = 6$, $m_{23} = 0$, $n_{23} = 8$, $\alpha_{23} = 0$, $\beta_{23} = 7$.

Step 7 Putting the values of $m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}$ in $\widetilde{x}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, the fuzzy optimal solution is $\widetilde{x}_{11} = (9.30, 9.30, 0.30, 0)_{LR}$, $\widetilde{x}_{12} = (0.70, 0.70, 0.70, 0)_{LR}$, $\widetilde{x}_{13} = (80, 82, 9, 3)_{LR}$, $\widetilde{x}_{21} = (30.70, 40.70, 4.70, 15)_{LR}$, $\widetilde{x}_{22} = (29.30, 39.30, 5.30, 6)_{LR}$, $\widetilde{x}_{23} = (0, 8, 0, 7)_{LR}$.

Step 8 Putting the values of $\widetilde{x}_{11}, \widetilde{x}_{12}, \widetilde{x}_{13}, \widetilde{x}_{21}, \widetilde{x}_{22}, \widetilde{x}_{23}$ in $\Big((14, 16, 1, 3)_{LR} \otimes \widetilde{x}_{11} \oplus (29, 31, 2, 2)_{LR} \otimes \widetilde{x}_{12} \oplus (18, 19, 1, 4)_{LR} \otimes \widetilde{x}_{13} \oplus (19, 21, 3, 2)_{LR} \otimes \widetilde{x}_{21} \oplus (14, 16, 2, 2)_{LR} \otimes \widetilde{x}_{22} \oplus (15, 17, 4, 3)_{LR} \otimes \widetilde{x}_{23}\Big)$ and in $\Big((15, 16, 2, 4)_{LR} \otimes \widetilde{x}_{11} \oplus (5, 6, 2, 2)_{LR} \otimes \widetilde{x}_{12} \oplus (8, 9, 3, 4)_{LR} \otimes \widetilde{x}_{13} \oplus (7, 8, 1, 1)_{LR} \otimes \widetilde{x}_{21} \oplus (13, 14, 2, 2)_{LR} \otimes \widetilde{x}_{22} \oplus (11, 12, 3, 6)_{LR} \otimes \widetilde{x}_{23}\Big)$, the fuzzy optimal values of first and second objective are $(2584, 3348, 556, 1203.30)_{LR}$ and $(1378.80, 1862.80, 486.80, 929.90)_{LR}$ respectively.

5.5.2 Fuzzy optimal solution of fully fuzzy multi-objective transportation problem chosen in Example 5.4

Step 1 Total fuzzy availability = $(150, 162, 20, 8)_{LR}$ and total fuzzy demand = $(120, 150, 14, 31)_{LR}$. Since, total fuzzy availability $\neq$ total fuzzy demand; So, it is an unbalanced fully fuzzy transportation problem. So, using Case (c) of Step 1 of the proposed method, the chosen unbalanced fully fuzzy multi-objective transportation problem can be converted into a balanced fully fuzzy multi-objective transportation problem by introducing a dummy source S_3 with fuzzy availability $(0, 18, 0, 23)_{LR}$ and a dummy destination D_4 with fuzzy demand $(30, 30, 6, 0)_{LR}$.

Step 2 Assuming the fuzzy penalties $(\widetilde{c}^k_{ij})$ for transporting a unit quantity of the product from dummy source S_3 to all destinations and from all sources to dummy destination D_4, as zero LR flat fuzzy numbers, i.e., $\widetilde{c}^1_{31} = \widetilde{c}^1_{32} = \widetilde{c}^1_{33} = \widetilde{c}^1_{14} =$

$\widetilde{c}^1_{24} = \widetilde{c}^1_{34} = (0,0,0,0)_{LR}$ and $\widetilde{c}^2_{31} = \widetilde{c}^2_{32} = \widetilde{c}^2_{33} = \widetilde{c}^2_{14} = \widetilde{c}^2_{24} = \widetilde{c}^2_{34} = (0,0,0,0)_{LR}$, the balanced fully fuzzy multi-objective transportation problem, obtained in Step 1, can be formulated into the following fully fuzzy multi-objective linear programming problem:

$Minimize \Big((14,16,1,3)_{LR}\otimes\widetilde{x}_{11}\oplus(29,31,2,2)_{LR}\otimes\widetilde{x}_{12}\oplus(18,19,1,4)_{LR}\otimes\widetilde{x}_{13}\oplus (0,0,0,0)_{LR}\otimes\widetilde{x}_{14}\oplus(19,21,3,2)_{LR}\otimes\widetilde{x}_{21}\oplus(14,16,2,2)_{LR}\otimes\widetilde{x}_{22}\oplus(15,17,4,3)_{LR}\otimes \widetilde{x}_{23}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{24}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{31}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{32}\oplus(0,0,0,0)_{LR}\otimes \widetilde{x}_{33}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{34}\Big)$

$Minimize \Big((15,16,2,4)_{LR}\otimes\widetilde{x}_{11}\oplus(5,6,2,2)_{LR}\otimes\widetilde{x}_{12}\oplus(8,9,3,4)_{LR}\otimes\widetilde{x}_{13}\oplus (0,0,0,0)_{LR}\otimes\widetilde{x}_{14}\oplus(7,8,1,1)_{LR}\otimes\widetilde{x}_{21}\oplus(13,14,2,2)_{LR}\otimes\widetilde{x}_{22}\oplus(11,12,3,6)_{LR}\otimes \widetilde{x}_{23}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{24}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{31}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{32}\oplus(0,0,0,0)_{LR}\otimes \widetilde{x}_{33}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{34}\Big)$

subject to

$$\widetilde{x}_{11}\oplus\widetilde{x}_{12}\oplus\widetilde{x}_{13}\oplus\widetilde{x}_{14} = (90,92,10,3)_{LR}$$

$$\widetilde{x}_{21}\oplus\widetilde{x}_{22}\oplus\widetilde{x}_{23}\oplus\widetilde{x}_{24} = (60,70,10,5)_{LR}$$

$$\widetilde{x}_{31}\oplus\widetilde{x}_{32}\oplus\widetilde{x}_{33}\oplus\widetilde{x}_{34} = (0,18,0,23)_{LR}$$

$$\widetilde{x}_{11}\oplus\widetilde{x}_{21}\oplus\widetilde{x}_{31} = (40,50,5,15)_{LR}$$

$$\widetilde{x}_{12}\oplus\widetilde{x}_{22}\oplus\widetilde{x}_{32} = (30,40,6,6)_{LR}$$

$$\widetilde{x}_{13}\oplus\widetilde{x}_{23}\oplus\widetilde{x}_{33} = (50,60,3,10)_{LR}$$

$$\widetilde{x}_{14}\oplus\widetilde{x}_{24}\oplus\widetilde{x}_{34} = (30,30,6,0)_{LR}$$

$\widetilde{x}_{11},\widetilde{x}_{12},\widetilde{x}_{13},\widetilde{x}_{14},\widetilde{x}_{21},\widetilde{x}_{22},\widetilde{x}_{23},\widetilde{x}_{24},\widetilde{x}_{31},\widetilde{x}_{32},\widetilde{x}_{33},\widetilde{x}_{34}$ are non-negative LR flat fuzzy numbers.

Step 3 Using Step 3 to Step 7, of the method, proposed in Section 5.4 and Remark 5.5, the fully fuzzy multi-objective linear programming problem, obtained in Step

2, can be converted into the following crisp multi-objective linear programming problem:

$Minimize\ \Big(\frac{1}{6}(40m_{11}+54n_{11}-26\alpha_{11}+38\beta_{11}+83m_{12}+97n_{12}-54\alpha_{12}+66\beta_{12}+52m_{13}+65n_{13}-34\alpha_{13}+46\beta_{13}+51m_{21}+67n_{21}-32\alpha_{21}+46\beta_{21}+38m_{22}+52n_{22}-24\alpha_{22}+36\beta_{22}+37m_{23}+57n_{23}-22\alpha_{23}+40\beta_{23})\Big)$

$Minimize\ \Big(\frac{1}{6}(41m_{11}+56n_{11}-26\alpha_{11}+40\beta_{11}+11m_{12}+22n_{12}-6\alpha_{12}+16\beta_{12}+18m_{13}+35n_{13}-10\alpha_{13}+26\beta_{13}+19m_{21}+26n_{21}-12\alpha_{21}+18\beta_{21}+35m_{22}+46n_{22}-22\alpha_{22}+32\beta_{22}+27m_{23}+48n_{23}-16\alpha_{23}+36\beta_{23})\Big)$

subject to

$$
\begin{array}{llll}
\sum_{j=1}^{4} m_{1j} = 90, & \sum_{j=1}^{4} n_{1j} = 92, & \sum_{j=1}^{4} \alpha_{1j} = 10, & \sum_{j=1}^{4} \beta_{1j} = 3 \\
\sum_{j=1}^{4} m_{2j} = 60, & \sum_{j=1}^{4} n_{2j} = 70, & \sum_{j=1}^{4} \alpha_{2j} = 10, & \sum_{j=1}^{4} \beta_{2j} = 5 \\
\sum_{j=1}^{4} m_{3j} = 0, & \sum_{j=1}^{4} n_{3j} = 18, & \sum_{j=1}^{4} \alpha_{3j} = 0, & \sum_{j=1}^{4} \beta_{3j} = 23 \\
\sum_{i=1}^{3} m_{i1} = 40, & \sum_{i=1}^{3} n_{i1} = 50, & \sum_{i=1}^{3} \alpha_{i1} = 5, & \sum_{i=1}^{3} \beta_{i1} = 15 \\
\sum_{i=1}^{3} m_{i2} = 30, & \sum_{i=1}^{3} n_{i2} = 40, & \sum_{i=1}^{3} \alpha_{i2} = 6, & \sum_{i=1}^{3} \beta_{i2} = 6 \\
\sum_{i=1}^{3} m_{i3} = 50, & \sum_{i=1}^{3} n_{i3} = 60, & \sum_{i=1}^{3} \alpha_{i3} = 3, & \sum_{i=1}^{3} \beta_{i3} = 10 \\
\sum_{i=1}^{3} m_{i4} = 30, & \sum_{i=1}^{3} n_{i4} = 30, & \sum_{i=1}^{3} \alpha_{i4} = 6, & \sum_{i=1}^{3} \beta_{i4} = 0
\end{array}
$$

$$m_{ij}-\alpha_{ij},\ n_{ij}-m_{ij},\ \alpha_{ij},\ \beta_{ij} \geq 0,\ \forall\ i=1,2,3;\ j=1,2,3,4.$$

Step 4 Using fuzzy programming technique [165], the optimal solution of the crisp multi-objective linear programming problem, obtained in Step 3, is:

$m_{11}=7.61,\ n_{11}=7.61,\ \alpha_{11}=0,\ \beta_{11}=0,\ m_{12}=2.39,\ n_{12}=2.39,\ \alpha_{12}=1,\ \beta_{12}=0,\ m_{13}=50,\ n_{13}=52,\ \alpha_{13}=3,\ \beta_{13}=3,\ m_{14}=30,\ n_{14}=30,\ \alpha_{14}=6,\ \beta_{14}=0,\ m_{21}=32.39,\ n_{21}=42.39,\ \alpha_{21}=5,\ \beta_{21}=5,\ m_{22}=27.61,\ n_{22}=27.61,\ \alpha_{22}=5,\ \beta_{22}=0,\ m_{23}=0,\ n_{23}=0,\ \alpha_{23}=0,\ \beta_{23}=0,\ m_{24}=0,\ n_{24}=$

$0,\ \alpha_{24}=0,\ \beta_{24}=0,\ m_{31}=0,\ n_{31}=0,\ \alpha_{31}=0,\ \beta_{31}=10,\ m_{32}=0,\ n_{32}=10,\ \alpha_{32}=0,\ \beta_{32}=6,\ m_{33}=0,\ n_{33}=8,\ \alpha_{33}=0,\ \beta_{33}=7,\ m_{34}=0,\ n_{34}=0,\ \alpha_{34}=0,\ \beta_{34}=0.$

Step 5 Putting the values of $m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}$ in $\widetilde{x}_{ij}=(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, the fuzzy optimal solution is $\widetilde{x}_{11}=(7.61, 7.61, 0, 0)_{LR}$, $\widetilde{x}_{12}=(2.39, 2.39, 1, 0)_{LR}$, $\widetilde{x}_{13}=(50, 52, 3, 3)_{LR}$, $\widetilde{x}_{14}=(30, 30, 6, 0)_{LR}$, $\widetilde{x}_{21}=(32.39, 42.39, 5, 5)_{LR}$, $\widetilde{x}_{22}=(27.61, 27.61, 5, 0)_{LR}$, $\widetilde{x}_{23}=(0,0,0,0)_{LR}$, $\widetilde{x}_{24}=(0,0,0,0)_{LR}$, $\widetilde{x}_{31}=(0,0,0,10)_{LR}$, $\widetilde{x}_{32}=(0,10,0,6)_{LR}$, $\widetilde{x}_{33}=(0,8,0,7)_{LR}$, $\widetilde{x}_{34}=(0,0,0,0)_{LR}$.

Step 6 Putting the values of $\widetilde{x}_{11}, \widetilde{x}_{12}, \widetilde{x}_{13}, \widetilde{x}_{14}, \widetilde{x}_{21}, \widetilde{x}_{22}, \widetilde{x}_{23}, \widetilde{x}_{24}, \widetilde{x}_{31}, \widetilde{x}_{32}, \widetilde{x}_{33}, \widetilde{x}_{34}$ in $\Big((14,16,1,3)_{LR}\otimes\widetilde{x}_{11}\oplus(29,31,2,2)_{LR}\otimes\widetilde{x}_{12}\oplus(18,19,1,4)_{LR}\otimes\widetilde{x}_{13}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{14}\oplus(19,21,3,2)_{LR}\otimes\widetilde{x}_{21}\oplus(14,16,2,2)_{LR}\otimes\widetilde{x}_{22}\oplus(15,17,4,3)_{LR}\otimes\widetilde{x}_{23}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{24}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{31}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{32}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{33}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{34}\Big)$ and in $\Big((15,16,2,4)_{LR}\otimes\widetilde{x}_{11}\oplus(5,6,2,2)_{LR}\otimes\widetilde{x}_{12}\oplus(8,9,3,4)_{LR}\otimes\widetilde{x}_{13}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{14}\oplus(7,8,1,1)_{LR}\otimes\widetilde{x}_{21}\oplus(13,14,2,2)_{LR}\otimes\widetilde{x}_{22}\oplus(11,12,3,6)_{LR}\otimes\widetilde{x}_{23}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{24}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{31}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{32}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{33}\oplus(0,0,0,0)_{LR}\otimes\widetilde{x}_{34}\Big)$, the fuzzy optimal values of first and second objective are $(2077.80, 2515.80, 432.78, 559.61)_{LR}$ and $(1111.76, 1329.76, 360.61, 424.83)_{LR}$ respectively.

5.6 Comparative study

To show the advantages of the method proposed in this chapter over the existing method [68], the results of the existing problems [68] and the problems chosen in this chapter obtained with the use of the existing method [68] and the method proposed in this chapter are compared in Table 5.9.

Table 5.9: Comparison of results obtained by using existing method [68] and proposed method

Example	Existing method [68]	Proposed method
5.1	Fuzzy optimal value of 1^{st} objective is $= (70.99, 113.99, 181.99, 325.99)$ Fuzzy optimal value of 2^{nd} objective is $= (102.01, 149.01, 210.01, 369.76)$	Fuzzy optimal value of 1^{st} objective is $= (113.99, 181.99, 43, 144)_{LR}$ Fuzzy optimal value of 2^{nd} objective is $= (149.01, 210.01, 47, 159.75)_{LR}$ $L(x) = R(x) = \{0, 1 - x\}$
5.2	Fuzzy optimal value of 1^{st} objective is $= (69.35, 111.25, 167.79, 309.79)$ Fuzzy optimal value of 2^{nd} objective is $= (104.75, 151.75, 200.75, 341)$	Fuzzy optimal value of 1^{st} objective is $= (111.25, 167.79, 41.89, 142)_{LR}$ Fuzzy optimal value of 2^{nd} objective is $= (151.75, 200.75, 47, 140.25)_{LR}$ $L(x) = R(x) = \{0, 1 - x\}$
5.3	Not applicable	Fuzzy optimal value of 1^{st} objective $(2584, 3384, 556, 1203.30)_{LR}$ Fuzzy optimal value of 2^{nd} objective is $= (1378.80, 1862.80, 486.80, 929.90)_{LR}$
5.4	Not applicable	Fuzzy optimal value of 1^{st} objective is $= (2077.80, 2515.80, 432.78, 559.61)_{LR}$ Fuzzy optimal value of 2^{nd} objective is $= (1111.76, 1329.76, 360.61, 424.83)_{LR}$

The results, shown in Table 5.9, can be explained as follows:

(1) The existing method [68], can be used only for solving such fully fuzzy multi-objective transportation problems in which all the parameters are either represented by triangular fuzzy numbers or trapezoidal fuzzy numbers. Since, in the fully fuzzy multi-objective transportation problems, chosen in Example 5.1 and Example 5.2, all the parameters are represented by trapezoidal fuzzy numbers; So, these problems can be solved by using the existing method

[68]. However, in the fully fuzzy multi-objective transportation problems, chosen in Example 5.3 and Example 5.4, all the parameters are represented by such LR flat fuzzy numbers which are neither triangular fuzzy numbers nor trapezoidal fuzzy numbers. So, the fully fuzzy multi-objective transportation problem, chosen in Examples 5.3 and 5.4, can not be solved by using the existing method [68].

(2) The method, proposed in this chapter, can be used for solving all such fully fuzzy multi-objective transportation problems in which all the parameters are represented by LR flat fuzzy numbers. As discussed in Remark 5.3 that the trapezoidal fuzzy numbers are particular types of LR flat fuzzy numbers; So, all the fully fuzzy multi-objective transportation problems, chosen in Examples 5.1 to 5.4, can be solved by using the method proposed in this chapter.

(3) It is obvious from the results shown in Table 5.9 that the solutions obtained by using the existing method [68] and the proposed method for Example 5.1 and Example 5.2 are same.

5.7 Conclusion

On the basis of the presented study, it can be concluded that the problems which can be solved by using the existing method [68], can also be solved by the method proposed in this chapter. However, as discussed in Section 5.3, there exist several problems which can be solved by the method, proposed in this chapter, but can not be solved by using the existing method [68]. Hence, it is better to use the method, proposed in this chapter, as compared to the existing method [68].

Chapter 6

Optimal Solution Of Fuzzy Multi-objective Transportation Problems With Parameters As Interval-valued LR Flat Fuzzy Numbers

There are several methods in the literature for solving transportation problems where the parameters are fuzzy numbers. Chiang [44] pointed out that it is better to represent the parameters as level (λ, ρ) interval-valued triangular fuzzy numbers instead of fuzzy numbers and proposed a method to find the optimal solution of single objective transportation problems by representing the availability and demand as level (λ, ρ) interval-valued triangular fuzzy numbers. In this chapter, the limitations of this existing method are pointed out and to overcome these limitations a new method is proposed. To illustrate the proposed method numerical examples have been considered. A comparative study has been given to show the advantages of the proposed method over the existing method.

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6.1 Preliminaries

In this section, some basic definitions and arithmetic operations of level (λ, ρ) interval-valued LR flat fuzzy numbers are presented [44].

6.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 6.1 An interval-valued fuzzy set $\widetilde{A}$ on $\mathbb{R}$ is given by

$\widetilde{A} = \{(x, [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)]) : x \in \mathbb{R}\}$ where $\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x) \in [0,1]$ and $\mu_{\widetilde{A}^L}(x) \leq \mu_{\widetilde{A}^U}(x)$ $\forall$ $x \in \mathbb{R}$ and is denoted as $\widetilde{A} = [\widetilde{A}^L, \widetilde{A}^U]$. This means that the grade of membership of x belongs to the interval $[\mu_{\widetilde{A}^L}(x),\ \mu_{\widetilde{A}^U}(x)]$, the least grade of membership at x is $\mu_{\widetilde{A}^L}(x)$ and greatest grade of membership at x is $\mu_{\widetilde{A}^U}(x)$

Let

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} \lambda L(\frac{m-x}{\alpha}), & x \leq m \\ \lambda, & m \leq x \leq n \\ \lambda R(\frac{x-n}{\beta}), & x \geq n \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu_{\widetilde{A}^U}(x) = \begin{cases} \rho L(\frac{m-x}{\alpha'}), & x \leq m \\ \rho, & m \leq x \leq n \\ \rho R(\frac{x-n}{\beta'}), & x \geq n \\ 0, & \text{otherwise} \end{cases}$$

where, α, β, α' and β' are non-negative real numbers satisfying $0 \leq \alpha \leq \alpha' \leq m$, $0 \leq \beta \leq \beta'$ and $0 < \lambda < \rho \leq 1$.

Then, interval-valued fuzzy set $\widetilde{A} = \{(x, [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)]) : x \in \mathbb{R}\}$ is called a level (λ, ρ) interval-valued LR flat fuzzy number and is denoted as $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$.

Definition 6.2 A level (λ, ρ) interval-valued LR flat fuzzy number

$\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is said to be a non-negative level (λ, ρ) interval-valued LR flat fuzzy number if and only if $m - \alpha' \geq 0$.

Definition 6.3 A level (λ, ρ) interval-valued LR flat fuzzy number $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is said to be a zero level (λ, ρ) interval-valued LR flat fuzzy number if and only if $m = 0$, $n = 0$, $\alpha = 0$, $\beta = 0$, $\alpha' = 0$ and $\beta' = 0$.

Definition 6.4 Two level (λ, ρ) interval-valued LR flat fuzzy numbers $\widetilde{A}_1 = [(m_1, n_1, \alpha_1, \beta_1; \lambda)_{LR}, (m_1, n_1, \alpha'_1, \beta'_1; \rho)_{LR}]$ and $\widetilde{A}_2 = [(m_2, n_2, \alpha_2, \beta_2; \lambda)_{LR}, (m_2, n_2, \alpha'_2, \beta'_2; \rho)_{LR}]$ are said to be equal if and only if $m_1 = m_2$, $n_1 = n_2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\alpha'_1 = \alpha'_2$ and $\beta'_1 = \beta'_2$.

Remark 6.1 If $m = n$, then a level (λ, ρ) interval-valued LR flat fuzzy number $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is said to be a level (λ, ρ) interval-valued LR fuzzy number and is denoted as $[(m, m, \alpha, \beta; \lambda)_{LR}, (m, m, \alpha', \beta'; \rho)_{LR}]$ or $[(m, \alpha, \beta; \lambda)_{LR}, (m, \alpha', \beta'; \rho)_{LR}]$ or $[(n, n, \alpha, \beta; \lambda)_{LR}, (n, n, \alpha', \beta'; \rho)_{LR}]$ or $[(n, \alpha, \beta; \lambda)_{LR}, (n, \alpha', \beta'; \rho)_{LR}]$.

Remark 6.2 If $m \neq n$ and $L(x) = R(x) = Maximum\{0, 1 - x\}$, then a level (λ, ρ) interval-valued LR flat fuzzy number $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is said to be a level (λ, ρ) interval-valued trapezoidal fuzzy number and is denoted as $[(a, b, c, d; \lambda), (p, b, c, r; \rho)]$ where, $a = m - \alpha$, $b = m$, $c = n$, $d = n + \beta$, $p = m - \alpha'$, $r = n + \beta'$ and $p \leq a \leq b \leq c \leq d \leq r$.

Remark 6.3 If $m = n$ and $L(x) = R(x) = Maximum\{0, 1 - x\}$, then a level (λ, ρ) interval-valued LR flat fuzzy number $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is said to be a level (λ, ρ) interval-valued triangular fuzzy number and is denoted as $[(a, b, c; \lambda), (p, b, r; \rho)]$ where, $a = m - \alpha$ (or $n - \alpha$), $b = m$ (or n), $c = m + \beta$ (or

$n + \beta$), $p = m - \alpha'$ (or $n - \alpha'$), $r = m + \beta'$ (or $n + \beta'$) and $p \leq a \leq b \leq c \leq r$.

Remark 6.4 If $\alpha = \alpha'$, $\beta = \beta'$ and $\lambda = \rho$, then a level (λ, ρ) interval-valued LR flat fuzzy number $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is called a level λ LR flat fuzzy number and is denoted as $\widetilde{A} = (m, n, \alpha, \beta; \lambda)_{LR}$.

Remark 6.5 If $m = n$ and $\alpha = \beta = 0$, then a level λ LR flat fuzzy number $\widetilde{A} = (m, n, \alpha, \beta; \lambda)_{LR}$ is said to be a real number and is denoted as $\widetilde{A} = (m, m, 0, 0; \lambda)_{LR}$.

6.1.2 Arithmetic operations

In this section, addition and multiplication operations of level (λ, ρ) interval-valued LR flat fuzzy numbers are presented.

(i) Let $\widetilde{A}_1 = [(m_1, n_1, \alpha_1, \beta_1; \lambda)_{LR}, (m_1, n_1, \alpha'_1, \beta'_1; \rho)_{LR}]$ and $\widetilde{A}_2 = [(m_2, n_2, \alpha_2, \beta_2; \lambda)_{LR}, (m_2, n_2, \alpha'_2, \beta'_2; \rho)_{LR}]$ be two level (λ, ρ) interval-valued LR flat fuzzy numbers. Then,

$\widetilde{A}_1 \oplus \widetilde{A}_2 = [(m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2; \lambda)_{LR}, (m_1 + m_2, n_1 + n_2, \alpha'_1 + \alpha'_2, \beta'_1 + \beta'_2; \rho)_{LR}]$.

(ii) Let $\widetilde{A}_1 = [(m_1, n_1, \alpha_1, \beta_1; \lambda)_{LR}, (m_1, n_1, \alpha'_1, \beta'_1; \rho)_{LR}]$ and $\widetilde{A}_2 = [(m_2, n_2, \alpha_2, \beta_2; \lambda)_{LR}, (m_2, n_2, \alpha'_2, \beta'_2; \rho)_{LR}]$ be two non-negative level (λ, ρ) interval-valued LR flat fuzzy numbers. Then,

$\widetilde{A}_1 \otimes \widetilde{A}_2 \simeq [(m_1m_2, n_1n_2, m_1m_2 - (m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2) - n_1n_2)_{LR}, (m_1m_2, n_1n_2, m_1m_2 - (m_1 - \alpha'_1)(m_2 - \alpha'_2), (n_1 + \beta'_1)(n_2 + \beta'_2) - n_1n_2)_{LR}]$.

(iii) Let $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ be a level (λ, ρ) interval-valued LR flat fuzzy number and $k \in \mathbb{R}$ be a real number. Then,

$$k\widetilde{A} = [(km, kn, k\alpha, k\beta; \lambda)_{LR}, (km, kn, k\alpha', k\beta'; \rho)_{LR}],\ k \geq 0$$

$$k\widetilde{A} = [(kn, km, -k\beta, -k\alpha; \lambda)_{RL}, (kn, km, -k\beta', -k\alpha'; \rho)_{RL}],\ k < 0$$

6.1.3 Ranking of level (λ, ρ) interval-valued LR flat fuzzy numbers

Let $\widetilde{A}_1 = [(m_1, n_1, \alpha_1, \beta_1; \lambda)_{LR}, (m_1, n_1, \alpha'_1, \beta'_1; \rho)_{LR}]$ and $\widetilde{A}_2 = [(m_2, n_2, \alpha_2, \beta_2; \lambda)_{LR}, (m_2, n_2, \alpha'_2, \beta'_2; \rho)_{LR}]$ be two level (λ, ρ) interval-valued LR flat fuzzy numbers. Then, $\widetilde{A}_1$ and $\widetilde{A}_2$ can be compared as [44]:

(i) $\widetilde{A}_1 \prec \widetilde{A}_2$ if and only if $\Re(\widetilde{A}_1) < \Re(\widetilde{A}_2)$

(ii) $\widetilde{A}_1 \preceq \widetilde{A}_2$ if and only if $\Re(\widetilde{A}_1) \leq \Re(\widetilde{A}_2)$

(iii) $\widetilde{A}_1 \approx \widetilde{A}_2$ if and only if $\Re(\widetilde{A}_1) = \Re(\widetilde{A}_2)$

where,

$$\Re(\widetilde{A}_1) = \frac{1}{4\lambda}\Big[\int_0^{\lambda} \Big(2m_1+2n_1-\alpha_1 L^{-1}\Big(\frac{\delta}{\lambda}\Big)-\alpha'_1 L^{-1}\Big(\frac{\delta}{\rho}\Big)+\beta_1 R^{-1}\Big(\frac{\delta}{\lambda}\Big)+\beta'_1 R^{-1}\Big(\frac{\delta}{\rho}\Big)\Big)d\delta\Big] + \frac{1}{2(\rho-\lambda)}\Big[\int_{\lambda}^{\rho} \Big(m_1 + n_1 - \alpha'_1 L^{-1}\Big(\frac{\delta}{\rho}\Big) + \beta'_1 R^{-1}\Big(\frac{\delta}{\rho}\Big)\Big)d\delta\Big]$$

$$\Re(\widetilde{A}_2) = \frac{1}{4\lambda}\Big[\int_0^{\lambda} \Big(2m_2+2n_2-\alpha_2 L^{-1}\Big(\frac{\delta}{\lambda}\Big)-\alpha'_2 L^{-1}\Big(\frac{\delta}{\rho}\Big)+\beta_2 R^{-1}\Big(\frac{\delta}{\lambda}\Big)+\beta'_2 R^{-1}\Big(\frac{\delta}{\rho}\Big)\Big)d\delta\Big] + \frac{1}{2(\rho-\lambda)}\Big[\int_{\lambda}^{\rho} \Big(m_2 + n_2 - \alpha'_2 L^{-1}\Big(\frac{\delta}{\rho}\Big) + \beta'_2 R^{-1}\Big(\frac{\delta}{\rho}\Big)\Big)d\delta\Big]$$

Remark 6.6 If $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is a level (λ, ρ) interval-valued LR flat fuzzy number and $L(x) = R(x) = Maximum\{0, \sqrt{1-x}\ \}$, then

$$\Re(\widetilde{A}) = \frac{1}{12}\Big(12m + 12n + 2(\beta - \alpha) + \frac{1}{\rho^2}(\beta' - \alpha')(7\rho^2 - 2\lambda\rho - 3\lambda^2)\Big)$$

Remark 6.7 If $\widetilde{A} = [(m, n, \alpha, \beta; \lambda)_{LR}, (m, n, \alpha', \beta'; \rho)_{LR}]$ is a level (λ, ρ) interval-valued trapezoidal fuzzy number, then

$$\Re(\widetilde{A}) = m + n + \frac{1}{8}\Big[(\beta - \alpha) + (4 - \frac{3\lambda}{\rho})(\beta' - \alpha')\Big]$$

Remark 6.8 If $\widetilde{A} = [(m, \alpha, \beta; \lambda)_{LR}, (m, \alpha', \beta'; \rho)_{LR}]$ is a level (λ, ρ) interval-valued triangular fuzzy number, then

$$\Re(\widetilde{A}) = 2m + \frac{1}{8}\Big[(\beta - \alpha) + (4 - \frac{3\lambda}{\rho})(\beta' - \alpha')\Big]$$

6.2 Applicability of the existing method

The existing method [44] can be used for solving such fuzzy single objective transportation problems of the type $(P_{6.1})$ in which the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued triangular fuzzy numbers.

$Minimize \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij}x_{ij}$

subject to

$$\begin{aligned}
&\sum_{j=1}^{q} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i = 1, 2, \ldots, p \\
&\sum_{i=1}^{p} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, \ldots, q \\
&\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j \\
&x_{ij} \geq 0, \quad i = 1, 2, \ldots, p; \; j = 1, 2, \ldots, q
\end{aligned} \qquad (P_{6.1})$$

where,

$\widetilde{1} = [(1, 0, 0; \lambda)_{LR}, (1, 0, 0; \rho)_{LR}]$

$\widetilde{a}_i = [(m_i, \alpha_{1i}, \alpha_{2i}; \lambda)_{LR}, (m_i, \alpha_{3i}, \alpha_{4i}; \rho)_{LR}]$

$\widetilde{b}_j = [(m'_j, \beta_{1j}, \beta_{2j}; \lambda)_{LR}, (m'_j, \beta_{3j}, \beta_{4j}; \rho)_{LR}]$

$L(x) = R(x) = Maximum\{0, 1 - x\}.$

Example 6.1 Solve the fuzzy single objective transportation problem for which the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued triangular fuzzy numbers.

$Minimize\ (2x_{11}+4x_{12}+14x_{13}+14x_{14}+2x_{21}+18x_{22}+6x_{23}+8x_{24}+16x_{31}+18x_{32}+ 8x_{33}+12x_{34})$

subject to

$$\sum_{j=1}^{4} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i=1,2,3$$
$$\sum_{i=1}^{3} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j=1,2,3,4$$
$$\sum_{i=1}^{3} \widetilde{a}_i = \sum_{j=1}^{4} \widetilde{b}_j$$
$$x_{ij} \geq 0, \quad i=1,2,3;\ j=1,2,3,4$$

where the values of $\widetilde{a}_i$ for $i=1,2,3$ and $\widetilde{b}_j$ for $j=1,2,3,4$ are given as:

$\widetilde{a}_1 = [(7,1,5;0.6)_{LR},(7,2,8;0.9)_{LR}]$

$\widetilde{a}_2 = [(20,3,1;0.6)_{LR},(20,9,2;0.9)_{LR}]$

$\widetilde{a}_3 = [(16,1,5;0.6)_{LR},(16,2,8;0.9)_{LR}]$

$\widetilde{b}_1 = [(11,1,1;0.6)_{LR},(11,3,2;0.9)_{LR}]$

$\widetilde{b}_2 = [(3,2,4;0.6)_{LR},(3,3,7;0.9)_{LR}]$

$\widetilde{b}_3 = [(14,1,1;0.6)_{LR},(14,3,3;0.9)_{LR}]$

$\widetilde{b}_4 = [(15,1,5;0.6)_{LR},(15,4,6;0.9)_{LR}]$

where, $L(x)=R(x)=Maximum\{0,1-x\}$

6.3 Limitations of the existing method

In this section, the limitations of the existing method [44] are pointed out.

(i) The existing method [44] cannot be used for solving fuzzy single objective transportation problems of the type $(P_{6.2})$ in which the parameters $\widetilde{c}_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ,ρ) interval-valued LR flat fuzzy numbers.

$$Minimize\ \sum_{i=1}^{p}\sum_{j=1}^{q} \widetilde{c}_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{q} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i = 1, 2, \ldots, p$$
$$\sum_{i=1}^{p} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, \ldots, q \qquad (P_{6.2})$$
$$\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j$$
$$x_{ij} \geq 0, \quad i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q$$

where,

$$\widetilde{1} = [(1, 1, 0, 0; \lambda)_{LR}, (1, 1, 0, 0; \rho)_{LR}]$$

$$\widetilde{c}_{ij} = [(m_{ij}, n_{ij}, \Delta_{1ij}, \Delta_{2ij}; \lambda)_{LR}, (m_{ij}, n_{ij}, \Delta_{3ij}, \Delta_{4ij}; \rho)_{LR}]$$

$$\widetilde{a}_i = [(m_i, n_i, \alpha_{1i}, \alpha_{2i}; \lambda)_{LR}, (m_i, n_i, \alpha_{3i}, \alpha_{4i}; \rho)_{LR}]$$

$$\widetilde{b}_j = [(m'_j, n'_j, \beta_{1j}, \beta_{2j}; \lambda)_{LR}, (m'_j, n'_j, \beta_{3j}, \beta_{4j}; \rho)_{LR}]$$

The fuzzy single objective transportation problem, chosen in Example 6.2, cannot be solved by using the existing method [44] as the parameters $\widetilde{c}_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

Example 6.2 Solve the fuzzy single objective transportation problem where the parameters $\widetilde{c}_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

$$Minimize \ \sum_{i=1}^{2} \sum_{j=1}^{3} \widetilde{c}_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{3} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i = 1, 2$$
$$\sum_{i=1}^{2} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, 3$$
$$\sum_{i=1}^{2} \widetilde{a}_i = \sum_{j=1}^{3} \widetilde{b}_j$$
$$x_{ij} \geq 0, \quad i = 1, 2; \ j = 1, 2, 3$$

The values of $\widetilde{c}_{ij}$ for $i = 1, 2; \ j = 1, 2, 3$ are:

$$\widetilde{c}_{11} = [(14, 16, 1, 3; 0.8)_{LR}, (14, 16, 2, 4; 1)_{LR}]$$

$\widetilde{c}_{12} = [(29, 31, 2, 2; 0.8)_{LR}, (29, 31, 4, 4; 1)_{LR}]$

$\widetilde{c}_{13} = [(18, 19, 1, 4; 0.8)_{LR}, (18, 19, 3, 6; 1)_{LR}]$

$\widetilde{c}_{21} = [(19, 21, 3, 2; 0.8)_{LR}, (19, 21, 4, 3; 1)_{LR}]$

$\widetilde{c}_{22} = [(14, 16, 2, 2; 0.8)_{LR}, (14, 16, 4, 3; 1)_{LR}]$

$\widetilde{c}_{23} = [(15, 17, 4, 3; 0.8)_{LR}, (15, 17, 5, 6; 1)_{LR}]$

The values of $\widetilde{a}_i$ for $i = 1, 2$ and $\widetilde{b}_j$ for $j = 1, 2, 3$ are given as:

$\widetilde{a}_1 = [(90, 92, 10, 4; 0.8)_{LR}, (90, 92, 20, 8; 1)_{LR}]$

$\widetilde{a}_2 = [(60, 88, 10, 28; 0.8)_{LR}, (60, 88, 20, 42; 1)_{LR}]$

$\widetilde{b}_1 = [(40, 50, 8, 4; 0.8)_{LR}, (40, 50, 10, 8; 1)_{LR}]$

$\widetilde{b}_2 = [(30, 40, 6, 12; 0.8)_{LR}, (30, 40, 10, 12; 1)_{LR}]$

$\widetilde{b}_3 = [(80, 90, 6, 16; 0.8)_{LR}, (80, 90, 20, 30; 1)_{LR}]$

$L(x) = R(x) = Maximum\{0, \sqrt{1-x}\,\}$.

(ii) The existing method [44] cannot be used for solving fuzzy multi-objective transportation problems of the type $(P_{6.3})$ where the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

$$Minimize \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij}^{k} x_{ij}, \quad k = 1, 2, \ldots, K$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{q} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i = 1, 2, \ldots, p \\
&\sum_{i=1}^{p} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, \ldots, q \\
&\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j \\
&x_{ij} \geq 0, \quad i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q
\end{aligned} \qquad (P_{6.3})$$

where,

$\widetilde{1} = [(1, 1, 0, 0; \lambda)_{LR}, (1, 1, 0, 0; \rho)_{LR}]$

$\widetilde{a}_i = [(m_i, n_i, \alpha_{1i}, \alpha_{2i}; \lambda)_{LR}, (m_i, n_i, \alpha_{3i}, \alpha_{4i}; \rho)_{LR}]$

$\widetilde{b}_j = [(m'_j, n'_j, \beta_{1j}, \beta_{2j}; \lambda)_{LR}, (m'_j, n'_j, \beta_{3j}, \beta_{4j}; \rho)_{LR}]$

The fuzzy multi-objective transportation problem, chosen in Example 6.3, cannot be solved by using the existing method [44] as the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ $\widetilde{x}_{ij}$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

Example 6.3 Solve the fuzzy multi-objective transportation problem for which the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

$Minimize\ (30.91x_{11} + 60x_{12} + 38.37x_{13} + 39.54x_{21} + 29.71x_{22} + 32.12x_{23})$

$Minimize\ (32.49x_{11} + 11.29x_{12} + 17.46x_{13} + 15.29x_{21} + 27.29x_{22} + 24.37x_{23})$

subject to

$$\sum_{j=1}^{3} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i = 1, 2$$
$$\sum_{i=1}^{2} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, 3$$
$$\sum_{i=1}^{2} \widetilde{a}_i = \sum_{j=1}^{3} \widetilde{b}_j$$
$$x_{ij} \geq 0, \quad i = 1, 2; \; j = 1, 2, 3$$

The values of $\widetilde{a}_i$ for $i = 1, 2$ and $\widetilde{b}_j$ for $j = 1, 2, 3$ are given as:

$\widetilde{a}_1 = [(90, 92, 10, 4; 0.8)_{LR}, (90, 92, 20, 8; 1)_{LR}]$

$\widetilde{a}_2 = [(60, 88, 10, 28; 0.8)_{LR}, (60, 88, 20, 42; 1)_{LR}]$

$\widetilde{b}_1 = [(40, 50, 10, 4; 0.8)_{LR}, (40, 50, 20, 8; 1)_{LR}]$

$\widetilde{b}_2 = [(30, 40, 4, 4; 0.8)_{LR}, (30, 40, 10, 10; 1)_{LR}]$

$\widetilde{b}_3 = [(80, 90, 6, 24; 0.8)_{LR}, (80, 90, 10, 32; 1)_{LR}]$

$L(x) = R(x) = Maximum\{0, \sqrt{1-x}\ \}$

(iii) The existing method [44] cannot be used for solving fuzzy multi-objective transportation problems of the type $(P_{6.4})$ in which the parameters $\widetilde{c}_{ij}^k$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

$$Minimize \quad \sum_{i=1}^{p}\sum_{j=1}^{q} \widetilde{c}_{ij}^k x_{ij}, \quad k = 1, 2, \ldots, K$$

subject to

$$\begin{aligned} &\sum_{j=1}^{q} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \quad i = 1, 2, \ldots, p \\ &\sum_{i=1}^{p} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, \ldots, q \\ &\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j \\ &x_{ij} \geq 0, \quad i = 1, 2, \ldots, p; \; j = 1, 2, \ldots, q \end{aligned} \qquad (P_{6.4})$$

where,

$$\widetilde{1} = [(1, 1, 0, 0; \lambda)_{LR}, (1, 1, 0, 0; \rho)_{LR}]$$

$$\widetilde{c}_{ij}^k = [(m_{ij}^k, n_{ij}^k, \Delta_{1ij}^k, \Delta_{2ij}^k; \lambda)_{LR}, (m_{ij}^k, n_{ij}^k, \Delta_{3ij}^k, \Delta_{4ij}^k; \rho)_{LR}]$$

$$\widetilde{a}_i = [(m_i, n_i, \alpha_{1i}, \alpha_{2i}; \lambda)_{LR}, (m_i, n_i, \alpha_{3i}, \alpha_{4i}; \rho)_{LR}]$$

$$\widetilde{b}_j = [(m'_j, n'_j, \beta_{1j}, \beta_{2j}; \lambda)_{LR}, (m'_j, n'_j, \beta_{3j}, \beta_{4j}; \rho)_{LR}]$$

The fuzzy multi-objective transportation problem, chosen in Example 6.4, cannot be solved by using the existing method [44] as the parameters $\widetilde{c}_{ij}^k$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

Example 6.4 Solve the fuzzy multi-objective transportation problem in which the parameters $\widetilde{c}_{ij}^k$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers.

$$\begin{aligned} &Minimize \quad \sum_{i=1}^{2}\sum_{j=1}^{3} \widetilde{c}_{ij}^1 x_{ij} \\ &Minimize \quad \sum_{i=1}^{2}\sum_{j=1}^{3} \widetilde{c}_{ij}^2 x_{ij} \end{aligned}$$

subject to

$$\sum_{j=1}^{3} x_{ij}\widetilde{1} \approx \widetilde{a}_i, \qquad i = 1, 2$$

$$\sum_{i=1}^{2} x_{ij}\widetilde{1} \approx \widetilde{b}_j, \quad j = 1, 2, 3$$
$$\sum_{i=1}^{2} \widetilde{a}_i = \sum_{j=1}^{3} \widetilde{b}_j$$
$$x_{ij} \geq 0, \quad i = 1, 2; \ j = 1, 2, 3$$

The values of $\widetilde{c}^1_{ij}$, and $\widetilde{c}^2_{ij}$, for $i = 1, 2; \ j = 1, 2, 3$ are:

$\widetilde{c}^1_{11} = [(14, 16, 1, 3; 0.8)_{LR}, (14, 16, 2, 4; 1)_{LR}]$

$\widetilde{c}^1_{12} = [(29, 31, 2, 2; 0.8)_{LR}, (29, 31, 4, 4; 1)_{LR}]$

$\widetilde{c}^1_{13} = [(18, 19, 1, 4; 0.8)_{LR}, (18, 19, 3, 6; 1)_{LR}]$

$\widetilde{c}^1_{21} = [(19, 21, 3, 2; 0.8)_{LR}, (19, 21, 4, 3; 1)_{LR}]$

$\widetilde{c}^1_{22} = [(14, 16, 2, 2; 0.8)_{LR}, (14, 16, 4, 3; 1)_{LR}]$

$\widetilde{c}^1_{23} = [(15, 17, 4, 3; 0.8)_{LR}, (15, 17, 5, 6; 1)_{LR}]$

$\widetilde{c}^2_{11} = [(15, 16, 2, 4; 0.8)_{LR}, (15, 16, 4, 8; 1)_{LR}]$

$\widetilde{c}^2_{12} = [(5, 6, 2, 2; 0.8)_{LR}, (5, 6, 3, 4; 1)_{LR}]$

$\widetilde{c}^2_{13} = [(8, 9, 3, 4; 0.8)_{LR}, (8, 9, 5, 6; 1)_{LR}]$

$\widetilde{c}^2_{21} = [(7, 8, 1, 1; 0.8)_{LR}, (7, 8, 2, 3; 1)_{LR}]$

$\widetilde{c}^2_{22} = [(13, 14, 2, 2; 0.8)_{LR}, (13, 14, 3, 4; 1)_{LR}]$

$\widetilde{c}^2_{23} = [(11, 12, 3, 6; 0.8)_{LR}, (11, 12, 5, 8; 1)_{LR}]$

The values of $\widetilde{a}_i$ for $i = 1, 2$ and $\widetilde{b}_j$ for $j = 1, 2, 3$ are given as:

$\widetilde{a}_1 = [(90, 92, 10, 4; 0.8)_{LR}, (90, 92, 20, 8; 1)_{LR}]$

$\widetilde{a}_2 = [(60, 88, 10, 28; 0.8)_{LR}, (60, 88, 20, 42; 1)_{LR}]$

$\widetilde{b}_1 = [(40, 50, 10, 4; 0.8)_{LR}, (40, 50, 20, 8; 1)_{LR}]$

$\widetilde{b}_2 = [(30, 40, 4, 4; 0.8)_{LR}, (30, 40, 10, 10; 1)_{LR}]$

$\widetilde{b}_3 = [(80, 90, 6, 24; 0.8)_{LR}, (80, 90, 10, 32; 1)_{LR}]$

$L(x) = R(x) = Maximum\{0, \sqrt{1-x}\ \}$

6.4 Proposed method

In this section, to overcome the limitations of existing method [44], discussed in Section 6.3, a new method is proposed to find the optimal solution of fuzzy multi-objective transportation problem ($P_{6.4}$).

The steps of proposed method are as follows:

Step 1 Convert the fuzzy multi-objective transportation problem ($P_{6.4}$) into the crisp multi-objective linear programming problem as:

$Minimize\ \Re(\sum_{i=1}^{p}\sum_{j=1}^{q}\widetilde{c}_{ij}^{k}x_{ij}), \quad k=1,2,\ldots,K$

subject to

$$\Re(\sum_{j=1}^{q}x_{ij}\widetilde{1})=\Re(\widetilde{a}_i), \quad i=1,2,\ldots,p$$
$$\Re(\sum_{i=1}^{p}x_{ij}\widetilde{1})=\Re(\widetilde{b}_j), \quad j=1,2,\ldots,q$$
$$x_{ij}\geq 0, \quad i=1,2,\ldots,p;\ j=1,2,\ldots,q$$

Step 2 Solve the crisp multi-objective linear programming problem, obtained in Step 1, by using any classical multi-objective linear programming approach to find the optimal solution $\{x_{ij}\}$.

Step 3 Find the fuzzy optimal value of each objective function by putting the values of x_{ij} in $\sum_{i=1}^{p}\sum_{j=1}^{q}\widetilde{c}_{ij}^{k}x_{ij}$.

6.4.1 Advantages of the proposed method

The existing method [44] can be used only to solve such fuzzy single objective transportation problems ($P_{6.1}$) in which the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ,ρ) interval valued triangular fuzzy numbers but cannot be used for solving fuzzy single objective transportation problem ($P_{6.2}$) and fuzzy multi-objective transportation problems $(P_{6.3})$ and $(P_{6.4})$ in which the parameters are represented by level (λ, ρ) interval-valued LR flat fuzzy numbers. Since, level (λ, ρ) interval valued triangular fuzzy numbers, level (λ, ρ) interval valued trapezoidal fuzzy numbers and level (λ, ρ) interval valued LR fuzzy numbers are particular types of level (λ, ρ) interval valued LR flat fuzzy numbers; So, the method proposed in this chapter can be used for solving all such fuzzy single and multi-objective transportation problems in which the parameters are represented by level (λ, ρ) interval valued triangular fuzzy numbers or level (λ, ρ) interval valued trapezoidal fuzzy numbers or level (λ, ρ) interval valued LR fuzzy numbers.

6.5 Illustrative example

In this section, to illustrate the proposed method, the fuzzy multi-objective transportation problem, chosen in Example 6.4, is solved by using the proposed method.

Step 1 Consider the fuzzy linear programming formulation of fuzzy multi-objective transportation problem, chosen in Example 6.4:

$$Minimize \Big([(14,16,1,3;0.8)_{LR},(14,16,2,4;1)_{LR}]\ x_{11} \oplus [(29,31,2,2;0.8)_{LR}, (29,31,4,4;1)_{LR}]\ x_{12} \oplus [(18,19,1,4;0.8)_{LR},(18,19,3,6;1)_{LR}]\ x_{13} \oplus [(19,21,3,2;0.8)_{LR}, (19,21,4,3;1)_{LR}]\ x_{21} \oplus [(14,16,2,2;0.8)_{LR},(14,16,4,3;1)_{LR}]\ x_{22} \oplus [(15,17,4,3;0.8)_{LR}, (15,17,5,6;1)_{LR}]\ x_{23}\Big)$$

$$Minimize \Big([(15,16,2,4;0.8)_{LR},(15,16,4,8;1)_{LR}]\ x_{11} \oplus [(5,6,2,2;0.8)_{LR}, (5,6,3,4;1)_{LR}]\ x_{12} \oplus [(8,9,3,4;0.8)_{LR},(8,9,5,6;1)_{LR}]\ x_{13} \oplus [(7,8,1,1;0.8)_{LR}, (7,8,2,3;1)_{LR}]\ x_{21} \oplus [(13,14,2,2;0.8)_{LR},(13,14,3,4;1)_{LR}]\ x_{22} \oplus [(11,12,3,6;0.8)_{LR}, (11,12,5,8;1)_{LR}]\ x_{23}\Big)$$

subject to

$$\sum_{j=1}^{3} x_{1j}\widetilde{1} \approx [(90, 92, 10, 4; 0.8)_{LR}, (90, 92, 20, 8; 1)_{LR}]$$
$$\sum_{j=1}^{3} x_{2j}\widetilde{1} \approx [(60, 88, 10, 28; 0.8)_{LR}, (60, 88, 20, 42; 1)_{LR}]$$
$$\sum_{i=1}^{2} x_{i1}\widetilde{1} \approx [(40, 50, 10, 4; 0.8)_{LR}, (40, 50, 20, 8; 1)_{LR}]$$
$$\sum_{i=1}^{2} x_{i2}\widetilde{1} \approx [(30, 40, 4, 4; 0.8)_{LR}, (30, 40, 10, 10; 1)_{LR}]$$
$$\sum_{i=1}^{2} x_{i1}\widetilde{1} \approx [(80, 90, 6, 24; 0.8)_{LR}, (80, 90, 10, 32; 1)_{LR}]$$
$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

Using Step 1 of proposed method, the above problem can be converted into the following crisp multi-objective linear programming problem:

$Minimize\ (30.91\ x_{11} + 60\ x_{12} + 38.37\ x_{13} + 39.54\ x_{21} + 29.71\ x_{22} + 32.12\ x_{23})$

$Minimize\ (32.49\ x_{11} + 11.29\ x_{12} + 17.46\ x_{13} + 15.29\ x_{21} + 27.29\ x_{22} + 24.37\ x_{23})$

subject to

$$x_{11} + x_{12} + x_{13} = 88.76$$
$$x_{21} + x_{22} + x_{23} = 78.69$$
$$x_{11} + x_{21} = 42.76$$
$$x_{12} + x_{22} = 35$$
$$x_{13} + x_{23} = 89.69$$
$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

Step 2 Using fuzzy programming technique [165], the optimal solution of the crisp multi-objective linear programming problem, obtained in Step 1, is:

$x_{11} = 6.97,\ x_{12} = 0,\ x_{13} = 81.79,\ x_{21} = 35.79,\ x_{22} = 35,\ x_{23} = 7.90$

Step 3 Putting the values of $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}$ in

$[(14, 16, 1, 3; 0.8)_{LR}, (14, 16, 2, 4; 1)_{LR}]\ x_{11} \oplus [(29, 31, 2, 2; 0.8)_{LR}, (29, 31, 4, 4; 1)_{LR}]\ x_{12} \oplus$

$[(18,19,1,4;0.8)_{LR},(18,19,3,6;1)_{LR}]\ x_{13}\oplus[(19,21,3,2;0.8)_{LR},(19,21,4,3;1)_{LR}]\ x_{21}\oplus$

$[(14,16,2,2;0.8)_{LR},(14,16,4,3;1)_{LR}]\ x_{22}\oplus[(15,17,4,3;0.8)_{LR},(15,17,5,6;1)_{LR}]\ x_{23}$

and in

$[(15,16,2,4;0.8)_{LR},(15,16,4,8;1)_{LR}]\ x_{11}\oplus[(5,6,2,2;0.8)_{LR},(5,6,3,4;1)_{LR}]\ x_{12}\oplus$

$[(8,9,3,4;0.8)_{LR},(8,9,5,6;1)_{LR}]\ x_{13}\oplus[(7,8,1,1;0.8)_{LR},(7,8,2,3;1)_{LR}]\ x_{21}\oplus$

$[(13,14,2,2;0.8)_{LR},(13,14,3,4;1)_{LR}]\ x_{22}\oplus[(11,12,3,6;0.8)_{LR},(11,12,5,8;1)_{LR}]\ x_{23}$,

the fuzzy optimal values of first and second objectives are

$[(2856.79,3109.74,297.49,513.19;0.8)_{LR},(2856.79,3109.74,581.65,778.15;1)_{LR}]$

and

$[(1550.74,1718.11,388.72,508.15;0.8)_{LR},(1550.74,1718.11,652.75,856.83;1)_{LR}]$

6.6 Comparative study

To show the advantages of the method, proposed in this chapter, over the existing method [44], the results of the existing problem [44] and the problems chosen in this chapter, obtained by using the existing method [44] and method proposed in this chapter, are compared in Table 6.1.

Table 6.1: Comparison of results obtained by using existing method [44] and proposed method

Example	Existing method [44]	Method proposed in this chapter
6.1	$x_{11} = 4.375$, $x_{12} = 3.625$, $x_{13} = 0$, $x_{14} = 0$, $x_{21} = 6.500$, $x_{22} = 0$, $x_{23} = 0$, $x_{24} = 12.500$, $x_{31} = 0$, $x_{32} = 0$, $x_{33} = 14$, $x_{34} = 3$, Optimal value of objective function is =284.25	$x_{11} = 4.375$, $x_{12} = 3.625$, $x_{13} = 0$, $x_{14} = 0$, $x_{21} = 6.500$, $x_{22} = 0$, $x_{23} = 0$, $x_{24} = 12.500$, $x_{31} = 0$, $x_{32} = 0$, $x_{33} = 14$, $x_{34} = 3$ Optimal value of objective function is =284.250
6.2	Not applicable	$x_{11} = 44.375$, $x_{12} = 0$, $x_{13} = 44.385$, $x_{21} = 0$, $x_{22} = 35.790$, $x_{23} = 42.900$, Fuzzy optimal value of objective function is = $[(2564.700, 2855.300, 331.900, 510.900; 0.8)_{LR}$, $(2564.700, 2855.300, 579.500, 808.500; 1)_{LR}]$
6.3	Not applicable	$x_{11} = 6.970$, $x_{12} = 0$, $x_{13} = 81.970$, $x_{21} = 35.790$, $x_{22} = 35$, $x_{23} = 7.900$, Optimal value of 1^{st} objective function is = 6062.460 Optimal value of 2^{nd} objective function is = 3349.411
6.4	Not applicable	$x_{11} = 6.970$, $x_{12} = 0$, $x_{13} = 81.790$, $x_{21} = 35.790$, $x_{22} = 35$, $x_{23} = 7.900$ Fuzzy optimal value of 1^{st} objective function is = $[(2856.790, 3109.740, 297.490, 513.190; 0.8)_{LR}$, $(2856.790, 3109.740, 581.650, 778.150; 1)_{LR}]$ Fuzzy optimal value of 2^{nd} objective function is = $[(1550.740, 1718.110, 388.720, 508.150; 0.8)_{LR}$, $(1550.740, 1718.110, 652.750, 856.830; 1)_{LR}]$

The results, shown in Table 6.1, can be explained as follows:

(1) The existing method [44], can solve such fuzzy single objective transportation problems in which the parameters $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued triangular fuzzy numbers. So, the problem chosen in Example 6.1 can be solved by using the existing method [44]. However, Example

6.2 wherein the fuzzy single objective transportation problem having parameters $\widetilde{c}_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ represented by level (λ, ρ) interval-valued LR flat fuzzy numbers and Example 6.3 and Example 6.4 wherein the fuzzy multi-objective transportation problems having parameters $\widetilde{c}^k_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ represented by level (λ, ρ) interval-valued LR flat fuzzy numbers cannot be solved by the use of the existing method [44].

(2) The method, proposed in this chapter, can be used for solving all such fuzzy single and multi-objective transportation problems in which the parameters $\widetilde{c}^k_{ij}$, $\widetilde{a}_i$ and $\widetilde{b}_j$ are represented by level (λ, ρ) interval-valued LR flat fuzzy number. As discussed in Remark 6.1, the level (λ, ρ) interval-valued triangular fuzzy numbers are special type of level (λ, ρ) interval-valued LR flat fuzzy numbers; So all such fuzzy single and multi-objective transportation problems, chosen in Examples 6.1 to 6.4, can be solved by using the method, proposed in this chapter.

(3) It is obvious from the results shown in Table 6.1 that the solution obtained by using the existing method [44] and proposed method for Example 6.1 are same.

6.7 Conclusion

On the basis of the presented study, it can be concluded that the problems which can be solved by using the existing method [44], can also be solved by the method proposed in this chapter. However, as discussed in Section 6.3, there exist several problems which can be solved by the method, proposed in this chapter, but can not be solved by using the existing method [44]. Hence, it is better to use the method, proposed in this chapter, as compared to the existing method [44].

Chapter 7

Conclusion And Future Scope

In this chapter, we have reviewed in brief the work presented in earlier chapters. The limitations and shortcomings of the existing methods for the solution of time minimizing transportation problems and transportation problems in fuzzy environment in earlier chapters have been discussed. Advantages of the proposed methods for these problems have been given.

A real life problem to select the warehouse sites for clustering ration shops to them has been considered in fuzzy environment in Chapter 2 and a method is proposed to find the efficient solutions of this problem.

The existing method [56] provides a unique optimal solution to time minimizing transportation problem. However, it has been shown that the problem contains infinite optimal solutions. In Chapter 3, a new method is proposed to find all the optimal solutions. The method is also applicable to the problem considered in Chapter 2 and provides the same efficient solutions.

In Chapter 4, the shortcomings of the existing formulation of fuzzy time minimizing transportation problem is pointed out and to overcome these shortcomings, a new formulation is suggested. Also on the basis of the proposed formulation, a new method is proposed for solving time minimizing transportation problem in fuzzy environment.

In Chapter 5, a new method is proposed for solving fully fuzzy multi-objective transportation problem with LR flat fuzzy numbers and it is shown that the proposed method is better than existing method [68].

In Chapter 6, a new method is proposed for solving fuzzy multi-objective transportation problem with level (λ, ρ) interval-valued LR flat fuzzy numbers and it is shown that the proposed method is better than existing method [44].

The methods proposed in this thesis can be beneficial in a myriad of business situations in various ways. The optimal solutions of business models in fuzzy environment give the decision maker a wide choice respecting his subjective preferences and allowing optimal use of the resources at his disposal.

The proposed methods are quite versatile and can be adapted to number of other situations by varying the constraints and objective functions. With the help of computer programming, proposed methods can be applied to even large size problems.

In future, the methods proposed can be further generalized and extended to involve more many types of transportation problems. For example, Chapter 6 provides optimal solution of fuzzy multi-objective transportation problems with parameters as (λ, ρ) interval-valued LR flat fuzzy numbers. However, this method may not be applicable to single objective transportation problem $(P_{7.1})$ and multi-objective transportation problem $(P_{7.2})$:

$$Minimize \sum_{i=1}^{p}\sum_{j=1}^{q} \widetilde{c}_{ij} \otimes \widetilde{x}_{ij}$$

subject to

$$\sum_{j=1}^{q} \widetilde{x}_{ij} = \widetilde{a}_i, \quad i = 1, 2, \ldots, p$$

$$\sum_{i=1}^{p} \widetilde{x}_{ij} = \widetilde{b}_j, \quad j = 1, 2, \ldots, q \qquad (P_{7.1})$$

$$\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j$$

$\widetilde{x}_{ij}$ is a non-negative level (λ, ρ) interval-valued LR flat fuzzy number

$$Minimize \sum_{i=1}^{p} \sum_{j=1}^{q} \widetilde{c}_{ij}^{k} \otimes \widetilde{x}_{ij}, \quad k = 1, 2, \ldots, K$$

subject to

$$\sum_{j=1}^{q} \widetilde{x}_{ij} = \widetilde{a}_i, \quad i = 1, 2, \ldots, p$$

$$\sum_{i=1}^{p} \widetilde{x}_{ij} = \widetilde{b}_j, \quad j = 1, 2, \ldots, q \qquad (P_{7.2})$$

$$\sum_{i=1}^{p} \widetilde{a}_i = \sum_{j=1}^{q} \widetilde{b}_j$$

$\widetilde{x}_{ij}$ is a non-negative level (λ, ρ) interval-valued LR flat fuzzy number

In future, it may be tried to modify this method to find solutions of such above mentioned fuzzy single and multi-objective transportation problems.

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