ANALYSIS AND DESIGN OF FRACTIONAL DOMAIN FILTERING UTILIZING FRACTIONAL TOOLS

Thesis submitted towards the partial fulfillment of requirement for the award of degree of

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DECLARATION

I hereby declare that the work presented here, entitled ***Analysis And Design Of Fractional Domain Filtering Utilizing Fractional Tools*** is an authentic record of my study carried out as requirement for the award of degree of ME (Wireless Communication Engineering) at Thapar University, Patiala, under the supervision of **Dr. Sanjay Kumar (Assistant Professor)**, Electronics and Communication Engineering Department.

The work presented in this thesis has not been submitted in any other Institute/University for the award of the degree.

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It is certified that the above statement made by the student is correct to the best of my knowledge and belief.

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ABSTRACT

In mobile communication, normally communication is intended to establish between the transmitter and receiver under the scenario where the transmitter / receiver or both having their respective movements, the consequence of the relative motion of receiver is to introduce a Doppler shift in the received frequency which in turn making the frequency components received as time-variant. This non-stationary behavior of the signal enforced the analysis of the signal to be performed in time-frequency plane. The fractional Fourier transform (FrFT), which is also known as generalization of Fourier transform (FT), has been established as a better mathematical tool to tackle this state of affair of the signal.

The proposed work can be divided into two broader segments. The first segment includes the efforts made in establishing the translation invariance concept of fractional convolution and correlation to show that these are no more partial invariant by expanding some convolution and correlation related identities.

The second segment comprises the filtering application in FrFT. The beneficial role of the FrFT in the filtering application lies in the capability of the FrFT in localizing the non-stationary (chirp) signals in time-frequency plane. This ascertains the superiority of FrFT domain filtering over time-domain and frequency domain filtering in the case of overlapping band limited signal and noise. Thus, FrFT proved to be a better technique in the context to other transformation techniques.

TABLE OF CONTENTS

CHAPTER 4

Fractional Translation and its Effect on Fractional Convolution and Correlation

CHAPTER 5

CHAPTER 6

LIST OF FIGURES

LIST OF ABBREVIATIONS

DSP	Digital Signal Processing
FT	Fourier Transform
IT	Integral Transform
LT	Laplace Transform
STFT	Short-Time Fourier Transform
WDF	Wigner Distribution Function
FrFT	Fractional Fourier Transform
AFT	Angular Fourier Transform
DFT	Discrete Fourier Transform
FFT	Fast Fourier Transform
DFrFT	Discrete Fractional Fourier Transform
LTI	Linear Time Invariant
LFSI	Linear Fractional Shift Invariant System
FIR	Finite Impulse Response
LPF	Low Pass Filter
MSE	Mean Square Error
SNR	Signal to Noise Ratio

CHAPTER 1

INTRODUCTION

In last few decades, Digital Signal Processing (DSP) has marked its contribution in various areas one way or another. However interpretation of stationary behavior of signal can be expected theoretically but practical realization allows only the non stationary nature of the signal to analyze. This enhances time-frequency method as compare to conventional Fourier based technique to make it suitable for the realization of non stationary signals.

Though the term fractional is not confined to Fourier transform (FT) but also widely used in almost every area: optics, mechanics, signal processing, calculus, numerical analysis but our concern is only related to the signal processing.

The transform is an approach to convert a signal from one domain into another domain when it becomes quite difficult to abstract the information in first domain. One of the vital classes of the transform family is "Integral Transform". As a matter of fact, integral transform is nothing but an operator used to transform a signal into its corresponding form using a 'kernel' function by integrating a kernel multiplied signal. Although the whole integration process is known as "integral transform". Mathematically representation of integral transform can be implemented as:

$$F(s) = \int_{-\infty}^{\infty} f(t)k(s,t)dt \tag{1.1}$$

where $F(s)$ is a transformed signal and $k(s,t)$ is a kernel function associated with the corresponding transform. The class of integral transform consists of various transform like Laplace Transform (LT), cosine transform, FT, Fractional Fourier transform (FrFT), Hilbert transform etc.

1.1 Historical Development of FrFT

FrFT initially proposed in 1929 to solve certain classes of differential equations in quantum mechanics. Though it was originally realized as 'transform method' by the mathematical bodies after the Namias work in 1980 in which the FrFT concept had been introduced by considering the fractional power of ordinary FT. But it gained popularity in 90's after its numerous publications in signal processing and optics.

This transform is not only helpful in solving mathematical problems but also valuable in quantum physics, optics and signal processing. As the FrFT is likely valuable tool for signal processing and many methods for implementing FrFT has been developed but direct computation of FrFT in computer has grown an important issue. So the need of discrete form arises. However, many algorithms are commenced to define DFrFT but the closed form DFrFT excels the all other algorithms.

1.2 Time-Frequency Analysis tool

There are basically three ways to examine the information of any signal:

- *Time domain representation:* Time-domain representation of the signal explains the signal's amplitude varying with respect to time. But it hides all the information regarding the frequency component of the signal because it considers the both variables time and frequency mutually exclusive and orthogonal to each other.

- *Frequency domain representation:* Any practical signal can be represented in frequency domain by its Fourier transform. It reveals out the information regarding the frequency component of the signal along with the amplitude associated with each frequency component. But it hides out all the information regarding the time of interval of those frequency, as FT does not deal with the variable time.

- *Time-Frequency representation:* As time and frequency domain representation is inadequate to provide all the information about the signal individually. So there seeks a representation of signal as two variable function whose domain is two dimensional time-frequency space and this representation is known as time-frequency distribution(TFD) and the space in which this occurs is called time-frequency plane.

FT provides a mapping of one-dimensional time domain signal to one dimensional frequency domain signal. However, it provides the details of the signal's spectral component but fails to tell the time location of spectral component which is quite important in case of non-stationary signals. For this time-frequency representation is needed which maps one-dimensional signal into two-dimensional time and frequency.

There exist so many time-frequency distribution functions. Some of methods are- Short Time Fourier Transform(STFT), Wigner Distribution(WD) and FrFT. Based upon some factors like clarity, cross-term, mathematical properties and computational complexity, the better techniques for representing signal in time-frequency plane can be anticipated.

As STFT is having worst clarity, no existence of cross-term, worst mathematical properties and lower computational complexity whereas WDF is good on all terms but causes a cross-term problem for multi-component signals. Thus, WDF is a better approach for a single term signal but for a multi-component, FrFT will be better choice. Hence for practical purpose where signal is multi-component and non-stationary in nature, FrFT will be a promising technique for representing signal in time-frequency plane.

1.3 Applications of FrFT

The FrFT has been found to perform a vital and significant role in Fourier optics along with application in optical information processing. It has stated a generalization of the notional concept of frequency domain and increased our perceptive of time-frequency plane, fundamental concepts in signal analysis and signal processing. It is assumed to have a large deeper impact in every unique and advanced application in which FT plays an integral role.

The other applications which have attained significant attention of this transform is detection, pattern recognition, correlation, radar, multiplexing data compression and linear FM detection. This transform has also found its way in context of microscopic application also. However, many signal and image processing application can be implemented optically. Though many of the mentioned applications been applied in optical context firstly. Believing that these might be the few of applications of this transform because the way it had broadens its concept and left its significance, it may have greater impact on other areas of science and engineering where Fourier concept is used.

1.4 Organization of Thesis

The work reported in this dissertation is completed in six chapters. The details are outlined below:

Chapter 1 covers the introduction of FrFT along with the significance of FrFT over other time-frequency distribution functions.

Chapter 2 presents an literature review of the work done so far in the field of FrFT and in the context of its application.

Chapter 3 covers the thorough definition of FrFT and its aspects along with the simulation results, testifying the nature of LFSI system.

Chapter 4 covers the elongated mathematics covering the translational concept of fractional convolution and correlation which is further analyzed using triangular and rectangular function.

Chapter 5 covers the filtering in fractional domain using optimal method and contains the comparison of different filtering techniques.

Chapter 6 concludes the brief discussion of the reported work along with the future scope of this dissertation.

CHAPTER 2

LITERATURE SURVEY

The literature review outlines the work been done so far in order to relate the ongoing research to establish that knowledge. However literature review may have some argument and may need some additional research.

Mendlovic *et al.* [3] proposed the brief introduction to the approach of FrFT following a discussion of chirp transform and wavelet transform. Convolution and filtering in fractional domain are among the featured plot and revealed that under certain circumstances one can have advancement of these special cases in the regular space and frequency domain. And because of ease of execution of FrFT, it could have significant operations for optical information processing. Although the desired signal and noise may collapse in conventional space and frequency domain but not in a precise fractional domain and if they somehow overlap then binary amplitude masking allow to split the noise suitably.

Almeida *et al.* [4] addressed the FrFT and some of its properties which remained in dark from so long and however found its potential usefulness. FrFT was described as rotation in time-frequency transform, supporting its analysis as rotational operator. The FrFT was later shown to prove the illustration of signal on orthonormal basis guided by chirps, which are basically the shifted form of one another. And later presented a work which opens wide area for advanced research in future and would have some helping hand in further studying the properties of FrFT and developing some time-frequency transforms, which better suit some specific application.

Mendlovic *et al.* [5] explained fractional correlation operator in two unusual ways. The work presented here defined the various ways of describing the definition of fractional correlation but somehow two definitions made out of it and each of them reflected a special case in which each mathematical expression of fractional correlation comes under single integral which later summarized that fractional correlation operator is anyhow not a shift-invariant operator.

Santhanam *et al.* [6] stated that continuous time angular Fourier transform (AFT) performed a rotation in continuous time-frequency domain. Along with this discrete edition of AFT, it also featured a rotation in discrete time-frequency space just like in case of continuous time-frequency domain. The derived transform is basically a generalization of Discrete Fourier Transform (DFT). In the end composed Eigen value structure is presented for the efficient computation of this transform.

Ozaktas *et al.* [7] defined an FrFT a subclass of generalized form of integral transform explained by quadratic complex kernels. It has greater computational complexity *O(NlogN)* and more numerical inaccuracy and also there lies a need for more memory space. Later work progresses with the definition of discrete fractional Fourier transform (DFrFT) which however produced a result that made an approximation of continuous FrFT definition. In the end, certain applications regarding DFrFT being discussed.

Xia *et al.* [8] studied the band limited signals in terms of FrFT and realized that the signal with non zero value can't be band limited with two distinct angles α and β at the same time, where $\beta \neq \pm\alpha + n\pi$ for every integer n. Although this is a generalized fact that a signal with non zero value can't be time limited and even band limited.

Zayed *et al.* [9] abstracted that the FrFT is nothing but the generalized form of FT which can rotate in time-frequency domain varying some particular parameter and many of its properties, just like inversion formula and sampling theorems can be obtained from the conventional FT simply by changing the variable name.

Almeida *et al.* [10] addressed the product and convolution theorems and their transform in fractional domain and found many applications in areas like signal processing and optics. However many of its properties are already came under limelight but further an extension of FT's convolution is missing still apart. The obtained result could have been extended to wider domain but the derived execution doesn't follow when the transform angle considers the value which are multiples of $\pi/2$. However these cases are duly noted in the end for simplicity purpose.

Zayed [11] proposed a new convolution structure for the FrFT that retained the convolution and product theorem for the FrFT. However Almeida and Mendlovic had already assigned the product and convolution theorem but it somewhat and somewhere lacked some properties which made these definitions unable to generalize the classical results of the FT.

Although the intended convolution structure is easy to implement in designing of filters. As passing the output through the chirp multiplier yield only the frequency spectrum in FrFT in the particular region which reflects its competency to have useful application in signal processing and in optics also.

Pei *et al.* [12] introduced the definitions of discrete fractional Hartley transform and DFrFT. Initially, the study included the exploration of the Eigen values and the Eigen vectors of the discrete Fourier and discrete Hartley transform. Then the definition of the discrete fractional Hartley transform and DFrFT are described using the consequences of the Eigen decomposition of the discrete Fourier and Discrete Hartley transform. And in the end, a filtering procedure in fractional domain is being defined to eliminate the noise introduced in the chirp as the conventional time and frequency domain won't be able to extract the noise from the chirp as it becomes quite difficult to separate the distribution in these domains.

Akay *et al.* [13] introduced the unitary fractional operator in FrFT after being motivated by the use of unitary operator method in signal analysis which gets popularized after some serious efforts. This fractional operator defines the unitary time and frequency shift operator by illustrating shift at random orientation in time-frequency plane. And then established the link between the FrFT by evolving two signal transformations, one is said to invariant and the another one is covariant to newly derived unitary fractional shift operator. Also the generalization of Hermitian time-frequency operators is being made by using duality concept and using stone's theorem.

Pei *et al.* [14] analyzed that the continuous FrFT performed a spectrum rotation of input signal in the time-frequency domain and has become the significant for the time varying non stationary signal analysis. Although the DFrFT developed by the Santhanam and McClellan does not equalizes the results to the corresponding continuous FrFT derived by the Pie. Hence, proposed a new definition for the DFrFT which has Eigen vectors and preserves the Eigen function-Eigen value relation as continuous FrFT.

But to obtain the Eigenvectors, two orthogonal projection techniques are being introduced which offered the corresponding transform properties as well as the rotational properties as of continuous FrFT. However, the relation between continuous FrFT and the DFrFT is being derived as the same way as the conventional FT and DFT.

Pei *et al.* [15] proposed a DFrFT a generalization of DFT. Although many types of DFrFT came into an existence defining its uses in signal processing applications. But Pie introduced a distinct type of DFrFT which includes unitary, reversibility and flexibility and in addition to this, closed form expression also obtained. However it works similar in a manner like continuous FrFT.

Later on, two types of DFrFT and DAFT being derived out. Type 1 contained the similarity to continuous FrFT and AFT and could be used for computing FrFT and AFT. Whereas type 2 is the efficient and better form of type 1 and also be implemented for the other applications of signal processing. Although there lies a series of properties that kept under the closed form DFrFT and DAFT. The introduced closed form DFrFT would have low computational complexity among all the discussed DFrFT's which are similar to continuous FrFT.

Akay *et al.* [16] concluded that using fractional shift operator within the theory of operator methods, the definitions of fractional convolution and fractional correlation can be derived. This theory allowed to derive explicit formulas for fractional operators by manipulating operator equations. Also, important relationship linking fractional autocorrelation and Ambiguity Function (AF) also is being derived out. Based upon one of the alternative formulations, a new fast discrete time implementation for fractional autocorrelation was proposed which determines that the examining results calculated through this method approach the approximation of AF.

Akay *et al.* [17] formulated the continuous Linear Fractional Shift-Invariant System (LFSI) system which illustrates the LTI system through an angle parameterα. LTI system is observed as a special case under LFSI system when $\alpha = 0$. Although LTI system adjusts with the time shifts, whereas LFSI system exchanges itself with the fractional shift in time-frequency domain.

Saxena *et al.* [20] showed that the FrFT lead to the concept of generalization of time and frequency domain. This significant aspect of the FT proposed that FrFT would have many applications in signal processing analysis. And the greater advantage of FrFT is that it is not only richer in theory but also have deeper impact on its application and on its low cost implementation. The most influential thing is that it has proved its significance in time-varying non stationary signal where FT fails to do so.

The FrFT additional extent of freedom to parameter α provides applications not only in signal processing but also in optics, mathematics and in physics also. Although the significance of FT is being discontinued a far way back in the area of mobile communication but the progression of FrFT has given a ray of hope to step forward in this area.

Sharma *et al.* [21] proposed a methodology for the on line tuning of the transition bandwidth of the FIR digital filter based upon windowing method using FrFT. Thus nominating FrFT order as a tuning parameter between the FrFT of windowing function and ideal frequency response helps in attaining the transition width of desirable frequency response.

Here comparison is performed considering the two tuning scheme for varying the transition width, one approaches the direct tuning in which sharpness of transition width can be modified over a broader range through filter order. But for the tuning method covered by FrFT approach, there requires a smaller filter order which is however efficient in saving computations for lengthy sequences. Also the tuning method through FrFT is easier to implement as no initial changes are required.

Gudadhe *et al.* [22] emerged a new definition of LFSI system. Although it was never considered to be shift invariant before, only the linearity of FrFT were followed.

Expression for LFSI system can be taken place in terms of convolution of two functions which formulate its response in terms of unit impulse. Linearity and time invariance is supposed to be the two basic properties for any system to follow. And in a same manner if FrFT is considered, then for the system to get linear and shift invariant, it would also have to acknowledge these two properties. However the complete description of LFSI system can be made in terms of its response to unit impulse.

Torres *et al.* [23] proposed the relation of translation invariance of fractional convolution and fractional correlation supported by fractional translational operator which until somehow was not defined till yet. So far, fractional convolution and correlation sustained the partial invariance properties which restrict its physical meaning in many applications.

Then Torres concluded that the translation of fractional convolution and correlation can be exhibited if translation operator is replaced by fractional translation operator. And to make it more convenient and efficient one dimensional integral expression was represented because such expressions are proved to be useful in practical applications. Although the derived fractional correlation is different from what Zayed introduced earlier as well as from the akay's definition.

Singh *et al.* [25] established the some of the properties related to FrFT which were still not proposed. Although many of the properties of FrFT were known but still the expressions for transform of auto and cross-correlation theorem in fractional domain were not examined. This paper consists of the mathematical expressions of transform of auto and cross-correlation theorem in FrFT domain satisfying all the properties of auto and cross-correlation functions.

The prospective theorem not only fascinates the variable dependability and FT alteration but also appeal all the properties which are the classical entities of FT.

A simulation based study of respective definition of correlation theorem is also being a part of discussion which further summarized that auto-correlation theorem can play a vital role in calculating the power spectral density of FM signals.

Kumar *et al.* [26] proposed a mathematical model for attaining FrFT of Dirichlet and commenced a generalized Hamming window function. The FrFT contains an adjustable parameter with which a main lobe width can be controlled along with the minimization of stop band attenuation. This paper concluded an analysis of Dirichlet and examined Hamming window function for different values of angle α which is related by $\alpha = a\pi/2$, whereas a lies within 0 and 1.

The simulated results depicts that the side lobe levels can be reduced by increasing the value of parameter a which results in broadening the main lobe width, thus reduced resolution. It is further conceded that increasing the value of parameter a for the Dirichlet window, increases maximum side lobe level, half main lobe width and side lobe fall off rate.

Joaquim *et al.* [27] presented a simple method for designing linear-phase FIR digital filter which is purely based upon steepest-descent method. This paper proposed an ideology that minimizes not only the mean square error between the desired frequency response and the approximated response. Although it is possible to control separately the absolute error for any band of desired filter. Furthermore, there is no need to calculate the filter order in advance as the algorithm boosts the filter order if the calculated error is more than the specified error. And this algorithm is purely based upon steepest descent method of local minimization and serves the alternate body to design the digital FIR. The developed algorithm is simple, produces low error and easy to implement.

Mishra *et al.* [29] presented a low pass filtering designing using convolution theorem for FrFT considering Blackman window and observed that FrFT domain filtering is quite far better than the FT domain filtering and time-domain filtering. This paper proposed a new consolidated structure of FrFT which broaden the convolution theorem of FT to FrFT. Filtering is the most compelling application of convolution, although frequency domain filtering is been quite extensive tool but has some certain limitations as it can't deal with the time-frequency plane noise.

For this FrFT domain filtering is being used in place of FT using proposed FrFT convolution theorem. The proposed designed filtering structures assists in filtering non-stationary signals as the components of the signal may interact in frequency and in time-domain which makes it difficult for them to separate out. The derived convolution structure is being a part of smoothing the signal also. Comparative analysis of output attained from the classical filtering method and the proposed FrFT filtering structure showed that the later provided the better performance with optimum value of parameter α amongst all the obtained results.

Chinchilla *et al.* [34] proposed a time varying filtering technique based upon the notion of fractional convolution and its affiliation with α –Wigner-Ville distribution. Further, some properties of α –Wigner-Ville distribution are studied which demonstrated that current theory on Fourier analysis is a distinct case of the theory proposed in this paper. Finally this paper proposes a formulation of α –Wigner-Ville distribution in terms of fractional power which is supposed to be the key for proving the relation between the fractional convolution and α –Wigner-Ville distribution along with the simulation results.

CHAPTER 3

FRACTIONAL FOURIER TRANSFORM AND ITS ASPECTS

Time and frequency are two central physical variables of signal analysis and processing. The FT, which represents a mapping between time and frequency domain of a signal can be known as a reformulation of the time domain signal. Thus, continuous time definition of FT can be concluded as:

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt \qquad (3.1)$$

As widely known, time and frequency form the orthogonal coordinates of the time-frequency plane, which imposes a fact that the FT of a signal can be considered as $\pi/2$ radian rotation of the signal in anticlockwise direction in time-frequency plane.

Following this analysis, the FrFT was defined as a generalization of FT with an angle parameter α. It is one of the promising and rising fields of effective research due to its wide variety of applications and its uses in other scientific practices and its low cost implementation. This transform is likely to have application in every area where FT concept has applied upon. The FT does not provide effective results in case of applications with moving subject as the received signal attains a Doppler shift, but in case of FrFT it plays a significant role as it can deal very finely with the signal having time varying frequency components.

3.1 Definition of FrFT

The *a*th order FrFT is defined by an integral [4]

$$F_\alpha(u) = \int_{-\infty}^{\infty} f(t) k_\alpha(t,u) dt \qquad (3.2)$$

$$F_\alpha(u) = \begin{cases} e^{\frac{j}{2}u^2 cot\alpha}\sqrt{\frac{1-jcot\alpha}{2\pi}}\int_{-\infty}^{\infty} f(t) e^{\frac{j}{2}t^2 cot\alpha} e^{-jutcsc\alpha} dt & ;\alpha \neq multiple\ of\ \pi \\ f(t) & ;\alpha = multiple\ of\ 2\pi \\ f(-t) & ;\alpha + \pi = multiple\ of\ 2\pi \end{cases}$$

The above mentioned definition is the most direct and particular one. It has some remarkable properties and some of them are quoted below:

- The FrFT with angle parameter $\alpha = \pi/2$ corresponds to Fourier transform (FT).
- The FrFT with $\alpha = 0$ tallies an identity operator.
- Two subsequent FrFT's with angle parameter α and β proportionate new FrFT with parameter $\alpha + \beta$.
- The inverse of FrFT with angle parameter α is identical to FrFT with angle parameter $-\alpha$.

Computations of FrFT implemented using following steps:

1. Multiplication by a chirp.
2. A Fourier transform (argument scaled by cosecα).
3. Another multiplication by a chirp.
4. Multiplication by an amplitude factor.

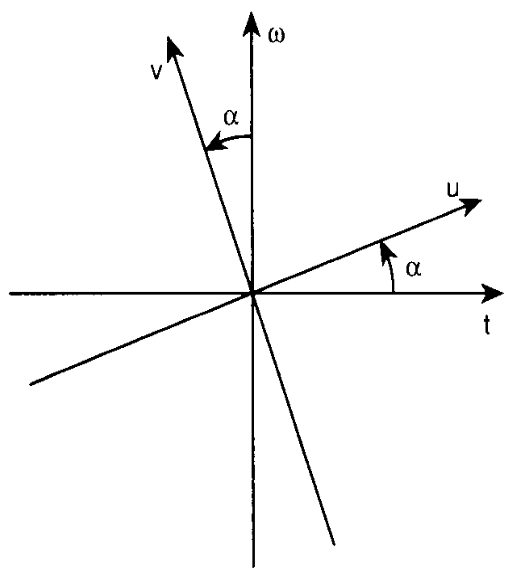


Fig. 3.1 Time-frequency plane and a set of coordinates (u, v) rotated by a parameter α corresponding to original coordinate (t, ω). **[4]**

Considering example, if the signal in the time domain is rectangular, it will become a sinc in the frequency domain.

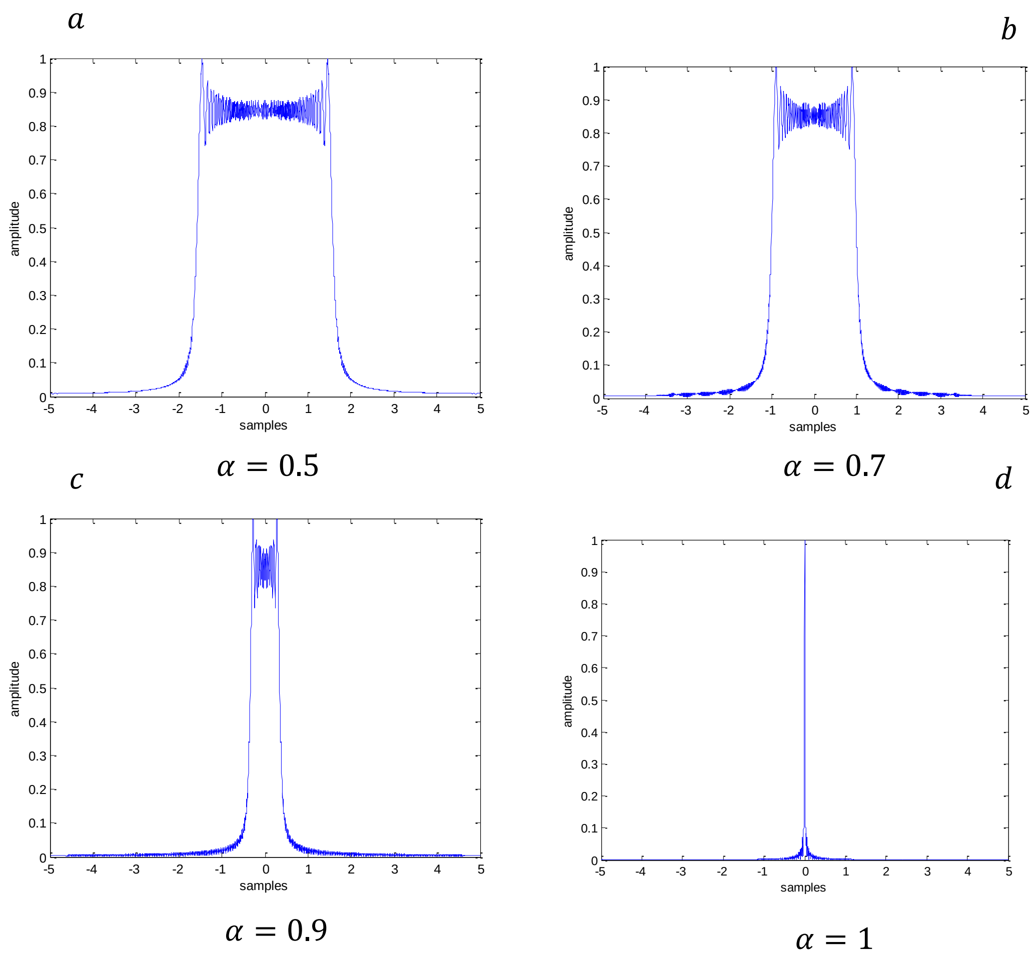


Fig. 3.2 FrFT of rectangular pulse computed by varying an angle parameter α

(Above quoted graphs are obtained in absolute form)

But if we apply the FrFT to the rectangular signal with varying angle parameter α, the transformed output will be in the domain between time and frequency.

Another illustration of FrFT comes along with the implementation of triangular pulse by varying a parameter in time-frequency domain.

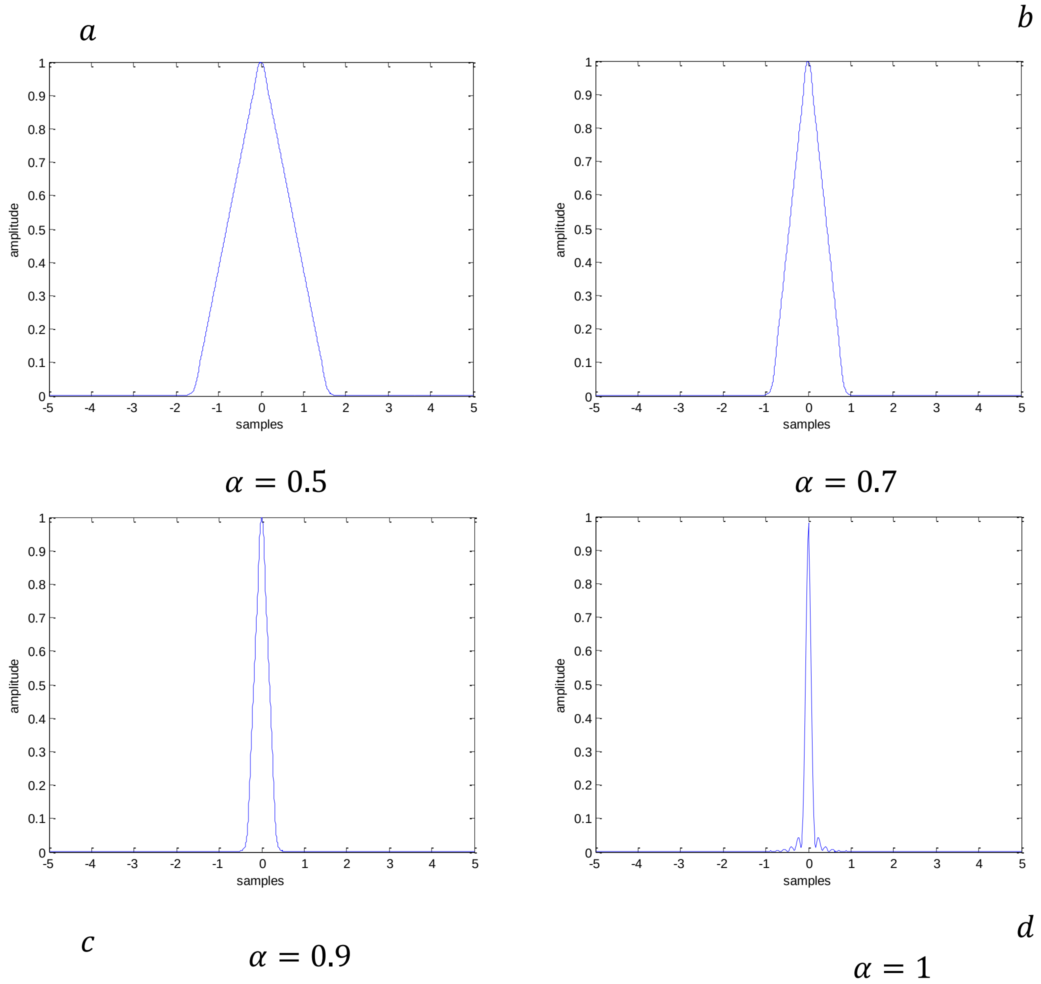


Fig. 3.3 FrFT of triangular pulse computed by varying an angle parameter α

(Above quoted graphs are obtained in absolute form)

3.2 Discrete Fractional Fourier Transform (DFrFT)

After the continuous FrFT, DFrFT came into existence. Many form of DFrFT's were derived but only closed form DFrFT [15] drawn the similar properties comparing to continuous FrFT. Some of the DFrFT's form are shown below:

3.2.1 Direct form of DFrFT

The easiest way to apply the DFrFT is sampling the continuous FrFT and computes it directly, but computing it directly followed a discrete transform with losing some of its main properties. And the most concerned problem of this type of DFrFT is that it won't be unitary and reversible. Adjoining, it lacks closed form properties and not additive in nature. Thus, its application area is supposed to be limited in certain extent.

3.2.2 Improved sampling-type DFrFT

This method contained an appropriate way to sample the continuous FrFT. Despite DFrFT work is similarly to the continuous one and has fast algorithm but somewhat the transformed kernel won't be orthogonal and additive in nature. Besides all this many constraints including input signal should satisfy.

3.2.3 Linear combination-type DFrFT

This form of DFrFT is derived using the linear combination of DFT, time inverse operation and IDFT. In this form kernel is orthogonal and additive in nature and follow the reversibility that satisfies this form of DFrFT. Although the obtained result doesn't imply the continuous FrFT. Although this will work in a manner identical to original FT and loses important characteristics of fractionalization.

3.2.4 Eigenvectors decomposition-type DFrFT

Another type of DFrFT definition come into existence through searching the Eigen values and Eigen vectors of DFT matrix and then computes it to the fractional power. This sort of DFrFT would work identical to the continuous form and executes orthogonality, additivity and reversibility. In this manner they somehow improved the existing DFrFT definition by modifying their Eigen vector and values in a manner identical to continuous Hermite function, which are supposed to be the Eigen function of the FrFT. Although these sort of DFrFT lacks fast algorithm and the Eigen vector can't be expressed in the closed form.

3.2.5 Group theory-type DFrFT

This sort of DFrFT is based upon group theory and DFrFT. The derived form of this type of DFrFT will satisfy the rotation property on Wigner distribution and also follow the reversible and the additive property. Although this type of DFrFT could be derived only when the fractional order of DFrFT equals some mentioned angles and when the value of '*N*' won't be prime.

3.2.6 Impulse train-type DFrFT

This type of DFrFT came into realization recently with a consideration as a special case of continuous form of FrFT. Input function in this case is supposed to be periodic and equally spaced impulse train. However, it will compete the properties of FrFT and will have fast algorithm but it lacks due to too many constraints and can't be defined considering all the values of α.

3.3 The Closed-Form Discrete Fractional Fourier Transform

The DFrFT introduced here is the sampling of the continuous FrFT. However, sampling the continuous FrFT took place by some equal proper intervals and therefore concluded that the transformed matrix will be orthogonal and reversible in nature. It needs to be reported in closed form such that many properties can be derived and could have fast algorithm.

$$x(n) = f(n.\Delta t)\ , X_\alpha(m) = F_\alpha(m.\Delta u) \quad (3.3)$$

$$where\ n = -N, -N+1, \dots\dots, N\ and\ m = -M, -M+1, \dots\dots\dots, M$$

$$X_\alpha(m) = \sqrt{\frac{1-jcot\alpha}{2\pi}}.\Delta t.\,e^{\frac{j}{2}(cot\alpha)m^2\Delta u^2} \sum_{n=-N}^{N} e^{-jcsc\alpha.n.m\Delta u\Delta t}.\,e^{\frac{j}{2}.cot\alpha.n^2\Delta t^2}.\,x(n) \quad (3.4)$$

$$X_\alpha(m) = \sqrt{\frac{sin\alpha - jcos\alpha}{2M+1}}\,e^{\frac{j}{2}cot\alpha m^2\Delta u^2} \sum_{n=-N}^{N} e^{-j\frac{2\pi.n.m}{2M+1}}.\,e^{\frac{j}{2}cot\alpha.n^2\Delta t^2}.\,x(n)$$

$$when\ \alpha \in 2D\pi + (0,\pi), D\ is\ integer\ (i.e.\,sin\alpha > 0)(3.5)$$

$$X_\alpha(m) = \sqrt{\frac{-sin\alpha + jcos\alpha}{2M+1}}\,e^{\frac{j}{2}cot\alpha m^2\Delta u^2} \sum_{n=-N}^{N} e^{j\frac{2\pi.n.m}{2M+1}}.\,e^{\frac{j}{2}cot\alpha.n^2\Delta t^2}.\,x(n)$$

$$when\ \alpha \in 2D\pi + (-\pi,0), D\ is\ integer\ (i.e.\,sin\alpha < 0)\quad (3.6)$$

Although DFrFT appears precise in concept and give up the additive property but quite convenient for the practical application due to its simple definition of fractional convolution and correlation and main dominant thing among other DFrFT's that the closed form would have lowest complexity and properties analogous to continuous form.

Due to the assimilation of the practical usage, it is advisable for the rest of the applications of the DFrFT such as filter design, pattern recognition. There lies only one disadvantage of the closed form DFrFT that it won't satisfy the additive property which however has not much impact on practical applications.

3.4 Linear Fractional Shift Invariant (LFSI) Systems [22]

In signal processing FrFT is one of the effective fields of the active research due to its applications in the stationary and non stationary environment.

According to FrFT definition [4]

$$\{F^{\alpha}f(t)\}(s) = \sqrt{\frac{1-icot\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{\frac{i}{2}(s^2+t^2)cot\alpha - istcsc\alpha} f(t)dt\,, where\ \alpha = \frac{a\pi}{2} \qquad (3.7)$$

It is well known consideration that FrFT is linear but not shift invariant as that of conventional FT. LTI system can be expressed in terms of convolution of two functions similarly LFSI can be analyzed in terms of its response to a unit impulse response. It is derived that if $x(t)$, $y(t)$ are input and output of LTI system with impulse response $h(t)$ then,

$$y(t) = h(t) * x(t), \qquad (3.8)$$

where $*$ denotes convolution operator.

LFSI system can be expressed as fractional convolution of two functions, one is $e^{it\tau cot\alpha}x(t)$ and other is $h(t)$, where $h(t)$ is impulse response [22].

$$y(t) = e^{it\tau cot\alpha}x(t) *^{\alpha} h(t) \qquad (3.9)$$

Just like LTI system, LFSI system can be formed by two inherent basic properties of a system linearity and shift invariance.

3.4.1 Linearity

A system is said to be **linear** if and only if it satisfies superposition principle which simply states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals, i.e.

$$y(t) = ay_1(t) + by_2(t) \qquad (3.10)$$

Now,

$$y(t) = e^{it\tau cot\alpha} x(t) *^{\alpha} h(t) \quad (3.11)$$

For any arbitrary input sequences $x_1(t)$, $x_2(t)$ and any arbitrary constants a and b, the corresponding outputs are

$$y_1(t) = e^{it\tau cot\alpha} x_1(t) *^{\alpha} h(t) \quad (3.12)$$

$$y_2(t) = e^{it\tau cot\alpha} x_2(t) *^{\alpha} h(t) \quad (3.13)$$

A linear combination of two input sequences results in the output

$$y(t) = e^{it\tau cot\alpha} (ax_1(t) + bx_2(t)) *^{\alpha} h(t) \quad (3.14)$$

Following distributive law,

$$\left(x_1(t) + x_2(t)\right) * h(t) = \left(x_1(t) * h(t)\right) + (x_2(t) * h(t)) \quad (3.15)$$

Thus we obtained,

$$y(t) = a(e^{it\tau cot\alpha} x_1(t) *^{\alpha} h(t)) + b(e^{it\tau cot\alpha} x_2(t)) *^{\alpha} h(t)) \quad (3.16)$$

By comparing Eq. (3.12 and 3.13) with (3.16) we conclude

$$y(t) = ay_1(t) + by_2(t) \quad (3.17)$$

Hence the system is linear.

3.4.2 Shift-Invariant

A system is said to be **shift invariant** if any changes in the input corresponds to the congruent changes in the output.

If $h(t)$ is the impulse response at t, then fractional convolution of $h(t)$ with $e^{it\tau cot\alpha} x(t)$ is defined by [22]

Now,

$$y(t) = e^{it\tau cot\alpha} x(t) *^{\alpha} h(t) \quad (3.18)$$

Considering corresponding changes in input, we have

$$y(t) = e^{it\tau cot\alpha} x(t - t_0) *^{\alpha} h(t) \quad (3.19)$$

Now by using convolution theorem in fractional domain [11].

$$\{F^{\alpha} y(t)\}(s) = e^{\frac{-i}{2}s^2 cot\alpha} \left\{F^{\alpha}\left(e^{it\tau cot\alpha} x(t-t_0)\right)\right\}(s).\{F^{\alpha} h(t)\}(s) \quad (3.20)$$

Using properties of the fractional Fourier transform [4]

$$\{F^{\alpha}\left(e^{it\tau cot\alpha} x(t-t_0)\right)(s) = X_{\alpha}(u - t_0 cos\alpha - \tau cos\alpha)$$

$$\times\, e^{-i\tau^2 \frac{cot^2\alpha}{2} sin\alpha cos\alpha + iu\tau cos\alpha cot\alpha + j\frac{t_0{}^2}{2} sin\alpha cos\alpha - jut_0 sin\alpha} \quad (3.21a)$$

$$\{F^{\alpha}\delta(t)\}(s) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2}(s^2+\tau^2)cot\alpha - js\tau csc\alpha} \quad (3.21b)$$

Let’s say $Y_1 = \{F^{\alpha} y(t)\}(s)$

Putting (3.21a) and (3.21b) into Eq. (3.20)

$$Y_1 = \left(X_{\alpha}(u - t_0 cos\alpha - \tau cos\alpha) e^{-i\tau^2 \frac{cot^2\alpha}{2} sin\alpha cos\alpha + iu\tau cos\alpha cot\alpha + j\frac{t_0{}^2}{2} sin\alpha cos\alpha - jut_0 sin\alpha}\right)$$

$$\times \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2}(s^2+\tau^2)cot\alpha - js\tau csc\alpha} \quad (3.22)$$

And taking magnitude, we get

$$|Y_1| = \sqrt{\frac{1-icot\alpha}{2\pi}}\, X_{\alpha}(u - (\tau + t_0)cos\alpha) \quad (3.23)$$

Now, conceding equivalent changes in output, we get

$$y(t-t_0) = e^{i(t-t_0)\tau cot\alpha} x(t-t_0) *^{\alpha} h(t-t_0) \quad (3.24)$$

$$y(t-t_0) = e^{-i\tau t_0 cos\alpha}\{e^{it\tau cot\alpha} x(t-t_0) *^{\alpha} h(t-t_0)\} \quad (3.25)$$

Now by using convolution theorem in fractional domain [11]

$$\{F^{\alpha} y(t-t_0)\}(s) = e^{\frac{-i}{2}s^2 cot\alpha} e^{-i\tau t_0 cos\alpha} \quad (3.26)$$

$$\times \left\{F^{\alpha}\left(e^{it\tau cot\alpha} x(t-t_0)\right)\right\}(s).\{F^{\alpha} h(t-t_0)\}(s)$$

Using properties of the FrFT [4]

$$\{F^{\alpha}\left(e^{it\tau cot\alpha}x(t-t_0)\right)(s) = X_{\alpha}(u - t_0cos\alpha - \tau cos\alpha) \tag{3.27a}$$

$$\times\, e^{-i\tau^2\frac{cot^2\alpha}{2}sin\alpha cos\alpha + iu\tau cos\alpha cot\alpha + j\frac{{t_0}^2}{2}sin\alpha cos\alpha - jut_0 sin\alpha}$$

$$\{F^{\alpha}\delta(t-t_0)\}(s) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2}(s^2+(\tau+t_0)^2)cot\alpha - js(\tau+t_0)csc\alpha} \tag{3.27b}$$

Let's say $$Y_2 = \{F^{\alpha}y(t-t_0)\}(s) \tag{3.28}$$

Putting (3.21a) and (3.21b) into (3.20)

$$Y_2 = \left(X_{\alpha}(u - t_0cos\alpha - \tau cos\alpha)\, e^{-i\tau^2\frac{cot^2\alpha}{2}sin\alpha cos\alpha + iu\tau cos\alpha cot\alpha + j\frac{{t_0}^2}{2}sin\alpha cos\alpha - jut_0 sin\alpha}\right)$$

$$\times \sqrt{\frac{1-icot\alpha}{2\pi}}\; e^{\frac{i}{2}(s^2+(\tau+t_0)^2)cot\alpha - js(\tau+t_0)csc\alpha} \tag{3.29}$$

And taking magnitude, we get

$$|Y_2| = \sqrt{\frac{1-icot\alpha}{2\pi}}\, X_{\alpha}(u - (\tau+t_0)cos\alpha) \tag{3.30}$$

As we can see from the above calculation work $Y_1 \neq Y_2$

But $$|Y_1| = |Y_2| \tag{3.31}$$

Thus preceding properties shows that the LFSI is linear and shift invariant in magnitude but not in phase. Though LFSI system is linear and shift-invariant in fractional domain $= a * \pi/2$, but the underneath results depicts simulation for an random value $a = 0.05$.

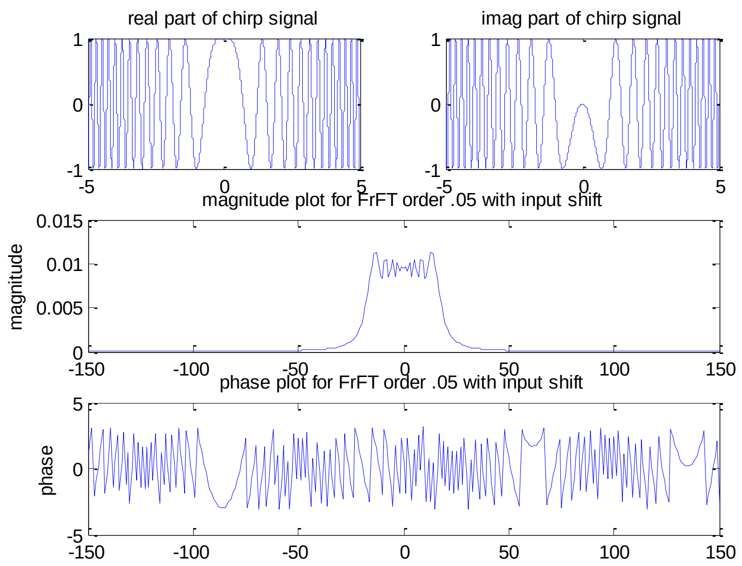


Fig. 3.4(a) Magnitude and phase plot of a FrFT convolution of input (without shift) with impulse response for order a=0.05

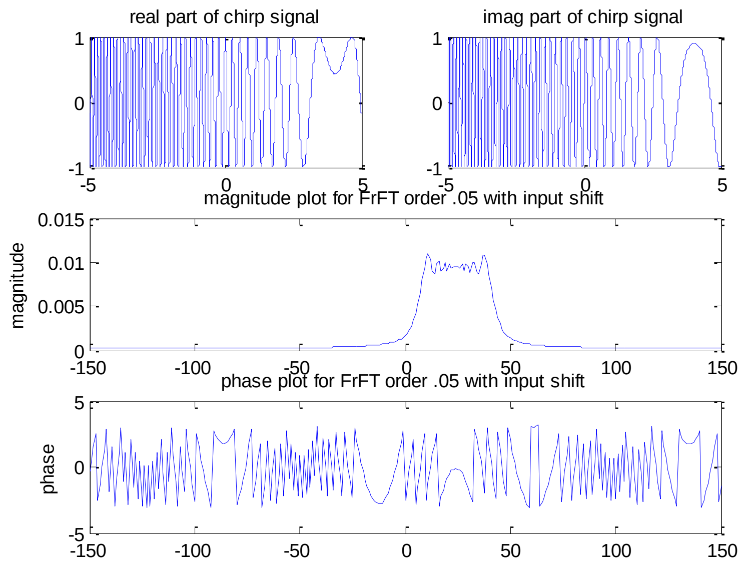


Fig. 3.4(b) Magnitude and phase plot of a FrFT convolution of input (with shift) with impulse response for order a=0.05

CHAPTER 4

FRACTIONAL TRANSLATION AND ITS EFFECT ON FRACTIONAL CONVOLUTION AND CORRELATION

4.1 Fractional Convolution

Consider two signals x and h. When we talk about the convolution of these two signals in the ath fractional domain, then we mean the new signal y which is a convolution of $x_\alpha(u)$ and $h_\alpha(u)$. The expected fractional domain representation of new signal y is:

$$y_\alpha(u) = x_\alpha(u) * h_\alpha(u) \tag{4.1}$$

Furthermore, when we talk about multiplication of two signals in fractional domain, then the corresponding representation means:

$$y_\alpha(u) = x_\alpha(u) h_\alpha(u) \tag{4.2}$$

Apparently convolution (or say multiplication) in $\alpha = 0th$ domain is standard convolution (or multiplication) definition and convolution (or multiplication) in $\alpha = 1st$ domain is standard multiplication (or convolution). More specifically, convolution (or say multiplication) in αth domain is multiplication (or convolution) in $(\alpha \pm 1)th$ domain (orthogonal to αth domain) and convolution (or say multiplication) in αth domain is still the convolution (or multiplication) in $(\alpha \pm 2)th$ domain (countered version of αth domain).

As convolution and FT of function are strongly associated together, similarly in a manner fractional convolution is accompanied by the notional concept of FrFT. Considering FrFT, the fractional convolution is seemed to be more generic than the usual one [11]. The fractional convolution of two functions comprises both usual convolution as well as their product as their exceptional cases, certainly the shift invariance property of the usual convolution is disappeared and fractional convolution is only partial invariant through translation.

4.1.1 First form of fractional convolution

Consider two signals x and h, then the fractional convolution of order α can be defined as:

$$x *^{\alpha} h = F_{-\alpha}[F_{\alpha}[x]F_{\alpha}[h]] \tag{4.3}$$

where $x *^{\pi/2} h = x * h$ and $x *^{0} h = xh$

However the consequences reveal that the translation (shift) invariance of usual convolution can be expressed through integral expressions and this fact guides to believe that shift invariance of usual convolution needs to be obtained through integral form of fractional convolution. More likely, the FrFT definition been implicated through integral expression but the above fractional convolution triple integral definition needs to be deduce to single integral to make it effective for practical applications.

4.1.2 Another form of fractional convolution

The later convolution definition is accompanied by linear phase shift operator. Let's say S_a $(a \in \mathbb{R})$ a linear shift operator can be defined as:

$$S_a x(t) = e^{-i2\pi at} x(t) \tag{4.4}$$

Recalling some properties and basic notations in order to make the comparison, the axis reversed function $\tilde{x}$ of a function x is defined by $\tilde{x}(t) = x(-t)$. We denote $T_a x$ or $T_a[x]$ the function x translated by a $(a \in \mathbb{R})$ is defined by

$$T_a x(t) = T_a[x](t) = x(t-a) \tag{4.5}$$

Where the T_a is a translation (or shift) operator.

If $(a, b) \in \mathbb{R}$ and consider the function $S_b T_a x$, such that

$$S_b T_a x(t) = e^{-i2\pi bt} x(t-a) \tag{4.6}$$

Using (4.5) and (4.6) we get

$$F_{\alpha}[\,S_b T_a x](\acute{t}) = C_{\alpha} e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-i\pi (t-a)^2 cot\alpha}\, e^{\frac{i2\pi t(t-a)}{sin\alpha}} e^{i2\pi b(t-a)} x(t-a) dt \tag{4.7}$$

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-i\pi t^2 cot\alpha}\, e^{\frac{j2\pi t\acute{t}}{sin\alpha}} e^{-i2\pi bt} x(t-a)dt \quad (4.8)$$

Putting $(t - a = u)$ in Eq. (4.8) we get

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-i\pi (u+a)^2 cot\alpha}\, e^{\frac{i2\pi \acute{t}(u+a)}{sin\alpha}} e^{-i2\pi b(u+a)} x(u)du \quad (4.9)$$

Replacing u by $'t'$ using similarity property, we get

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-i\pi (t+a)^2 cot\alpha}\, e^{\frac{i2\pi \acute{t}(t+a)}{sin\alpha}} e^{-i2\pi b(t+a)} x(t)dt \quad (4.10)$$

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-i\pi t^2 cot\alpha}\, e^{-i\pi a^2 cot\alpha} e^{\frac{-i2\pi at cot\alpha}{sin\alpha}} e^{\frac{j2\pi a\acute{t}}{sin\alpha}} \quad (4.11)$$

$$\times\, e^{-i2\pi bt} e^{-i2\pi ab} x(t)dt$$

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} e^{-i\pi a^2 cot\alpha} e^{\frac{i2\pi \acute{t}a}{sin\alpha}} e^{-i2\pi ba} \int e^{-i\pi t^2 cot\alpha}\, e^{\frac{i2\pi \acute{t}t}{sin\alpha}} \quad (4.12)$$

$$\times\, e^{-i2\pi at cot\alpha} e^{-i2\pi bt} x(t)dt$$

$$= C_\alpha e^{i\pi b^2(sin\alpha cos\alpha - sin^2\alpha cot\alpha)} e^{-i\pi a^2 (sin\alpha cos\alpha + cos^2\alpha cot\alpha)} \quad (4.13)$$

$$\times\, e^{i2\pi \acute{t}a(sin\alpha + cos\alpha cot\alpha)} e^{-i2\pi \acute{t}b(cos\alpha - sin\alpha cot\alpha)}$$

$$\times\, e^{\frac{i2\pi \acute{t}t}{sin\alpha}} e^{-i2\pi at cot\alpha} e^{-i2\pi bt} x(t)dt$$

$$\times\, e^{-i2\pi ab} e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-i\pi t^2 cot\alpha}$$

$$= C_\alpha e^{i\pi(b^2-a^2) sin\alpha cos\alpha} e^{i2\pi \acute{t}(a sin\alpha - b cos\alpha)} e^{-i2\pi ab} e^{-i\pi \acute{t}^2 cot\alpha} e^{-i2\pi bt} x(t)$$

$$\times\, e^{-i\pi a^2 (cos^2\alpha cot\alpha)} e^{-i\pi b^2 (sin^2\alpha cot\alpha)} e^{i2\pi \acute{t}a(cos\alpha cot\alpha)}$$

$$\times\, e^{i2\pi \acute{t}b(sin\alpha cot\alpha)} \int e^{-i\pi t^2 cot\alpha}\, e^{\frac{i2\pi \acute{t}t}{sin\alpha}} e^{-i2\pi at cot\alpha} dt \quad (4.14)$$

$$= C_\alpha e^{i\pi(b^2-a^2) sin\alpha cos\alpha} e^{i2\pi \acute{t}(a sin\alpha - b cos\alpha)} e^{-i2\pi ab sin^2\alpha} e^{-i\pi \acute{t}^2 cot\alpha}$$

$$\times\, e^{-i\pi b^2 (sin^2\alpha cot\alpha)} e^{-i2\pi ab cos\alpha sin\alpha cot\alpha} e^{i2\pi \acute{t}a(cos\alpha cot\alpha)} \quad (4.15)$$

$$\times\, e^{i2\pi \acute{t}b(sin\alpha cot\alpha)} \int e^{-i\pi t^2 cot\alpha}\, e^{\frac{i2\pi \acute{t}t}{sin\alpha}} e^{-i2\pi at cot\alpha} e^{-i2\pi bt} x(t)dt$$

$$= C_\alpha e^{i\pi(b^2-a^2)sin\alpha cos\alpha} e^{i2\pi \acute{t}(asin\alpha - bcos\alpha)} e^{-i2\pi absin^2\alpha} e^{-i\pi \acute{t}^2 cot\alpha} \quad (4.16)$$

$$\times e^{-i\pi(acos\alpha+bsin\alpha)^2 cot\alpha} e^{i2\pi \acute{t}(acos\alpha+bsin\alpha)cot\alpha} \int e^{-i\pi t^2 cot\alpha}$$

$$\times e^{\frac{i2\pi \acute{t} t}{sin\alpha}} e^{\frac{-i2\pi(acos\alpha+bsin\alpha)t}{sin\alpha}} x(t)dt$$

$$= C_\alpha e^{i\pi(b^2-a^2)sin\alpha cos\alpha} e^{i2\pi \acute{t}(asin\alpha - bcos\alpha)} e^{-i2\pi absin^2\alpha} \quad (4.17)$$

$$\times \int e^{-i\pi t^2 cot\alpha} e^{\frac{i2\pi t(\acute{t}-(acos\alpha+bsin\alpha))}{sin\alpha}} x(t) e^{-i\pi(t-(acos\alpha+bsin\alpha))^2 cot\alpha} dt$$

$$= e^{i\pi(b^2-a^2)sin\alpha cos\alpha} e^{i2\pi \acute{t}(asin\alpha - bcos\alpha)} e^{-i2\pi absin^2\alpha} \quad (4.18)$$

$$\times F_\alpha[x](\acute{t} - (acos\alpha + bsin\alpha))$$

Although both $F_\alpha[S_b T_a x]$ and $F_\alpha[x]$ depend upon the same argument if we chose a and b in such a manner that

$$acos\alpha + bsin\alpha = 0 \quad (4.19)$$

When Eq. (4.19) holds true then Eq. (4.18) becomes

$$F_\alpha[\, S_{-acot\alpha} T_a x](\acute{t}) = e^{i\pi a^2 cot\alpha} e^{\frac{i2\pi a \acute{t}}{sin\alpha}} F_\alpha[x](\acute{t}) \quad (4.20)$$

Using the integral form of $F_\alpha[x]$ and rewriting Eq. (4.3) as

$$F_\alpha[x *^\alpha h](\acute{t}) = F_\alpha[h](\acute{t}) C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-j\pi u^2 cot\alpha} e^{\frac{i2\pi u \acute{t}}{sin\alpha}} x(u)du \quad (4.21)$$

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-j\pi u^2 cot\alpha} e^{\frac{i2\pi u \acute{t}}{sin\alpha}} F_\alpha[h](\acute{t}) x(u)du \quad (4.22)$$

Using the relation shown in Eq. (4.20) we get

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-j\pi u^2 cot\alpha} e^{-j\pi u^2 cot\alpha} F_\alpha[\, S_{-ucot\alpha} T_u h](\acute{t})$$

$$\times x(u)du \quad (4.23)$$

$$= C_\alpha e^{-i\pi \acute{t}^2 cot\alpha} \int e^{-j2\pi u^2 cot\alpha} F_\alpha[\, S_{-ucot\alpha} T_u h](\acute{t}) x(u)du \quad (4.24)$$

Using (4.6), we get

$$= C_\alpha e^{-i\pi\acute{t}^2 cot\alpha} \int e^{-j2\pi u^2 cot\alpha}$$

$$\times F_\alpha \left(e^{i2\pi utcot\alpha} h(t-u) \right) x(u)du \quad (4.25)$$

$$= C_\alpha e^{-i\pi\acute{t}^2 cot\alpha} \int e^{-j2\pi u^2 cot\alpha} (C_\alpha e^{-i\pi\acute{t}^2 cot\alpha} \int e^{-j2\pi t^2 cot\alpha}$$

$$\times e^{\frac{i2\pi t\acute{t}}{sin\alpha}} e^{i2\pi utcot\alpha} h(t-u)x(u)dtdu \quad (4.26)$$

$$= C^2{}_\alpha e^{-i\pi\acute{t}^2 cot\alpha} \int e^{-i\pi\acute{t}^2 cot\alpha} e^{-j2\pi u^2 cot\alpha} x(u) \int e^{-j\pi t^2 cot\alpha}$$

$$\times e^{\frac{i2\pi t\acute{t}}{sin\alpha}} e^{i2\pi utcot\alpha} h(t-u)dtdu \quad (4.27)$$

Let $m(t) = C_\alpha \int x(u)h(t-u)e^{-j2\pi u^2 cot\alpha} e^{i2\pi utcot\alpha} du$ (4.28)

$$= e^{-i\pi\acute{t}^2 cot\alpha} [C_\alpha e^{-i\pi\acute{t}^2 cot\alpha} \int e^{-j\pi t^2 cot\alpha} e^{\frac{i2\pi t\acute{t}}{sin\alpha}} (C_\alpha \int x(u)h(t-u)$$

$$\times e^{-j2\pi u^2 cot\alpha} e^{i2\pi utcot\alpha} du)dt] \quad (4.29)$$

$$= e^{-i\pi\acute{t}^2 cot\alpha} \left[C_\alpha e^{-i\pi\acute{t}^2 cot\alpha} \int e^{-j\pi t^2 cot\alpha} e^{\frac{i2\pi t\acute{t}}{sin\alpha}} m(t)dt \right] \quad (4.30)$$

$$= e^{-i\pi\acute{t}^2 cot\alpha} F_\alpha[m](\acute{t}) \quad (4.31)$$

Thus concluding that m might be the good choice to be a new fractional convolution of x and h. Hence for the reason being, $*_\alpha$ a new fractional convolution.

$$[x *_\alpha h](\acute{y}) = C_\alpha F_{-\alpha}[F_\alpha[x](s)F_\alpha[h](s)e^{i\pi s^2 cot\alpha}](\acute{y}) \quad (4.32)$$

$$F_\alpha[x *_\alpha h](\acute{y}) = C_\alpha[F_\alpha[x](s)F_\alpha[h](s)e^{i\pi s^2 cot\alpha}] \quad (4.33)$$

$$= C_\alpha F_\alpha[h](s)e^{i\pi s^2 cot\alpha} F_\alpha[x](s) \quad (4.34)$$

$$= C_\alpha F_\alpha[h](s)e^{i\pi s^2 cot\alpha}$$

$$\times \left(C_\alpha e^{-i\pi s^2 cot\alpha} \int e^{-j\pi u^2 cot\alpha} e^{\frac{i2\pi us}{sin\alpha}} x(u)du \right) \quad (4.35)$$

$$= C_\alpha F_\alpha[h](s)e^{i\pi s^2 cot\alpha} C_\alpha e^{-i\pi s^2 cot\alpha} \int x(u) \quad (4.36)$$

$$\times\, e^{-j\pi u^2 cot\alpha(2-(cos^2\alpha+sin^2\alpha))} e^{i2\pi us(cos\alpha\frac{cos\alpha}{sin\alpha}+sin\alpha)} du$$

$$= C_\alpha F_\alpha[h](s) e^{i\pi s^2 cot\alpha} C_\alpha e^{-i\pi s^2 cot\alpha} \int e^{-i\pi u^2(2cot\alpha)} \quad (4.37)$$

$$\times\, e^{i\pi u^2(cos^2\alpha cot\alpha+cos\alpha sin\alpha)} e^{i2\pi us(cos\alpha cot\alpha+sin\alpha)} x(u)du$$

$$= C_\alpha F_\alpha[h](s) e^{i\pi s^2 cot\alpha} C_\alpha e^{-i\pi s^2 cot\alpha} \int e^{i\pi u^2(cos^2\alpha cot\alpha-2cot\alpha)} \quad (4.38)$$

$$\times\, e^{i\pi u^2 cos\alpha sin\alpha} e^{i2\pi us(cos\alpha cot\alpha+sin\alpha)} x(u)du$$

$$= C_\alpha F_\alpha[h](s) e^{i\pi s^2 cot\alpha} C_\alpha e^{-i\pi s^2 cot\alpha} \int e^{i\pi u^2(cos^2\alpha cot\alpha-2cot\alpha)} \quad (4.39)$$

$$\times\, e^{i\pi u^2(-sin\alpha cos\alpha+2cos\alpha sin\alpha)} e^{i2\pi us(cos\alpha cot\alpha)}$$

$$\times\, e^{i2\pi susin\alpha} x(u)du$$

$$= C^2{}_\alpha F_\alpha[h](s) e^{-i\pi s^2 cot\alpha} \int e^{i\pi(u^2cos^2\alpha+s^2+2uscos\alpha)cot\alpha} \quad (4.40)$$

$$\times\, e^{-i\pi 2u^2 cot\alpha - i\pi u^2 sin\alpha cos\alpha} e^{i2\pi u^2 cos\alpha sin\alpha}$$

$$\times\, e^{i2\pi susin\alpha} x(u)du$$

$$= C^2{}_\alpha e^{-i\pi s^2 cot\alpha} \int x(u) e^{i\pi(ucos\alpha+s)^2 cot\alpha} e^{-i\pi 2u^2 cot\alpha} \quad (4.41)$$

$$\times\, e^{-i\pi u^2 sin\alpha cos\alpha} e^{i2\pi(ucos\alpha+s)sin\alpha} F_\alpha[h](s)du$$

$$F_\alpha[\,S_b T_a h](\acute{t}) = e^{i\pi(b^2-a^2)sin\alpha cos\alpha} e^{i2\pi\acute{t}(asin\alpha-bcos\alpha)} \quad (4.42)$$

$$\times\, e^{-i2\pi ab sin^2\alpha} F_\alpha[h]\big(\acute{t}-(acos\alpha+bsin\alpha)\big)$$

Put $= 0$, a$=u$ and $t \acute{=} z$

$$F_\alpha[T_u h](z) = e^{-i\pi(u^2)sin\alpha cos\alpha} e^{i2\pi zusin\alpha} F_\alpha[h](z-ucos\alpha) \quad (4.43)$$

Put $z = ucos\alpha + s$

$$F_\alpha[T_u h](ucos\alpha+s) = e^{-i\pi(u^2)sin\alpha cos\alpha} e^{i2\pi(ucos\alpha+s)usin\alpha} F_\alpha[h](s) \quad (4.44)$$

Thus Eq. (4.41) becomes

$$= C^2{}_\alpha e^{-i\pi s^2 cot\alpha} \int x(u) e^{i\pi(ucos\alpha+s)^2 cot\alpha} e^{-i\pi 2u^2 cot\alpha}$$

$$\times F_\alpha[T_u h](ucos\alpha + s)du \tag{4.45}$$

$$= C^2{}_\alpha e^{-i\pi s^2 cot\alpha} \int x(u) e^{i\pi(ucos\alpha+s)^2 cot\alpha}\, e^{-i\pi 2u^2 cot\alpha} \tag{4.46}$$

$$\times (C_\alpha e^{-i\pi(ucos\alpha+s)^2 cot\alpha} \int h(\acute{t} - u)\, e^{-i\pi\acute{t}^2 cot\alpha}$$

$$\times e^{i2\pi\acute{t}\frac{(ucos\alpha+s)}{sin\alpha}} d\acute{t})du$$

$$= C^2{}_\alpha C_\alpha e^{-i\pi s^2 cot\alpha} \int (e^{-i\pi(ucos\alpha+s)^2 cot\alpha} \tag{4.47}$$

$$\times e^{i\pi(ucos\alpha+s)^2 cot\alpha}) x(u)$$

$$\times \int e^{i2\pi\acute{t}\frac{(ucos\alpha+s)}{sin\alpha}}\, e^{-i\pi\left(2u^2+\acute{t}\right)^2 cot\alpha} h(\acute{t} - u)d\acute{t})du$$

$$= C^2{}_\alpha C_\alpha e^{-i\pi s^2 cot\alpha} \int x(u) \int e^{i2\pi\acute{t}\frac{(ucos\alpha+s)}{sin\alpha}}\, e^{-i\pi\left(2u^2+\acute{t}\right)^2} \tag{4.48}$$

$$\times h(\acute{t} - u)d\acute{t}du$$

$$= C^2{}_\alpha C_\alpha e^{-i\pi s^2 cot\alpha} \int e^{-i\pi\acute{t}^2 cot\alpha}\, e^{\frac{i2\pi\acute{t}s}{sin\alpha}} \int x(u)h(\acute{t} - u)$$

$$\times e^{i2\pi u(\acute{t}-u)cot\alpha} dud\acute{t} \tag{4.49}$$

$$= C_\alpha e^{-i\pi s^2 cot\alpha} \int e^{-i\pi\acute{t}^2 cot\alpha}\, e^{\frac{i2\pi\acute{t}s}{sin\alpha}} \int C^2{}_\alpha \left(x(u)h(\acute{t} - u)\right)$$

$$\times e^{i2\pi u(\acute{t}-u)cot\alpha} dud\acute{t} \tag{4.50}$$

$$F_\alpha[x *_\alpha h](\acute{t}) = F_\alpha[C^2{}_\alpha \int x(u)h(\acute{t} - u)\, e^{i2\pi u(\acute{t}-u)cot\alpha} du](\acute{t}) \tag{4.51}$$

$$[\, x *_\alpha h](\acute{t}) = C^2{}_\alpha \int x(u)h(\acute{t} - u)\, e^{i2\pi u(\acute{t}-u)cot\alpha} du \tag{4.52}$$

Indeed the new fractional convolution definition expressed in single integral form is promising solution although the former definition didn't have much concerns, the only area where it lacked was application. Due to its triple integral form, the former definition demands significant attempts to make it possible for practical realization which is not cost-effective.

4.1.3 Translation invariance of convolution

Convolution and FT of function are strongly associated together. Although convolution is translation (shift) invariant in nature but when it passes through fractional domain it loses some of its properties until or unless it is not supported by translation fractional operator (which will be discussed later).

Recalling some properties and basic notations in order to make the comparison, the axis reversed function $\tilde{x}$ of a function x is defined by $\tilde{x}(t) = x(-t)$. We denote $T_a x$ or $T_a[x]$ the function x translated by a ($a \in \mathbb{R}$) is defined by

$$T_a x(t) = T_a[x](t) = x(t-a) \quad (4.53)$$

Where the T_a is a translation (or shift) operator.

The convolution of two functions x and h is defined as:

$$x * h(t) = \int x(u)h(t-u)du \quad (4.54)$$

$$x * h(t) = \int x(u)T_u h(t)\, du \quad (4.55)$$

Translation invariance of usual convolution means

$$T_a[x*h] = T_a(\int x(u)h(t-u)du) \quad (4.56)$$

$$= (\int (T_a x(u))h(t-u)du) \quad (4.57)$$

$$= \left(T_a x(t)\right) * h(t) \quad (4.58)$$

$$T_a[x*h] = T_a(\int h(t-u)x(u)du) \quad (4.59)$$

$$= \int T_a(h(t-u))x(u)du \quad (4.60)$$

$$= \int (h(t-u-a))x(u)du \quad (4.61)$$

$$= \int x(u)(h(t-u-a))du \quad (4.62)$$

$$= x(t) * T_a h(t) \quad (4.63)$$

Comparing Eqs. (4.53) and (4.58), we get

$$T_a[x*h] = \left(T_a x(t)\right) * h(t) = x(t) * T_a h(t) \quad (4.64)$$

This can also be written as

$$T_a[x*h](t) = T_a(\int h(t-u)x(u)du) \quad (4.64)$$

$$= \int T_a(h(t-u))x(u)du \quad (4.65)$$

$$= \int (h(t-u-a))x(u)du \quad (4.66)$$

$$= \int x(u)(h(t-(u+a)))du \quad (4.67)$$

$$= \int x(u)\, T_{a+u}h(t)du \quad (4.68)$$

$$T_a[x*h](t) = T_a(\int x(u)h(t-u)du) \quad (4.69)$$

$$= (\int (T_a x(u))h(t-u)du) \quad (4.70)$$

$$= \int (T_a x(u))T_u h(t)du \quad (4.71)$$

Comparing Eqs. (4.58) and (4.63), we get

$$T_a[x*h](t) = \int (T_a x(u))T_u h(t)du = \int x(u)\, T_{a+u}h(t)du \quad (4.73)$$

4.1.4 Fractional translation and its effect on fractional convolution

4.1.4.1 Fractional translation

In order to make the fractional convolution shift invariant, the notional concept of fractional translation is being introduced. Thus fractional translation of order α and value a is the transform which generalizes the fact that shifting the function by an amount a is equal to perform the convolution of function and the Dirac distribution δ_a.

$$T_{a:\alpha}[x] = \frac{1}{C^2{}_\alpha}(x *_\alpha \delta_a)e^{i\pi a^2 cot\alpha} \quad (4.74)$$

A précised definition obtained by applying an Eq. (4.16) is

$$x *_\alpha \delta_a(t) = C^2{}_\alpha \int x(t-u)\delta(u-a)e^{i2\pi a(t-a)cot\alpha}du \quad (4.75)$$

$$x *_\alpha \delta_a(t) = C^2{}_\alpha x(t-a)e^{i2\pi at cot\alpha}e^{-i2\pi a^2 cot\alpha} \quad (4.76)$$

$$T_{a:\alpha}[x] = \frac{1}{C^2{}_\alpha}(\, x *_\alpha \delta_a)e^{i\pi a^2 cot\alpha} \quad (4.77)$$

$$= \frac{1}{C^2{}_\alpha}\left(C^2{}_\alpha x(t-a)e^{i2\pi atcot\alpha}e^{-i2\pi a^2cot\alpha}\right)e^{i\pi a^2cot\alpha} \quad (4.78)$$

$$= x(t-a)e^{i2\pi a(t-\frac{a}{2})cot\alpha} \quad (4.79)$$

$$T_{a:\pi/2}[x] = x(t-a) = T_a[x] \quad (4.80)$$

4.1.4.2 Property of fractional translation operator

The composition law for translation operator can be defined as:

$$T_{\acute{a}:\alpha}{}^{\circ}T_{a:\alpha} = T_{a+\acute{a}:\alpha} \quad (4.81)$$

Let's say $h = T_{a:\alpha}[x]$

$$T_{\acute{a}:\alpha}[h](t) = h(t-\acute{a})e^{i2\pi a(t-\frac{\acute{a}}{2})cot\alpha} \quad (4.82)$$

$$= x\big(t-(a+\acute{a})\big)e^{i2\pi a(t-\frac{(a+\acute{a})}{2})cot\alpha}e^{i2\pi\acute{a}(t-\frac{(\acute{a})}{2})cot\alpha} \quad (4.83)$$

$$= x\big(t-(a+\acute{a})\big)e^{i2\pi cot\alpha(at-a\acute{a}-\frac{a^2}{2}+\acute{a}t-\frac{\acute{a}}{2})} \quad (4.84)$$

$$= x\big(t-(a+\acute{a})\big)e^{i2\pi cot\alpha((a+\acute{a})t-\frac{(a+\acute{a})^2}{2})} \quad (4.85)$$

$$= x\big(t-(a+\acute{a})\big)e^{i2\pi cot\alpha(a+\acute{a})(t-\frac{(a+\acute{a})}{2})} \quad (4.86)$$

$$= T_{a+\acute{a}:\alpha}[x](t) \quad (4.87)$$

4.1.4.3 Fractional translation and fractional convolution

Considering the effect of fractional translation operator on fractional convolution, Eq. (4.42) can be written as

$$[\,x *_\alpha h](t) = C^2{}_\alpha \int x(u)h(t-u)\, e^{i2\pi u(t-u)cot\alpha}du \quad (4.88)$$

$$= C^2{}_\alpha \int x(u)T_{u:\alpha}[h](t)e^{-i\pi u^2cot\alpha}du \quad (4.89)$$

$$x *_\alpha h(t-a) = C^2{}_\alpha \int x(u)h(t-a-u)\, e^{i2\pi u(t-a-u)cot\alpha}du \quad (4.90)$$

$$= C^2{}_\alpha \int x(u-a)h(t-u)\, e^{i2\pi(u-a)(t-u)cot\alpha}du \quad (4.91)$$

$$= C^2{}_\alpha \int x(u-a)h(t-u)\, e^{i2\pi(-at-u^2+au+ut)cot\alpha}du \quad (4.92)$$

$$= C^2{}_\alpha \int x(u-a)h(t-u) \quad (4.93)$$

$$\times e^{i2\pi a(u-t)cot\alpha} e^{i2\pi u(t-u)cot\alpha}\, du$$

$$= C^2{}_\alpha \int x(u-a)e^{i2\pi a(u-t)cot\alpha}\, du \quad (4.94)$$

$$\times h(t-u)e^{i2\pi u(t-u)cot\alpha} du$$

$$x *_\alpha h(t-a)e^{i2\pi atcot\alpha} e^{-i\pi u^2 cot\alpha} = C^2{}_\alpha \int x(u-a)e^{i2\pi atcot\alpha} e^{-i\pi a^2 cot\alpha}$$

$$\times e^{i2\pi a(u-t)cot\alpha} h(t-u)e^{i2\pi u(t-u)cot\alpha} du \quad (4.95)$$

$$= C^2{}_\alpha \int x(u-a)e^{i2\pi aucot\alpha} e^{-i\pi a^2 cot\alpha} h(t-u)$$

$$\times e^{i2\pi u(t-u)cot\alpha} du \quad (4.96)$$

$$= C^2{}_\alpha \int x(u-a)\, e^{i2\pi a(u-\frac{a}{2})cot\alpha}$$

$$\times h(t-u)e^{i2\pi u(t-u)cot\alpha} du \quad (4.97)$$

$$= C^2{}_\alpha \int x(u-a)\, e^{i2\pi a(u-\frac{a}{2})cot\alpha} h(t-u) \quad (4.98)$$

$$\times e^{i2\pi u(t-\frac{u}{2})cot\alpha} e^{-i\pi u^2 cot\alpha} du$$

Substituting the value from Eq. (4.79) in Eq. (4.98), we get

$$= C^2{}_\alpha \int T_{a:\alpha}[x](u) T_{u:\alpha}[h](t)\, e^{-i\pi u^2 cot\alpha} du \quad (4.99)$$

Following composition law from Eq. (4.80)

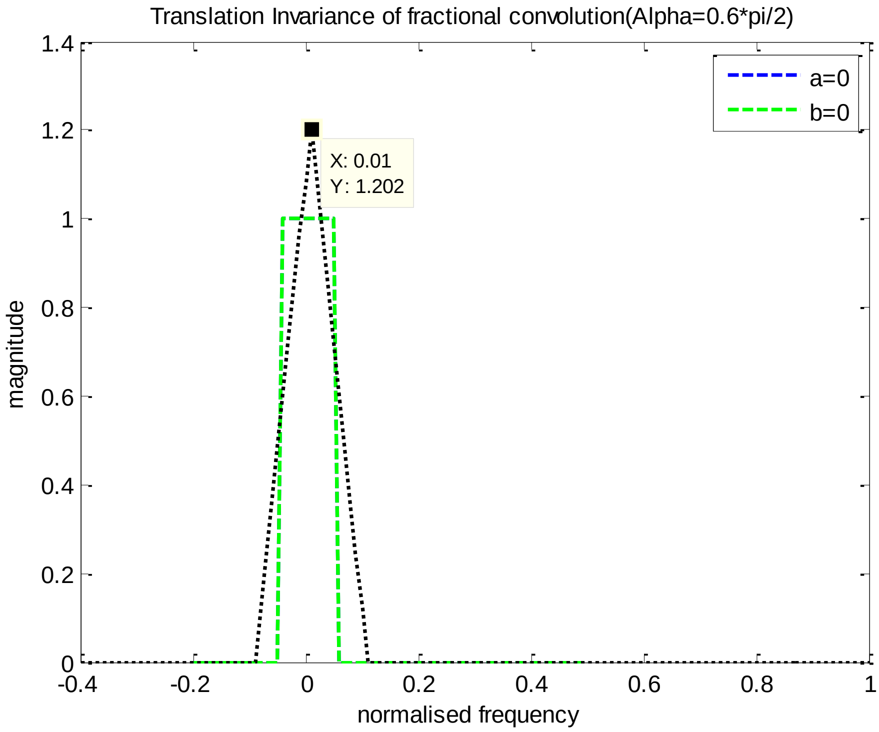
$$= C^2{}_\alpha \int x(u)\, T_{a+u:\alpha}[h](t) e^{-i\pi u^2 cot\alpha} du \quad (4.100)$$

Thus

$$T_{a:\alpha}[x *_\alpha h](t) = C^2{}_\alpha \int x(u)\, T_{a+u:\alpha}[h](t) e^{-i\pi u^2 cot\alpha} du \quad (4.101)$$

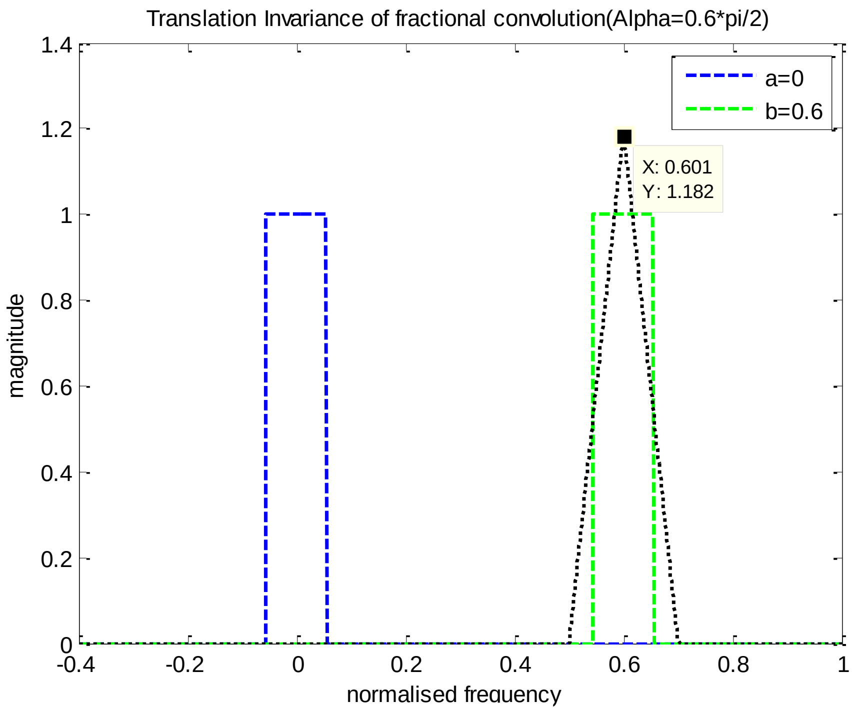
4.1.4.4 Simulation results

We interpret the translation invariance of new fractional convolution. Computing $y = x *_\alpha h$ for $x = T_{a:\alpha} rect_T = rect(t-a)e^{i(2\pi atcot\alpha - \pi a^2 cot\alpha)}$ and $h = T_{b:\alpha} rect_T = rect(t-b)e^{i(2\pi btcot\alpha - \pi b^2 cot\alpha)}$. As x and h are fractional translated illustration of $rect_T$ function, which include quadratic phase factor $((2\pi a t cot\alpha - \pi a^2 cot\alpha)$ for x and $(2\pi b t cot\alpha - \pi b^2 cot\alpha)$ for $h)$, for which these are represented graphically in magnitude form $|x|$ and $|h|$. Thus in magnitude form x and h are simple rectangular function which are similar to usual translation operator function $T_{a:\pi/2} rect_T = rect(t-a)$. The simulation results in Fig. 4.1 are made with the $\alpha = 0.6 * \pi/2$.



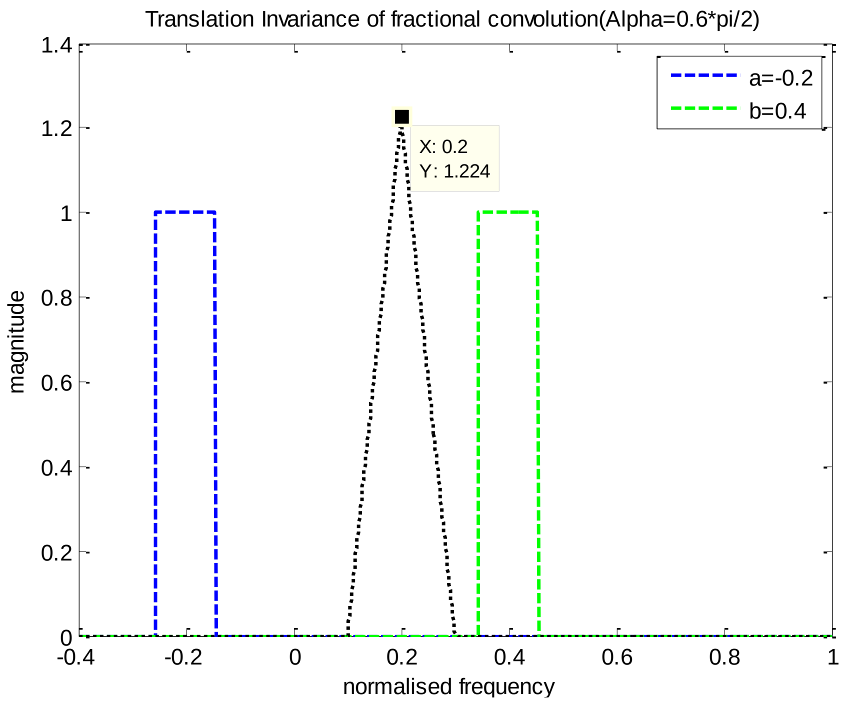
$Fig. 4.1(a)$

$Fig. 4.1(a)$ shows the fractional convolution $(a = 0$ and $b = 0)$ of order $\alpha = 0.6 * \pi/2$ of rectangular function. Following Eq. (4.101), the convolution function will be located at abscissa $x = b + a$. As $a = 0$ and $b = 0$, thus convolution function is at $x = 0$, which clearly illustrates the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is triangular



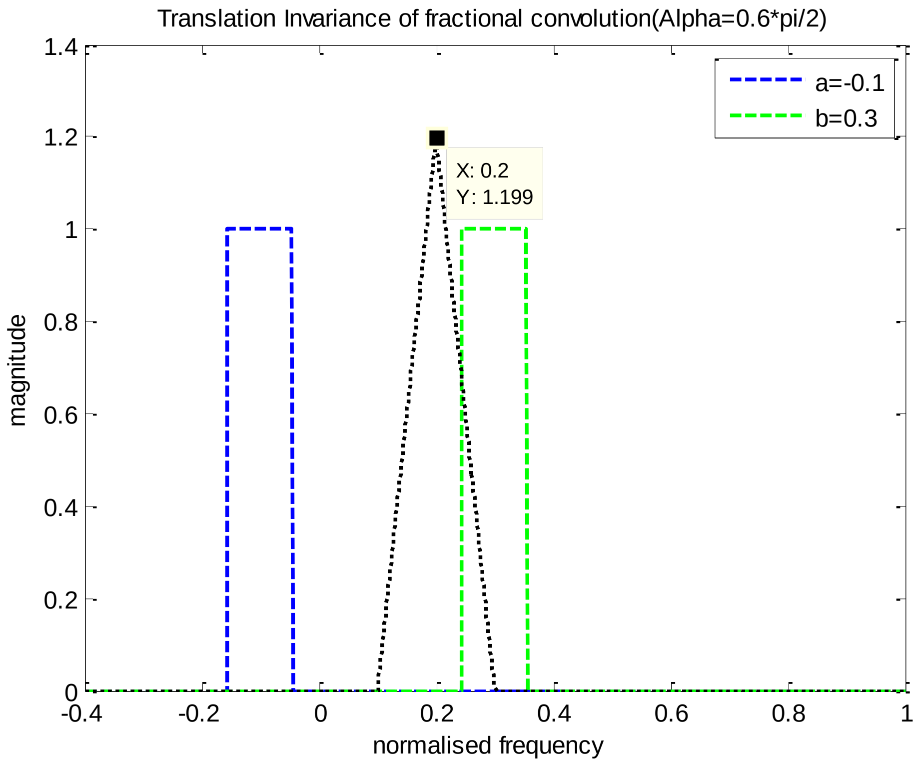
$Fig. 4.1(b)$

$Fig. 4.1(b)$ represents the simulation of fractional convolution ($a = 0$ and $b = 0.6$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The convolution function will be located at abscissa $x = b + a$. As $a = 0$ and $b = 0.6$, thus convolution function is at $x = 0.6$, which clearly emphasize the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is triangular function which will be located at 0.6.



$Fig. 4.1(c)$

$Fig. 4.1(c)$ shows the simulation of fractional convolution ($a = -0.2$ and $b = 0.4$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The convolution function will be located at abscissa $x = b + a$. As $a = -0.2$ and $b = 0.4$, thus convolution function clearly at $x = 0.2$, which clearly represents the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is triangular function which will be located at 0.2.



$Fig. 4.1(d)$

$Fig. 4.1(d)$ shows the simulation of fractional convolution ($a = -0.1$ and $b = 0.3$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The convolution function will be located at abscissa $x = b + a$. As $a = -0.1$ and $b = 0.3$, thus convolution function clearly at $x = 0.2$, which clearly represents the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is triangular function which will be located at 0.2.

The above simulation is repeated with the triangular pulse. Computing $y = x *_\alpha h$ for $x = T_{a:\alpha} tri_T = tri(t-a)e^{i(2\pi atcot\alpha - \pi a^2 cot\alpha)}$ and $h = T_{b:\alpha} tri_T = tri(t-b)e^{i(2\pi btcot\alpha - \pi b^2 cot\alpha)}$. As x and h are fractional translated version of tri_T which include quadratic phase factor $((2\pi atcot\alpha - \pi a^2 cot\alpha)$ for x and $(2\pi btcot\alpha - \pi b^2 cot\alpha)$ for h), for which these are represented in magnitude form $|x|$ and $|h|$. Thus in magnitude form x and h are simple triangular function which are similar to usual translation operator function $T_{a:\pi/2}tri_T = tri(t-a)$. The simulation results in Fig. 4.2 made with the $\alpha = 0.6 * \pi/2$.

Following Eq. (4.52) and Eq. (4.101), when $a = 0$ and $b = 0$, it becomes standard convolution between two triangular functions which leads to a Gaussian pulse. From Eq. (4.52), we have

$$[\, x *_\alpha h](\acute{t}) = C^2{}_\alpha \int x(u)h(\acute{t}-u)\, e^{i2\pi u(\acute{t}-u)cot\alpha}du \qquad (4.102)$$

$y = x *_\alpha h$ for $x = T_{a:\alpha}tri_T = tri(t-a)e^{i(2\pi atcot\alpha - \pi a^2 cot\alpha)}$ and $h = T_{b:\alpha}tri_T = tri(t-b)e^{i(2\pi btcot\alpha - \pi b^2 cot\alpha)}$, where triangular pulse is of width 2 and centre lies at the origin. If $a = 0$ and $b = 0$ then there is no shifting in the triangular pulses and the convolution process proceeds in a standard way.

$$y(t) = [\, x *_\alpha h](\acute{t}) = \int x(u)h(t-u)du \qquad (4.103)$$

when $t + 1 < -1 \; or \; t < -2$, there is no overlap

$$y(t) = 0$$

when $-2 \le t < -1$,

$$y(t) = \int_{-1}^{t+1}(1+u)(1-u+t)du \qquad (4.104)$$

solving we get, $\qquad = \frac{1}{6}(t+2)^3$

when $-1 \le t < 0$,

$$y(t) = \int_{-1}^{t}(1+u)(1+u-t)du + \int_{t}^{0}(1+u)(1-u+t)du + \int_{0}^{t+1}(1-u)(1-u+t)du \qquad (4.105)$$

solving we get,

$$= \frac{1}{3}(-t^3+3t+2) + \left(-\frac{1}{6}t\right)(t^2+6t+6) \qquad (4.106)$$

when $0 \le t < 1$,

$$y(t) = \int_{t-1}^{0} (1+u)(1+u-t)du + \int_{0}^{t} (1-u)(1+u-t)du + \int_{t}^{1} (1-u)(1-u+t)du \quad (4.107)$$

solving we get,

$$= \frac{1}{3}(t^3 - 3t + 2) + \left(\frac{1}{6}t\right)(t^2 - 6t + 6) \quad (4.108)$$

when $1 \le t < 2$,

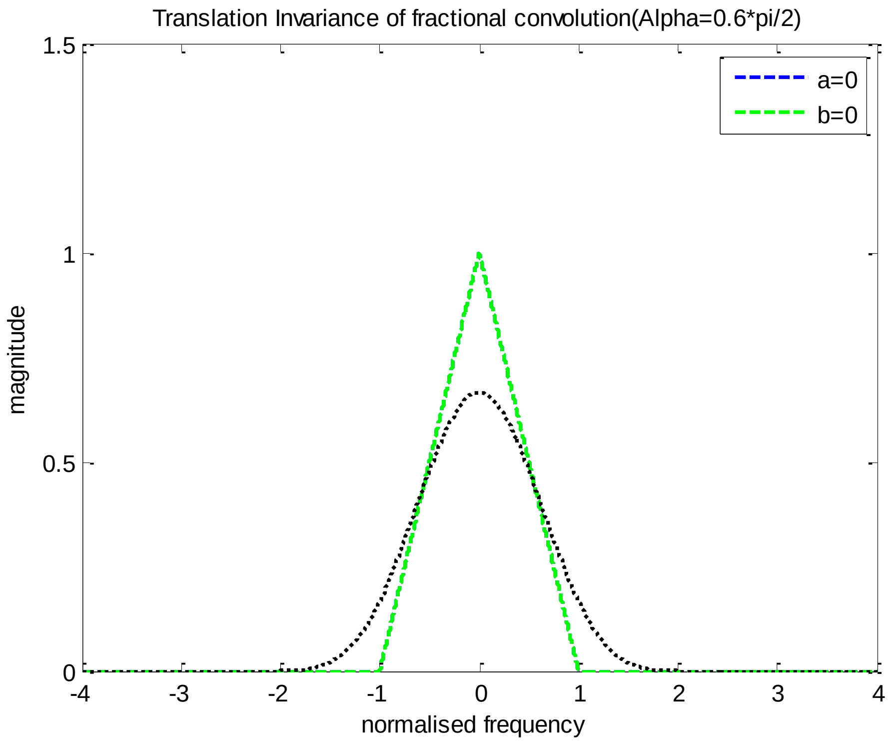
$$y(t) = \int_{t-1}^{1} (1-u)(1+u-t)du \quad (4.109)$$

$$= -\frac{1}{6}(t-2)^3$$

when $t \ge 2$,

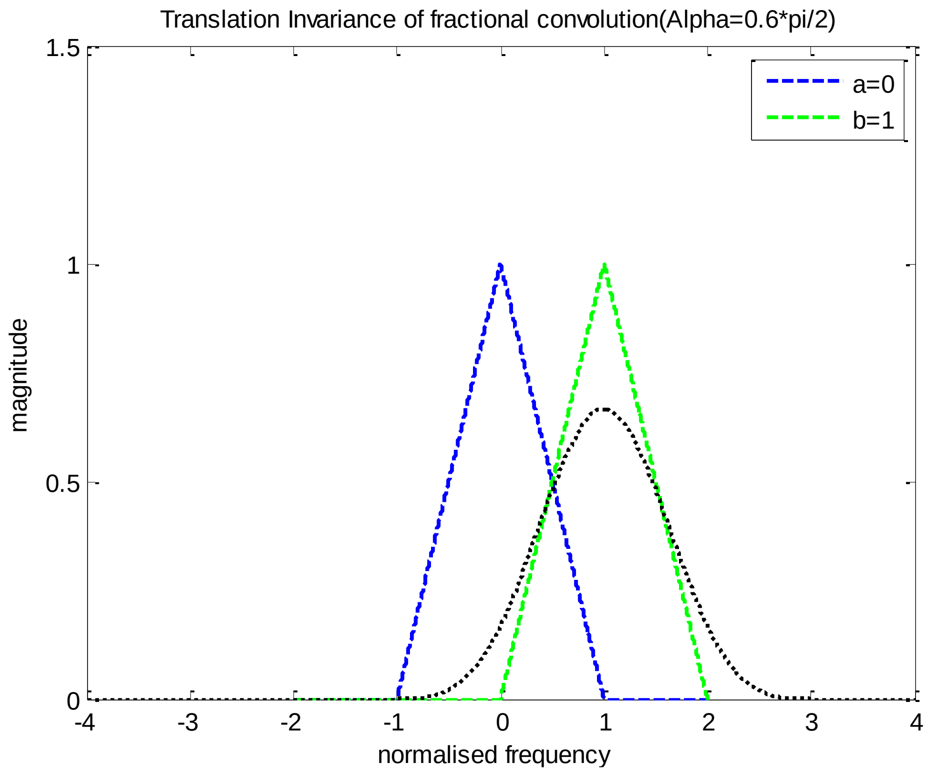
$$y(t) = 0$$

Thus we have,
$$y(t) = \begin{cases} \frac{1}{6}(t+2)^3; & -2 \le t < -1 \\ \frac{1}{3}(-t^3+3t+2) + \left(-\frac{1}{6}t\right)(t^2+6t+6); & -1 \le t < 0 \\ \frac{1}{3}(t^3-3t+2) + \left(\frac{1}{6}t\right)(t^2-6t+6); & 0 \le t < 1 \\ -\frac{1}{6}(t-2)^3; & 1 \le t < 2 \end{cases} \quad (4.110)$$



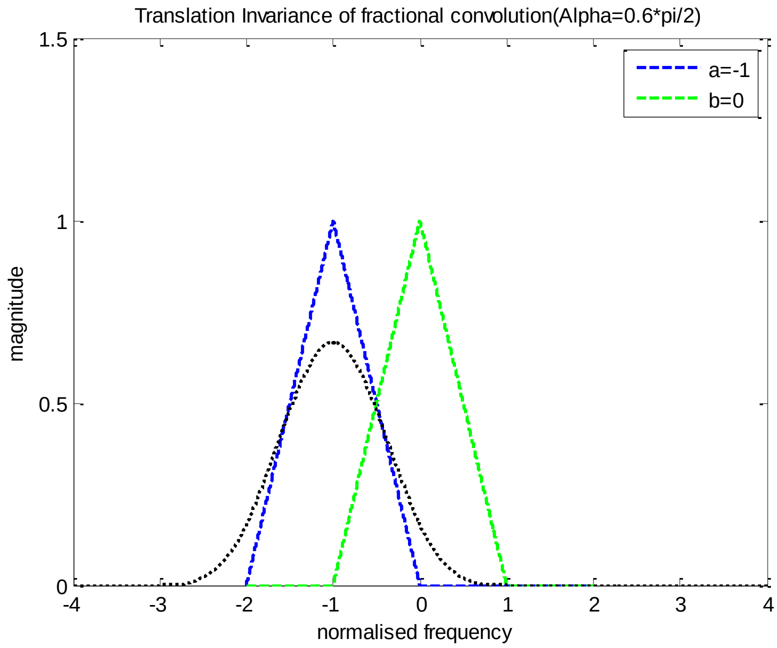
$Fig. 4.2(a)$

$Fig. 4.2(a)$ shows the simulation of fractional convolution ($a = 0$ and $b = 0$) of order $\alpha = 0.6 * \pi/2$ of triangular function. The convolution function will be located at abscissa $x = b + a$. As $a = 0$ and $b = 0$, thus convolution function clearly at $x = 0$, which clearly represents the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is Gaussian function and will be located at 0.



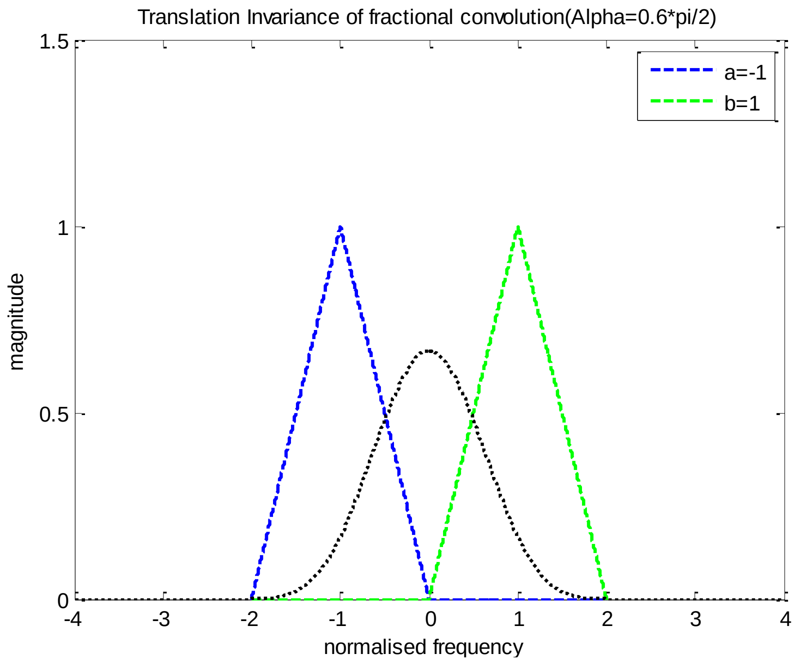
$Fig. 4.2(b)$

In a similar manner $Fig. 4.2(b)$ represents the simulation of fractional convolution ($a = 0$ and $b = 1$) of order $\alpha = 0.6 * \pi/2$ of triangular function. The convolution function will be located at abscissa $x = b + a$. As $a = 0$ and $b = 1$, thus convolution function is at $x = 1$, which clearly emphasize the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is Gaussian function and will be located at 1.



$Fig. 4.2(c)$

$Fig. 4.2(c)$ shows the simulation of fractional convolution ($a = -1$ and $b = 0$) of order $\alpha = 0.6 * \pi/2$ of triangular function. The convolution function will be located at abscissa $x = b + a$. As $a = -1$ and $b = 0$, thus convolution function clearly at $x = -1$, which clearly represents the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is Gaussian function and will be located at 0.2.



$Fig. 4.2(d)$

$Fig. 4.2(d)$ shows the simulation of fractional convolution ($a = -1$ and $b = 1$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The convolution function will be located at abscissa $x = b + a$. As $a = -1$ and $b = 1$, thus convolution function clearly at $x = 0$, which clearly represents the invariance property of fractional convolution under fractional translation. The convolution function $|h|$ obtained is Gaussian function which will be located at 0.

4.2 Fractional correlation

The correlation is basically for comparing two signals, or to recognizing a smaller pattern in a larger signal. It is somewhat identical as convolution but depicted differently. When smaller signal is to be correlated with the larger signal, it is normally more adequate to determine it directly in either time or space domain, but when larger signal needs to be correlated then it is advisable to exploit the FFT and multiply the FT of one with the complex conjugate of FT of other to employ more efficiency. Eventually correlation followed convolution, acknowledging its intrinsic shift-invariant property but in fractional domain correlation operation exhibit partial translation which needs to be replaced by fractional translation.

Let $g(t)$ be an ordinary correlation of an input $x(t)$ with the reference $h(t)$

$$x(t) \circledast h(t) = \int x(u)\overline{T_x g(u) du} \quad (4.111)$$

Although correlation of two functions is nothing but the convolution after one of the two functions is axis-reversed and complex conjugated.

$$x(t) * \overline{\tilde{h}(t)} = \int x(u)\overline{\tilde{h}(t-u)}du \quad (4.112)$$

$$= \int x(u)\overline{h(-t+u)}du \quad (4.113)$$

$$= \int x(u)\overline{h(u-t)}du \quad (4.114)$$

$$x(t) * \overline{\tilde{h}(t)} = x(t) \circledast h(t) \quad (4.115)$$

Now the simple integral form of fractional correlation can be represented as:

$$x(t) \circledast_\alpha h(t) = C_\alpha \overline{C_\alpha} \int x(u)\, \overline{h(u-t)} e^{-i2\pi t(u-t)cot\alpha} du \quad (4.116)$$

Taking FrFT of Eq. (4.115)

$$F_\alpha[x \circledast_\alpha h](t) = C^2{}_\alpha \overline{C_\alpha} e^{-i\pi t^2 cot\alpha} \int e^{-i\pi \acute{t}^2 cot\alpha} e^{i2\pi \acute{t} t/sin\alpha} \int x(u)\overline{h(u-\acute{t})}$$

$$\times\, e^{-i2\pi \acute{t}(u-\acute{t})cot\alpha} du d\acute{t} \qquad (4.117)$$

$$= C^2{}_\alpha \overline{C_\alpha} e^{-i\pi t^2 cot\alpha} \int x(u) \int e^{\frac{i2\pi \acute{t}(t-ucos\alpha)}{sin\alpha}} \qquad (4.118)$$

$$\times\, e^{-i\pi \acute{t}^2 cot\alpha}\overline{h(u-\acute{t})} d\acute{t} du$$

$$= C^2{}_\alpha \overline{C_\alpha} e^{-i\pi t^2 cot\alpha} \int x(u)\, e^{-i\pi(t-ucos\alpha)^2 cot\alpha} e^{i\pi(t-ucos\alpha)^2 cot\alpha}$$

$$\times \int e^{i\pi \acute{t}^2 cot\alpha}\, e^{\frac{i2\pi \acute{t}(t-ucos\alpha)}{sin\alpha}} \overline{h(u-\acute{t})} d\acute{t} du \qquad (4.119)$$

$$= C^2{}_\alpha e^{-i\pi t^2 cot\alpha} \int x(u)\, e^{-i\pi(t-ucos\alpha)^2 cot\alpha} \overline{C_\alpha} e^{i\pi(t-ucos\alpha)^2 cot\alpha} \qquad (4.120)$$

$$\times \int e^{i\pi \acute{t}^2 cot\alpha}\, e^{\frac{-i2\pi \acute{t}(ucos\alpha-t)}{sin\alpha}} \overline{h(u-\acute{t})} d\acute{t} du$$

Considering the definition of FrFT from Eq. (3.2) and taking the conjugate, we get

$$\overline{F_\alpha[x]}(\acute{t}) = \overline{C_\alpha} e^{i\pi \acute{t}^2 cot\alpha} \int e^{i\pi t^2 cot\alpha}\, e^{\frac{-i2\pi \acute{t} t}{sin\alpha}} \overline{x(t)} dt \qquad (4.121)$$

Thus it becomes

$$\overline{F_\alpha\left[T_u \tilde{h}\right]}(ucos\alpha - t) = \overline{C_\alpha} e^{i\pi(t-ucos\alpha)^2 cot\alpha} \int e^{i\pi \acute{t}^2 cot\alpha}\, e^{\frac{-i2\pi \acute{t}(ucos\alpha-t)}{sin\alpha}} \overline{h(u-\acute{t})} d\acute{t} du$$

Put this value in Eq. (4.116) and it becomes

$$F_\alpha[x \circledast_\alpha h](t) = C^2{}_\alpha e^{-i\pi t^2 cot\alpha} \int x(u)\, e^{-i\pi(t-ucos\alpha)^2 cot\alpha} \qquad (4.122)$$

$$\times\, \overline{F_\alpha\left[T_u \tilde{h}\right]}(ucos\alpha - t) du$$

Put $b = 0$, $a = u$ and $\acute{t} = z$ in Eq. (4.42), we get

$$F_\alpha\left[T_u \tilde{h}\right](z) = e^{-i\pi u^2 sin\alpha cos\alpha} e^{i2\pi z u sin\alpha} F_\alpha\left[\tilde{h}\right](z - ucos\alpha) \qquad (4.123)$$

Put $z = ucos\alpha - t$ in Eq. (4.114)

$$\overline{F_\alpha\left[\tilde{h}\right]}(ucos\alpha - t) = e^{i\pi u^2 sin\alpha cos\alpha} e^{-i2\pi(ucos\alpha - t)usin\alpha} \overline{F_\alpha\left[\tilde{h}\right](-t)} \qquad (4.124)$$

Put Eq. (4.124) in Eq. (4.122) and we get

$$F_\alpha[x \circledast_\alpha h](t) = C^2{}_\alpha e^{-i\pi t^2 cot\alpha} \int x(u)\, e^{-i\pi(t-ucos\alpha)^2 cot\alpha} \tag{4.125}$$

$$\times\, e^{i\pi u^2 sin\alpha cos\alpha} e^{-i2\pi(ucos\alpha-t)usin\alpha} \overline{F_\alpha[h](t)}$$

$$= C^2{}_\alpha \overline{F_\alpha[h](t)} e^{-i2\pi t^2 cot\alpha} \int e^{-i\pi u^2 cot\alpha cos^2\alpha}\, e^{i\pi 2tucos\alpha cot\alpha} \tag{4.126}$$

$$\times\, e^{i\pi u^2 sin\alpha cos\alpha} e^{-i2\pi u^2 cos\alpha sin\alpha} e^{i\pi 2tusin\alpha} x(u)du$$

$$= C^2{}_\alpha \overline{F_\alpha[h](t)} e^{-i2\pi t^2 cot\alpha} \int e^{-i\pi u^2 cos\alpha sin\alpha}\, e^{-i\pi u^2 cot\alpha cos^2\alpha}$$

$$\times\, e^{i\pi 2tu\left(sin\alpha + \frac{cos\alpha cos\alpha}{sin\alpha}\right)} x(u)du \tag{4.127}$$

$$= C^2{}_\alpha \overline{F_\alpha[h](t)} e^{-i2\pi t^2 cot\alpha} \int e^{-i\pi u^2 cot\alpha}\, e^{\frac{i2\pi ut}{sin\alpha}} x(u)du \tag{4.128}$$

$$F_\alpha[x \circledast_\alpha h](t) = C_\alpha \overline{F_\alpha[h](t)} e^{-i\pi t^2 cot\alpha} F_\alpha[x](t) \tag{4.129}$$

Thus Eq. (4.116) is equivalent to Eq. (4.129).

4.2.1 Translation invariance of correlation

Although correlation is translation (shift) invariant in nature but when it passes through fractional domain it loses some of its properties until or unless it is not supported by translation fractional operator.

$$T_a[x \circledast h](t) = \int T_a x(u) \overline{T_x h(u)} du \tag{4.130}$$

$$= \int x(u)\, \overline{T_{x-a} h(u)} du \tag{4.131}$$

$$= T_a \int x(u) \overline{h(u-t)} du \tag{4.132}$$

$$= \int T_a x(u)\, \overline{h(u-t)} du \tag{4.133}$$

$$T_a[x \circledast h](t) = \int T_a x(u) \overline{T_t h(u)} du \tag{4.134}$$

Also

$$T_a[x \circledast h](t) = T_a \int x(u) \overline{h(u-t)} du \tag{4.135}$$

$$= \int x(u) T_a \overline{h(u-t)} du \quad (4.136)$$

$$= \int x(u) \overline{h(u-t+a)} du \quad (4.137)$$

$$= \int x(u) \overline{h(u-(t-a))} du \quad (4.138)$$

$$= \int x(u) \overline{T_{x-a} h(u)} du \quad (4.139)$$

Competing Eq. (4.115) and Eq. (4.116), we get

$$T_a[x \circledast h](t) = \int T_a x(u) \overline{T_t h(u)} du = \int x(u) \overline{T_{x-a} h(u)} du \quad (4.140)$$

4.2.2 Effect of fractional translation on fractional correlation

Considering the effect of fractional translation, fractional correlation can be staged as:

$$x(t) \circledast_\alpha h(t) = C_\alpha \overline{C_\alpha} \int x(u)\, \overline{h(u-t)} e^{-i2\pi t(u-t)cot\alpha} du \quad (4.141)$$

Following the invariance of usual correlation and Eq. (4.79), we get

$$T_{a:\alpha}[x \circledast_\alpha h](t) = x \circledast_\alpha T_{a:\alpha} h(t) \quad (4.142)$$

$$= x \circledast_\alpha h(t-a) e^{i2\pi atcot\alpha} e^{-i\pi a^2 cot\alpha} \quad (4.143)$$

$$x \circledast_\alpha h(t-a) = C_\alpha \overline{C_\alpha} \int x(u)\, \overline{h(u-t+a)} e^{-i2\pi(t-a)(u-t+a)cot\alpha} du \quad (4.144)$$

$$= C_\alpha \overline{C_\alpha} \int x(u-a)\, \overline{h(u-t)} e^{-i2\pi(t-a)(u-t)cot\alpha} du \quad (4.145)$$

$$= C_\alpha \overline{C_\alpha} \int x(u-a)\, \overline{h(u-t)} e^{i2\pi(tu-t^2-au+at)cot\alpha} du \quad (4.146)$$

$$= C_\alpha \overline{C_\alpha} \int x(u-a)\, \overline{h(u-t)} e^{i2\pi a(u-t)cot\alpha}$$

$$\times\, e^{-i2\pi t(u-t)cot\alpha} du \quad (4.147)$$

$$= C_\alpha \overline{C_\alpha} \int x(u-a)\, e^{i2\pi a(u-t)cot\alpha} \overline{h(u-t)}$$

$$\times\, e^{-i2\pi t(u-t)cot\alpha} du \quad (4.148)$$

$$x \circledast_\alpha h(t-a) e^{i2\pi atcot\alpha} = C_\alpha \overline{C_\alpha} \int x(u-a)\, e^{i2\pi aucot\alpha} \quad (4.149)$$

$$\times\, \overline{h(u-t)} e^{-i2\pi t(u-t)cot\alpha} du$$

$$x \circledast_\alpha h(t-a)e^{i2\pi atcot\alpha}e^{-i\pi a^2cot\alpha} = C_\alpha\overline{C_\alpha}\int x(u-a)\,e^{i2\pi au}e^{-i\pi a^2cot\alpha}\overline{h(u-t)}$$

$$\times\, e^{-i2\pi t(u-t)cot\alpha}e^{i\pi t^2cot\alpha}du \qquad (4.150)$$

Following Eq. (4.143) and Eq. (4.79), we obtain

$$T_{a:\alpha}[x \circledast_\alpha h](t) = C_\alpha\overline{C_\alpha}\int T_{a:\alpha}[x]\,\overline{T_{t:\alpha}[h](u)}e^{i\pi t^2cot\alpha}du \qquad (4.151)$$

Now considering again the fractional correlation definition

$$x(t) \circledast_\alpha h(t) = C_\alpha\overline{C_\alpha}\int x(u)\,\overline{h(u-t)}e^{-i2\pi t(u-t)cot\alpha}du \qquad (4.152)$$

$$x \circledast_\alpha h(t-a) = C_\alpha\overline{C_\alpha}\int x(u)\,\overline{T_{t-a:\alpha}[h](u)}e^{i\pi(t-a)^2cot\alpha}du \quad (4.153)$$

$$x \circledast_\alpha h(t-a)e^{i2\pi atcot\alpha}e^{-i\pi a^2cot\alpha} = C_\alpha\overline{C_\alpha}\int x(u)\,\overline{T_{t-a:\alpha}[h](u)}e^{i\pi(t-a)^2cot\alpha}$$

$$\times\, e^{i2\pi atcot\alpha}e^{-i\pi a^2cot\alpha}du \qquad (4.154)$$

$$= C_\alpha\overline{C_\alpha}\int x(u)\,\overline{T_{t-a:\alpha}[h](u)}e^{i\pi t^2cot\alpha}du \qquad (4.155)$$

From Eq. (4.143) we get

$$T_{a:\alpha}[x \circledast_\alpha h](t) = C_\alpha\overline{C_\alpha}\int x(u)\,\overline{T_{t-a:\alpha}[h](u)}e^{i\pi t^2cot\alpha}du \qquad (4.156)$$

Comparing Eq. (4.151) and Eq. (4.156) we get

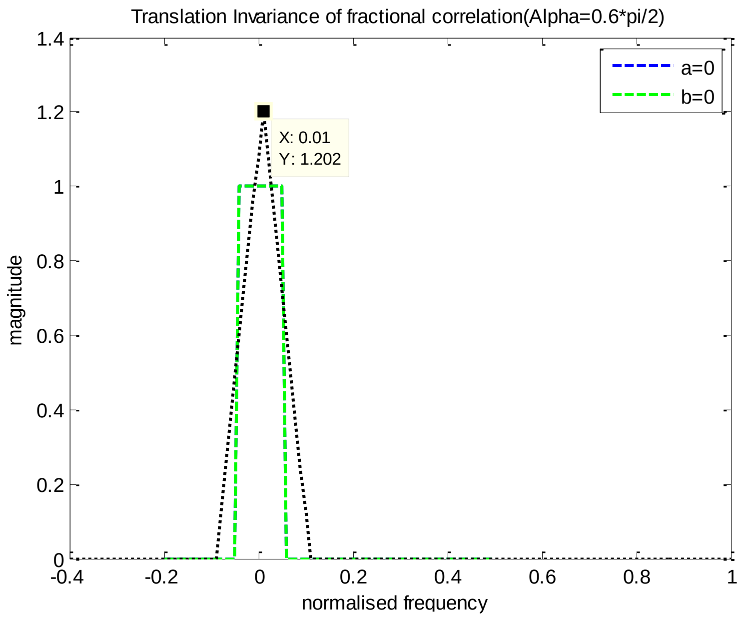
$$T_{a:\alpha}[x \circledast_\alpha h](t) = C_\alpha\overline{C_\alpha}\int T_{a:\alpha}[x]\,\overline{T_{t:\alpha}[h](u)}e^{i\pi t^2cot\alpha}du = C_\alpha\overline{C_\alpha}\int x(u) \qquad (4.157)$$

$$\times\,\overline{T_{t-a:\alpha}[h](u)}e^{i\pi t^2cot\alpha}du$$

4.2.3 Simulation results

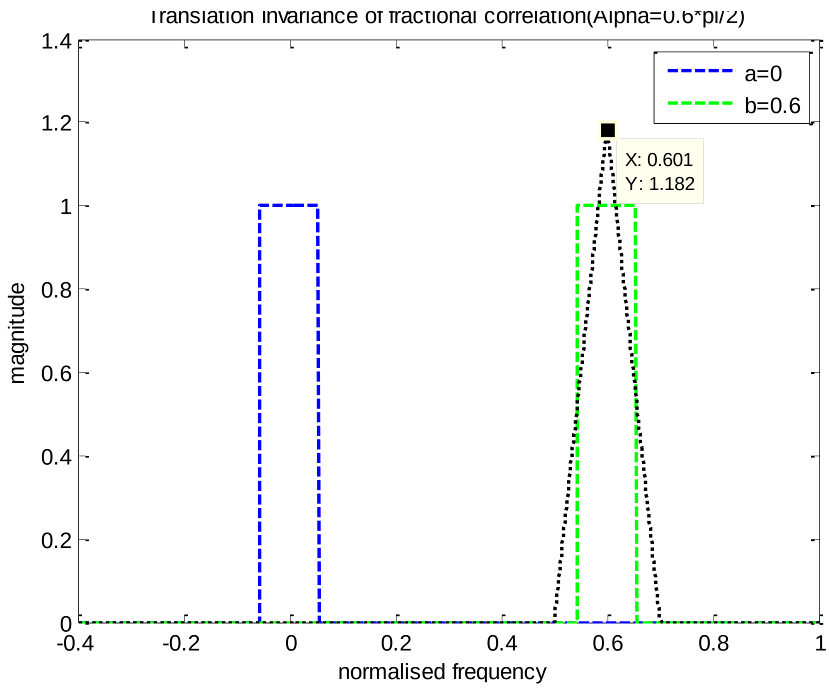
We interpret the translation invariance of new fractional correlation. Computing $y = x \circledast_\alpha h$ for $x = T_{a:\alpha}rect_T = rect(t-a)e^{i(2\pi atcot\alpha-\pi a^2cot\alpha)}$ and $h = T_{b:\alpha}rect_T = rect(t-b)e^{i(2\pi btcot\alpha-\pi b^2cot\alpha)}$. As x and h are fractional translated illustration of $rect_T$ function, which include quadratic phase factor (($2\pi atcot\alpha - \pi a^2cot\alpha$) for x and ($2\pi btcot\alpha - \pi b^2cot\alpha$) for h), for which these are represented graphically in magnitude form $|x|$ and $|h|$. Thus in magnitude form x and h are simple rectangular function which are similar to usual translation operator function

$T_{a:\pi/2}rect_T = rect(t-a)$. The simulation results in Fig. 4.3 are made with the $\alpha = 0.6 * \pi/2$.



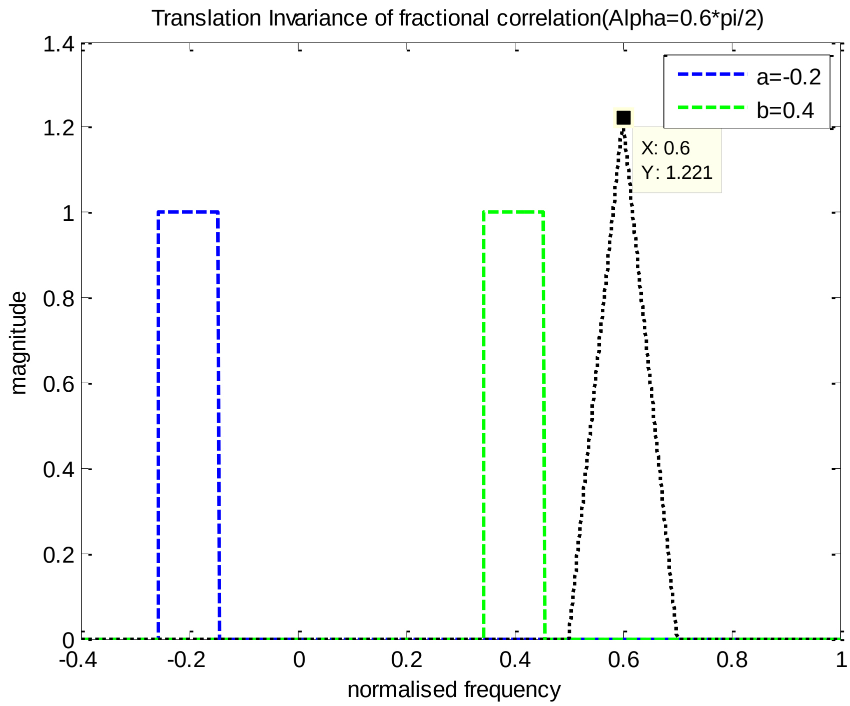
$Fig. 4.3(a)$

$Fig. 4.3(a)$ shows the fractional correlation ($a = 0$ and $b = 0$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. Following Eq. (4.101), the correlation function will be located at abscissa $x = b - a$. As $a = 0$ and $b = 0$, thus correlation function is at $x = 0$, which clearly illustrates the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is triangular function.



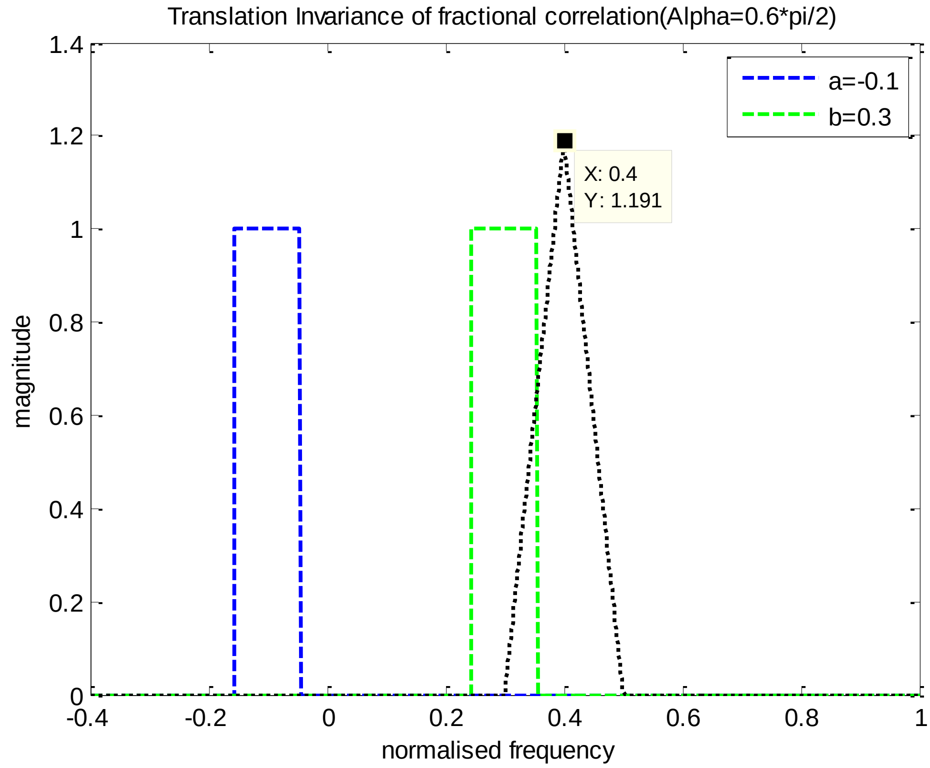
$Fig. 4.3(b)$

$Fig. 4.3(b)$ represents the simulation of fractional correlation ($a = 0$ and $b = 0.6$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The correlation function will be located at abscissa $x = b - a$. As $a = 0$ and $b = 0.6$, thus correlation function is at $x = 0.6$, which clearly emphasize the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is triangular function which will be located at 0.6.



$Fig. 4.3(c)$

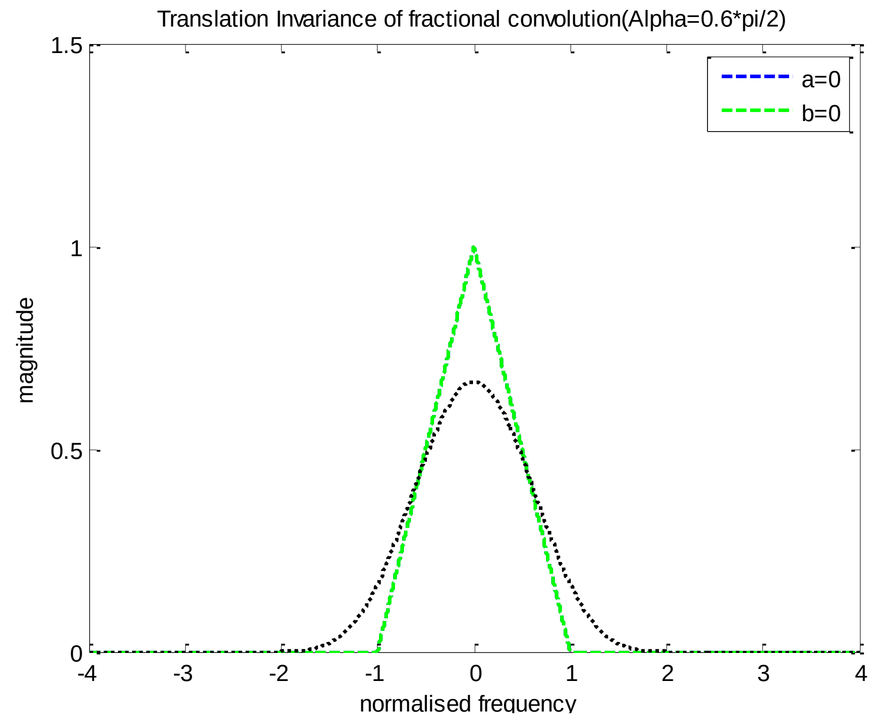
$Fig. 4.3(c)$ shows the simulation of fractional correlation ($a = -0.2$ and $b = 0.4$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The correlation function will be located at abscissa $x = b - a$. As $a = -0.2$ and $b = 0.4$, thus correlation function clearly at $x = 0.6$, which clearly represents the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is triangular function which will be located at 0.6.



$Fig.4.3(d)$

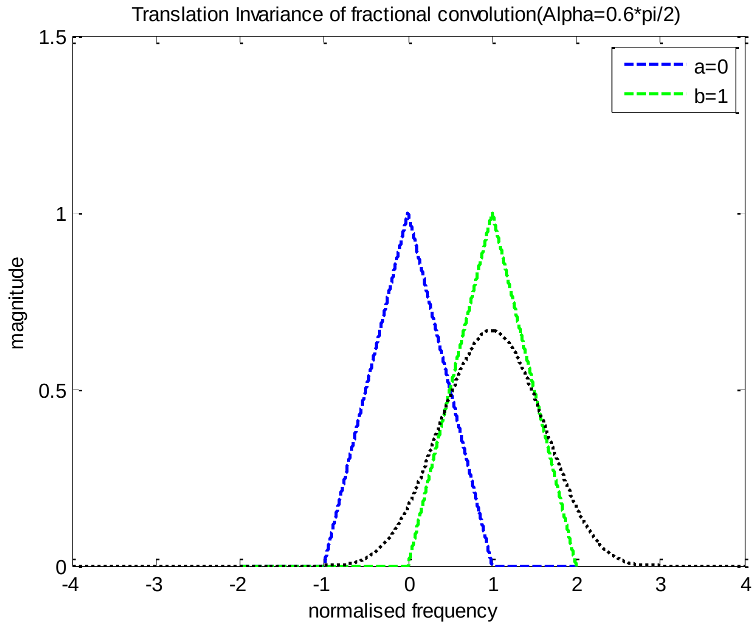
$Fig.4.3(d)$ shows the simulation of fractional correlation ($a = -0.1$ and $b = 0.3$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The correlation function will be located at abscissa $x = b - a$. As $a = -0.1$ and $b = 0.3$, thus correlation function clearly at $x = 0.4$, which clearly represents the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is triangular function which will be located at 0.4.

The above simulation is repeated with the triangular pulse. Computing $y = x \circledast_\alpha h$ for $x = T_{a:\alpha} tri_T = tri(t-a)e^{i(2\pi atcot\alpha - \pi a^2 cot\alpha)}$ and $h = T_{b:\alpha} tri_T = tri(t - b)e^{i(2\pi btcot\alpha - \pi b^2 cot\alpha)}$. As x and h are fractional translated version of tri_T which include quadratic phase factor ($(2\pi atcot\alpha - \pi a^2 cot\alpha)$ for x and $(2\pi btcot\alpha - \pi b^2 cot\alpha)$ for h), for which these are represented in magnitude form $|x|$ and $|h|$. Thus in magnitude form x and h are simple triangular function which are similar to usual translation operator function $T_{a:\pi/2} tri_T = tri(t-a)$. The simulation results in Fig. 4.4 made with the $\alpha = 0.6 * \pi/2$.



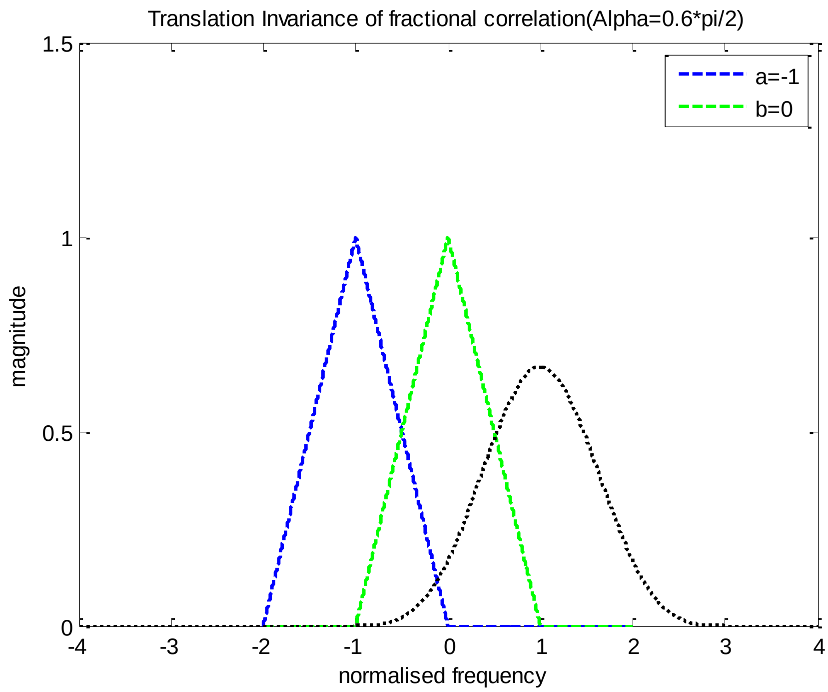
$Fig. 4.4(a)$

$Fig. 4.4(a)$ shows the simulation of fractional correlation ($a = 0$ and $b = 0$) of order $\alpha = 0.6 * \pi/2$ of triangular function. The correlation function will be located at abscissa $x = b - a$. As $a = 0$ and $b = 0$, thus correlation function clearly at $x = 0$, which clearly represents the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is Gaussian function and will be located at 0.



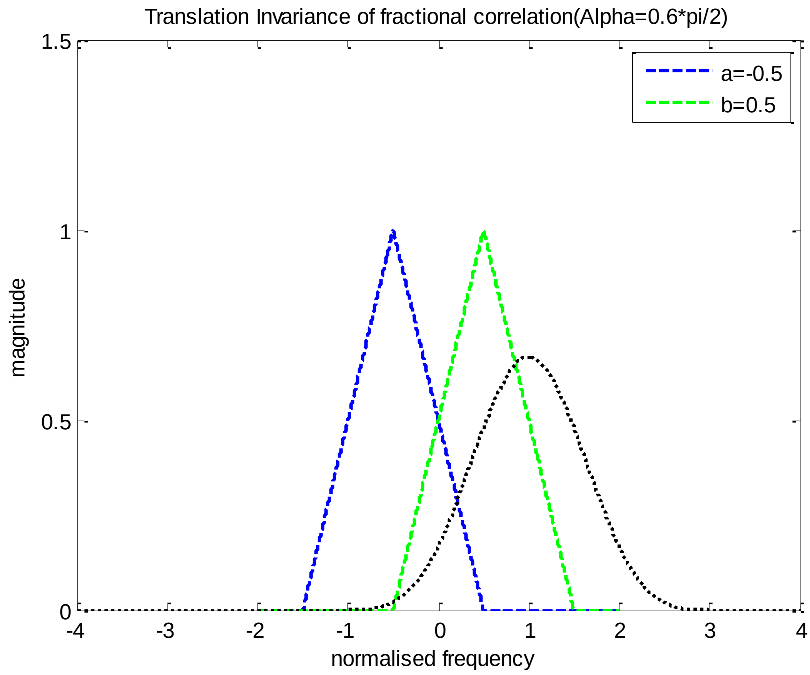
$Fig. 4.4(b)$

In a similar manner $Fig. 4.4(b)$ represents the simulation of fractional correlation ($a = 0$ and $b = 1$) of order $\alpha = 0.6 * \pi/2$ of triangular function. The correlation function will be located at abscissa $x = b - a$. As $a = 0$ and $b = 1$, thus correlation function is at $x = 1$, which clearly emphasize the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is Gaussian function and will be located at 1.



$Fig. 4.4(c)$

$Fig. 4.4(c)$ shows the simulation of fractional correlation ($a = -1$ and $b = 0$) of order $\alpha = 0.6 * \pi/2$ of triangular function. The correlation function will be located at abscissa $x = b - a$. As $a = -1$ and $b = 0$, thus correlation function clearly at $x = 1$, which clearly represents the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is Gaussian function and will be located at 1.



$Fig. 4.4(d)$

$Fig. 4.4(d)$ shows the simulation of fractional correlation ($a = -0.5$ and $b = 0.5$) of order $\alpha = 0.6 * \pi/2$ of rectangular function. The correlation function will be located at abscissa $x = b - a$. As $a = -0.5$ and $b = 0.5$, thus correlation function clearly at $x = 0$, which clearly represents the invariance property of fractional correlation under fractional translation. The correlation function $|h|$ obtained is Gaussian function which will be located at 1.

CHAPTER 5

FILTERING IN FRACTIONAL DOMAIN

5.1 Filtering in fractional domain

A filter is a system that is constructed to pass the spectral component of the desired signal in a stated band of frequencies. In alter words, the filter designs the transfer function to form a window by which only the desired part is allowed to pass. Although the time-domain and frequency domain filtering have convinced the filtering scenarios for stationary signal, but for non-stationary signal it leaves the contradictory remarks as the interaction of signal components with noise components makes it difficult to separate out in frequency and time-domain.

Filter designing in fractional domain is influenced by many methods like windows, frequency sampling and optimal method but somehow optimal method provides an easy and efficient way of computing filter coefficients because of an existence of an excellent design algorithm which provides the total control of filter specifications.

It is noticeable in the window method that in the process of calculating suitable filter coefficients, there is a problem of finding a suitable approximation to a desired or ideal frequency response. The ripples are distributed more evenly equal over the passband and stopband, which may end up with the designing of either too small passband ripple or too large stop band attenuation. Thus because of this, optimal method which is based on the concept of equiripple passband and stopband is applied upon. It is more flexible and having excellent designing algorithm.

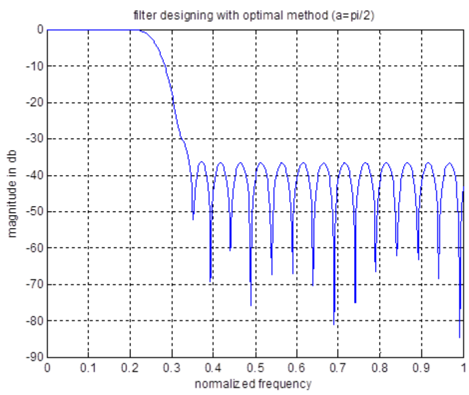
Filtering designing using optimal method in fractional domain can be favorable because after obtaining the equiripple along passband and stopband, transition width of FIR filter can be controlled by FrFT parameter $'\alpha'$.

To design filter in fractional domain, the very first thing need to compute is filter coefficients which can be obtained by optimal method. Optimal method involves the following key steps:

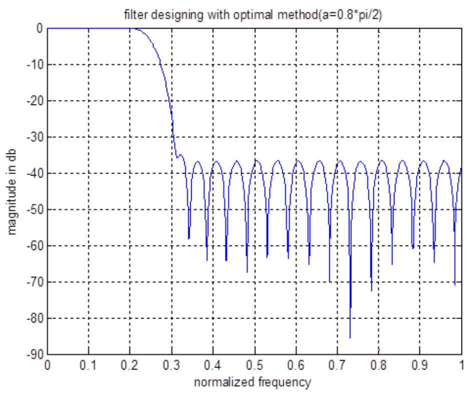
- Use the remez algorithm to find the optimum set of external frequencies.
- Determine the frequency response using the extremal frequencies.
- Obtain the filter coefficients.

After obtaining the filter coefficients, the response of the filter in fractional domain can be calculated. This method provides an optimum equiripple approximation to the desired frequency response and has become the dominant method for optimum design of FIR filters.

In traditional method, order of filter is varied for improvement or sharpening the transition width. But with the FrFT in filter designing, the transition width can be modified using FrFT order with no need of redesigning new filter specifications after once the order of filter is set. Here order of filter is taken to be 27.



$Fig. 5.1(a)$



$Fig. 5.1(b)$

Comparison of filter frequency response is observed for FrFT order 1, 0.8 in figures but there lies a number of values of a $(0 < a \leq 1)$ for which filter frequency response can be monitored, thus to obtain the optimum result a_{opt} value can be calculated which needs to give more efficient result comparing all.

$$\alpha_{opt} = a_{opt} * \frac{\pi}{2}$$

$$a_{opt} = \frac{2}{\pi} * \tan^{-1}\left(\frac{\frac{\delta f}{\delta t}}{2\mu}\right) \quad (5.1)$$

where, δf =frequency resolution=$\frac{f_s}{N}$ and

δt =time resolution=$\frac{1}{f_s}$

f_s =sampling frequency

N =total number of samples

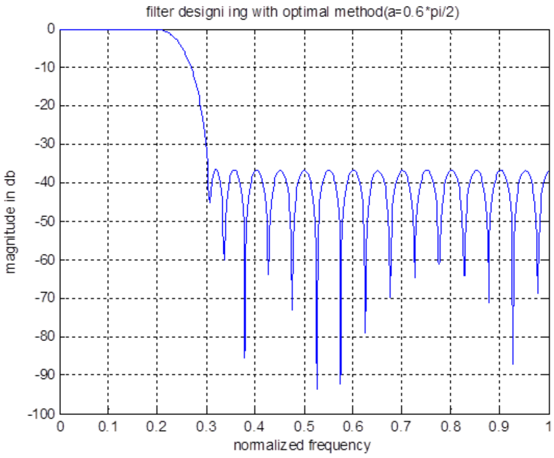
μ =chirp rate=0.5

thus, $$a_{opt} = \frac{2}{\pi} * \tan^{-1}(\frac{{f_s}^2}{N*2\mu}) \quad (5.2)$$

which is used to find the optimal order FrFT with chirp rate 0.5.

Considering $f_s = 1$khz, $N = 7.2 \times 10^5$, $\mu = 0.5$, we get

$$a_{opt} = 0.6$$



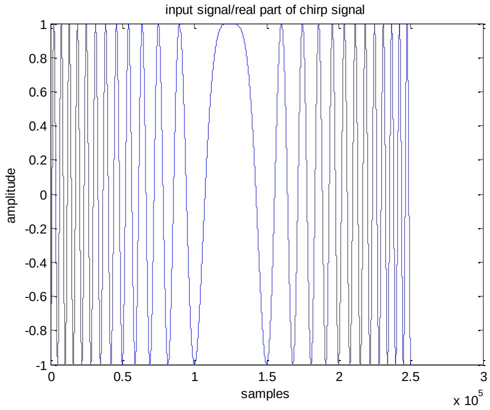
$Fig. 5.1(c)$

Frequency filter response for FrFT order 0.6 can be observed which shows a reduction in transition width as the FrFT order is reduced to a_{opt}.

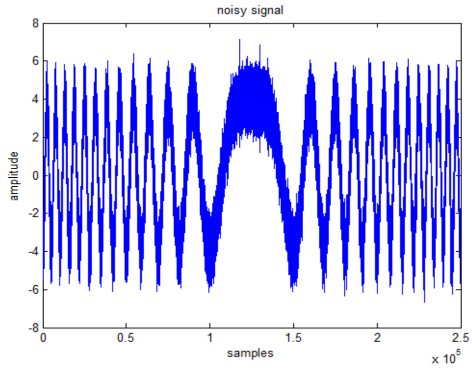
5.1.1 Discussion

The above graph depicts that the transition width of the filter response gradually decreases as the order of FrFT decreases but it follows up to some certain extent (say a_{opt}), as there is not much enhancement in stopband and passband ripples anymore, which further leading to the formation of prominent filter designing.

5.1.2 Comparison

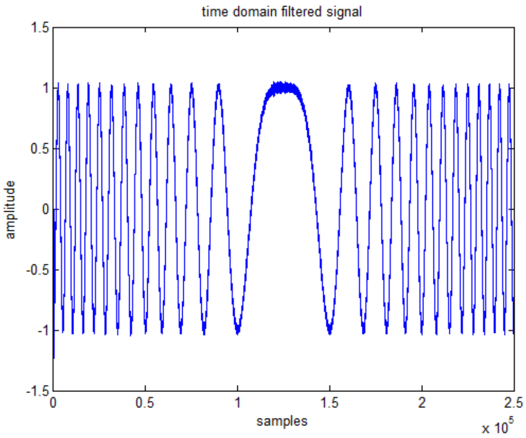


$Fig. 5.2(a)$

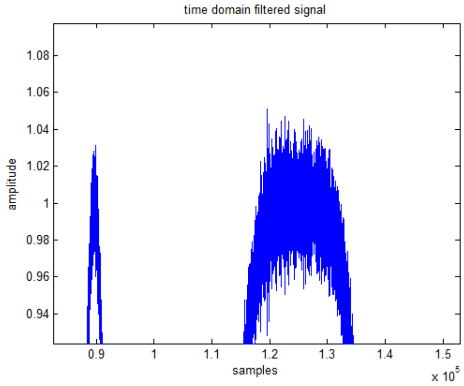


$Fig. 5.2(b)$

Fig. 5.2(a) contains the input chirp signal of second order $e^{-j2\pi\mu t^2}$ with chirp rate 0.5 whereas Fig. 5.2(b) contains the input signal affected by random noise

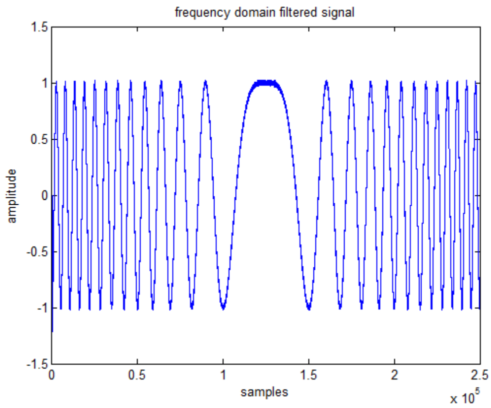


$Fig. 5.2(c)$

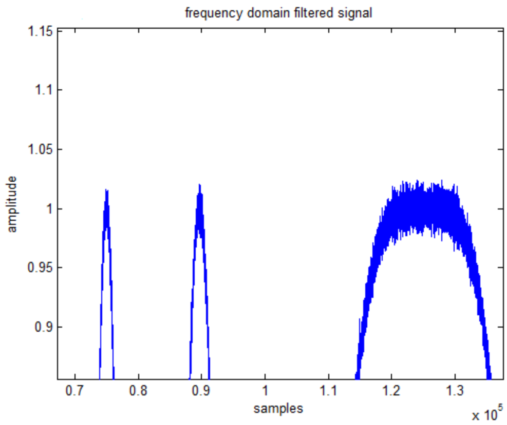


$Fig. 5.2(d)$

Fig. 5.2 (c-d) illustrates the time-domain filtering of the chirp signal affected by random noise.

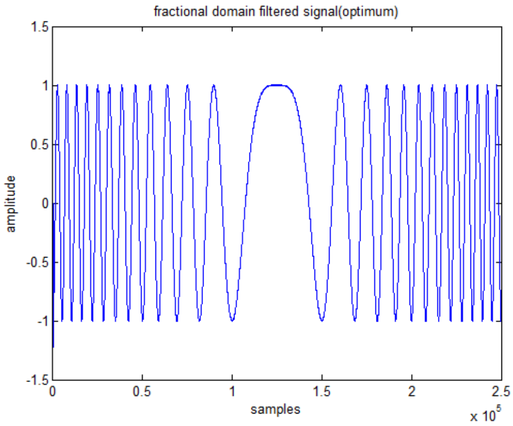


$Fig. 5.2(e)$

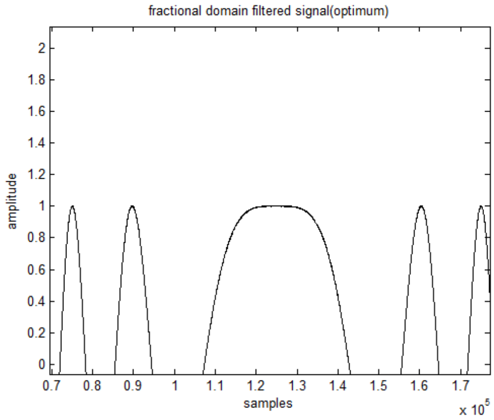


$Fig. 5.2(f)$

Fig. 5.2 (e-f) contains the frequency-domain filtering of the chirp signal affected by random noise.



$Fig. 5.2(g)$



$Fig. 5.2(h)$

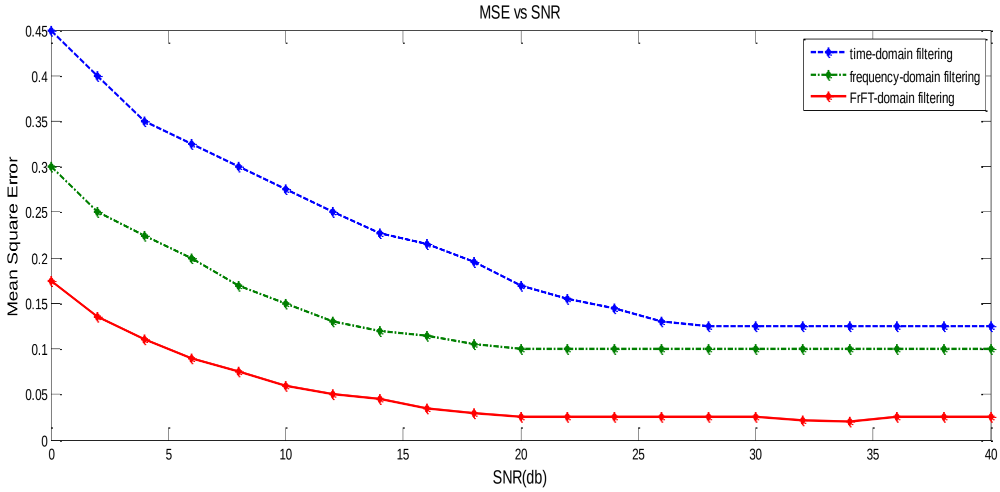
Fig. 5.2 (g-h) contains the FrFT-domain filtering at optimum value $\alpha = 0.6 * \frac{\pi}{2}$

Any value attain by measurement contain two components, one is the signal which carries the information of interest and the other one is noise which is superimposed on the first component. The errors which are of course the undesired quantity affect the precision and accuracy of the signal of interest, needs to be diminished. Thus here chirp signal affected by noise is carried out by filtering operation to correlate the performance of time-domain, frequency-domain and FrFT domain filtering. The criterion used for optimal filtering is mean square error (MSE) between original input signal and the recovered signal for describing the filtering outline. Eq. (5.2) is used for calculating this optimal FrFT order for a linear chirp second order signal. At this optimal value, it reflects the best results, comparing original signal and the filtered signal.

As MSE is arguably the most important criterion for evaluating the performance at particular SNR value.

$$MSE = \frac{1}{N} \sum_{i=1}^{N} (x_{original}(i) - x_{recovered}(i))^2$$

And because of square, large error have relatively high influence on MSE than do the smaller ones and if the error is not squared then negative and positive value may cancel out each other and in that case negative error will be underestimated.



$Fig. 5.3(a)$

Fig. 5.3(a) represents the MSE between the original input signal and the recovered signal for distinct values of signal to noise ratio (SNR) which further concludes that the FrFT domain filtered signal yields better results comparing to time and frequency-domain filtered signal as it yields maximally with the original signal as compared to time-domain filtered signal and frequency-domain filtered signal.

CHAPTER 6

CONCLUSION AND FUTURE SCOPE OF WORK

This section includes the conclusive remarks of the analytical and detailed study reported in previous chapters along with the future scope of the work.

6.1 Summary

The FrFT has been considered to be the noteworthy mathematical tool for interpreting the non stationary signal in the time-frequency plane. But in order to realize it practical feasibility, the numerical interpretation of FrFT is mandatory which favored the need of DFrFT. Due to nonexistent of one clean and complete definition of DFrFT, many algorithms are commenced to define DFrFT but a comparative debate has revealed that closed form DFrFT excels the other algorithm. Apart from this, FrFT definition is deduced out along with the simulation of FrFT of rectangular pulse and triangular function.

LFSI system generalizing LTI system is presented along with the proof which shows that LFSI system is linear and shift-invariant in magnitude but not in phase. Also the simple integral definition of fractional convolution and fractional correlation is presented and further contained the translational concept of fractional convolution and fractional correlation using fractional translational operator which shows that fractional convolution and correlation is no more partial invariant. This whole concept of translation invariance of fractional convolution and correlation is presented along with the simulation results.

In the end, filter designing using optimal method in fractional domain is abstracted and later filtering operation is being performed out to correlate the performance of time-domain, frequency-domain and FrFT domain filtering.

6.2 Future Scope

This dissertation reports the results of the work carried out on translation invariance of fractional convolution and correlation. But it does not foreclose further work which can be taken out. Some of the possible expectations of the reported work can be outlined as:

- The proposed theoretical concept of translational invariance of fractional convolution and correlation may be favorable in correlation and detection application where it may be helpful to have distinct degree of shift invariance for different section of image or a signal. As being capable to spatially accept the degree of shift invariance can raise our discrimination capability in such situations.
- This prominent aspect of shift invariance fractional convolution and correlation can be valuable to some real time application like radar and biomedical.

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