FINITE ELEMENT APPROXIMATION FOR LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS

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Submitted by

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CERTIFICATE

This is to certify that the thesis entitled "**FINITE ELEMENT APPROXIMATION FOR LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS**", **being presented in partial fulfilment of the requirements for the award of the degree of Master of Science in the School of Mathematics, Thapar Institute of Engineering & Technology, Patiala is a bonafide work and study carried out under the supervision of Dr Vivek Sangwan.**

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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(Jashanpreet Kaur)

ABSTRACT

The dissertation entitled "FINITE ELEMENT APPROXIMATION FOR LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS" focuses on finding the numerical solutions of linear and nonlinear differential equations. Differential equations arise in almost all areas of science and engineering. But differential equations governing physical phenomenon are in general complex and nonlinear in nature and have complicated coefficients. Generally, analytical methods are not applicable on these type of problems. Therefore, we need numerical techniques to approximate the solutions of such type of problems. For this purpose, we have used finite element technique to approximate the solutions of the differential equations.

To brief, the first chapter of the dissertation focuses on the basic concepts related to differential equations. One example of each analytical and numerical techniques have been explained with the help of numerical example. Second chapter discusses the finite element formulation for linear differential equations. Proposed method has been explained in detail. Numerical examples have been solved using proposed method and MATLAB code has been generated for solving linear differential equations using finite element technique. In the third chapter, nonlinear differential equations have been considered. Quasilinearization technique has been employed to handle the nonlinearity. Convergence analysis of quasilinearization technique has also been discussed. Numerical results presented depicts that finite element method is capable of approximating the solution very nicely both for linear and nonlinear differential equations.

Contents

Chapter 1

Differential Equations - An Overview

1.1 Introduction

A differential equation is a relation that relates dependent variables with its derivatives with respect to the independent variables. In applications, the functions can represent their rate of change with respect to the independent variables and the differential equation defines the relationship between the two.
In mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions and the set of solutions that satisfy the equation. But very few differential equations are solvable by analytical methods, however some properties of solutions of given differential equations may be determined without finding their exact solutions.

1.2 Differential Equations

An equation involving independent and dependent variables and the derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.
Generally, differential equations arise while modeling various physical phenomena

arising in all most all areas of science and engineering. Physical problem is modeled in terms of differential equations and then the differential equation is solved using analytical or numerical techniques. Finally the solution is simulated and analyzed to find the solution of the physical phenomenon. Differential equations first came into existence with invention of calculus by Newton and Leibnitz.

1.3 Applications Of Differential Equations

Differential equations governs various phenomena arising in different fields like

1. In medicine for modeling cancer growth or spread of disease.
2. In engineering for describing the movement of electricity.
3. In physics, to describe the motion of waves, pendulums or chaotic system. It is also used in Newton's Second law of motion etc.

Differential equations can be categorized into two main categories:

1.3.1 Ordinary Differential Equations

An ordinary differential equation contains only one independent variable and the derivatives with respect to the independent variable. An ordinary differential equation of order 'n' is called linear if it is of below mentioned form:

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_1(x)y' + a_0(x)y = Q(x).$$

Generally, an n-th order ordinary differential equation has n-linearly independent solutions. There are many general methods for solving the ordinary differential equations. Ordinary differential equations deal with functions of single variable and their derivatives.

1.3.2 Partial Differential Equations

In mathematics, a partial differential equation is a differential equation that contains multi-variable functions and their partial derivatives with respect to the independent variables. PDE's are used to formulate problems involving functions of several variables, and are either solved by analytically or numerically.

Partial differential equations can be used to describe a wide variety of phenomena such as sound, electrostatics, heat, diffusion, electrodynamics, elasticity, fluid dynamics or quantum mechanics. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. Partial differential equations find their generalization in stochastic partial differential equations and many more areas.

1.3.3 General Solution Of Differential Equations

General solution of an n-th order differential equation contains n linearly independent solutions. One necessarily have to introduce an arbitrary constant as soon as integration is performed once, thus, we can say that the general solution of a first order differential equation contains one arbitrary constant after simplification.

Similarly, the general solution of second order differential equation will contain two necessary arbitrary constants and so on. The general solution geometrically represents an n-parameter family of curves.

1.4 Solution Methodology

1. Analytical Methods

2. Numerical Methods

1.4.1 Analytical Methods

Analytical solution denotes exact solution that can be used to study the behavior of the system with varying properties. Unfortunately, analytical methods can be applicable to very few practical model problems and are applicable to a very restricted class of differential equations. That is why one needs numerical approach.

1.4.2 Example of Analytical Approach

Example: Find the solution of the differential equation using operator method

$$\frac{d^2u}{dy^2} - 2\frac{du}{dy} + u = y. \tag{1.1}$$

Solution: The auxiliary equation of the given differential equation is given by

$$D^2 - 2D + 1 = 0$$

On solving, we get

$$(D-1)^2 = 0$$

$$D = 1, 1$$

$$C.F. = u_h = (C_1 + C_2 y)e^y$$

$$P.I. = u_p = \frac{1}{(D-1)^2} y = 2 + y$$

Now, the complete solution is given by

$$u = C.S = u_h + u_p$$

$$u = (C_1 + C_2 y)e^y + 2 + y$$

1.4.3 Numerical Methods

Numerical techniques provide us numerical solutions. Numerical solutions are those solutions that cannot be expressed in the form of complete mathematical equations. Numerical solutions has no closed forms. Analytically, it is hard to

solve the differential equations, but numerically we can solve those problems with much ease. Numerical methods are capable of handling large systems of equations. Numerical methods can handle complicated physical geometries which are often impossible to solve analytically. Below we will discuss one of the numerical techniques:

1.4.4 Galerkin's Method

In this method, first of all an appropriate form of functional is sought corresponding to given differential equations. Then like residual method , an appropriate solution is assumed which is linear combination of basis function involving unknown parameters. This solution is substituted in the functional which is extremised with respect to the unknown parameters. As a result, as many linear simultaneous equations are obtained as their are unknown. The solution of simultaneous equations gives the value of unknown which are put in assumed solution to provide the final solution of given differential equations.

Example: Solve the problem (1.1) using Galerkin's method with boundary conditions

$$u(0) = u(1) = 0.$$

Solution: The basis functions satisfying the given boundary conditions are given by

$$\begin{aligned}
\phi &= x(x-1), x^2(x-1) \\
\bar{u} &= a_1 x(x-1) + a_2 x^2(x-1) \\
\bar{u'} &= a_1(2x-1) + a_2(3x^2-2x) \\
\bar{u''} &= 2a_1 + a_2(6x-2).
\end{aligned}$$

Substituting first derivative and second derivative in the differential equation, we get

$$\int_0^1 Re\ x(x-1)dx = 0$$

$$a_1[2-2(2x-1)+(x^2-x)]dx + a_2[(6x-2)-2(3x^2-2x)+(x^3-x^2)]dx = x$$

$$a_1[x^2-5x+4] + a_2[x^3-7x^2+10x-2] = x$$

$$\int_0^1 [a_1(x^2-5x+4)(x^2-x) + a_2(x^3-7x^2+10x-2)(x^2-x)]dx = \int_0^1 x(x^2-x)dx$$

$$\int_0^1 [a_1(x^4-6x^3+9x^2-4x) + a_2(x^5-8x^4+17x^3-12x^2+2x)]dx = \int_0^1 (x^3-x^2)dx$$

$$a_1[\frac{1}{5}-\frac{3}{2}+1] + a_2[\frac{1}{6}-\frac{8}{5}+\frac{17}{4}-3] = -\frac{1}{12}$$

$$0.3a_1 + 0.1833a_2 = 0.0833$$

$$\int_0^1 Re(x^2(x-1))dx = 0$$

$$\int_0^1 [a_1(x^2-5x+4)(x^3-x^2) + a_2(x^3-7x^2+10x-2)(x^3-x^2)]dx = \int_0^1 [x(x^3-x^2)]dx$$

$$\int_0^1 [a_1(x^5-6x^4+9x^3-4x^2) + a_2(x^6-8x^5+17x^4-12x^3+2x^2)]dx = \int_0^1 (x^4-x^3)dx$$

$$a_1(0.1167) + a_2(0.1238) = 0.05$$

Solving these, we get the following two equations:

$$0.3a_1 + 0.1833a_2 = 0.0833$$

$$0.1167a_1 + 0.1238a_2 = 0.05$$

Apply elimination method on these two equations, we get

$$a_1 = 0.0728$$

$$a_2 = 0.3353$$

Hence the Galerkin's solution is given by

$$\bar{u} = 0.0723013x(x-1) + 0.335068x^2(x-1).$$

For the present work, we will use one of the famous numerical technique known as 'finite element technique' to solve the linear and nonlinear problems.

Chapter 2

FINITE ELEMENT APPROXIMATION FOR LINEAR PROBLEMS

2.1 Introduction of Finite Element Method

Finite element method(FEM) is a numerical technique to solve problems arising in engineering and mathematical sciences. Problems which are typical can be solved using this method. The analytical solution of such problem generally can not be derived. The finite element formulation of such problems result in algebraic equations. The method yields approximate values of unknowns at discrete number of points in the domain.
To solve the problem, it subdivides large problem into smaller and simpler parts that are known as finite elements. Dividing the problem of the domain into a collection of sub-domains, with each sub-domain represented by set of elements to original problem. The element equations that are formed using finite elements are then assembled into larger system of equations that model the entire problem. Systematically, all these element equations are assembled into global system for final calculation. The global system of equations are then solved after incorporating the initial or boundary conditions.

The subdivision of whole domain into simpler parts has several advantages:

1) Accurate representations of complex geometries.

2) Inclusion of dissimilar material properties.

3) Easy representation of total solution etc.

2.1.1 Literature Review

The word “finite element method” was first used by Clough in his paper in the Proceedings of Second American Society of Civil Engineering conference on Electronic Computation held in 1960. Clough extended the matrix method of structural analysis, used essentially for frame-like structures, two-dimensional continuum domains by dividing the domain into triangular elements. Then he obtained the stiffness matrices of these elements from the strain energy expressions by assuming a linear variation for the displacements over the elements. Clough called this method the ‘finite element method’ because the domain is to be divided into elements of finite size. An element of infinitesimal size is used when a physical statement of some balance law needs to be converted into a mathematical equation, usually a differential equation. Argyris, during the same time, developed similar technique in Germany. But, the idea of dividing the domain into finite number of elements for the structural analysis is older. Firstly this idea was used by Courant in 1943 when he was solving the problem of torsion of non-circular shafts. Courant used the integral form of the balance law, namely the expression for the total potential energy instead of the differential form (i.e., the equilibrium equation). He divided the shaft cross-section into triangular elements and assumed a linear variation for the primary variable (i.e., the stress function) over the domain. The unknown constants in the linear variation were obtained by minimizing the total potential energy expression. The Courants technique is known as applied mathematicians version of Finite element method where as that of Clough and Argyris is said to be engineers version of Finite element method.

2.1.2 Model of Finite Element Method

While solving the differential equations using finite elements, the following basic steps are used:

Step 1: Week Formulation

Firstly, the governing differential equation of the problem is converted into an integral form. These are two techniques to achieve this:

(i) Variational Technique and

(ii) Weighted Residual Technique.

In variational technique, the calculus of variation is used to obtain the integral form corresponding to the given differential equation. This integral needs to be minimized to obtain the solution of the problem. In weighted residual technique, the integral form is constructed as a weighted integral of the governing differential equation where the weight functions are known and arbitrary except that they satisfy certain boundary conditions. By this we reduce the continuity requirement of this solution.

Step 2: Domain Discretization

In this step we divide the whole region of interest into finite number of subdomains. The domain of the problem is divided into several parts, known as elements, using intermediate and end points. These points are known as nodes. The division of the domain into the elements and the nodes is called mesh. When all these elements are of the same size, it is known as the uniform mesh. Based on the shapes, elements can be classified as:

1) Linear elements, quadratic elements etc. in one dimension

2) Rectangular elements, triangular elements etc in two dimensions etc.

Step 3: Element Equations

In this step, we take a typical element. Then for this typical element, a suitable approximation is carried out for the primary variable of the problem using shape functions and the unknown values of the primary variable at some pre-selected points of the element. Usually we choose polynomial functions as the shape functions. For linear elements, there are at least two nodes placed at the end-points. for quadratic and higher order shape functions, more number of nodes are placed in the interior of the element. In two and three dimensions, the nodes are placed at the vertices (3 nodes for triangular elements, 4 nodes for rectangular or quadrilateral or tetrahedral elements and minimum 8 nodes for parallelepiped shaped elements). Additional nodes are placed on the boundaries or placed in the interior.

The values of the unknown variable at the nodes are known as the degrees of freedom. To acquire the exact solution, the expression for the primary variable must contain a complete set of polynomials (i.e., infinite terms) or if it contains only the infinite number of terms, then the number of elements should be infinite. In either case, this results into an infinite set of algebraic equations. To make the problem tractable, either only a finite number of elements are used or expressions with only finite number of terms are used. Solving this finite resulting system, we get an approximate finite element solution. The accuracy of the approximate solution, however, can be improved either by increasing the number of terms in the approximation or by increasing the number of elements.

Step 4: Assembly of Element Equations

In the step number four, the approximation for the unknown variable is substituted into the integral form. If the integral form is of variational type, then it is minimized to get the algebraic equations for the unknown nodal values of the primary variable. If the integral form is in the form of the weighted residual type, then it is set to zero to obtain the algebraic equations. In each case, the algebraic equations are obtained element wise first known as the element equations and then they are assembled with all the elements to obtain the algebraic equations for the whole domain known as the global system.

Step 5: Imposition of Boundary Conditions

In this step, the algebraic equations are modified in such a way so that they satisfy the imposed boundary conditions. These modified algebraic equations are then solved to find the nodal values of the dependent variable or variables.

Step 6: Postprocessing of the Solution

In this step, the post-processing of the solution is carried out. Secondary variables of the problem are calculated from the solution.

2.1.3 Applications of Finite Element Method

Finite element technique is used in many fields of engineering and research and allows efficient and precise modeling the behavior of mechanical, thermal or other complex systems. This method separates a complex geometry into a network of nodes and elements of similar shape and equations, called a mesh.

A variety of specializations under the umbrella of the mechanical engineering discipline (such as automotive and aeronautical industries etc.) commonly use integrated FEM in designing and developing their products. Several modern finite element packages include specific components such as thermal, electromagnetic, fluid and structural working environments. In structural stimulations, FEM helps tremendously in producing stiffness and strength visualizations and also in minimizing weight, materials and costs.

Finite element method allows detailed visualization of various structures bends or twist, and indicates the distribution of displacements and stresses. Finite element method provides a wide range of simulation options for controlling the complexity of both modeling and analyzing the systems.

2.1.4 Finite Element Formulation

Finite element methods are numerical schemes for approximating the solutions of mathematical problems.

A finite element method is characterized by variational formulation, a discretization strategy, one or more solution algorithms and post processing procedures. Galerkin method, the discontinuous Galerkin method, mixed method etc. are some of the examples of variational formulations.

Consider the differential equation

$$\frac{d^2y}{dx^2} - y + x = 0$$

with initial conditions $y(0) = 1$ **and at** $\frac{dy}{dx}(1) = 1.$

We are interested in solving this problem using the finite element method. For this, we first divide the domain $[0, 1]$ **into finite number of subintervals say four equal elements here. Hence there will be 5 number of nodes, i.e. N=5.**
Therefore, the nodes will be at $x = 0.2, 0.5, 0.8, 1.0.$
Let the finite element solution be given by:

$$\bar{y} = \sum_{j=1}^{5} \phi_j y_j, j = 1(1)5 \tag{2.1}$$

$$\int_0^1 (\frac{d^2\bar{y}}{dx^2} - \bar{y} + x)\phi_i x dx = 0, i = 1 \tag{2.2}$$

$$\int_0^1 \phi_i x \frac{d^2\bar{u}}{dx^2} dx - \int_0^1 \phi_i x \bar{u} dx + \int_0^1 x\phi_i x dx = 0 \tag{2.3}$$

$$\begin{aligned}\int_0^1 \phi_i x \frac{d^2y}{dx^2} dx &= \phi_i x \frac{d\bar{y}}{dx} - \int_0^1 \frac{d\phi_i}{dx}\frac{d\bar{y}}{dx} dx \\ &= -\phi_i x_5 Q_5 + \phi_i x_1 Q_1 - \int_0^1 \frac{dy}{dx}\end{aligned}$$

where $Q = \frac{d\bar{y}}{dx}$.

Using integration by parts, equation (3) becomes

$$\int_0^1 \frac{d\phi_i}{dx}\frac{d\bar{y}}{dx}dx + \int_0^1 \phi_i(x)\bar{y}dx = \int_0^1 x\phi_i xdx - \phi_i x_5 Q_5 + \phi_i x_1 Q_1$$

$$= \sum_{j=1}^{5} u_j \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx + \sum_{j=1}^{5} y_j \int_0^1 \phi_i x\phi_j xdx$$

$$= \int_0^1 x\phi_i x\phi_j xdx - [\phi_i x_5 Q_5 - \phi_i x_1 Q_1], i = 1(1)5$$

Integrating elementwise,

$$\sum_{r=1}^{4}\sum_{j=1}^{5} y_j \int_{e_r} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx + \sum_{r=1}^{4}\sum_{j=1}^{5} y_j \int_{e_r} \phi_i x\phi_j xdx$$

$$= (R - T), i = 1(1)5 \tag{2.4}$$

where $R = \sum_{r=1}^{4} x\phi_i dx$ **and**

$$T = \phi_i x_5 Q_5 - \phi_i x_1 Q_1. \tag{2.5}$$

In equation (5), there are two terms on left side containing the unknowns. Consider first term on the left side of equation (5) and denote it in matrix form as:

$$AU = \sum_{r=1}^{4}\sum_{j=1}^{5} y_j \int_{e_r} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx, i = 1(1)5$$

First let us evaluate the integral over an element e **having local nodes 1 and 2, that is :**

$$\int_e \frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}dx$$

for i = 1,2 and j = 1,2

$$\int_e \frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}dx$$

for i = 1,2 and j = 1,2

$$\phi_1^e = 1 - \xi, \phi_2^e = \xi$$

$$\xi = \frac{x - x_1}{x_2 - x_1} = \frac{1}{L}x - x_1 andLd\xi = dx$$

$$\frac{d\phi^e}{dx} = \frac{d\phi^e}{d\xi}\frac{d\xi}{dx} = \frac{1}{L}\frac{d\phi^e}{d}\xi$$

$$A_{11}^e = \int_{x_1}^{x_2} \frac{d\phi_1^e}{dx}\frac{d\phi_1^e}{dx}dx$$
$$= \int_0^1 (-\frac{1}{L}\frac{d\phi_1^e}{d\xi})(-\frac{1}{L}\frac{d\phi_1^e}{d\xi})L_1 d\xi = \frac{1}{L}$$

$$A_{12}^e = \int_{x_1}^{x_2} \frac{d\phi_1^e}{d\xi}\frac{d\phi_2^e}{dx}dx$$
$$= \int_0^1 (-\frac{1}{L}\frac{d\phi_1^e}{d\xi})(\frac{1}{L}\frac{d\phi_2^e}{d\xi})L_1 d\xi = -\frac{1}{L}$$

$$A_{22}^e = \int_{x_1}^{x_2} \frac{d\phi_2^e}{dx}\frac{d\phi_2^e}{dx}$$
$$= \int_0^1 (\frac{1}{L}\frac{d\phi_2^e}{d\xi})(\frac{1}{L}\frac{d\phi_2^e}{d\xi})L_1 d\xi = \frac{1}{L}$$

Now by symmetry

$$A_{21}^e = a_{21}^e.$$

For an element e, the subscript 1 and 2 will be replaced by r and $r+1$ respectively, $r = 1(1)4$.

The coefficients computed above will be inserted in the r-th and $(r+1)$-th the rows under r-th and $(r+1)$-th columns of the assembly table. The matrix $A^{(e)}$ is given by

$$A^e = \begin{pmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{pmatrix}$$

The second term on the left hand side of equation (5) can be denoted as

$$BU = \int_{r=1}^{4} \int_{j=1}^{5} y_j \int_{e_r} \phi_i x \phi_j x dx, i = 1(1)5$$

Again over the element e,

$$B_{11}^e = \int_{x_1}^{x_2} \phi^e x \phi^e x dx = \int_0^1 (1-\xi)^2 L d\xi = \frac{L}{3}$$

$$B_{12}^e = \int_{x_1}^{x_2} \phi_1^e x \phi_2^e x dx = \int_0^1 (1-\xi)\xi L d\xi = \frac{L}{6}$$

$$B_{22}^e = \int_{x_1}^{x_2} \phi_2^e x \phi_2^e x dx = \int_0^1 \xi^2 L d\xi = \frac{L}{3}$$

$$B_{21}^e = B_{12}^e$$

$$B^e = \begin{pmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{pmatrix}$$

The values of different L_i's are given by $L_1 = 0.2, L_2 = 0.3, L_3 = 0.3, L_4 = 0.2$

Values of P_1, P_2, P_3 and P_4 to be inserted in appropriate rows and columns.

The i-th term of the column vector R of equation (5) is given by

$$R_i = \int_0^1 x\phi_i(x)dx, i = 1(1)5$$

$$R_i^e = \sum_{r=1}^{4} \int_{e_r} x\phi_i(x)dx.$$

Let us evaluate the above integral over an element e having nodes 1 and 2.

$$\begin{aligned} R_i^e &= \int_{x_1}^{x_2} x\phi_1 x dx = \int_{x_1}^{x_2} x\frac{x-x_2}{x_1-x_2}dx \\ &= -\frac{1}{L}\int_{x_1}^{x_2} x(x-x_2)dx \\ &= -\frac{1}{L}x\frac{(x-x_2)^2}{2} - \frac{(x-x_2)^3}{6} \\ &= -\frac{1}{L}(-x_1\frac{L^2}{2} - \frac{L^3}{6}) = \frac{1}{2}x_1 L + \frac{L^3}{6} \\ R_2^e &= \int_{x_1}^{x_2} x\phi_2 x dx = \int_{x_1}^{x_2} x\frac{x-x_1}{x_2-x_1}dx \\ &= \frac{1}{L}x\frac{(x-x_1)^2}{2} - \frac{(x-x_1)^3}{6} = \frac{1}{2}x_2 L - \frac{L^2}{6} \end{aligned}$$

$$R = \begin{pmatrix} \frac{1}{2}x_1 L_1 - \frac{L_1^2}{6} \\ \frac{1}{2}x_2 L_1 - \frac{L_1^2}{6} + 12 x_2 L_2 - \frac{L_2^2}{6} \\ \frac{1}{2}x_3 L_2 - \frac{L_2^2}{6} + \frac{1}{2}x_3 L_3 - \frac{L_3^2}{6} \\ \frac{1}{2}x_4 L_3 - \frac{L_3^2}{6} + \frac{1}{2}x_4 L_4 - \frac{L_4^2}{6} \\ \frac{1}{2}x_5 L_4 - \frac{L_4^2}{6} \end{pmatrix} = \begin{pmatrix} \frac{0.02}{3} \\ \frac{0.35}{6} \\ \frac{0.90}{6} \\ \frac{1.15}{6} \\ \frac{0.56}{6} \end{pmatrix}$$

$$x_1 = 0, x_2 = 0.2, x_3 = 0.5, x_4 = 0.8, x_5 = 1.0$$

$$T_i = -\phi_i(x_5)Q_5 + \phi_i(x_1)Q_1.$$

Using $Q_5 = -1$ and insert the values of R and T, we get

$$\begin{pmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2.1.5 Assembly Table

On assembling the above element level equations, we get the following assembly table: On solving this global system, we get the finite element solution.

ϕ_i	y_1	y_2	y_3	y_4	y_5	R	T
ϕ_1	$\frac{15.2}{3}$	$-\frac{14.9}{3}$	0	0	0	$\frac{0.02}{3}$	$-Q_1$
ϕ_2	$-\frac{14.9}{3}$	$\frac{25.5}{3}$	$-\frac{19.7}{6}$	0	0	$\frac{0.35}{6}$	0
ϕ_3	0	$-\frac{19.7}{6}$	$\frac{20.6}{3}$	$-\frac{19.7}{6}$	0	$\frac{0.90}{6}$	0
ϕ_4	0	0	$-\frac{19.7}{6}$	$\frac{25.5}{3}$	$-\frac{14.9}{3}$	$\frac{1.15}{6}$	0
ϕ_5	0	0	0	$-\frac{14.9}{3}$	$\frac{15.2}{3}$	$\frac{0.56}{6}$	1

2.2 Numerical Results

The problem under consideration is solved in MATLAB by generating the finite element code. For different number of nodal points, the problem has been solved and the numerical solution is compared with the analytic solution. The finite element solution and exact solution have been plotted in Figures 2.1, 2.2, and 2.3 for $N = 32, \ , 64$ **and** 128 **elements. One can notice that as the number of elements are increasing, the finite element solution is converging to the exact solution. It can be easily seen that the numerical results are in good agreement with the analytical solution.**

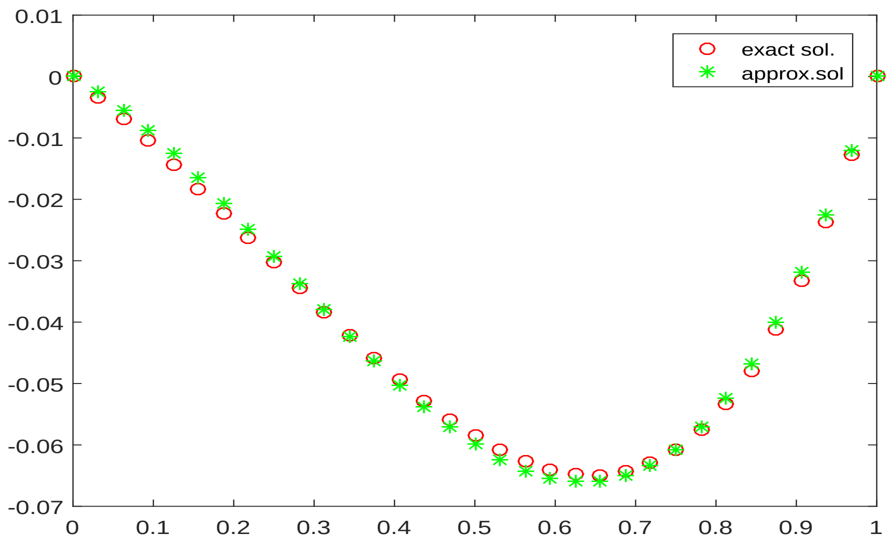

Figure 2.1: Finite element solution and exact solution for 32 elements

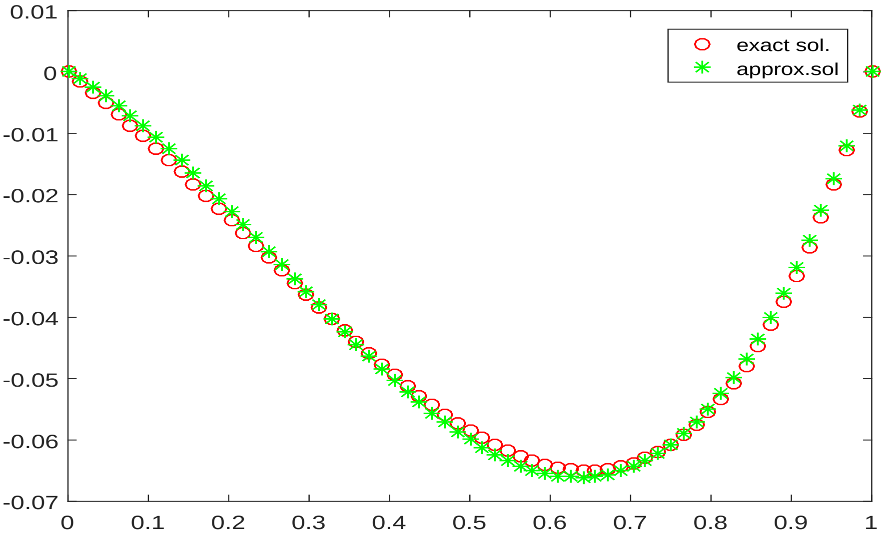

Figure 2.2: Finite element solution and exact solution for 64 elements

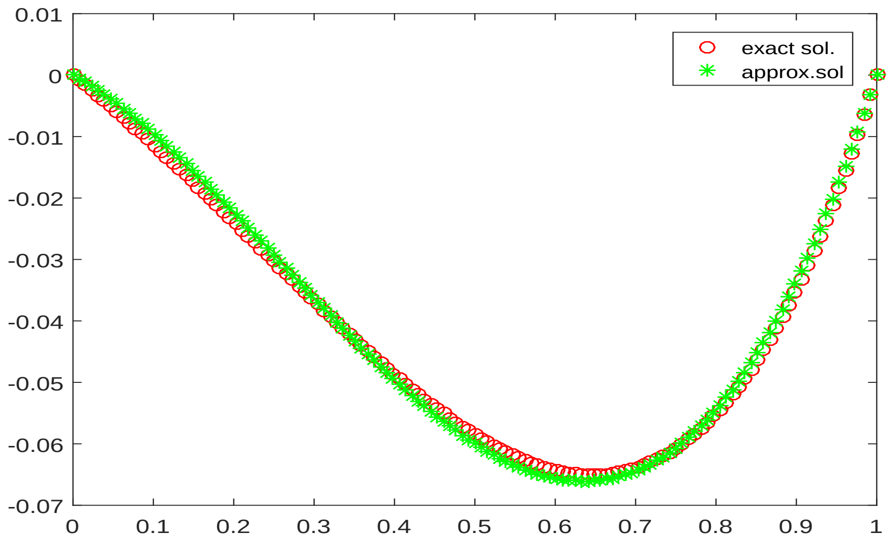

Figure 2.3: Finite element solution and exact solution for 128 elements

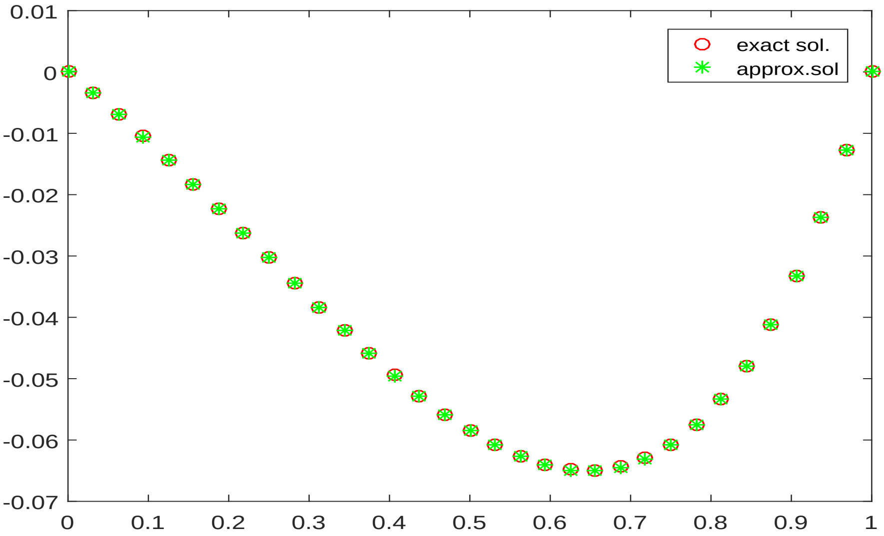

Figure 2.4: Finite element solution and exact solution

Chapter 3

FINITE ELEMENT APPROXIMATION FOR NONLINEAR PROBLEMS

3.1 Introduction

In mathematical sciences, for nonlinear systems, change in the output is not in proportion to the change in input. Nonlinear problems are much in use as the equations are widely used by engineers, biologists, mathematicians and scientists. Since nonlinear equations are difficult to solve, therefore, these equations are approximated by linear equations(linearization) mostly. A system of differential equations is nonlinear if atleast one of the problem in the system is nonlinear.
We cannot get the analytical solution of every nonlinear differential equation, therefore, we use various numerical techniques to solve such equations.
Nonlinear differential equations are less in number which have exact solutions, but they are important in applications. Nonlinear equations can be linearized by doing the expansion of equation in which the nonlinear terms are discarded.When nonlinear terms contributes itself ,this is not possible but sometimes its enough to retain a few 'small'ones.Then perturbation theory sometimes helps to obtain the solution. Sometimes solution of differential equation is approximated by 'small'

nonlinearities then that differential equation solution is valid to different ranges of parameters.

3.1.1 Applications of Nonlinear Differential Equations

Nonlinear differential equations arise in many areas and are very useful, for instance

1) deterministic and stochastic ordinary and partial differential equation
2) finite and infinite dimensional dynamical systems.
3) qualitative analysis of solution.
4) complex dynamics and pattern formation.
5) approximation and numerical aspects.

Now in this particular chapter, we will use quasilinearization technique to solve the nonlinear differential equations. After linearization, the resulted problem will further be solved using finite element methodology.

3.1.2 Quasilinearization Technique

This is a well known method which helps in obtaining approximate numerical solution of differential equation.This method was generally introduced by Bellman and Kalaba which is the generalization of Newton Raphson method to solve the system of nonlinear ordinary and partial differential equations.Quasilinerization technique results in approximation solution and this approximation solution converges monotonically and quadratically to the solution.
To yield a numerical technique for boundary value problem first of all linearize the nonlinear problem into linear problem by the process of quasilineraization method.
This procedure takes place in two steps:

1) Firstly, linearize the nonlinear terms present in the differential equation, satisfying the boundary conditions.
2) Solve the linearized boundary value problem, in which the solution of the k-th iteration boundary value problem satisfies the specific boundary conditions and is taken as the nominal profile solution for the linear boundary value problem at (k+1)th iteration.

In quasilinearization technique, process repeats recursively to solve the nonlinear problem into linear differential equation.The main advantages of quasilinearization are:
1)Solution of nonlinear differential equation is approximated by treating nonlinear terms as perturbation about the linear ones and is not based,unlike the theories of perturbation,on the existence of kind of small parameter.
2)There is continuation in convergence of sequence at some intervals once the iterations of quasilinearization technique starts.But in the asymptotic perturbation ,the method results in precision the moment the initial guess generates convergence.
3)If the process converges, then the quadratic solution converges.Then the error in the (n+1)th iteration is in in proportion to the square of the error in the nth iteration.

3.2 Continuous Problem

Consider the nonlinear differential equation

$$y'' + a(x)y'(x) + b(x)y = f(x, y)$$

with boundary conditions

$$y(c) = A, \ y(d) = B$$

where A and B are the constants.

In order to discuss the quasilinearization technique, we will consider the following differential equation

$$y''(x) + \frac{10}{x}y'(x) + y^{10}(x) = x^{100} + 190x^8 \tag{3.1}$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$.

3.3 Convergence Analysis of Quasilinearization Technique

Consider the nonlinear differential equation

$$(y'') = F(y), 0 \leq x \leq 1$$

Using quasilinearization, we obtain a sequence of linear differential equations determined by

$$(y'')_{k+1} = F(y)_k + (y_{k+1} - y_k)\frac{\partial F}{\partial y}|_k$$

where the initial guess y_0 is so chosen to satisfy the boundary conditions

$$y_{k+1}(c) = A, \quad y_{k+1}(d) = B.$$

Rewriting the above recurrence relations at previous iteration step, we get

$$(y'')_k = F(y_{k-1}) + (y_k - yk - 1)\frac{\partial F}{\partial y}|_{k-1}$$

Subtracting the above equations, we get

$$(y''_{k+1} - y''_k) = F(y_k) - F(y_{k-1}) + (y_{k+1} - y_k)\frac{\partial F}{\partial y}|_k - (y_k - y_{k-1})\frac{\partial F}{\partial y}|_{k-1}$$

The above differential equation is of second order in $(y_{k+1} - y_k)$.

Converting into integral equation by using Green's function, we obtain

$$(y_{k+1}-y_k)=\int_0^1 G(x,s)[F(y,k)-F(y_{k-1})+(y_{k+1}-y_k)\frac{\partial F}{\partial y}|_k-(y_k-y_{k-1})\frac{\partial F}{\partial y}|_{k-1}]$$

where Green's function is defined as

$$G(x,s)=\{x(s-1),0\leq x\leq s\leq 1,(x-1)s,0\leq s\leq x\leq 1\}$$

and max $=\frac{1}{4}$

Using mean value theorem

$$F(y_k)-F(y_{k-1})\approx(y_k-y-k-1)\frac{\partial F}{\partial y})|_{k-1}$$

$$(y_{k+1}-y_k)=\int_0^1 G(x,s)[(y_k-y_{k-1})\frac{\partial F}{\partial y}|_{k-1}+(y_{k+1}-yk)\frac{\partial F}{\partial y}-(y_k-y_{k-1})\frac{\partial F}{\partial y}|_{k-1}]$$

Taking

$$\max|\frac{\partial F}{\partial y}(y_k)|=Q$$

Taking maximum norm over the partial domain and using equations.

$$\|y_{k+1}-y_k\|\leq\frac{1}{4}\int_0^1[Q\|y_{k+1}-yk\|]ds.$$

Simplifying the inequality, we get

$$\|y_{k+1}-y_k\|\leq C\|y_k-y_{k-1}\|.$$

3.4 Construction of Finite Element Scheme

The given nonlinear problem is

$$y''(x)+\frac{10}{x}(y'(x)+y^{10}(x)=x^{100}+190x^8.$$

Now applying the quasilinearization technique, the linearized problem is given by:

$$(y'')^{k+1} + \frac{10}{x}(y')^{k+1} + (y^9)^k.y^{k+1} = f(x).$$

Weak formulation of the above problem is given by

$$\int_0^1 [(y'')^{k+1} + \frac{10}{x}(y')^{k+1} + (y^9)^k.(y)^{k+1}]\phi_i(x)dx = 0$$

$$\int_0^1 (y'')^{k+1}\phi_i(x)dx + \int_0^1 \frac{10}{x}(y')^{k+1}\phi_i(x)dx + \int_0^1 (y^9)^k.(y)^{k+1}\phi_i(x)dx = 0.$$

Now the approximate solution is given by

$$\bar{y} = \sum_{j=0}^{N} \phi_j y_j,$$

where ϕ_j is a basis function and basis function satisfies the Kronicker delta property.

$$\phi_i(x)(\bar{y'})^{k+1} - \int_0^1 \phi_{i'}(x)(\bar{y'})^{k+1}dx + \int_0^1 \frac{10}{x}(\bar{y'})^{k+1}\phi_i(x)dx + \int_0^1 (\bar{y}^9)^k(\bar{y})^{k+1}\phi_i(x)dx$$

$$\sum_{j=0}^{N}[-\int_0^1 \phi_i'(x)\phi_j' + \int_0^1 (\frac{10}{x}\phi_i(x)\phi_j' + (y^9)^k\phi_i(x)\phi_j'(x)](y_j)^{k+1} = \int_0^1 f(x)\phi_i - y'\phi_i|_0^1$$

where,

$$\phi_j(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}}, x \in [x_{j-1}, x_j]$$
$$\frac{x - x_{j-1}}{x_j - x_{j-1}}, x \in [x_j, x_{j+1}]$$

Now are final solution will be in the form as:

$$A^k Y^{k+1} = F$$

3.5 Numerical Examples

3.5.1 Example 1

For problem 1, we have considered equation (3.1). The exact solution of the problem 1 is x^{10}.

3.5.2 Error table

The maximum error table for problem (3.1) **is given below in Table [3.1]:**

Iterations	Errors
1	$2.4716 + 10^8$
2	0.0039
3	0.0038
4	0.0038
5	0.0038

Table 3.1: Maximum nodal errors for Example 1 per iteration

3.5.3 Example 2

For second example, consider the differential equation

$$u'' + \frac{10}{x}u' + u^2 = x^{20} + 190x^8$$

with boundary conditions y(0)=0 and y(1)=1.
The exact solution of this problem is x^{10}**.**

3.5.4 Error table

The maximum nodal errors at each iteration level for the Example 2 are given in Table [3.2].

3.5.5 Example 3

For third example, we have considered the nonlinear problem

$$u'' - 3x^2u' + u^2 = f(x)$$

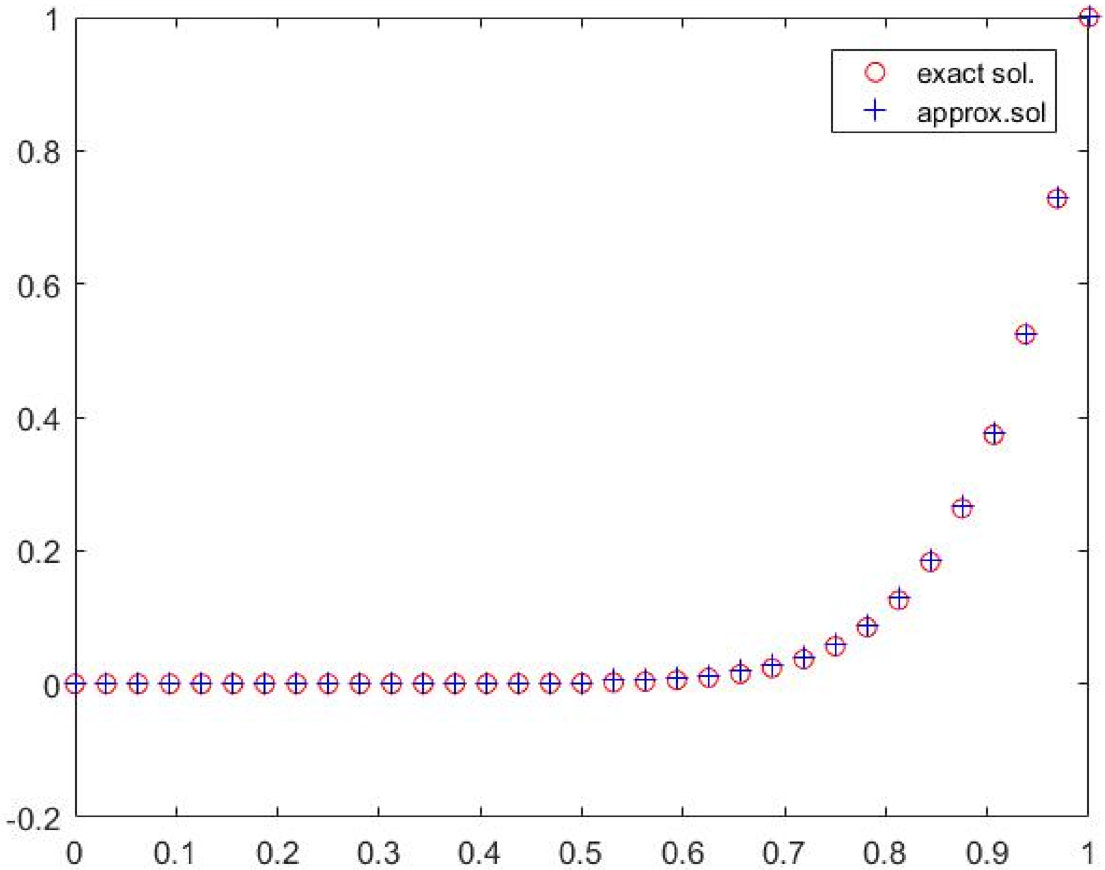

Figure 3.1: Exact solution and Finite Element solution for Example 1

where $f(x) = x^6 - 5x^4 - 2x^2 + 6x$.

The boundary conditions are y(0)=0, y(1)=3. The exact solution of this problem is not known. The problem has been linearized using quasilinearization technique as discussed above. Further, the problem has been solved using finite element technique with linear shape functions. Computed solution at different nodal points is given in Table 3.3.

It can be easily seen that numerical solution is in very much agreement with the exact solution.

Iterations	**Errors**
1	$1.6709 + 10^8$
2	0.0321
3	0.0065
4	0.0052
5	0.0045

Table 3.2: Maximum nodal errors for Example 2 per iteration

Nodal Points	**Solution**
0	0
0.1	0.1920
0.2	0.3902
0.3	0.6006
0.4	0.8298
0.5	1.0842
0.6	1.3709
0.7	1.6972
0.8	2.0711
0.9	2.5019
1.0	2.000

Table 3.3: Finite Element Solution at different nodal points for Example 3

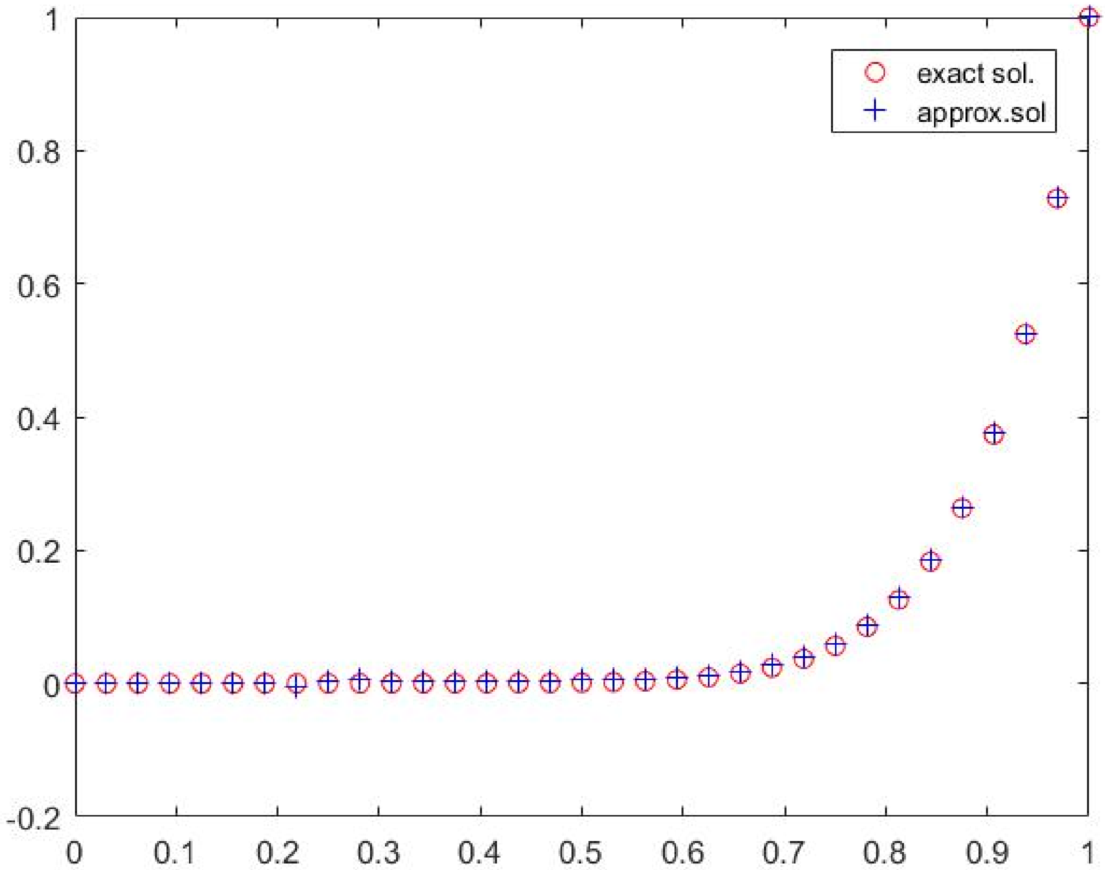

Figure 3.2: Exact solution and Finite Element solution for Example 2

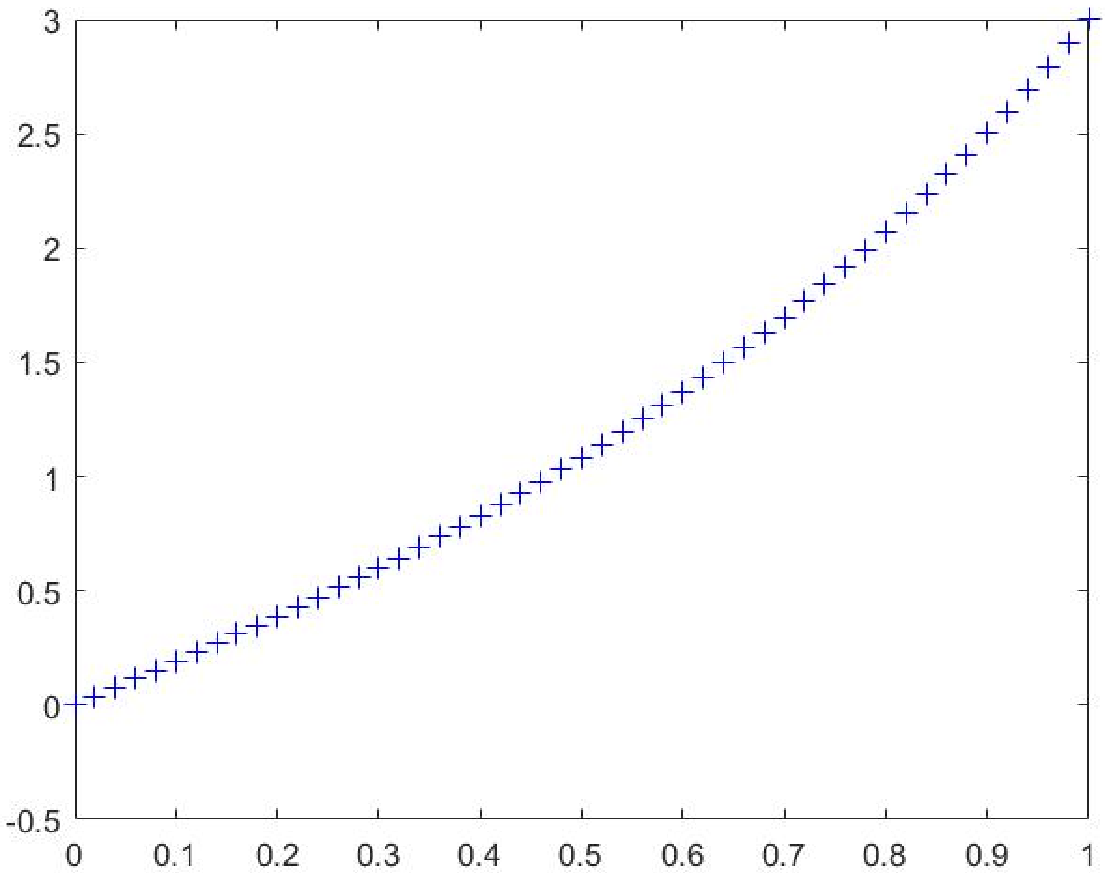

Figure 3.3: Finite element solution for Example 3

Chapter 4

Conclusion

4.1 Conclusion

The differential equations governing the physical phenomenon and various real life problems are very complicated and complex in nature and hence can not be solved by using analytical techniques in general. Therefore, one has to depend on the numerical techniques for approximating the solutions of such problems. In the present work, finite element method has been employed for approximating the solutions of both the linear and nonlinear differential equations. Standard linear hat functions have been used for test and trial functions. For linear problems, the proposed method works very well and captures the solution very nicely. For nonlinear terms, quasilinearization technique has been used to handle the nonlinear terms. Convergence of the quasilinearization process has been discussed. Numerical results have been presented for both linear and nonlinear problems. The finite element numerical results are in good agreement with exact results. To conclude, finite element technique can be applied to approximate the solutions of linear and nonlinear problems.

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