METHODS FOR SOLVING LINEAR PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

A THESIS SUBMITTED TO

THAPAR UNIVERSITY
PATIALA (PUNJAB), INDIA

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

JAGDEEP KAUR

(Registration No. : 900911012)

SCHOOL OF MATHEMATICS & COMPUTER APPLICATIONS

THAPAR UNIVERSITY - 147 004 (PUNJAB), INDIA
October - 2012

CERTIFICATE

This is to certify that the thesis entitled, **"Methods for Solving Linear Programming Problems with Fuzzy Parameters"**, submitted by Ms. Jagdeep Kaur in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisor

Dr. Amit Kumar
Assistant Professor
School of Mathematics and Computer Applications
Thapar University
Patiala-147004
INDIA

DECLARATION

It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.

Jagdeep Kaur

Jagdeep Kaur

Attestation by supervisor

Dr. Amit Kumar
Assistant Professor
School of Mathematics and Computer Applications
Thapar University
Patiala-147004
INDIA

DEDICATED

TO

THE ALMIGHTY

&

MY PARENTS

Acknowledgements

I thank the Almighty God, who gave me the opportunity and strength to carry out this work.

I feel privileged to express my sincere regards and gratitude to my supervisor Dr. Amit Kumar, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala for his expert guidance, cool and benign temperament, valuable suggestions, support, advice and continuous encouragement throughout the course of my research work. The critical comments, rendered by him during the discussions are deeply appreciated.

I want to acknowledge the innocent blessings of Mehar. My supervisor, Dr. Amit Kumar, believes that Mehar is an angel for him and without Mehar's blessings it would not have been possible to attain and fulfill the ideas presented in this thesis. Mehar is a lovely daughter of Parmpreet Kaur (Research Scholar under the supervision of Dr. Amit Kumar).

I am highly obliged to Dr. S.S. Bhatia, Head, School of Mathematics and Computer Applications, all the members of Doctoral Committee, faculty members and non-teaching staff for providing me encouragement and necessary facilities for carrying out my research work.

I am also grateful to Dr. P. K. Bajpai, Dean of Research and Sponsored

Project, for his constant encouragement that was of great importance in the completion of the thesis.

I extend my thanks to Dr. Abhijit Mukherjee, Director, for his valuable support that made me consistent performer.

The financial support given by the University Grant Commission under Maulana Azad National Fellowship Scheme is also gratefully acknowledged.

I am also grateful to my friends Manjot Kaur, Sneh Lata, Amarpreet Kaur, Parmpreet Kaur and all other Research Scholars of School of Mathematics and Computer Applications for their timely help and the moral support they provided me during my research work.

I feel a deep sense of gratitude for my parents who formed part of my initial vision and inculcated in me the real value of education which really matters in life. I would like to give my special thanks to my brother for his constant encouragement that was of great importance in the completion of the thesis.

Jagdeep Kaur

Patiala
(Jagdeep Kaur)

Date: October 30, 2012

Abstract

In this thesis, the shortcomings and limitations of some existing methods for solving linear programming problems with fuzzy parameters are pointed out and new methods are proposed to overcome the shortcomings and limitations of existing methods.

The chapter wise summary of the thesis is as follows:

In **Chapter 1**, a brief review of the work done in the area of linear programming problems with fuzzy parameters is presented.

Chapter 2

Lotfi et al. [80] pointed out that there is no method in the literature for solving fully fuzzy linear programming problems with equality constraints and proposed a method for the same. In this chapter, the limitations and shortcomings of the existing method [80] are pointed out and to overcome the limitations as well as to resolve the shortcomings, a new method is proposed to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints.

Chapter 3

In this chapter, the limitations of the method, proposed in Chapter 2, are pointed out and to overcome these limitations, a new method is proposed for solving

fully fuzzy linear programming problems with equality constraints. To show the application of the proposed method a real life problem, which cannot be solved by using the method, proposed in Chapter 2, is solved by using the proposed method.

Chapter 4

To the best of our knowledge, till now no one have defined the product of such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. Due to non-existence of such product, till now there is no method in the literature for solving such fully fuzzy linear programming problems in which some or all the parameters are represented by such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. In this chapter, the product of such fuzzy numbers is proposed and also the limitations of the method, proposed in Chapter 3, are pointed out. To overcome the limitations of the method, proposed in Chapter 3, a new method is proposed to find the fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints.

Chapter 5

In this chapter, limitations of the existing methods for solving fuzzy linear programming problems and fully fuzzy linear programming problems with inequality constraints are pointed out. To overcome the limitations of the existing methods, two new methods are proposed for solving fully fuzzy linear programming problems with inequality constraints. The advantages of the proposed methods over the existing methods are also discussed.

Chapter 6

In this chapter, it is shown that the fuzzy optimal value, obtained by using the method, proposed in Chapter 4, is not necessarily a unique fuzzy number. So, it does not conform to the uniqueness property of fuzzy optimal value. To overcome this limitation of the method, proposed in Chapter 4, a new method is proposed for solving fully fuzzy linear programming problems with equality constraints.

Chapter 7

In this chapter, it is pointed out that the existing general from of fully fuzzy linear programming problems is valid only if there does not exist subtraction of fuzzy numbers in the fully fuzzy linear programming problems. However, if there exist subtraction of fuzzy numbers in the fully fuzzy linear programming problems then either the existing general form of fully fuzzy linear programming problems is not valid or the obtained fuzzy optimal solution is not genuine.

Chapter 8

Finally, in this chapter, based on the present study, future work has been suggested.

List of Research Papers

1. **J. Kaur**, A. Kumar, Exact fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted fuzzy variables, Applied Intelligence, 37 (2012) 145-154.
2. **J. Kaur**, A. Kumar, Unique fuzzy optimal value of fully fuzzy linear programming problems, Control and Cybernetics, 41 (2012) 171-182 .
3. **J. Kaur**, A. Kumar, A new method to find the unique fuzzy optimal value of fuzzy linear programming problems, Journal of Optimization Theory and Applications. (Accepted)
4. A. Kumar, **J. Kaur**, P. Singh, A new method for solving fully fuzzy linear programming problems, Applied Mathematical Modelling, 35 (2011) 817-823.
5. A. Kumar, **J. Kaur**, A new method for solving fuzzy linear programs with trapezoidal fuzzy numbers, Journal of Fuzzy Set Valued Analysis, 2011 (2011) 1-12.
6. A. Kumar, **J. Kaur**, P. Singh, Fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints, International Journal of Applied Mathematics and Computer Sciences, 6 (2010) 37-41.

7. A. Kumar, **J. Kaur**, P. Singh, Fuzzy linear programming problems with fuzzy Parameters, Journal of Advanced Research in Scientific Computing, 2 (2010) 1-12.

8. A. Kumar, **J. Kaur**, Mehar's methods for solving fully fuzzy linear programming problems with *LR* fuzzy parameters, Applied Mathematical Modelling. (Communicated)

9. A. Kumar, **J. Kaur**, Fuzzy optimal solution of fully fuzzy linear programming problems using ranking function, Journal of Intelligent and Fuzzy Systems. (Communicated)

10. A. Kumar, **J. Kaur**, General form of fully fuzzy linear programming problems, Journal of Optimization Theory and Applications. (Communicated)

Paper Presented in International Congress

1. A. Kumar, **J. Kaur**, A new method for solving fuzzy linear programs with trapezoidal fuzzy numbers, International Congress on Productivity, Quality, Reliability, Optimization and Modelling (ICPQROM), Feb 7-8, 2011.

Table of Contents

Chapter 1

INTRODUCTION

Decision making is the process of identifying and choosing alternatives based on the values and preferences of the decision maker. Making a decision implies that there are alternative choices to be considered, and in such a case it is desired not only to identify as many of these alternatives as possible but to choose the one that (1) has the highest probability of success or effectiveness and (2) best fits with our goals, desires, lifestyle, values and so on. Decision making is also the process of sufficiently reducing uncertainty and doubts about alternatives to allow a reasonable choice to be made from among them.

Optimization is a kind of the decision making in which decisions have to be taken to optimize one or more objectives under some prescribed set of circumstances. The problems which often arise as a result of mathematical modeling of many real life situations are called optimization problems. Optimization problems are frequently encountered in disciplines, such as water resources management, finance, engineering optimization, manufacturing systems, management, business, physical sciences and agriculture etc.

The most general form of optimization problems is:

$$\text{Minimize/Maximize } f^n(X), X = (x_1, x_2, ..., x_n)$$

subject to

$$g_j(X) \leq 0 \;\; \forall \; j = 1, 2, ..., k$$

$$h_j(X) = 0 \;\; \forall \; j = 1, 2, ..., m$$

where $f, g_1, g_2, ..., g_k, h_1, h_2, ..., h_m$ are real valued functions defined on $\mathbb{R}^n$ and n is a positive integer. The function f is usually called the objective function. Each of the constraints $g_j(X) \leq 0$ is called an inequality constraint, and each of the constraints $h_j(X) = 0$ is called an equality constraint. Constraints of the type $g_j(X) \geq 0$ may be written as $-g_j(X) \leq 0$ and therefore it is not mentioned separately. $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is called a decision vector and $x_1, x_2, ..., x_n$ are called decision variables or unknown variables. If $n = 1$ and all the functions $f^n(X), g_j(X)$ and $h_j(X)$ are linear then the problem is called a single-objective linear programming problem.

Over the last several decades optimization models have been primarily developed in a deterministic, crisp environment. In such models the objectives and constraints are formulated in a 'hard' crisp manner leaving no scope for uncertainty and vagueness. However, in real life situations there may exist uncertainty. In such situations it is practical to develop the mathematical models of the realistic problem in fuzzy environment and the optimization problem under consideration becomes a fuzzy programming problem.

Any linear programming model representing real world situations involves a lot of parameters whose values are assigned by experts. However, both experts and decision maker frequently do not precisely know the value of those parameters. Therefore, it is useful to consider the knowledge of experts about the parameters as fuzzy data [143].

1.1 Literature review

Fuzzy linear programming is an application of fuzzy set theory in linear decision making problems. Fuzzy linear programming problems cannot be uniquely defined as crisp linear programming, since it will very much dependent upon the type of fuzziness and its specification as prescribed by the decision maker. Therefore the class of fuzzy linear programming problems can be broadly classified as:

(i) Linear programming problem with fuzzy inequalities and crisp objective function.

(ii) Linear programming problems with crisp inequalities and fuzzy objective function.

(iii) Linear programming problems with fuzzy inequalities and fuzzy objective function.

(iv) Linear programming problems with fuzzy resources and fuzzy coefficient, also termed as linear programming problems with fuzzy parameters.

In this section, a brief review of the work done in the area of linear programming problems with fuzzy parameters is presented.

Rommelfanger et al. [119] presented a new method for solving linear programming problems with fuzzy parameters in the objective function. Nakahara and Gen [96] proposed an approach for comparing triangular fuzzy numbers and used it for solving linear programming problems with coefficients as triangular fuzzy numbers.

Cadenas and Verdegay [17] studied a linear programming problem in which all its elements are defined as fuzzy sets. Wang [136] proposed an inexact approach to solve linear programming problems with fuzzy objective and resources. Fang et al. [38] presented a method for solving linear programming with fuzzy coefficients in constraints. Guu and Wu [48] proposed a two phase approach to solve the fuzzy linear programming problem. Sakawa et al. [121] presented interactive fuzzy programming for two-level linear fractional programming problems with fuzzy parameters. Tanaka et al. [127] studied several kinds of possibility distributions of fuzzy variables in possibilistic linear programming problems to reflect the inherent fuzziness in fuzzy decision problems.

Buckley and Feuring [16] introduced a method to find the solution of the linear programming problems with all the parameters and variables as fuzzy numbers by changing the objective function into a multiobjective fuzzy linear programming problem. Maleki et al. [91] solved the linear programming problems in which all decision parameters are fuzzy numbers by the comparison of fuzzy numbers.

Zhang et al. [145] proposed a method for solving fuzzy linear programming problems which involve fuzzy numbers in coefficients of objective functions. Nehi et al. [108] defined the concept of optimality for linear programming problem with fuzzy parameters by transforming fuzzy linear programming problem into multiobjective linear programming problem.

Nasseri and Ardil [99] proposed a new method for solving fuzzy variable linear programming directly using linear ranking. Nasseri et al. [100] presented a new method for solving fuzzy number linear programming problems. Mahdavi and Nasseri [86] introduced the dual of fuzzy variable linear programming problem and then deduced some important duality results.

Ganesan and Veeramani [41] proposed an approach to solve a fuzzy linear programming problem involving symmetric trapezoidal fuzzy numbers without converting it to crisp linear programming problem. Hashemi et al. [49] proposed a two phase approach to find the optimal solutions of class of fuzzy linear programming problems called fully fuzzified linear programming, where all decision parameters and variables are fuzzy numbers.

Jimenez et al. [58] proposed a method for solving linear programming problems in which all the coefficients are fuzzy numbers and used a ranking method to rank the fuzzy objective values and to deal with the inequality relation on constraints. Mahdavi and Nasseri [88] established some duality results for linear programming problems with treapezoidal fuzzy variables. Cai et al. [18] considered a class of fuzzy linear programming problems in which the coefficients of the objective function are characterized by LR fuzzy intervals or LR fuzzy numbers. Allahviranloo et al. [4] proposed a new method for solving fully fuzzy linear programming problems by the use of ranking function.

Nasseri [98] proposed a method for solving fuzzy linear programming problems by solving the classical linear programming. Gani et al. [42] dealt with a kind of fuzzy linear programming problems where all the parameters and variables are LR fuzzy numbers. Lotfi et al. [80] pointed out that there is no method in literature for solving fully fuzzy linear programming problems with equality constraints and proposed a method for the same. Ebrahimnejad and Nasseri [33] used the complementary slackness property to solve fuzzy linear programming problem with fuzzy parameters without the need of a simplex tableau.

Stanojevic and Stanojevic [124] proposed an algorithm for solving fuzzy linear programming problems with trapezoidal fuzzy numbers. Mahdavi et al. [89] considered two classes of fuzzy linear programming problems: Fuzzy number linear programming and linear programming with trapezoidal fuzzy variables problems and developed fuzzy primal simplex algorithms for solving these problems. Ebrahimnejad et al. [35] pointed out that Ganesan and Veeramani's approach [41] is not efficient for situations in which some or all variables are restricted to lie within fuzzy lower and fuzzy upper bounds and proposed a natural extension of their approach to overcome this shortcoming.

Ebrahimnejad and Nasseri [34] introduced a new method called the bounded dual simplex method for bounded fuzzy number linear programming problems for the situations in which some or all variables are restricted to lie within fuzzy lower and fuzzy upper bounds. Nasseri and Ebrahimnejad [101] proposed a fuzzy dual algorithm for solving linear programming problems with fuzzy numbers by using the duality results.

Nasseri and Ebrahimnejad [102] proposed a fuzzy primal simplex algorithm for solving the flexible linear programming problem and then suggested the fuzzy primal simplex method to solve the flexible linear programming problems directly without solving any auxiliary problem. Nasseri et al. [103] introduced the dual of the linear programming problem with symmetric trapezoidal fuzzy numbers and established some duality results.

Nasseri and Khabiri [104] proposed a revised fuzzy simplex algorithm for solving linear programming problems with fuzzy variables. Nasseri and Khabiri [105] proposed a revised simplex algorithm for solving fuzzy number linear programming problems using linear ranking functions. Ebrahimnejad et al. [36] proposed a new primal-dual algorithm for solving linear programming problems with fuzzy variables by using duality results. Ebrahimnejad et al. [37] proposed a new method for solving such linear programming problems with fuzzy cost coefficients in which some or all variables are restricted to lie within lower and upper bounds.

Ebrahimnejad [32] developed a novel approach namely the primal-dual simplex algorithm to solve fuzzy linear programming problems involving symmetric trapezoidal fuzzy numbers without converting them to crisp linear programming problems. Ebrahimnejad [31] proved that in the absence of degeneracy, the method proposed by Ganesan and Veermani [41] stops in a finite number of iterations and then proposed a revised kind of their method that is more efficient and robust in practice.

Darvishi et al. [24] discussed a real life problem of diet formulation and used linear fuzzy model in formulation of dairy cow ration to control cost in dairy cattle industries along with carefully planned feed requirements. Baykasoglu and Gocken [7] proposed a direct solution approach for solving fuzzy mathematical programming problems with fuzzy decision variables by using fuzzy ranking procedure for fuzzy numbers.

After reviewing the literature, it is found that there are some limitations and shortcomings in the existing methods for solving linear programming problems with fuzzy parameters. In this thesis, these limitations and shortcomings are pointed out and to overcome the limitations as well as to resolve the shortcomings of the existing methods, new methods are proposed.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Lotfi et al. [80] pointed out that there is no method in the literature for solving fully fuzzy linear programming problems with equality constraints and proposed a method for the same. In this chapter, the limitations and shortcomings of this existing method are pointed out and to overcome the limitations as well as to resolve the shortcomings, a new method is proposed for solving fully fuzzy linear programming problems with equality constraints.

Chapter 3

Lotfi et al. [80] pointed out that there is no method in the literature for solving fully fuzzy linear programming problems with equality constraints and proposed a method for the same. In this chapter, the limitations and shortcomings of the existing method [80] are pointed out and to overcome the limitations as well as to resolve the shortcomings, a new method is proposed to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints.

Chapter 4

To the best of our knowledge, till now no one have defined the product of such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. Due to non-existence of such product, till now there is no method in the literature for solving such fully fuzzy linear programming problems in which some or all the parameters are represented by such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. In this chapter, the product of such fuzzy numbers is proposed and also the limitations of the method, proposed in Chapter 3, are pointed out. To overcome the limitations of the method, proposed in Chapter 3, a new method is proposed to find the fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints.

Chapter 5

In this chapter, limitations of the existing methods for solving fuzzy linear programming problems and fully fuzzy linear programming problems with inequality constraints are pointed out. To overcome the limitations of the existing methods, two new methods are proposed for solving fully fuzzy linear programming problems with inequality constraints. The advantages of the proposed methods over the existing methods are also discussed.

Chapter 6

In this chapter, it is shown that the fuzzy optimal value, obtained by using the method, proposed in Chapter 4, is not necessarily a unique fuzzy number. So, it does not conform to the uniqueness property of fuzzy optimal value. To overcome this limitation of the method, proposed in Chapter 4, a new method is proposed for solving fully fuzzy linear programming problems with equality constraints.

Chapter 7

In this chapter, it is pointed out that the existing general from of fully fuzzy linear programming problems is valid only if there does not exist subtraction of fuzzy numbers in the fully fuzzy linear programming problems. However, if there exist subtraction of fuzzy numbers in the fully fuzzy linear programming problems then either the existing general form of fully fuzzy linear programming problems is not valid or the obtained fuzzy optimal solution is not genuine.

Chapter 8

Finally, in this chapter, based on the present study, future work has been suggested.

Chapter 2

A New Method To Find The Non-negative Fuzzy Optimal Solution Of Fully Fuzzy Linear Programming Problems With Equality Constraints

Lotfi et al. [80] pointed out that there is no method in the literature for solving fully fuzzy linear programming problems with equality constraints and proposed a method for the same. In this chapter, the limitations and shortcoming of the existing method [80] are pointed out and to overcome the limitations as well as to resolve the shortcoming, a new method is proposed to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints.

2.1 Preliminaries

In this section, some basic definitions and arithmetic operations for trapezoidal fuzzy numbers are presented [63].

The contents of this chapter are published in *Applied Mathematical Modelling* 35 (2011) 817-823.

2.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 2.1 Let X be a classical set of objects. Then, the set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0, 1]$, is called a fuzzy set in X. The evaluation function $\mu_{\tilde{A}}(x)$ is called the membership function.

Definition 2.2 Let $\tilde{A}$ be a fuzzy set in X and $\lambda \in [0, 1]$ be a real number. Then, a classical set $A^{\lambda} = \{x \in X : \mu_{\tilde{A}}(x) \geq \lambda\}$ is called an λ-level set or λ-cut or parametric form of $\tilde{A}$.

Definition 2.3 A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a normalized fuzzy set if and only if $\underset{x \in X}{\text{Supremum}}\{\mu_{\tilde{A}}(x)\} = 1$.

Definition 2.4 A fuzzy set $\tilde{A}$ is called a convex fuzzy set if and only if $\mu_{\tilde{A}}(\alpha x_1 + (1 - \alpha)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\},\ \forall\ x_1, x_2 \in X,\ \alpha \in [0, 1]$.

Definition 2.5 A convex normalized fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a fuzzy number if and only if $\mu_{\tilde{A}}(x)$ is piecewise continuous in X.

Definition 2.6 A fuzzy number $\tilde{A}$ is said to be a non-negative fuzzy number if and only if $\mu_{\tilde{A}}(x) = 0,\ \forall\ x < 0$.

Definition 2.7 A fuzzy number $\tilde{A}$ defined on the universal set of real numbers $\mathbb{R}$, denoted as $\tilde{A} = (a, b, c, d)$, is said to be a trapezoidal fuzzy number if its membership function, $\mu_{\tilde{A}}(x)$, is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)} & a \leq x < b \\ 1 & b \leq x \leq c \\ \frac{(x-d)}{(c-d)} & c < x \leq d \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.8 Let $\tilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then, its λ-cut A^λ is defined as follows:

$$A^\lambda = [a + (b - a)\lambda, d - (d - c)\lambda],\ 0 \leq \lambda \leq 1$$

Definition 2.9 A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is said to be symmetric trapezoidal fuzzy number if and only if $b - a = d - c$, otherwise $\tilde{A}$ is said to be an asymmetric trapezoidal fuzzy number.

A symmetric trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ can be denoted as (b, c, σ), where $[b, c]$ is the core and $\sigma = b - a = d - c$ is the spread of the symmetric trapezoidal fuzzy number $\tilde{A}$.

Definition 2.10 A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is said to be non-negative trapezoidal fuzzy number if and only if $a \geq 0$ and is said to be non-positive trapezoidal fuzzy number if and only if $a \leq 0$.

Definition 2.11 A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is said to be unrestricted trapezoidal fuzzy number if and only if a is a real number.

Definition 2.12 Two trapezoidal fuzzy numbers $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2$ and $d_1 = d_2$.

2.1.2 Arithmetic operations

In this section, the arithmetic operations for trapezoidal fuzzy numbers and intervals are presented.

2.1.2.1 Arithmetic operations for trapezoidal fuzzy numbers

In this section, some arithmetic operations for two trapezoidal fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are presented.

(i) Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$.

(ii) Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$.

(iii) Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two non-negative trapezoidal fuzzy numbers. Then, $\tilde{A}_1 \otimes \tilde{A}_2 = (a_1a_2, b_1b_2, c_1c_2, d_1d_2)$.

(iv) Let $\tilde{A} = (a, b, c, d)$ be any trapezoidal fuzzy number. Then,

$$\gamma\tilde{A} = \begin{cases} (\gamma a, \gamma b, \gamma c, \gamma d) & \gamma \geq 0 \\ (\gamma d, \gamma c, \gamma b, \gamma a) & \gamma \leq 0 \end{cases}$$

2.1.2.2 Arithmetic operations for intervals

In this section, some arithmetic operations for two intervals are presented.

(i) Let $A_1 = [a_1, b_1]$ and $A_2 = [a_2, b_2]$ be two intervals. Then,

$A_1 + A_2 = [a_1 + a_2, b_1 + b_2]$

(ii) Let $A_1 = [a_1, b_1]$ and $A_2 = [a_2, b_2]$ be two non-negative intervals. Then,

$A_1A_2 = [a_1a_2, b_1b_2]$

Remark 2.1 An interval $A = [a, b]$ is said to be non-negative interval if and only if $a \geq 0$.

Remark 2.2 If $b = c$ then a trapezoidal fuzzy number (a, b, c, d) is said to be triangular fuzzy number and is denoted as (a, b, b, d) or (a, c, c, d) or (a, b, d) or (a, c, d).

Remark 2.3 [80] Let $\tilde{a}^\lambda = [\underline{a}(\lambda), \overline{a}(\lambda)]$ be a parametric form of an asymmetric triangular fuzzy number $\tilde{a}$ then its nearest symmetric triangular fuzzy number is (a_0, σ), where the core 'a_0' and the spread 'σ' of the symmetric triangular fuzzy number can be obtained as $a_0 = \frac{3}{2}\int\limits_0^1 (\overline{a}(\lambda) - \underline{a}(\lambda))(1-\lambda)d\lambda$, $\sigma = \frac{1}{2}\int\limits_0^1 (\overline{a}(\lambda) + \underline{a}(\lambda))d\lambda$.

Remark 2.4 In the entire thesis 'minimum' and 'maximum' are represented by 'min' and 'max' respectively.

2.2 Existing method for solving fully fuzzy linear programming problems with equality constraints

Lotfi et al. [80] pointed out that there is no method in literature for solving fully fuzzy linear programming problems with equality constraints and proposed the following method to find the fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints ($P_{2.1}$):

$$\begin{aligned}
&\text{Maximize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{2.1}) \\
&\qquad \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \;\; \forall\; i = 1, 2, ..., m
\end{aligned}$$

where $\tilde{c}_j$, $\tilde{x}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ are non-negative triangular fuzzy numbers.

Step 1 Assuming $\tilde{c}_j = (p_j, q_j, r_j), \tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}), \tilde{x}_j = (x_j, y_j, z_j)$ and $\tilde{b}_i = (b_i, g_i, h_i)$ the fully fuzzy linear programming problem ($P_{2.1}$) can be written as:

$$\begin{aligned}
&\text{Maximize } \sum_{j=1}^{n} (p_j, q_j, r_j) \otimes (x_j, y_j, z_j) \\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{2.2}) \\
&\qquad \sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i) \;\; \forall\; i = 1, 2, ..., m
\end{aligned}$$

where (x_j, y_j, z_j) is a non-negative triangular fuzzy number.

Step 2 Using Definition 2.8 and Definition 2.10, the fully fuzzy linear programming problem ($P_{2.2}$) can be converted into problem ($P_{2.3}$):

$$\text{Maximize } \sum_{j=1}^{n} [p_j + (q_j - p_j)\lambda, r_j - (r_j - q_j)\lambda][x_j + (y_j - x_j)\lambda, z_j - (z_j - y_j)\lambda]$$

subject to $(P_{2.3})$

$$\sum_{j=1}^{n}[a_{ij}+(b_{ij}-a_{ij})\lambda, c_{ij}-(c_{ij}-b_{ij})\lambda][x_j+(y_j-x_j)\lambda, z_j-(z_j-y_j)\lambda]=[b_i+(g_i-b_i)\lambda, h_i-(h_i-g_i)\lambda] \quad \forall\ i=1,2,...,m$$

$$x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0 \quad \forall\ j=1,2,...,n$$

Step 3 Using the arithmetic operations of intervals, defined in Section 2.1.2.2, the problem $(P_{2.3})$ can be converted into the problem $(P_{2.4})$:

Maximize $[\sum_{j=1}^{n}(p_jx_j+\lambda(p_jy_j+q_jx_j-2p_jx_j)+\lambda^2(q_j-p_j)(y_j-x_j)), \sum_{j=1}^{n}(r_jz_j-\lambda(2r_jz_j-r_jy_j-q_jz_j)+\lambda^2(r_j-q_j)(z_j-y_j))]$

subject to $(P_{2.4})$

$$\sum_{j=1}^{n}[a_{ij}x_j+\lambda(a_{ij}y_j+b_{ij}x_j-2a_{ij}x_j)+\lambda^2(b_{ij}-a_{ij})(y_j-x_j), c_{ij}z_j-\lambda(2c_{ij}z_j-c_{ij}y_j-b_{ij}z_j)+\lambda^2(c_{ij}-b_{ij})(z_j-y_j)]=[b_i+\lambda(g_i-b_i), h_i-\lambda(h_i-g_i)] \quad \forall\ i=1,2,...,m$$

$$x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0 \quad \forall\ j=1,2,...,n$$

Step 4 Using Remark 2.3, the problem $(P_{2.4})$ can be converted into the problem $(P_{2.5})$:

Maximize $(\sum_{j=1}^{n}(\frac{1}{3}q_jy_j+\frac{1}{12}q_jz_j+\frac{1}{12}r_jy_j+\frac{1}{6}r_jz_j+\frac{1}{12}q_jx_j+\frac{1}{12}p_jy_j+\frac{1}{6}p_jx_j), \sum_{j=1}^{n}(\frac{1}{8}q_jz_j+\frac{1}{8}r_jy_j+\frac{3}{8}r_jz_j-\frac{1}{8}q_jx_j-\frac{1}{8}p_jy_j-\frac{3}{8}p_jx_j))$

subject to $(P_{2.5})$

$$\sum_{j=1}^{n}(\tfrac{1}{3}b_{ij}y_j+\tfrac{1}{12}b_{ij}z_j+\tfrac{1}{12}c_{ij}y_j+\tfrac{1}{6}c_{ij}z_j+\tfrac{1}{12}b_{ij}x_j+\tfrac{1}{12}a_{ij}y_j+\tfrac{1}{6}a_{ij}x_j, \tfrac{1}{8}b_{ij}z_j+\tfrac{1}{8}c_{ij}y_j+\tfrac{3}{8}c_{ij}z_j-\tfrac{1}{8}b_{ij}x_j-\tfrac{1}{8}a_{ij}y_j-\tfrac{3}{8}a_{ij}x_j)=(\tfrac{1}{4}b_i+\tfrac{1}{2}g_i+\tfrac{1}{4}h_i, \tfrac{1}{2}h_i-\tfrac{1}{2}b_i) \quad \forall\ i=1,2,...,m$$

$$x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0 \quad \forall\ j=1,2,...,n$$

Step 5 Using Definition 2.12, convert the obtained fully fuzzy linear programming $(P_{2.5})$ into the crisp linear programming problem $(P_{2.6})$ to maximize the core:

Maximize $\sum_{j=1}^{n}(\frac{1}{3}q_jy_j+\frac{1}{12}q_jz_j+\frac{1}{12}r_jy_j+\frac{1}{6}r_jz_j+\frac{1}{12}q_jx_j+\frac{1}{12}p_jy_j+\frac{1}{6}p_jx_j)$

subject to

$\sum\limits_{j=1}^{n}(\frac{1}{3}b_{ij}y_j+\frac{1}{12}b_{ij}z_j+\frac{1}{12}c_{ij}y_j+\frac{1}{6}c_{ij}z_j+\frac{1}{12}b_{ij}x_j+\frac{1}{12}a_{ij}y_j+\frac{1}{6}a_{ij}x_j)=\frac{1}{4}b_i+\frac{1}{2}g_i+\frac{1}{4}h_i \;\; \forall\, i=1,2,...,m$ $(P_{2.6})$

$\sum\limits_{j=1}^{n}(\frac{1}{8}b_{ij}z_j+\frac{1}{8}c_{ij}y_j+\frac{3}{8}c_{ij}z_j-\frac{1}{8}b_{ij}x_j-\frac{1}{8}a_{ij}y_j-\frac{3}{8}a_{ij}x_j)=\frac{1}{2}h_i-\frac{1}{2}b_i \;\; \forall\, i=1,2,...,m$

$x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0 \;\; \forall\, j=1,2,...,n$

Step 6 If the crisp linear programming problem $(P_{2.6})$ has a unique optimal solution x_j^*, y_j^* and z_j^* then the fuzzy optimal solution of $(P_{2.1})$ will be (x_j^*, y_j^*, z_j^*). If it has alternative optimal solutions then solve the crisp linear programming problem $(P_{2.7})$ to minimize spread:

Minimize $\sum\limits_{j=1}^{n}(\frac{1}{8}q_jz_j+\frac{1}{8}r_jy_j+\frac{3}{8}r_jz_j-\frac{1}{8}q_jx_j-\frac{1}{8}p_jy_j-\frac{3}{8}p_jx_j)$

subject to

$\sum\limits_{j=1}^{n}(\frac{1}{3}b_{ij}y_j+\frac{1}{12}b_{ij}z_j+\frac{1}{12}c_{ij}y_j+\frac{1}{6}c_{ij}z_j+\frac{1}{12}b_{ij}x_j+\frac{1}{12}a_{ij}y_j+\frac{1}{6}a_{ij}x_j)=\frac{1}{4}b_i+\frac{1}{2}g_i+\frac{1}{4}h_i \;\; \forall\, i=1,2,...,m$ $(P_{2.7})$

$\sum\limits_{j=1}^{n}(\frac{1}{8}b_{ij}z_j+\frac{1}{8}c_{ij}y_j+\frac{3}{8}c_{ij}z_j-\frac{1}{8}b_{ij}x_j-\frac{1}{8}a_{ij}y_j-\frac{3}{8}a_{ij}x_j)=\frac{1}{2}h_i-\frac{1}{2}b_i \;\; \forall\, i=1,2,...,m$

$\sum\limits_{j=1}^{n}(\frac{1}{3}q_jy_j+\frac{1}{12}q_jz_j+\frac{1}{12}r_jy_j+\frac{1}{6}r_jz_j+\frac{1}{12}q_jx_j+\frac{1}{12}p_jy_j+\frac{1}{6}p_jx_j)=a^*$

$x_j\geq 0, y_j-x_j\geq 0, z_j-y_j\geq 0 \;\; \forall\, j=1,2,...,n$

where a^* is the optimal value of the crisp linear programming problem $(P_{2.6})$.

Step 7 Let x_j^*, y_j^* and z_j^* be the optimal solution of crisp linear programming problem $(P_{2.7})$. Then, the fuzzy optimal solution of fully fuzzy linear programming problem $(P_{2.1})$ is (x_j^*, y_j^*, z_j^*).

2.3 Limitations and shortcoming of the existing method

In this section, the limitations and shortcoming of the existing method [80] are pointed out.

2.3.1 Limitations of the existing method

In this section, the limitations of the existing method [80] are pointed out.

The existing method [80] can be used to find the non-negative fuzzy optimal solution of such fully fuzzy linear programming problems with equality constraints in which all the parameters are represented by non-negative triangular fuzzy numbers. However, the existing method [80] cannot be used to find the non-negative fuzzy optimal solution of the following problems:

(i) Fully fuzzy linear programming problems with equality constraints in which all the parameters are represented by non-negative trapezoidal fuzzy numbers:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_{2.8}) \\ &\quad \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \ \ \forall\, i = 1, 2, ..., m \end{aligned}$$

where $\tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i$ and $\tilde{x}_j$ are non-negative trapezoidal fuzzy numbers.

Example 2.1 Maximize $((1,2,3,4) \otimes \tilde{x}_1 \oplus (2,3,4,5) \otimes \tilde{x}_2)$

subject to

$$(0,1,2,3) \otimes \tilde{x}_1 \oplus (1,2,3,4) \otimes \tilde{x}_2 = (2,10,24,44)$$

$$(1,2,3,4) \otimes \tilde{x}_1 \oplus (0,1,2,3) \otimes \tilde{x}_2 = (1,8,21,40)$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative trapezoidal fuzzy numbers.

(ii) Fully fuzzy linear programming problems with equality constraints in which some or all the coefficients are either represented by unrestricted triangular or trapezoidal fuzzy numbers:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_{2.9}) \end{aligned}$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \ \ \forall\, i = 1, 2, ..., m$$

where $\tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i$ are unrestricted triangular or trapezoidal fuzzy numbers and $\tilde{x}_j$ is a non-negative triangular or trapezoidal fuzzy number.

Example 2.2 Maximize $((1, 6, 9, 12) \otimes \tilde{x}_1 \oplus (2, 3, 8, 9) \otimes \tilde{x}_2)$

subject to

$(2, 3, 4, 5) \otimes \tilde{x}_1 \oplus (1, 2, 3, 4) \otimes \tilde{x}_2 = (6, 16, 30, 48)$

$(-1, 1, 2, 3) \otimes \tilde{x}_1 \oplus (1, 3, 4, 6) \otimes \tilde{x}_2 = (0, 17, 30, 54)$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative trapezoidal fuzzy numbers.

2.3.2 Shortcoming of the existing method

In this section, the shortcoming of the existing method [80] is pointed out.

For solving the fully fuzzy linear programming problems by using the existing method [80] there is a need to approximate all the coefficients into its nearest symmetric fuzzy numbers. Due to this conversion, the obtained solutions are approximate and do not satisfy the constraints exactly e.g., on solving the fully fuzzy linear programming problem, chosen in Example 2.3, by using the existing method [80], the obtained fuzzy optimal solution is $\tilde{x}_1 = (1.45, 1.45, 3.31)$ and $\tilde{x}_2 = (0.88, 4.32, 4.32)$ which does not satisfy the constraints exactly.

Example 2.3 Maximize $((0, 1, 4) \otimes \tilde{x}_1 \oplus (2, 4, 5) \otimes \tilde{x}_2)$

subject to

$(2, 3, 7) \otimes \tilde{x}_1 \oplus (2, 4, 5) \otimes \tilde{x}_2 = (6, 18, 46)$

$(0, 2, 4) \otimes \tilde{x}_1 \oplus (3, 5, 8) \otimes \tilde{x}_2 = (6, 19, 52)$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative triangular fuzzy numbers.

2.4 Proposed product

For solving the fully fuzzy linear programming problems, there is a need to find the product of fuzzy coefficients and fuzzy variables. Since, in the fully fuzzy linear programming problems, the fuzzy coefficients are known and the product of fuzzy numbers depends upon the nature of fuzzy numbers. So, in this section, on the basis of nature of fuzzy coefficients new product, with the help of existing product of two fuzzy numbers [63], is proposed.

Let $\tilde{A} = (a, b, c, d)$ be an unrestricted trapezoidal fuzzy number and $\tilde{X} = (x, y, z, w)$ be a non-negative trapezoidal fuzzy number. Then

$$\tilde{A} \otimes \tilde{X} = \begin{cases} (ax, by, cz, dw) & a \geq 0 \\ (aw, by, cz, dw) & a < 0 \text{ and } b \geq 0 \\ (aw, bz, cz, dw) & b < 0 \text{ and } c \geq 0 \\ (aw, bz, cy, dw) & c < 0 \text{ and } d \geq 0 \\ (aw, bz, cy, dx) & \text{otherwise.} \end{cases}$$

2.5 Proposed method to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints

In this section, to overcome the limitations as well as to resolve the shortcoming of the existing method [80], discussed in Section 2.3, a new method is proposed to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints ($P_{2.9}$).

The steps of the proposed method are as follows:

Step 1 Assuming $\tilde{c}_j = (p_j, q_j, r_j, s_j)$, $\tilde{x}_j = (x_j, y_j, z_j, w_j)$, $\tilde{a} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ and

$\tilde{b} = (b_i, g_i, h_i, k_i)$ the fully fuzzy linear programming problem $(P_{2.9})$ can be converted into $(P_{2.10})$:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} (p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j) \\ &\text{subject to} \qquad\qquad\qquad\qquad (P_{2.10}) \\ &\sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}, d_{ij}) \otimes (x_j, y_j, z_j, w_j) = (b_i, g_i, h_i, k_i) \ \ \forall\ i = 1, 2, ..., m \end{aligned}$$

where (x_j, y_j, z_j, w_j) is a non-negative trapezoidal fuzzy number.

Step 2 Using the product of trapezoidal fuzzy numbers, proposed in Section 2.4 and assuming $(a_{ij}, b_{ij}, c_{ij}, d_{ij}) \otimes (x_j, y_j, z_j, w_j) = (a'_{ij}, b'_{ij}, c'_{ij}, d'_{ij})$, the fully fuzzy linear programming problem $(P_{2.10})$ can be converted into $(P_{2.11})$:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} (p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j) \\ &\text{subject to} \qquad\qquad\qquad\qquad (P_{2.11}) \\ &\sum_{j=1}^{n} (a'_{ij}, b'_{ij}, c'_{ij}, d'_{ij}) = (b_i, g_i, h_i, k_i) \ \ \forall\ i = 1, 2, ..., m \end{aligned}$$

where (x_j, y_j, z_j, w_j) is a non-negative trapezoidal fuzzy number.

Step 3 Using arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fully fuzzy linear programming problem $(P_{2.11})$ can be converted into $(P_{2.12})$:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} (p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j) \\ &\text{subject to} \\ &\sum_{j=1}^{n} a'_{ij} = b_i \ \ \forall\ i = 1, 2, ..., m \\ &\sum_{j=1}^{n} b'_{ij} = g_i \ \ \forall\ i = 1, 2, ..., m \qquad\qquad (P_{2.12}) \\ &\sum_{j=1}^{n} c'_{ij} = h_i \ \ \forall\ i = 1, 2, ..., m \\ &\sum_{j=1}^{n} d'_{ij} = k_i \ \ \forall\ i = 1, 2, ..., m \\ &x_j \geq 0, y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \ \ \forall\ j = 1, 2, ..., n \end{aligned}$$

Step 4 Suppose the fuzzy linear programming problem $(P_{2.12})$ have 'l' basic feasible solutions and $\{x_j^t, y_j^t, z_j^t, w_j^t\}$ is the t^{th} basic feasible solution then our aim is to find that basic feasible solution out of all 'l' basic feasible solutions corresponding to which the value of objective function is maximum (or minimum) i.e., our aim is to find $\max\limits_{1\le t\le l}$ (or min)$\{\sum\limits_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^t,y_j^t,z_j^t,w_j^t)\}$. Liou and Wang [78] proposed the concept that if $\max\limits_{1\le t\le l}$ (or min)$\{\Re(\sum\limits_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^t,y_j^t,z_j^t,w_j^t))\}$ is $\Re(\sum\limits_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^\theta,y_j^\theta,z_j^\theta,w_j^\theta))$ then $\max\limits_{1\le t\le l}$ (or min)$\{\sum\limits_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^t,y_j^t,z_j^t,w_j^t)\}$ will also be $\sum\limits_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j^\theta,y_j^\theta,z_j^\theta,w_j^\theta)$, where $\Re(a,b,c,d)=\frac{1}{4}(a+b+c+d)$, i.e., according to the existing method [78], the fuzzy optimal solution of $(P_{2.12})$ can be obtained by solving the crisp linear programming problem $(P_{2.13})$:

$$\text{Maximize/Minimize } \Re(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} a'_{ij}=b_i \quad \forall\, i=1,2,...,m \\
&\sum_{j=1}^{n} b'_{ij}=g_i \quad \forall\, i=1,2,...,m \\
&\sum_{j=1}^{n} c'_{ij}=h_i \quad \forall\, i=1,2,...,m \\
&\sum_{j=1}^{n} d'_{ij}=k_i \quad \forall\, i=1,2,...,m \\
&x_j\ge 0, y_j-x_j\ge 0, z_j-y_j\ge 0, w_j-z_j\ge 0 \quad \forall\, j=1,2,...,n
\end{aligned} \qquad (P_{2.13})$$

Step 5 Assuming $(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j)=(p'_j,q'_j,r'_j,s'_j)$ the crisp linear programming problem $(P_{2.13})$ can be written as $(P_{2.14})$:

$$\text{Maximize/Minimize } \Re(\sum_{j=1}^{n}(p'_j,q'_j,r'_j,s'_j))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} a'_{ij}=b_i \quad \forall\, i=1,2,...,m \\
&\sum_{j=1}^{n} b'_{ij}=g_i \quad \forall\, i=1,2,...,m \\
&\sum_{j=1}^{n} c'_{ij}=h_i \quad \forall\, i=1,2,...,m \\
&\sum_{j=1}^{n} d'_{ij}=k_i \quad \forall\, i=1,2,...,m
\end{aligned} \qquad (P_{2.14})$$

$$x_j \geq 0, y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \;\; \forall\, j = 1, 2, ..., n$$

Step 6 Using the linearity property $\Re(\sum\limits_{j=1}^{n} \tilde{A}_i) = \sum\limits_{j=1}^{n} \Re(\tilde{A}_i)$, where $\tilde{A}_i$ is a fuzzy number, the crisp linear programming problem $(P_{2.14})$ can be converted into $(P_{2.15})$:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} \Re(p'_j, q'_j, r'_j, s'_j) \\ &\text{subject to} \\ &\qquad \sum_{j=1}^{n} a'_{ij} = b_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n} b'_{ij} = g_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n} c'_{ij} = h_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n} d'_{ij} = k_i \;\; \forall\, i = 1, 2, ..., m \\ &x_j \geq 0, y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \;\; \forall\, j = 1, 2, ..., n \end{aligned} \qquad (P_{2.15})$$

Step 7 Using $\Re(a, b, c, d) = \frac{1}{4}(a + b + c + d)$ the crisp linear programming problem $(P_{2.15})$ can be converted into $(P_{2.16})$:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} \tfrac{1}{4}(p'_j + q'_j + r'_j + s'_j) \\ &\text{subject to} \\ &\qquad \sum_{j=1}^{n} a'_{ij} = b_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n} b'_{ij} = g_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n} c'_{ij} = h_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n} d'_{ij} = k_i \;\; \forall\, i = 1, 2, ..., m \\ &x_j \geq 0, y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \;\; \forall\, j = 1, 2, ..., n \end{aligned} \qquad (P_{2.16})$$

Step 8 Solve the crisp linear programming problem $(P_{2.16})$ by using an appropriate existing method [125] to find the optimal solution $\{x^*_j, y^*_j, z^*_j, w^*_j\}$.

Step 9 Find the fuzzy optimal solution $\{\tilde{x}^*_j\}$ of the fully fuzzy linear programming problem $(P_{2.9})$ by putting the values of x^*_j, y^*_j, z^*_j and w^*_j in $\tilde{x}^*_j = (x^*_j, y^*_j, z^*_j, w^*_j)$.

Step 10 Find the fuzzy optimal value by putting the values of $\tilde{x}_j^*$, obtained from Step 9, in $\sum\limits_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j^*$.

2.6 Illustrative examples

In this section, the proposed method is illustrated with the help of fully fuzzy linear programming problems, chosen in Example 2.1 and Example 2.2, which cannot be solved by using the existing method [80]. Moreover, a fully fuzzy linear programming problem, which can be solved by using the existing method [80], is also solved by using the proposed method.

2.6.1 Fuzzy optimal solution of the chosen fully fuzzy linear programming problems

In this section, fully fuzzy linear programming problems, chosen in Example 2.1, Example 2.2 and Example 2.3, are solved by using the proposed method.

2.6.1.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 2.1

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 2.1, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$ the fully fuzzy linear programming problem, chosen in Example 2.1, can be written as:

Maximize $((1,2,3,4) \otimes (x_1, y_1, z_1, w_1) \oplus (2,3,4,5) \otimes (x_2, y_2, z_2, w_2))$

subject to

$(0,1,2,3) \otimes (x_1, y_1, z_1, w_1) \oplus (1,2,3,4) \otimes (x_2, y_2, z_2, w_2) = (2,10,24,44)$

$(1,2,3,4) \otimes (x_1, y_1, z_1, w_1) \oplus (0,1,2,3) \otimes (x_2, y_2, z_2, w_2) = (1,8,21,40)$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are non-negative trapezoidal fuzzy numbers.

Step 2 Using the product, proposed in Section 2.4, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

$$\text{Maximize } ((x_1, 2y_1, 3z_1, 4w_1) \oplus (2x_2, 3y_2, 4z_2, 5w_2))$$

subject to

$$(0x_1, y_1, 2z_1, 3w_1) \oplus (x_2, 2y_2, 3z_2, 4w_2) = (2, 10, 24, 44)$$

$$(x_1, 2y_1, 3z_1, 4w_1) \oplus (0x_2, y_2, 2z_2, 3w_2) = (1, 8, 21, 40)$$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are non-negative trapezoidal fuzzy numbers.

Step 3 Using the arithmetic operations defined in Section 2.1.2.1 and Definition 2.12, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

$$\text{Maximize } (x_1 + 2x_2, 2y_1 + 3y_2, 3z_1 + 4z_2, 4w_1 + 5w_2)$$

subject to

$$0x_1 + x_2 = 2$$

$$x_1 + 0x_2 = 1$$

$$y_1 + 2y_2 = 10$$

$$2y_1 + y_2 = 8$$

$$2z_1 + 3z_2 = 24$$

$$3z_1 + 2z_2 = 21$$

$$3w_1 + 4w_2 = 44$$

$$4w_1 + 3w_2 = 40$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

$$\text{Maximize } \tfrac{1}{4}(x_1 + 2x_2 + 2y_1 + 3y_2 + 3z_1 + 4z_2 + 4w_1 + 5w_2)$$

subject to

$$0x_1 + x_2 = 2$$

$$x_1 + 0x_2 = 1$$

$$y_1 + 2y_2 = 10$$

$$2y_1 + y_2 = 8$$

$$2z_1 + 3z_2 = 24$$

$$3z_1 + 2z_2 = 21$$

$$3w_1 + 4w_2 = 44$$

$$4w_1 + 3w_2 = 40$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 5 The optimal solution of the crisp linear programming problem, obtained in Step 4, is $x_1 = 1, y_1 = 2, z_1 = 3, w_1 = 4, x_2 = 2, y_2 = 4, z_2 = 6$ and $w_2 = 8$.

Step 6 Putting the values of $x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2 in $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (1, 2, 3, 4), \tilde{x}_2 = (2, 4, 6, 8)$.

Step 7 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 6, in the objective function the fuzzy optimal value of the fully fuzzy linear programming problem is $(5, 16, 33, 56)$.

2.6.1.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 2.2

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 2.2, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$ the fully fuzzy linear programming problem, chosen in Example 2.2, can be written as:

$$\text{Maximize } ((1,6,9,12) \otimes (x_1, y_1, z_1, w_1) \oplus (2,3,8,9) \otimes (x_2, y_2, z_2, w_2))$$

subject to

$$(2,3,4,5) \otimes (x_1, y_1, z_1, w_1) \oplus (1,2,3,4) \otimes (x_2, y_2, z_2, w_2) = (6,16,30,48)$$

$$(-1,1,2,3) \otimes (x_1, y_1, z_1, w_1) \oplus (1,3,4,6) \otimes (x_2, y_2, z_2, w_2) = (0,17,30,54)$$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are non-negative trapezoidal fuzzy numbers.

Step 2 Using the product, proposed in Section 2.4, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

$$\text{Maximize } ((x_1, 6y_1, 9z_1, 12w_1) \oplus (2x_2, 3y_2, 8z_2, 9w_2))$$

subject to

$$(2x_1, 3y_1, 4z_1, 5w_1) \oplus (x_2, 2y_2, 3z_2, 4w_2) = (6,16,30,48)$$

$$(-w_1, y_1, 2z_1, 3w_1) \oplus (x_2, 3y_2, 4z_2, 6w_2) = (0,17,30,54)$$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are non-negative trapezoidal fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fuzzy linear programming problem, obtained in Step 2, can be written as:

$$\text{Maximize } (x_1 + 2x_2, 6y_1 + 3y_2, 9z_1 + 8z_2, 12w_1 + 9w_2)$$

subject to

$$2x_1 + x_2 = 6$$

$$-w_1 + x_2 = 0$$

$$3y_1 + 2y_2 = 16$$

$$y_1 + 3y_2 = 17$$

$$4z_1 + 3z_2 = 30$$

$$2z_1 + 4z_2 = 30$$

$$5w_1 + 4w_2 = 48$$

$$3z_1 + 6z_2 = 54$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\frac{1}{4}(x_1 + 2x_2 + 6y_1 + 3y_2 + 9z_1 + 8z_2 + 12w_1 + 9w_2)$

subject to

$$2x_1 + x_2 = 6$$

$$-w_1 + x_2 = 0$$

$$3y_1 + 2y_2 = 16$$

$$y_1 + 3y_2 = 17$$

$$4z_1 + 3z_2 = 30$$

$$2z_1 + 4z_2 = 30$$

$$5w_1 + 4w_2 = 48$$

$$3z_1 + 6z_2 = 54$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 5 The optimal solution of the crisp linear programming problem, obtained in Step 4, is $x_1 = 1, y_1 = 2, z_1 = 3, w_1 = 4, x_2 = 4, y_2 = 5, z_2 = 6$ and $w_2 = 7$.

Step 6 Putting the values of $x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2 in $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (1, 2, 3, 4), \tilde{x}_2 = (4, 5, 6, 7)$.

Step 7 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 6, in the objective function the fuzzy optimal value of the fully fuzzy linear programming problem is $(9, 27, 75, 111)$.

2.6.1.3 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 2.3

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 2.3, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2)$ the fully fuzzy linear programming problem, chosen in Example 2.3, can be written as:

$$\text{Maximize } ((0,1,4) \otimes (x_1, y_1, z_1) \oplus (2,4,5) \otimes (x_2, y_2, z_2))$$

$$\text{subject to}$$

$$(2,3,7) \otimes (x_1, y_1, z_1) \oplus (2,4,5) \otimes (x_2, y_2, z_2) = (6, 18, 46)$$

$$(0,2,4) \otimes (x_1, y_1, z_1) \oplus (3,5,8) \otimes (x_2, y_2, z_2) = (6, 19, 52)$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are non-negative triangular fuzzy numbers.

Step 2 Using the product, proposed in Section 2.4, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

$$\text{Maximize } ((0x_1, y_1, 4z_1) \oplus (2x_2, 4y_2, 5z_2))$$

$$\text{subject to}$$

$$(2x_1, 3y_1, 7z_1) \oplus (2x_2, 4y_2, 5z_2) = (6, 18, 46)$$

$$(0x_1, 2y_1, 4z_1) \oplus (3x_2, 5y_2, 8z_2) = (6, 19, 52)$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are non-negative triangular fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fuzzy linear programming problem, obtained in Step 2, can be written as:

$$\text{Maximize } (0x_1 + 2x_2, y_1 + 4y_2, 4z_1 + 5z_2)$$

subject to

$$2x_1 + 2x_2 = 6$$

$$0x_1 + 3x_2 = 6$$

$$3y_1 + 4y_2 = 18$$

$$2y_1 + 5y_2 = 19$$

$$7z_1 + 5z_2 = 46$$

$$4z_1 + 8z_2 = 52$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

$$\text{Maximize } \frac{1}{4}(0x_1 + 2x_2 + 2y_1 + 8y_2 + 4z_1 + 5z_2)$$

subject to

$$2x_1 + 2x_2 = 6$$

$$0x_1 + 3x_2 = 6$$

$$3y_1 + 4y_2 = 18$$

$$2y_1 + 5y_2 = 19$$

$$7z_1 + 5z_2 = 46$$

$$4z_1 + 8z_2 = 52$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0$$

Step 5 The optimal solution of the crisp linear programming problem, obtained in Step 4, is $x_1 = 1, y_1 = 2, z_1 = 3, x_2 = 2, y_2 = 3$ and $z_2 = 5$.

Step 6 Putting the values of x_1, y_1, z_1, x_2, y_2 and z_2 in $\tilde{x}_1 = (x_1, y_1, z_1)$ and $\tilde{x}_2 =$

(x_2, y_2, z_2), the exact fuzzy optimal solution is $\tilde{x}_1 = (1, 2, 3), \tilde{x}_2 = (2, 3, 5)$.

Step 7 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 6, in the objective function the fuzzy optimal value of the fully fuzzy linear programming problem is $(4, 14, 37)$.

2.7 Advantages of the proposed method

In this section, the advantages of proposed method over the existing method [80] are discussed.

(i) It is easy to apply the proposed method as compared to the existing method [80].

(ii) The fuzzy optimal solution, obtained by using the existing method [80], does not exactly satisfy the constraints of the fully fuzzy linear programming problems while the fuzzy optimal solution, obtained by using the proposed method, exactly satisfy the constraints of the fully fuzzy linear programming problems.

(iii) The existing method [80] can be used to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems $(P_{2.1})$ but cannot be used to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems $(P_{2.8})$ and $(P_{2.9})$. However, the proposed method can be used to find the non-negative fuzzy optimal solution of all the fully fuzzy linear programming problems $(P_{2.1})$, $(P_{2.8})$ and $(P_{2.9})$.

2.8 Comparative study

The results of the chosen fully fuzzy linear programming problems, obtained by using the existing method [80] and proposed method, are shown in Table 2.1.

Table 2.1 Results of the chosen fully fuzzy linear programming problems

Example	Fuzzy optimal value	
	Existing method [80]	Proposed method
2.1	Not Applicable	$(5, 16, 33, 56)$
2.2	Not Applicable	$(9, 27, 75, 111)$
2.3	$(1.76, 18.75, 34.86)$	$(4, 14, 37)$

The results, presented in Table 2.1 can be explained as follows:

(i) In the problems, chosen in Example 2.1 and Example 2.2, all the coefficients are not non-negative triangular fuzzy numbers. So, due to the limitations of the existing method [80], discussed in Section 2.3.1, none of these problems can be solved by using the existing method [80]. However, in the problem, chosen in Example 2.3, all the coefficients are represented by non-negative triangular fuzzy numbers. So, as discussed in Section 2.3.1, it can be solved by using the existing method [80] but due to the shortcoming of the existing method [80], discussed in Section 2.3.2, the obtained results are not exact.

(ii) The proposed method can be used to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted coefficients. So, all the problems, chosen in Example 2.1, Example 2.2 and Example 2.3, can be solved by using the proposed method. Also, as discussed in Section 2.7, the results obtained by using the proposed method are exact.

2.9 Conclusions

On the basis of presented study, it can be concluded that it is better to use the proposed method as compared to the existing method [80] for solving fully fuzzy linear programming problems with equality constraints.

Chapter 3

A New Method For Solving Fully Fuzzy Linear Programming Problems With Equality Constraints

In this chapter, the limitations of the method, proposed in Chapter 2, are pointed out and to overcome these limitations, a new method is proposed for solving fully fuzzy linear programming problems with equality constraints. To show the application of proposed method a real life problem, which cannot be solved by using the method, proposed in Chapter 2, is solved by using the proposed method.

3.1 Limitations of the previous proposed method

The method, proposed in Chapter 2, can be used to find the exact fuzzy optimal solution of the following type of problems:

(i) Fully fuzzy linear programming problems with equality constraints having non-negative fuzzy coefficients and non-negative fuzzy variables.

(ii) Fully fuzzy linear programming problems with equality constraints having unrestricted fuzzy coefficients and non-negative fuzzy variables.

The contents of this chapter are published in *Applied Intelligence* 37 (2012) 145-154.

i.e., the method, proposed in Chapter 2, can be used to find the exact fuzzy optimal solution of such fully fuzzy linear programming problems with equality constraints in which all the variables are represented by non-negative fuzzy numbers.

However, the method, proposed in Chapter 2, cannot be used for solving the following type of problems:

(iii) Fully fuzzy linear programming problems with equality constraints having non-negative fuzzy coefficients and unrestricted fuzzy variables:

$$\begin{aligned}&\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j\\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_{3.1})\\ &\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \;\; \forall\, i = 1, 2, ..., m\end{aligned}$$

where $\tilde{c}_j, \tilde{a}_{ij}$ are non-negative trapezoidal fuzzy numbers and $\tilde{b}_i, \tilde{x}_j$ are unrestricted trapezoidal fuzzy numbers.

Example 3.1 Maximize $((1,2,3,4) \otimes \tilde{x}_1 \oplus (2,4,6,8) \otimes \tilde{x}_2)$

subject to

$(0,1,2,3) \otimes \tilde{x}_1 \oplus (1,3,5,7) \otimes \tilde{x}_2 = (-8,2,27,57)$

$(2,4,7,9) \otimes \tilde{x}_1 \oplus (2,3,5,6) \otimes \tilde{x}_2 = (-25,-8,34,81)$

where $\tilde{x}_1$ and $\tilde{x}_2$ are unrestricted trapezoidal fuzzy numbers.

(iv) Fully fuzzy linear programming problems with equality constraints having unrestricted fuzzy coefficients and unrestricted fuzzy variables:

$$\begin{aligned}&\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j\\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_{3.2})\\ &\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \;\; \forall\, i = 1, 2, ..., m\end{aligned}$$

where $\tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i$ and $\tilde{x}_j$ are unrestricted trapezoidal fuzzy numbers.

Example 3.2 Maximize $((-2,-1,3,6) \otimes \tilde{x}_1 \oplus (2,4,6,8) \otimes \tilde{x}_2)$

subject to

$(-3,2,5,7)\otimes\tilde{x}_1\oplus(-2,-1,5,7)\otimes\tilde{x}_2=(-42,-17,45,91)$

$(1,2,3,4)\otimes\tilde{x}_1\oplus(-4,-3,-2,-1)\otimes\tilde{x}_2=(-48,-27,-4,28)$

where $\tilde{x}_1$ and $\tilde{x}_2$ are unrestricted trapezoidal fuzzy numbers.

i.e., the method, proposed in Chapter 2, cannot be used for solving such fully fuzzy linear programming problems with equality constraints in which all or some of the variables are represented by unrestricted fuzzy numbers.

3.2 Proposed product

In this section, the product of two unrestricted trapezoidal fuzzy numbers is proposed.

Let $\tilde{A}=(a,b,c,d)$ and $\tilde{X}=(x,y,z,w)$ be two unrestricted trapezoidal fuzzy numbers. Then,

$$\tilde{A}\otimes\tilde{X}=\begin{cases}(\min\{ax,dx\},\min\{by,cy\},\max\{bz,cz\},\max\{aw,dw\}) & a\geq 0\\(\min\{aw,dx\},\min\{by,cy\},\max\{bz,cz\},\max\{ax,dw\}) & a<0 \text{ and } b\geq 0\\(\min\{aw,dx\},\min\{bz,cy\},\max\{by,cz\},\max\{ax,dw\}) & b<0 \text{ and } c\geq 0\\(\min\{aw,dx\},\min\{bz,cz\},\max\{by,cy\},\max\{ax,dw\}) & c<0 \text{ and } d\geq 0\\(\min\{aw,dw\},\min\{bz,cz\},\max\{by,cy\},\max\{ax,dx\}) & \text{otherwise.}\end{cases}$$

3.2.1 Particular cases of the proposed product

In this section, all the particular cases of the product, proposed in Section 3.2, are discussed.

(i) Let $\tilde{A}=(a,b,c,d)$ be an unrestricted trapezoidal fuzzy number and $\tilde{X}=(x,y,z,w)$ be a non-negative trapezoidal fuzzy number. Then,

$$\tilde{A} \otimes \tilde{X} = \begin{cases} (ax, by, cz, dw) & a \geq 0 \\ (aw, by, cz, dw) & a < 0 \text{ and } b \geq 0 \\ (aw, bz, cz, dw) & b < 0 \text{ and } c \geq 0 \\ (aw, bz, cy, dw) & c < 0 \text{ and } d \geq 0 \\ (aw, bz, cy, dx) & \text{otherwise.} \end{cases}$$

(ii) Let $\tilde{A} = (a, b, c, d)$ be a non-negative trapezoidal fuzzy number and $\tilde{X} = (x, y, z, w)$ be an unrestricted trapezoidal fuzzy number. Then,

$$\tilde{A} \otimes \tilde{X} = (\min\{ax, dx\}, \min\{by, cy\}, \max\{bz, cz\}, \max\{aw, dw\})$$

(iii) Let $\tilde{A} = (a, b, c, d)$ and $\tilde{X} = (x, y, z, w)$ be two non-negative trapezoidal fuzzy number. Then,

$$\tilde{A} \otimes \tilde{X} = (ax, by, cz, dw)$$

3.3 Proposed method for solving fully fuzzy linear programming problems with equality constraints

In this section, to overcome all the limitations of the method, proposed in Chapter 2, a new method is proposed to find the exact fuzzy optimal solution of fully fuzzy linear programming problems ($P_{3.2}$) in which all the parameters are represented by unrestricted trapezoidal fuzzy numbers.

The steps of the proposed method are as follows:

Step 1 Assuming $\tilde{c}_j = (p_j, q_j, r_j, s_j)$, $\tilde{x}_j = (x_j, y_j, z_j, w_j)$, $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ and $\tilde{b}_i = (b_i, g_i, h_i, k_i)$ the fully fuzzy linear programming problem ($P_{3.2}$) can be written as:

Maximize/Minimize $\sum\limits_{j=1}^{n} (p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j)$

subject to $\qquad\qquad (P_{3.3})$

$$\sum_{j=1}^{n}(a_{ij}, b_{ij}, c_{ij}, d_{ij}) \otimes (x_j, y_j, z_j, w_j) = (b_i, g_i, h_i, k_i) \;\; \forall\, i = 1, 2, ..., m$$

where (x_j, y_j, z_j, w_j) is an unrestricted trapezoidal fuzzy number.

Step 2 Using the product of trapezoidal fuzzy numbers, proposed in Section 3.2, the fully fuzzy linear programming problem $(P_{3.3})$ can be written as:

Maximize/Minimize $\sum_{j=1}^{n}(p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j)$

subject to $\qquad (P_{3.4})$

$\sum_{j=1}^{n}(\min\{a'_{ij}, a''_{ij}\},\min\{b'_{ij}, b''_{ij}\},\max\{c'_{ij}, c''_{ij}\},\max\{d'_{ij}, d''_{ij}\}) = (b_i, g_i, h_i, k_i) \;\; \forall\, i = 1, 2, ..., m$

where (x_j, y_j, z_j, w_j) is an unrestricted trapezoidal fuzzy number and

$$(\min\{a'_{ij}, a''_{ij}\},\min\{b'_{ij}, b''_{ij}\},\max\{c'_{ij}, c''_{ij}\},\max\{d'_{ij}, d''_{ij}\}) =$$
$$\begin{cases} (\min\{a_{ij}x_j, d_{ij}x_j\},\min\{b_{ij}y_j, c_{ij}y_j\},\max\{b_{ij}z_j, c_{ij}z_j\},\max\{a_{ij}w_j, d_{ij}w_j\}) & a_{ij} \geq 0 \\ (\min\{a_{ij}w_j, d_{ij}x_j\},\min\{b_{ij}y_j, c_{ij}y_j\},\max\{b_{ij}z_j, c_{ij}z_j\},\max\{d_{ij}w_j, a_{ij}x_j\}) & a_{ij} < 0 \text{ and } b_{ij} \geq 0 \\ (\min\{a_{ij}w_j, d_{ij}x_j\},\min\{b_{ij}z_j, c_{ij}y_j\},\max\{b_{ij}y_j, c_{ij}z_j\},\max\{a_{ij}x_j, d_{ij}w_j\}) & b_{ij} < 0 \text{ and } c_{ij} \geq 0 \\ (\min\{a_{ij}w_j, d_{ij}x_j\},\min\{b_{ij}z_j, c_{ij}z_j\},\max\{b_{ij}y_j, c_{ij}y_j\},\max\{a_{ij}x_j, d_{ij}w_j\}) & c_{ij} < 0 \text{ and } d_{ij} \geq 0 \\ (\min\{a_{ij}w_j, d_{ij}w_j\},\min\{b_{ij}z_j, c_{ij}z_j\},\max\{b_{ij}y_j, c_{ij}y_j\},\max\{a_{ij}x_j, d_{ij}x_j\}) & \text{otherwise.} \end{cases}$$

Step 3 Using arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fully fuzzy linear programming problem $(P_{3.4})$ can be written as:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n}(p_j, q_j, r_j, s_j) \otimes (x_j, y_j, z_j, w_j) \\ &\text{subject to} \\ &\qquad \sum_{j=1}^{n}\min\{a'_{ij}, a''_{ij}\} = b_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n}\min\{b'_{ij}, b''_{ij}\} = g_i \;\; \forall\, i = 1, 2, ..., m \qquad\qquad (P_{3.5}) \\ &\qquad \sum_{j=1}^{n}\max\{c'_{ij}, c''_{ij}\} = h_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad \sum_{j=1}^{n}\max\{d'_{ij}, d''_{ij}\} = k_i \;\; \forall\, i = 1, 2, ..., m \\ &\qquad y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \;\; \forall\, j = 1, 2, ..., n \end{aligned}$$

Step 4 As discussed in Step 4 of Section 2.5, the fuzzy optimal solution of $(P_{3.5})$

can be obtained by solving the crisp non-linear programming problem ($P_{3.6}$):

$$\text{Maximize/Minimize } \Re(\sum_{j=1}^{n}(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\min\{a'_{ij},a''_{ij}\}=b_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\min\{b'_{ij},b''_{ij}\}=g_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\max\{c'_{ij},c''_{ij}\}=h_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\max\{d'_{ij},d''_{ij}\}=k_i \;\; \forall\, i=1,2,...,m\\
&y_j-x_j\geq 0, z_j-y_j\geq 0, w_j-z_j\geq 0 \;\; \forall\, j=1,2,...,n
\end{aligned} \qquad (P_{3.6})$$

Step 5 Assuming $(p_j,q_j,r_j,s_j)\otimes(x_j,y_j,z_j,w_j)=(\min\{p'_j,p''_j\},\min\{q'_j,q''_j\},\max\{r'_j,r''_j\},\max\{s'_j,s''_j\})$ the crisp non-linear programming problem ($P_{3.6}$) can be written as:

$$\text{Maximize/Minimize } \Re(\sum_{j=1}^{n}(\min\{p'_j,p''_j\},\min\{q'_j,q''_j\},\max\{r'_j,r''_j\},\max\{s'_j,s''_j\}))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\min\{a'_{ij},a''_{ij}\}=b_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\min\{b'_{ij},b''_{ij}\}=g_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\max\{c'_{ij},c''_{ij}\}=h_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\max\{d'_{ij},d''_{ij}\}=k_i \;\; \forall\, i=1,2,...,m\\
&y_j-x_j\geq 0, z_j-y_j\geq 0, w_j-z_j\geq 0 \;\; \forall\, j=1,2,...,n
\end{aligned} \qquad (P_{3.7})$$

Step 6 Using the linearity property $\Re(\sum_{j=1}^{n}\tilde{A}_i)=\sum_{j=1}^{n}\Re(\tilde{A}_i)$, where $\tilde{A}_i$ is a fuzzy number, problem ($P_{3.7}$) can be converted into ($P_{3.8}$):

$$\text{Maximize/Minimize } (\sum_{j=1}^{n}\Re(\min\{p'_j,p''_j\},\min\{q'_j,q''_j\},\max\{r'_j,r''_j\},\max\{s'_j,s''_j\}))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n}\min\{a'_{ij},a''_{ij}\}=b_i \;\; \forall\, i=1,2,...,m\\
&\sum_{j=1}^{n}\min\{b'_{ij},b''_{ij}\}=g_i \;\; \forall\, i=1,2,...,m
\end{aligned} \qquad (P_{3.8})$$

$$\begin{aligned}
&\sum_{j=1}^{n}\max\{c'_{ij}, c''_{ij}\} = h_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}\max\{d'_{ij}, d''_{ij}\} = k_i \ \ \forall\ i = 1, 2, ..., m\\
&y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \ \ \forall\ j = 1, 2, ..., n
\end{aligned}$$

Step 7 Using $\max(a, b) = \frac{a+b}{2} + |\frac{a-b}{2}|$ and $\min(a, b) = \frac{a+b}{2} - |\frac{a-b}{2}|$ problem $(P_{3.8})$ can be converted into $(P_{3.9})$:

Maximize/Minimize $\sum_{j=1}^{n}\Re(\frac{p'_j+p''_j}{2}-|\frac{p'_j-p''_j}{2}|, \frac{q'_j+q''_j}{2}-|\frac{q'_j-q''_j}{2}|, \frac{r'_j+r''_j}{2}+|\frac{r'_j-r''_j}{2}|, \frac{s'_j+s''_j}{2}+|\frac{s'_j-s''_j}{2}|)$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n}(\frac{a'_{ij}+a''_{ij}}{2} - |\frac{a'_{ij}-a''_{ij}}{2}|) = b_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}(\frac{b'_{ij}+b''_{ij}}{2} - |\frac{b'_{ij}-b''_{ij}}{2}|) = g_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}(\frac{c'_{ij}+c''_{ij}}{2} + |\frac{c'_{ij}-c''_{ij}}{2}|) = h_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}(\frac{d'_{ij}+d''_{ij}}{2} + |\frac{d'_{ij}-d''_{ij}}{2}|) = k_i \ \ \forall\ i = 1, 2, ..., m\\
&y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \ \ \forall\ j = 1, 2, ..., n
\end{aligned} \qquad (P_{3.9})$$

Step 8 Using $\Re(a, b, c, d) = \frac{1}{4}(a + b + c + d)$ problem $(P_{3.9})$ can be converted into $(P_{3.10})$:

Maximize/Minimize $\frac{1}{4}(\sum_{j=1}^{n}(\frac{p'_j+p''_j}{2} - |\frac{p'_j-p''_j}{2}| + \frac{q'_j+q''_j}{2} - |\frac{q'_j-q''_j}{2}| + \frac{r'_j+r''_j}{2} + |\frac{r'_j-r''_j}{2}| + \frac{s'_j+s''_j}{2} + |\frac{s'_j-s''_j}{2}|))$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n}(\frac{a'_{ij}+a''_{ij}}{2} - |\frac{a'_{ij}-a''_{ij}}{2}|) = b_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}(\frac{b'_{ij}+b''_{ij}}{2} - |\frac{b'_{ij}-b''_{ij}}{2}|) = g_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}(\frac{c'_{ij}+c''_{ij}}{2} + |\frac{c'_{ij}-c''_{ij}}{2}|) = h_i \ \ \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n}(\frac{d'_{ij}+d''_{ij}}{2} + |\frac{d'_{ij}-d''_{ij}}{2}|) = k_i \ \ \forall\ i = 1, 2, ..., m\\
&y_j - x_j \geq 0, z_j - y_j \geq 0, w_j - z_j \geq 0 \ \ \forall\ j = 1, 2, ..., n
\end{aligned} \qquad (P_{3.10})$$

Step 9 Solve the crisp non-linear programming problem $(P_{3.10})$ by using an appropriate existing method [125] to find the optimal solution $\{x_j^*, y_j^*, z_j^*, w_j^*\}$.

Step 10 Find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ of the fully fuzzy linear programming problem $(P_{3.2})$ by putting the values of x_j^*, y_j^*, z_j^* and w_j^* in $\tilde{x}_j^* = (x_j^*, y_j^*, z_j^*, w_j^*)$.

Step 11 Find the fuzzy optimal value by putting the values of $\tilde{x}_j^*$, obtained from Step 10, in $\sum\limits_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j^*$.

3.4 Illustrative examples

In this section, the proposed method is illustrated with the help of fully fuzzy linear programming problems, chosen in Example 3.1 and Example 3.2, which cannot be solved by using the method, proposed in Chapter 2. Moreover, the fully fuzzy linear programming problem, chosen in Example 2.1, which can be solved by using the method, proposed in Chapter 2, is also solved by using the proposed method.

3.4.1 Fuzzy optimal solution of the chosen fully fuzzy linear programming problems

In this section, fully fuzzy linear programming problems, chosen in Example 3.1, Example 3.2 and Example 2.1, are solved by using the proposed method.

3.4.1.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 3.1

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 3.1, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$ the fully fuzzy linear programming problem, chosen in Example 3.1, can be written as:

$$\text{Maximize } ((1,2,3,4) \otimes (x_1, y_1, z_1, w_1) \oplus (2,4,6,8) \otimes (x_2, y_2, z_2, w_2))$$

subject to

$(0,1,2,3) \otimes (x_1, y_1, z_1, w_1) \oplus (1,3,5,7) \otimes (x_2, y_2, z_2, w_2) = (-8, 2, 27, 57)$

$(2,4,7,9) \otimes (x_1, y_1, z_1, w_1) \oplus (2,3,5,6) \otimes (x_2, y_2, z_2, w_2) = (-25, -8, 34, 81)$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are unrestricted fuzzy numbers.

Step 2 Using the product, proposed in Section 3.2, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((\min\{x_1, 4x_1\}, \min\{2y_1, 3y_1\}, \max\{2z_1, 3z_1\}, \max\{w_1, 4w_1\}) \oplus (\min\{2x_2, 8x_2\}, \min\{4y_2, 6y_2\}, \max\{4z_2, 6z_2\}, \max\{2w_2, 8w_2\}))$

subject to

$(\min\{0, 3x_1\}, \min\{y_1, 2y_1\}, \max\{z_1, 2z_1\}, \max\{0, 3w_1\}) \oplus (\min\{x_2, 7x_2\}, \min\{3y_2, 5y_2\}, \max\{3z_2, 5z_2\}, \max\{w_2, 7w_2\}) = (-8, 2, 27, 57)$

$(\min\{2x_1, 9x_1\}, \min\{4y_1, 7y_1\}, \max\{4z_1, 7z_1\}, \max\{2w_1, 9w_1\}) \oplus (\min\{2x_2, 6x_2\}, \min\{3y_2, 5y_2\}, \max\{3z_2, 5z_2\}, \max\{2w_2, 6w_2\}) = (-25, -8, 34, 81)$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 3 Using the arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $(\min\{x_1, 4x_1\} + \min\{2x_2, 8x_2\}, \min\{2y_1, 3y_1\} + \min\{4y_2, 6y_2\}, \max\{2z_1, 3z_1\} + \max\{4z_2, 6z_2\}, \max\{w_1, 4w_1\} + \max\{2w_2, 8w_2\})$

subject to

$$\min\{0, 3x_1\} + \min\{x_2, 7x_2\} = -8$$

$$\min\{y_1, 2y_1\} + \min\{3y_2, 5y_2\} = 2$$

$$\max\{z_1, 2z_1\} + \max\{3z_2, 5z_2\} = 27$$

$$\max\{0, 3w_1\}+\max\{w_2, 7w_2\} = 57$$

$$\min\{2x_1, 9x_1\}+\min\{2x_2, 6x_2\} = -25$$

$$\min\{4y_1, 7y_1\}+\min\{3y_2, 5y_2\} = -8$$

$$\max\{4z_1, 7z_1\}+\max\{3z_2, 5z_2\} = 34$$

$$\max\{2w_1, 9w_1\}+\max\{2w_2, 6w_2\} = 81$$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\Re(\min\{x_1, 4x_1\}+\min\{2x_2, 8x_2\}, \min\{2y_1, 3y_1\}+\min\{4y_2, 6y_2\}, \max\{2z_1, 3z_1\}$ $+\max\{4z_2, 6z_2\}, \max\{w_1, 4w_1\}+\max\{2w_2, 8w_2\})$

subject to

$$\min\{0, 3x_1\}+\min\{x_2, 7x_2\} = -8$$

$$\min\{y_1, 2y_1\}+\min\{3y_2, 5y_2\} = 2$$

$$\max\{z_1, 2z_1\}+\max\{3z_2, 5z_2\} = 27$$

$$\max\{0, 3w_1\}+\max\{w_2, 7w_2\} = 57$$

$$\min\{2x_1, 9x_1\}+\min\{2x_2, 6x_2\} = -25$$

$$\min\{4y_1, 7y_1\}+\min\{3y_2, 5y_2\} = -8$$

$$\max\{4z_1, 7z_1\}+\max\{3z_2, 5z_2\} = 34$$

$$\max\{2w_1, 9w_1\}+\max\{2w_2, 6w_2\} = 81$$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 5 Using Step 7 and Step 8 of the proposed method, the problem, obtained in Step 4, can be converted into the following crisp non-linear programming problem:

Maximize $\frac{1}{4}(\frac{5}{2}x_1 - \frac{3}{2}|x_1| + 5x_2 - 3|x_2| + \frac{5}{2}y_1 - \frac{1}{2}|y_1| + 5y_2 - |y_2| + \frac{5}{2}z_1 + \frac{1}{2}|z_1| + 5z_2 + |z_2| + \frac{5}{2}w_1 + \frac{3}{2}|w_1| + 5w_2 + 3|w_2|)$

subject to

$$\frac{3}{2}x_1 - \frac{3}{2}|x_1| + 4x_2 - 3|x_2| = -8$$

$$\frac{3}{2}y_1 - \frac{1}{2}|y_1| + 4y_2 - |y_2| = 2$$

$$\frac{3}{2}z_1 + \frac{1}{2}|z_1| + 4z_2 + |z_2| = 27$$

$$\frac{3}{2}w_1 + \frac{3}{2}|w_1| + 4w_2 + 3|w_2| = 57$$

$$\frac{11}{2}x_1 - \frac{7}{2}|x_1| + 4x_2 - 2|x_2| = -25$$

$$\frac{11}{2}y_1 - \frac{3}{2}|y_1| + 4y_2 - |y_2| = -8$$

$$\frac{11}{2}z_1 + \frac{3}{2}|z_1| + 4z_2 + |z_2| = 34$$

$$\frac{11}{2}w_1 + \frac{7}{2}|w_1| + 4w_2 + 2|w_2| = 81$$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 6 The optimal solution of the crisp non-linear programming problem, obtained in Step 5, is $x_1 = -3, y_1 = -2, z_1 = \frac{7}{5}, w_1 = 5, x_2 = 1, y_2 = 2, z_2 = \frac{121}{25}$ and $w_2 = 6$.

Step 7 Putting the values of $x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2 in $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (-3, -2, \frac{7}{5}, 5), \tilde{x}_2 = (1, 2, \frac{121}{25}, 6)$.

Step 8 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 7, in the objective function the fuzzy optimal value is $(-10, 2, \frac{831}{25}, 68)$.

3.4.1.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 3.2

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 3.2, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$ the fully fuzzy linear programming problem, chosen in Example 3.2, can be written as:

Maximize $((-2,-1,3,6) \otimes (x_1, y_1, z_1, w_1) \oplus (2,4,6,8) \otimes (x_2, y_2, z_2, w_2))$

subject to

$(-3,2,5,7) \otimes (x_1, y_1, z_1, w_1) \oplus (-2,-1,5,7) \otimes (x_2, y_2, z_2, w_2) = (-42,-17,45,91)$

$(1,2,3,4) \otimes (x_1, y_1, z_1, w_1) \oplus (-4,-3,-2,-1) \otimes (x_2, y_2, z_2, w_2) = (-48,-27,-4,28)$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are unrestricted trapezoidal fuzzy numbers.

Step 2 Using the product, proposed in Section 3.2, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((\min\{-2w_1, 6x_1\}, \min\{-z_1, 3y_1\}, \max\{-y_1, 3z_1\}, \max\{-2x_1, 6w_1\}) \oplus (\min\{2x_2, 8x_2\}, \min\{4y_2, 6y_2\}, \max\{4z_2, 6z_2\}, \max\{2w_2, 8w_2\}))$

subject to

$(\min\{-3w_1, 7x_1\}, \min\{2y_1, 5y_1\}, \max\{2z_1, 5z_1\}, \max\{7w_1, -3x_1\}) \oplus (\min\{-2w_2, 7x_2\}, \min\{-z_2, 5y_2\}, \max\{-y_2, 5z_2\}, \max\{-2x_2, 7w_2\}) = (-42,-17,45,91)$

$(\min\{x_1, 4x_1\}, \min\{2y_1, 3y_1\}, \max\{2z_1, 3z_1\}, \max\{w_1, 4w_1\}) \oplus (\min\{-4w_2, -w_2\}, \min\{-3z_2, -2z_2\}, \max\{-3y_2, -2y_2\}, \max\{-4x_2, -x_2\}) = (-48,-27,-4,28)$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 3 Using the arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $(\min\{-2w_1, 6x_1\} + \min\{2x_2, 8x_2\}, \min\{-z_1, 3y_1\} + \min\{4y_2, 6y_2\}, \max\{-y_1, 3z_1\} + \max\{4z_2, 6z_2\}, \max\{-2x_1, 6w_1\} + \max\{2w_2, 8w_2\})$

subject to

$$\min\{-3w_1, 7x_1\}+\min\{-2w_2, 7x_2\} = -42$$

$$\min\{2y_1, 5y_1\}+\min\{-z_2, 5y_2\} = -17$$

$$\max\{2z_1, 5z_1\}+\max\{-y_2, 5z_2\} = 45$$

$$\max\{7w_1, -3x_1\}+\max\{-2x_2, 7w_2\} = 91$$

$$\min\{x_1, 4x_1\}+\min\{-4w_2, -w_2\} = -48$$

$$\min\{2y_1, 3y_1\}+\min\{-3z_2, -2z_2\} = -27$$

$$\max\{2z_1, 3z_1\}+\max\{-3y_2, -2y_2\} = -4$$

$$\max\{w_1, 4w_1\}+\max\{-4x_2, -x_2\} = 28$$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\Re(\min\{-2w_1, 6x_1\}+\min\{2x_2, 8x_2\},\min\{-z_1, 3y_1\}+\min\{4y_2, 6y_2\},\max\{-y_1, 3z_1\}+\max\{4z_2, 6z_2\},\max\{-2x_1, 6w_1\}+\max\{2w_2, 8w_2\})$

subject to

$$\min\{-3w_1, 7x_1\}+\min\{-2w_2, 7x_2\} = -42$$

$$\min\{2y_1, 5y_1\}+\min\{-z_2, 5y_2\} = -17$$

$$\max\{2z_1, 5z_1\}+\max\{-y_2, 5z_2\} = 45$$

$$\max\{7w_1, -3x_1\}+\max\{-2x_2, 7w_2\} = 91$$

$$\min\{x_1, 4x_1\}+\min\{-4w_2, -w_2\} = -48$$

$$\min\{2y_1, 3y_1\}+\min\{-3z_2, -2z_2\} = -27$$

$$\max\{2z_1, 3z_1\}+\max\{-3y_2, -2y_2\} = -4$$

$$\max\{w_1, 4w_1\}+\max\{-4x_2, -x_2\} = 28$$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 5 Using Step 7 and Step 8 of the proposed method, the problem, obtained in Step 4, can be converted into the following crisp non-linear programming problem:

Maximize $\frac{1}{4}(-w_1+3x_1-|-w_1-3x_1|+5x_2-|-3x_2|-\frac{1}{2}z_1+\frac{3}{2}y_1-|-\frac{1}{2}z_1-\frac{3}{2}y_1|+5y_2-|-y_2|-\frac{1}{2}y_1+\frac{3}{2}z_1+|-\frac{1}{2}y_1-\frac{3}{2}z_1|+5z_2+|-z_2|-x_1+3w_1+|-x_1-3w_1|+5w_2+|-3w_2|)$

subject to

$$-\tfrac{3}{2}w_1 + \tfrac{7}{2}x_1 - |-\tfrac{3}{2}w_1 - \tfrac{7}{2}x_1| - w_2 + \tfrac{7}{2}x_2 - |-w_2 - \tfrac{7}{2}x_2| = -42$$

$$\tfrac{7}{2}y_1 - \tfrac{3}{2}|y_1| - \tfrac{1}{2}z_2 + \tfrac{5}{2}y_2 - |-\tfrac{1}{2}z_2 - \tfrac{5}{2}y_2| = -17$$

$$\tfrac{7}{2}z_1 + \tfrac{3}{2}|z_1| - \tfrac{1}{2}y_2 + \tfrac{5}{2}z_2 + |-\tfrac{1}{2}y_2 - \tfrac{5}{2}z_2| = 45$$

$$\tfrac{7}{2}w_1 - \tfrac{3}{2}x_1 + |\tfrac{7}{2}w_1 + \tfrac{3}{2}x_1| - x_2 + \tfrac{7}{2}w_2 + |-x_2 - \tfrac{7}{2}w_2| = 91$$

$$\tfrac{5}{2}x_1 - \tfrac{3}{2}|x_1| - \tfrac{5}{2}w_2 - \tfrac{3}{2}|w_2| = -48$$

$$\tfrac{5}{2}y_1 - \tfrac{1}{2}|y_1| - \tfrac{5}{2}z_2 - \tfrac{1}{2}|z_2| = -27$$

$$\tfrac{5}{2}z_1 + \tfrac{1}{2}|z_1| - \tfrac{5}{2}y_2 + \tfrac{1}{2}|y_2| = -4$$

$$\tfrac{5}{2}w_1 + \tfrac{3}{2}|w_1| - \tfrac{5}{2}x_2 + \tfrac{3}{2}|x_2| = 28$$

$$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 6 The optimal solution of the crisp non-linear programming problem, obtained in Step 5, is $x_1 = -2, y_1 = -2, z_1 = 2, w_1 = 3, x_2 = -4, y_2 = 5, z_2 = 7$ and $w_2 = 10$.

Step 7 Putting the values of $x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2 in $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (-2, -2, 2, 3), \tilde{x}_2 = (-4, 5, 7, 10)$.

Step 8 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 7, in the objective function the fuzzy optimal value is $(-44, 14, 48, 98)$.

3.4.1.3 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 2.1

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 2.1, can also be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$ the fully fuzzy linear programming problem, chosen in Example 2.1, can be written as:

Maximize $((1, 2, 3, 4) \otimes (x_1, y_1, z_1, w_1) \oplus (2, 3, 4, 5) \otimes (x_2, y_2, z_2, w_2))$

subject to

$(0, 1, 2, 3) \otimes (x_1, y_1, z_1, w_1) \oplus (1, 2, 3, 4) \otimes (x_2, y_2, z_2, w_2) = (2, 10, 24, 44)$

$(1, 2, 3, 4) \otimes (x_1, y_1, z_1, w_1) \oplus (0, 1, 2, 3) \otimes (x_2, y_2, z_2, w_2) = (1, 8, 21, 40)$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are non-negative trapezoidal fuzzy numbers.

Step 2 Using the product, proposed in Section 3.2, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((\min\{x_1, 4x_1\},\min\{2y_1, 3y_1\},\max\{2z_1, 3z_1\},\max\{w_1, 4w_1\}) \oplus (\min\{2x_2, 5x_2\},\min\{3y_2, 4y_2\},\max\{3z_2, 4z_2\},\max\{2w_2, 5w_2\}))$

subject to

$(\min\{0x_1, 3x_1\},\min\{y_1, 2y_1\},\max\{z_1, 2z_1\},\max\{0w_1, 3w_1\})\oplus(\min\{x_2, 4x_2\},\min\{2y_2, 3y_2\},\max\{2z_2, 3z_2\},\max\{w_2, 4w_2\}) = (2, 10, 24, 44)$

$(\min\{x_1, 4x_1\},\min\{2y_1, 3y_1\},\max\{2z_1, 3z_1\},\max\{w_1, 4w_1\})\oplus(\min\{0x_2, 3x_2\},\min\{y_2, 2y_2\},\max\{z_2, 2z_2\},\max\{0w_2, 3w_2\}) = (1, 8, 21, 40)$

where (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) are non-negative trapezoidal fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 2.1.2.1 and Definition 2.12, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $(\min\{x_1,4x_1\}+\min\{2x_2,5x_2\},\min\{2y_1,3y_1\}+\min\{3y_2,4y_2\},\max\{2z_1,3z_1\}$

$+\max\{3z_2,4z_2\},\max\{w_1,4w_1\}+\max\{2w_2,5w_2\})$

subject to

$$\min\{0x_1,3x_1\}+\min\{x_2,4x_2\}=2$$

$$\min\{y_1,2y_1\}+\min\{2y_2,3y_2\}=10$$

$$\max\{z_1,2z_1\}+\max\{2z_2,3z_2\}=24$$

$$\max\{0w_1,3w_1\}+\max\{w_2,4w_2\}=44$$

$$\min\{x_1,4x_1\}+\min\{0x_2,3x_2\}=1$$

$$\min\{2y_1,3y_1\}+\min\{y_2,2y_2\}=8$$

$$\max\{2z_1,3z_1\}+\max\{z_2,2z_2\}=21$$

$$\max\{w_1,4w_1\}+\max\{0w_2,3w_2\}=40$$

$$x_1\geq 0, y_1-x_1\geq 0, z_1-y_1\geq 0, w_1-z_1\geq 0$$

$$x_2\geq 0, y_2-x_2\geq 0, z_2-y_2\geq 0, w_2-z_2\geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\Re(\min\{x_1,4x_1\}+\min\{2x_2,5x_2\},\min\{2y_1,3y_1\}+\min\{3y_2,4y_2\},\max\{2z_1,3z_1\}$

$+\max\{3z_2,4z_2\},\max\{w_1,4w_1\}+\max\{2w_2,5w_2\})$

subject to

$$\min\{0x_1,3x_1\}+\min\{x_2,4x_2\}=2$$

$$\min\{y_1,2y_1\}+\min\{2y_2,3y_2\}=10$$

$$\max\{z_1,2z_1\}+\max\{2z_2,3z_2\}=24$$

$$\max\{0w_1,3w_1\}+\max\{w_2,4w_2\}=44$$

$$\min\{x_1,4x_1\}+\min\{0x_2,3x_2\}=1$$

$$\min\{2y_1,3y_1\}+\min\{y_2,2y_2\}=8$$

$$\max\{2z_1, 3z_1\}+\max\{z_2, 2z_2\} = 21$$

$$\max\{w_1, 4w_1\}+\max\{0w_2, 3w_2\} = 40$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 5 Using Step 7 and Step 8 of the proposed method, the problem, obtained in Step 3, can be written as:

Maximize $\frac{1}{4}(\frac{5x_1}{2} - |-\frac{3x_1}{2}| + \frac{7x_2}{2} - |-\frac{3x_2}{2}| + \frac{5y_1}{2} - |-\frac{y_1}{2}| + \frac{7y_2}{2} - |-\frac{y_2}{2}| + \frac{5z_1}{2} + |-\frac{z_1}{2}| + \frac{7z_2}{2} + |-\frac{z_2}{2}| + \frac{5w_1}{2} + |-\frac{3w_1}{2}| + \frac{7z_2}{2} + |-\frac{3z_2}{2}|)$

subject to

$$\frac{3x_1}{2} - |-\frac{3x_1}{2}| + \frac{5x_2}{2} - |-\frac{3x_2}{2}| = 2$$

$$\frac{3y_1}{2} - |-\frac{y_1}{2}| + \frac{5y_2}{2} - |-\frac{y_2}{2}| = 10$$

$$\frac{3z_1}{2} + |-\frac{z_1}{2}| + \frac{5z_2}{2} + |-\frac{z_2}{2}| = 24$$

$$\frac{3w_1}{2} - |-\frac{3w_1}{2}| + \frac{5w_2}{2} + |-\frac{3w_2}{2}| = 44$$

$$\frac{5x_1}{2} - |-\frac{3x_1}{2}| + \frac{3x_2}{2} - |-\frac{3x_2}{2}| = 1$$

$$\frac{5y_1}{2} - |-\frac{y_1}{2}| + \frac{3y_2}{2} - |-\frac{y_2}{2}| = 8$$

$$\frac{3z_1}{2} + |-\frac{z_1}{2}| + \frac{3z_2}{2} + |-\frac{z_2}{2}| = 21$$

$$\frac{5w_1}{2} - |-\frac{3w_1}{2}| + \frac{3w_2}{2} - |-\frac{3w_2}{2}| = 40$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 6 Since, $x_1 \geq 0, y_1 \geq 0, z_1 \geq 0, w_1 \geq 0, x_2 \geq 0, y_2 \geq 0, z_2 \geq 0$ and $w_2 \geq 0$ so the problem, obtained in Step 5, can be written as:

Maximize $\frac{1}{4}(x_1 + 2x_2 + 2y_1 + 3y_2 + 3z_1 + 4z_2 + 4w_1 + 5w_2)$

subject to

$$0x_1 + x_2 = 2$$

$$x_1 + 0x_2 = 1$$

$$y_1 + 2y_2 = 10$$

$$2y_1 + y_2 = 8$$

$$2z_1 + 3z_2 = 24$$

$$3z_1 + 2z_2 = 21$$

$$3w_1 + 4w_2 = 44$$

$$4w_1 + 3w_2 = 40$$

$$x_1 \geq 0, y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, w_1 - z_1 \geq 0$$

$$x_2 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0, w_2 - z_2 \geq 0$$

Step 7 The optimal solution of the crisp linear programming problem, obtained in Step 6, is $x_1 = 1, y_1 = 2, z_1 = 3, w_1 = 4, x_2 = 2, y_2 = 4, z_2 = 6$ and $w_2 = 8$.

Step 8 Putting the values of $x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2 in $\tilde{x}_1 = (x_1, y_1, z_1, w_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2, w_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (1, 2, 3, 4), \tilde{x}_2 = (2, 4, 6, 8)$.

Step 9 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 8, in the objective function the fuzzy optimal value is $(5, 16, 33, 56)$.

3.5 Advantages of the proposed method

In this section, the advantages of proposed method over the method, proposed in Chapter 2, are discussed.

(i) The fully fuzzy linear programming problems $(P_{2.1})$ and $(P_{2.9})$ which can be solved by using the the method, proposed in Chapter 2, can also be solved by using the method proposed in this chapter and the obtained results are same.

(ii) The fully fuzzy linear programming problems $(P_{3.1})$ and $(P_{3.2})$ which cannot be solved by using the method, proposed in Chapter 2, can be solved by using the method proposed in this chapter.

3.6 Real life application of the proposed method

Kaur and Kumar [65] proposed a new method, based on tabular representation of transportation problems, to find the fuzzy optimal solution of uncapacitated fully fuzzy transportation problems and solved the real life uncapacitated fully fuzzy transportation problem, chosen in Section 3.6.1, to show the application of the proposed method.

In the existing method [65], it is assumed that the rank $(\frac{x_{ij}+2y_{ij}+z_{ij}}{4})$ of fuzzy variables $\tilde{x}_{ij} = (x_{ij}, y_{ij}, z_{ij})$, representing the optimal amount of the product that should be transported from i^{th} source to j^{th} destination, should be greater than or equal to zero. For several unrestricted triangular fuzzy numbers, the rank can be greater than or equal to zero e.g., the rank of unrestricted triangular fuzzy number $(-2, -1, 6)$ is positive. So, to find the fuzzy optimal solution of the chosen real life problem by using its fuzzy linear programming formulation, there is a need to solve a fully fuzzy linear programming problem with unrestricted fuzzy variables.

As discussed in Section 3.1, there is no method in the literature to solve fully fuzzy linear programming problems with unrestricted fuzzy variables. So, the real life problem, chosen in Section 3.6.1, cannot be solved by any of the existing methods. Although, the uncapacitated fully fuzzy transportation problem can be solved by using the tabular method [65]. However, there may exist several problems which can be solved only by using fuzzy linear programming approach e.g., there is no tabular method in the literature to solve the crisp capacitated minimal cost flow problems which can be extended to propose a new method for solving fully fuzzy capacitated minimal cost flow problems but such problems can be formulated as a fuzzy linear programming problem and then can be solved by using the proposed method.

In this section, to show the application of proposed method, the real life problem, chosen in Section 3.6.1, is solved by using the proposed method and it is concluded that the results, obtained by using the existing method [65] and proposed method are same.

3.6.1 Description of the problem

Dali Company is the leading producer of soft drinks and low-temperature foods in Taiwan. Currently, Dali plans to develop the South-East Asian market and broaden the visibility of Dali products in the Chinese market. Notably, following the entry of Taiwan to the World Trade Organization, Dali plans to seek strategic alliance with prominent international companies and introduced international bread to lighten the embedded future impact . In the domestic soft drinks market, Dali produces tea beverages to meet demand from four distribution centers in Taichung, Chiayi, Kaohsiung and Taipei, with production being based at three plants in Changhua, Touliu and Hsinchu. According to the preliminary environmental information, Table 3.1 summarizes the potential supply available from these three plants, the forecast demand from the four distribution centers and the unit transportation costs for each route used by Dali for the upcoming season.

The environmental coefficients and related parameters generally are imprecise numbers with triangular possibility distributions over the planning horizon due to incomplete or unobtainable information. For example, the available supply of the Changhua plant is $(\$7.2, \$8, \$8.8)$ thousand dozen bottles, the forecast demand of the Taichung distribution center is $(\$6.2, \$7, \$7.8)$ thousand dozen bottles and the transportation cost per dozen bottles from Changhua to Taichung is $(\$8, \$10, \$10.8)$. Due to transportation costs being a major expense, the management of Dali is initiating a study to reduce these costs as much as possible.

Table 3.1 Summarized data in the Dali case (in U.S. dollar)

Source	Destination				Supply(000 dozen bottles)
	Taichung	Chiayi	Kaohsiung	Taipei	
Changhua	$(\$8, \$10, \$10.8)$	$(\$20.4, \$22, \$24)$	$(\$8, \$10, \$10.6)$	$(\$18.8, \$20, \$22)$	$(7.2, 8, 8.8)$
Touliu	$(\$14, \$15, \$16)$	$(\$18.2, \$20, \$22)$	$(\$10, \$12, \$13)$	$(\$6, \$8, \$8.8)$	$(12, 14, 16)$
Hsinchu	$(\$18.4, \$20, \$21)$	$(\$9.6, \$12, \$13)$	$(\$7.8, \$10, \$10.8)$	$(\$14, \$15, \$16)$	$(10.2, 12, 13.8)$
Demand(000 dozen bottles)	$(6.2, 7, 7.8)$	$(8.9, 10, 11.1)$	$(6.5, 8, 9.5)$	$(7.8, 9, 10.2)$	

Solution : The chosen real life problem [65] can be formulated into the following fully fuzzy linear programming problem:

Minimize $((\$8, \$10, \$10.8) \otimes \tilde{x}_{11} \oplus (\$20.4, \$22, \$24) \otimes \tilde{x}_{12} \oplus (\$8, \$10, \$10.6) \otimes \tilde{x}_{13} \oplus (\$18.8, \$20, \$22) \otimes \tilde{x}_{14} \oplus (\$14, \$15, \$16) \otimes \tilde{x}_{21} \oplus (\$18.2, \$20, \$22) \otimes \tilde{x}_{22} \oplus (\$10, \$12, \$13) \otimes \tilde{x}_{23} \oplus (\$6, \$8, \$8.8) \otimes \tilde{x}_{24} \oplus (\$18.4, \$20, \$21) \otimes \tilde{x}_{31} \oplus (\$9.6, \$12, \$13) \otimes \tilde{x}_{32} \oplus (\$7.8, \$10, \$10.8) \otimes \tilde{x}_{33} \oplus (\$14, \$15, \$16) \otimes \tilde{x}_{34})$

subject to

$$\tilde{x}_{11} \oplus \tilde{x}_{12} \oplus \tilde{x}_{13} \oplus \tilde{x}_{14} = (7.2, 8, 8.8)$$

$$\tilde{x}_{21} \oplus \tilde{x}_{22} \oplus \tilde{x}_{23} \oplus \tilde{x}_{24} = (12, 14, 16)$$

$$\tilde{x}_{31} \oplus \tilde{x}_{32} \oplus \tilde{x}_{33} \oplus \tilde{x}_{34} = (10.2, 12, 13.8)$$

$$\tilde{x}_{11} \oplus \tilde{x}_{21} \oplus \tilde{x}_{31} = (6.2, 7, 7.8)$$

$$\tilde{x}_{12} \oplus \tilde{x}_{22} \oplus \tilde{x}_{32} = (8.9, 10, 11.1)$$

$$\tilde{x}_{13} \oplus \tilde{x}_{23} \oplus \tilde{x}_{33} = (6.5, 8, 9.5)$$

$$\tilde{x}_{14} \oplus \tilde{x}_{24} \oplus \tilde{x}_{34} = (7.8, 9, 10.2)$$

$$\tilde{x}_{ij} \succeq \tilde{0} \;\; \forall\, i = 1, ..., 3,\; j = 1, ..., 4$$

$\tilde{x}_{ij} \succeq \tilde{0} \Rightarrow \tilde{x}_{ij}$ is an unrestricted fuzzy number.

Assuming $\tilde{x}_{ij} = (x_{ij}, y_{ij}, z_{ij})$ and applying the proposed method, the obtained minimum total fuzzy transportation cost is $(\$\frac{20439}{100}, \$\frac{7047}{20}, \$\frac{45651}{100})$.

Remark 3.1 In all the existing methods [4,16,36,49,80,90,91] and also in the proposed method it is assumed that if a fuzzy number $\tilde{A}$ is the fuzzy optimal solution of a problem and if there exist any fuzzy number $\tilde{B}$ such that $\Re(\tilde{A}) = \Re(\tilde{B})$ then $\tilde{B}$ will also be a fuzzy optimal solution of the same problem. Although, the fuzzy numbers, representing the minimum total fuzzy transportation cost, for the real life problem, chosen in Section 3.6.1, obtained by using the existing method [65] and proposed methods are $(\$\frac{2796}{10}, \$352, \$382)$ and $(\$\frac{20439}{100}, \$\frac{7047}{20}, \$\frac{45651}{100})$ respectively but $\Re(\$\frac{2796}{10}, \$352, \$382) = \Re(\$\frac{20439}{100}, \$\frac{7047}{20}, \$\frac{45651}{100})$ which implies that there exist alternative fuzzy optimal solution of the chosen real life problem.

3.7 Comparative study

The results of the chosen fully fuzzy linear programming problems, obtained by using the method, proposed in Chapter 2, and the method proposed in this chapter, are shown in Table 3.2.

Table 3.2 Results of the chosen fully fuzzy linear programming problems

Example	Fuzzy optimal value	
	Method proposed in Chapter 2	Method proposed in this chapter
2.1	$(5, 16, 33, 56)$	$(5, 16, 33, 56)$
2.2	$(9, 27, 75, 111)$	$(9, 27, 75, 111)$
3.1	Not Applicable	$(-10, 2, \frac{831}{25}, 68)$
3.2	Not Applicable	$(-44, 14, 48, 98)$
Real life problem	Not Applicable	$(\$\frac{20439}{100}, \$\frac{7047}{20}, \$\frac{45651}{100})$

The results, presented in Table 3.2, can be explained as follows:

(i) In the problems, chosen in Example 2.1 and Example 2.2, all the decision variables are represented by non-negative trapezoidal fuzzy numbers. While, in the problems, chosen in Example 3.1, Example 3.2 and existing real life problem [65] all the decision variables are represented by unrestricted fuzzy numbers. So, as discussed in Section 3.1, the problems, chosen in Example 2.1 and Example 2.2, can be solved by using the method proposed in Chapter 2 but the problems, chosen in Example 3.1, Example 3.2 and existing real life problem [65], cannot be solved by using the method proposed in Chapter 2.

(ii) Since, the proposed method can be used to find the fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted fuzzy parameters. So, all the chosen problems as well as the existing real life problem [65] can be solved by using the proposed method.

3.8 Conclusions

On the basis of presented study, it can be concluded that it is better to use the method, proposed in this chapter, as compared to the method, proposed in previous chapter, for solving fully fuzzy linear programming problems with equality constraints.

Chapter 4

AN EFFICIENT METHOD FOR SOLVING FULLY FUZZY LINEAR PROGRAMMING PROBLEMS WITH EQUALITY CONSTRAINTS

In the literature [30] it is pointed out that there may exist several real life problems in which it is not always possible to represent all the parameters as triangular or trapezoidal fuzzy numbers and due to the same reason several authors [30,51,75,112,116] have represented the parameters as LR flat fuzzy numbers instead of triangular or trapezoidal fuzzy numbers.

To the best of our knowledge, till now no one have defined the product of such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. Due to non-existence of such product, till now there is no method in the literature for solving such fully fuzzy linear programming problems in which some or all the parameters are represented by such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. In this chapter, the product of such fuzzy numbers is proposed and also the limitations of the method, proposed in Chapter 3, are pointed out. To overcome the limitations of the method,

---

The contents of this chapter are communicated in *Applied Mathematical Modelling.*

proposed in Chapter 3, a new method is proposed to find the fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints.

4.1 Preliminaries

In this section, some basic definitions and arithmetic operations of LR flat fuzzy numbers are presented [30].

4.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 4.1 [30] A function $L : [0, \infty) \to [0, 1]$ (or $R : [0, \infty) \to [0, 1]$) is said to be reference function of fuzzy number if and only if

(i) $L(0) = 1$ (or $R(0) = 1$)

(ii) L (or R) is non-increasing on $[0, \infty)$.

Definition 4.2 [30] A fuzzy number $\widetilde{A}$, defined on universal set of real numbers $\mathbb{R}$, denoted as $(m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & x \le m,\ \alpha > 0 \\ R(\frac{x-n}{\beta}) & x \ge n,\ \beta > 0 \\ 1 & m \le x \le n \end{cases}$$

Definition 4.3 [30] Let $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ be an LR flat fuzzy number and λ be a real number in the interval $[0, 1]$. Then, the crisp set $A^{\lambda} = \{x \in X : \mu_{\widetilde{A}}(x) \ge \lambda\} = [m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)]$, is said to be λ-cut of $\widetilde{A}$.

Definition 4.4 [30] An LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be zero LR flat fuzzy number if and only if $m = 0, n = 0, \alpha = 0$ and $\beta = 0$.

Definition 4.5 [30] Two LR flat fuzzy numbers $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 =$

$(m_2, n_2, \alpha_2, \beta_2)_{LR}$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $m_1 = m_2, n_1 = n_2, \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Definition 4.6 [25] An LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be non-negative LR flat fuzzy number if and only if $m - \alpha \geq 0$ and is said to be non-positive LR flat fuzzy number if and only if $m - \alpha \leq 0$.

Definition 4.7 An LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be unrestricted LR flat fuzzy number if and only if $m - \alpha$ is a real number.

Remark 4.1 If $m = n$ then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be an LR fuzzy number and is denoted as $(m, m, \alpha, \beta)_{LR}$ or $(n, n, \alpha, \beta)_{LR}$ or $(m, \alpha, \beta)_{LR}$ or $(n, \alpha, \beta)_{LR}$.

Remark 4.2 If $m = n$ and $L(x) = R(x) = \max\{0, 1 - x\}$ then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number and is denoted as (a, b, c) where $a = m - \alpha, b = m(\text{or } n), c = m + \beta(\text{or } n + \beta)$.

Remark 4.3 If $m \neq n$ and $L(x) = R(x) = \max\{0, 1 - x\}$ then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a trapezoidal fuzzy number and is denoted as (a, b, c, d) where $a = m - \alpha, b = m, c = n, d = n + \beta$.

4.1.2 Arithmetic operations

In this section, the arithmetic operations between LR flat fuzzy numbers are presented [30].

Let $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$, $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be any LR flat fuzzy numbers and $\tilde{A}_3 = (m_3, n_3, \alpha_3, \beta_3)_{RL}$ be any RL flat fuzzy number. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$

(ii) $\tilde{A}_1 \ominus \tilde{A}_3 = (m_1 - n_3, n_1 - m_3, \alpha_1 + \beta_3, \beta_1 + \alpha_3)_{LR}$

(iii) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-negative, then

$$\tilde{A}_1 \otimes \tilde{A}_2 = (m_1m_2, n_1n_2, m_1\alpha_2 + \alpha_1m_2 - \alpha_1\alpha_2, n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2)_{LR}$$

(iv) If $\tilde{A}_1$ is non-positive and $\tilde{A}_2$ is non-negative, then

$$\tilde{A}_1 \otimes \tilde{A}_2 = (m_1n_2, n_1m_2, \alpha_1n_2 - m_1\beta_2 + \alpha_1\beta_2, \beta_1m_2 - n_1\alpha_2 - \beta_1\alpha_2)_{LR}$$

(v) If $\tilde{A}_1$ is non-negative and $\tilde{A}_2$ is non-positive, then

$$\tilde{A}_1 \otimes \tilde{A}_2 = (n_1m_2, m_1n_2, n_1\alpha_2 - \beta_1m_2 + \beta_1\alpha_2, m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2)_{LR}$$

(vi) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-positive, then

$$\tilde{A}_1 \otimes \tilde{A}_2 = (n_1n_2, m_1m_2, -n_1\beta_2 - \beta_1n_2 - \beta_1\beta_2, -m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2)_{LR}$$

(vii) $\lambda\tilde{A}_1 = \begin{cases} (\lambda m_1, \lambda n_1, \lambda\alpha_1, \lambda\beta_1)_{LR} & \lambda \geq 0 \\ (\lambda n_1, \lambda m_1, -\lambda\beta_1, -\lambda\alpha_1)_{RL} & \lambda \leq 0 \end{cases}$

There also exist another formula [30] for the product of such LR flat fuzzy numbers in which the spreads $\alpha_1, \alpha_2, \beta_1$ and β_2 are smaller as compared to the mean values m_1 and m_2:

(i) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-negative, then

$$\tilde{A}_1 \odot \tilde{A}_2 = (m_1m_2, n_1n_2, m_1\alpha_2 + \alpha_1m_2, n_1\beta_2 + \beta_1n_2)_{LR}$$

(ii) If $\tilde{A}_1$ is non-positive and $\tilde{A}_2$ is non-negative, then

$$\tilde{A}_1 \odot \tilde{A}_2 = (m_1n_2, n_1m_2, \alpha_1n_2 - m_1\beta_2, \beta_1m_2 - n_1\alpha_2)_{LR}$$

(iii) If $\tilde{A}_1$ is non-negative and $\tilde{A}_2$ is non-positive, then

$$\tilde{A}_1 \odot \tilde{A}_2 = (n_1m_2, m_1n_2, n_1\alpha_2 - \beta_1m_2, m_1\beta_2 - \alpha_1n_2)_{LR}$$

(iv) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-positive, then

$$\tilde{A}_1 \odot \tilde{A}_2 = (n_1n_2, m_1m_2, -n_1\beta_2 - \beta_1n_2, -m_1\alpha_2 - \alpha_1m_2)_{LR}$$

Remark 4.4 Using Yager's ranking approach [141], the value of Yager's ranking index $\Re(\tilde{A})$ for any parameter, represented by LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is as follows:

(i) If $L(x) = R(x) = \max\{0, 1-x\}$ then $\Re(\tilde{A}) = \frac{1}{2}(m+n) + \frac{1}{4}(\beta - \alpha)$

(ii) If $L(x) = \max\{0, 1-x\}$ and $R(x) = \max\{0, 1-x^2\}$ then $\Re(\tilde{A}) = \frac{1}{2}(m+n) + \frac{1}{3}\beta - \frac{1}{4}\alpha$

(iii) If $L(x) = \max\{0, 1-x^2\}$ and $R(x) = \max\{0, 1-x\}$ then $\Re(\tilde{A}) = \frac{1}{2}(m+n) + \frac{1}{4}\beta - \frac{1}{3}\alpha$

To find the ranking index for LR fuzzy numbers put $m = n$ in the all of the above obtained formulas of LR flat fuzzy numbers.

4.2 Proposed product

The existing product rule, presented in Section 4.1.2 can be used to find the product of such LR fuzzy numbers or LR flat fuzzy numbers which are either non-negative or non-positive. To the best of our knowledge, till now no one have defined the product of such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive e.g., if $\tilde{A}_1 = (1, 3, 4, 2)_{LR}$ and $\tilde{A}_2 = (2, 4, 5, 3)_{LR}$ are two LR flat fuzzy numbers then there is no product rule to find the value of $\tilde{A}_1 \otimes \tilde{A}_2$ or $\tilde{A}_1 \odot \tilde{A}_2$. Due to non-existence of such product, till now there is no method in the literature for solving such fully fuzzy linear programming problems in which some or all the parameters are represented by such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive. In this section, corresponding to the existing product rules, presented in Section 4.1.2, new product rules are introduced.

4.2.1 New product corresponding to the existing product $\otimes$

In this section, new product corresponding to the existing product $\otimes$, is introduced.

Proposition 4.1 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 - \alpha_1 < 0$ and $m_1 \geq 0$ then $\tilde{A}_1 \otimes \tilde{A}_2 = (m_1', n_1', \alpha_1', \beta_1')_{LR}$, where $m_1' = \min\{m_1m_2, n_1m_2\}, n_1' = \max\{m_1n_2, n_1n_2\}, \alpha_1' = \min\{m_1m_2, n_1m_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_1' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - \max\{m_1n_2, n_1n_2\}$.

Proof: Let $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers such that $m_1 - \alpha_1 < 0$ and $m_1 \geq 0$. Then, using the Definition 4.3, $A_1^\lambda = [m_1 - \alpha_1 L^{-1}(\lambda), n_1 + \beta_1 R^{-1}(\lambda)]$ and $A_2^\lambda = [m_2 - \alpha_2 L^{-1}(\lambda), n_2 + \beta_2 R^{-1}(\lambda)]$. Since $m_1 - \alpha_1 < 0$ and $m_1 \geq 0$ so $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ for $\lambda \geq L(\frac{m_1}{\alpha_1})$ and $m_1 - \alpha_1 L^{-1}(\lambda) \geq 0$ for $\lambda \leq L(\frac{m_1}{\alpha_1})$ and $n_1 + \beta_1 R^{-1}(\lambda) \geq 0$ for all λ, so to find the product of $\tilde{A}_1$ and $\tilde{A}_2$ there is a need to consider the following five cases:

Case (i) If $m_2 - \alpha_2 \geq 0$ then $m_2 - \alpha_2 L^{-1}(\lambda) \geq 0$ and $n_2 + \beta_2 R^{-1}(\lambda) \geq 0$ for all λ so the following two subcases may arise to find the product of A_1^λ and A_2^λ:

(a) If $m_1 - \alpha_1 L^{-1}(\lambda) \geq 0$ then

$$A_1^\lambda A_2^\lambda = [(m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$$

Putting $\lambda = 1$,

$$A_1^\lambda A_2^\lambda = [m_1m_2, n_1n_2]. \tag{4.1}$$

(b) If $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ then

$$A_1^\lambda A_2^\lambda = [(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$$

Putting $\lambda = 0$,

$$A_1^\lambda A_2^\lambda = [m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2]. \tag{4.2}$$

Now combining (4.1) and (4.2):

$\tilde{A}_1 \otimes \tilde{A}_2 = (m_1'', n_1'', \alpha_1'', \beta_1'')_{LR}$

where $m_1'' = m_1m_2, n_1'' = n_1n_2, \alpha_1'' = m_1m_2 - m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, \beta_1'' =$

$n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2 - n_1n_2$.

Case (ii) If $m_2 - \alpha_2 < 0, m_2 \geq 0$ then $m_2 - \alpha_2L^{-1}(\lambda) \geq 0$ for $\lambda \leq L(\frac{m_2}{\alpha_2})$, $m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ for $\lambda \geq L(\frac{m_2}{\alpha_2})$ and $n_2 + \beta_2R^{-1}(\lambda) \geq 0$ for all λ so, the four subcases may arise to find the product of A_1^λ and A_2^λ. Since, the aim is to find the product of A_1^λ and A_2^λ corresponding to $\lambda = 0$ and $\lambda = 1$ so there is a need to consider only the following two subcases:

(a) If $m_1 - \alpha_1L^{-1}(\lambda) \leq 0$ and $m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ then

$A_1^\lambda A_2^\lambda = [\min\{(m_1 - \alpha_1L^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda))\}, \max\{(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))\}]$

Putting $\lambda = 0$,

$$A_1^\lambda A_2^\lambda = [\min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\}]. \tag{4.3}$$

(b) If $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ and $m_2 - \alpha_2L^{-1}(\lambda) \geq 0$ then

$A_1^\lambda A_2^\lambda = [(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))]$

Putting $\lambda = 1$,

$$A_1^\lambda A_2^\lambda = [m_1m_2, n_1n_2]. \tag{4.4}$$

Now combining (4.3) and (4.4):

$\tilde{A}_1 \otimes \tilde{A}_2 = (m_2'', n_2'', \alpha_2'', \beta_2'')_{LR}$

where $m_2'' = m_1m_2, n_2'' = n_1n_2, \alpha_2'' = m_1m_2 - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2 - m_1m_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_2'' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - n_1n_2$.

Case (iii) If $m_2 < 0, n_2 \geq 0$ then $m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ and $n_2 + \beta_2R^{-1}(\lambda) \geq 0$ for all λ so the following two subcases may arise to find the product of A_1^λ and

A_2^λ:

(a) If $m_1 - \alpha_1 L^{-1}(\lambda) \geq 0$ then

$A_1^\lambda A_2^\lambda = [(n_1+\beta_1 R^{-1}(\lambda))(m_2-\alpha_2 L^{-1}(\lambda)), (n_1+\beta_1 R^{-1}(\lambda))(n_2+\beta_2 R^{-1}(\lambda))]$

Putting $\lambda = 1$,

$$A_1^\lambda A_2^\lambda = [n_1 m_2, n_1 n_2]. \tag{4.5}$$

(b) If $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ then

$A_1^\lambda A_2^\lambda = [\min\{(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))\}, \max\{(m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))\}]$

Putting $\lambda = 0$,

$$A_1^\lambda A_2^\lambda = [\min\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2, n_1 m_2 + \beta_1 m_2 - n_1\alpha_2 - \beta_1\alpha_2\}, \max\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\}]. \tag{4.6}$$

Now combining (4.5) and (4.6):

$\tilde{A}_1 \otimes \tilde{A}_2 = (m_3'', n_3'', \alpha_3'', \beta_3'')_{LR}$

where $m_3'' = n_1 m_2, n_3'' = n_1 n_2, \alpha_3'' = n_1 m_2 - \min\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2, n_1 m_2 + \beta_1 m_2 - n_1\alpha_2 - \beta_1\alpha_2\}, \beta_3'' = \max\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\} - n_1 n_2$.

Case (iv) If $n_2 < 0, n_2 + \beta_2 \geq 0$ then $m_2 - \alpha_2 L^{-1}(\lambda) \leq 0$ for all λ and $n_2 + \beta_2 R^{-1}(\lambda) \leq 0$ for $\lambda \leq R(-\frac{n_2}{\beta_2})$, $n_2 + \beta_2 R^{-1}(\lambda) \geq 0$ for $\lambda \geq R(-\frac{n_2}{\beta_2})$ so the four subcases may arise to find the product of A_1^λ and A_2^λ. Since, the aim is to find the product of A_1^λ and A_1^λ corresponding to $\lambda = 0$ and $\lambda = 1$ so there is a need to consider only the following two subcases:

(a) If $m_1 - \alpha_1 L^{-1}(\lambda) \geq 0$ and $n_2 + \beta_2 R^{-1}(\lambda) \leq 0$ then

$A_1^\lambda A_2^\lambda = [(n_1+\beta_1 R^{-1}(\lambda))(m_2-\alpha_2 L^{-1}(\lambda)), (m_1-\alpha_1 L^{-1}(\lambda))(n_2+\beta_2 R^{-1}(\lambda))]$

Putting $\lambda = 1$,

$$A_1^\lambda A_2^\lambda = [n_1m_2, m_1n_2]. \tag{4.7}$$

(b) If $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ and $n_2 + \beta_2 R^{-1}(\lambda) \geq 0$ then

$A_1^\lambda A_2^\lambda = [\min\{(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))\}, \max\{(m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

Putting $\lambda = 0$,

$$A_1^\lambda A_2^\lambda = [\min\{m_1n_2 + m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2\}, \max\{ m_1m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\}]. \tag{4.8}$$

Now combining (4.7) and (4.8):

$\tilde{A}_1 \otimes \tilde{A}_2 = (m_4'', n_4'', \alpha_4'', \beta_4'')_{LR}$

where $m_4'' = n_1m_2, n_4'' = m_1n_2, \alpha_4'' = n_1m_2 - \min\{m_1n_2 + m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2\}, \beta_4'' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\} - m_1n_2$.

Case(v) If $n_2 + \beta_2 < 0$ then $m_2 - \alpha_2 L^{-1}(\lambda) \leq 0$ and $n_2 + \beta_2 R^{-1}(\lambda) \leq 0$ for all λ so the following two subcases may arise:

(a) If $m_1 - \alpha_1 L^{-1}(\lambda) \geq 0$ then

$A_1^\lambda A_2^\lambda = [(n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

Putting $\lambda = 1$,

$$A_1^\lambda A_2^\lambda = [n_1m_2, m_1n_2]. \tag{4.9}$$

(b) If $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ then

$A_1^\lambda A_2^\lambda = [(n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))]$

Putting $\lambda = 0$,

$$A_1^\lambda A_2^\lambda = [n_1m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2, m_1m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2]. \tag{4.10}$$

Now combining (4.9) and (4.10):

$\tilde{A}_1 \otimes \tilde{A}_2 = (m_5'', n_5'', \alpha_5'', \beta_5'')_{LR}$

where $m_5'' = n_1m_2, n_5'' = m_1n_2, \alpha_5'' = n_1m_2 - n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2, \beta_5'' = m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2 - m_1n_2$.

Combining the results of all five cases the following result is obtained:

If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 - \alpha_1 < 0, m_1 \geq 0$ and $\tilde{A}_2$ is any LR flat fuzzy number, then

$\tilde{A}_1 \otimes \tilde{A}_2 = (m_1', n_1', \alpha_1', \beta_1')_{LR}$,

where $m_1' = \min\{m_1m_2, n_1m_2\}, n_1' = \max\{m_1n_2, n_1n_2\}, \alpha_1' = \min\{m_1m_2, n_1m_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_1' = \mathrm{ma}\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - \max\{m_1n_2, n_1n_2\}$.

Proposition 4.2 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 < 0$ and $n_1 \geq 0$ then $\tilde{A}_1 \otimes \tilde{A}_2 = (m_2', n_2', \alpha_2', \beta_2')_{LR}$, where $m_2' = \min\{m_1n_2, n_1m_2\}, n_2' = \max\{m_1m_2, n_1n_2\}, \alpha_2' = \min\{m_1n_2, n_1m_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_2' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - \max\{m_1m_2, n_1n_2\}$.

Proof: Similar to Proposition 4.1.

Proposition 4.3 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $n_1 < 0$ and $n_1 + \beta_1 \geq 0$ then $\tilde{A}_1 \otimes \tilde{A}_2 = (m_3', n_3', \alpha_3', \beta_3')_{LR}$, where $m_3' = \min\{m_1n_2, n_1n_2\}, n_3' = \max\{n_1m_2, m_1m_2\}, \alpha_3' = \min\{m_1n_2, n_1n_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_3' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - \max\{n_1m_2, m_1m_2\}$.

Proof: Similar to Proposition 4.1.

Proposition 4.4 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $n_1 + \beta_1 < 0$ then $\tilde{A}_1 \otimes \tilde{A}_2 = (m'_4, n'_4, \alpha'_4, \beta'_4)_{LR}$, where $m'_4 = \min\{m_1n_2, n_1n_2\}, n'_4 = \max\{m_1m_2, n_1m_2\}, \alpha'_4 = \min\{m_1n_2, n_1n_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\}, \beta'_4 = \max\{n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2, m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2\} - \max\{m_1m_2, n_1m_2\}$.

Proof: Similar to Proposition 4.1.

Proposition 4.5 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 - \alpha_1 \geq 0$ then $\tilde{A}_1 \otimes \tilde{A}_2 = (m'_5, n'_5, \alpha'_5, \beta'_5)_{LR}$, where $m'_5 = \min\{m_1m_2, n_1m_2\}, n'_5 = \max\{m_1n_2, n_1n_2\}, \alpha'_5 = \min\{m_1m_2, n_1m_2\} - \min\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta'_5 = \max\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - \max\{m_1n_2, n_1n_2\}$.

Proof: Similar to Proposition 4.1.

4.2.2 New product corresponding to the existing product $\odot$

In this section, new product corresponding to the existing product $\odot$, is introduced.

Proposition 4.6 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 - \alpha_1 \geq 0$ then $\tilde{A}_1 \odot \tilde{A}_2 = (m'_1, n'_1, \alpha'_1, \beta'_1)_{LR}$, where $m'_1 = \min\{m_1m_2, n_1m_2\}, n'_1 = \max\{m_1n_2, n_1n_2\}, \alpha'_1 = \min\{m_1m_2, n_1m_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \beta'_1 = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\} - \max\{m_1n_2, n_1n_2\}$.

Proof: The proposed results may be obtained by considering the following five cases:

Case (i) Neglecting the terms $\alpha_1\beta_2$ and $\beta_1\beta_2$ from the results obtained in Case (i) of Proposition 4.1, $A_1^\lambda A_2^\lambda = [m_1m_2, n_1n_2]$ for $\lambda = 1$ and $A_1^\lambda A_2^\lambda = [m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + n_1\beta_2 + \beta_1n_2]$ for $\lambda = 0$. Combining the both,

$\tilde{A}_1 \odot \tilde{A}_2 = (m_1'', n_1'', \alpha_1'', \beta_1'')_{LR}$

where $m_1'' = m_1m_2, n_1'' = n_1n_2, \alpha_1'' = m_1m_2 - m_1n_2 + m_1\beta_2 - \alpha_1n_2, \beta_1'' = n_1n_2 + n_1\beta_2 + \beta_1n_2 - n_1n_2$.

Case (ii) Neglecting the terms $\alpha_1\beta_2$, $\beta_1\alpha_2$, $\alpha_1\alpha_2$ and $\beta_1\beta_2$ from the results obtained in Case (ii) of Proposition 4.1, $A_1^\lambda A_2^\lambda = [\min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\}]$ for $\lambda = 0$ and $A_1^\lambda A_2^\lambda = [m_1m_2, n_1n_2]$ for $\lambda = 1$. Combining the both,

$\tilde{A}_1 \odot \tilde{A}_2 = (m_2'', n_2'', \alpha_2'', \beta_2'')_{LR}$

where $m_2'' = m_1m_2, n_2'' = n_1n_2, \alpha_2'' = m_1m_2 - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - m_1m_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \beta_2'' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\} - n_1n_2$.

Case (iii) Neglecting the terms $\alpha_1\beta_2$, $\beta_1\alpha_2$, $\alpha_1\alpha_2$ and $\beta_1\beta_2$ from the results obtained in Case (iii) of Proposition 4.1, $A_1^\lambda A_2^\lambda = [n_1m_2, n_1n_2]$ for $\lambda = 1$ and $A_1^\lambda A_2^\lambda = [\min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 + \beta_1m_2 - n_1\alpha_2\}, \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\}]$ for $\lambda = 0$. Combining the both,

$\tilde{A}_1 \odot \tilde{A}_2 = (m_3'', n_3'', \alpha_3'', \beta_3'')_{LR}$

where $m_3'' = n_1m_2, n_3'' = n_1n_2, \alpha_3'' = n_1m_2 - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 + \beta_1m_2 - n_1\alpha_2\}, \beta_3'' = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\} - n_1n_2$.

Case (iv) Neglecting the terms $\alpha_1\beta_2$, $\beta_1\alpha_2$, $\alpha_1\alpha_2$ and $\beta_1\beta_2$ from the results obtained in Case (iv) of Proposition 4.1, $A_1^\lambda A_2^\lambda = [n_1m_2, m_1n_2]$ for $\lambda = 1$ and $A_1^\lambda A_2^\lambda = [\min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \max\{m_1m_2 - m_1\alpha_2 -

$\alpha_1 m_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2\}]$ for $\lambda = 0$. Combining the both,

$\tilde{A}_1 \odot \tilde{A}_2 = (m''_4, n''_4, \alpha''_4, \beta''_4)_{LR}$

where $m''_4 = n_1 m_2, n''_4 = m_1 n_2, \alpha''_4 = n_1 m_2 - \min\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2, n_1 m_2 - n_1\alpha_2 + \beta_1 m_2\}, \beta''_4 = \max\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2\} - m_1 n_2$.

Case (v) Neglecting the terms $\beta_1\alpha_2$ and $\alpha_1\alpha_2$ from the results obtained in Case (v) of Proposition 4.1, $A_1^\lambda A_2^\lambda = [n_1 m_2, m_1 n_2]$ for $\lambda = 1$ and $A_1^\lambda A_2^\lambda = [n_1 m_2 - n_1\alpha_2 + \beta_1 m_2, m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2]$ for $\lambda = 0$. Combining the both,

$\tilde{A}_1 \odot \tilde{A}_2 = (m''_5, n''_5, \alpha''_5, \beta''_5)_{LR}$

where $m''_5 = n_1 m_2, n''_5 = m_1 n_2, \alpha''_5 = n_1 m_2 - n_1 m_2 - n_1\alpha_2 + \beta_1 m_2, \beta''_5 = m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 - m_1 n_2$.

Combining the results of all five cases the following result is obtained:

If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 - \alpha_1 < 0, m_1 \geq 0$ and $\tilde{A}_2$ is any LR flat fuzzy number, then

$\tilde{A}_1 \odot \tilde{A}_2 = (m'_1, n'_1, \alpha'_1, \beta'_1)_{LR}$

where $m'_1 = \min\{m_1 m_2, n_1 m_2\}, n'_1 = \max\{m_1 n_2, n_1 n_2\}, \alpha'_1 = \min\{m_1 m_2, n_1 m_2\} - \min\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2, n_1 m_2 - n_1\alpha_2 + \beta_1 m_2\}, \beta'_1 = \max\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2\} - \max\{m_1 n_2, n_1 n_2\}$.

Proposition 4.7 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 < 0$ and $n_1 \geq 0$ then $\tilde{A}_1 \odot \tilde{A}_2 = (m'_2, n'_2, \alpha'_2, \beta'_2)_{LR}$, where $m'_2 = \min\{m_1 n_2, n_1 m_2\}, n'_2 = \max\{m_1 m_2, n_1 n_2\}, \alpha'_2 = \min\{m_1 n_2, n_1 m_2\} - \min\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2, n_1 m_2 - n_1\alpha_2 + \beta_1 m_2\}, \beta'_2 = \max\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2\} - \max\{m_1 m_2, n_1 n_2\}$.

Proof: Similar to Proposition 4.6.

Proposition 4.8 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $n_1 < 0$ and $n_1 + \beta_1 \geq 0$ then $\tilde{A}_1 \odot \tilde{A}_2 = (m'_3, n'_3, \alpha'_3, \beta'_3)_{LR}$, where $m'_3 = \min\{m_1n_2, n_1n_2\}, n'_3 = \max\{n_1m_2, m_1m_2\}, \alpha'_3 = \min\{m_1n_2, n_1n_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \beta'_3 = \max\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\} - \max\{n_1m_2, m_1m_2\}$.

Proof: Similar to Proposition 4.6.

Proposition 4.9 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $n_1 + \beta_1 < 0$ then $\tilde{A}_1 \odot \tilde{A}_2 = (m'_4, n'_4, \alpha'_4, \beta'_4)_{LR}$, where $m'_4 = \min\{m_1n_2, n_1n_2\}, n'_4 = \max\{m_1m_2, n_1m_2\}, \alpha'_4 = \min\{m_1n_2, n_1n_2\} - \min\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\}, \beta'_4 = \max\{n_1m_2 - n_1\alpha_2 + \beta_1m_2, m_1m_2 - m_1\alpha_2 - \alpha_1m_2\} - \max\{m_1m_2, n_1m_2\}$.

Proof: Similar to Proposition 4.6.

Proposition 4.10 If $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are two LR flat fuzzy numbers such that $m_1 - \alpha_1 \geq 0$ then $\tilde{A}_1 \odot \tilde{A}_2 = (m'_5, n'_5, \alpha'_5, \beta'_5)_{LR}$, where $m'_5 = \min\{m_1m_2, n_1m_2\}, n'_5 = \max\{m_1n_2, n_1n_2\}, \alpha'_5 = \min\{m_1m_2, n_1m_2\} - \min\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \beta'_5 = \max\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\} - \max\{m_1n_2, n_1n_2\}$.

Proof: Similar to Proposition 4.6.

4.3 Limitations of previous proposed method

The method, proposed in Chapter 3, can be used only for solving such fully fuzzy linear programming problems with equality constraints in which all the parameters are either represented by triangular fuzzy numbers or trapezoidal fuzzy numbers. However, the same method cannot be used for solving fully fuzzy linear programming problems ($P_{4.1}$) and ($P_{4.2}$) in which the parameters are represented by LR fuzzy numbers or LR flat fuzzy numbers:

$$\begin{aligned} &\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{4.1}) \\ &\qquad \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i \;\; \forall\, i = 1, 2, ..., m \end{aligned}$$

where $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, y_j, \alpha''_j, \beta''_j)_{LR}$, $\tilde{b}_i = (b_i, g_i, \gamma_i, \delta_i)_{LR}$ and $\tilde{c}_j = (p_j, q_j, \alpha'_j, \beta'_j)_{LR}$ are LR flat fuzzy numbers.

Example 4.1 Maximize $((-1, 3, 1, 3)_{LR} \otimes \tilde{x}_1 \oplus (4, 6, 2, 2)_{LR} \otimes \tilde{x}_2)$

subject to

$$(2, 5, 5, 2)_{LR} \otimes \tilde{x}_1 \oplus (-1, 5, 1, 2)_{LR} \otimes \tilde{x}_2 = (-17, 45, 25, 46)_{LR}$$

$$(2, 3, 1, 1)_{LR} \otimes \tilde{x}_1 \oplus (-3, -2, 1, 1)_{LR} \otimes \tilde{x}_2 = (-27, -4, 21, 32)_{LR}$$

where $\tilde{x}_1, \tilde{x}_2$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1 - x\}$, $R(x) = \max\{0, 1 - x^2\}$.

$$\begin{aligned} &\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{4.2}) \\ &\qquad \sum_{j=1}^{n} \tilde{a}_{ij} \odot \tilde{x}_j = \tilde{b}_i \;\; \forall\, i = 1, 2, ..., m \end{aligned}$$

where $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, y_j, \alpha''_j, \beta''_j)_{LR}$, $\tilde{b}_i = (b_i, g_i, \gamma_i, \delta_i)_{LR}$ and $\tilde{c}_j = (p_j, q_j, \alpha'_j, \beta'_j)_{LR}$ are LR flat fuzzy numbers.

Example 4.2 Maximize $((1, 2, 2, 2)_{LR} \odot \tilde{x}_1 \oplus (1, 1, 1, 1)_{LR} \odot \tilde{x}_2)$

subject to

$$(2, 4, 6, 2)_{LR} \odot \tilde{x}_2 = (2, 8, 10, 4)_{LR}$$

$$(2, 3, 1, 1)_{LR} \odot \tilde{x}_1 \oplus (-3, -2, 1, 1)_{LR} \odot \tilde{x}_2 = (-2, 7, 6, 7)_{LR}$$

where $\tilde{x}_1, \tilde{x}_2$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1 - x^2\}$, $R(x) =$

$\max\{0, 1-x\}$.

4.4 Proposed method for solving fully fuzzy linear programming problems with equality constraints having LR flat fuzzy numbers

In this section, to overcome the limitations of the method, proposed in Chapter 3, a new method is proposed for solving fully fuzzy linear programming problems ($P_{4.1}$) in which all the parameters are represented as LR flat fuzzy numbers. The same method can also be used to find the fuzzy optimal solution of the proposed fully fuzzy linear programming problems ($P_{4.2}$) by replacing $\otimes$ by $\odot$.

The steps of the proposed method for solving fully fuzzy linear programming problem ($P_{4.1}$) are as follows:

Step 1 Assuming $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, y_j, \alpha''_j, \beta''_j)_{LR}$, $\tilde{b}_i = (b_i, g_i, \gamma_i, \delta_i)_{LR}$ and $\tilde{c}_j = (p_j, q_j, \alpha'_j, \beta'_j)_{LR}$ the fully fuzzy linear programming problem ($P_{4.1}$) can be written as:

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n}((p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR})\\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{4.3})\\
&\sum_{j=1}^{n}(a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR} = (b_i, g_i, \gamma_i, \delta_i)_{LR} \quad \forall\ i = 1, 2, ..., m
\end{aligned}$$

where $(x_j, y_j, \alpha''_j, \beta''_j)_{LR}$ is an LR flat fuzzy number.

Step 2 Using the product, proposed in Section 4.2.1 and assuming $(a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ $\otimes$ $(x_j, y_j, \alpha''_j, \beta''_j)_{LR} = (m_{ij}, n_{ij}, \gamma'_{ij}, \delta'_{ij})_{LR}$ the fully fuzzy linear programming problem ($P_{4.3}$) can be written as:

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n}((p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR})\\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{4.4})
\end{aligned}$$

$$\sum_{j=1}^{n}(m_{ij},n_{ij},\gamma'_{ij},\delta'_{ij})_{LR}=(b_i,g_i,\gamma_i,\delta_i)_{LR}\ \ \forall\ i=1,2,...,m$$

where $(x_j,y_j,\alpha''_j,\beta''_j)_{LR}$ is an LR flat fuzzy number.

Step 3 Using arithmetic operations, defined in Section 4.1.2 and Definition 4.5, the fully fuzzy linear programming problem $(P_{4.4})$ can be written as:

$$\text{Maximize/Minimize } \sum_{j=1}^{n}((p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j,y_j,\alpha''_j,\beta''_j)_{LR})$$

subject to

$$\begin{aligned}&\sum_{j=1}^{n}m_{ij}=b_i\ \ \forall\ i=1,2,...,m\\&\sum_{j=1}^{n}n_{ij}=g_i\ \ \forall\ i=1,2,...,m\\&\sum_{j=1}^{n}\gamma'_{ij}=\gamma_i\ \ \forall\ i=1,2,...,m\\&\sum_{j=1}^{n}\delta'_{ij}=\delta_i\ \ \forall\ i=1,2,...,m\\&x_j\le y_j,\alpha''_j\ge 0,\beta''_j\ge 0\ \ \forall\ j=1,2,...,n\end{aligned}\qquad (P_{4.5})$$

Step 4 Suppose the fuzzy linear programming problem $(P_{4.5})$ have 'l' basic feasible solutions and $\{x_j^t,y_j^t,\alpha_j^{''t},\beta_j^{''t}\}$ is the t^{th} basic feasible solution then our aim is to find that basic feasible solution out of all 'l' basic feasible solutions corresponding to which the value of objective function is maximum (or minimum) i.e., our aim is to find $\max\limits_{1\le t\le l}$ (or min)$\{\sum\limits_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j^t,y_j^t,\alpha_j^{''t},\beta_j^{''t})_{LR}\}$.

Yager [141] has proposed the concept that if $\max\limits_{1\le t\le l}$ (or min)$\{\text{Rank}(\sum\limits_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j^t,y_j^t,\alpha_j^{''t},\beta_j^{''t})_{LR})\}$ is $\text{Rank}(\sum\limits_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j^\theta,y_j^\theta,\alpha_j^{''\theta},\beta_j^{''\theta})_{LR})$ then $\max\limits_{1\le t\le l}$ (or min)$\{\sum\limits_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j^t,y_j^t,\alpha_j^{''t},\beta_j^{''t})_{LR}\}$ will also be $\sum\limits_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j^\theta,y_j^\theta,\alpha_j^{''\theta},\beta_j^{''\theta})_{LR})$, i.e. according to the existing method [141] the fuzzy optimal solution of $(P_{4.5})$ can be obtained by solving the problem $(P_{4.6})$:

$$\text{Maximize/Minimize } \Re(\sum_{j=1}^{n}((p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j,y_j,\alpha''_j,\beta''_j)_{LR}))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} m_{ij} = b_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} n_{ij} = g_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} \gamma'_{ij} = \gamma_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} \delta'_{ij} = \delta_i \ \ \forall\ i = 1,2,...,m\\
&x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0 \ \ \forall\ j = 1,2,...,n
\end{aligned} \qquad (P_{4.6})$$

Step 5 Assuming $(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR} = (s_j, t_j, \alpha'''_j, \beta'''_j)_{LR}$ the problem $(P_{4.6})$ can be written as:

Maximize/Minimize $\Re(\sum_{j=1}^{n}(s_j, t_j, \alpha'''_j, \beta'''_j)_{LR})$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} m_{ij} = b_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} n_{ij} = g_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} \gamma'_{ij} = \gamma_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} \delta'_{ij} = \delta_i \ \ \forall\ i = 1,2,...,m\\
&x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0 \ \ \forall\ j = 1,2,...,n
\end{aligned} \qquad (P_{4.7})$$

Step 6 Using the linearity property $\Re(\sum_{j=1}^{n} \tilde{A}_i) = \sum_{j=1}^{n} \Re(\tilde{A}_i)$, where $\tilde{A}_i$ is a fuzzy number, problem $(P_{4.7})$ can be converted into $(P_{4.8})$:

Maximize/Minimize $\sum_{j=1}^{n} \Re(s_j, t_j, \alpha'''_j, \beta'''_j)_{LR}$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} m_{ij} = b_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} n_{ij} = g_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} \gamma'_{ij} = \gamma_i \ \ \forall\ i = 1,2,...,m\\
&\sum_{j=1}^{n} \delta'_{ij} = \delta_i \ \ \forall\ i = 1,2,...,m\\
&x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0 \ \ \forall\ j = 1,2,...,n
\end{aligned} \qquad (P_{4.8})$$

Step 7 Using the existing formula [141] $\Re(m,n,\alpha,\beta)=\frac{1}{2}(\int_0^1(m-\alpha L^{-1}(\lambda))\;d\lambda+\int_0^1(n+\beta R^{-1}(\lambda))\;d\lambda)$ problem $(P_{4.8})$ can be converted into $(P_{4.9})$:

Maximize/Minimize $\sum\limits_{j=1}^{n}(\frac{1}{2}(\int_0^1(s_j-\alpha_j'''L^{-1}(\lambda))\;d\lambda+\int_0^1(t_j+\beta_j'''R^{-1}(\lambda))\;d\lambda))$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} m_{ij}=b_i \;\;\forall\; i=1,2,...,m\\
&\sum_{j=1}^{n} n_{ij}=g_i \;\;\forall\; i=1,2,...,m\\
&\sum_{j=1}^{n} \gamma'_{ij}=\gamma_i \;\;\forall\; i=1,2,...,m\\
&\sum_{j=1}^{n} \delta'_{ij}=\delta_i \;\;\forall\; i=1,2,...,m\\
&x_j\le y_j, \alpha_j''\ge 0, \beta_j''\ge 0 \;\;\forall\; j=1,2,...,n
\end{aligned} \qquad (P_{4.9})$$

Step 8 Solve the crisp non-linear programming problem $(P_{4.9})$ by using an appropriate existing method [125] to find the optimal solution $\{x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*}\}$.

Step 9 Find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ of the fully fuzzy linear programming problem $(P_{4.1})$ by putting the values of $x_j^*, y_j^*, \alpha_j^{''*}$ and $\beta_j^{''*}$ in $\tilde{x}_j^*=(x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*})$.

Step 10 Find the fuzzy optimal value of the fully fuzzy linear programming problems $(P_{4.1})$ by putting the values of $\tilde{x}_j^*$, obtained from Step 9, in $\sum\limits_{j=1}^{n}\tilde{c}_j\otimes\tilde{x}_j^*$.

4.5 Illustrative examples

In this section, the proposed method is illustrated with the help of fully fuzzy linear programming problems, chosen in Example 4.1 and Example 4.2, which cannot be solved by using the method, proposed in Chapter 3.

4.5.1 Fuzzy optimal solution of the chosen fully fuzzy linear programming problems

In this section, fully fuzzy linear programming problems, chosen in Example 4.1 and Example 4.2, are solved by using the proposed method.

4.5.1.1 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 4.1

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 4.1, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{LR}$ the fully fuzzy linear programming problem, chosen in Example 4.1, can be written as:

Maximize $((-1, 3, 1, 3)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (4, 6, 2, 2)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR})$

subject to

$(2, 5, 5, 2)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (-1, 5, 1, 2)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR} = (-17, 45, 25, 46)_{LR}$

$(2, 3, 1, 1)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (-3, -2, 1, 1)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR} = (-27, -4, 21, 32)_{LR}$

where $(x_1, y_1, \alpha_1, \beta_1)_{LR}, (x_2, y_2, \alpha_2, \beta_2)_{LR}$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1 - x\}$, $R(x) = \max\{0, 1 - x^2\}$.

Step 2 Using the product, proposed in Section 4.2.1, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((\min\{-y_1, 3x_1\}, \max\{-x_1, 3y_1\}, \min\{-y_1, 3x_1\} - \min\{-2y_1 - 2\beta_1, 6x_1 - 6\alpha_1\}, \max\{-2x_1 + 2\alpha_1, 6y_1 + 6\beta_1\} - \max\{-x_1, 3y_1\})_{LR} \oplus (\min\{4x_2, 6x_2\}, \max\{4y_2, 6y_2\}, \min\{4x_2, 6x_2\} - \min\{2x_2 - 2\alpha_2, 8x_2 - 8\alpha_2\}, \max\{2y_2 + 2\beta_2, 8y_2 + 8\beta_2\} - \max\{4y_2, 6y_2\})_{LR})$

subject to

$(\min\{2x_1, 5x_1\}, \max\{2y_1, 5y_1\}, \min\{2x_1, 5x_1\} - \min\{-3y_1 - 3\beta_1, 7x_1 - 7\alpha_1\}, \max\{7y_1 + 7\beta_1, -3x_1 + 3\alpha_1\} - \max\{2y_1, 5y_1\})_{LR} \oplus (\min\{-y_2, 5x_2\}, \max\{-x_2, 5y_2\}, \min\{-y_2, 5x_2\} - \min\{-2y_2 - 2\beta_2, 7x_2 - 7\alpha_2\}, \max\{-2x_2 + 2\alpha_2, 7y_2 + 7\beta_2\} - \max\{-x_2, 5y_2\})_{LR} = (-17, 45, 25, 46)_{LR}$

$(\min\{2x_1, 3x_1\}, \max\{2y_1, 3y_1\}, \min\{2x_1, 3x_1\} - \min\{x_1 - \alpha_1, 4x_1 - 4\alpha_1\}, \max\{y_1 + \beta_1, 4y_1 + 4\beta_1\} - \max\{2y_1, 3y_1\})_{LR} \oplus (\min\{-3y_2, -2y_2\}, \max\{-3x_2, -2x_2\}, \min\{-3y_2, -2y_2\}$

$-\min\{-4y_2-4\beta_2,-y_2-\beta_2\}$,$\max\{-4x_2+4\alpha_2,-x_2+\alpha_2\}-\max\{-3x_2,-2x_2\})_{LR} =$ $(-27,-4,21,32)_{LR}$

where $(x_1,y_1,\alpha_1,\beta_1)_{LR}$ and $(x_2,y_2,\alpha_2,\beta_2)_{LR}$ are LR flat fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 4.1.2 and Definition 4.5, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $(\min\{-y_1,3x_1\}+\min\{4x_2,6x_2\}$,$\max\{-x_1,3y_1\}+\max\{4y_2,6y_2\}$,$\min\{-y_1,3x_1\}-\min\{-2y_1-2\beta_1,6x_1-6\alpha_1\}+\min\{4x_2,6x_2\}-\min\{2x_2-2\alpha_2,8x_2-8\alpha_2\}$,$\max\{-2x_1+2\alpha_1,6y_1+6\beta_1\}-\max\{-x_1,3y_1\}+\max\{2y_2+2\beta_2,8y_2+8\beta_2\}-\max\{4y_2,6y_2\})_{LR}$

subject to

$$\min\{2x_1,5x_1\}+\min\{-y_2,5x_2\}=-17$$

$$\max\{2y_1,5y_1\}+\max\{-x_2,5y_2\}=45$$

$$\min\{2x_1,3x_1\}+\min\{-3y_2,-2y_2\}=-27$$

$$\max\{2y_1,3y_1\}+\max\{-3x_2,-2x_2\}=-4$$

$$\min\{2x_1,3x_1\}-\min\{x_1-\alpha_1,4x_1-4\alpha_1\}+$$
$$\min\{-3y_2,-2y_2\}-\min\{-4y_2-4\beta_2,-y_2-\beta_2\}=21$$

$$\max\{y_1+\beta_1,4y_1+4\beta_1\}-\max\{2y_1,3y_1\})_{LR}+$$
$$\max\{-4x_2+4\alpha_2,-x_2+\alpha_2\}-\max\{-3x_2,-2x_2\}=32$$

$$\min\{2x_1,5x_1\}-\min\{-3y_1-3\beta_1,7x_1-7\alpha_1\}+$$
$$\min\{-y_2,5x_2\}-\min\{-2y_2-2\beta_2,7x_2-7\alpha_2\}=25$$

$$\max\{7y_1+7\beta_1,-3x_1+3\alpha_1\}-\max\{2y_1,5y_1\}+$$
$$\max\{-2x_2+2\alpha_2,7y_2+7\beta_2\}-\max\{-x_2,5y_2\})_{LR}=46$$

$$x_1\leq y_1,\alpha_1\geq 0,\beta_1\geq 0,x_2\leq y_2,\alpha_2\geq 0,\beta_2\geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\Re(\min\{-y_1, 3x_1\}+\min\{4x_2, 6x_2\},\max\{-x_1, 3y_1\}+\max\{4y_2, 6y_2\},\min\{-y_1,$ $3x_1\}-\min\{-2y_1-2\beta_1, 6x_1-6\alpha_1\}+\min\{4x_2, 6x_2\}-\min\{2x_2-2\alpha_2, 8x_2-8\alpha_2\},\max\{-2x_1$ $+ 2\alpha_1, 6y_1 + 6\beta_1\}-\max\{-x_1, 3y_1\}+\max\{2y_2 + 2\beta_2, 8y_2 + 8\beta_2\}-\max\{4y_2, 6y_2\})_{LR}$

subject to

$$\min\{2x_1, 5x_1\}+\min\{-y_2, 5x_2\} = -17$$

$$\max\{2y_1, 5y_1\}+\max\{-x_2, 5y_2\} = 45$$

$$\min\{2x_1, 3x_1\}+\min\{-3y_2, -2y_2\} = -27$$

$$\max\{2y_1, 3y_1\}+\max\{-3x_2, -2x_2\} = -4$$

$$\min\{2x_1, 3x_1\}-\min\{x_1 - \alpha_1, 4x_1 - 4\alpha_1\} +$$

$$\min\{-3y_2, -2y_2\}-\min\{-4y_2 - 4\beta_2, -y_2 - \beta_2\} = 21$$

$$\max\{y_1 + \beta_1, 4y_1 + 4\beta_1\}-\max\{2y_1, 3y_1\})_{LR} +$$

$$\max\{-4x_2 + 4\alpha_2, -x_2 + \alpha_2\}-\max\{-3x_2, -2x_2\} = 32$$

$$\min\{2x_1, 5x_1\}-\min\{-3y_1 - 3\beta_1, 7x_1 - 7\alpha_1\} +$$

$$\min\{-y_2, 5x_2\}-\min\{-2y_2 - 2\beta_2, 7x_2 - 7\alpha_2\} = 25$$

$$\max\{7y_1 + 7\beta_1, -3x_1 + 3\alpha_1\}-\max\{2y_1, 5y_1\} +$$

$$\max\{-2x_2 + 2\alpha_2, 7y_2 + 7\beta_2\}-\max\{-x_2, 5y_2\})_{LR} = 46$$

$$x_1 \le y_1, \alpha_1 \ge 0, \beta_1 \ge 0, x_2 \le y_2, \alpha_2 \ge 0, \beta_2 \ge 0$$

Step 5 Using Step 6 and Step 7 of the proposed method, the fuzzy linear programming problem, obtained in Step 4, can be written as:

Maximize $(\frac{17}{24}x_1 - \frac{5}{12}\alpha_1 + \frac{5}{2}x_2 - \frac{5}{4}\alpha_2 + \frac{7}{8}y_1 + \frac{3}{4}\beta_1 + \frac{5}{2}y_2 + \frac{5}{3}\beta_2 - \frac{1}{8}|y_1 + 3x_1| + \frac{1}{12}|x_1 + 3y_1| +$ $\frac{1}{3}|3y_1 + 3\beta_1 + x_1 - \alpha_1| - \frac{1}{4}|y_1 + \beta_1 + 3x_1 - 3\alpha_1| - \frac{1}{4}|x_2| + \frac{1}{6}|y_2| + |y_2 + \beta_2| - \frac{3}{4}|x_2 - \alpha_2|)$

subject to

$$\frac{7}{2}x_1 - \frac{3}{2}|x_1| - \frac{1}{2}y_2 + \frac{5}{2}x_2 - \frac{1}{2}|y_2 + 5x_2| = -17$$

$$\frac{7}{2}y_1 + \frac{3}{2}|y_1| - \frac{1}{2}x_2 + \frac{5}{2}y_2 + \frac{1}{2}|x_2 + 5y_2| = 45$$

$$\tfrac{5}{2}x_1 - \tfrac{1}{2}|x_1| - \tfrac{5}{2}y_2 - \tfrac{1}{2}|y_2| = -27$$

$$\tfrac{5}{2}y_1 + \tfrac{1}{2}|y_1| - \tfrac{5}{2}x_2 + \tfrac{1}{2}|x_2| = -4$$

$$-\tfrac{1}{2}|x_1| - \tfrac{1}{2}|y_2| + \tfrac{5}{2}\alpha_1 + \tfrac{3}{2}|x_1 - \alpha_1| + \tfrac{5}{2}\beta_2 + \tfrac{3}{2}|y_2 + \beta_2| = 21$$

$$\tfrac{5}{2}\beta_1 + \tfrac{3}{2}|y_1 + \beta_1| + \tfrac{5}{2}\alpha_2 + \tfrac{3}{2}|x_2 - \alpha_2| - \tfrac{1}{2}|y_1| - \tfrac{1}{2}|x_2| = 32$$

$$-\tfrac{3}{2}|x_1| + \tfrac{1}{2}y_2 - x_2 - \tfrac{1}{2}|5x_2 + y_2| + \tfrac{3}{2}y_1 + \tfrac{3}{2}\beta_1 + \tfrac{7}{2}\alpha_1 +$$

$$\tfrac{1}{2}|7x_1 + 3y_1 - 7\alpha_1 + 3\beta_1| + \tfrac{7}{2}\alpha_2 + \beta_2 + |\tfrac{7}{2}x_2 + y_2 - \tfrac{7}{2}\alpha_2 + \beta_2| = 25$$

$$\tfrac{7}{2}\beta_1 - \tfrac{3}{2}x_1 + \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}|3x_1 + 7y_1 - 3\alpha_1 + 7\beta_1| - \tfrac{1}{2}x_2 + \alpha_2 + y_2 +$$

$$\tfrac{7}{2}\beta_2 + |x_2 + \tfrac{7}{2}y_2 - \alpha_2 + \tfrac{7}{2}\beta_2| - \tfrac{3}{2}|y_1| - \tfrac{1}{2}|x_2 + 5y_2| = 46$$

$$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

Step 6 The optimal solution of the crisp non-linear programming problem, obtained in Step 5, is $x_1 = -2, y_1 = 2, \alpha_1 = 0, \beta_1 = 1, x_2 = 5, y_2 = 7, \alpha_2 = 9$ and $\beta_2 = 3$.

Step 7 Putting the values of $x_1, y_1, \alpha_1, \beta_1, x_2, y_2, \alpha_2$ and β_2 in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (-2, 2, 0, 1)_{LR}, \tilde{x}_2 = (5, 7, 9, 3)_{LR}$.

Step 8 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 7, in the objective function the fuzzy optimal value is $(14, 48, 58, 50)_{LR}$.

4.5.1.2 Fuzzy optimal solution of the fully fuzzy linear programming problem chosen in Example 4.2

The fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 4.2, can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{LR}$ the fully fuzzy linear programming problem, chosen in Example 4.1, can be written as:

Maximize $((1, 2, 2, 2)_{LR} \odot (x_1, y_1, \alpha_1, \beta_1) \oplus (1, 1, 1, 1)_{LR} \odot (x_2, y_2, \alpha_2, \beta_2))$

subject to

$(2,4,6,2)_{LR} \odot (x_2,y_2,\alpha_2,\beta_2) = (2,8,10,4)_{LR}$

$(2,3,1,1)_{LR} \odot (x_1,y_1,\alpha_1,\beta_1) \oplus (-3,-2,1,1)_{LR} \odot (x_2,y_2,\alpha_2,\beta_2) = (-2,7,6,7)_{LR}$

where $(x_1,y_1,\alpha_1,\beta_1)_{LR}, (x_2,y_2,\alpha_2,\beta_2)_{LR}$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1-x^2\}$, $R(x) = \max\{0, 1-x\}$.

Step 2 Using the product, proposed in Section 4.2.2, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((\min\{x_1,2x_1\},\max\{y_1,2y_1\},\min\{x_1,2x_1\}-\min\{-y_1+\beta_1,4x_1-2\alpha_1\},\max\{-x_1-\alpha_1,4y_1+2\beta_1\}-\max\{y_1,2y_1\})_{LR}\oplus(\min\{x_2,x_2\},\max\{y_2,y_2\},\min\{x_2,x_2\}-\min\{-\alpha_2,2x_2-\alpha_2\},\max\{\beta_2,2y_2+\beta_2\}-\max\{y_2,y_2\})_{LR})$

subject to

$(\min\{2x_2,4x_2\},\max\{2y_2,4y_2\},\min\{2x_2,4x_2\}-\min\{-4y_2+2\beta_2,6x_2-4\alpha_2\},\max\{-4x_2-2\alpha_2,6y_2+4\beta_2\}-\max\{2y_2,4y_2\})_{LR} = (2,8,10,4)_{LR}$

$(\min\{2x_1,3x_1\},\max\{2y_1,3y_1\},\min\{2x_1,3x_1\}-\min\{x_1-2\alpha_1,4x_1-3\alpha_1\},\max\{y_1+2\beta_1,4y_1+3\beta_1\}-\max\{2y_1,3y_1\})_{LR}\oplus(\min\{-3y_2,-2y_2\},\max\{-3x_2,-2x_2\},\min\{-3y_2,-2y_2\}-\min\{-4y_2-3\beta_2,-y_2-2\beta_2\},\max\{-4x_2+3\alpha_2,-x_2+2\alpha_2\}-\max\{-3x_2,-2x_2\})_{LR} = (-2,7,6,7)_{LR}$

where $(x_1,y_1,\alpha_1,\beta_1)_{LR}$ and $(x_2,y_2,\alpha_2,\beta_2)_{LR}$ are LR flat fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 4.1.2 and Definition 4.5, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $(\min\{x_1,2x_1\}+\min\{x_2,x_2\},\max\{y_1,2y_1\}+\max\{y_2,y_2\},\min\{x_1,2x_1\}-\min\{-y_1+\beta_1,4x_1-2\alpha_1\}+\min\{x_2,x_2\}-\min\{-\alpha_2,2x_2-\alpha_2\},\max\{-x_1-\alpha_1,4y_1+2\beta_1\}-\max\{y_1,2y_1\}+\max\{\beta_2,2y_2+\beta_2\}-\max\{y_2,y_2\})_{LR}$

subject to

$$\min\{2x_2, 4x_2\} = 2$$

$$\max\{2y_2, 4y_2\} = 8$$

$$\min\{2x_2, 4x_2\} - \min\{-4y_2 + 2\beta_2, 6x_2 - 4\alpha_2\} = 10$$

$$\max\{-4x_2 - 2\alpha_2, 6y_2 + 4\beta_2\} - \max\{2y_2, 4y_2\} = 4$$

$$\min\{2x_1, 3x_1\} + \min\{-3y_2, -2y_2\} = -2$$

$$\max\{2y_1, 3y_1\} + \max\{-3x_2, -2x_2\} = 7$$

$$\min\{2x_1, 3x_1\} - \min\{x_1 - 2\alpha_1, 4x_1 - 3\alpha_1\} +$$

$$\min\{-3y_2, -2y_2\} - \min\{-4y_2 - 3\beta_2, -y_2 - 2\beta_2\} = 6$$

$$\max\{y_1 + 2\beta_1, 4y_1 + 3\beta_1\} - \max\{2y_1, 3y_1\} +$$

$$\max\{-4x_2 + 3\alpha_2, -x_2 + 2\alpha_2\} - \max\{-3x_2, -2x_2\} = 7$$

$$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

Step 4 Using Step 4 of the proposed method, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\Re(\min\{x_1, 2x_1\} + \min\{x_2, x_2\}, \max\{y_1, 2y_1\} + \max\{y_2, y_2\}, \min\{x_1, 2x_1\} - \min\{-y_1 + \beta_1, 4x_1 - 2\alpha_1\} + \min\{x_2, x_2\} - \min\{-\alpha_2, 2x_2 - \alpha_2\}, \max\{-x_1 - \alpha_1, 4y_1 + 2\beta_1\} - \max\{y_1, 2y_1\} + \max\{\beta_2, 2y_2 + \beta_2\} - \max\{y_2, y_2\})_{LR}$

subject to

$$\min\{2x_2, 4x_2\} = 2$$

$$\max\{2y_2, 4y_2\} = 8$$

$$\min\{2x_2, 4x_2\} - \min\{-4y_2 + 2\beta_2, 6x_2 - 4\alpha_2\} = 10$$

$$\max\{-4x_2 - 2\alpha_2, 6y_2 + 4\beta_2\} - \max\{2y_2, 4y_2\} = 4$$

$$\min\{2x_1, 3x_1\} + \min\{-3y_2, -2y_2\} = -2$$

$$\max\{2y_1, 3y_1\} + \max\{-3x_2, -2x_2\} = 7$$

$$\min\{2x_1, 3x_1\} - \min\{x_1 - 2\alpha_1, 4x_1 - 3\alpha_1\} +$$

$$\min\{-3y_2, -2y_2\} - \min\{-4y_2 - 3\beta_2, -y_2 - 2\beta_2\} = 6$$

$$\max\{y_1 + 2\beta_1, 4y_1 + 3\beta_1\} - \max\{2y_1, 3y_1\} +$$

$$\max\{-4x_2 + 3\alpha_2, -x_2 + 2\alpha_2\} - \max\{-3x_2, -2x_2\} = 7$$

$$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

Step 5 Using Step 6 and Step 7 of the proposed method, the fuzzy linear programming problem, obtained in Step 4, can be written as:

Maximize $(\frac{19}{24}x_1 + \frac{17}{24}y_1 - \frac{11}{24}\alpha_1 + \frac{5}{12}\beta_1 + \frac{1}{2}x_2 + \frac{1}{2}y_2 - \frac{1}{3}\alpha_2 + \frac{1}{4}\beta_2 - \frac{1}{12}|x_1| + \frac{1}{8}|y_1| - \frac{1}{3}|x_2| + \frac{1}{4}|y_2| + \frac{1}{4}|2y_1 + \beta_1 + \frac{1}{2}x_1 + \frac{1}{2}\alpha_1| - \frac{1}{3}|2x_1 - \alpha_1 + \frac{1}{2}y_1 - \frac{1}{2}\beta_1|)$

subject to

$$3x_2 - |x_2| = 2$$

$$3y_2 + |y_2| = 8$$

$$-|x_2| + 2y_2 - \beta_2 + 2\alpha_2 + |2y_2 - \beta_2 + 3x_2 - 2\alpha_2| = 10$$

$$-2x_2 - \alpha_2 + 2\beta_2 + |3y_2 + 2\beta_2 + 2x_2 + \alpha_2| - |y_2| = 4$$

$$\frac{5}{2}x_1 - \frac{1}{2}|x_1| - \frac{5}{2}y_2 - \frac{1}{2}|y_2| = -2$$

$$\frac{5}{2}y_1 + \frac{1}{2}|y_1| - \frac{5}{2}x_2 + \frac{1}{2}|x_2| = 7$$

$$\frac{5}{2}\alpha_1 - \frac{1}{2}|x_1| + \frac{1}{2}|3x_1 - \alpha_1| - \frac{1}{2}|y_2| + \frac{5}{2}\beta_2 + \frac{1}{2}|3y_2 + \beta_2| = 6$$

$$\frac{5}{2}\beta_1 + \frac{1}{2}|3y_1 + \beta_1| - \frac{1}{2}|y_1| + \frac{5}{2}\alpha_2 + \frac{1}{2}|3x_2 - \alpha_2| - \frac{1}{2}|x_2| = 7$$

$$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

Step 6 The optimal solution of the crisp non-linear programming problem, obtained in Step 5, is $x_1 = 2, y_1 = 3, \alpha_1 = 1, \beta_1 = 1, x_2 = 1, y_2 = 2, \alpha_2 = 0$ and $\beta_2 = 0$.

Step 7 Putting the values of $x_1, y_1, \alpha_1, \beta_1, x_2, y_2, \alpha_2$ and β_2 in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (2, 3, 1, 1)_{LR}, \tilde{x}_2 = (1, 2, 0, 0)_{LR}$.

Step 8 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 7, in the objective function the fuzzy optimal value is $(3, 8, 5, 10)_{LR}$.

4.6 Advantages of the proposed method

The main advantage of the proposed method is that all the fully fuzzy linear programming problems which can be solved by using the methods, proposed in previous chapters, can also be solved by using the method proposed in this chapter. However, there exist several fully fuzzy linear programming problems which cannot be solved by using the methods, proposed in previous chapters, but can be solved by using the method, proposed in this chapter.

4.7 Comparative study

The results of the chosen fully fuzzy linear programming problems, obtained by using the method, proposed in Chapter 3 and the method proposed in this chapter, are shown in Table 4.1.

Table 4.1. Results of the chosen fully fuzzy linear programming problems

Example	Fuzzy optimal value	
	Method proposed in Chapter 3	Method proposed in this chapter
3.1	$(-10, 2, \frac{831}{25}, 68)$	$(2, \frac{831}{25}, 12, \frac{869}{25})_{LR}$
3.2	$(-44, 14, 48, 98)$	$(14, 48, 58, 50)_{LR}$
4.1	Not Applicable	$(14, 48, 58, 50)_{LR}$
4.2	Not Applicable	$(3, 8, 5, 10)_{LR}$

The results, presented in Table 4.1 can be explained as follows:

(i) In the problems, chosen in Example 3.1 and Example 3.2, all the parameters are represented by trapezoidal fuzzy numbers while in the problems, chosen in Example 4.1 and Example 4.2, all the parameters are represented by unrestricted LR flat fuzzy numbers. So, as discussed in Section 4.3 the problems, chosen in Example 3.1 and Example 3.2, can be solved by using the method, proposed in Chapter 3. However, the problems, chosen in Example 4.1 and Example 4.2, cannot be solved by using the method, proposed in Chapter 3.

(ii) Since, the method proposed in this chapter can be used to find the fuzzy optimal solution of fully fuzzy linear programming problems with unrestricted *LR* flat fuzzy parameters. So, all the chosen problems can be solved by using the method proposed in this chapter.

4.8 Conclusions

On the basis of presented study, it can be concluded that the method, proposed in this chapter, can be used to solve all the fully fuzzy linear programming problems with equality constraints. Hence, it is better to use the method, proposed in this chapter, as compared to the methods, proposed in previous chapters for solving fully fuzzy linear programming problems with equality constraints.

Chapter 5

New Methods For Solving Fully Fuzzy Linear Programming Problems With Inequality Constraints

In this chapter, limitations of the existing methods for solving fuzzy linear programming problems and fully fuzzy linear programming problems with inequality constraints are pointed out. To overcome the limitations of the existing methods, two new methods are proposed for solving fully fuzzy linear programming problems with inequality constraints. The advantages of the proposed methods over the existing methods are also discussed.

5.1 Existing method for solving fully fuzzy linear programming problems with inequality constraints

Allahviranloo et al. [4] proposed the following method for solving fully fuzzy linear programming problems ($P_{5.1}$):

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j$$

$$\text{subject to} \qquad (P_{5.1})$$

The contents of this chapter are communicated in *Journal of Intelligent and Fuzzy Systems.*

$$\sum_{j=1}^{n} \tilde{a}_{ij} \odot \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i \quad \forall\ i = 1, 2, ..., m$$

where $\tilde{x}_j$ is a non-negative LR fuzzy number and $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ are non-negative or non-positive LR fuzzy numbers.

Step 1 Assuming $N_1 = \{j\ :\ 1 \leq j \leq n$ and $\tilde{c}_j$ is non-negative LR fuzzy number$\}$, $N_2 = \{j\ :\ 1 \leq j \leq n$ and $\tilde{c}_j$ is non-positive LR fuzzy number$\}$, $N_3 = \{j\ :\ 1 \leq j \leq n$ and $\tilde{a}_{ij}$ is non-negative LR fuzzy number$\}$, $N_4 = \{j\ :\ 1 \leq j \leq n$ and $\tilde{a}_{ij}$ is non-positive LR fuzzy number$\}$ where $N_1 \cup N_2 = N, N_3 \cup N_4 = N, N_1 \cap N_2 = \phi, N_3 \cap N_4 = \phi$, the fully fuzzy linear programming problem $(P_{5.1})$ can be written as:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j:j\in N_1} \tilde{c}_j \odot \tilde{x}_j \oplus \sum_{j:j\in N_2} \tilde{c}_j \odot \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_{5.2}) \\ &\sum_{j:j\in N_3} \tilde{a}_{ij} \odot \tilde{x}_j \oplus \sum_{j:j\in N_4} \tilde{a}_{ij} \odot \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i \quad \forall\ i = 1, 2, ..., m \end{aligned}$$

where $\tilde{x}_j$ is a non-negative LR fuzzy number.

Step 2 Assuming $\tilde{c}_j = (p_j, \alpha'_j, \beta'_j)_{LR}, \tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}, \tilde{x}_j = (x_j, \alpha''_j, \beta''_j)_{LR}$ and $\tilde{b}_i = (b_i, \gamma_i, \delta_i)_{LR}$ the fully fuzzy linear programming problem $(P_{5.2})$ can be written as:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j\in N_1} (p_j, \alpha'_j, \beta'_j)_{LR} \odot (x_j, \alpha''_j, \beta''_j)_{LR} \oplus \sum_{j\in N_2} (p_j, \alpha'_j, \beta'_j)_{LR} \odot (x_j, \alpha''_j, \beta''_j)_{LR} \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (P_{5.3}) \\ &\sum_{j\in N_3} (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR} \odot (x_j, \alpha''_j, \beta''_j)_{LR} \oplus \sum_{j\in N_4} (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR} \odot (x_j, \alpha''_j, \beta''_j)_{LR} \preceq, \approx, \succeq \\ &(b_i, \gamma_i, \delta_i)_{LR} \quad \forall\ i = 1, 2, ..., m \end{aligned}$$

where $\tilde{x}_j$ is a non-negative LR fuzzy number.

Step 3 Using the arithmetic operations, presented in Section 4.1.2, the fully fuzzy linear programming problem $(P_{5.3})$ can be written as:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j:j\in N_1} (p_j x_j, p_j \alpha''_j + \alpha'_j x_j, p_j \beta''_j + \beta'_j x_j)_{LR} \oplus \sum_{j:j\in N_2} (p_j x_j, \alpha'_j x_j - \\ &p_j \alpha''_j, \beta'_j x_j - p_j \beta''_j)_{LR} \end{aligned}$$

subject to $\quad (P_{5.4})$

$\sum\limits_{j:j\in N_3}(a_{ij}x_j, a_{ij}\alpha_j''+\alpha_{ij}x_j, a_{ij}\beta_j''+\beta_{ij}x_j)_{LR}\oplus \sum\limits_{j:j\in N_4}(a_{ij}x_j, \alpha_{ij}x_j-a_{ij}\alpha_j'', \beta_{ij}x_j-a_{ij}\beta_j'')_{LR} \preceq, \approx, \succeq (b_i, \gamma_i, \delta_i)_{LR} \ \ \forall\ i = 1, 2, ..., m$

where $\tilde{x}_j$ is a non-negative LR fuzzy number.

Step 4 Using the existing ranking approach [141], the fully fuzzy linear programming problem $(P_{5.4})$ can be written as:

Maximize/Minimize $\Re(\sum\limits_{j:j\in N_1}(p_jx_j, p_j\alpha_j'' + \alpha_j'x_j, p_j\beta_j'' + \beta_j'x_j)_{LR} \oplus \sum\limits_{j:j\in N_2}(p_jx_j, \alpha_j'x_j - p_j\alpha_j'', \beta_j'x_j - p_j\beta_j'')_{LR})$

subject to $\quad (P_{5.5})$

$\Re(\sum\limits_{j:j\in N_3}(a_{ij}x_j, a_{ij}\alpha_j'' + \alpha_{ij}x_j, a_{ij}\beta_j'' + \beta_{ij}x_j)_{LR} \oplus \sum\limits_{j:j\in N_4}(a_{ij}x_j, \alpha_{ij}x_j - a_{ij}\alpha_j'', \beta_{ij}x_j - a_{ij}\beta_j'')_{LR}) \leq, =, \geq \Re(b_i, \gamma_i, \delta_i)_{LR} \ \ \forall\ i = 1, 2, ..., m$

$$x_j - \alpha_j'' \geq 0, \alpha_j'' \geq 0, \beta_j'' \geq 0 \ \ \forall\ j \in N$$

Step 5 Using the linearity property $\Re(\sum\limits_{j=1}^{n}\tilde{A}_i) = \sum\limits_{j=1}^{n}\Re(\tilde{A}_i)$, where $\tilde{A}_i$ is a fuzzy number, the fully fuzzy linear programming problem $(P_{5.5})$ can be written as:

Maximize/Minimize $\sum\limits_{j:j\in N_1}\Re(p_jx_j, p_j\alpha_j''+\alpha_j'x_j, p_j\beta_j''+\beta_j'x_j)_{LR}+\sum\limits_{j:j\in N_2}\Re(p_jx_j, \alpha_j'x_j - p_j\alpha_j'', \beta_j'x_j - p_j\beta_j'')_{LR}$

subject to $\quad (P_{5.6})$

$\sum\limits_{j:j\in N_3}\Re(a_{ij}x_j, a_{ij}\alpha_j'' + \alpha_{ij}x_j, a_{ij}\beta_j'' + \beta_{ij}x_j)_{LR} + \sum\limits_{j:j\in N_4}\Re(a_{ij}x_j, \alpha_{ij}x_j - a_{ij}\alpha_j'', \beta_{ij}x_j - a_{ij}\beta_j'')_{LR} \leq, =, \geq \Re(b_i, \gamma_i, \delta_i)_{LR} \ \ \forall\ i = 1, 2, ..., m$

$$x_j - \alpha_j'' \geq 0, \alpha_j'' \geq 0, \beta_j'' \geq 0 \ \ \forall\ j \in N$$

Step 6 Solve the crisp linear programming problem $(P_{5.6})$ to find the optimal solution $\{x_j^*, \alpha_j''^*, \beta_j''^*\}$.

Step 7 Find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ of the fully fuzzy linear programming problem ($P_{5.1}$) by putting the values of $x_j^*, \alpha_j^{''*}$ and $\beta_j^{''*}$ in $\tilde{x}_j^* = (x_j^*, \alpha_j^{''*}, \beta_j^{''*})$.

Step 8 Find the fuzzy optimal value by putting the values of $\tilde{x}_j^*$, obtained from Step 7, in $\sum\limits_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j^*$.

5.2 Applicability of the existing methods

In this section, different types of fuzzy linear programming problems and fully fuzzy linear programming problems, which can be solved by using the existing methods [4, 41, 87, 88, 91, 100], are discussed:

(i) The existing methods [87, 91] can be used for solving the following type of fuzzy linear programming problems:

$$\begin{aligned} &\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j x_j \\ &\text{subject to} \\ &\sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq, \approx, \succeq \tilde{b}_i \quad \forall\ i = 1, 2, ..., m \\ &x_j \geq 0 \quad \forall\ j = 1, 2, ..., n \end{aligned} \qquad (P_{5.7})$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}_j$ are trapezoidal fuzzy numbers.

Example 5.1 [87, 91] Maximize $((2,3,1,1)x_1 \oplus (3,4,1,2)x_2)$

$$\begin{aligned} &\text{subject to} \\ &(1,2,1,1)x_1 \oplus (2,3,1,2)x_2 \preceq (5,6,2,2) \\ &(2,3,1,3)x_1 \oplus (1,2,1,1)x_2 \preceq (4,6,2,1) \\ &x_1, x_2 \geq 0 \end{aligned}$$

(ii) Nasseri et al. [100] proposed a method for solving the following type of fuzzy linear programming problems:

$$\begin{aligned} &\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j x_j \\ &\text{subject to} \end{aligned} \qquad (P_{5.8})$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i \quad \forall\ i = 1, 2, ..., m$$

$$x_j \geq 0 \quad \forall\ j = 1, 2, ..., n$$

where a_{ij}, b_i are real numbers and $\tilde{c}_j$ is a trapezoidal fuzzy number.

Example 5.2 [100] Maximize $((5, 8, 2, 5)x_1 \oplus (6, 10, 2, 6)x_2)$

subject to

$$2x_1 + 3x_2 \leq 6$$

$$5x_1 + 4x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

(iii) In the existing methods [87,91,100] either the cost coefficients are assumed as crisp numbers or the decision variables are assumed as crisp numbers i.e., in none of the existing methods [87,91,100] cost coefficients as well as decision variables are assumed as fuzzy numbers. Ganesan and Veeramani [41] pointed out that for the ranking function ($\Re$) neither the property $\Re(\tilde{c} \otimes \tilde{x}) = \Re(\tilde{c})\Re(\tilde{x})$ nor $\Re(\tilde{c} \odot \tilde{x}) = \Re(\tilde{c})\Re(\tilde{x})$ is satisfied. To overcome this limitation they have defined a new product ($\odot_{GV}$) for symmetric trapezoidal fuzzy numbers for which the property $\Re(\tilde{c} \odot_{GV} \tilde{x}) = \Re(\tilde{c})\Re(\tilde{x})$ is satisfied and proposed a new method for solving the following type of fuzzy linear programming problems:

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j \odot_{GV} \tilde{x}_j$$

$$\text{subject to} \qquad (P_{5.9})$$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i \quad \forall\ i = 1, 2, ..., m$$

$$\tilde{x}_{ij} \succeq \tilde{0} \quad \forall\ j = 1, 2, ..., n$$

where $\tilde{x}_j$, $\tilde{c}_j$ and $\tilde{b}_i$ are symmetric trapezoidal fuzzy numbers.

Example 5.3 [41]

Maximize $((13, 15, 2, 2) \odot_{GV} \tilde{x}_1 \oplus (12, 14, 3, 3) \odot_{GV} \tilde{x}_2 \oplus (15, 17, 2, 2) \odot_{GV} \tilde{x}_3)$

subject to

$$12\tilde{x}_1 + 13\tilde{x}_2 + 12\tilde{x}_3 \preceq (475, 505, 6, 6)$$

$$14\tilde{x}_1 + 13\tilde{x}_3 \preceq (460, 480, 8, 8)$$

$$12\tilde{x}_1 + 15\tilde{x}_2 \preceq (465, 495, 5, 5)$$

$$\tilde{x}_1 \succeq \tilde{0}, \tilde{x}_2 \succeq \tilde{0}, \tilde{x}_3 \succeq \tilde{0}$$

where $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ are symmetric trapezoidal fuzzy numbers.

(iv) Mahadavi and Nasseri [88] proposed a fuzzy dual simplex method for solving the following type of fuzzy linear programming problems:

$$\begin{aligned} &\text{Minimize } \sum_{j=1}^{n} c_j \tilde{x}_j \\ &\text{subject to} \\ &\quad \sum_{j=1}^{n} a_{ij}\tilde{x}_j \succeq \tilde{b}_i \quad \forall\, i = 1, 2, ..., m \\ &\quad \tilde{x}_j \succeq \tilde{0}, c_j \geq 0 \quad \forall\, j = 1, 2, ..., n \end{aligned} \qquad (P_{5.10})$$

where a_{ij} is a real number and $\tilde{x}_j$, $\tilde{b}_i$ are trapezoidal fuzzy numbers.

Example 5.4 [88] Minimize $(6\tilde{x}_1 \oplus 10\tilde{x}_2)$

subject to

$$2\tilde{x}_1 \oplus 5\tilde{x}_2 \succeq (5, 8, 2, 5)$$

$$3\tilde{x}_1 \oplus 4\tilde{x}_2 \succeq (6, 10, 2, 6)$$

$$\tilde{x}_1 \succeq \tilde{0}, \tilde{x}_2 \succeq \tilde{0}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are trapezoidal fuzzy numbers.

(v) The existing method [4] can be used for solving the following type of fully fuzzy linear programming problems:

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\ &\text{subject to} \end{aligned} \qquad (P_{5.11})$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i \;\; \forall\, i = 1, 2, ..., m$$

where $\tilde{x}_j$ is a non-negative LR fuzzy number and $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ are non-negative or non-positive LR fuzzy numbers.

Example 5.5 [4] Minimize $((1,1,1)_{LR} \otimes \tilde{x}_1 \oplus (2,1,2)_{LR} \otimes \tilde{x}_2)$

subject to

$(4,1,0)_{LR} \otimes \tilde{x}_1 \oplus (-3,2,1)_{LR} \otimes \tilde{x}_2 \succeq (2,1,2)_{LR}$

$(-3,1,2)_{LR} \otimes \tilde{x}_1 \oplus (2,1,1)_{LR} \otimes \tilde{x}_2 \succeq (1,0,1)_{LR}$

where $\tilde{x}_1$, $\tilde{x}_2$ are non-negative LR fuzzy numbers and $L(x) = R(x) = \max\{0, 1-x\}$.

5.3 Limitations of the existing methods

In this section, on the basis of applicability of existing methods, discussed in Section 5.2, the limitations of existing methods [4, 41, 87, 88, 91, 100] are pointed out.

5.3.1 Limitations of the existing methods for solving fuzzy linear programming problems

In this section, the limitations of the existing methods [41,87,88,91,100] for solving fuzzy linear programming problems are pointed out.

The existing methods [41, 87, 88, 91, 100] can be used only to find the fuzzy optimal solution of the following type of fuzzy linear programming problems:

(i) The existing methods [87, 91] can be used for solving such fuzzy linear programming problems in which the decision variables are represented by crisp numbers and the remaining parameters are represented by triangular or trapezoidal fuzzy numbers.

(ii) The existing method [100] can be used for solving such fuzzy linear programming problems in which the cost coefficients are represented by triangular or trapezoidal fuzzy numbers and the remaining parameters are represented by crisp numbers.

(iii) The existing method [41] can be used for solving such fuzzy linear programming problems in which the coefficients of the constraints are represented by crisp numbers and the remaining parameters are represented by symmetric triangular or trapezoidal fuzzy numbers.

(iv) The existing method [88] can be used for solving such fuzzy linear programming problems in which the cost coefficients and the coefficients of the constraints are represented by crisp numbers and the remaining parameters are represented by triangular or trapezoidal fuzzy numbers.

However, the existing methods [41,87,88,91,100] cannot be used to find the fuzzy optimal solution of the following type of fuzzy linear programming problems:

(i) The fuzzy linear programming problems in which the decision variables are represented by crisp numbers and the remaining parameters are represented by *LR* fuzzy numbers or *LR* flat fuzzy numbers.

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j x_j \\
&\text{subject to} \qquad\qquad\qquad\qquad (P_{5.12}) \\
&\sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq, \approx, \succeq \tilde{b}_i \ \ \forall\ i = 1, 2, ..., m \\
&x_j \geq 0 \ \ \forall\ j = 1, 2, ..., n
\end{aligned}$$

where x_j is a real number and $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ are *LR* flat fuzzy numbers.

Example 5.6 Maximize $((2, 3, 3, 1)_{LR} x_1 \oplus (0, 4, 1, 2)_{LR} x_2)$

$$\text{subject to}$$

$$(1,2,1,1)_{LR}x_1 \oplus (2,3,1,2)_{LR}x_2 \preceq (5,6,2,2)_{LR}$$

$$(2,3,1,3)_{LR}x_1 \oplus (1,2,1,1)_{LR}x_2 \preceq (4,6,2,1)_{LR}$$

$$x_1, x_2 \geq 0$$

where $L(x) = R(x) = \max\{0, 1 - x^2\}$.

(ii) The fuzzy linear programming problems in which the cost coefficients are represented by LR fuzzy numbers or LR flat fuzzy numbers and the remaining parameters are represented by crisp numbers.

$$\begin{aligned} &\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j x_j \\ &\text{subject to} \\ &\sum_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i \quad \forall\, i = 1, 2, ..., m \\ &x_j \geq 0 \quad \forall\, j = 1, 2, ..., n \end{aligned} \qquad (P_{5.13})$$

where a_{ij}, b_i, x_j are real numbers and $\tilde{c}_j$ is an LR flat fuzzy number.

Example 5.7 Maximize $((3,8,5,4)_{LR}x_1 \oplus (4,10,6,6)_{LR}x_2)$

$$\text{subject to}$$

$$2x_1 + 3x_2 \leq 6$$

$$5x_1 + 4x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

where $L(x) = R(x) = \max\{0, 1 - x^4\}$.

(iii) The fuzzy linear programming problems in which the coefficients of the constraints are represented by crisp numbers and the remaining are represented by symmetric LR flat fuzzy numbers.

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j$$

subject to $(P_{5.14})$

$$\sum_{j=1}^{n}(a_{ij}\tilde{x}_j) \preceq, \approx, \succeq \tilde{b}_i \;\; \forall\; i = 1, 2, ..., m$$

$$\tilde{x}_j \succeq \tilde{0} \;\; \forall\; j = 1, 2, ..., n$$

where $\tilde{x}_j$, $\tilde{c}_j$ and $\tilde{b}_i$ are symmetric LR flat fuzzy numbers.

Example 5.8 Maximize $((2, 5, 3, 3)_{LR} \otimes \tilde{x}_1 \oplus (3, 4, 4, 4)_{LR} \otimes \tilde{x}_2)$

subject to

$2\tilde{x}_1 + 3\tilde{x}_2 \preceq (10, 12, 2, 2)_{LR}$

$3\tilde{x}_1 + 4\tilde{x}_2 \preceq (20, 30, 5, 5)_{LR}$

where $\tilde{x}_1$, $\tilde{x}_2$ are non-negative symmetric trapezoidal fuzzy numbers and $L(x) = R(x) = \max\{0, 1 - x^4\}$.

(iv) The fuzzy linear programming problems in which the cost coefficients and the coefficients of the constraints are represented by crisp numbers and the remaining parameters are represented by LR fuzzy numbers or LR flat fuzzy numbers.

$$\text{Minimize} \sum_{j=1}^{n} c_j\tilde{x}_j$$

subject to $(P_{5.15})$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \succeq \tilde{b}_i \;\; \forall\; i = 1, 2, ..., m$$

$$\tilde{x}_j \succeq \tilde{0}, c_j \geq 0 \;\; \forall\; j = 1, 2, ..., n$$

where a_{ij} is a real number and $\tilde{x}_j$, $\tilde{b}_i$ are LR flat fuzzy numbers.

Example 5.9 Minimize $(6\tilde{x}_1 \oplus 10\tilde{x}_2)$

subject to

$2\tilde{x}_1 \oplus 5\tilde{x}_2 \succeq (5, 8, 6, 5)_{LR}$

$3\tilde{x}_1 \oplus 4\tilde{x}_2 \succeq (6, 10, 2, 6)_{LR}$

$\tilde{x}_1 \succeq \tilde{0}, \tilde{x}_2 \succeq \tilde{0}$

where $\tilde{x}_1$, $\tilde{x}_2$ are LR flat fuzzy numbers and $L(x) = R(x) = \max\{0, 1 - x^2\}$.

5.3.2 Limitations of the existing method for solving fully fuzzy linear programming problems

In this section, the limitations of the existing method [4] for solving fully fuzzy linear programming problems with inequality constraints are pointed out:

The existing method [4] can be used for solving such fully fuzzy linear programming problems with inequality constraints in which all the coefficients are represented by either non-negative LR fuzzy numbers or non-positive LR fuzzy numbers and all the decision variables are represented by non-negative LR fuzzy numbers.

However, the existing method [4] cannot be used to find the fuzzy optimal solution of such fully fuzzy linear programming problems in which some or all the parameters are represented by such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive.

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{5.16}) \\
&\quad \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, =, \succeq \tilde{b}_i \quad \forall\ i = 1, 2, ..., m
\end{aligned}$$

where $\tilde{a}_{ij}$, $\tilde{x}_j$, $\tilde{b}_i$ and $\tilde{c}_j$ are LR flat fuzzy numbers.

Example 5.10 Maximize $((4,4,0,0)_{LR} \otimes \tilde{x}_1 \oplus (1,1,1,1)_{LR} \otimes \tilde{x}_2)$

subject to

$$(2,5,5,2)_{LR} \otimes \tilde{x}_1 \oplus (-1,5,1,2)_{LR} \otimes \tilde{x}_2 \preceq (-17,45,25,46)_{LR}$$

$$(1,2,1,1)_{LR} \otimes \tilde{x}_1 \oplus (3,5,2,2)_{LR} \otimes \tilde{x}_2 = (7,24,6,24)_{LR}$$

where $\tilde{x}_1, \tilde{x}_2$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1-x\}$, $R(x) = \max\{0, 1-x^2\}$.

$$\begin{aligned} &\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{5.17}) \\ &\quad \sum_{j=1}^{n} \tilde{a}_{ij} \odot \tilde{x}_j \preceq, =, \succeq \tilde{b}_i \ \ \forall\ i = 1, 2, ..., m \end{aligned}$$

where $\tilde{a}_{ij}$, $\tilde{x}_j$, $\tilde{b}_i$ and $\tilde{c}_j$ are LR flat fuzzy numbers.

Example 5.11 Maximize $((1, 1, 1, 1)_{LR} \odot \tilde{x}_1 \oplus (4, 4, 0, 0)_{LR} \odot \tilde{x}_2)$

subject to

$$(2, 3, 1, 1)_{LR} \odot \tilde{x}_1 \oplus (-3, -2, 1, 1)_{LR} \odot \tilde{x}_2 \succeq (-27, -4, 21, 32)_{LR}$$

$$(1, 2, 1, 1)_{LR} \odot \tilde{x}_1 \oplus (3, 5, 2, 2)_{LR} \odot \tilde{x}_2 = (11, 39, \tfrac{193}{5}, \tfrac{168}{5})_{LR}$$

where $\tilde{x}_1, \tilde{x}_2$ are LR flat fuzzy numbers and $L(x) = R(x) = \max\{0, 1 - x\}$.

5.4 Proposed methods for solving fully fuzzy linear programming problems with inequality constraints having LR flat fuzzy numbers

On the basis of the limitations of the existing methods [4,41,87,88,91,100], discussed in Section 5.3, it can be concluded that till now in literature, there is no method to find the fuzzy optimal solution of such fuzzy linear programming problems and fully fuzzy linear programming problems in which some of all the parameters are represented by such LR fuzzy numbers or LR flat fuzzy numbers which are neither non-negative nor non-positive.

In this section, to overcome the limitations of the existing methods two new methods are proposed to find the fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints.

5.4.1 Proposed method

In this section, a new method to find the fuzzy optimal solution of fully fuzzy linear programming problems of type $(P_{5.16})$ with inequality constraints, is proposed. The same method can also be used to find the fuzzy optimal solution of the proposed fully fuzzy linear programming problems $(P_{5.17})$ by replacing $\otimes$ by $\odot$ and also other fuzzy linear programming problems $(P_{5.7})$ to $(P_{5.15})$ by replacing the fuzzy parameters by crisp parameters.

The steps of the proposed method for solving fully fuzzy linear programming problems $(P_{5.16})$ are as follows:

Step 1 Dividing all the constraints into three categories i.e., $\sum\limits_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \preceq \tilde{b}_p \ \ \forall\, p \in N_1$, $\sum\limits_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \succeq \tilde{b}_q \ \ \forall\, q \in N_2$ and $\sum\limits_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j = \tilde{b}_r \ \ \forall\, r \in N_3$ the fully fuzzy linear programming problem $(P_{5.16})$ can be written as:

$$\text{Maximize/Minimize} \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \preceq \tilde{b}_p && \forall\, p \in N_1 \\ &\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \succeq \tilde{b}_q && \forall\, q \in N_2 \\ &\sum_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j = \tilde{b}_r && \forall\, r \in N_3 \end{aligned} \qquad (P_{5.18})$$

where $\tilde{x}_j$ is an LR flat fuzzy number and $N_1 = \{i \ : \ 1 \le i \le m \text{ and } \sum\limits_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \preceq \tilde{b}_i\}$, $N_2 = \{i \ : \ 1 \le i \le m \text{ and } \sum\limits_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \succeq \tilde{b}_i\}$, $N_3 = \{i \ : \ 1 \le i \le m \text{ and } \sum\limits_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i\}$.

Step 2 Convert the inequality constraints $\sum\limits_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \preceq \tilde{b}_p \ \forall\, p \in N_1$ into equality constraints by introducing fuzzy variable $\tilde{S}_p$ to the left side and $\tilde{S'}_p$ to the right side of the constraint i.e.,

$$\sum_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \oplus \tilde{S}_p = \tilde{b}_p \oplus \tilde{S'}_p \ \ \forall\, p \in N_1$$

where $\Re(\tilde{S}_p) - \Re(\tilde{S'}_p) \ge 0$.

Convert the inequality constraints $\sum\limits_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \succeq \tilde{b}_q \ \forall \ q \in N_2$ into equality constraints by introducing fuzzy variable $\tilde{S}_q$ to the left side and $\tilde{S'}_q$ to the right side of the constraint i.e.,

$$\sum_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{S}_q = \tilde{b}_q \oplus \tilde{S'}_q \quad \forall \ q \in N_2$$

where $\Re(\tilde{S}_q) - \Re(\tilde{S'}_q) \leq 0$.

Step 3 The fully fuzzy linear programming problem, obtained from Step 2, can be written as:

$$\begin{array}{ll} \text{Maximize/Minimize } \sum\limits_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j & \\ \text{subject to} & \\ \quad \sum\limits_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \oplus \tilde{S}_p = \tilde{b}_p \oplus \tilde{S'}_p & \forall \ p \in N_1 \\ \quad \sum\limits_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{S}_q = \tilde{b}_q \oplus \tilde{S'}_q & \forall \ q \in N_2 \\ \quad \sum\limits_{j=1}^{n} \tilde{a}_{rj} \otimes \tilde{x}_j = \tilde{b}_r & \forall \ r \in N_3 \\ \quad \Re(\tilde{S}_p) - \Re(\tilde{S'}_p) \geq 0 & \forall \ p \in N_1 \\ \quad \Re(\tilde{S}_q) - \Re(\tilde{S'}_q) \leq 0 & \forall \ q \in N_2 \end{array} \qquad (P_{5.19})$$

where $\tilde{x}_j$, $\tilde{S}_p$, $\tilde{S'}_p$, $\tilde{S}_q$ and $\tilde{S'}_q$ are LR flat fuzzy number.

Step 4 Assuming $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, y_j, \alpha''_j, \beta''_j)_{LR}$, $\tilde{b}_i = (b_i, g_i, \gamma_i, \delta_i)_{LR}$, $\tilde{c}_j = (p_j, q_j, \alpha'_j, \beta'_j)_{LR}$, $\tilde{S}_i = (s_i, t_i, \eta_i, \rho_i)_{LR}$ and $\tilde{S}'_i = (s'_i, t'_i, \eta'_i, \rho'_i)_{LR}$ the fully fuzzy linear programming problem $(P_{5.19})$ can be written as:

$$\text{Maximize/Minimize } \sum_{j=1}^{n} (p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}$$

subject to

$$\sum_{j=1}^{n} (a_{pj}, b_{pj}, \alpha_{pj}, \beta_{pj})_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR} \oplus (s_p, t_p, \eta_p, \rho_p)_{LR} = (b_p, g_p, \gamma_p, \delta_p)_{LR} \oplus (s'_p, t'_p, \eta'_p, \rho'_p)_{LR} \quad \forall \ p \in N_1$$

$$\sum_{j=1}^{n} (a_{qj}, b_{qj}, \alpha_{qj}, \beta_{qj})_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR} \oplus (s_q, t_q, \eta_q, \rho_q)_{LR} = (b_q, g_q, \gamma_q, \delta_q)_{LR} \oplus (s'_q, t'_q, \eta'_q, \rho'_q)_{LR} \quad \forall \ q \in N_2 \qquad (P_{5.20})$$

$\sum_{j=1}^{n}(a_{rj},b_{rj},\alpha_{rj},\beta_{rj})_{LR}\otimes(x_j,y_j,\alpha''_j,\beta''_j)_{LR}=(b_r,g_r,\gamma_r,\delta_r)_{LR} \quad \forall\ r\in N_3$

$\Re(s_p,t_p,\eta_p,\rho_p)_{LR}-\Re(s'_p,t'_p,\eta'_p,\rho'_p)_{LR}\geq 0 \quad \forall\ p\in N_1$

$\Re(s_q,t_q,\eta_q,\rho_q)_{LR}-\Re(s'_q,t'_q,\eta'_q,\rho'_q)_{LR}\leq 0 \quad \forall\ q\in N_2$

where $(x_j,y_j,\alpha''_j,\beta''_j)_{LR}$, $(s_p,t_p,\eta_p,\rho_p)_{LR}$, $(s'_p,t'_p,\eta'_p,\rho'_p)_{LR}$, $(s_q,t_q,\eta_q,\rho_q)_{LR}$ and $(s'_q,t'_q,\eta'_q,\rho'_q)_{LR}$ are LR flat fuzzy number.

Step 5 Using the product, proposed in Section 4.2.1 and assuming $(a_{ij},b_{ij},\alpha_{ij},\beta_{ij})_{LR}\otimes(x_j,y_j,\alpha''_j,\beta''_j)_{LR}=(m_{ij},n_{ij},\gamma'_{ij},\delta'_{ij})_{LR}$ the fully fuzzy linear programming problem $(P_{5.20})$ can be written as:

Maximize/Minimize $\sum_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j,y_j,\alpha''_j,\beta''_j)_{LR}$

subject to

$\sum_{j=1}^{n}(m_{pj},n_{pj},\gamma'_{pj},\delta'_{pj})_{LR}\oplus(s_p,t_p,\eta_p,\rho_p)_{LR}=(b_p,g_p,\gamma_p,\delta_p)_{LR}\oplus(s'_p,t'_p,\eta'_p,$
$\rho'_p)_{LR} \quad \forall\ p\in N_1$

$\sum_{j=1}^{n}(m_{qj},n_{qj},\gamma'_{qj},\delta'_{qj})_{LR}\oplus(s_q,t_q,\eta_q,\rho_q)_{LR}=(b_q,g_q,\gamma_q,\delta_q)_{LR}\oplus(s'_q,t'_q,\eta'_q,$
$\rho'_q)_{LR} \quad \forall\ q\in N_2 \qquad (P_{5.21})$

$\sum_{j=1}^{n}(m_{rj},n_{rj},\gamma'_{rj},\delta'_{rj})_{LR}=(b_r,g_r,\gamma_r,\delta_r)_{LR} \quad \forall\ r\in N_3$

$\Re(s_p,t_p,\eta_p,\rho_p)_{LR}-\Re(s'_p,t'_p,\eta'_p,\rho'_p)_{LR}\geq 0 \quad \forall\ p\in N_1$

$\Re(s_q,t_q,\eta_q,\rho_q)_{LR}-\Re(s'_q,t'_q,\eta'_q,\rho'_q)_{LR}\leq 0 \quad \forall\ q\in N_2$

where $(x_j,y_j,\alpha''_j,\beta''_j)_{LR}$, $(s_p,t_p,\eta_p,\rho_p)_{LR}$, $(s'_p,t'_p,\eta'_p,\rho'_p)_{LR}$, $(s_q,t_q,\eta_q,\rho_q)_{LR}$ and $(s'_q,t'_q,\eta'_q,\rho'_q)_{LR}$ are LR flat fuzzy number.

Step 6 Using arithmetic operations, defined in Section 4.1.2 and Definition 4.5, the fully fuzzy linear programming problem $(P_{5.21})$ can be written as:

Maximize/Minimize $\sum_{j=1}^{n}(p_j,q_j,\alpha'_j,\beta'_j)_{LR}\otimes(x_j,y_j,\alpha''_j,\beta''_j)_{LR}$

subject to

$$\begin{array}{llll}
\sum\limits_{j=1}^{n} m_{pj} + s_p = b_p + s'_p \ \ \forall\ p \in N_1, & \sum\limits_{j=1}^{n} n_{pj} + t_p = g_p + t'_p \ \ \forall\ p \in N_1 & \\
\sum\limits_{j=1}^{n} \gamma'_{pj} + \eta_p = \gamma_p + \eta'_p \ \ \forall\ p \in N_1, & \sum\limits_{j=1}^{n} \delta'_{pj} + \rho_p = \delta_p + \rho'_p \ \ \forall\ p \in N_1 & (P_{5.22}) \\
\sum\limits_{j=1}^{n} m_{qj} + s_q = b_q + s'_q \ \ \forall\ q \in N_2, & \sum\limits_{j=1}^{n} n_{qj} + t_q = g_q + t'_q \ \ \forall\ q \in N_2 & \\
\sum\limits_{j=1}^{n} \gamma'_{qj} + \eta_q = \gamma_q + \eta'_q \ \ \forall\ q \in N_2, & \sum\limits_{j=1}^{n} \delta'_{qj} + \rho_q = \delta_q + \rho'_q \ \ \forall\ q \in N_2 & \\
\sum\limits_{j=1}^{n} m_{rj} = b_r \qquad \forall\ r \in N_3, & \sum\limits_{j=1}^{n} n_{rj} = g_r \qquad \forall\ r \in N_3 & \\
\sum\limits_{j=1}^{n} \gamma'_{rj} = \gamma_r \qquad \forall\ r \in N_3, & \sum\limits_{j=1}^{n} \delta'_{rj} = \delta_r \qquad \forall\ r \in N_3 &
\end{array}$$

$$\Re(s_p, t_p, \eta_p, \rho_p)_{LR} - \Re(s'_p, t'_p, \eta'_p, \rho'_p)_{LR} \geq 0 \ \ \forall\ p \in N_1$$

$$\Re(s_q, t_q, \eta_q, \rho_q)_{LR} - \Re(s'_q, t'_q, \eta'_q, \rho'_q)_{LR} \leq 0 \ \ \forall\ q \in N_2$$

$$x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0,\ s_i \leq t_i, \eta_i \geq 0, \rho_i \geq 0,\ s'_i \leq t'_i, \eta'_i \geq 0, \rho'_i \geq 0$$

$$\forall\ i = 1, 2, ..., m;\ \ j = 1, 2, ..., n$$

Step 7 As discussed in Step 4 of Section 4.4 the fuzzy optimal solution of $(P_{5.16})$ can be obtained by solving the problem $(P_{5.23})$:

Maximize/Minimize $\Re(\sum\limits_{j=1}^{n} (p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR})$

subject to

$$\begin{array}{llll}
\sum\limits_{j=1}^{n} m_{pj} + s_p = b_p + s'_p \ \ \forall\ p \in N_1, & \sum\limits_{j=1}^{n} n_{pj} + t_p = g_p + t'_p \ \ \forall\ p \in N_1 & \\
\sum\limits_{j=1}^{n} \gamma'_{pj} + \eta_p = \gamma_p + \eta'_p \ \ \forall\ p \in N_1, & \sum\limits_{j=1}^{n} \delta'_{pj} + \rho_p = \delta_p + \rho'_p \ \ \forall\ p \in N_1 & (P_{5.23}) \\
\sum\limits_{j=1}^{n} m_{qj} + s_q = b_q + s'_q \ \ \forall\ q \in N_2, & \sum\limits_{j=1}^{n} n_{qj} + t_q = g_q + t'_q \ \ \forall\ q \in N_2 & \\
\sum\limits_{j=1}^{n} \gamma'_{qj} + \eta_q = \gamma_q + \eta'_q \ \ \forall\ q \in N_2, & \sum\limits_{j=1}^{n} \delta'_{qj} + \rho_q = \delta_q + \rho'_q \ \ \forall\ q \in N_2 & \\
\sum\limits_{j=1}^{n} m_{rj} = b_r \qquad \forall\ r \in N_3, & \sum\limits_{j=1}^{n} n_{rj} = g_r \qquad \forall\ r \in N_3 & \\
\sum\limits_{j=1}^{n} \gamma'_{rj} = \gamma_r \qquad \forall\ r \in N_3, & \sum\limits_{j=1}^{n} \delta'_{rj} = \delta_r \qquad \forall\ r \in N_3 &
\end{array}$$

$$\Re(s_p, t_p, \eta_p, \rho_p)_{LR} - \Re(s'_p, t'_p, \eta'_p, \rho'_p)_{LR} \geq 0 \ \ \forall\ p \in N_1$$

$$\Re(s_q, t_q, \eta_q, \rho_q)_{LR} - \Re(s'_q, t'_q, \eta'_q, \rho'_q)_{LR} \leq 0 \ \ \forall\ q \in N_2$$

$$x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0,\ s_i \leq t_i, \eta_i \geq 0, \rho_i \geq 0,\ s'_i \leq t'_i, \eta'_i \geq 0, \rho'_i \geq 0$$

$\forall\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$

Step 8 Assuming $(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR} = (p'_j, q'_j, \alpha'''_j, \beta'''_j)_{LR}$ the problem $(P_{5.23})$ can be written as:

Maximize/Minimize $\Re(\sum_{j=1}^{n}(p'_j, q'_j, \alpha'''_j, \beta'''_j)_{LR})$

subject to

$$\begin{array}{lll}
\sum_{j=1}^{n} m_{pj} + s_p = b_p + s'_p \ \ \forall\ p \in N_1, & \sum_{j=1}^{n} n_{pj} + t_p = g_p + t'_p \ \ \forall\ p \in N_1 & \\
\sum_{j=1}^{n} \gamma'_{pj} + \eta_p = \gamma_p + \eta'_p \ \ \forall\ p \in N_1, & \sum_{j=1}^{n} \delta'_{pj} + \rho_p = \delta_p + \rho'_p \ \ \forall\ p \in N_1 & (P_{5.24}) \\
\sum_{j=1}^{n} m_{qj} + s_q = b_q + s'_q \ \ \forall\ q \in N_2, & \sum_{j=1}^{n} n_{qj} + t_q = g_q + t'_q \ \ \forall\ q \in N_2 & \\
\sum_{j=1}^{n} \gamma'_{qj} + \eta_q = \gamma_q + \eta'_q \ \ \forall\ q \in N_2, & \sum_{j=1}^{n} \delta'_{qj} + \rho_q = \delta_q + \rho'_q \ \ \forall\ q \in N_2 & \\
\sum_{j=1}^{n} m_{rj} = b_r \qquad \forall\ r \in N_3, & \sum_{j=1}^{n} n_{rj} = g_r \qquad \forall\ r \in N_3 & \\
\sum_{j=1}^{n} \gamma'_{rj} = \gamma_r \qquad \forall\ r \in N_3, & \sum_{j=1}^{n} \delta'_{rj} = \delta_r \qquad \forall\ r \in N_3 &
\end{array}$$

$\Re(s_p, t_p, \eta_p, \rho_p)_{LR} - \Re(s'_p, t'_p, \eta'_p, \rho'_p)_{LR} \geq 0 \ \ \forall\ p \in N_1$

$\Re(s_q, t_q, \eta_q, \rho_q)_{LR} - \Re(s'_q, t'_q, \eta'_q, \rho'_q)_{LR} \leq 0 \ \ \forall\ q \in N_2$

$x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0,\ s_i \leq t_i, \eta_i \geq 0, \rho_i \geq 0,\ s'_i \leq t'_i, \eta'_i \geq 0, \rho'_i \geq 0$

$\forall\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$

Step 9 Using the linearity property $\Re(\sum_{j=1}^{n} \tilde{A}_i) = \sum_{j=1}^{n} \Re(\tilde{A}_i)$, where $\tilde{A}_i$ is a fuzzy number, the problem $(P_{5.24})$ can be converted into $(P_{5.25})$:

Maximize/Minimize $\sum_{j=1}^{n} \Re(p'_j, q'_j, \alpha'''_j, \beta'''_j)_{LR}$

subject to

$$\begin{array}{lll}
\sum_{j=1}^{n} m_{pj} + s_p = b_p + s'_p \ \ \forall\ p \in N_1, & \sum_{j=1}^{n} n_{pj} + t_p = g_p + t'_p \ \ \forall\ p \in N_1 & \\
\sum_{j=1}^{n} \gamma'_{pj} + \eta_p = \gamma_p + \eta'_p \ \ \forall\ p \in N_1, & \sum_{j=1}^{n} \delta'_{pj} + \rho_p = \delta_p + \rho'_p \ \ \forall\ p \in N_1 & (P_{5.25}) \\
\sum_{j=1}^{n} m_{qj} + s_q = b_q + s'_q \ \ \forall\ q \in N_2, & \sum_{j=1}^{n} n_{qj} + t_q = g_q + t'_q \ \ \forall\ q \in N_2 & \\
\sum_{j=1}^{n} \gamma'_{qj} + \eta_q = \gamma_q + \eta'_q \ \ \forall\ q \in N_2, & \sum_{j=1}^{n} \delta'_{qj} + \rho_q = \delta_q + \rho'_q \ \ \forall\ q \in N_2 &
\end{array}$$

$$\begin{array}{llll}
\sum_{j=1}^{n} m_{rj} = b_r & \forall\, r \in N_3, & \sum_{j=1}^{n} n_{rj} = g_r & \forall\, r \in N_3 \\
\sum_{j=1}^{n} \gamma'_{rj} = \gamma_r & \forall\, r \in N_3, & \sum_{j=1}^{n} \delta'_{rj} = \delta_r & \forall\, r \in N_3
\end{array}$$

$\Re(s_p, t_p, \eta_p, \rho_p)_{LR} - \Re(s'_p, t'_p, \eta'_p, \rho'_p)_{LR} \geq 0 \;\; \forall\, p \in N_1$

$\Re(s_q, t_q, \eta_q, \rho_q)_{LR} - \Re(s'_q, t'_q, \eta'_q, \rho'_q)_{LR} \leq 0 \;\; \forall\, q \in N_2$

$x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0,\ s_i \leq t_i, \eta_i \geq 0, \rho_i \geq 0,\ s'_i \leq t'_i, \eta'_i \geq 0, \rho'_i \geq 0$

$\forall\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$

Step 10 Using $\Re(m, n, \alpha, \beta) = \frac{1}{2}(\int_0^1 (m - \alpha L^{-1}(\lambda))\, d\lambda + \int_0^1 (n + \beta R^{-1}(\lambda))\, d\lambda)$ problem $(P_{5.25})$ can be converted into $(P_{5.26})$:

Maximize/Minimize $\sum_{j=1}^{n} (\frac{1}{2}(\int_0^1 (p'_j - \alpha'''_j L^{-1}(\lambda))\, d\lambda + \int_0^1 (q'_j + \beta'''_j R^{-1}(\lambda))\, d\lambda))$

subject to

$$\begin{array}{lll}
\sum_{j=1}^{n} m_{pj} + s_p = b_p + s'_p \;\; \forall\, p \in N_1, & \sum_{j=1}^{n} n_{pj} + t_p = g_p + t'_p \;\; \forall\, p \in N_1 & \\
\sum_{j=1}^{n} \gamma'_{pj} + \eta_p = \gamma_p + \eta'_p \;\; \forall\, p \in N_1, & \sum_{j=1}^{n} \delta'_{pj} + \rho_p = \delta_p + \rho'_p \;\; \forall\, p \in N_1 & (P_{5.26}) \\
\sum_{j=1}^{n} m_{qj} + s_q = b_q + s'_q \;\; \forall\, q \in N_2, & \sum_{j=1}^{n} n_{qj} + t_q = g_q + t'_q \;\; \forall\, q \in N_2 & \\
\sum_{j=1}^{n} \gamma'_{qj} + \eta_q = \gamma_q + \eta'_q \;\; \forall\, q \in N_2, & \sum_{j=1}^{n} \delta'_{qj} + \rho_q = \delta_q + \rho'_q \;\; \forall\, q \in N_2 & \\
\sum_{j=1}^{n} m_{rj} = b_r \qquad \forall\, r \in N_3, & \sum_{j=1}^{n} n_{rj} = g_r \qquad \forall\, r \in N_3 & \\
\sum_{j=1}^{n} \gamma'_{rj} = \gamma_r \qquad \forall\, r \in N_3, & \sum_{j=1}^{n} \delta'_{rj} = \delta_r \qquad \forall\, r \in N_3 &
\end{array}$$

$\frac{1}{2}(\int_0^1 (s_p - \eta_p L^{-1}(\lambda))\, d\lambda + \int_0^1 (t_p + \rho_p R^{-1}(\lambda))\, d\lambda) - \frac{1}{2}(\int_0^1 (s'_p - \eta'_p L^{-1}(\lambda))\, d\lambda + \int_0^1 (t'_p + \rho'_p R^{-1}(\lambda))\, d\lambda) \geq 0 \;\; \forall\, p \in N_1$

$\frac{1}{2}(\int_0^1 (s_q - \eta_q L^{-1}(\lambda))\, d\lambda + \int_0^1 (t_q + \rho_q R^{-1}(\lambda))\, d\lambda) - \frac{1}{2}(\int_0^1 (s'_q - \eta'_q L^{-1}(\lambda))\, d\lambda + \int_0^1 (t'_q + \rho'_q R^{-1}(\lambda))\, d\lambda) \leq 0 \;\; \forall\, p \in N_2$

$x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0,\ s_i \leq t_i, \eta_i \geq 0, \rho_i \geq 0,\ s'_i \leq t'_i, \eta'_i \geq 0, \rho'_i \geq 0 \;\; \forall\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$

Step 11 Solve the crisp non-linear programming problem $(P_{5.26})$ by using an appropriate existing method [125] to find the optimal solution $\{x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*}\}$.

Step 12 Find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ of the fully fuzzy linear programming problem $(P_{5.16})$ by putting the values of $x_j^*, y_j^*, \alpha_j^{''*}$ and $\beta_j^{''*}$ in $\tilde{x}_j^* = (x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*})$.

Step 13 Find the fuzzy optimal value by putting the values of $\tilde{x}_j^*$, obtained from Step 12, in $\sum\limits_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j^*$.

5.4.2 Alternative proposed method

In this section, an alternative method, to find the fuzzy optimal solution of fully fuzzy linear programming problems of type $(P_{5.16})$ with inequality constraints, is proposed. The same method can be used to find the fuzzy optimal solution of the fully fuzzy linear programming problems and fuzzy linear programming problems which can be solved by using the method, proposed in Section 5.4.1.

The steps of the proposed method for solving fully fuzzy linear programming problem $(P_{5.16})$ are as follows:

Step 1 Assuming $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, y_j, \alpha_j'', \beta_j'')_{LR}$, $\tilde{b}_i = (b_i, g_i, \gamma_i, \delta_i)_{LR}$ and $\tilde{c}_j = (p_j, q_j, \alpha_j', \beta_j')_{LR}$ the fully fuzzy linear programming problem $(P_{5.18})$ can be written as:

Maximize/Minimize $\sum\limits_{j=1}^{n} (p_j, q_j, \alpha_j', \beta_j')_{LR} \otimes (x_j, y_j, \alpha_j'', \beta_j'')_{LR}$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} (a_{pj}, b_{pj}, \alpha_{pj}, \beta_{pj})_{LR} \otimes (x_j, y_j, \alpha_j'', \beta_j'')_{LR} \preceq (b_p, g_p, \gamma_p, \delta_p)_{LR} \quad \forall\ p \in N_1 \\ &\sum_{j=1}^{n} (a_{qj}, b_{qj}, \alpha_{qj}, \beta_{qj})_{LR} \otimes (x_j, y_j, \alpha_j'', \beta_j'')_{LR} \succeq (b_q, g_q, \gamma_q, \delta_q)_{LR} \quad \forall\ q \in N_2 \qquad (P_{5.27}) \\ &\sum_{j=1}^{n} (a_{rj}, b_{rj}, \alpha_{rj}, \beta_{rj})_{LR} \otimes (x_j, y_j, \alpha_j'', \beta_j'')_{LR} = (b_r, g_r, \gamma_r, \delta_r)_{LR} \quad \forall\ r \in N_3 \end{aligned}$$

where $(x_j, y_j, \alpha_j'', \beta_j'')_{LR}$ is an LR flat fuzzy number.

Step 2 Using the product, proposed in Section 4.2.1 and assuming $(a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR} \otimes$

$(x_j, y_j, \alpha''_j, \beta''_j)_{LR} = (m_{ij}, n_{ij}, \gamma'_{ij}, \delta'_{ij})_{LR}$ the fully fuzzy linear programming problem $(P_{5.27})$ can be written as:

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}\\
&\text{subject to}\\
&\qquad \sum_{j=1}^{n}(m_{pj}, n_{pj}, \gamma'_{pj}, \delta'_{pj})_{LR} \preceq (b_p, g_p, \gamma_p, \delta_p)_{LR} \quad \forall\ p \in N_1\\
&\qquad \sum_{j=1}^{n}(m_{qj}, n_{qj}, \gamma'_{qj}, \delta'_{qj})_{LR} \succeq (b_q, g_q, \gamma_q, \delta_q)_{LR} \quad \forall\ q \in N_2 \qquad\qquad (P_{5.28})\\
&\qquad \sum_{j=1}^{n}(m_{rj}, n_{rj}, \gamma'_{rj}, \delta'_{rj})_{LR} = (b_r, g_r, \gamma_r, \delta_r)_{LR} \quad \forall\ r \in N_3
\end{aligned}$$

where $(x_j, y_j, \alpha''_j, \beta''_j)_{LR}$ is an LR flat fuzzy number.

Step 3 Using arithmetic operations, defined in Section 4.1.2 and Definition 4.5, the fully fuzzy linear programming problem $(P_{5.28})$ can be written as:

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}\\
&\text{subject to}\\
&\qquad \sum_{j=1}^{n}(m_{pj}, n_{pj}, \gamma'_{pj}, \delta'_{pj})_{LR} \preceq (b_p, g_p, \gamma_p, \delta_p)_{LR} \quad \forall\ p \in N_1\\
&\qquad \sum_{j=1}^{n}(m_{qj}, n_{qj}, \gamma'_{qj}, \delta'_{qj})_{LR} \succeq (b_q, g_q, \gamma_q, \delta_q)_{LR} \quad \forall\ q \in N_2 \qquad\qquad (P_{5.29})\\
&\qquad \sum_{j=1}^{n} m_{rj} = b_r \quad \forall\ r \in N_3, \qquad \sum_{j=1}^{n} n_{rj} = g_r \quad \forall\ r \in N_3\\
&\qquad \sum_{j=1}^{n} \gamma'_{rj} = \gamma_r \quad \forall\ r \in N_3, \qquad \sum_{j=1}^{n} \delta'_{rj} = \delta_r \quad \forall\ r \in N_3\\
&\qquad x_j \le y_j, \alpha''_j \ge 0, \beta''_j \ge 0 \ \ \forall\ j = 1, 2, ..., n
\end{aligned}$$

Step 4 As discussed in Step 4 of the Section 4.4 the fuzzy optimal solution of $(P_{5.16})$ can be obtained by solving the problem $(P_{5.30})$:

$$\begin{aligned}
&\text{Maximize/Minimize } \Re\Big(\sum_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}\Big)\\
&\text{subject to}\\
&\qquad \Re\Big(\sum_{j=1}^{n}(m_{pj}, n_{pj}, \gamma'_{pj}, \delta'_{pj})_{LR}\Big) \le \Re(b_p, g_p, \gamma_p, \delta_p)_{LR} \quad \forall\ p \in N_1\\
&\qquad \Re\Big(\sum_{j=1}^{n}(m_{qj}, n_{qj}, \gamma'_{qj}, \delta'_{qj})_{LR}\Big) \ge \Re(b_q, g_q, \gamma_q, \delta_q)_{LR} \quad \forall\ q \in N_2 \qquad\qquad (P_{5.30})
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{n} m_{rj} = b_r \quad \forall\, r \in N_3, && \sum_{j=1}^{n} n_{rj} = g_r \quad \forall\, r \in N_3\\
&\sum_{j=1}^{n} \gamma'_{rj} = \gamma_r \quad \forall\, r \in N_3, && \sum_{j=1}^{n} \delta'_{rj} = \delta_r \quad \forall\, r \in N_3\\
&x_j \le y_j, \alpha''_j \ge 0, \beta''_j \ge 0 \ \ \forall\, j = 1,2,...,n
\end{aligned}$$

Step 5 Assuming $(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR} = (p'_j, q'_j, \alpha'''_j, \beta'''_j)_{LR}$ the problem $(P_{5.30})$ can be written as:

$$\begin{aligned}
&\text{Maximize/Minimize } \Re(\sum_{j=1}^{n} (p'_j, q'_j, \alpha'''_j, \beta'''_j)_{LR})\\
&\text{subject to}\\
&\quad \Re(\sum_{j=1}^{n} (m_{pj}, n_{pj}, \gamma'_{pj}, \delta'_{pj})_{LR}) \le \Re(b_p, g_p, \gamma_p, \delta_p)_{LR} \quad \forall\, p \in N_1\\
&\quad \Re(\sum_{j=1}^{n} (m_{qj}, n_{qj}, \gamma'_{qj}, \delta'_{qj})_{LR}) \ge \Re(b_q, g_q, \gamma_q, \delta_q)_{LR} \quad \forall\, q \in N_2 \qquad\qquad (P_{5.31})\\
&\quad \sum_{j=1}^{n} m_{rj} = b_r \quad \forall\, r \in N_3, \qquad \sum_{j=1}^{n} n_{rj} = g_r \quad \forall\, r \in N_3\\
&\quad \sum_{j=1}^{n} \gamma'_{rj} = \gamma_r \quad \forall\, r \in N_3, \qquad \sum_{j=1}^{n} \delta'_{rj} = \delta_r \quad \forall\, r \in N_3\\
&\quad x_j \le y_j, \alpha''_j \ge 0, \beta''_j \ge 0 \ \ \forall\, j = 1,2,...,n
\end{aligned}$$

Step 6 Using the linearity property $\Re(\sum_{j=1}^{n} \tilde{A}_i) = \sum_{j=1}^{n} \Re(\tilde{A}_i)$, where $\tilde{A}_i$ is a fuzzy number, problem $(P_{5.31})$ can be converted into $(P_{5.32})$:

$$\begin{aligned}
&\text{Maximize/Minimize } \sum_{j=1}^{n} \Re(p'_j, q'_j, \alpha'''_j, \beta'''_j)_{LR}\\
&\text{subject to}\\
&\quad \Re(\sum_{j=1}^{n} (m_{pj}, n_{pj}, \gamma'_{pj}, \delta'_{pj})_{LR}) \le \Re(b_p, g_p, \gamma_p, \delta_p)_{LR} \quad \forall\, p \in N_1\\
&\quad \Re(\sum_{j=1}^{n} (m_{qj}, n_{qj}, \gamma'_{qj}, \delta'_{qj})_{LR}) \ge \Re(b_q, g_q, \gamma_q, \delta_q)_{LR} \quad \forall\, q \in N_2 \qquad\qquad (P_{5.32})\\
&\quad \sum_{j=1}^{n} m_{rj} = b_r \quad \forall\, r \in N_3, \qquad \sum_{j=1}^{n} n_{rj} = g_r \quad \forall\, r \in N_3\\
&\quad \sum_{j=1}^{n} \gamma'_{rj} = \gamma_r \quad \forall\, r \in N_3, \qquad \sum_{j=1}^{n} \delta'_{rj} = \delta_r \quad \forall\, r \in N_3\\
&\quad x_j \le y_j, \alpha''_j \ge 0, \beta''_j \ge 0 \ \ \forall\, j = 1,2,...,n
\end{aligned}$$

Step 7 Using $\Re(m, n, \alpha, \beta) = \frac{1}{2}(\int_0^1 (m - \alpha L^{-1}(\lambda))\, d\lambda + \int_0^1 (n + \beta R^{-1}(\lambda))\, d\lambda)$ problem $(P_{5.32})$ can be converted into $(P_{5.33})$:

Maximize/Minimize $\sum\limits_{j=1}^{n}(\frac{1}{2}(\int_0^1(p'_j - \alpha'''_j L^{-1}(\lambda))\ d\lambda + \int_0^1(q'_j + \beta'''_j R^{-1}(\lambda))\ d\lambda))$

subject to

$$\sum_{j=1}^{n} \tfrac{1}{2}(\int_0^1(m'_{pj} - \gamma'_{pj} L^{-1}(\lambda))\ d\lambda + \int_0^1(n_{pj} + \delta'_{pj} R^{-1}(\lambda))\ d\lambda) \le \tfrac{1}{2}(\int_0^1(b_p - \gamma_p L^{-1}(\lambda))\ d\lambda + \int_0^1(g_p + \delta_p R^{-1}(\lambda))\ d\lambda)\ \forall\, p \in N_1 \qquad (P_{5.33})$$

$$\sum_{j=1}^{n} \tfrac{1}{2}(\int_0^1(m'_{qj} - \gamma'_{qj} L^{-1}(\lambda))\ d\lambda + \int_0^1(n_{qj} + \delta'_{qj} R^{-1}(\lambda))\ d\lambda) \ge \tfrac{1}{2}(\int_0^1(b_q - \gamma_q L^{-1}(\lambda))\ d\lambda + \int_0^1(g_q + \delta_q R^{-1}(\lambda))\ d\lambda)\ \ \forall\ q \in N_2$$

$$\sum_{j=1}^{n} m_{rj} = b_r\ \ \forall\ r \in N_3,\ \sum_{j=1}^{n} n_{rj} = g_r\ \ \forall\ r \in N_3$$

$$\sum_{j=1}^{n} \gamma'_{rj} = \gamma_r\ \ \forall\ r \in N_3,\ \sum_{j=1}^{n} \delta'_{rj} = \delta_r\ \ \forall\ r \in N_3$$

$$x_j \le y_j, \alpha''_j \ge 0, \beta''_j \ge 0\ \ \forall\ j = 1, 2, ..., n$$

Step 8 Solve the crisp non-linear programming problem ($P_{5.33}$) by using an appropriate existing method [125] to find the optimal solution $\{x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*}\}$.

Step 9 Find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ of the fully fuzzy linear programming problem ($P_{5.16}$) by putting the values of $x_j^*, y_j^*, \alpha_j^{''*}$ and $\beta_j^{''*}$ in $\tilde{x}_j^* = (x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*})$.

Step 10 Find the fuzzy optimal value by putting the values of $\tilde{x}_j^*$, obtained from Step 9, in $\sum\limits_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j^*$.

5.4.3 Verification of the proposed methods

In this section, it is depicted that the methods, proposed in Section 5.4.1 and Section 5.4.2, are equivalent and the solution obtained by using any of these two methods will be same.

If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy numbers such that $\tilde{A} = \tilde{B}$ then $\Re(\tilde{A}) = \Re(\tilde{B})$ so the 1st and 2nd constraints i.e., $\sum\limits_{j=1}^{n} \tilde{a}_{pj} \otimes \tilde{x}_j \oplus \tilde{S}_p = \tilde{b}_p \oplus \tilde{S'}_p$ and $\sum\limits_{j=1}^{n} \tilde{a}_{qj} \otimes \tilde{x}_j \oplus \tilde{S}_q = \tilde{b}_q \oplus \tilde{S'}_q$ of problem ($P_{5.19}$) can be written as:

$$\Re(\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j\oplus\tilde{S}_p)=\Re(\tilde{b}_p\oplus\tilde{S'}_p)\ \forall\, p\in N_1 \tag{5.1}$$
$$\Re(\sum_{j=1}^{n}\tilde{a}_{qj}\otimes\tilde{x}_j\oplus\tilde{S}_q)=\Re(\tilde{b}_q\oplus\tilde{S'}_q)\ \forall\, q\in N_2 \tag{5.2}$$

Since, the existing ranking function [141] $\Re$ satisfies the linearity property, so equation (5.1) and equation (5.2) can be written as:

$$\Re(\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j)+\Re(\tilde{S}_p)=\Re(\tilde{b}_p)+\Re(\tilde{S'}_p)\ \forall\, p\in N_1 \tag{5.3}$$
$$\Re(\sum_{j=1}^{n}\tilde{a}_{qj}\otimes\tilde{x}_j)+\Re(\tilde{S}_q)=\Re(\tilde{b}_q)+\Re(\tilde{S'}_q)\ \forall\, q\in N_2 \tag{5.4}$$

Equation (5.3) and equation (5.4) can be written as:

$$\Re(\tilde{b}_p)-\Re(\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j)=\Re(\tilde{S}_p)-\Re(\tilde{S'}_p)\ \forall\, p\in N_1 \tag{5.5}$$
$$\Re(\tilde{b}_q)-\Re(\sum_{j=1}^{n}\tilde{a}_{qj}\otimes\tilde{x}_j)=\Re(\tilde{S}_q)-\Re(\tilde{S'}_q)\ \forall\, q\in N_2 \tag{5.6}$$

Using 4^{th} and 5^{th} constraint of problem $(P_{5.19})$ i.e., $\Re(\tilde{S}_p)-\Re(\tilde{S'}_p)\geq 0$ and $\Re(\tilde{S}_q)-\Re(\tilde{S'}_q)\leq 0$ equation (5.5) and equation (5.6) can be written as:

$$\Re(\tilde{b}_p)-\Re(\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j)\geq 0\ \ \forall\ p\in N_1$$
$$\Re(\tilde{b}_q)-\Re(\sum_{j=1}^{n}\tilde{a}_{qj}\otimes\tilde{x}_j)\leq 0\ \ \forall\ q\in N_2$$

i.e.,

$$\Re(\sum_{j=1}^{n}\tilde{a}_{pj}\otimes\tilde{x}_j)\leq\Re(\tilde{b}_p)\ \ \forall\ p\in N_1$$
$$\Re(\sum_{j=1}^{n}\tilde{a}_{qj}\otimes\tilde{x}_j)\geq\Re(\tilde{b}_q)\ \ \forall\ q\in N_2$$

So, the methods, proposed in Section 5.4.1 and Section 5.4.2 are equivalent hence the solution obtained by using any of these two methods will be same.

5.5 Illustrative Example

In this section, the proposed methods are illustrated with the help of fully fuzzy linear programming problem, chosen in Example 5.10, which cannot be solved by using the existing method [4].

5.5.1 Fuzzy optimal solution of the chosen problem by using the proposed method

Using the method, proposed in Section 5.4.1, the fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 5.10, can be obtained as follows:

Step 1 Using Step 2 of the method, proposed in Section 5.4.1, the fully fuzzy linear programming problem, chosen in Example 5.10, can be written as:

Maximize $((4,4,0,0)_{LR} \otimes \tilde{x}_1 \oplus (1,1,1,1)_{LR} \otimes \tilde{x}_2)$

subject to

$(2,5,5,2)_{LR} \otimes \tilde{x}_1 \oplus (-1,5,1,2)_{LR} \otimes \tilde{x}_2 \oplus \tilde{S} = (-17,45,25,46)_{LR} \oplus \tilde{S}'$

$(1,2,1,1)_{LR} \otimes \tilde{x}_1 \oplus (3,5,2,2)_{LR} \otimes \tilde{x}_2 = (7,24,6,24)_{LR}$

where $\tilde{x}_1, \tilde{x}_2, \tilde{S}, \tilde{S}'$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1-x\}$, $R(x) = \max\{0, 1-x^2\}$.

Step 2 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{LR}$, $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{LR}$, $\tilde{S} = (s,t,\eta,\rho)_{LR}$ and $\tilde{S}' = (s',t',\eta',\rho')_{LR}$ the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((4,4,0,0)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (1,1,1,1)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR})$

subject to

$(2,5,5,2)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (-1,5,1,2)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR} \oplus (s,t,\eta,\rho)_{LR} =$
$(-17,45,25,46)_{LR} \oplus (s',t',\eta',\rho')_{LR}$

$(1,2,1,1)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (3,5,2,2)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR} = (7,24,6,24)_{LR}$

where $(x_1, y_1, \alpha_1, \beta_1)_{LR}, (x_2, y_2, \alpha_2, \beta_2)_{LR}, (s,t,\eta,\rho)_{LR}$ and $(s',t',\eta',\rho')_{LR}$ are LR flat fuzzy numbers.

Step 3 Using the product, proposed in Section 4.2.1, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $((\min\{4x_1,4x_1\},\max\{4y_1,4y_1\},\min\{4x_1,4x_1\}-\min\{4x_1-4\alpha_1,4x_1-4\alpha_1\},$ $\max\{4y_1+4\beta_1,4y_1+4\beta_1\}-\max\{4y_1,4y_1\})_{LR}\oplus(\min\{x_2,x_2\},\max\{y_2,y_2\},\min\{x_2,x_2\}$ $-\min\{0,2x_2-2\alpha_2\},\max\{0,2y_2+2\beta_2\}-\max\{y_2,y_2\})_{LR})$

subject to

$(\min\{2x_1,5x_1\},\max\{2y_1,5y_1\},\min\{2x_1,5x_1\}-\min\{-3y_1-3\beta_1,7x_1-7\alpha_1\},\max\{7y_1+$ $7\beta_1,-3x_1+3\alpha_1\}-\max\{2y_1,5y_1\})_{LR}\oplus(\min\{-y_2,5x_2\},\max\{-x_2,5y_2\},\min\{-y_2,5x_2\}$ $-\min\{-2y_2-2\beta_2,7x_2-7\alpha_2\},\max\{-2x_2+2\alpha_2,7y_2+7\beta_2\}-\max\{-x_2,5y_2\})_{LR}\oplus$ $(s,t,\eta,\rho)_{LR}=(-17,45,25,46)_{LR}\oplus(s',t',\eta',\rho')_{LR}$

$(\min\{x_1,2x_1\},\max\{y_1,2y_1\},\min\{x_1,2x_1\}-\min\{0,3x_1-3\alpha_1\},\max\{0,3y_1+3\beta_1\}-$ $\max\{y_1,2y_1\})_{LR}\oplus(\min\{3x_2,5x_2\},\max\{3y_2,5y_2\},\min\{3x_2,5x_2\}-\min\{x_2-\alpha_2,7x_2-$ $7\alpha_2\},\max\{y_2+\beta_2,7y_2+7\beta_2\}-\max\{3y_2,5y_2\})_{LR}=(7,24,6,24)_{LR}$

where $(x_1,y_1,\alpha_1,\beta_1)_{LR}$, $(x_2,y_2,\alpha_2,\beta_2)_{LR}$, $(s,t,\eta,\rho)_{LR}$ and $(s',t',\eta',\rho')_{LR}$ are LR flat fuzzy numbers.

Step 4 Using the arithmetic operations, defined in Section 4.1.2 and Definition 4.5, the fully fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $((\min\{4x_1,4x_1\}+\min\{x_2,x_2\},\max\{4y_1,4y_1\}+\max\{y_2,y_2\},\min\{4x_1,4x_1\}$ $-\min\{4x_1-4\alpha_1,4x_1-4\alpha_1\}+\min\{x_2,x_2\}-\min\{0,2x_2-2\alpha_2\},\max\{4y_1+4\beta_1,4y_1+$ $4\beta_1\}-\max\{4y_1,4y_1\}+\max\{0,2y_2+2\beta_2\}-\max\{y_2,y_2\})_{LR})$

subject to

$$\min\{2x_1,5x_1\}+\min\{-y_2,5x_2\}+s=-17+s'$$

$$\max\{2y_1,5y_1\}+\max\{-x_2,5y_2\}+t=45+t'$$

$$\min\{x_1,2x_1\}+\min\{3x_2,5x_2\}=7$$

$$\max\{y_1, 2y_1\}+\max\{3y_2, 5y_2\} = 24$$

$\min\{2x_1, 5x_1\}-\min\{-3y_1-3\beta_1, 7x_1-7\alpha_1\}+\min\{-y_2, 5x_2\}-\min\{-2y_2-2\beta_2, 7x_2-7\alpha_2\} + \eta = 25 + \eta'$

$\max\{7y_1+7\beta_1, -3x_1+3\alpha_1\}-\max\{2y_1, 5y_1\}+\max\{-2x_2+2\alpha_2, 7y_2+7\beta_2\}-\max\{-x_2, 5y_2\} + \rho = 46 + \rho'$

$\min\{x_1, 2x_1\}-\min\{0, 3x_1 - 3\alpha_1\}+\min\{3x_2, 5x_2\}-\min\{x_2 - \alpha_2, 7x_2 - 7\alpha_2\} = 6$

$\max\{0, 3y_1 + 3\beta_1\}-\max\{y_1, 2y_1\}+\max\{y_2 + \beta_2, 7y_2 + 7\beta_2\}-\max\{3y_2, 5y_2\} = 24$

$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$

$s \leq t, \eta \geq 0, \rho \geq 0, s' \leq t', \eta' \geq 0, \rho \geq 0$

Step 5 Using Step 7 of the proposed method, the fuzzy linear programming problem, obtained in Step 4, can be written as:

Maximize $\Re((\min\{4x_1, 4x_1\}+\min\{x_2, x_2\}, \max\{4y_1, 4y_1\}+\max\{y_2, y_2\}, \min\{4x_1, 4x_1\} -\min\{4x_1 - 4\alpha_1, 4x_1 - 4\alpha_1\}+\min\{x_2, x_2\}-\min\{0, 2x_2 - 2\alpha_2\}, \max\{4y_1 + 4\beta_1, 4y_1 + 4\beta_1\}-\max\{4y_1, 4y_1\}+\max\{0, 2y_2 + 2\beta_2\}-\max\{y_2, y_2\})_{LR})$

subject to

$$\min\{2x_1, 5x_1\}+\min\{-y_2, 5x_2\} + s = -17 + s'$$

$$\max\{2y_1, 5y_1\}+\max\{-x_2, 5y_2\} + t = 45 + t'$$

$$\min\{x_1, 2x_1\}+\min\{3x_2, 5x_2\} = 7$$

$$\max\{y_1, 2y_1\}+\max\{3y_2, 5y_2\} = 24$$

$\min\{2x_1, 5x_1\}-\min\{-3y_1-3\beta_1, 7x_1-7\alpha_1\}+\min\{-y_2, 5x_2\}-\min\{-2y_2-2\beta_2, 7x_2-7\alpha_2\} + \eta = 25 + \eta'$

$\max\{7y_1+7\beta_1, -3x_1+3\alpha_1\}-\max\{2y_1, 5y_1\}+\max\{-2x_2+2\alpha_2, 7y_2+7\beta_2\}-\max\{-x_2, 5y_2\} + \rho = 46 + \rho'$

$\min\{x_1, 2x_1\}-\min\{0, 3x_1 - 3\alpha_1\}+\min\{3x_2, 5x_2\}-\min\{x_2 - \alpha_2, 7x_2 - 7\alpha_2\} = 6$

$\max\{0, 3y_1 + 3\beta_1\} - \max\{y_1, 2y_1\} + \max\{y_2 + \beta_2, 7y_2 + 7\beta_2\} - \max\{3y_2, 5y_2\} = 24$

$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$

$s \leq t, \eta \geq 0, \rho \geq 0, s' \leq t', \eta' \geq 0, \rho \geq 0$

Step 6 Using Step 9 and Step 10 of the proposed method, the fuzzy linear programming problem, obtained in Step 5, can be written as:

Maximize $\frac{1}{3}|y_2 + \beta_2| - \frac{1}{4}|x_2 - \alpha_2| - \alpha_1 + 2x_1 - \frac{1}{4}\alpha_2 + \frac{1}{2}x_2 + 2y_1 + \frac{4}{3}\beta_1 + \frac{1}{2}y_2 + \frac{1}{3}\beta_2$

subject to

$$\frac{7}{2}x_1 - \frac{3}{2}|x_1| - \frac{1}{2}y_2 + \frac{5}{2}x_2 - \frac{1}{2}|y_2 + 5x_2| + s = -17 + s'$$

$$\frac{7}{2}y_1 + \frac{3}{2}|y_1| - \frac{1}{2}x_2 + \frac{5}{2}y_2 + \frac{1}{2}|x_2 + 5y_2| + t = 45 + t'$$

$$\frac{3}{2}x_1 - \frac{1}{2}|x_1| + 4x_2 - |x_2| = 7$$

$$\frac{3}{2}y_1 + \frac{1}{2}|y_1|_4 y_2 + |y_2| = 24$$

$-\frac{3}{2}|x_1| + \frac{3}{2}y_1 + \frac{3}{2}\beta_1 + \frac{7}{2}\alpha_1 + \frac{1}{2}|3y_1 + 3\beta_1 + 7x_1 - 7\alpha_1| + \frac{1}{2}y_2 - x_2 - \frac{1}{2}|y_2 + 5x_2| + \beta_2 + \frac{7}{2}\alpha_2 + \frac{1}{2}|2y_2 + 2\beta_2 + 7x_2 - 7\alpha_2| + \eta = 25 + \eta'$

$\frac{7}{2}\beta_1 - \frac{3}{2}x_1 + \frac{3}{2}\alpha_1 + \frac{1}{2}|7y_1 + 7\beta_1 + 3x_1 - 3\alpha_1| - \frac{3}{2}|y_1| - \frac{1}{2}x_2 + \alpha_2 + y_2 + \frac{7}{2}\beta_2 + \frac{1}{2}|7y_2 + 7\beta_2 + 2x_2 - 2\alpha_2| - \frac{1}{2}|x_2 + 5y_2| + \rho = 46 + \rho'$

$$-\frac{1}{2}|x_1| + \frac{3}{2}\alpha_1 + \frac{3}{2}|x_1 - \alpha_1| - |x_2| + 4\alpha_2 + 3|x_2 - \alpha_2| = 6$$

$$\frac{3}{2}\beta_1 + \frac{3}{2}|y_1 + \beta_1| - \frac{1}{2}|y_1| + 4\beta_2 + 3|y_2 + \beta_2| - |y_2| = 24$$

$$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

$$s \leq t, \eta \geq 0, \rho \geq 0, s' \leq t', \eta' \geq 0, \rho \geq 0$$

Step 7 The optimal solution of the crisp non-linear programming problem, obtained in Step 6, is $x_1 = 4, y_1 = 4, \alpha_1 = 0, \beta_1 = \frac{51}{335}, x_2 = 1, y_2 = \frac{16}{5}, \alpha_2 = 0$ and $\beta_2 = \frac{629}{335}$.

Step 8 Putting the values of $x_1, y_1, \alpha_1, \beta_1, x_2, y_2, \alpha_2$ and β_2 in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (4, 4, 0, \frac{51}{335})_{LR}, \tilde{x}_2 =$

$(1, \frac{16}{5}, 0, \frac{629}{335})_{LR}$.

Step 9 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 8, in the objective function the fuzzy optimal value is $(17, \frac{96}{5}, 1, \frac{2534}{335})_{LR}$.

5.5.2 Fuzzy optimal solution of the chosen problem by using the alternative proposed method

Using the method, proposed in Section 5.4.2, the fuzzy optimal solution of the fully fuzzy linear programming problem, chosen in Example 5.10, can be obtained as follows:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{LR}$ the fully fuzzy linear programming problem, chosen in Example 5.10, can be written as:

Maximize $((4, 4, 0, 0)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (1, 1, 1, 1)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR})$

subject to

$(2, 5, 5, 2)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (-1, 5, 1, 2)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR} \preceq (-17, 45, 25, 46)_{LR}$

$(1, 2, 1, 1)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1)_{LR} \oplus (3, 5, 2, 2)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)_{LR} = (7, 24, 6, 24)_{LR}$

where $(x_1, y_1, \alpha_1, \beta_1)_{LR}, (x_2, y_2, \alpha_2, \beta_2)_{LR}$ are LR flat fuzzy numbers and $L(x) = \max\{0, 1 - x\}$, $R(x) = \max\{0, 1 - x^2\}$.

Step 2 Using the product, proposed in Section 4.2.1, the fully fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $((\min\{4x_1, 4x_1\}, \max\{4y_1, 4y_1\}, \min\{4x_1, 4x_1\} - \min\{4x_1 - 4\alpha_1, 4x_1 - 4\alpha_1\}, \max\{4y_1 + 4\beta_1, 4y_1 + 4\beta_1\} - \max\{4y_1, 4y_1\})_{LR} \oplus (\min\{x_2, x_2\}, \max\{y_2, y_2\}, \min\{x_2, x_2\} - \min\{0, 2x_2 - 2\alpha_2\}, \max\{0, 2y_2 + 2\beta_2\} - \max\{y_2, y_2\})_{LR})$

subject to

$(\min\{2x_1, 5x_1\}, \max\{2y_1, 5y_1\}, \min\{2x_1, 5x_1\} - \min\{-3y_1 - 3\beta_1, 7x_1 - 7\alpha_1\}, \max\{7y_1 +$

$7\beta_1, -3x_1+3\alpha_1\}-\max\{2y_1, 5y_1\})_{LR}\oplus(\min\{-y_2, 5x_2\},\max\{-x_2, 5y_2\},\min\{-y_2, 5x_2\} -\min\{-2y_2 - 2\beta_2, 7x_2 - 7\alpha_2\},\max\{-2x_2 + 2\alpha_2, 7y_2 + 7\beta_2\}-\max\{-x_2, 5y_2\})_{LR} \preceq (-17, 45, 25, 46)_{LR}$

$(\min\{x_1, 2x_1\},\max\{y_1, 2y_1\},\min\{x_1, 2x_1\}-\min\{0, 3x_1 - 3\alpha_1\},\max\{0, 3y_1 + 3\beta_1\} - \max\{y_1, 2y_1\})_{LR} \oplus (\min\{3x_2, 5x_2\},\max\{3y_2, 5y_2\},\min\{3x_2, 5x_2\}-\min\{x_2 - \alpha_2, 7x_2 - 7\alpha_2\},\max\{y_2 + \beta_2, 7y_2 + 7\beta_2\}-\max\{3y_2, 5y_2\})_{LR} = (7, 24, 6, 24)_{LR}$

where $(x_1, y_1, \alpha_1, \beta_1)_{LR}$ and $(x_2, y_2, \alpha_2, \beta_2)_{LR}$ are LR flat fuzzy numbers.

Step 3 Using Step 4 of the proposed method, the fully fuzzy linear programming problem, obtained in Step 2, can be written as:

Maximize $\Re((\min\{4x_1, 4x_1\},\max\{4y_1, 4y_1\},\min\{4x_1, 4x_1\}-\min\{4x_1-4\alpha_1, 4x_1-4\alpha_1\}, \max\{4y_1+4\beta_1, 4y_1+4\beta_1\}-\max\{4y_1, 4y_1\})_{LR}\oplus(\min\{x_2, x_2\},\max\{y_2, y_2\},\min\{x_2, x_2\} -\min\{0, 2x_2 - 2\alpha_2\},\max\{0, 2y_2 + 2\beta_2\}-\max\{y_2, y_2\})_{LR})$

subject to

$\Re((\min\{2x_1, 5x_1\},\max\{2y_1, 5y_1\},\min\{2x_1, 5x_1\}-\min\{-3y_1-3\beta_1, 7x_1-7\alpha_1\},\max\{7y_1+ 7\beta_1, -3x_1+3\alpha_1\}-\max\{2y_1, 5y_1\})_{LR}\oplus(\min\{-y_2, 5x_2\},\max\{-x_2, 5y_2\},\min\{-y_2, 5x_2\} -\min\{-2y_2 - 2\beta_2, 7x_2 - 7\alpha_2\},\max\{-2x_2 + 2\alpha_2, 7y_2 + 7\beta_2\}-\max\{-x_2, 5y_2\})_{LR}) \leq \Re(-17, 45, 25, 46)_{LR}$

$$\min\{x_1, 2x_1\}+\min\{3x_2, 5x_2\} = 7$$

$$\max\{y_1, 2y_1\}+\max\{3y_2, 5y_2\} = 24$$

$\min\{x_1, 2x_1\}-\min\{0, 3x_1 - 3\alpha_1\}+\min\{3x_2, 5x_2\}-\min\{x_2 - \alpha_2, 7x_2 - 7\alpha_2\} = 6$

$\max\{0, 3y_1 + 3\beta_1\}-\max\{y_1, 2y_1\}+\max\{y_2 + \beta_2, 7y_2 + 7\beta_2\}-\max\{3y_2, 5y_2\})_{LR} = 24$

$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$

Step 4 Using Step 6 and Step 7 of the proposed method, the fully fuzzy linear programming problem, obtained in Step 3, can be written as:

Maximize $\frac{1}{3}|y_2+\beta_2| - \frac{1}{4}|x_2-\alpha_2| - \alpha_1 + 2x_1 - \frac{1}{4}\alpha_2 + \frac{1}{2}x_2 + 2y_1 + \frac{4}{3}\beta_1 + \frac{1}{2}y_2 + \frac{1}{3}\beta_2$

subject to

$-\frac{3}{2}|5x_2+y_2| + |x_2+5y_2| - \frac{3}{2}|-7x_1+7\alpha_1-3y_1-3\beta_1| + 2|-3x_1+3\alpha_1-7y_1-7\beta_1| - \frac{9}{2}|x_1| + 3|y_1| + 2|2\alpha_2-2x_2-7y_2-7\beta_2| - \frac{3}{2}|7\alpha_2-7x_2-2y_2-2\beta_2| + 15x_1 - \frac{9}{2}\alpha_1 - \frac{13}{2}\alpha_2 + 13x_2 + \frac{33}{2}y_1 + \frac{19}{2}\beta_1 + \frac{29}{2}y_2 + 11\beta_2 \leq 277$

$$\frac{3}{2}x_1 - \frac{1}{2}|x_1| + 4x_2 - |x_2| = 7$$

$$\frac{3}{2}y_1 + \frac{1}{2}|y_1|_4y_2 + |y_2| = 24$$

$$-\frac{1}{2}|x_1| + \frac{3}{2}\alpha_1 + \frac{3}{2}|x_1-\alpha_1| - |x_2| + 4\alpha_2 + 3|x_2-\alpha_2| = 6$$

$$\frac{3}{2}\beta_1 + \frac{3}{2}|y_1+\beta_1| - \frac{1}{2}|y_1| + 4\beta_2 + 3|y_2+\beta_2| - |y_2| = 24$$

$$x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

Step 5 The optimal solution of the crisp non-linear programming problem, obtained in Step 4, is $x_1 = 4, y_1 = 4, \alpha_1 = 0, \beta_1 = \frac{51}{335}, x_2 = 1, y_2 = \frac{16}{5}, \alpha_2 = 0$ and $\beta_2 = \frac{629}{335}$.

Step 6 Putting the values of $x_1, y_1, \alpha_1, \beta_1, x_2, y_2, \alpha_2$ and β_2 in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)$, the exact fuzzy optimal solution is $\tilde{x}_1 = (4, 4, 0, \frac{51}{335})_{LR}, \tilde{x}_2 = (1, \frac{16}{5}, 0, \frac{629}{335})_{LR}$.

Step 7 Putting the values of $\tilde{x}_1$ and $\tilde{x}_2$, obtained from Step 6, in the objective function, the fuzzy optimal value is $(17, \frac{96}{5}, 1, \frac{2534}{335})_{LR}$.

5.6 Advantages of the proposed methods

In this section, the advantages of the proposed methods over the existing methods [4, 41, 87, 88, 91, 100] and the methods, proposed in previous chapters are discussed.

The main advantage of the methods, proposed in this chapter, is that all the fully fuzzy linear programming problems which can be solved by using the existing methods [4, 41, 87, 88, 91, 100] and the methods proposed in previous chapters can also be solved by using the methods proposed in this chapter. Furthermore, on solving these problems by using the existing methods [4, 41, 87, 88, 91, 100], methods proposed in previous chapters and the methods proposed in this chapter, same results are obtained.

However, there exist several problems which cannot be solved by using the existing methods [4,41,87,88,91,100] and the methods, proposed in previous chapters but can be solved by using the methods, proposed in this chapter.

5.7 Comparative study

To compare the existing methods [4, 41, 87, 88, 91, 100] and the methods proposed in this chapter, the results of fuzzy linear programming problems and fully fuzzy linear programming problems, obtained by using the existing methods and the methods proposed in this chapter, are shown in Table 5.1.

Table 5.1 Results of chosen problems by proposed and existing methods

Example	Fuzzy optimal value						
	Existing Method [91]	Existing Method [87]	Existing Method [100]	Existing Method [41]	Existing Method [88]	Existing Method [4]	Proposed Methods
5.1	$(\frac{151}{24}, \frac{277}{32}, \frac{227}{96}, \frac{377}{96})$	$(\frac{151}{24}, \frac{277}{32}, \frac{227}{96}, \frac{377}{96})$	Not Applicable	Not Applicable	Not Applicable	$(\frac{151}{24}, \frac{277}{32}, \frac{227}{96}, \frac{377}{96})$	$(\frac{151}{24}, \frac{277}{32}, \frac{227}{96}, \frac{377}{96})$
5.2	$(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7})$	$(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7})$	$(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7})$	Not Applicable	Not Applicable	$(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7})$	$(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7})$
5.3	Not Applicable	Not Applicable	Not Applicable	$(\frac{94235}{169}, \frac{120265}{169}, \frac{19819}{169}, \frac{19819}{169})$	Not Applicable	$(\frac{8250}{13}, \frac{8250}{13}, 0, 0)$	$(\frac{8250}{13}, \frac{8250}{13}, 0, 0)$
5.4	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(\frac{-62}{7}, \frac{300}{7}, \frac{360}{7}, \frac{418}{7})$	$(\frac{267}{14}, \frac{267}{14}, 0, 0)$	$(\frac{267}{14}, \frac{267}{14}. 0, 0)$
5.5	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(8.25, 16.5, 79.25)_{LR}$	Infeasible solution
5.6	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(\frac{179}{121}, \frac{2037}{242}, \frac{456}{121}, \frac{929}{242})_{LR}$
5.7	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(\frac{58}{7}, \frac{148}{7}, \frac{90}{7}, \frac{84}{7})_{LR}$
5.8	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(0, 55, 11, 33)_{LR}$
5.9	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(\frac{168}{9}, \frac{168}{9}, 0, 0)_{LR}$
5.10	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(17, \frac{96}{5}, 1, \frac{2534}{335})_{LR}$
5.11	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable	$(\frac{2579}{110}, \frac{2579}{110}, \frac{67}{110}, \frac{67}{110})_{LR}$

It is obvious from the results, shown in Table 5.1, that the fuzzy linear programming problems and fully fuzzy linear programming problems, chosen in Example 5.6 to Example 5.11, cannot be solved by using any of the existing methods but all these problems can be solved by using proposed methods. Also, all the fuzzy linear programming problems and fully fuzzy linear programming problems, chosen in Example 5.1 to Example 5.5, for which there is a need to apply the different existing methods can be easily solved by using proposed methods i.e., all the fuzzy linear programming problems and fully fuzzy linear programming problems which may or may not be solved by using the existing methods can be solved by using proposed methods.

Remark 5.1 Let $\tilde{A} = (m_1, \alpha_1, \beta_1)_{LR}$ be a non-positive LR fuzzy number and $\tilde{B} = (m_2, \alpha_2, \beta_2)_{LR}$ be a non-negative LR fuzzy number. Then,

(i) According to the existing product [4],

$$\tilde{A} \otimes \tilde{B} = (m_1 m_2, \alpha_1 m_2 - m_1 \alpha_2, \beta_1 m_2 - m_1 \beta_2)_{LR}$$

(ii) According to the existing product [30],

$$\tilde{A} \otimes \tilde{B} = (m_1 m_2, \alpha_1 m_2 - m_1 \beta_2, \beta_1 m_2 - m_1 \alpha_2)_{LR}$$

Using the product (i) Example 5.5 has a fuzzy optimal solution but by using product (ii) and by using the proposed product Example 5.5 has infeasible solution.

5.8 Conclusions

On the basis of the presented study, it can be concluded that it is better to use the methods, proposed in this chapter as compared to the existing methods [4, 41, 87, 88, 91, 100] and methods proposed in previous chapters for solving fully fuzzy linear programming problems.

Chapter 6

A New Method To Find The Unique Fuzzy Optimal Value Of Fully Fuzzy Linear Programming Problems With Equality Constraints

In this chapter, it is shown that the fuzzy optimal value, obtained by using the method, proposed in Chapter 4, is not necessarily a unique fuzzy number. So, it does not conform to the uniqueness property of fuzzy optimal value. To overcome this limitation of the method, proposed in Chapter 4, a new method is proposed for solving fully fuzzy linear programming problems with equality constraints.

6.1 Limitations of the previous proposed method

In this section, limitations of the method for solving fully fuzzy linear programming problems with equality constraints, proposed in Chapter 4, are pointed out.

Let $\{x_j\}$ and A be the optimal solution and optimal value of a linear programming problem respectively. If there exist any feasible solution $\{y_j\}$ of the same

---

The contents of this chapter are published in *Control and Cybernetics* 41 (2012) 171-182 which are the generalization of the contents of the paper accepted for publication in *Journal of Optimization Theory and Applications*.

linear programming problem such that the value of the objective function of the linear programming problem corresponding to $\{y_j\}$ is also A then $\{y_j\}$ is said to be an alternative optimal solution of the same linear programming problem i.e. corresponding to all alternative optimal solutions the values of objective function should be same.

In this section, a fully fuzzy linear programming problem, chosen in Example 6.1, is solved to show that the results of the fully fuzzy linear programming problem, chosen in Example 6.1, obtained by using the method, proposed in Chapter 4, do not conform to the property of alternative optimal solutions.

Example 6.1 Maximize $((2,2,2,2)_{LR} \otimes \tilde{x}_1 \oplus (7,12,2,2)_{LR} \otimes \tilde{x}_2 \oplus (7,8,2,2)_{LR} \otimes \tilde{x}_3)$

subject to

$$\tilde{x}_1 \oplus \tilde{x}_2 = (1,1,0,0)_{LR}$$

$$\tilde{x}_1 = \tilde{x}_3$$

$$\tilde{x}_2 \oplus \tilde{x}_3 = (1,1,0,0)_{LR}$$

where $\tilde{x}_1, \tilde{x}_2,\ \tilde{x}_3$ are non-negative LR flat fuzzy numbers and $L(x) = R(x) = \max\{0, 1-x\}$.

Solution: On solving the chosen fully fuzzy linear programming problem by using the method, proposed in Chapter 4, it is found that all the fuzzy feasible solutions $\tilde{x}_1 = \tilde{x}_3 = (a,a,0,0)_{LR}$ and $\tilde{x}_2 = (1-a, 1-a, 0, 0)_{LR}$, $0 \leq a \leq 1$ are fuzzy optimal solutions of the chosen fully fuzzy linear programming problem. Putting $\tilde{x}_1 = \tilde{x}_3 = (a,a,0,0)_{LR}$ and $\tilde{x}_2 = (1-a, 1-a, 0, 0)_{LR}$ in the objective function the fuzzy optimal value is $(7+2a, 12-2a, 2+2a, 2+2a)_{LR}$. Since, the fuzzy optimal value is depending upon 'a' so for the chosen fully fuzzy linear programming problem infinite fuzzy numbers, representing the fuzzy optimal values, can be obtained.

Since, the fuzzy optimal values of the fully fuzzy linear programming problem, chosen in Example 6.1, corresponding to alternative fuzzy optimal solutions are not equal i.e., the results obtained by using the method, proposed in Chapter 4, do not conform to the uniqueness property of fuzzy optimal value of a fully fuzzy linear programming problem. So, it is not genuine to apply the method, proposed in Chapter 4, for solving fully fuzzy linear programming problems.

6.2 Proposed method based on RMDS approach

Kumar et al. [66] used four parameters Rank , Mode, Divergence and Left spread for comparing LR flat fuzzy numbers. It can be easily seen that if $\tilde{A}$ and $\tilde{B}$ are two LR flat fuzzy numbers such that Rank($\tilde{A}$) = Rank($\tilde{B}$), Mode($\tilde{A}$) = Mode($\tilde{B}$), Divergence($\tilde{A}$) = Divergence($\tilde{B}$) and Left spread($\tilde{A}$) = Left spread($\tilde{B}$) then $\tilde{A} = \tilde{B}$.

In this section, to overcome the limitations of the method, proposed in Chapter 4, a new method, based on RMDS approach [66], is proposed for solving fully fuzzy linear programming problems.

6.2.1 RMDS approach

In this section, the existing ranking approach [66] for comparing LR flat fuzzy numbers is presented.

Let $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers then use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1 Find Rank($\tilde{A}$) $= \frac{1}{2}[\int_0^1 m_1 d\lambda - \int_0^1 \alpha_1 L^{-1}(\lambda) d\lambda + \int_0^1 n_1 d\lambda + \int_0^1 \beta_1 R^{-1}(\lambda) d\lambda]$ and Rank($\tilde{B}$) $= \frac{1}{2}[\int_0^1 m_2 d\lambda - \int_0^1 \alpha_2 L^{-1}(\lambda) d\lambda + \int_0^1 n_2 d\lambda + \int_0^1 \beta_2 R^{-1}(\lambda) d\lambda]$

Case (i) If Rank($\tilde{A}$) $>$ Rank($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$

Case (ii) If Rank($\tilde{A}$) $<$ Rank($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$

Case (iii) If Rank($\tilde{A}$) = Rank($\tilde{B}$) then go to Step 2.

Step 2 Find Mode($\tilde{A}$) $= \frac{1}{2}\int_0^1 (m_1 + n_1) d\lambda$ and Mode($\tilde{B}$) $= \frac{1}{2}\int_0^1 (m_2 + n_2) d\lambda$

Case (i) If Mode($\tilde{A}$) > Mode($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$

Case (ii) If Mode($\tilde{A}$) < Mode($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$

Case (iii) If Mode($\tilde{A}$) = Mode($\tilde{B}$) then go to Step 3

Step 3 Find Divergence($\tilde{A}$) $= \int_0^1 n_1 d\lambda + \int_0^1 \beta_1 R^{-1}(\lambda) d\lambda - \int_0^1 m_1 d\lambda + \int_0^1 \alpha_1 L^{-1}(\lambda) d\lambda$ and Divergence($\tilde{B}$) $= \int_0^1 n_2 d\lambda + \int_0^1 \beta_2 R^{-1}(\lambda) d\lambda - \int_0^1 m_2 d\lambda + \int_0^1 \alpha_2 L^{-1}(\lambda) d\lambda$

Case (i) If Divergence($\tilde{A}$) > Divergence($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$

Case (ii) If Divergence($\tilde{A}$) < Divergence($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$

Case (iii) If Divergence($\tilde{A}$) = Divergence($\tilde{B}$) then go to Step 4

Step 4 Find Left spread($\tilde{A}$) $= \int_0^1 \alpha_1 L^{-1}(\lambda) d\lambda$ and Left spread($\tilde{B}$) $= \int_0^1 \alpha_2 L^{-1}(\lambda) d\lambda$

Case (i) If Left spread($\tilde{A}$) > Left spread($\tilde{B}$) then $\tilde{A} \succ \tilde{B}$

Case (ii) If Left spread($\tilde{A}$) < Left spread($\tilde{B}$) then $\tilde{A} \prec \tilde{B}$

Case (iii) If Left spread($\tilde{A}$) = Left spread($\tilde{B}$) then $\tilde{A} = \tilde{B}$.

6.2.2 Proposed method

In this section, to overcome the limitations of the method, proposed in Chapter 4, a new method is proposed for solving fully fuzzy linear programming problems ($P_{4.1}$).

Step 1 Use Step 1 to Step 8 of the method, proposed in Chapter 4, and check that an alternative optimal solution of crisp linear programming problem ($P_{4.9}$) exist or not.

Case (i) If an alternative optimal solution does not exist then the fuzzy optimal solution, obtained by using the method, proposed in Chapter 4, is exact fuzzy optimal solution of the fully fuzzy linear programming problem ($P_{4.1}$).

Case (ii) If alternative optimal solution exist then Go to Step 2.

Step 2 Let the optimal value of the problem ($P_{4.9}$) be ‘a’ and it occurs corresponding to ‘p’ basic feasible solutions $\{x_j^k, y_j^k, \alpha_j^{''k}, \beta_j^{''k}\}$ where $k = 1, ..., p$. Now, our aim is to find $\underset{1\leq k\leq p}{\max \text{ (or min)}}\{\sum\limits_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j^k, y_j^k, \alpha_j^{''k}, \beta_j^{''k})_{LR}\}$. Since, Rank$(\sum\limits_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j^k, y_j^k, \alpha_j^{''k}, \beta_j^{''k})_{LR}) = a \;\forall\; k = 1, ..., p$ so using Step 2 of the RMDS approach , discussed in Section 6.2.1 if $\underset{1\leq t\leq l}{\max \text{ (or min)}}\{\text{Mode}(\sum\limits_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j^k, y_j^k, \alpha_j^{''k}, \beta_j^{''k})_{LR})\}$ is Mode$(\sum\limits_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j^{\phi}, y_j^{\phi}, \alpha_j^{''\phi}, \beta_j^{''\phi})_{LR})$ then $\underset{1\leq t\leq l}{\max \text{ (or min)}}\{\sum\limits_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j^k, y_j^k, \alpha_j^{''k}, \beta_j^{''k})_{LR}\}$ will also be $\sum\limits_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j^{\phi}, y_j^{\phi}, \alpha_j^{''\phi}, \beta_j^{''\phi})_{LR}$ i.e., the fuzzy optimal solution of ($P_{4.1}$) can be obtained by solving the problem ($P_{6.1}$):

$$\text{Maximize/Minimize } (\text{Mode}(\sum_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}))$$

subject to

$$\left.\begin{aligned}
&\sum_{j=1}^{n} m_{ij} = b_i \;\;\forall\; i = 1, 2, ..., m\\
&\sum_{j=1}^{n} n_{ij} = g_i \;\;\forall\; i = 1, 2, ..., m\\
&\sum_{j=1}^{n} \gamma'_{ij} = \gamma_i \;\;\forall\; i = 1, 2, ..., m\\
&\sum_{j=1}^{n} \delta'_{ij} = \delta_i \;\;\forall\; i = 1, 2, ..., m\\
&\text{Rank}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = a\\
&x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0 \;\;\forall\; j = 1, 2, ..., n
\end{aligned}\right. \qquad (P_{6.1})$$

Case (i) If there does not exist any alternative optimal solution then the solution obtained from problem ($P_{6.1}$) is the optimal solution. Put the values of $x_j^*, y_j^*, \alpha_j^{''*}$ and $\beta_j^{''*}$ in $\tilde{x}_j^* = (x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*})_{LR}$ to find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ and find the fuzzy optimal value $\sum\limits_{j=1}^{n}(\tilde{c}_j \otimes \tilde{x}_j^*)$ by putting the values of $\tilde{x}_j^*$.

Case (ii) If alternative solution exist then Go to Step 3.

Step 3 On the same direction, as discussed in Step 2, solve the problem ($P_{6.2}$) and check that alternative optimal solution exist or not:

$$\text{Maximize/Minimize (Divergence}(\sum_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} m_{ij} = b_i \quad \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n} n_{ij} = g_i \quad \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n} \gamma'_{ij} = \gamma_i \quad \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n} \delta'_{ij} = \delta_i \quad \forall\ i = 1, 2, ..., m\\
&\text{Rank}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = a\\
&\text{Mode}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = b\\
&x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0 \quad \forall\ j = 1, 2, ..., n
\end{aligned} \qquad (P_{6.2})$$

where 'b' is the optimal value of the problem ($P_{6.1}$).

Case (i) If there does not exist any alternative optimal solution then the solution obtained from problem ($P_{6.2}$) is the optimal solution. Put the values of $x_j^*, y_j^*, \alpha_j^{''*}$ and $\beta_j^{''*}$ in $\tilde{x}_j^* = (x_j^*, y_j^*, \alpha_j^{''*}, \beta_j^{''*})_{LR}$ to find the fuzzy optimal solution $\{\tilde{x}_j^*\}$ and find the fuzzy optimal value $\sum_{j=1}^{n}(\tilde{c}_j \otimes \tilde{x}_j^*)$ by putting the values of $\tilde{x}_j^*$.

Case (ii) If alternative solution exist then Go to Step 4.

Step 4 On the same direction, as discussed in Step 2, solve the problem ($P_{6.3}$):

$$\text{Maximize/Minimize (Left spread}(\sum_{j=1}^{n}(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{LR}))$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} m_{ij} = b_i \quad \forall\ i = 1, 2, ..., m\\
&\sum_{j=1}^{n} n_{ij} = g_i \quad \forall\ i = 1, 2, ..., m
\end{aligned}$$

$$\sum_{j=1}^{n} \gamma'_{ij} = \gamma_i \;\; \forall\; i = 1, 2, ..., m \qquad (P_{6.3})$$
$$\sum_{j=1}^{n} \delta'_{ij} = \delta_i \;\; \forall\; i = 1, 2, ..., m$$
$$\text{Rank}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = a$$
$$\text{Mode}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = b$$
$$\text{Divergence}(\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j) = c$$
$$x_j \leq y_j, \alpha''_j \geq 0, \beta''_j \geq 0 \;\; \forall\; j = 1, 2, ..., n$$

where 'c' is the optimal value of the problem $(P_{6.2})$.

Find the fuzzy optimal solution $\{\tilde{x}^*_j\}$ by putting the values of $x^*_j, y^*_j, \alpha''^*_j$ and β''^*_j in $\tilde{x}^*_j = (x^*_j, y^*_j, \alpha''^*_j, \beta''^*_j)_{LR}$ and fuzzy optimal value $\sum_{j=1}^{n}(\tilde{c}_j \otimes \tilde{x}^*_j)$ by putting the values of $\tilde{x}^*_j$.

6.3 Illustrative example

In this section, to illustrate the proposed method fully fuzzy linear programming problem, chosen in Example 6.1, is solved. The exact fuzzy optimal solution of the chosen problem can be obtained by using the following steps:

Step 1 Assuming $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{LR}$, $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{LR}$ and $\tilde{x}_3 = (x_3, y_3, \alpha_3, \beta_3)_{LR}$ and using Step 1 to Step 8 of the method, proposed in Chapter 4, the fully fuzzy linear programming problem, chosen in Example 6.1, can be written as:

Maximize $\frac{1}{4}(2x_1 + 6y_1 + 4\beta_1 + 12x_2 + 26y_2 - 5\alpha_2 + 14\beta_2 + 12x_3 + 18y_3 - 5\alpha_3 + 10\beta_3)$

subject to

$$x_1 + x_2 = 1,\; y_1 + y_2 = 1,\; \alpha_1 + \alpha_2 = 0,\; \beta_1 + \beta_2 = 0$$

$$x_1 = x_3,\; y_1 = y_3,\; \alpha_1 = \alpha_3,\; \beta_1 = \beta_3$$

$$x_2 + x_3 = 1,\; y_2 + y_3 = 1,\; \alpha_2 + \alpha_3 = 0,\; \beta_2 + \beta_3 = 0$$

$$x_1 \geq 0, x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0$$

$$x_2 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

$$x_3 \geq 0, x_3 \leq y_3, \alpha_3 \geq 0, \beta_3 \geq 0$$

Step 2 Since, on solving the crisp linear programming problem, obtained in Step 1, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem is $\frac{19}{2}$ so, using Step 2 of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

Maximize $(x_1 + \frac{7}{2}x_2 + \frac{7}{2}x_3 + y_1 + 6y_2 + 4y_3)$

subject to

$$x_1 + x_2 = 1,\ y_1 + y_2 = 1,\ \alpha_1 + \alpha_2 = 0,\ \beta_1 + \beta_2 = 0$$

$$x_1 = x_3,\ y_1 = y_3,\ \alpha_1 = \alpha_3,\ \beta_1 = \beta_3$$

$$x_2 + x_3 = 1,\ y_2 + y_3 = 1,\ \alpha_2 + \alpha_3 = 0,\ \beta_2 + \beta_3 = 0$$

$$2x_1 + 6y_1 + 4\beta_1 + 12x_2 + 26y_2 - 5\alpha_2 + 14\beta_2$$

$$+ 12x_3 + 18y_3 - 5\alpha_3 + 10\beta_3 = 38$$

$$x_1 \geq 0, x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0$$

$$x_2 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

$$x_3 \geq 0, x_3 \leq y_3, \alpha_3 \geq 0, \beta_3 \geq 0$$

Step 3 Since, on solving the crisp linear programming problem, obtained in Step 2, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem is $\frac{19}{2}$ so, using Step 3 of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

Maximize $(4y_1 + 14y_2 + 10y_3 + 4\beta_1 + 14\beta_2 + 10\beta_3 - 5x_2 - 5x_3 + 5\alpha_2 + \alpha_3)$

subject to

$$x_1 + x_2 = 1,\ y_1 + y_2 = 1,\ \alpha_1 + \alpha_2 = 0,\ \beta_1 + \beta_2 = 0$$

$$x_1 = x_3,\ y_1 = y_3,\ \alpha_1 = \alpha_3,\ \beta_1 = \beta_3$$

$$x_2 + x_3 = 1,\ y_2 + y_3 = 1,\ \alpha_2 + \alpha_3 = 0,\ \beta_2 + \beta_3 = 0$$

$$2x_1 + 6y_1 + 4\beta_1 + 12x_2 + 26y_2 - 5\alpha_2 + 14\beta_2$$

$$+ 12x_3 + 18y_3 - 5\alpha_3 + 10\beta_3 = 38$$

$$x_1 + \tfrac{7}{2}x_2 + \tfrac{7}{2}x_3 + y_1 + 6y_2 + 4y_3 = \tfrac{19}{2}$$

$$x_1 \geq 0, x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0$$

$$x_2 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

$$x_3 \geq 0, x_3 \leq y_3, \alpha_3 \geq 0, \beta_3 \geq 0$$

Step 4 Since, on solving the crisp linear programming problem, obtained in Step 3, alternative optimal solution is obtained and the optimal value of the crisp linear programming problem is 9 so, using Step 4 of the proposed method the solution of the chosen problem can be obtained by solving the following crisp linear programming problem:

Maximize $(2x_1 + 2x_2 + 2x_3 + 5\alpha_2 + 5\alpha_3)$

subject to

$$x_1 + x_2 = 1,\ y_1 + y_2 = 1,\ \alpha_1 + \alpha_2 = 0,\ \beta_1 + \beta_2 = 0$$

$$x_1 = x_3,\ y_1 = y_3,\ \alpha_1 = \alpha_3,\ \beta_1 = \beta_3$$

$$x_2 + x_3 = 1,\ y_2 + y_3 = 1,\ \alpha_2 + \alpha_3 = 0,\ \beta_2 + \beta_3 = 0$$

$$2x_1 + 6y_1 + 4\beta_1 + 12x_2 + 26y_2 - 5\alpha_2 + 14\beta_2$$

$$+ 12x_3 + 18y_3 - 5\alpha_3 + 10\beta_3 = 38$$

$$x_1 + \tfrac{7}{2}x_2 + \tfrac{7}{2}x_3 + y_1 + 6y_2 + 4y_3 = \tfrac{19}{2}$$

$$4y_1 + 14y_2 + 10y_3 + 4\beta_1 + 14\beta_2 + 10\beta_3 - 5x_2 - 5x_3 + 5\alpha_2 + \alpha_3 = 9$$

$$x_1 \geq 0, x_1 \leq y_1, \alpha_1 \geq 0, \beta_1 \geq 0$$

$$x_2 \geq 0, x_2 \leq y_2, \alpha_2 \geq 0, \beta_2 \geq 0$$

$$x_3 \geq 0, x_3 \leq y_3, \alpha_3 \geq 0, \beta_3 \geq 0$$

The obtained optimal solution is $x_1 = 1, y_1 = 1, \alpha_1 = 0, \beta_1 = 0, x_2 = 0, y_2 = 0, \alpha_2 = 0, \beta_2 = 0, x_3 = 1, y_3 = 1, \alpha_3 = 0$ and $\beta_3 = 0$. Using Step 4 of the proposed method the fuzzy optimal solution is $\tilde{x}_1 = (1,1,0,0)_{LR}, \tilde{x}_2 = (0,0,0,0)_{LR}$, $\tilde{x}_3 = (1,1,0,0)_{LR}$ and the fuzzy optimal value is $(9,10,4,4)_{LR}$.

6.4 Advantages of the proposed method

As discussed in Section 6.1, by using the method, proposed in Chapter 4, a unique fuzzy number, representing the fuzzy optimal value, is not obtained which do not conform to the uniqueness property of fuzzy optimal value of fully fuzzy linear programming problems. While, by using the proposed method always a unique fuzzy number, representing the fuzzy optimal value, will be obtained i.e., on solving the fully fuzzy linear programming problems by using the proposed method the uniqueness property of fuzzy optimal value will always be preserved.

6.5 Comparative study

The results of the fully fuzzy linear programming problems, chosen in Example 6.1, Example 4.1 and Example 4.2, obtained by using the method, proposed in Chapter 4, and the method proposed in this chapter, are shown in Table 6.1.

Table 6.1 Results of the chosen fully fuzzy linear programming problems

Example	Fuzzy optimal value	
	Method proposed in Chapter 4	Method proposed in this chapter
4.1	$(14, 48, 58, 50)_{LR}$	$(14, 48, 58, 50)_{LR}$
4.2	$(3, 8, 5, 10)_{LR}$	$(3, 8, 5, 10)_{LR}$
6.1	$(7+2a, 12-2a, 2+2a, 2+2a)_{LR}, 0 \leq a \leq 1$	$(9, 10, 4, 4)_{LR}$

It is obvious from the results, shown in Table 6.1, that on solving the fully fuzzy linear programming problems, chosen in Example 4.1 and Example 4.2, by using the method, proposed in Chapter 4, a unique fuzzy number, representing the fuzzy optimal value, is obtained. While, on solving the fully fuzzy linear programming problem, chosen in Example 6.1, by using the method, proposed in Chapter 4, infinite fuzzy numbers, representing the fuzzy optimal value of the same problem, are obtained which do not conform to the uniqueness property of fuzzy optimal value of a fully fuzzy linear programming problems. However, on solving all the chosen fully fuzzy linear programming problems by using the method proposed in this chapter a unique fuzzy number, representing the fuzzy optimal, is obtained.

6.6 Conclusions

On the basis of presented study, it can be concluded that it is better to use the method, proposed in this chapter, as compared to the method, proposed in Chapter 4, for solving fully fuzzy linear programming problems with equality constraints.

Chapter 7

GENERAL FORM OF FULLY FUZZY LINEAR PROGRAMMING PROBLEMS

In this chapter, it is pointed out that the existing general from of fully fuzzy linear programming problems is valid only if there does not exist subtraction of fuzzy numbers in the fully fuzzy linear programming problems. However, if there exist subtraction of fuzzy numbers in the fully fuzzy linear programming problems then either the existing general form of fully fuzzy linear programming problems is not valid or the obtained fuzzy optimal solution is not genuine.

7.1 Existing general form of fully fuzzy linear programming problems

In the existing methods [4, 36, 87, 88, 91, 98–102, 104] it is assumed that any fully fuzzy linear programming problem is obtained by replacing the crisp parameters c_j, a_{ij}, b_j and x_j by fuzzy parameters $\tilde{c}_j$, $\tilde{a}_{ij}$, $\tilde{b}_j$ and $\tilde{x}_j$ respectively. The general form of crisp linear programming problems is:

$$\text{Maximize/Minimize } \Big(\sum_{j:j\in N_1} c_j x_j - \sum_{j:j\in N_2} c_j x_j\Big)$$

$$\text{subject to} \qquad (P_{7.1})$$

The contents of this chapter are communicated in *Journal of Optimization Theory and Applications*.

$$\sum_{j:j\in N_3} a_{ij}x_j - \sum_{j:j\in N_4} a_{ij}x_j \leq, =, \geq b_i \quad \forall\ i = 1, 2, ..., m$$

where x_j, a_{ij}, b_i, c_j are any real numbers and $N_1 \cup N_2 = \{1, 2, ..., n\}, N_3 \cup N_4 = \{1, 2, ..., n\}, N_1 \cap N_2 = \phi, N_3 \cap N_4 = \phi$.

If it is assumed that there is a need to represent all the parameters of crisp linear programming problem $(P_{7.1})$ by fuzzy parameters then it will be converted into fully fuzzy linear programming problem:

$$\begin{array}{ll} \text{Maximize/Minimize } (\sum\limits_{j:j\in N_1} \tilde{c}_j \otimes \tilde{x}_j \ominus \sum\limits_{j:j\in N_2} \tilde{c}_j \otimes \tilde{x}_j) & \\ \text{subject to} & (P_{7.2}) \\ \sum\limits_{j:j\in N_3} \tilde{a}_{ij} \otimes \tilde{x}_j \ominus \sum\limits_{j:j\in N_4} \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, =, \succeq \tilde{b}_i \quad \forall\ i = 1, 2, ..., m & \end{array}$$

where $\tilde{x}_j$, $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ are unrestricted fuzzy numbers and $N_1 \cup N_2 = \{1, 2, ..., n\}, N_3 \cup N_4 = \{1, 2, ..., n\}, N_1 \cap N_2 = \phi, N_3 \cap N_4 = \phi$.

7.2 Shortcomings of existing general form of fully fuzzy linear programming problems

It is not genuine to use the general form of fully fuzzy linear programming problem $(P_{7.2})$ due to the following reasons.

In the literature [30], it is pointed out that only a RL flat fuzzy number $\tilde{A}_2$ can be subtracted from an LR flat fuzzy number $\tilde{A}_1$ i.e., if $\tilde{A}_1$ and $\tilde{A}_2$ both are LR flat fuzzy numbers such that $L(\cdot) \neq R(\cdot)$ then $\tilde{A}_1 \ominus \tilde{A}_2$ does not exist. So, if all the parameters of the fully fuzzy linear programming problem $(P_{7.2})$ are represented by such LR flat fuzzy numbers for which $L(\cdot) \neq R(\cdot)$ then due to the existence of $\ominus$ in $\sum\limits_{j:j\in N_1} \tilde{c}_j \otimes \tilde{x}_j \ominus \sum\limits_{j:j\in N_2} \tilde{c}_j \otimes \tilde{x}_j$ and $\sum\limits_{j:j\in N_3} \tilde{a}_{ij} \otimes \tilde{x}_j \ominus \sum\limits_{j:j\in N_4} \tilde{a}_{ij} \otimes \tilde{x}_j$, the fully fuzzy linear programming problem $(P_{7.2})$ is not valid.

To show the shortcoming of the existing general form of fully fuzzy linear programming problems, the problem, chosen in Example 7.1, is formulated into fully fuzzy linear programming problem and it is shown that the results obtained by solving the formulated fully fuzzy linear programming problem are not appropriate according to the real life situations.

Example 7.1 A manufacturer of biscuits is considering four types of gift packs containing three types of biscuits: orange cream (OC), chocolate cream (CC) and wafers (W). Market research conducted recently according to preferences of the consumers shows the following types of assortments of to be in good demand.

Table 7.1 Most preferred assortments of biscuits

Assortments	Contents	Fuzzy selling price per kg. (Rs.)
A	Not less than 40% of OC, not more than 20% of CC, any quantity of W	$(10, 30, 10, 10)_{LR}$
B	Not less than 20% of OC, not more than 40% of CC, any quantity of W	$(20, 30, 10, 10)_{LR}$
C	Not less than 50% of OC, not more than 10% of CC, any quantity of W	$(22, 22, 12, 12)_{LR}$
D	No restrictions	$(10, 14, 2, 2)_{LR}$

For the biscuits, the fuzzy manufacturing capacity and fuzzy costs are given below:

Table 7.2 Fuzzy manufacturing capacity and fuzzy costs for biscuits

Biscuit variety	OC	CC	W
Fuzzy plant capacity (kg/day)	$(150, 250, 50, 50)_{LR}$	$(180, 220, 20, 20)_{LR}$	$(100, 200, 50, 50)_{LR}$
Fuzzy manufacturing cost (Rs./kg)	$(8, 8, 8, 8)_{LR}$	$(9, 10, 4, 2)_{LR}$	$(5, 9, 5, 5)_{LR}$

Formulate a fuzzy linear programming model to find the production schedule which maximizes the fuzzy profit using that there are no market restrictions.

7.2.1 Existing fuzzy linear programming formulation of the chosen problem

The problem, chosen in Example 7.1, can be formulated into the following fully fuzzy linear programming problem:

Maximize $((10, 30, 10, 10)_{LR} \otimes (\tilde{x}_{A1} \oplus \tilde{x}_{A2} \oplus \tilde{x}_{A3}) \oplus (20, 30, 10, 10)_{LR} \otimes (\tilde{x}_{B1} \oplus \tilde{x}_{B2} \oplus$

$\tilde{x}_{B3}) \oplus (22, 22, 12, 12)_{LR} \otimes (\tilde{x}_{C1} \oplus \tilde{x}_{C2} \oplus \tilde{x}_{C3}) \oplus (10, 14, 2, 2)_{LR} \otimes (\tilde{x}_{D1} \oplus \tilde{x}_{D2} \oplus \tilde{x}_{D3}) \ominus$ $(8, 8, 8, 8)_{LR} \otimes (\tilde{x}_{A1} \oplus \tilde{x}_{B1} \oplus \tilde{x}_{C1} \oplus \tilde{x}_{D1}) \ominus (9, 10, 4, 2)_{LR} \otimes (\tilde{x}_{A2} \oplus \tilde{x}_{B2} \oplus \tilde{x}_{C2} \oplus \tilde{x}_{D2}) \ominus$ $(5, 9, 5, 5)_{LR} \otimes (\tilde{x}_{A3} \oplus \tilde{x}_{B3} \oplus \tilde{x}_{C3} \oplus \tilde{x}_{D3}))$

subject to

$$\begin{aligned}
\tilde{x}_{A1} &\succeq 0.40(\tilde{x}_{A1} \oplus \tilde{x}_{A2} \oplus \tilde{x}_{A3}) \\
\tilde{x}_{B1} &\succeq 0.20(\tilde{x}_{B1} \oplus \tilde{x}_{B2} \oplus \tilde{x}_{B3}) \\
\tilde{x}_{C1} &\succeq 0.50(\tilde{x}_{C1} \oplus \tilde{x}_{C2} \oplus \tilde{x}_{C3}) \\
\tilde{x}_{A2} &\preceq 0.20(\tilde{x}_{A1} \oplus \tilde{x}_{A2} \oplus \tilde{x}_{A3}) \\
\tilde{x}_{B2} &\preceq 0.40(\tilde{x}_{B1} \oplus \tilde{x}_{B2} \oplus \tilde{x}_{B3}) \\
\tilde{x}_{C2} &\preceq 0.10(\tilde{x}_{C1} \oplus \tilde{x}_{C2} \oplus \tilde{x}_{C3}) \\
\tilde{x}_{A1} \oplus \tilde{x}_{B1} \oplus \tilde{x}_{C1} \oplus \tilde{x}_{D1} &\preceq (150, 250, 50, 50)_{LR} \\
\tilde{x}_{A2} \oplus \tilde{x}_{B2} \oplus \tilde{x}_{C2} \oplus \tilde{x}_{D2} &\preceq (180, 220, 20, 20)_{LR} \\
\tilde{x}_{A3} \oplus \tilde{x}_{B3} \oplus \tilde{x}_{C3} \oplus \tilde{x}_{D3} &\preceq (100, 200, 50, 50)_{LR}
\end{aligned} \qquad (P_{7.3})$$

where $\tilde{x}_{ij}$ $(i = A, B, C, D; j = 1, 2, 3)$ is a non-negative LR flat fuzzy number and

(i) $\tilde{x}_{A1}, \tilde{x}_{A2}, \tilde{x}_{A3}$ denote the fuzzy quantity in kg. of OC, CC and W type of biscuits for the gift pack A.

(ii) $\tilde{x}_{B1}, \tilde{x}_{B2}, \tilde{x}_{B3}$ denote the fuzzy quantity in kg. of OC, CC and W type of biscuits for the gift pack B.

(iii) $\tilde{x}_{C1}, \tilde{x}_{C2}, \tilde{x}_{C3}$ denote the fuzzy quantity in kg. of OC, CC and W type of biscuits for the gift pack C.

(iv) $\tilde{x}_{D1}, \tilde{x}_{D2}, \tilde{x}_{D3}$ denote the fuzzy quantity in kg. of OC, CC and W type of biscuits for the gift pack D.

7.2.2 Drawbacks of the obtained optimal value

The fuzzy optimal value, obtained by using the existing method [4] and method, proposed in Chapter 5, are shown in Table 7.3.

Table 7.3 Results of the chosen problem using existing formulation

Method	Optimal value	$\Re$(Optimal value)	$L(\cdot)\&R(\cdot)$
Existing method [4] using ($P_{7.3}$)	Not Applicable	Not Applicable	$L(x)=\max\{0,1-x\}$ $R(x)=\max\{0,1-x^2\}$
Method proposed in Chapter 5 using ($P_{7.3}$)	Not Applicable	Not Applicable	$L(x)=\max\{0,1-x\}$ $R(x)=\max\{0,1-x^2\}$
Existing method [4] using ($P_{7.3}$)	$(0,0,7700,22000)_{LR}$	3575	$L(x)=\max\{0,1-x\}$ $R(x)=\max\{0,1-x\}$
Method proposed in Chapter 5 using ($P_{7.3}$)	$(0,0,7700,22000)_{LR}$	3575	$L(x)=\max\{0,1-x\}$ $R(x)=\max\{0,1-x\}$

In the objective function of fuzzy linear programming problem ($P_{7.3}$) subtraction of two LR flat fuzzy numbers is occurring. So, if $L(\cdot) \neq R(\cdot)$ then the fuzzy linear programming problem ($P_{7.3}$) is not valid i.e, it is not possible to find the fuzzy optimal solution of the chosen problem by using the fuzzy linear programming problem ($P_{7.3}$). Although, if $L(\cdot) = R(\cdot)$ then existing fully fuzzy linear programming problem ($P_{7.3}$) is valid but it is obvious from the results, shown in Table 7.3, that on using the existing fully fuzzy linear programming problem ($P_{7.3}$) the total profit is not a non-negative fuzzy number which is not genuine according to the real life situations.

7.3 Proposed general form of fully fuzzy linear programming problems

In this section, to resolve the shortcomings of existing general form of fully fuzzy linear programming problems, pointed out in Section 7.2, a new general form of the fully fuzzy linear programming problems is proposed.

The general form of crisp linear programming problems $(P_{7.1})$ can be written as:

$$\begin{aligned}&\text{Maximize/Minimize } S\\&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{7.4})\\&\quad\sum_{j:j\in N_1} c_j x_j = S + \sum_{j:j\in N_2} c_j x_j\\&\quad\sum_{j:j\in N_3} a_{ij} x_j \leq, =, \geq b_i + \sum_{j:j\in N_4} a_{ij} x_j \quad \forall\ i = 1, 2, ..., m\end{aligned}$$

where x_j, a_{ij}, b_i, c_j and S are real numbers.

Replacing all the parameters of the crisp linear programming problem $(P_{7.4})$ by fuzzy parameters, the general form of the fully fuzzy linear programming problem $(P_{7.5})$ is obtained:

$$\begin{aligned}&\text{Maximize/Minimize } \tilde{S}\\&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P_{7.5})\\&\quad\sum_{j:j\in N_1} \tilde{c}_j \otimes \tilde{x}_j = \tilde{S} \oplus \sum_{j:j\in N_2} \tilde{c}_j \otimes \tilde{x}_j\\&\quad\sum_{j:j\in N_3} \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, =, \succeq \tilde{b}_i \oplus \sum_{j:j\in N_4} \tilde{a}_{ij} \otimes \tilde{x}_j \quad \forall\ i = 1, 2, ..., m\end{aligned}$$

where $\tilde{x}_j$, $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ and $\tilde{S}$ are LR flat fuzzy numbers.

7.4 Advantages of the proposed general form of fully fuzzy linear programming problems

Since, in the proposed general form, subtraction of LR flat fuzzy numbers is not occurring so using the proposed general form, the shortcomings of the existing general form, pointed out in Section 7.2, are resolved.

To show the advantage of the proposed general form $(P_{7.5})$ over the existing general form $(P_{7.2})$ it is shown that if the problem, chosen in Example 7.1, is formulated by using proposed general form then all the shortcomings, pointed out in Section 7.2, are resolved.

Using the proposed general form the problem, chosen in Example 7.1, can be formulated into fully fuzzy linear programming problem $(P_{7.6})$:

Maximize $\tilde{P}$

subject to

$$(10,30,10,10)_{LR}(\tilde{x}_{A1}\oplus\tilde{x}_{A2}\oplus\tilde{x}_{A3})\oplus(20,30,10,10)_{LR}(\tilde{x}_{B1}\oplus\tilde{x}_{B2}\oplus\tilde{x}_{B3})\oplus(22,22,12,12)_{LR}(\tilde{x}_{C1}\oplus\tilde{x}_{C2}\oplus\tilde{x}_{C3})\oplus(10,14,2,2)_{LR}(\tilde{x}_{D1}\oplus\tilde{x}_{D2}\oplus\tilde{x}_{D3}) = \tilde{P}\oplus(8,8,8,8)_{LR}(\tilde{x}_{A1}\oplus\tilde{x}_{B1}\oplus\tilde{x}_{C1}\oplus\tilde{x}_{D1})\oplus(9,10,4,2)_{LR}(\tilde{x}_{A2}\oplus\tilde{x}_{B2}\oplus\tilde{x}_{C2}\oplus\tilde{x}_{D2})\oplus(5,9,5,5)_{LR}(\tilde{x}_{A3}\oplus\tilde{x}_{B3}\oplus\tilde{x}_{C3}\oplus\tilde{x}_{D3})$$

$$\begin{aligned}
&\tilde{x}_{A1} \succeq 0.40(\tilde{x}_{A1}\oplus\tilde{x}_{A2}\oplus\tilde{x}_{A3})\\
&\tilde{x}_{B1} \succeq 0.20(\tilde{x}_{B1}\oplus\tilde{x}_{B2}\oplus\tilde{x}_{B3})\\
&\tilde{x}_{C1} \succeq 0.50(\tilde{x}_{C1}\oplus\tilde{x}_{C2}\oplus\tilde{x}_{C3})\\
&\tilde{x}_{A2} \preceq 0.20(\tilde{x}_{A1}\oplus\tilde{x}_{A2}\oplus\tilde{x}_{A3})\\
&\tilde{x}_{B2} \preceq 0.40(\tilde{x}_{B1}\oplus\tilde{x}_{B2}\oplus\tilde{x}_{B3})\\
&\tilde{x}_{C2} \preceq 0.10(\tilde{x}_{C1}\oplus\tilde{x}_{C2}\oplus\tilde{x}_{C3})\\
&\tilde{x}_{A1}\oplus\tilde{x}_{B1}\oplus\tilde{x}_{C1}\oplus\tilde{x}_{D1} \preceq (150,250,50,50)_{LR}\\
&\tilde{x}_{A2}\oplus\tilde{x}_{B2}\oplus\tilde{x}_{C2}\oplus\tilde{x}_{D2} \preceq (180,220,20,20)_{LR}\\
&\tilde{x}_{A3}\oplus\tilde{x}_{B3}\oplus\tilde{x}_{C3}\oplus\tilde{x}_{D3} \preceq (100,200,50,50)_{LR}
\end{aligned} \qquad (P_{7.6})$$

where $\tilde{x}_{ij}$ $(i = A, B, C, D; j = 1, 2, 3)$ is a non-negative LR flat fuzzy number and $\tilde{P}$ is an LR flat fuzzy number.

Fuzzy optimal values of the formulated problem $(P_{7.6})$ by the using the existing method [4] and method, proposed in Chapter 5, are shown in Table 7.4.

Table 7.4 Results of the chosen problem using proposed formulation

Method	Optimal value	$\Re$(Optimal value)	$L(\cdot)\&R(\cdot)$
Existing method [4] using ($P_{7.6}$)	$(14555, 14555, 0, 0)_{LR}$	14555	$L(x) = \max\{0, 1-x\}$ $R(x) = \max\{0, 1-x^2\}$
Method proposed in Chapter 5 using ($P_{7.6}$)	$(0, 0, 0, 43665)_{LR}$	14555	$L(x) = \max\{0, 1-x\}$ $R(x) = \max\{0, 1-x^2\}$
Existing method [4] using ($P_{7.6}$)	$(3575, 3575, 0, 0)_{LR}$	3575	$L(x) = \max\{0, 1-x\}$ $R(x) = \max\{0, 1-x\}$
Method proposed in Chapter 5 using ($P_{7.6}$)	$(0, 0, 0, 14300)_{LR}$	3575	$L(x) = \max\{0, 1-x\}$ $R(x) = \max\{0, 1-x\}$

It is obvious from Table 7.4 that if $L(\cdot) \neq R(\cdot)$ then the fully fuzzy linear programming problem, chosen in Example 7.1, can be solved by using the proposed general form ($P_{7.6}$). Also, if $L(\cdot) = R(\cdot)$ then the obtained total profit is a non-negative fuzzy number. So, it is better to use the proposed general form ($P_{7.5}$) as compared to the existing general form ($P_{7.2}$).

7.5 Conclusions

In this chapter, the shortcomings of the existing general form of fully fuzzy linear programming problems are pointed out and new general form of fully fuzzy linear programming problems is proposed. Also, it is shown that it is better to use the proposed general form of fully fuzzy linear programming problems as compared to the existing general form of fully fuzzy linear programming problems.

Chapter 8

FUTURE SCOPE

(i) The method proposed in Chapter 6 can be used to find the unique fuzzy optimal value of fully fuzzy linear programming problems with equality constraints. However, this method cannot be used to find the unique fuzzy optimal value of fully fuzzy linear programming problems with inequality constraints. In future, it may be tried to develop a method for the same.

(ii) To find the fuzzy optimal solution of the fully fuzzy linear programming problems by using the proposed methods, there is a need to convert the fully fuzzy linear programming problems into crisp linear programming problems. In future, it may be tried to develop a method which can be used directly to find the fuzzy optimal solution of the fully fuzzy linear programming problems without converting it into crisp linear programming problems.

Bibliography

[1] I. Ahmad, T.R. Gulati, Mixed type duality for multiobjective variational problems with generalized $(F, \rho)-$ convexity, Journal of Mathematical Analysis and Applications, 306 (2005) 669-683.

[2] I. Ahmad, Z. Husain, On multiobjective second order symmetric duality with cone constraints, European Journal of Operational Research, 204 (2010) 402-409.

[3] T. Allahviranloo, F.H. Lotfi, L. Alizadeh, N. Kiani, Degeneracy in fuzzy linear programming problems, The Journal of Fuzzy Mathematics, 17 (2009) 389-402.

[4] T. Allahviranloo, F.H. Lotfi, M.K. Kiasary, N.A. Kiani, L. Alizadeh, Solving fully fuzzy linear programming problem by the ranking functhion, Applied Matematical Sciences, 2 (2008) 19-32.

[5] T. Allahviranloo, N. Mikaeilvand, F.H. Lotfi, M.K. Kiasari, Fully fuzzy linear programming problem with positive or negative core, Far East Journal of Applied Mathematics, 33 (2008) 337-350.

[6] T. Allahviranloo, K.H. Shamsolkotabi, N.A. Kiani, L. Alizadeh, Fuzzy integer linear programming problems, International Journal of Contemporary Mathematical Sciences, 2 (2007) 167-181.

[7] A. Baykasoglu, T. Gocken, A direct solution approach to fuzzy mathematical programs with fuzzy decision variables, Expert Systems with Applications, 39 (2012) 1972-1978.

[8] M.S. Bazaraa, J.J Jarvis, H.D Sherali, Linear Programming and Network Flows, Second Edition, Wiley, New York, 1990.

[9] C.R. Bector, S. Chandra, On duality in linear programming under fuzzy environment, Fuzzy Sets and Systems, 125 (2002) 317-325.

[10] C.R. Bector, S. Chandra, Fuzzy Mathematical Programming and Fuzzy Matrix Games, Springer-Verlag, 2005.

[11] R.E. Bellman, L.A. Zadeh, Decision making in a fuzzy environment, Management Science, 17 (1970) 141-164.

[12] J.J. Buckley, Possibilistic linear programming with triangular fuzzy numbers, Fuzzy Sets and Systems, 26 (1988) 135-138.

[13] J.J. Buckley, Solving possibilistic linear programming problems, Fuzzy Sets and Systems, 31 (1989) 329-341.

[14] J.J. Buckley, Fuzzy genetics algorithm and applications, Fuzzy Sets and Systems, 61 (1994) 129-136.

[15] J.J. Buckley, Joint solution to fuzzy programming problems, Fuzzy Sets and Systems, 72 (1995) 215-220.

[16] J. Buckley, T. Feuring, Evolutionary algorithm solution to fuzzy problems: fuzzy linear programming, Fuzzy Sets and Systems, 109 (2000) 35-53.

[17] J.M. Cadenas, J.L. Verdegay, Using fuzzy numbers in linear programming, IEEE Transactions on Systems, Man and Cybernetics-Part B: Cybernetics, 27 (1997) 1016-1022.

[18] Q. Cai, Z. Hao, S. Pan, The solution of linear programming with LR fuzzy numbers in objective function, Proceedings Second International Conference on Fuzzy Information and Engineering, 40 (2007) 988-999.

[19] L. Campos, J.L. Verdegay, Linear programming problems and ranking of fuzzy numbers, Fuzzy Sets and Systems, 32 (1989) 1-11.

[20] C. Carlsson, P. Korhonen, A parametric approach to fuzzy linear programming, Fuzzy Sets and Systems, 20 (1986) 17-30.

[21] S. Chanas, The use of parametric programming in fuzzy linear programming, Fuzzy Sets and Systems, 11 (1983) 229-241.

[22] S. Chanas, Fuzzy programming in multiobjective linear programming - A parametric approach, Fuzzy Sets and Systems, 29 (1989) 303-313.

[23] G.B. Dantzig, Linear Programming and Extensions, Princeton University Press, New Jersey, 1963.

[24] D. Darvishi, A.Teimouri, S.H. Nasseri, Application of fuzzy optimization in diet formulation, The Journal of Mathematics and Computer Science, 2 (2011) 459-468.

[25] M. Dehghan, B. Hashemi, M. Ghatee, Computational methods for solving fully fuzzy linear systems, Applied Mathematics and Computation, 179 (2006) 328-343.

[26] D. Delgado, J. Kacprzyk, J.L. Verdegy, M.A. Villa, Fuzzy Optimization: Recent Advances, Physica Verlag, Germany, 1994.

[27] M. Delgado, J.L. Verdegay, M.A. Vila, A general model for fuzzy linear programming, Fuzzy Sets and Systems, 29 (1989) 21-29.

[28] S. Dempe, A. Ruziyeva, On the calculation of a membership function for the solution of a fuzzy linear optimization problem, Fuzzy Sets and Systems, 188 (2012) 58-67.

[29] U. Diwekar, Introduction to Applied Optimization, Second Edition, Springer-Verlag, Cambridge, MA, 2008.

[30] D. Dubois, H. Prade, Fuzzy Sets and Systems, Theory and Applications, Academic Press, New York, 1980.

[31] A. Ebrahimnejad, Some new results in linear programs with trapezoidal fuzzy numbers: Finite convergence of the Ganesan and Veeramani's method and a fuzzy revised simplex method, Applied Mathematical Modelling, 35 (2011) 4526-4540.

[32] A. Ebrahimnejad, A primal-dual simplex algorithm for solving linear programming problems with symmetric trapezoidal fuzzy numbers, Applied Mathematics, 2 (2011) 676-684.

[33] A. Ebrahimnejad, S.H. Nasseri, Using complementary slackness property to solve linear programming with fuzzy parameters, Fuzzy Information and Engineering, 1 (2009) 233-245.

[34] A. Ebrahimnejad, S.H. Nasseri, A dual simplex method for bounded linear programmes with fuzzy numbers, International Journal of Mathematics in Operational Research, 2 (2010) 762-779.

[35] A. Ebrahimnejad, S.H. Nasseri, F.H. Lotfi, Bounded linear programs with trapezoidal fuzzy numbers, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 18 (2010) 269-286.

[36] A. Ebrahimnejad, S.H. Nasseri, F.H. Lotfi, M. Soltanifar, A primal-dual method for linear programming problems with fuzzy variables, European Journal of Industrial Engineering, 4 (2010) 189-209.

[37] A. Ebrahimnejad, S.H. Nasseri, S.M. Mansourzadeh, Bounded primal simplex algorithm for bounded linear programming with fuzzy cost coefficients, International Journal of Operations Research and Information Systems, 2 (2011) 100-124.

[38] S.C. Fang, C.F. Hu, H.F. Wang, S.Y. Wu, Linear programming with fuzzy coefficents in constraints, Computers and Mathematics with Applications, 37 (1999) 63-76.

[39] B. Farhadinia, Ranking fuzzy numbers based on lexicographical ordering, International Journal of Mathematical and Computer Sciences, 5 (2009) 220-223.

[40] R. Fuller, H.J. Zimmermann, Fuzzy reasoning for solving fuzzy mathematical programming problems, Fuzzy Sets and Systems, 60 (1993) 121-133.

[41] K. Ganesan, P. Veeramani, Fuzzy linear programs with trapezoidal fuzzy numbers, Annals of Operations Research, 143 (2006) 305-315.

[42] A.N. Gani, C. Duraisamy, C. Veeramani, A note on fuzzy linear programming problem using *LR* fuzzy number, International Journal of Algorithms, Computing and Mathematics, 2 (2009) 93-106.

[43] A.N. Gani, S.N. Mohamed Assarudeen, A new operation on triangular fuzzy number for solving fuzzy linear programming problem, Applied Mathematical Sciences, 6 (2012) 525-532.

[44] M. Gavalec, K. Zimmermann, Duality of optimization problems with generalized fuzzy relation equation and inequality constraints, Information Sciences, 2011, DOI:10.1016/j.ins.2011.04.019.

[45] T.R. Gulati, S.K. Gupta, I. Ahmad, Second-order symmetric duality with cone constraints, Journal of Computational and Applied Mathematics, 220 (2008) 347-354.

[46] P. Gupta, M.K. Mehlawat, An algorithm for a fuzzy transportation problem to select a new type of coal for a steel manufacturing unit, TOP, 15 (2007) 114-137.

[47] P. Gupta, M.K. Mehlawat, Bector-Chandra type duality in fuzzy linear programming with exponential membership functions, Fuzzy Sets and Systems, 160 (2009) 3290-3308.

[48] S.M. Guu, Y.K. Wu, Two phase approach for solving the linear programming problems, Fuzzy Sets and Systems, 107 (1999) 191-195.

[49] S.M. Hashemi, M. Modarres, E. Nasrabadi, M. M. Nasrabadi, Fully fuzzified linear programming, solution and duality, Journal of Intelligent and Fuzzy Systems, 17 (2006) 253-261.

[50] A. Hatami-Marbini, M. Tavana, An extension of the linear programming method with fuzzy parameters, International Journal of Mathematics in Operational Research, 3 (2011) 44-55.

[51] S. Heilpern, Representation and application of fuzzy numbers, Fuzzy Sets and Systems, 91 (1997) 259-268.

[52] F. Herrera, J.L. Verdegay, Three models of fuzzy integer linear programming, European Journal of Operational Research, 83 (1995) 581-593.

[53] N.V. Hop, Solving linear programming problems under fuzziness and randomness environment using attainment values, Information Sciences, 177 (2007) 2971-2984.

[54] M. Inuiguchi, H. Ichihashi, Y. Kume, A solution algorithm for fuzzy linear programming with piecewise linear membership function, Fuzzy Sets and Systems, 34 (1990) 15-31.

[55] M. Inuiguchi, J. Ramik, Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, Fuzzy Sets and Systems, 111 (2000) 3-28.

[56] R. Jain, Decision making in the presence of fuzzy variables, IEEE Transactions on Systems, Man and Cybernetics, 6 (1976) 698-703.

[57] S. Jain, A. Mangal, P.R. Parihar, Solution of fuzzy linear fractional programming problem, OPSEARCH, 48 (2011) 129-135.

[58] M. Jimenez, M. Arenas, A. Bilbao, M. V. Rodrguez, Linear programming with fuzzy parameters: An interactive method resolution, European Journal of Operational Research, 177 (2007) 1599-1609.

[59] M. Jimenez, J.M. Cadenas, J.L. Verdegay, G. Sanchez, Solving fuzzy optimization problems by evolutionary algorithms, Information Sciences, 152 (2003) 303-311.

[60] X. Jiuping, A kind of fuzzy linear programming problems based on interval-valued fuzzy sets, Applied Mathematics-A Journal of Chinese Universities, 15 (2000) 65-72.

[61] B. Julien, An extension to possibilistic linear programming, Fuzzy Sets and Systems, 64 (1994) 195-206.

[62] H. S. Kasana, K. D. Kumar, Introductory Operations Research: Theory and Applications, Springer-Verlag, Berlin Heidelberg, New York, 2004.

[63] A. Kaufmann, M.M. Gupta, Introduction to Fuzzy Arithmetic Theory and Applications, Van Nostrand Reinhold, New York, 1985.

[64] A. Kaufmann, M.M. Gupta, Fuzzy Mathematical Models in Engineering and Management Science, Amsterdam Netherland, Elsevier, 1988.

[65] A. Kaur, A. Kumar, A new method for solving fuzzy transportation problems using ranking function, Applied Mathematical Modelling, 35 (2011) 5652-5661.

[66] P. Kaur, A. Kumar, A new ranking approach and its application for solving fuzzy critical path problems, South African Journal of Industrial Engineering (accepted).

[67] Kavita, S.P. Yadav, S. Kumar, A multi-criteria interval-valued intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, Lecture Notes in Computer Science, 5908 (2009) 303-312.

[68] F.I. Khan, R. Sadiq, Risk-based prioritization of air pollution monitoring using fuzzy synthetic evaluation technique, Environmental Monitoring and Assessment, 105 (2005) 261-283.

[69] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice-Hall, Inc., New Jersey, USA, 1996.

[70] D.S. Kseibat, G.S. Mittal, O.A. Basir, Predicting safety and quality of thermally processed canned foods using a neural network, Transactions of the Institue of Measurement and Control, 26 (2004) 55-68.

[71] A. Kumar, P. Kaur, Exact optimal solution of fuzzy critical path problems, Applications and Applied Mathematics: An International Journal, 6 (2011) 1992-2008.

[72] Y.J. Lai, C.L. Hwang, Interactive fuzzy linear programming, Fuzzy Sets and Systems, 45 (1992) 169-183.

[73] Y.J. Lai, C.L. Hwang, Fuzzy Mathematical Programming: Methods and Applications, Springer-Verlag, Heidelberg, 1992.

[74] Y. Leung, Spatial Analysis and Planning under Imprecision, North-Holland, Amsterdam, 1988.

[75] T.F. Liang, Project management decisions using fuzzy linear programming, International Journal of Systems Science, 37 (2006) 1141-1152.

[76] F.T. Lin, M.G. Lee, Fuzzy constraints linear programming: A genetic algorithm approach, International Journal of Innovative Computing, Information and Control, 3 (2007) 71-83.

[77] Y.C. Lin, Y.C. Lin, K.L. Su, W.C. Lin, T.H. Chen, Fuzzy programming for mixed-integer optimization problems, Artificial Life and Robotics, 16 (2011) 174-177.

[78] T.S. Liou, M.J. Wang, Ranking fuzzy numbers with integral value, Fuzzy Sets and Systems, 50 (1992) 247-255.

[79] X. Liu, Measuring the satisfaction of constraints in fuzzy linear programming, Fuzzy Sets and Systems, 122 (2001) 263-275.

[80] F.H. Lotfi, T. Allahviranloo, M. A. Jondabeha, L. Alizadeh, Solving a fully fuzzy linear programming using lexicography method and fuzzy approximate solution, Applied Mathematical Modelling, 33 (2009) 3151-3156.

[81] M.K. Luhandjula, Linear programming under randomness and fuzziness, Fuzzy Sets and Systems, 10 (1983) 45-55.

[82] M.K. Luhandjula, Linear programming with a possibilistic objective function, European Journal of Operational Research, 31 (1987) 110-117.

[83] M.K. Luhandjula, Fuzzy optimization: an appraisal, Fuzzy Sets and Systems, 30 (1989) 257-282.

[84] B. Lv, H. He, X. Li, Z. Zhang, W. Huang, M. Li, G. Lu, Structural and functional deficits in human amblyopia, Neuroscience Letters, 437 (2008) 5-9.

[85] B. Lv, J. Li, H. He, M. Li, M. Zhao, L. Ai, F. Yan, J. Xian, Z. Wang, Gender consistency and difference in healthy adults revealed by cortical thickness, NeuroImage, 53 (2010) 373-382.

[86] N. Mahdavi-Amiri, S.H. Nasseri, Duality in fuzzy variable linear programming, World Academy of Science, Engineering and Technology, 6 (2005) 115-117.

[87] N. Mahdavi-Amiri, S.H. Nasseri, Duality in fuzzy number linear programming by the use of a certain linear ranking function, Applied Mathematics and Computation, 180 (2006) 206-216.

[88] N. Mahdavi-Amiri, S.H. Nasseri, Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables, Fuzzy Sets and Systems, 158 (2007) 1961-1978.

[89] N. Mahdavi-Amiri, S.H. Nasseri, A. Yazdani, Fuzzy primal simplex algorithms for solving fuzzy linear programming problems, Iranian Journal of Operations Research, 1 (2009) 68-84.

[90] H.R. Maleki, Ranking functions and their applications to fuzzy linear programming, Far East Journal of Mathematical Sciences, 4 (2002) 283-301.

[91] H.R. Maleki, M. Tata, M. Mashinchi, Linear programming with fuzzy variables, Fuzzy Sets and Systems, 109 (2000) 21-33.

[92] C. Mohan, S.K. Verma, Interactive algorithms using fuzzy concepts for solving mathematical models of real life optimization problems, In J.L. Verdegay (Ed.), Fuzzy Sets Based Heuristics for Optimization, Springer-Verlag, Berlin, Heidelberg, (2003) 123-137.

[93] B.K. Mohanty, A procedure for measuring uncertainties due to lack of information in a fuzzy game theory problem, International Journal of Systems Science, 25 (1994) 2309-2317.

[94] B.K. Mohanty, A game theoretic procedure for ranking efficient points of a fuzzy multiobjective programming problem, The Journal of Fuzzy Mathematics, 3 (1995) 529-541.

[95] B.K. Mohanty, A preference structure on efficient points in a multiple criteria decision making problem-A fuzzy approach, The Journal of Fuzzy Mathematics, 18 (2010) 903-912.

[96] Y. Nakahara, M. Gen, A method for solving linear programming problems with triangular fuzzy coefficients using new ranking index, Computers and Industrial Engineering, 25 (1993) 1-4.

[97] K. Nakamura, Some extensions of fuzzy linear programming, Fuzzy Sets and Systems, 14 (1984) 211-219.

[98] S.H. Nasseri, A new method for solving fuzzy linear programming by solving linear programming, Applied Mathematical Sciences, 2 (2008) 2473-2480.

[99] S.H. Nasseri, E. Ardil, Simplex method for fuzzy variable linear programming problems, Proceedings of World Academy of Science, Engineering and Technology, 8 (2005) 198-202.

[100] S.H. Nasseri, E. Ardil, A. Yazdani, R. Zaefarian, Simplex method for solving linear programming problems with fuzzy numbers, Proceedings of World Academy of Science, Engineering and Technology, 10 (2005) 284-288.

[101] S.H. Nasseri, A. Ebrahimnejad, A fuzzy dual simplex method for a fuzzy number linear programming problem, Advances in Fuzzy Sets and Systems, 5 (2010) 81-95.

[102] S.H. Nasseri, A. Ebrahimnejad, A fuzzy primal simplex algorithm and its application for solving flexible linear programming problems, European Journal of Industrial Engineering, 4 (2010) 372-389.

[103] S.H. Nasseri, A. Ebrahimnejad, S. Mizuno, Duality in fuzzy linear programming with symmetric trapezoidal numbers, Applications and Applied Mathematics, 5 (2010) 1467-1482.

[104] S.H. Nasseri, B. Khabiri, Revised fuzzy simplex algorithm for linear programming problems with fuzzy variables using linear ranking functions, International Journal of Mathematics and Computation, 6 (2010) 44-54.

[105] S.H. Nasseri, B. Khabiri, A revised simplex algorithm for fuzzy number linear programming problems using linear ranking functions, International Journal of Mathematics and Computation, 8 (2010) 114-126.

[106] S.H. Nasseri, N. Mahdavi-Amiri, Some duality results on linear programming problems with symmetric fuzzy numbers, Fuzzy Information and Engineering, 1 (2009) 61-68.

[107] C.V. Negoita, M. Sularia, On fuzzy mathematical programming and tolerances in planning, Economic Computation and Economic Cybernetics Studies and Research, 3 (1976) 3-15.

[108] H.M. Nehi, H.R. Maleki, M. Mashinchi, Solving fuzzy number linear programming problem by lexicographic ranking function, Italian Journal of Pure and Applied Mathematics, 15 (2004) 9-20.

[109] R. Ostermark, Fuzzy linear constraints in the capital asset pricing model, Fuzzy Sets and Systems, 30 (1989) 93-102.

[110] P. Pandian, M. Jayalakshmi, A new method for solving integer linear programming problems with fuzzy variables, Applied Mathematical Sciences, 4 (2010) 997-1004.

[111] J. Ramik, Duality in fuzzy linear programming: some new concepts and results, Fuzzy Optimization and Decision Making, 4 (2005) 25-39.

[112] J. Ramik, J. Rimanek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, Fuzzy Sets and Systems, 16 (1985) 123-138.

[113] L.V. Reddy, R.N. Mukherjee, Composite nonsmooth multiobjective programs with $(V, \rho)-$ invexity, Journal of Mathematical Analysis and Applications, 235 (1999) 567-577.

[114] L.V. Reddy, R.N. Mukherjee, Some results on mathematical programming with generalized ratio invexity, Journal of Mathematical Analysis and Applications, 240 (1999) 299-310

[115] W. Rodder, H.J. Zimmerman, Duality in fuzzy linear programming, in: A.V. Fiacco, K.O. Kortanek (Eds.), Extremal Methods and System Analysis, Berlin, NewYork, 1980, 415-429.

[116] H. Rommelfanger, Interactive decision making in fuzzy linear optimization problems, European Journal of Operational Research, 41 (1989) 210-217.

[117] H. Rommelfanger, Fuzzy linear programming and applications, European Journal of Operational Research, 92 (1996) 512-527.

[118] H. Rommelfanger, A general concept for solving linear multicriteria programming problems with crisp, fuzzy or stochastic values, Fuzzy Sets and Systems, 158 (2007) 1892-1904.

[119] H. Rommelfanger, R. Hanuscheck, J. Wolf, Linear programming with fuzzy objectives, Fuzzy Sets and Systems, 29 (1989) 31-48.

[120] M.R. Safi, H.R. Maleki, E. Zaeimazad, A note on the Zimmermann method for solving fuzzy linear programming problems, Iranian Journal of Fuzzy Systems, 4 (2007) 31-45.

[121] M. Sakawa, I. Nishizaki, Y. Uemura, Interactive fuzzy programming for two-level linear fractional programming problems with fuzzy parameters, Fuzzy Sets and Systems, 115 (2000) 93-103.

[122] T. Shaocheng, Interval number and fuzzy number linear programmings, Fuzzy Sets and Systems, 66 (1994) 301-306.

[123] I.M. Stancu-Minasian, B. Pop, On a fuzzy set approach to solving multiple objective linear fractional programming problem, Fuzzy Sets and Systems, 134 (2003) 397-405.

[124] B. Stanojevic, M. Stanojevic, Penalty method for fuzzy linear programming with trapezoidal numbers, Yugoslav Journal of Operations Research, 19 (2009) 149-156.

[125] H.A. Taha, Operations Research: An Introduction, Prentice-Hall, New Jersey, 2003.

[126] H. Tanaka, K. Asai, Fuzzy linear programming problems with fuzzy numbers, Fuzzy Sets and Systems, 13 (1984) 1-10.

[127] H. Tanaka, P. Guo, H.J. Zimmermann, Possibility distributions of fuzzy decision variables obtained from possibilistic linear programming problems, Fuzzy Sets and Systems, 113 (2000) 323-332.

[128] H. Tanaka, H. Ichihashi, K. Asai, A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers, Control and Cybernetics, 13 (1984) 185-194.

[129] H. Tanaka, T. Okuda, K. Asai, On fuzzy mathematical programming, Journal of Cybernetics, 3 (1973) 37-46.

[130] P.M. Vasant, Application of fuzzy linear programming in production planning, Fuzzy Optimization and Decision Making, 2 (2003) 229-241.

[131] P. Vasant, N.N. Barsoum, Fuzzy optimization of units products in mix-product selection problem using fuzzy linear programming, Soft Computing, 10 (2006) 144-151.

[132] J.L. Verdegay, A dual approach to solve the fuzzy linear programming problems, Fuzzy Sets and Systems, 14 (1984) 131-141.

[133] J.L. Verdegay, Application of fuzzy optimization in operational research, Control and Cybernetics, 13 (1984) 229-239.

[134] A.K. Verma, A. Srividya, B.C. Deka, Composite system reliability assessment using fuzzy linear programming, Electric Power Systems Research, 73 (2005) 143-149.

[135] A.K. Verma, A. Srividya, R.S.P. Gaonkar, Fuzzy-Reliability Engineering: Concepts and Applications, Narosa Publishing House, New Delhi, 2007.

[136] D. Wang, An inexact approach for linear programming problems with fuzzy objective and resources, Fuzzy Sets and Systems, 89 (1997) 61-68.

[137] B. Werners, An interactive fuzzy programming system, Fuzzy Sets and Systems, 23 (1987) 131-147.

[138] H.C. Wu, Duality theory in fuzzy linear programming problems with fuzzy coefficients, Fuzzy Optimization and Decision Making, 2 (2003) 61-73.

[139] H.C. Wu, Optimality conditions for linear programming problems with fuzzy coefficients, Computers and Mathematics with Applications, 55 (2008) 2807-2822.

[140] R.R. Yager, Mathematical programmming with fuzzy constraints and preference on the objective, Kybernetes, 8 (1979) 285-291.

[141] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, Information Sciences, 24 (1981) 143-161.

[142] S.X. Yang, H. Li, M.Q.H. Meng, P.X. Liu, An embedded fuzzy controller for a behavior-based mobile robot with guaranteed performance, IEEE Transactions on Fuzzy Systems, 12 (2004) 436-446.

[143] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

[144] L.A. Zadeh, Fuzzy algorithms, Information and Control, 12 (1968) 94-102.

[145] Q. Zhang, S.X. Yang, G.S. Mittal, S. Yi, Prediction of performance indices and optimal parameters of rough rice drying using neural networks, Biosystems Engineering, 83 (2002) 281-290.

[146] G. Zhang, Y.H. Wu, M. Remias, J. Lu, Formulation of fuzzy linear programming problems as four-objective constrained optimization problems, Applied Mathematics and Computation, 139 (2003) 383-399.

[147] R. Zhao, R. Govind, Symmetrical fuzzy linear programming and its application to chemical engineering problems, Proceedings IEEE First International Symposium on Uncertainty Modeling and Analysis, 1990, 500-505.

[148] A. Zhu, S.X. Yang, Neurofuzzy-based approach to mobile robot navigation in unknown environments, IEEE Transactions on Systems, Man and Cybernetics, Part C: Applications and Reviews, 37 (2007) 610-621.

[149] H.J. Zimmermann, Description and optimization of fuzzy systems, International Journal of General Systems, 2 (1976) 209-215.

[150] H.J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems, 1 (1978) 45-55.

[151] H.J. Zimmermann, Fuzzy Sets, Decision Making and Expert Systems, Kluwer Academic Publishers, Boston, MA, 1987.

[152] H.J. Zimmermann, Fuzzy Set Theory and Its Applications, Fourth Edition, Kluwer Academic Publisher, Dordrecht, 2001.