SYMMETRY REDUCTION METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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under

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to the

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CERTIFICATE

I hereby certify that the work presented in the thesis entitled ***"SYMMETRY REDUCTION METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS"*** which is being submitted for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Rajesh Kumar Gupta.** The matter presented in this thesis has not submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

This Thesis entitled ***“SYMMETRY REDUCTION METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS”*** is aimed to derive exact solutions of some nonlinear partial differential equations (PDEs) by the using Symmetry reduction methods. By using this method, symmetries are obtained which helps us to construct optimal system of generators. After that partial differential equations are reduced to ordinary differential equations (ODEs) and exact solutions corresponding to these ordinary differential equations are obtained.

The thesis is divided into five chapters. The brief outlines of the research work presented chapter wise in the thesis are as follows:

In the chapter 1, we have explained the introduction of nonlinear partial differential equations and exact solutions. In this chapter, we discussed preliminary material and relevant literature of Lie group of transformations. Symmetry of a system of differential equations is a transformation that maps any solution to another solution of the system. In Lie’s framework such transformations are groups that depend on continuous parameters and consists of either point transformations, acting on the system’s space of independent and dependent variables, or, more generally, contact transformations, acting on the space of independent and dependent variables as well as on all first derivatives of the dependent variables.

In the chapter 2, a detailed study of Symmetry reduction method based on Frechet derivatives of differential operators. From 1870 Sophus Lie's work put the theory of differential equations on a more satisfactory foundation. He showed that the integration theories of the older mathematicians can, by the introduction of what are now called Lie groups, be referred to a common source; and that ordinary differential equations which admit the same infinitesimal transformations present comparable difficulties of integration. He also emphasized the subject of transformations of contact.

A general approach to solve PDEs uses the symmetry property of differential equations, the continuous infinitesimal transformations of solutions to solutions (Lie theory). Continuous group theory, Lie algebras and differential geometry are used to understand the structure of

linear and nonlinear partial differential equations for generating integrable equations. Symmetry methods have been recognized to study differential equations arising in mathematics, physics, engineering, and many other disciplines.

In the chapter 3, by using symmetry method, we deduce the Lie symmetries and find reduced ODEs and the exact solutions of Dym equation.

The Dym equation first appeared in Kruskal [21]. The Dym equation represents a system in which dispersion and nonlinearity are coupled together. The Dym equation has strong links to Korteweg-de Vries equation.

In the chapter 4, by using symmetry method, we constructed the Lie symmetries and found reduced ODEs and the exact solutions of Benjamin-Bona-Mahony equation. We first studied the classical Lie symmetries of the Benjamin-Bona-Mahony (BBM) equation which were obtained through the Lie group method of infinitesimal transformations. Secondly using the classical symmetries of the equation, similarity reductions are obtained.

In chapter 5, Ordinary differential equations and solutions of Burger`s equation with symmetry method of Frechet derivatives techniques are obtained. The Burger equation was first studied by Cole [14] who gave a theoretical solution based on Fourier series analysis, using the appropriate initial and boundary conditions. Another theoretical solution given by Madsen and Sincovec [20], based on "test and trial" method, using approximate initial and boundary conditions. In Benton and Platzman, are mentioned almost 35 distinct solution of Burger equation and Agas tried to get approximate solution of Burger equation using numerical analysis. The Burgers equation was developed to describe the immediate, localized reaction of a tiny, incremental element of mass to given external forces and momentum and energy inputs. In physics terms, the integrated result is expressed in body-centred coordinates whose origin moves with the subject particle of mass. At any given instant of time and for given local conditions, the integrated result indicates how the particle of mass will move next within the body-centred frame.

It is worth to mention that all the solutions of nonlinear problems constructed in the thesis are checked by software Maple .

CHAPTER 1

INTRODUCTION

Partial differential equations (PDEs) arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. The nonlinear phenomena are encountered in a variety of situations in physics as well as in other natural and applied sciences. Most of these phenomena are governed by nonlinear partial differential equations. The study of these systems of differential equations is often regarded as a difficult and confusing endeavour due various limitations posed by the intrinsic nonlinearity. When compared with the variety of techniques available in linear system theory, the tools for analysis and design of nonlinear systems are limited to some very special categories. In a sense nonlinear systems are in their full complexity, and so it is not surprising that there exist no general method for solving them.

Like ordinary differential equations (ODEs), partial differential equations are equations to be solved in which the unknown element is a function, but in PDEs the function is one of several variables, and so of course the known information relates the function and its partial derivatives with respect to the several variables. Modern approaches seek methods applicable to non-linear PDEs as well as linear ones. In this context existence and uniqueness results, and theorems concerning the regularity of solutions, are more difficult. Since it is unlikely that explicit solutions can be obtained for any but the most special of problems, methods of "solving" the PDEs involve analysis within the appropriate function space - for example, seeking convergence of a sequence of functions which can be shown to approximately solve the PDE, or describing the sought-for function as a fixed point under a self-map on the function space, or as the point at which some real-valued function is minimized. Some of these approaches may be modified to give algorithms for estimating numerical solutions to a PDE. Historically, comparatively little was known about the extraordinary range of behaviour exhibited by the solutions to nonlinear partial differential equations. Many of the most fundamental phenomena that now drive modern-day research, including solitons, chaos, stability, blow-up and singularity formation, asymptotic properties, etc., remained undetected or at best dimly perceived in the pre-computer era. The last sixty years has witnessed a remarkable blossoming in our understanding, due in large part to the insight offered by the availability of high performance computers coupled with great advances in the understanding and development of suitable numerical approximation schemes. New analytical methods, new mathematical theories, coupled with new computational algorithms have precipitated this revolution in our understanding and study of nonlinear systems, an activity that continues to grow in intensity and breadth. Each leap in computing power coupled with theoretical advances has led to yet deeper understanding of nonlinear phenomena, while simultaneously demonstrating how far we have yet to go. To make sense of this bewildering variety of methods, equations, and results, it is essential build upon a firm foundation on, first of all, linear systems theory, and secondly, nonlinear algebraic equations and nonlinear ordinary differential equations.

Theory of differential provides the knowledge about the physical behaviour of the nonlinear problems in real world. Some of the standard techniques for solving partial differential equations can be categorized as:

(i) By using continuous group of transformations of nonlinear differential equations can be linearized.
(ii) Numerical simulation.
(iii) To construct exact solutions of nonlinear partial differential equations.

In this thesis applications of Lie symmetry reduction method are used. Symmetry reduction method is one of the useful methods in group theoretic techniques for solving partial differential equations.

In this thesis, three nonlinear partial differential equations considered for exact solutions are as follows:

(i) Dym equation. It is a third order partial differential equation and written as follows:

$$u_t = u^3 u_{xxx}.$$

(ii) Benjamin-Bona-Mahony. It is an improvement of Korteweg-de Vries equation for modelling long waves of small amplitudes and written as follows:

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

(iii) Burger equation. It is of the form:

$$u_t + uu_x - ku_{xx} = 0.$$

It is worth to mention here that all the solutions obtained for these nonlinear partial differential equations are exact analytic solutions. For the correctness of these solution, software MAPLE has been used.

1.1 Literature Survey

Sophus Lie introduced the notion of a continuous group of transformations to put order to the hodgepodge of techniques for solving ordinary differential equations (ODEs). He was motivated by the lectures of his fellow Norwegian, Sylow, on the works of Abel and Galois on solving algebraic equations. In the latter part of the nineteenth century, Sophus Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend various specialized methods for solving ordinary differential equations (ODEs). A Lie group of transformations admitted by a differential equation corresponds to a mapping of each of its solutions to another solution of the same differential equation. There are an infinite number of ways of representing such a mapping by allowing an arbitrary change of independent variables. The representation is unique if the independent variables are kept fixed. This point of view is essential when one extends Lie's algorithm to the computation and use of higher-order local transformations admitted by differential equations as well as when one extends Lie's work on integrating factors for first-order ODEs to higher-order ODEs. Lie groups represent the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical physics. They provide a natural framework for analysing the continuous symmetries of differential equations.

Although today Sophus Lie is rightfully recognized as the creator of the theory of continuous groups, a major stride in the development of their structure theory, which was to have a profound influence on subsequent development of mathematics. To every Lie group, we can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Informally we can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity, and the Lie bracket is something to do with the commutator of two such infinitesimal elements. Lie's work systematically related a miscellany of topics in ODEs including: integrating factor, separable equation, homogeneous equation, reduction of order, the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation, and the use of the Laplace transform. Lie (1881) also indicated that for linear partial differential equations (PDEs), invariance under a Lie group leads directly to superposition of solutions in terms of transforms. Lie devoted his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically based sciences.

The group theoretic methods are divided into two categories:

(i) Inspection methods.

(ii) Deductive methods.

In the class of deductive methods, some of the following techniques are included:

(i) Nonclassical method (Bluman and Cole [8]).

(ii) Classical Lie method (Olver [22]).

(iii) Symmetry reduction method (Steinberg [31]).

1.2 Basic Definitions

In this section, some basic definitions and fundamentals of present work are given (Bluman and Anco [7] and Bluman and Cole [8]).

One Parameter Lie Group of Transformations

Let $x = (x_1, x_2, x_3,....., x_n)$ lie in a region $D \subset R^n$ the set of transformation $x^* = X(x; \in)$ defined for each x in D and parameter $\in$ in set $S \subset R,$ with

$\phi(\in, \delta)$ defining a law of composition of parameter $\in$ and δ in S, forms a One-parameter group of transformations on D if the following hold:

(i) For each $\in$ in S the transformations are one to one onto D.

(ii) S with the law of composition ϕ forms a group G.

(iii) For each x in D, $x^* = x$ when $\in = \in_0$ corresponds to the identity e, that is,

$$X(x; \varepsilon_0) = x.$$

(iv) If $x^* = X(x; \varepsilon)$, $x^{**} = X(x^*; \delta)$ then $x^{**} = X(x; \phi(\varepsilon, \delta))$.

(v) ε is a continuous parameter.

(vi) X is infinitely differentiable with respect to x in D and an analytic function of ε in S .

(vii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and $\delta.$ where $\varepsilon, \delta \in S.$

Example:-

Consider $x^* = x + \varepsilon$

$y^* = y$, $\varepsilon \in R.$

And $\phi(\varepsilon, \delta) = \varepsilon + \delta,$ this forms a one-parameter lie group of transformations.

Infinitesimal Transformations

Consider a one-parameter Lie group of transformations

$$x^* = X(x; \varepsilon) \qquad (1.2.1)$$

with the identity $\varepsilon = 0$ and law of composition ϕ expanding (1) about $\varepsilon = 0$.

Hence, we get

$$x^* = x + \in \left(\frac{\partial X(x; \in)}{\partial \in} \right)_{\in=0} + \frac{1}{2} \in^2 \left(\frac{\partial^2 X(x; \in)}{\partial \in^2} \right)_{\in=0} + \ldots$$

$$x^* = x + \varepsilon \left(\frac{\partial X(x; \varepsilon)}{\partial \varepsilon} \right)_{\varepsilon=0} + O(\varepsilon^2).$$

Let $\xi(x) = \left(\frac{\partial X(x; \in)}{\partial \in} \right)_{\in=0}$. Then the transformations $x + \in \xi(x)$ is called the infinitesimal transformation of lie group of transformations (1.2.1).

Infinitesimal Generators

The infinitesimal generator of the one-parameter lie group of transformations (1.2.1) is the operator

$$X = X(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator $\nabla = \left(\frac{\partial}{\partial x_{1i}}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right).$

Lie Bracket

For an r-parameter Lie group of transformations with infinitesimal generators $X_\alpha, \alpha = 1,2,...,r$, the commutator (Lie Bracket) of X_α and X_β is a first order operator defined by

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = \sum_{i,j=1}^{n} \left[\left(\xi_{\alpha i}(x) \frac{\partial}{\partial x_i} \right) \left(\xi_{\beta j}(x) \frac{\partial}{\partial x_j} \right) - \left(\xi_{\beta i}(x) \frac{\partial}{\partial x_i} \right) \left(\xi_{\alpha j}(x) \frac{\partial}{\partial x_j} \right) \right]$$

$$= \sum_{j=1}^{n} \eta_j(x) \frac{\partial}{\partial x_j},$$

where

$$\eta_j(x) = \sum_{i=1}^{n} \left(\xi_{\alpha i}(x) \frac{\partial \xi_{\beta j}(x)}{\partial x_i} - \xi_{\beta i}(x) \frac{\partial \xi_{\alpha j}(x)}{\partial x_i} \right).$$

It immediately follows that

$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha].$

Adjoint Vector

Let G be a Lie group with Lie algebra L. For each vector $v \in L$, the adjoint vector ad v at $w \in L$ is

$$\text{ad } v\big|_w = [w, v] = -[v, w].$$

The adjoint representation Ad G of the underlying Lie group can be reconstructed either by integrating the system of linear ordinary differential equations

$$\frac{\partial w}{\partial \varepsilon} = \text{ad } v\big|_w, \; w(0) = w_0,$$

with solution

$$w(\varepsilon) = Ad(\exp(\varepsilon v)w_0,$$

$$Ad(\exp(\varepsilon v))w_0 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adv)^n(w_0)$$

$$= w_0 - \varepsilon[v, w_0] + \frac{\varepsilon^2}{2}[v,[v, w_0]] - \cdots.$$

Optimal system

Let G be a Lie group. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub algebras. An optimal system of one-parameter subgroups is a list of conjugacy in equivalent one-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. For one dimensional sub algebras, this classification problem is essential the same as the problem of classifying the orbits of the adjoint representation and subjecting it to various adjoint transformations so as to simplify it as much as possible.

CHAPTER 2

METHODOLOGY

Since 1989, there have been considerable developments in symmetry methods (group methods) for differential equations as evidenced by the number of research papers, books, and new symbolic manipulation software devoted to the subject. This is, no doubt, due to the inherent applicability of the methods to nonlinear differential equations. Symmetry methods for differential equations, originally developed by Sophus Lie in the latter half of the nineteenth century, are highly algorithmic and hence amenable to symbolic computation. These methods systematically unify and extend well-known adhoc techniques to construct explicit solutions for differential equations, especially for nonlinear differential equations. Often ingenious tricks for solving particular differential equations arise transparently from the symmetry point of view, and thus it remains somewhat surprising that symmetry methods are not more widely known. Nowadays it is essential to learn the methods presented in this book to understand existing symbolic manipulation software for obtaining analytical results for differential equations. For ordinary differential equations (ODEs), these include reduction of order through group invariance or integrating factors. For partial differential equations (PDEs), these include the construction of special solutions such as similarity solutions or nonclassical solutions, finding conservation laws, equivalence mappings, and linearization.

The applications of continuous groups to differential equations make no use of the global aspects of Lie groups. These applications use connected local Lie groups of transformations. Lie’s fundamental theorems show that such groups are completely characterized by their infinitesimal generators. In turn, these form a Lie algebra determined by structure constants. Lie groups, and hence their infinitesimal generators, can be naturally extended or “prolonged” to act on the space of independent variables, dependent variables, and derivatives of the dependent variables up to any finite order. As a consequence, the seemingly intractable nonlinear conditions for group invariance of a given system of differential equations reduce to linear homogeneous equations determining the infinitesimal generators of the group. Since these determining equations form an over determined system of linear homogeneous PDEs, one can usually determine the infinitesimal generators in explicit form. For a given system of differential equations, the setting up of the determining equations is entirely routine. Symbolic manipulations programs exist to set up the determining equations and in some cases explicitly solve them. As mentioned earlier the work comprising this thesis is based primarily on the applications of symmetry reduction method. The problems are dealt-with in two phases- in the first; the symmetries of the system under investigation are derived using Symmetry reduction method and then in the second phase, after successful deduction of the reduced systems of ordinary differential equations and exact solutions are found.

2.1 Symmetry Reduction Technique

Symmetry reduction method is an important are useful method for solving nonlinear partial differential equations found by Steinberg [31]. This technique involves systematic procedures followed by described below used by specialist and non-specialist in same way. This method gives the straight forward and clear concepts about symmetries of nonlinear partial differential equations.

Exact solutions of the nonlinear partial differential equations: Dym equation, Benjamin-Bona-Mahony equation, and burger`s equation in chapter 3-5 are derived by using this method. By using this method, it becomes possible to reduce nonlinear differential equations into ordinary differential equations by using similarity variables and similarity solutions. By this method, computed symmetries are more generalised than Lie classical method.

To apply this technique, we follow the steps given below:

a) Find the symmetries of the differential equations.

b) Determined the canonical coordinates for symmetry or assume a separable form for the differential equation.

c) Find the reduced problem in terms of the canonical coordinates.

To determine the symmetry operator of a system of differential equations, we have to follow some systematic steps as:

A system of k nonlinear partial differential equations in k dependent variables $\bar{u} = (u_1, u_2, ..., u_k)$ and $n+1$ independent variables $(t, \bar{x}) = (t, x_1, x_2, ..., x_n)$ is considered.

The system of nonlinear partial differential equations can be written in terms of nonlinear differential operator $\bar{N} = (N_1, N_2, ...N_k)$ as :

$$\bar{N}(\bar{u}) \equiv \frac{\partial^p \bar{u}}{\partial t^p} - \bar{H}(\bar{u}) = \bar{0}, \qquad (2.1.1)$$

where $\bar{u} = \bar{u}(t, \bar{x})$.

$\bar{H}$ may be defined on the space of $t, \bar{x}, \bar{u}$ and any derivative of $\bar{u}$ as long as the derivatives of $\bar{u}$ don`t contain more than $p-1$ derivatives of t. $\bar{H}$ can be nonlinear. The symmetry operator $\bar{S} = (S_1, S_2, ..., S_k)$ for the system (2.1.1) is defined. The infinitesimal symmetries are quasi-linear partial differential operators of first order and consequently must have the form given by:

$$\bar{S} = A(t, \bar{x}, \bar{u}) \frac{\partial \bar{u}}{\partial t} + \sum_{i=1}^{n} B_i(t, \bar{x}, \bar{u}) \frac{\partial \bar{u}}{\partial x_i} + \bar{C}(t, \bar{x}, \bar{u}), \qquad (2.1.2)$$

where $\bar{C} = (C_1, C_2, ..., C_k)$.

The Fréchet derivative $\bar{F} = (F_1, F_2, ..., F_k)$ of $\bar{N}$ at $\bar{u} = (u_1, u_2, ..., u_k)$ in the direction of $\bar{v} = (v_1, v_2, ..., v_k)$ is given by

$$\bar{F}(\bar{N}, \bar{u}, \bar{v}) = \frac{\partial}{\partial \varepsilon} [\bar{N}(\bar{u} + \varepsilon \bar{v})]\Big|_{\varepsilon=0}. \qquad (2.1.3)$$

Now, we determine the coefficient of $A, B_i = i = 1,2,...,n$ and $C_j, j = 1,2,...,k$ in the symmetry operator $\overline{S}$, using various steps which are given below:

(i) We first find Fréchet derivative $\overline{F} = (F_1, F_2,..., F_k)$ of $\overline{N}(\overline{u}) = (N_1, N_2,..., N_k)$ by the equation (2.1.3), then $\overline{v} = (v_1, v_2,..., v_k)$ is substituted by $\overline{S} = (S_1, S_2,..., S_k)$ in the direction of symmetry operator in order to evaluate it.

$$\overline{F}(\overline{N}, \overline{u}, \overline{S}) = \frac{\partial}{\partial \varepsilon}[\overline{N}(\overline{u} + \varepsilon \overline{S})]\Big|_{\varepsilon=0}. \tag{2.1.4}$$

(ii) For invariance of the system (2.1.1), we require that the Fréchet derivative (2.1.4) must vanish on the solution set of (2.1.1) in the direction of the symmetry operator $\overline{S}$. That is, we have

$$\overline{F}(\overline{N}, \overline{\eta}, \overline{S})\Big|_{\overline{N}=0} = \overline{0}. \tag{2.1.5}$$

(iii) Equating the various coefficient of these derivatives terms, we will get a set of linear partial differential equations called "determining equations" for the group infinitesimals A, $B_i, i = 1,2,...,n$ and $C_j, j = 1,2,...,k.$ Solve the resulting "determining equations" for symmetries of the system (2.1.1).

(iv) When coefficients of $\overline{S}$ is solved for nonlinear partial differential equations, then associated Lie algebra of infinitesimal symmetries of (2.1.1) is then the set of vector fields of the form as given below:

$$V \equiv A(t,\bar{x},\bar{u})\frac{\partial}{\partial t}+\sum_{i=1}^{n} B_i(t,\bar{x},\bar{u})\frac{\partial}{\partial x_i}-\sum_{j=1}^{k} C_j(t,\bar{x},\bar{u})\frac{\partial}{\partial u_j}. \qquad (2.1.6)$$

(v) On using the infinitesimal generators (2.1.6), one can obtain a reduction of system (2.1.1) to a system with number of independent variables one less than the original one. For this, first we solve the "characteristic equations

$$\frac{dt}{A}=\frac{dx_1}{B_1}=\frac{dx_2}{B_2}=\cdots=\frac{dx_n}{B_n}=\frac{du_1}{-C_1}=\frac{du_2}{-C_2}=\cdots=\frac{du_k}{-C_k}.$$

(vi) Above form of (2.1.1) is changed to get the reduced form of the problem.

CHAPTER 3

DYM EQUATION

The Dym equation first appeared in Kruskal [21]. In mathematics, and in particular in the theory of solitons, the Dym equation is third order partial differential equation:

$$u_t = u^3 u_{xxx}.$$

It is often written in the equivalent form

$$v_t = (v^{\frac{-1}{2}})_{xxx}.$$

The Dym equation represents a system in which dispersion and nonlinearity are coupled together. The Dym equation has strong links to Korteweg-de Vries equation.

Using some symmetry subalgebra of Dym equation, the thirteen types of significant similarity reduction are obtained by virtue of classical Lie approach. For six types of reduction the general solution can be obtained by means of Weierstrass elliptic function, Riemann's zeta function, Jacobi elliptic function and the solution of Riccati equation implicity. Three types of reduction can all be solved by means of the Painleve equation but with differential independent arguments. Some types of nontravelling singular solitary wave exist for Dym equation there are two types of non-singular nontravelling solitary wave solution for some suitable potential.

3.1 Determining Equations For Lie Symmetries

Let us consider the Dym equation

$$u_t = u^3 u_{xxx}. \tag{3.1.1}$$

Let the system (3.1.1) is defined in terms of non linear operators N_1 as follows:

$$N_1 \equiv u_t - u^3 u_{xxx} = 0. \qquad (3.1.2)$$

The symmetry operator $\overline{S} = (S_1)$ for the equation (3.1.1) is:

$$S_1(u) \equiv A(\overline{X}, \overline{\eta}) \frac{\partial u}{\partial t} + B(\overline{X}, \overline{\eta}) \frac{\partial u}{\partial x} + C(\overline{X}, \overline{\eta}), \qquad (3.1.3)$$

with $\overline{X} = (t, x), \quad \overline{\eta} = (u)$.

The Fréchet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\overline{S} = (S_1)$ are given, respectively, by the following:

$$\overline{F}_1(N_1, \overline{\eta}, \overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta} + \varepsilon \overline{S})]\Big|_{\varepsilon=0} = [S_1]_t - 3u^2[S_1]u_{xxx} - u^3[S_1]_{xxx} = 0. \qquad (3.1.4)$$

In equation (3.1.4), on replacing S_1 with the help of equation (3.1.3):

$$(Au_t + Bu_x + C)_t - 3u^2(Au_t + Bu_X + C)u_{xxx} - u^3(Au_t + Bu_x + C)_{xxx} = 0, \qquad (3.1.4A)$$

$$[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t - 3u^2 Au_t u_{xxx} - 3u^2 Bu_x u_{xxx} - 3u^2 Cu_{xxx} - u^3[A]_{xxx} u_t$$
$$- u^3 Au_{txxx} - u^3[B]_{xxx} u_x - u^3 Bu_{xxxx} - u^3[C]_{xxx} = 0,$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc in equation (3.1.4) are represent the total differential with respect to suffices and are given by the following expressions:

$$[A]_t = A_t + A_u u_t,$$
$$[A]_x = A_x + A_u u_x,$$
$$[A]_{xx} = A_{xx} + 2A_{ux} u_x + A_u u_{xx} + A_{uu} u_x{}^2,$$
$$[A]_{xxx} = A_{xxx} + 3A_{uxx} u_x + 3A_{ux} u_{xx} + A_u u_{xxx} + 3A_{uux} u_x{}^3 + 3A_{uu} u_{xx} u_x + A_{uuu} u_x{}^3.$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$. the invariance equation applied on equation (3.1.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C which are obtained after equating the coefficient of various derivative terms to zero:

$$
\begin{aligned}
&A_t = A_u = A_{xxx} = 0, \\
&B_u = B_x = B_{tt} = 0, \\
&C = \frac{-1}{3} u (B_t - 3A_x).
\end{aligned}
\tag{3.1.5}
$$

From these values, we get infinitesimals A, B and C:

$$
\begin{aligned}
&A = ax + b, \\
&B = lt + m, \\
&C = au - \frac{l}{3} u,
\end{aligned}
\tag{3.1.6}
$$

where a, b, l, m are arbitrary constants.

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations:

$$
\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}.
\tag{3.1.7}
$$

Thus, it is easily seen that the application of symmetry method to equation (3.1.1) leads to two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$
\begin{aligned}
&V_1 = \frac{\partial}{\partial x}, \\
&V_2 = \frac{\partial}{\partial t}, \\
&V_3 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \\
&V_4 = t \frac{\partial}{\partial t} - \frac{u}{3} \frac{\partial}{\partial u}.
\end{aligned}
\tag{3.1.8}
$$

3.2 Optimal System Of Sub Algebras

The commutator table-3.1 and adjoint table-3.2 for Lie algebra (3.1.7) can easily constructed as follows:

The commutator (Lie bracket) of X_α and X_β is a first order operator defined by

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha .$$

Commutator table-3.1

Comm.	V_1	V_2	V_3	V_4
V_1	0	0	V_1	0
V_2	0	0	0	V_2
V_3	$-V_1$	0	0	0
V_4	0	$-V_2$	0	0

Adjoint table using:-

$$Ad(\exp \varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] + \cdots$$

Adjoint table-3.2

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	$V_3 - \varepsilon V_1$	V_4
V_2	V_1	V_2	V_3	$V_4 - \varepsilon V_2$
V_3	$V_1 e^{\varepsilon}$	V_2	V_3	V_4
V_4	V_1	$V_2 e^{\varepsilon}$	V_3	V_4

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) $V_4 + a_3 V_3$

(ii) $V_3 + a_2 V_2$

(iii) $V_1 + a_1 V_1$

(iv) V_1 (3.2.1)

where a_1, a_2, a_3 are arbitrary constant.

3.3 The Use Of Lie Symmetries To Derive Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (3.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables F is obtained using the auxiliary equation (3.1.7). Some exact solutions of each reduced system are then attempted.

Vector field (i)

The vector field (i) in the optimal system defines the similarity variable and similarity solution as follows: V_1.

We use values of system (3.2.1), and use characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Then by solving this characteristic equation we have:

$\xi = t,$

$u(x,t) = F(\xi),$
$u(x,t) = F(t).$

Vector field (ii)

For this vector field the associated similarity variable and similarity solution as follows: $V_2 + a_1 V_1$.

Take $a_1 = 1$ and by using characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{0}.$$

Then by solving this characteristic equation , we have:

$$\xi = x - t,$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (3.1.1) reduces to following system of ODEs:

$$-F'(\xi) - F(\xi)^3 F'''(\xi) = 0. \qquad (3.2.2)$$

By using Maple software, we get exact solution of (3.2.2)

$$F(\xi) = \frac{1}{c_1}.$$

The solution of the system (3.1.1) is given by

$$u(x.t) = \frac{1}{c_1}.$$

Take $a_1 = 0$ and by using characteristic equation we have:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = x,$$
$$u(x,t) = F(\xi),$$
$$u(x,t) = F(x).$$

Vector field (iii)

For this vector field the associated similarity variable and similarity solution as follows:

$V_3 + a_2V_2$.

Take $a_2 = 1$ and by using characteristic equation we have:

$$\frac{dx}{x} = \frac{dt}{1} = \frac{du}{u}.$$

Then by solving this characteristic equation we have:

$$\xi = \log x - t,$$
$$u(x,t) = e^t F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (3.1.1) reduces to the following systems of ODEs:

$$-F(\xi) + F'(\xi) + F(\xi)^3 F'''(\xi) - 3F(\xi)^3 + F''(\xi) + 2F(\xi)^3 F'(\xi) = 0. \qquad (3.2.3)$$

By using Maple software, we get exact solution of (3.2.3):

$$F(\xi) = c_2 e^{c_1+\xi},$$
$$F(\xi) = c_3 e^{\left(c_1+\frac{1}{2}\xi\right)^2},$$
$$F(\xi) = c_4 e^{\left(c_1+\frac{1}{3}\xi\right)^3}.$$

Thus, following solution of the system (3.1.1) is obtained

$$u(x,t) = e^t c_2 e^{c_1+(\log x - t)},$$
$$u(x,t) = e^t c_3 e^{\left(c_1+\frac{1}{2}(\log x - t)\right)^2},$$
$$u(x,t) = e^t c_4 e^{\left(c_1+\frac{1}{3}(\log x - t)\right)^3}. \qquad (3.2.4)$$

Take $a_2 = 0$ and by using characteristic equation we have:

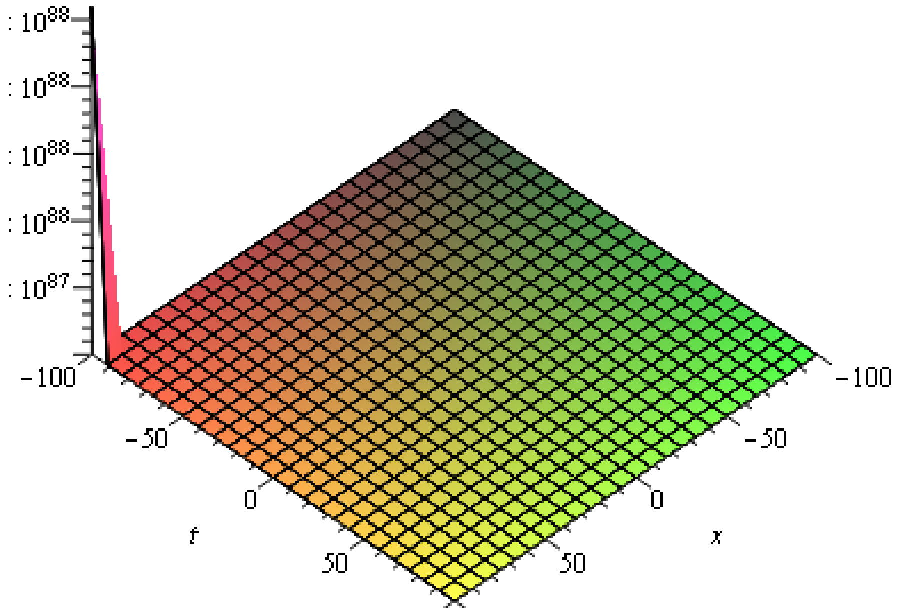
$$\frac{dx}{x} = \frac{dt}{0} = \frac{du}{0}.$$

Then by solving characteristic equation we have:

$$\xi = t,$$
$$u(x.t) = xF(\xi),$$
$$u(x.t) = xF(t).$$

3.4 Graph

For the solution of (3.2.4) with $c_1 = \frac{1}{2}, \quad c_4 = 2.$



Vector field (iv)

For this vector field the associated similarity variable and similarity solution as follows: $V_4 + a_3 V_3$.

Take $a_3 = 1$ and by using characteristic equation we have:

$$\frac{dx}{x} = \frac{dt}{t} = \frac{du}{u - \frac{u}{3}}.$$

Then by solving this characteristic equation, we have:

$$\xi = \frac{x}{t},$$
$$u(x,t) = t^{2/3} F(\xi).$$

The system of PDEs (3.1.1) reduces to following system of ODEs:

$$2F(\xi) - 3\xi F(\xi) - 3F(\xi)^3 F'''(\xi) = 0.$$

In this case we are able to find only reductions.

CHAPTR 4

BENJAMIN-BONA-MAHONY EQUATION

The Benjamin-Bona-Mahony (BBM) equation was introduced by Benjamin, Bona and Mahony [4]

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

in 1972 as an improvement of the Korteweg-de Vries equation for modelling long waves of small amplitude. They show the stability and uniqueness of solution to Benjamin-Bona-Mahony equation. This contrasts with the Korteweg-de Vries equation, which is unstable in its high wavenumber components. Further, which the Koreweg-de Vries equation has infinite number of integrals of motion, the Benjamin-Bona-Mahony equation only has three.

The nonlinear phenomena in scientific work or engineering fields are more and more attractive to scientists. To depict and analyze some nonlinear phenomena, the nonlinear evolutionary equation are playing an important role and their solitary wave solution are main interests of mathematicians and physicists.

To obtain the travelling wave solutions of these nonlinear evolution equations, many methods were attempted such as inverse scattering method, sine-cosine method, homogeneous balance method etc. With the aid of symbolic computation system many explicit solutions are easily obtained and many interesting works deeply promote research of nonlinear phenomena.

4.1 Symmetry Analysis

Consider Benjamin-Bona-Mahony equation

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{4.1.1}$$

Let the system (4.1.1) is defined in terms of non linear operator N_1 as follows:

$$N_1 \equiv u_t + u_x + uu_x - u_{xxt} = 0. \tag{4.1.2}$$

The symmetry operator $\overline{S} = (S_1)$ for the equation (4.1.1) is:

$$S_1(u) \equiv A(\overline{X}, \overline{\eta}) \frac{\partial u}{\partial t} + B(\overline{X}, \overline{\eta}) \frac{\partial u}{\partial x} + C(\overline{X}, \overline{\eta}),$$

with $\overline{X} = (t, x)$, $\overline{\eta} = (u)$.

The Fréchet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\overline{S} = (S_1)$ are given, respectively, by the following:

$$\overline{F}_1(N_1, \overline{\eta}, \overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta} + \varepsilon \overline{S})]\Big|_{\varepsilon=0} = [S_1]_t + [S_1]_x + [S_1]u_x + u[S_1]_x - [S_1]_{xxt} = 0. \tag{4.1.3}$$

In equation (4.1.1), on replacing S_1 with the help of equation(4.1.3):

$$(Au_t + Bu_x + C)_t + (Au_t + Bu_x + C)_x + (Au_t + Bu_x + C)u_x + u(Au_t + Bu_x + C)_x$$
$$- (Au_t + Bu_x + C)_{xxt} = 0. \tag{4.1.4}$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc in equation (4.1.4) are represent the total differential with respect to suffices and are given by the following expressions:

$$[A]_t = A_t + A_u u_t,$$
$$[A]_x = A_x + A_u u_x,$$
$$[A]_{xx} = A_{xx} + 2A_{ux}u_x + A_u u_{xx} + A_{uu}u_x{}^2,$$
$$[A]_{xxx} = A_{xxx} + 3A_{uxx}u_x + 3A_{ux}u_{xx} + A_u u_{xxx} + 3A_{uux}u_x{}^3 + 3A_{uu}u_{xx}u_x + A_{uuu}u_x{}^3.$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$. the invariance equation applied on equation (4.1.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C

which are obtained after equating the coefficient of various derivative terms to zero:

$$A_x = A_t = A_u = 0,$$
$$B_x = B_u = B_{tt} = 0, \qquad (4.1.5)$$
$$C = -B_t(u+1).$$

From these values, we get infinitesimals A,B and C:

$$A = a,$$
$$B = b + lt, \qquad (4.1.6)$$
$$C = -l(u+1),$$

where a,b,l are arbitrary constants.

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \qquad (4.1.7)$$

Thus, it is easily seen that the application of symmetry method to equation (4.1.1) leads to two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t}, \qquad (4.1.8)$$
$$V_3 = t\frac{\partial}{\partial t} - (u+1)\frac{\partial}{\partial u}.$$

For optimal system, the commutator table-4.1 and adjoint table-4.2 for Lie algebra (4.1.7) can easily constructed as follows:

The commutator (Lie bracket) of X_α and X_β is a first order operator defined by: $[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha$.

Commutator table-4.1

Comm.	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	V_2
V_3	0	$-V_2$	0

Adjoint table using:-

$$Ad(\exp \varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i[V_i, V_j]] + \cdots$$

Adjoint table-4.2

Ad	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 - \varepsilon V_2$
V_3	V_1	$V_2 e^{\varepsilon}$	V_3

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) $V_3 + a_1 V_1$

(ii) $V_2 + a_1 V_1$

(iii) V_1. (4.1.9)

where a_1 is arbitrary constant.

4.2 Group Invariant Solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (4.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables F is obtained using the auxiliary equation (4.1.7). Some exact solutions of each reduced system are then attempted.

Essential field (i)

The essential vector field (i) in the optimal system the similarity variable and similarity solution as follows: V_1

Using characteristic equation we have:

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{0}.$$

Then by solving characteristic equation, we have:

$$\xi = t,$$
$$u(x.t) = F(\xi),$$
$$u(x,t) = F(t).$$

Essential field (ii)

The similarity variable and similarity solution of this essential vector field as

follows: $V_3 + a_1 V_{1.}$.

Using characteristic equation we have:

$$\frac{dx}{a_1}=\frac{dt}{t}=\frac{du}{-(u+1)}.$$

Then by solving this characteristic equation, we have:

$$\xi = x - a_1 \log t,$$
$$u(x,t) = \frac{F(\xi)}{t} - 1.$$

Using the similarity variable, the forms of similarity solutions, the system of

PDEs (4.1.1) reduces to following system of ODEs:

$$-bF'(\xi)-F(\xi)+F(\xi)F'(\xi)+bF'''(\xi)+F''(\xi)=0. \qquad (4.2.1)$$

If b=0, then by using Maple software, we get exact solution of (4.2.1)

$$F(\xi)=c_3+\frac{c_1+c_2\xi}{c_2}.$$

Thus, the following solution of the system (4.1.1) is obtained

$$u(x,t)=\frac{c_3+\dfrac{c_1+c_2(x-a_1\log t)}{c_2}}{t}-1.$$

Essential field (iii)

The essential field (iii) in the optimal system defines the similarity variable and similarity solution as follows:

$$V_2+a_1V_1.$$

Using characteristic equation , we have:

$$\frac{dx}{1}=\frac{dt}{1}=\frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi=x-t,$$
$$u(x,t)=F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (4.1.1) reduces to following system of ODEs:

$$F(\xi)F'(\xi)+F'''(\xi)=0. \qquad (4.2.2)$$

By using Maple software, we get some exact solutions of (4.2.2):

$$F(\xi)=c_6,\quad F(\xi)=c_6+c_{12}\cos(c_5-\xi^{3/2}),\quad F(\xi)=c_6+c_{12}\cos(c_5+\xi^{3/2})$$

$$F(\xi)=c_6+c_{10}\cos(c_5-\frac{1}{2}\xi^{3/2})^2,\quad F(\xi)=c_6+c_{10}\cos(c_5+\frac{1}{2}\xi^{3/2})^2$$

$$F(\xi)=c_6-\frac{3}{4}c_7\cos\left(c_5-\frac{1}{3}\xi^{3/2}\right)+c_7\cos\left(c_5-\frac{1}{3}\xi^{3/2}\right)^3,$$

$$F(\xi)=c_6-\frac{3}{4}c_7\cos\left(c_5+\frac{1}{3}\xi^{3/2}\right)+c_7\cos\left(c_5+\frac{1}{3}\xi^{3/2}\right)^3$$

$$F(\xi)=c_6+c_{12}\cosh\left(-c_5+\sqrt{-\xi}\xi\right),\quad F(\xi)=c_6+c_{12}\cosh\left(c_5+\sqrt{-\xi}\xi\right),$$

$$F(\xi)=c_6+c_{10}\cosh\left(-c_5+\frac{1}{2}\sqrt{-\xi}\xi\right)^2,\quad F(\xi)=c_6+c_{10}\cosh\left(c_5+\frac{1}{2}\sqrt{-\xi}\xi\right)^2$$

$$F(\xi)=c_6-\frac{3}{4}c_7\cosh\left(-c_5+\frac{1}{3}\sqrt{-\xi}\xi\right)+c_7\cosh\left(-c_5+\frac{1}{3}\sqrt{-\xi}\xi\right)^3,$$

$$F(\xi)=c_6-\frac{3}{4}c_7\cosh\left(c_5+\frac{1}{3}\sqrt{-\xi}\xi\right)+c_7\cosh\left(c_5+\frac{1}{3}\sqrt{-\xi}\xi\right)^3$$

$$F(\xi)=c_6+c_{10}\sin\left(c_5-\frac{1}{2}\xi^{3/2}\right)^2,\quad F(\xi)=c_6+c_{10}\sin\left(c_5+\frac{1}{2}\xi^{3/2}\right)^2$$

$$F(\xi)=c_6-\frac{3}{4}c_7\sin\left(c_5-\frac{1}{3}\xi^{3/2}\right)+c_7\sin\left(c_5-\frac{1}{3}\xi^{3/2}\right)^3,$$

$$F(\xi)=c_6-\frac{3}{4}c_7\sin\left(c_5+\frac{1}{3}\xi^{3/2}\right)+c_7\sin\left(c_5+\frac{1}{3}\xi^{3/2}\right)^3$$

$$F(\xi)=c_{12},\quad F(\xi)=c_8,\quad F(\xi)=c_8+c_6\cos\left(c_9-\xi^{3/2}\right),\quad F(\xi)=c_8+c_6\cos\left(c_9+\xi^{3/2}\right)$$

$$F(\xi)=c_8+c_6\cosh\left(-c_9+\sqrt{-\xi}\xi\right),\quad F(\xi)=c_8+c_6\cosh\left(c_9+\sqrt{-\xi}\xi\right)$$

$$F(\xi)=c_8+c_{11}e^{c_9-\sqrt{-\xi}\xi}, \quad F(\xi)=c_8+c_{11}e^{c_9+\sqrt{-\xi}\xi}$$

$$F(\xi)=c_8+c_{12}\left(e^{c_9-\frac{1}{2}\sqrt{-\xi}\xi}\right)^2, \quad F(\xi)=c_8+c_{12}\left(e^{c_9+\frac{1}{2}\sqrt{-\xi}\xi}\right)^2$$

$$F(\xi)=c_8+c_{10}\left(e^{c_9-\frac{1}{3}\sqrt{-\xi}\xi}\right)^3, \quad F(\xi)=c_8+c_{10}\left(e^{c_9+\frac{1}{3}\sqrt{-\xi}\xi}\right)^3$$

$$F(\xi)=c_8+c_{11}\sin\left(c_9-\xi^{3/2}\right), \quad F(\xi)=c_8+c_{11}\sin\left(c_9+\xi^{3/2}\right)$$

$$F(\xi)=c_8-c_{11}\sinh\left(-c_9+\sqrt{-\xi}\xi\right), \quad F(\xi)=c_8+c_{11}\sinh\left(c_9+\sqrt{-\xi}\xi\right).$$

Thus, the following solutions of system (3.1.1) is obtained

$$u(x,t)=c_6, \quad u(x,t)=c_6+c_{12}\cos(c_5-(x-t)^{3/2}), \quad u(x,t)=c_6+c_{12}\cos(c_5+(x-t)^{3/2})$$

$$u(x,t)=c_6+c_{10}\cos(c_5-\frac{1}{2}(x-t)^{3/2})^2, \quad u(x,t)=c_6+c_{10}\cos(c_5+\frac{1}{2}(x-t)^{3/2})^2 \tag{4.2.3}$$

$$u(x,t)=c_6-\frac{3}{4}c_7\cos\left(c_5-\frac{1}{3}(x-t)^{3/2}\right)+c_7\cos\left(c_5-\frac{1}{3}(x-t)^{3/2}\right)^3,$$

$$u(x,t)=c_6+c_{12}\cosh\left(-c_5+\sqrt{-(x-t)}(x-t)\right),$$

$$u(x,t)=c_6-\frac{3}{4}c_7\cos\left(c_5+\frac{1}{3}(x-t)^{3/2}\right)+c_7\cos\left(c_5+\frac{1}{3}(x-t)^{3/2}\right)^3$$

$$u(x,t)=c_6+c_{12}\cosh\left(c_5+\sqrt{-(x-t)}(x-t)\right)$$

$$u(x,t)=c_6+c_{10}\cosh\left(-c_5+\frac{1}{2}\sqrt{-(x-t)}(x-t)\right)^2,$$

$$u(x,t)=c_6+c_{10}\cosh\left(c_5+\frac{1}{2}\sqrt{-(x-t)}(x-t)\right)^2$$

$$u(x,t)=c_6-\frac{3}{4}c_7\cosh\left(c_5+\frac{1}{3}\sqrt{-(x-t)}(x-t)\right)+c_7\cosh\left(c_5+\frac{1}{3}\sqrt{-(x-t)}(x-t)\right)^3$$

$$u(x,t)=t^{2/3}(c_6+c_{12}\sin(c_5-(x-t)^{3/2})),\quad u(x,t)=t^{2/3}(c_6+c_{12}\sin(c_5+(x-t)^{3/2}))$$

$$u(x,t)=c_6+c_{10}\sin\left(c_5-\frac{1}{2}(x-t)^{3/2}\right)^2,\quad u(x,t)=c_6+c_{10}\sin\left(c_5+\frac{1}{2}(x-t)^{3/2}\right)^2$$

$$u(x,t)=c_6-\frac{3}{4}c_7\sin\left(c_5-\frac{1}{3}(x-t)^{3/2}\right)+c_7\sin\left(c_5-\frac{1}{3}(x-t)^{3/2}\right)^3, \tag{4.2.4}$$

$$u(x,t)=c_6-\frac{3}{4}c_7\sin\left(c_5+\frac{1}{3}(x-t)^{3/2}\right)+c_7\sin\left(c_5+\frac{1}{3}(x-t)^{3/2}\right)^3$$

$$u(x,t)=c_8+c_6\cos(c_9-(x-t)^{3/2}), \tag{4.2.5}$$

$$u(x,t)=c_8+c_6\cos(c_9+(x-t)^{3/2})$$

$$u(x,t)=c_8+c_6\cosh(-c_9+\sqrt{-(x-t)}(x-t)),$$

$$u(x,t)=c_8+c_6\cosh(c_9+\sqrt{-(x-t)}(x-t))$$

$$u(x,t)=c_8+c_{12}\left(e^{c_9-\frac{1}{2}\sqrt{-(x-t)}(x-t)}\right)^2,\quad u(x,t)=c_8+c_{12}\left(e^{c_9+\frac{1}{2}\sqrt{-(x-t)}(x-t)}\right)^2$$

$$u(x,t)=c_8+c_{10}\left(e^{c_9-\frac{1}{3}\sqrt{-(x-t)}(x-t)}\right)^3,\quad u(x,t)=c_8+c_{10}\left(e^{c_9+\frac{1}{3}\sqrt{-(x-t)}(x-t)}\right)^3$$

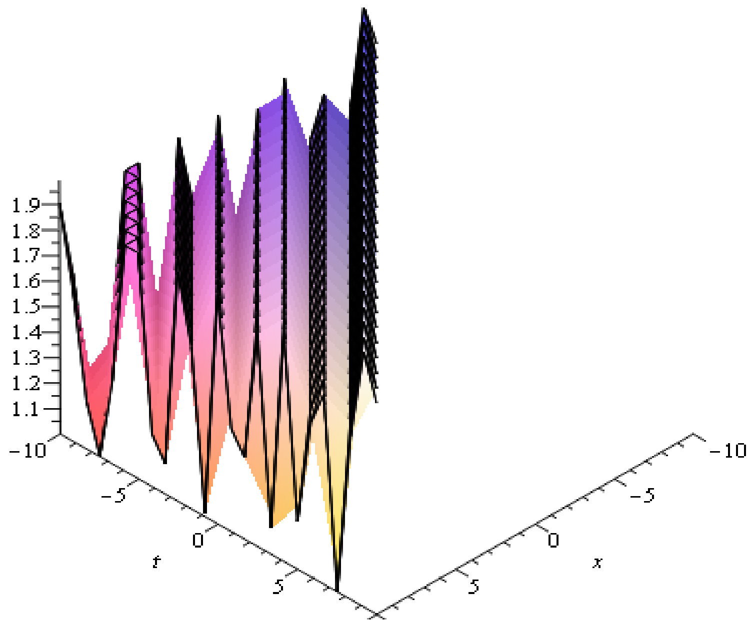
$$F(\xi)=c_8+c_{11}\sin(c_9-(x-t)^{3/2}),\qquad u(x,t)=c_8+c_{11}\sin(c_9+(x-t)^{3/2})$$

$$u(x,t)=c_8-c_{11}\sinh(-c_9+\sqrt{-(x-t)}(x-t)),$$

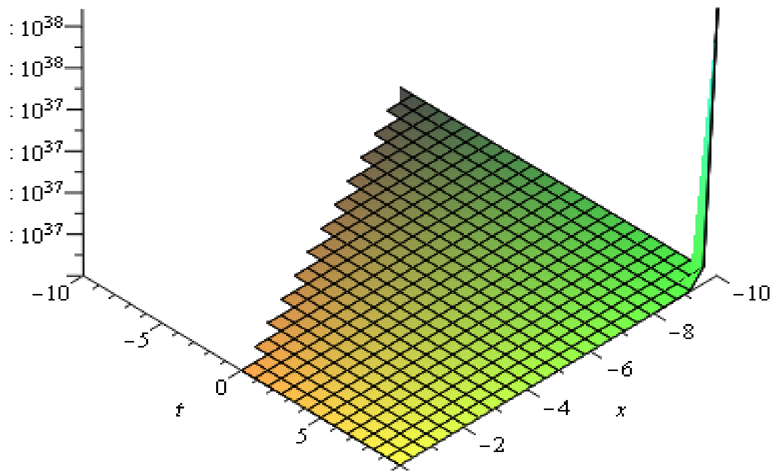
$$u(x,t)=c_8+c_{11}\sinh(c_9+\sqrt{-(x-t)}(x-t))\ . \tag{4.2.6}$$

4.3 Graphs

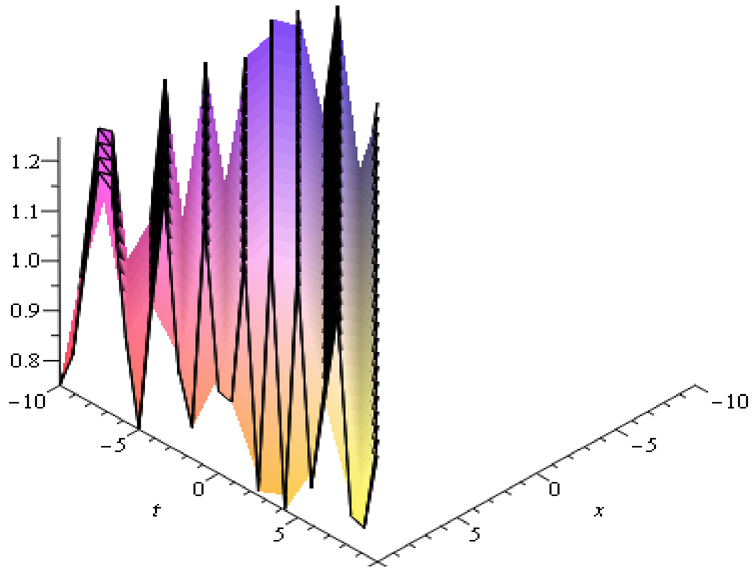
(i) For the solution of (4.2.3) with $c_5 = 1, c_6 = 1, c_{10} = 1$.



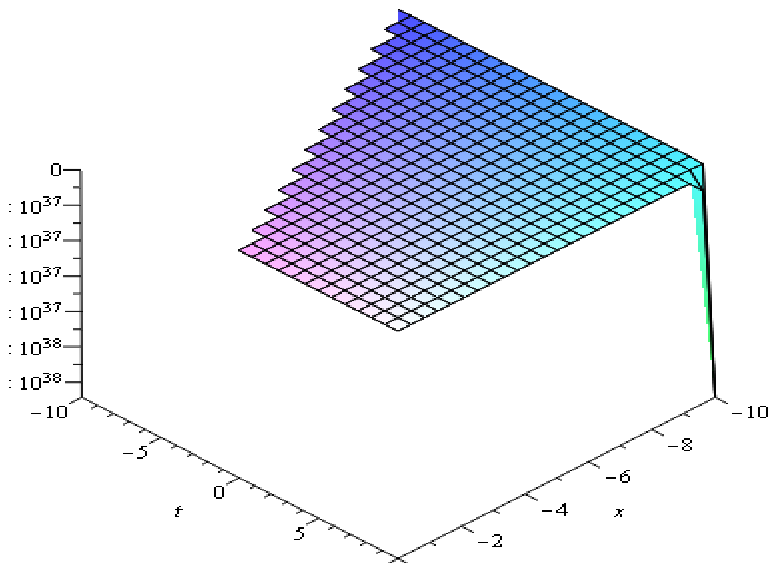
(ii) For the solution of (4.2.5) with $c_6 = 1, c_8 = 1, c_9 = 1$.



(iii) For the solution of (4.2.4) with $c_5 = 1, c_6 = 1, c_7 = 1$.



(iv) For the solution of (4.2.6) with $c_8 = 1, c_9 = 1, c_{11} = 1$.



CHAPTER 5

BURGER`S EQUATION

Burger's equation introduced by Burger [11] is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modelling of gas dynamics and traffic flow. It is named for Jchannes Martinus Burgera (1895-1981). It is of the form

$$u_t + uu_x - ku_{xx} = 0.$$

The one-dimensional non-linear differential equation, which is similar to one-dimensional Navier-Stokes equation without the stress term, and was presented for the first time in a paper in 1940 from Burger, is the model for solution of Navier-Stokes equation and is applied to laminar and turbulence floes as well. The Burger equation was first studied by Cole [14] who gave a theoretical solution based on Fourier series analysis, using the appropriate initial and boundary conditions. Another theoretical solution given by Madsen and Sincovec [20], based on "test and trial" method, using approximate initial and boundary conditions. In Benton and Platzman, are mentioned almost 35 distinct solution of Burger equation and Agas tried to get approximate solution of Burger equation using numerical analysis.

5.1 Determining Equations For Burger's Equation

Consider burger`s equation

$$u_t + uu_x - ku_{xx} = 0. \tag{5.1.1}$$

Let the system (5.1.1) is defined in terms of non linear operator N_1 as follows:

$$N_1 \equiv u_t + uu_x - ku_{xx} = 0. \tag{5.1.2}$$

The symmetry operator $\overline{S} = (S_1)$ for the equation (5.1.1) is:

$$S_1(u) \equiv A(\overline{X}, \overline{\eta}) \frac{\partial u}{\partial t} + B(\overline{X}, \overline{\eta}) \frac{\partial u}{\partial x} + C(\overline{X}, \overline{\eta}), \tag{5.1.3}$$

with $\overline{X} = (t, x)$, $\overline{\eta} = (u)$.

The Fréchet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\overline{S} = (S_1)$ are given, respectively, by the following:

$$F_1(N_1, \overline{\eta}, \overline{S}) = \frac{d}{d\varepsilon}[N_1(\overline{\eta} + \varepsilon \overline{S})]\Big|_{\varepsilon=0} = [S_1]_t + [S_1]u_x + u[S_1]_x - k[S_1]_{xx} = 0. \tag{5.1.4}$$

In equation (5.1.4), on replacing S_1 with the help of equation (5.1.3):

$$(Au_t + Bu_x + C)_t + (Au_t + Bu_x + C)u_x + u(Au_t + Bu_x + C)_x - k(Au_t + Bu_x + C)_{xx} = 0, \tag{5.1.5}$$

$$\begin{aligned}&[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t + Bu_x^{\,2} + Au_t u_x + Cu_x + u[A]_x u_t + uAu_{tx} + u[B]_x u_x \\ &uBu_{xx} + u[C]_x - k[A]_{xx} u_t - kAu_{txx} - k[B]_{xx} u_x - kBu_{xxx} - [C]_{xx} = 0,\end{aligned}$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc in equation (5.1.5) are represent the total differential with respect to suffices and are given by the following expressions:

$$\begin{aligned}
[A]_t &= A_t + A_u u_t, \\
[A]_x &= A_x + A_u u_x, \\
[A]_{xx} &= A_{xx} + 2A_{ux} u_x + A_u u_{xx} + A_{uu} u_x^{\,2}, \\
[A]_{xxx} &= A_{xxx} + 3A_{uxx} u_x + 3A_{ux} u_{xx} + A_u u_{xxx} + 3A_{uux} u_x^{\,3} + 3A_{uu} u_{xx} u_x + A_{uuu} u_x^{\,3}.
\end{aligned}$$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$. the invariance equation applied on equation (5.1.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C which are obtained after equating the coefficient of various derivative terms to zero:

$$\begin{aligned}
A_t &= \frac{1}{2} B_t u + C, \\
A_x &= \frac{B_t}{2}, \\
A_u &= 0, \\
B_u &= B_x = B_{ttt} = 0, \\
C_t &= \frac{-u}{2} B_{tt}, \\
C_u &= \frac{-B_t}{2}, \\
C_x &= \frac{B_{tt}}{2}.
\end{aligned} \tag{5.1.6}$$

From these values, we get infinitesimals A, B and C:

$$\begin{aligned}
A &= ax + bt + l, \\
B &= 2at + m, \\
C &= b - au,
\end{aligned} \tag{5.1.7}$$

where a, b, l, m are arbitrary constants.

5.2 Symmetry Generators

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \tag{5.2.1}$$

Thus, it is easily seen that the application of symmetry method to equation (5.1.1) leads to two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial x}, \\
V_2 &= \frac{\partial}{\partial t}, \\
V_3 &= t\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},
\end{aligned} \tag{5.2.2}$$

$$V_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

For optimal system, the commutator table-5.1 and adjoint table-5.2 for Lie algebra (5.1.7) can easily constructed as follows:

The commutator (Lie bracket) of X_α and X_β is a first order operator defined by: $[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha$.

Commutator table=5.1

Comm.	V_1	V_2	V_3	V_4
V_1	0	0	0	V_1
V_2	0	0	0	V_2
V_3	0	0	0	$-V_3$
V_4	$-V_1$	$-V_2$	V_3	0

Adjoint table using:-

$$Ad(\exp \varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i[V_i, V_j]] + \cdots$$

Adjoint table-5.2

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	$V_4 - \varepsilon V_1$
V_2	V_1	V_2	V_3	$V_4 - \varepsilon V_2$
V_3	V_1	V_2	V_3	$V_4 + \varepsilon V_3$
V_4	$V_1 e^{\varepsilon}$	$V_2 e^{\varepsilon}$	$V_3 e^{-\varepsilon}$	V_4

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) V_4

(ii) $V_3 + a_2 V_2 + a_1 V_1$

(iii) $V_2 + a_1 V_1$

(iv) V_1. (5.2.3)

where a_1, a_2 are arbitrary constants.

5.3 Analytic And Exact Solutions Of Burger's Equation

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of PDEs (5.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables F is obtained using the auxiliary equation (5.1.7). Some exact solutions of each reduced system are then attempted.

Essential field (i)

The essential vector field (i) in the optimal system the similarity variable and similarity solution as follows: V_1

Using characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Then by solving characteristic equation, we have:

$$\xi = t,$$
$$u(x.t) = F(\xi),$$
$$u(x,t) = F(t).$$

Essential field (ii)

The similarity variable and similarity solution of this essential vector field as follows: $V_3 + a_2 V_2 + a_1 V_1$.

Take $a_1 = a_2 = 1$ and by using characteristic equation we have:

$$\frac{dx}{t+1} = \frac{dt}{1} = \frac{du}{1}.$$

Then by solving this characteristic equation, we have:

$$\xi = x - \frac{t^2}{2} - t,$$
$$u(x,t) = t + F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (5.1.1) reduces to following system of ODEs:

$$1 - F^{I}(\xi) + F(\xi)F^{I}(\xi) - kF^{II}(\xi) = 0.$$

In this case we able to find only reductions.

Essential field (iii)

The essential field (iii) in the optimal system defines the similarity variable and similarity solution as follows:

$$V_2 + a_1 V_1.$$

Using characteristic equation , we have:

$$\frac{dx}{1} = \frac{dt}{a_1} = \frac{du}{0}.$$

Then by solving this characteristic equation, we have:

$$\xi = x - a_1 t,$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of similarity solutions, the systems of PDEs (5.1.1)

reduces to following system of ODEs:

$$-a_1 F'(\xi) + F(\xi)F'(\xi) - kF''(\xi) = 0.$$

By using Maple software, we get some exact solutions of above equation:

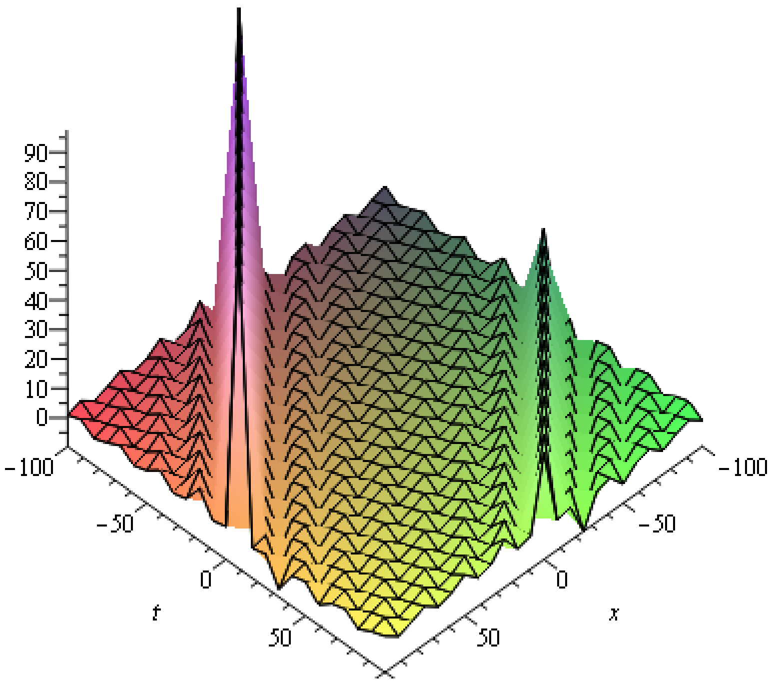
$$F(\xi) = \frac{1}{2}\left(\sqrt{2}a_1 + 2\tan\left(\frac{1}{2}\frac{\sqrt{c_1 k}(\xi + c_2)\sqrt{2}}{k}\right)\sqrt{c_1 k}\right)\sqrt{2}.$$

Thus, the following solution of system (5.1.1) is obtained

$$u(x,t) = \frac{1}{2}\left(\sqrt{2}a_1 + 2\tan\left(\frac{1}{2}\frac{\sqrt{c_1 k}((x - a_1 t) + c_2)\sqrt{2}}{k}\right)\sqrt{c_1 k}\right)\sqrt{2}. \quad (5.3.1)$$

5.4 Graph

For the solution of (5.3.1) with $c_1 = 1, c_2 = 1, a_1 = 1, k = 1$.



Essential field (iv)

The similarity variable and similarity solutions of this essential vector field as follows: V_4.

Using characteristic equation we have:

$$\frac{dx}{x}=\frac{dt}{2t}=\frac{du}{-u}.$$

Then by solving this by characteristic equation we have:

$$\xi=\frac{x}{t^{1/2}},$$
$$u(x,t)=\frac{F(\xi)}{t^{1/2}}.$$

Using the similarity variable, the forms of similarity solutions, the system of PDEs (5.1.1) reduces to following system of ODEs:

$$-\xi F(\xi)-F(\xi)+2F(\xi)F^{I}(\xi)-2kF^{II}(\xi)=0. \tag{5.3.2}$$

By using Maple software, we get some exact solutions of (5.3.2):

$$F(\xi)=-\frac{\left(2\left(\begin{array}{l}KummerM\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\xi^2}{k}\right)c_1+(1-c_1)KummerM\left(\frac{3}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\xi^2}{k}\right)\\ +kc_2\left(2KummerU\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\xi^2}{k}\right)+KummerU\left(\frac{3}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\xi^2}{k}\right)(-1+c_1)\right)c_1\end{array}\right)k\right)}{\left(\xi(2c_2kKummerU\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\xi^2}{k}\right)+KummerM\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\xi^2}{k}\right)\right)},$$

where the Kummer M (mu, nu, z) and Kummer U (mu, nu, z) solve the differential

equation zy''+(v-z)y'-μy=0, where mu, nu and z are algebraic expressions.

Thus, the following solution of the system (5.1.1) is obtained

$$u(x,t)=-\frac{\left(2\left(KummerM\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\left(\frac{x}{t^{1/2}}\right)^2}{k}\right)c_1+(1-c_1)KummerM\left(\frac{3}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\left(\frac{x}{t^{1/2}}\right)^2}{k}\right)+kc_2\left(2KummerU\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\left(\frac{x}{t^{1/2}}\right)^2}{k}\right)+KummerU\left(\frac{3}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\left(\frac{x}{t^{1/2}}\right)^2}{k}\right)(-1+c_1)\right)c_1\right)k\right)}{\left(\left(\frac{x}{t^{1/2}}\right)(2c_2kKummerU\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\left(\frac{x}{t^{1/2}}\right)^2}{k}\right)+KummerM\left(\frac{1}{2}-\frac{1}{2}c_1,\frac{3}{2},\frac{-1}{4}\frac{\left(\frac{x}{t^{1/2}}\right)^2}{k}\right)\right)\right)}t^{-1/2}.$$

RESULTS AND DISCUSSIONS

In this thesis we carried out the studies of nonlinear partial differential equations due to their immense use in the study of various fields i.e. in pure mathematics, in applied mathematics and in various physical phenomena. To derive the exact solution of nonlinear partial differential equations, Symmetry reduction method is used in this thesis. This method is based on Frechet derivative of the nonlinear operators and known as Symmetry reduction method. The reduced ordinary differential equations have been examined for various types of exact solutions via techniques which are essentially based on special functions such as Kummer U , Kummer M and hyperbolic functions.

Using above technique in a systematic manner step by step we formed first determining equations and with the use of these equations we find out the generators. Then optimal solutions are to be found with the help of generators and reduced the obtained partial differential equations to ordinary differential equations.

In mathematical point of view along with its applications, the group classification and the exact solution of nonlinear partial differential equation are examined. In fact, a systematic and detailed study of all these systems has been made and new solutions are obtained.

In all the three cases, the solutions obtained are of very specific nature and a proper search for the reduction of partial differential equations to ordinary differential equations and then reduction of order of ordinary differential equations based on Lie groups led to the symmetries. As the objective in present work was aimed to the applications of Symmetry reduction method with the view of deducing the symmetries and then attempting some exact analytic solutions.

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