FIXED CHARGE MULTI-OBJECTIVE TRANSPORTATION PROBLEM

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DEDICATED

TO

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AND

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CERTIFICATE

*I hereby certify that the work which is bring presented in the thesis entitled "**Fixed Charge Multi-Objective Transportation Problem**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Mahesh Kumar Sharma.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

The fixed charge transportation problem (FCTP) is an extension of the classical transportation problem in which a fixed cost is incurred, independent of the amount transported along with a variable cost that is proportional to the amount shipped. The introduction of fixed costs in addition to variable costs results in the objective function being a step function.

The fixed charge multi-objective transportation (FMOT) has been studied in the present work. In the FMOT a fixed cost called setup cost is incurred for every origin and the cost of transportation is directly proportional to the number of units transported, but on account of quantity discounts, price break etc.

The present thesis contains three Chapters. The Chapter 1 is introductory in nature. In Chapter 2, the algorithm proposed by Basu *et al* (1994) is modified in conjunction with the approach given by Gupta and Gupta (1982) for FMOT problem. In Chapter 3 FMOT problem is extended by including one more non-linear objective which is conflicting in nature and non dominated solutions are obtained for the same.

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CHAPTER 1

INTRODUCTION

Transportation is very important since it is considered a determining factor for economic growth. It facilitates distribution of raw materials and finished products from one point to another. It also improves time performance and increases productivity.

Transportation is needed because few economic resources—raw materials, fuels, food, manufactured goods—are located where they are wanted. Each region or place on Earth produces more than it consumes of some goods and services and less than it consumes of others. Through transportation, goods are moved from where there are surpluses to where there are shortages.

1.1 CLASSICAL TRANSPORTATION PROBLEM

One of the well structure problem in operations research is classical transportation problem and this problem has been extended in different versions in literature by many research workers. The transportation problem is the subclass of the linear programming problems and due to its special structure simple and practical computational procedures have been developed. The transportation problem is amongst the most important special LPP in terms of the frequency with it appears in the application and also in the simplicity of the procedure developed for its solution. The classical transportation problem received its name because it arises naturally in the context of determining optimum shipping pattern. For example a product may be transported from factories to retail stores. The factories are the sources and the stores are the destinations. The amount of product that is available is known and the demands are also known. The problem is that different legs of the network joining the sources to the destinations have different costs associated with them. The aim is to find the minimum cost routing of products from the supply point to the destination. The general transportation problem can be formulated as: A product is available at each of m origins and it is required that given quantities of the product be shipped to each of n destinations. The minimum cost of shipping a unit of the product from any origin to any destination is known. The shipping schedule which minimizes the total cost of shipment is to be determined.

The problem can be formulated as below:

$$\text{Minimize} \quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} \le a_i,\ a_i > 0$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j,\ b_j > 0$$

$$x_{ij} \ge 0,\ for\ i = 1,2,\ldots,m; j = 1,2,\ldots,n.$$

a_i is the quantity of the product available at origin i

b_j is the quantity of the product required at destination j

c_{ij} is the cost of shipping one unit from origin i to destination j

The aim is to minimize the objective function satisfying the above mentioned constraints. the classical transportation problem of linear programming, the traditional objective is one of minimizing the total cost.

1.2 TIME MINIMIZING TRANSPORTATION PROBLEM

In a time minimizing transportation problem, the time of transporting goods from m origins to *n* destinations is minimized, satisfying certain conditions in respect of availabilities at sources and requirements at the destinations. The time minimizing transportation problem are of importance when it is required to transport perishable goods or during war days, it is required to transport food and ornaments in the shortest possible time and in so many other similar situations. Thus, a time-minimizing transportation problem can be formulated as:

$$\text{Minimize} \quad [\max t_{ij} / x_{ij} > 0]$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i,\ a_i > 0$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j,\ b_j > 0$$

$$x_{ij} \ge 0,\ for\ i = 1,2,\ldots,m; j = 1,2,\ldots,n.$$

Here t_{ij} is the time of transporting goods from the i^{th} origin, where the availability is a_i to the j^{th} destination, where the requirement is b_j. For any given feasible solution, $X = [x_{ij}]$ satisfying the above constraints, the time of transportation is the maximum of t_{ij}*'s* among the cells in which there are positive allocations, i.e., corresponding to the solution *X*, the time of transportation is

$$Z = [\max t_{ij} / x_{ij} > 0]$$

The aim is to minimize this time of transportation. This time of transportation remains independent of the amount of commodity sent so long as $x_{ij} > 0$. It is assumed that (i) the carriers have sufficient capacity to carry goods from an origin to destination in a single trip, (ii) they start simultaneously from there respective origins. Thus, the basic difference between the cost minimizing transportation problem and the time minimizing problem is that whereas the cost of transportation changes with variations in the quantity of the commodity, the time involved remains unchanged, irrespective of the quantities of the commodity involved in the occupied cells in the time minimizing transportation problem. From a practical point of view, the cost minimizing transportation problem and the time minimizing transportation problem cannot be viewed as two independent problems, of one is interested in obtaining a solution which cost and time simultaneously. If the unit costs of transportation and the associated duration of transportation are given for each supply demand pair of points, then the cost-time trade-off solutions are of interest.

1.3 FIXED CHARGE TRANSPORTATION PROBLEM

The fixed charge transportation problem (FCTP) is an extension of classical transportation problem in which fixed cost is incurred for every origin. The fixed charge transportation problem was originally formulated by Hirsch and Danzig. Many distribution problems in practice can only be modeled as FCTPs. For example, rails, roads and trucks have invariable used freight rates which consist of a fixed cost and a variable cost. The fixed cost may represent the cost of renting a vehicle, landing fees at airport, set up costs for machines in manufacturing environment etc.

The problem can be formulated mathematically as:

$$\text{Minimize} \quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} F_i$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} \le a_i,\ a_i > 0$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ b_j > 0$$

$$x_{ij} \ge 0,\ for\ i = 1, 2, ..., m; j = 1, 2, ..., n.$$

where

i =1, 2,. . . ,m, are the origins,

j = 1, 2,. . .,n, are the destinations,

x_{ij} = the amount transported from the i^{th} origin to the j^{th} destination,

c_{ij} = the variable cost per unit amount transported from i^{th} origin to the j^{th} destination,

a_i = maximum capacity at origin i,

b_j = the demand at destination j,

F_i = the fixed cost associated with origin i.

1.4 MULTI-CRITERIA OPTIMIZATON

Multi-criteria optimization (or multi-objective programming), also known as multi-criteria or multi-attribute optimization, is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints.

Multi-criteria optimization problems can be found in various fields: product and process design, finance, aircraft design, the oil and gas industry, automobile design, or wherever optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives.

For nontrivial multi-criteria optimization problems, one cannot identify a single solution that simultaneously minimizes each objective to its fullest. While searching for solution, one reaches points such that, when attempting to improve an objective further, other objectives suffer as a result. A solution is called non-dominated if it cannot be eliminated from consideration because there is at least another solution which improves an objective without worsening another one. Finding such non-dominated solution and quantifying the trade offs in satisfying the different objectives, is the goal when setting up and solving a multi-criteria optimization problem.

In general a multi-criteria programming problem can be formulated as:

$$\text{Optimize} \quad f(x)=(f_1(x), f_2(x),..., f_k(x))$$

$$\text{Subject to} \quad g_j(x) \leq = \geq b_j, j=1,2,\dots m$$

$$x \geq 0$$

$$X = (x_1,x_2,\dots,x_n)^T$$

where, $f(x)$ is the objective function to optimize $f_1(x), f_2(x),..., f_k(x)$ are k number of distinct objective function subject to m constraints, X is a vector consists of decision variable $x_1,x_2,\dots,x_n.$

1.5 CONCEPT OF OPTIMAL AND EFFICIENT SOLUTIONS:

Optimal Solution

An optimal solution is the classical sense is one which attains the maximum value of all the objectives simultaneously. The solution x^* is optimal to the problem defined if and only if $x^* \in S$ and $f_l(x^*) \geq f_l(x)$ for all l and for all $x \in S$,where S is the feasible region.

In general, there is no optimal solution to a multi-objective problem. Therefore, optimality replaced by the concept of “satisfying” or the best compromise solution, which depends on the decision makers preferences with respect to object. Optimality is not an illusion only when the objectives are non-conflicting. Therefore one must be satisfied with obtaining efficient solutions in multi-objective problems.

Efficient or Non-Dominated Solutions

A set of solutions is said to be efficient if there exists no solutions that is superior to it with respect to at least one objective function but is not inferior to it with respect to any of the objective functions.

If x_1 and x_2 are two solution, then these can have any of two possibilities- one dominates the other or non-dominates the other. In a minimization problem, without the loss of generality, a solution x_1 dominates x_2 if the following two conditions are satisfied:

$$\forall\, i \in \{1,2,...,N_{obj}\} : f_i(x_1) \le f_i(x_2)$$
$$\exists\, j \in \{1,2,...,N_{obj}\} : f_j(x_1) \le f_j(x_2)$$

Where, $f\,(x_1)$ and $f(x_2)$ are the objective functions

If any the above conditions are violated, the solution x_1 does not dominate the solution x_2. If x_1 dominates the solution x_2 is called the non dominated solution with in the set $\{ x_1 , x_2\}$. The solutions that are non-dominated within the entire search space and denoted as pareto-optimal and constitute the pareto-optimal set or pareto-optimal front. From the entire set of efficient (non-dominated) solutions the decision maker can select the solution one believed most attractive.

1.6 LINEAR MULTI-OBJECTIVE TRANSPORTATION PROBLEM

In real world cases transportation problem can be formulated as a multi-objective transportation problem because the complexity of the social and economic environment requires the explicit consideration of criteria other than cost. Example of additional concerns include: average delivery of the commodities, reliability of transportation, accessibility to the users, product deterioration, among others.

Multi-objective problem is in which there are more than one objective satisfying a set of constraints and can be formulated as follows:

$$\text{Minimize} \quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{l} x_{ij};\ l = 1,2,...,k.$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i,\ a_i > 0$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ b_j > 0$$

$$x_{ij} \ge 0,\ for\ i = 1,2,....,m; j = 1,2,...,n.$$

where

i =1, 2,. . . ,m, are the origins,

j = 1, 2,. . .,n, are the destinations,

x_{ij} = the amount transported from the i^{th} origin to the j^{th} destination,

c_{ij}^{l} = the cost per unit amount transported from i^{th} origin to the j^{th} destination corresponding to k objectives i.e.$l=1,2,\ldots,k$.

a_i = capacity at origin i,

b_j = the demand at destination j.

1.7 LITERATURE SURVEY

Transportation problems are different types and the simplest of them is now standard in the literature was first presented by Hitchcock (1941). It usually aims to minimize the total transportation cost. Other objectives that can be set are a minimization of the total delivery time, a maximization of profits, etc. from the investigation; the entire existing objectives in single objective transportation model are represented by quantitative information. This may cause the negligence of some crucial points which cannot be described by quantitative data.

Later independently, by Koopman(1947), Koopman began to spearhead research on the potentialities of linear programs for the study of the problems in economics. His historic paper "optimum utilization of the transportation potations systems" was based on his war time experience. Because of this and the work done earlier by Hitchcock, the classical case is often referred as the Hitchcock-Koopman's transportation problem Kantorovich (1942) publishes the paper on a continuous version of the problem and later with Gavurin, an applied study of the capacitated transportation problem (Kantorovich and Gavurin 1949).

The time minimizing transportation problem has been studied by Hammer (1969), Garfinkel and Rao(1971) and Szwarc(1971). Hammer(1969) and Szwarc(1971) used labeling techniques to solve the problem. Garfunkel and Rao(1971) solved the problem by introducing a sufficiently large cost M on certain routes. Sometimes there may exist emergency situations such as those requiring police services, fire services, ambulance services, etc., when the time of transportation is of greater importance than cost of transportation. Some methods for minimizing the time of transportation have been established. Several methods for minimizing the time of transportation are also developed. Bhatia *et al.* (1975) developed a technique for minimizing time in a transportation problem. The procedure involved finite number of iterations and is based on moving from one basic feasible solution to another till the last solution is arrived at. The algorithm given by them consists of determination of an initial basic feasible solution which can be found by the methods applicable in the case of the common cost minimizing transportation problem and finding an adjacent better basic feasible solution, the procedure is repeated until no better adjacent basic feasible solution can be found. This repeated procedure deals with the determination of a cell not in the basis which, when introduced will either reduce the time of transportation or reduce the allocation in at least one of the cells $\in$ Q, where Q is the set of cell with positive allocations and corresponding time equal to the time of transportation.

The transportation problem with two-objectives known as the bi-criterion transportation problem has been studied by many research workers. In this type of problem there are two objectives- one primary and the other secondary. The primary objective is to minimize the total cost of transportation problem and the secondary objective is to minimize the duration of transportation.

Isermann's (1979) proposed an algorithm to obtain the set of all efficient solutions for a linear multi-objective transportation problem in different phases. The algorithm starts with an initial efficient basic feasible solution (Phase 1) and while passing from one efficient solution to another, to generate the entire set of basic feasible efficient solutions (Phase 2), has to solve a linear sub problem, at each step, to find which vector should enter the basis. Gupta and Gupta(1982) developed a multi-criteria simplex method for a linear multi-objective transportation problem which a direct generalization of the multi-criteria simplex method of Zeleny (1974) to the linear multi-objective transportation problem ,by which the set of all non-dominated basic feasible solutions are generated. They also proposed a technique to check the dominance or non-dominance of the solutions and shown that in some cases there is no need to solve the problem completely. However the approach given by Klingman and Russell (1975) is much and more simplified to check the non-dominance character of the solution but in that case the problem has to be solved completely.

In a classical transportation problem the cost of transportation is directly proportional to the number of units of the commodity transported. But in real world situations when a commodity is transported, a fixed cost is incurred in objective function. The fixed cost may represent the cost of renting a vehicle, landing fees in an airport, set up costs for machines in a manufacturing environment etc. such type of situation is formulated into a problem. This problem with bi-criterion transportation problem is called Fixed charge bi-criterion transportation problem.

The fixed charge transportation problem was originally formulated by Dantzig and Hirsch (1954). Then Murty (1968) solved the fixed charge problem by ranking the extreme points. After that several procedures for solving Fixed charge transportation problems were developed. Also Basu *et.al.*(1994) developed an algorithm for the optimum time-cost trade-off in a fixed charge linear transportation problem giving same priority to cost and time. The fixed-charge transportation problem (FCTP) is an extension of the classical transportation problem in which a fixed cost is incurred, independent of the amount transported, along with a variable cost that is proportional to the amount shipped. The introduction of fixed costs in addition to variable costs results in the objective function being a step function. Therfore, fixed-charge problem are usually solved using sophistyicated analytical or computer software.

The transportation problem considered in the classical transportation problem is generally a two-dimensional linear transportation problem. After fixed charge bi-criteria transportation problem Thirwani et al. (1997) review this algorithm and the gives the algorithm on Fixed charge bi-criteria transportation problem restricted flow is introduced. Fixed charge bi-criteria transportation problem with restricted flow which is an extension of the fixed charge bi-criteria transportation problem. In this type of problem, there is a restriction on the total flow. In the fixed charge bi-criterion transportation problem a fixed cost called the set up cost is incurred for every origin. In the bi-criterion transportation problem the cost of transportation is directly proportional to the number of units transported, but on account of quantity discounts, price breaks etc. in this problem we keep on solving with initial basic feasible solution to an optimal solution. When we reach to the optimality then we stop the criteria.

1.8 Present Work

In the present thesis a fixed charge multi-objective transportation problem is considered. An algorithm which is a modification of algorithm proposed by Basu *et. al. (*1994) in conjunction with approach given by Gupta and Gupta (1982). Also this problem is extended by including one more nonlinear objective with is conflicting in nature and an algorithm is proposed to find the efficient cost-time trade off pairs to given problem.

CHAPTER-2

AN ALGORITHM FOR FIXED CHARGE MULTI-OBJECTIVE TRANSPORTATION PROBLEM

2.1 Introduction

A fixed charge multi-objective transportation problem is an extension of bi-criterion fixed charge transportation problem. In this problem there are k objectives and a fixed cost is incurred for every origin. The fixed cost may represent the cost of renting a vehicle, landing fee, setup costs for machines in manufacturing environment.

Basu *et al.* (1994) developed an algorithm for fixed charge bi-criterion transportation problem. In this chapter this approach is modified in conjunction with the approach proposed by Gupta and Gupta (1982) for a linear multi-objective transportation problem.

2.2 Formulation of Fixed Charge Multi-objective Transportation Problem

Suppose there are m origins and n destinations, the quantities of a uniform product available at the origins and required at the destinations are given. The total quantity available at the sources is precisely the same as the total quantity required at the destinations and it is possible to transport to any destination from any origin. In this problem there are k objectives which have to be minimize. The formulation of the problem is as follows.

(P)

$$\text{Minimize} \quad Z=\{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}{}^{l} x_{ij} + \sum_{i=1}^{m} F_i\}; \quad l=1,2,...,k.$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} \le a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for \;\; i=1,2,\ldots,m; \quad j=1,2,\ldots,n.$$

$c_{ij}{}^{l}$= the units of cost of transportation of one unit of the product from origin i to destination j corresponding to k objectives i.e. $l=1,2,...,k$,

a_i = the units of the product available at origin i,

b_j = the units of the product required at destination j,

x_{ij} = the number of units of the product transported from origin i to destination j.

2.3 SOLUTION PROCEDURE

To solve the fixed charge multi-objective transportation problem a procedure given by Basu *et al* (1994) in conjunction with the algorithm proposed by Gupta and Gupta (1982) and given as below.

For formulation of F_i ($i = 1, 2,. . ., m$), it is assumed that F_i ($i = 1, 2,. . ., m$) has p-number of steps so that

$$F_i = \sum_{t=1}^{p} \delta_{it} F_{it}, \quad i = 1, 2, ..., m,$$

where

$$\delta_{it} = 1, \text{ if } \sum_{j=1}^{n} x_{ij} > A_{it},\ i = 1, 2, ..., m,\ t = 1, 2, ..., p$$
$$= 0, \text{otherwise}.$$

Here

$$0 = A_{i1} < A_{i2} < . . . < A_{ip}.$$

A_{i1} , A_{i2} , . . ., A_{ip} ($i = 1,2, . . . ,m$) are constants and F_{it} $(t = 1,2,. . .,p ; i = 1,2,. . .,m)$ are fixed costs. Since fixed cost at each origin is considered, unbalanced transportation problem is to be taken into account.

So, first we have to balance the problems (P_1):

(P_1)

$$\text{Minimize} \quad z = \{ \sum_{i=1}^{m} \sum_{j=1}^{n+1} c_{ij}{}^{l} x_{ij} + \sum_{i=1}^{m} F_i \} \quad l = 1,2,...,\text{k}$$

$$\text{Subject to} \quad \sum_{j=1}^{n+1} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \geq 0 \quad \text{for} \quad i = 1,2,...,m; \quad j = 1, 2, ..., n, n+1$$

In problems (P_1) the cost (variable cost and also fixed cost) with the dummy cells are all zero. Solving the (P_1) means finding the set of all non dominated solution of fixed charge multi objective transportation problem, in terms of the following definition.

Definition [Gupta and Gupta (1982)]: A feasible solution $\bar{X}=\bar{x}_{ij}$ is said to be a non-dominated solution of (P_1) if there does not exist any other feasible solution $\bar{X}=\bar{x}_{ij}$ of (P_1), such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}{}^{l} x_{ij} \le \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}{}^{l} \bar{x}_{ij} \ , \ l = 1, 2, ..., k$$

with strict inequality in at least one of the *k* inequalities.

Definition: A feasible solution $\bar{X}=\bar{x}_{ij}$ is said to be a non-dominated solution of (P_1) if there does not exist any other feasible solution $\bar{X}=\bar{x}_{ij}$ of (P_1), such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}{}^{l} x_{ij} + \sum_{i=1}^{m} F_i \le \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}{}^{l} \bar{x}_{ij} + \sum_{i=1}^{m} F_i \ , \ l = 1, 2, ..., k$$

with strict inequality in at least one of the k inequalities. The algorithm starts with any initial basic feasible solution and a simplex type iteration proposed by Gupta and Gupta (1994) is used to generate the set of all non- dominated basic feasible solutions while moving from one solution to the next, while ensure that new solution so obtained is not dominated by the previous one.

2.4 ALGORITHM

Step 1: Take *p*=1, where *p* is the number of iterations in the algorithm.

Step 2: Find $\bar{X}$ basic feasible solution of the problem (P_1) $l=1,2,\dots,k$ for each of the sub problems and denote this current basis with $\bar{B}$.

Step 3: Calculate the fixed cost of the current basic feasible solution and denote this by F^1 (current) where

$$\sum_{i=1}^{m} F_i$$

F^1 (current) =

Step 4: Calculate $(C_{ij}^{l}-u_i^{l}-v_j^{l})$, $l=1,2,\ldots,k$ for all $i, j \notin \overline{B}$ and denote it by $(M_{ij})_l^1$, where u_i^l, v_j^l are the dual variables associated with k sub problems for i=1,2,. . . ,m; j=1,2,. . .,n, n+1.

Step 5: Find $(A_{ij})_l^1 = (M_{ij})_l^1 \times (E_{ij})_l^1$ for all $i, j \notin \overline{B}$

where $(A_{ij})_l^1$ is the change in cost which occurs for introducing a non-basic cell (i, j) with value $(E_{ij})_l^1$ into the basis by making reallocation.

Step 6: Calculate F_{ij}^{1}(NB) is the total fixed cost involved for introducing the variable x_{ij} with values $(E_{ij})_l^1$ for all $i, j \notin \overline{B}$ into the current basis to form a new basis.

$$F_{ij}^{1}\text{ (Difference)} = F_{ij}^{1}\text{(NB)} - F^{1}\text{(Current)}$$

Step 7: Now add F_{ij}^{1} (Difference) and $(A_{ij})_l^1$ and denote it by $(\Delta_{ij})_l^1$ *i.e.*

$(\Delta_{ij})_l^1 = F_{ij}^{1}$ (Difference)+ $(A_{ij})_l^1$ for all $i, j \notin \overline{B}$

Step 8: Check for a dominated or non dominated solution as given in Appendix 2.1.

2.5 NUMERICAL EXAMPLE

The above algorithm is explained by considering the following 3×3 fixed-charge linear multi-objective transportation problem.

$$\text{Minimize} \quad \left\{ \sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}^{l} x_{ij} + \sum_{i=1}^{3} F_i \right\} \quad l = 1,2,3.$$

$$\text{Subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \quad i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \quad i = 1,2,3; \quad j = 1,2,3.$$

Table 2 gives the values of variable cost C_{ij}^{l} (i=1, 2, 3; j=1, 2, 3; l=1, 2, 3) . The fixed costs are:

F_{11}=100, F_{12}=50, F_{13} =50

F_{21}=150, F_{22}=50, F_{23}=50

F_{31}=200, F_{32}=50, F_{33}=50

TABLE 2.1

	c^1_{ij}			a_i
c^1_{ij}	5	9	10	9
	4	6	2	14
	4	2	3	17
b_j	5	12	8	

	c^2_{ij}		
c^2_{ij}	1	2	4
	3	7	4
	2	9	5

	c^3_{ij}		
c^3_{ij}	4	7	2
	1	5	8
	9	8	1

The above data can be written in one table which is given below.

TABLE 2.2

Destination j→ *Origin i↓*	1	2	3	a_i
1	5 1 4	9 2 7	10 4 2	9
2	4 3 1	6 7 5	2 4 8	14
3	4 2 9	2 9 8	3 5 1	17
b_j	5	12	8	

The total cost which is to be minimized is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}^{\,l} x_{ij} + \sum_{i=1}^{3} F_i$$

$$F_i = \sum_{l=1}^{3} \delta_{il} F_{il}\ , \qquad i = 1,2,3$$

where $\delta_{i1} = 1, if \ \sum_{j=1}^{3} x_{ij} > 0,\ i = 1,2,3,$

$= 0, \text{otherwise};$

$\delta_{i2} = 1, if \ \sum_{j=1}^{3} x_{ij} > 8,\ i = 1,2,3,$

$= 0, \text{otherwise}.$

$\delta_{i3} = \ 1,\ if \ \sum_{j=1}^{3} x_{ij} > 11,\ i = 1,2,3,$

$= 0, \text{ otherwise}.$

Here F_i (i=1,2, 3) has three steps. Introducing a dummy destination $j=4$ with zero cost in Table 2.2, we get Table 2.3.

TABLE 2.3

Destination j→ *Origin i↓*	1	2	3	4	a_i
1	5 1 4	9 2 7	10 4 2	0 0 0	9
2	4 3 1	6 7 5	2 4 8	0 0 0	14
3	4 2 9	2 9 8	3 5 1	0 0 0	17
b_j	5	12	8	15	

A basic feasible solution (using North–West Corner Rule) and the total fixed cost of the current solution of sub problem is given in Table 2.4.

TABLE 2.4

Destinationj→ *Origin i↓*	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	5 1 4 **5**	9 2 7 **4**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1	6 7 5 **8**	2 4 8 **6**	0 0 0	250	-3 5 -2	14
3	4 2 9	2 9 8	3 5 1 **2**	0 0 0 **15**	200	-2 6 -9	17
v_j^1 v_j^2 v_j^3	5 1 4	9 2 7	5 -1 10	2 -6 9	600		

b_j	5	12	8	15			

Applying Step 4, we get $C_{ij}^{l} - u_i^{l} - v_j^{l}$ values, for all $i, j \notin \bar{B}$ which are given in Table 2.5.

TABLE 2.5

ij	13	14	21	24	31	32
$C_{ij}^{1} - u_i^{1} - v_j^{1}$	5	-2	2	1	1	-5
$C_{ij}^{2} - u_i^{2} - v_j^{2}$	5	6	-3	1	-5	1
$C_{ij}^{3} - u_i^{3} - v_j^{3}$	-8	-9	-1	-7	14	10

Applying Step 5, we get the values of $(A_{ij})_l^1$, which are displayed in Table 2.6.

TABLE 2.6

ij	13	14	21	24	31	32
$(A_{ij})_1^1$	20	-8	10	6	2	-10
$(A_{ij})_2^1$	20	24	-15	6	-10	2
$(A_{ij})_3^1$	-32	-36	-5	-42	28	20

Applying Step 6, we get the following results which are displayed in Table 2.7.

TABLE 2.7

ij	13	14	21	24	31	32
1	150	100	150	150	150	150
2	250	250	250	150	250	250
3	200	200	200	200	200	200
$F_{ij}(NB)$	600	550	600	500	600	600
$F_{ij}(Difference)$	0	-50	0	-100	0	0

Applying Step 7, we get the values of $(\Delta_{ij})_l^1$, which are displayed in Table 2. 8.

TABLE 2.8

ij	13	14	21	24	31	32
$(\Delta_{ij})^1_1$	20	-58	10	-94	2	-10
$(\Delta_{ij})^1_2$	20	-26	-15	-94	-10	2
$(\Delta_{ij})^1_3$	-32	-86	-5	-142	28	20

From Table 2.4 $X^1 = (5,4,0,0,0,8,6,0,0,0,2,15)$, for which $Z = (727,703,738)$. Now, applying step 8(ii), Since $(\Delta_{ij})^1_l \le 0, l=1,2,3,$ the cell (2,4) enter to the basis and the new solution is given in Table 2.9.

TABLE 2.9

Destinationj→ *Origin i*↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	5 1 4 **5**	9 2 7 **4**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1	6 7 5 **8**	2 4 8	0 0 0 **6**	150	-3 5 -2	14
3	4 2 9	2 9 8	3 5 1 **8**	0 0 0 **9**	200	-3 5 -2	17
v_j^1 v_j^2 v_j^3	5 1 4	9 2 7	6 0 3	3 -5 2	500		
b_j	5	12	8	15			

Applying Step 4, we get the following results as shown in Table 2.10.

TABLE 2.10

ij	13	14	21	23	31	32
$C^1_{ij} - u^1_i - v^1_j$	4	-3	2	-1	2	-4
$C^2_{ij} - u^2_i - v^2_j$	4	5	-3	-1	-4	2
$C^3_{ij} - u^3_i - v^3_j$	-1	-2	-1	7	7	3

Applying Step 5, we get the values of $(A_{ij})_l^1$, which are given in Table 2.11.

TABLE 2.11

ij	13	14	21	23	31	32
$(A_{ij})_1^1$	16	-12	10	-6	10	-32
$(A_{ij})_2^1$	16	20	-15	-6	-20	16
$(A_{ij})_3^1$	-4	-8	-5	42	35	24

Applying Step 6, we get the following results which are displayed in Table 2.12.

TABLE 2.12

ij	13	14	21	23	31	32
1	150	100	150	150	150	150
2	250	250	150	250	150	0
3	200	200	200	200	300	300
$F_{ij}\,(NB)$	600	550	500	600	600	450
$F_{ij}\,(Difference)$	100	50	0	100	100	-50

Applying Step 7, we get the values of $(\Delta_{ij})_l^1$, which are displayed in Table 2.13.

TABLE 2.13

ij	13	14	21	23	31	32
$(\Delta_{ij})_1^1$	116	38	10	94	110	-82
$(\Delta_{ij})_2^1$	116	70	-15	94	80	-34
$(\Delta_{ij})_3^1$	96	42	-5	142	135	-26

From Table 2.9 $X^2=(5,4,0,0,0,8,0,6,0,0,8,9)$, for which $Z=(633,609,596)$ .Now , applying step 8(ii), Since $(\Delta_{ij})_l^1 \leq 0$,l=1,2,3, the cell (3,2) enter to the basis and the new solution is given in Table 2.14.

TABLE 2.14

Destination j→ *Origin i*↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	5 1 4 **5**	9 2 7 **4**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1	6 7 5	2 4 8	0 0 0 **14**	0	-7 7 1	14
3	4 2 9	2 9 8 **8**	3 5 1 **8**	0 0 0 **1**	300	-7 7 1	17
v_j^1 v_j^2 v_j^3	5 1 4	9 2 7	10 -2 0	7 -7 -1	450		
b_j	5	12	8	15			

Applying Step 4, we get the following results in Table 2.15.

TABLE 2.15

ij	13	14	21	22	23	31
$C_{ij}^1 - u_i^1 - v_j^1$	0	-7	6	4	-1	6
$C_{ij}^2 - u_i^2 - v_j^2$	6	7	-5	-2	-1	-6
$C_{ij}^3 - u_i^3 - v_j^3$	2	1	-4	-3	7	4

Applying Step 5, we get the values of $(A_{ij})_l^1$ which are tabulated in Table 2.16.

TABLE 2.16

ij	13	14	21	22	23	31
$(A_{ij})_1^1$	0	-7	30	32	-8	30
$(A_{ij})_2^1$	24	7	-25	-16	-8	-30
$(A_{ij})_3^1$	8	1	-20	-24	56	20

Applying Step 6, we get the following results which are displayed in Table 2.17.

TABLE 2.17

ij	13	14	21	22	23	31
1	150	100	150	150	150	150
2	0	0	150	150	150	0
3	300	300	250	200	200	300
$F_{ij}\,(NB)$	450	400	550	500	500	450
$F_{ij}\,(Difference)$	0	-50	100	50	50	0

Applying Step 7, we get the following values of $(\Delta_{ij})_l^1$ which are tabulated in Table 2.18

TABLE 2.18

ij	13	14	21	22	23	31
$(\Delta_{ij})_1^1$	0	-57	130	82	42	30
$(\Delta_{ij})_2^1$	24	-43	75	34	42	-30
$(\Delta_{ij})_3^1$	8	-49	80	26	106	20

From Table 2.14 $X^3=(5,4,0,0,0,0,0,14,0,8,8,1)$, for which $Z=(551,575,570)$.Now, applying step 8(ii), Since $(\Delta_{ij})_l^1 \le 0$,l=1,2,3, the cell (1,4) enter to the basis and the new solution is given in Table2.19.

TABLE 2.19

Destination ***j***→ *Origin* ***i***↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	5 1 4 **5**	9 2 7 **3**	10 4 2	0 0 0 **1**	100	0 0 0	9
2	4 3 1	6 7 5	2 4 8	0 0 0 **14**	0	0 0 0	14
3	4 2 9	2 9 8 **9**	3 5 1 **8**	0 0 0	300	-7 7 1	17
v_j^1 v_j^2 v_j^3	5 1 4	9 2 7	10 -2 0	0 0 0	400		
b_j	5	12	8	15			

Applying Step 4, we get the following results in Table 2.20.

TABLE 2.20

ij	13	21	22	23	31	34
$C_{ij}^1 - u_i^1 - v_j^1$	0	-1	-3	-8	6	7
$C_{ij}^2 - u_i^2 - v_j^2$	6	2	5	6	-6	-7
$C_{ij}^3 - u_i^3 - v_j^3$	2	-3	-2	8	4	-1

Applying Step 5, we get the values of $(A_{ij})_l^1$ which are tabulated in Table 2.21.

TABLE 2.21

ij	13	21	22	23	31	34
$(A_{ij})_1^1$	0	-5	-9	-24	30	7
$(A_{ij})_2^1$	18	10	15	18	-30	-7
$(A_{ij})_3^1$	6	-15	-6	24	20	-1

Applying Step 6, we get the following results which are displayed in Table 2.22.

TABLE 2.22

ij	13	21	22	23	31	34
1	100	100	100	100	100	150
2	0	150	150	150	0	0
3	300	300	300	300	300	300
$F_{ij}\,(NB)$	400	550	550	550	400	450
$F_{ij}\,(Difference)$	0	150	150	150	0	50

Applying Step 7, we get the following values of $(\Delta_{ij})_l^1$ which are tabulated in Table 2.23.

TABLE 2.23

ij	13	21	22	23	31	34
$(\Delta_{ij})_1^1$	0	145	141	126	30	57
$(\Delta_{ij})_2^1$	18	160	165	168	-30	43
$(\Delta_{ij})_3^1$	6	135	144	174	20	49

From Table 2.19 X^4=(5,3,0,1,0,0,0,14,0,9,8,0), for which Z=(494,532,521).

Since there is no cell for which $(\Delta_{ij})_l^1 \leq 0$ for $l=1,2,3$ with at least one inequality so it is must to check the character of this solution is dominate or non dominate .

Initially, $H^{-1}=I_3$, $s_1=-1$, $s_2=-1$, $s_3=-1$. As $\sum_{l=1}^{k} s_l (\Delta_{ij})_l^1$

Is most negative for the cell (3,1), therefore for this cell

$$Y_A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}30\\-30\\20\end{bmatrix}=\begin{bmatrix}30\\-30\\20\end{bmatrix}$$

Since, Y_A is not less than zero, hence X^4 is non dominate solution. Now R= (3,1)

applying step 8(ii), the cell (3,1) enter to the basis and the new solution is given in Table 2.24.

TABLE 2.24

Destinationj→ *Origin i↓*	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	5 1 4	9 2 7 **8**	10 4 2	0 0 0 **1**	100	0 0 0	9
2	4 3 1	6 7 5	2 4 8	0 0 0 **14**	0	0 0 0	14
3	4 2 9 **5**	2 9 8 **4**	3 5 1 **8**	0 0 0	300	-7 7 1	17
v_j^1 v_j^2 v_j^3	11 -5 8	9 2 7	10 -2 0	0 0 0	400		
b_j	5	12	8	15			

Applying Step 4, we get the following results in Table 2.25.

TABLE 2.25

ij	11	13	21	22	23	34
$C_{ij}^1 - u_i^1 - v_j^1$	-6	0	-7	-3	-8	7
$C_{ij}^2 - u_i^2 - v_j^2$	6	6	8	5	6	-7
$C_{ij}^3 - u_i^3 - v_j^3$	-4	2	-7	-2	8	-1

Applying Step 5, we get the values of $(A_{ij})_l^1$ which are tabulated in Table 2.26.

TABLE 2.26

ij	11	13	21	22	23	34
$(A_{ij})_1^1$	-30	0	-35	-24	-64	7
$(A_{ij})_2^1$	30	48	40	40	48	-7
$(A_{ij})_3^1$	-20	16	-35	-16	64	-1

Applying Step 6, we get the following results which are displayed in Table 2.27.

TABLE 2.27

ij	11	13	21	22	23	34
1	100	100	100	0	0	150
2	0	0	150	150	150	0
3	300	300	300	300	300	300
$F_{ij}(NB)$	400	400	550	450	450	450
$F_{ij}(Difference)$	0	0	150	50	50	50

Applying Step 7, we get the following values of $(\Delta_{ij})_l^1$ which are tabulated in Table 2.28.

TABLE 2.28

ij	11	13	21	22	23	34
$(\Delta_{ij})_1^1$	-30	0	115	26	-14	57
$(\Delta_{ij})_2^1$	30	48	190	90	98	43
$(\Delta_{ij})_3^1$	-20	16	115	34	114	49

From Table 2.24 $X^5 = (0,8,0,1,0,0,0,14,5,4,8,0)$, for which $Z = (524,502,541)$.

Since there is no cell for which $(\Delta_{ij})_l^1 \leq 0$ for $l=1,2,3$ with at least one inequality so it is must to check the character of this solution is dominate or non dominate .

Initially, $H^{-1} = I_3$, $s_1 = -1$, $s_2 = -1$, $s_3 = -1$. As $\sum_{l=1}^{k} s_l (\Delta_{ij})_l^1$

Is most positive for the cell (1,1), therefore for this cell

$$Y_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -30 \\ 30 \\ -20 \end{bmatrix} = \begin{bmatrix} -30 \\ 30 \\ -20 \end{bmatrix}$$

Since, the first component is most positive so, q_2 leaves the basis, hence $E_H^T = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}$

Performing the simplex iteration.

we get, $s_1 = -1$, $s_2 = 50$, $s_3 = -1$

and $H^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/30 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}$

Since, Y_A is not less than zero, hence X^5 is non dominate solution. Now R= ϕ

Thus we terminate, the set of non-dominated solution is given below:

$X^4 = (5,3,0,1,0,0,0,14,0,9,8,0)$, $Z = (494,532,521)$

$X^5 = (0,8,0,1,0,0,0,14,5,4,8,0)$, $Z = (524,502,541)$

Appendix 2.1

(i) $\bar{X}$ will dominate all other solutions X^* obtained by introducing the non-basic cell *(i, j)* for which $(\Delta_{ij})_l^1 \geq 0$ for $l = 1,.. ,k$, with at least one inequality strictly positive, and $x_{ij}^* = \theta_{ij} \leq 0$. Therefore, such a non-basic cell possibly cannot enter the basis.

(ii) If for a non-basic cell (*i, j*), $(\Delta_{ij})_l^1 \leq 0$ for $l = 1,...,k$, with at least one inequality, with a strictly negative sign, then the solution X^* obtained by introducing the cell *(i, j)* will dominate $\bar{X}$ if $x_{ij}^* = \theta_{ij} \leq 0$.

(iii) If there are non-basic cells *(i, j)* and *(h, k)* such that $\theta_{ij}(\Delta_{ij})_l^1 \leq \theta_{hk}(\Delta_{ij})_l^1$, $l = 1,2,...,k$ with strict inequality in at least one of the *k* inequalities, where $x_{cd}^* = \theta_{cd}$ if the cell (*c,d*) is introduced into the basis, then the solution obtained by introducing the cell (*h, k*) is dominated by the solution resulting from introducing the cell (*i, j*).

(iv) $\bar{X}$ is a non-dominated solution if for all non-basic cells *(i, j)*, $(\Delta_{ij})_l^1 > 0$ for at least one *l* i.e. $\bar{X}$ is a unique optimal solution for at least one of the k sub-problems (P^l), l = 1,…, k. Thus the above observations (ii) and (iv) help one analyzing the character of any basic feasible solution, but still, they do not cover all the possibilities. We may have the possibility when for at least one non-basic cell, s $(l \leq s < k)$ of the *k* quantities Δ_{ij}, $l = 1,...,k$ are negative, while the remaining *k-s* are positive, it being assumed that all the *k* quantities for the remaining non-basic cells are non-positive.

In this case, to find out whether $\bar{X}$ is dominated or non-dominated, we concentrate on the linear programming problem (P_2):

$$(P_2): \text{ Minimize } \ -\sum_{l=1}^{k} q_l$$

$$\text{Subject to } \quad \sum_{j=1}^{n} x_{ij} = a_i \qquad \text{....(2.1)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \text{....(2.2)}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{\ l} x_{ij} + q_l + \sum_{i=1}^{m} F_i \leq \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{\ l} \bar{x}_{ij} + \sum_{i=1}^{m} F_i \ , \ l = 1,2,...,k \qquad \text{....(2.3)}$$

$$q_l \geq 0, \qquad l = 1, .., k. \qquad \text{....(2.4)}$$

$$x_{ij} \geq 0, \quad for \ \ i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n.$$

and

Clearly $\bar{X}$ is a non-dominated solution if and only if the problem (P_2) has optimum and is dominated otherwise.

Klingman and Russell (1975) have given a technique for finding the optimal solution of problems of the type (P_2) above. But in our case, to check the non-dominance character of $\bar{X}$, their technique becomes much more simplified due to the special structure of (P_2). In some cases, we need not even solve the problem (P_2) completely, but the moment we discover that some q_i becomes positive, we conclude that it is a dominated solution.

Now, consider the dual of the problem (P_3) namely,

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j + \sum_{l=1}^{k} s_l \Big(\sum_{i=1}^{m} c_{ij}^l \sum_{j=1}^{n} \bar{x}_{ij} + \sum_{i=1}^{m} F_i\Big) \\ &\text{Subject to} \quad u_i + v_j + \sum_{l=1}^{k} s_l c_{ij}{}^l \le 0 \\ &\qquad\qquad s_l \le -1, \ for \ l = 1,2,\dots,k. \end{aligned}$$

Let (P_3) denote the problem (P_2) without the additional constraints (2.3) and (2.4). Let (X, q_1, $q_2,\dots,q_k$) be any basic feasible solution for (P_2) with basis $\hat{B}$.As has been remarked in Klingman and Russell (1975), the basis $\hat{B}$ can be initially partitioned in the form

$$\hat{B} = \begin{bmatrix} B & B_A \\ C_B & C_A \end{bmatrix}$$

where $\hat{B}$ is the basis for (P_3), C_B is a $k \times (m+n-1)$ matrix containing all those components c_{ij}^l for which $(i\ ,\ j) \in B$. Also

$B_A : (m\ + n) \times k$, zero matrix

and

$C_A : k \times k$, identity matrix, if $q_1, q_2, \dots. q_k$

are in the basis, otherwise they contain the appropriate columns.

Let (u_i^l , v_j^l), $l = 1,2,\dots.,k$ be determined from the equations

$\Delta_{ij}^l = 0$ for $(i,j) \in B$, $l=1,2,\dots.k$; ….(2.5)

The generalized left inverse $\hat{B}_G$ of $\hat{B}$, Ben Israel and Greville (1974), may be obtained as explained in Gupta (1977).

In order to find out the vector which enters the basis, consider the following system connecting the dual variables

$U = (u_i), V = (v_j), S = (s_l)$

i.e. $(U^T, V^T, S^T) \begin{bmatrix} B & B_A \\ C_B & C_A \end{bmatrix} = \begin{bmatrix} 0, E^T \end{bmatrix}$ …..(2.6)

where the l^{th} component of E^T is zero if q_i is not in the basis and is -1, if q_i is in the basis

for all $l=1,...,k$.

As has been shown in Klingman and Russell (1975), (2.6) can equivalently be expressed in the form

$$u_i + v_j + \sum_{l=1}^{k} s_l c_{ij}^{\ l} = 0 \text{ for } (i,j) \in B \qquad \text{….(2.7)}$$

$$S^T H = E_H^{\ T} \qquad \text{....(2.8)}$$

where H is a $k \times k$ non-singular matrix, so that H^{-1} is the last k columns of the last k rows of $\hat{B}_G$ and $E_H^{\ T}$ is a k component row vector such that each component of $E_H^{\ T}$; is a linear-combination of the right-hand side of (2.6), where exactly one component of E^T is multiplied by one and others by zero.

So, one can determine the dual variables from (2.7), and (2.8),by first finding s_l, $1 = 1, ..,k$ from (2.8) and then the remaining variables u_i*'s* and v_i*'s* can be found out by setting

$$u_i = -\sum_{l=1}^{k} s_l u_i^{\ l} \text{ and } v_j = -\sum_{l=1}^{k} s_l u_j^{\ l} \qquad \text{….(2.9)}$$

where u_i^l and v_j^l $(l=1, ..,k)$ are determined from (2.5).

The u_i's and v_i's in (2.9) satisfy (2.7). Therefore, once having known the dual variables, the non-basic cell (i, j) for which $\sum_{l=1}^{k} s_l (\Delta_{ij})_l^1$ is most positive, may be determined, so that it qualifies for entry into the basis.

Let this column be denoted by $\hat{P}_{hk} = [P_{hk}, c_{hk}]$. In terms of $\hat{B}, \hat{P}_{hk}$ expressed in the form

$$\begin{bmatrix} B & B_A \\ C_B & C_A \end{bmatrix} \begin{bmatrix} Y_{hk} \\ Y_A \end{bmatrix} = \begin{bmatrix} P_{hk} \\ c_{hk} \end{bmatrix} \quad \ldots(2.10)$$

Let $\hat{B}_k^G$ denote the last k rows of $\hat{B}^G$. Then

$$Y_A = \hat{B}_k^G \ [P_{hk}, C_{hk}]^T.$$

As explained in Gupta (1977),

$$Y_{A_q} = -\sum_{l=1}^{k} a_{ql}\, u_h^l - \sum_{l=1}^{k} a_{ql} v_k^l - \sum_{l=1}^{k} a_{ql} c_{hk}^l = -\sum_{l=1}^{k} a_{ql} (u_h^l + v_k^l - c_{hk}^l) \quad \ldots(2.11)$$

$$(q=1,2,\ldots,k)$$

where $H^{-1} = (a_{ij})$, and hence

$$Y_{A_q} = [Y_{A_1}, Y_{A_2}, \ldots, Y_{A_k}]^T$$

is determined completely.

It can be observed from the nature of the problem (P_2) that ($\bar{X}$, $q_1 = 0, q_2 = 0, \ldots,$ $q_k = 0)$ is the initial basic feasible solution, with $q_1, q_2, \ldots, q_k$ in the basis. Therefore initially, $H = I_k = H^{-1}$, where I_k is the $k \times k$ unit matrix,

$$E_H^T = [-1, -1, \ldots, -1] \text{ and } B = \bar{B} .$$

Then $s_1 = -1, s_2 = -1, \ldots, s_k = -1$. Using these values, the entering cell may be obtained as explained above and Y_A for this column can be determined from eqns. (2.11). It may be recalled that the only quantities we need to check the non-dominance character of $\bar{X}$, under the non-degeneracy assumption are Y_A and H^{-1}. It can be observed that if $[Y_{A_1}, Y_{A_2}, \ldots, Y_{A_k}]^T$ is non-positive, then $\hat{P}_{hk}$ when entered into the basis, will lead to a solution for which at least one of the k variables $q_1, q_2, \ldots q_k$ will be at positive level. Therefore $\bar{X}$ will be a dominated solution. However if at least one of the k, Y_{A_q}'s is positive, then the simplex-iteration with one of the rows of H^{-1} as the pivot row, can be made and at each step, one needs to update H^{-1} and $s_1, s_2, \ldots, s_k$ only. Thus proceeding in the above stated manner, we may encounter any one of the following possibilities:

a. Zero optimum for (P_2) is obtained, in which case $\bar{X}$ is a non-dominated solution.

b. There exists a column $\hat{P}_{hk}$ for which

   $\sum_{l=1}^{k} s_l \, (u_i^l + v_j^l - c_{ij}^l)$ is most +ve and (Y_A) is non-postive, in which case $\bar{X}$ is a dominated solution.

CHAPTER-3

FIXED CHARGE MULTI-OBJECTIVE TRANSPORTATION PROBLEM WITH ONE MORE NON-LINEAR OBJECTIVE

3.1 INTRODUCTION

The fixed charge multi-objective transportation problem (FMOTP) problem considered in chapter 2 has been extended by including one more objective which is conflicting in nature with other *k* objectives. In this Chapter it is considered the objective is that the total cost of first *k* objectives as well as time of transportation are to be minimized.

3.2 PROBLEM FORMULATION

The mathematical formulation of this problem is as follows.

$$\text{Minimize} \quad \{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{1} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{2} x_{ij} + \ldots + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} + \sum_{i=1}^{m} F_i\}$$

$$\text{Minimize} \quad \{\max[\mathrm{t_{ij}} / x_{ij} > 0]\}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} \le a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for\ \ i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n.$$

where

i =1, 2,. . . ,m, are the origins and j = 1, 2,. . .,n, are the destinations,

t_{ij} = the time of transportation of the product from i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported, so long as $x_{ij} > 0$,

c_{ij}^{l} = the variable cost per unit amount transported from i^{th} origin to the j^{th} destination corresponding to k objectives i.e. $l=1,2,\ldots,k$,

x_{ij} = the amount transported from the i^{th} origin to the j^{th} destination,

a_i = maximum capacity at origin i,

b_j = the demand at destination j,

F_i = the fixed cost associated with origin i.

The objective in fixed charge multi-objective transportation problem is to minimize the total cost which includes both the variable cost and fixed cost of first *k* objectives and the total time of transportation satisfying the above constraints.

3.3 SOLUTION PROCEDURE

To solve this problem in the algorithm find the total cost of the first *k* objectives (by using algorithm 2.4, chapter 2) has been used to optimize the first *k* objectives and it is assumed that the first priority is given to cost of first *k* objectives and then time is minimize with respect to the minimized total cost of *k* objectives. A re-optimization procedure Basu *et al* (1994) is used to fixed cost-time trade off pairs.

For re-optimization the above problem is separated into two problems (P_4) and (S), where

(P_4) Minimize the cost function $\{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{1} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{2} x_{ij} + \ldots + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} + \sum_{i=1}^{m} F_i\}$

Subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for \;\; i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n.$$

and

(S)

$$\text{Minimize}\, \mathrm{T} = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} [t_{ij} / x_{ij} > 0]$$

Subject to

$$\sum_{j=1}^{n+1} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for \;\; i = 1, 2, \ldots, m; \;\; j = 1, 2, \ldots, n, n+1.$$

Since fixed cost at each origin is considered, unbalanced transportation problem is to be taken into account. So, first we have to balance the problems (P_5) and (S_1) using destinations. Then we have

(P_5)

$$\text{Minimize} \quad Z = \{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{1} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{2} x_{ij} + \ldots + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} + \sum_{i=1}^{m} F_i\}$$

Subject to

$$\sum_{j=1}^{n+1} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0 \quad for \;\; i = 1, 2, \ldots, m; \;\; j = 1, 2, \ldots, n, n+1$$

(S_1) $$\text{Minimize } T = \max[t_{ij} / x_{ij} > 0]$$

$$\text{Subject to} \quad \sum_{j=1}^{n+1} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \geq 0, \quad for \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n, n+1.$$

In problems (P_4) and (S_1), the cost (variable cost and also fixed cost) and time associated respectively with the dummy cells are all zero. At first iteration, let Z_1 be the minimum total cost of the k sub problems (P_1) and T_1 be the optimal time of the problem (S_1) with respect to Z_1, then any schedule which is competed earlier than T_1 would cost more than Z_1. (Z_1, T_1) is called the time-cost trade-off pair at the first iteration.

Using re-optimization procedure (Basu *et al.* 1994), let after *q-th* iteration, the solution is infeasible. Therefore, we get the following complete set of time-cost trade-off pairs:

$$(Z_1, T_1), (Z_2, T_2), \ldots, (Z_q, T_q)$$

where

$$Z_1 < Z_2 < \ldots < Z_q \quad \text{and} \quad T_1 > T_2 > \ldots > T_q$$

3.4ALGORITHM

By using the steps 1 to 8 of algorithm 2.4 of Chapter 2 and considering that the first priority is given to the first k cost objectives, we get an optimal solution.

Step 1: Let Z_1 be the optimal cost of P_1 and X^1 be the solution corresponding to Z_1.

Step 2: Calculate T_1

$$T_1 = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} [t_{ij} / x_{ij} > 0 \text{ according to } X^1]$$

Then the corresponding pair (Z_1,T_1) is called the time-cost trade-off pair at the first iteration.

Step 3: Define $(C_{ij}^l)_k = M$, if $t_{ij} \geq T_k$

$$= (C_{ij}^l) \text{ , if } t_{ij} < T_k$$

where M is a sufficiently large positive number. Let (P_{p+1}) be the fixed charge multi objective transportation problem with variable cost $(C_{ij}^l)_k$.

Step 4: Find a basic feasible solution of the problem (P_{p+1}) with respect to the variable costs.

If the total variable cost ≥ M, then go to Step 13; otherwise go to Step 5

Step 5: Find the fixed cost of the current basic feasible solution of problem (P_{p+1}) and denote this by F^{p+1}(current). where

$$F^1 \text{ (current)} = \sum_{i=1}^{m} F_i$$

Step 6: Calculate (C_{ij}^l-u_i^l-v_j^l),$l=1,2,\ldots$,k for all $i, j \notin \overline{B}$ and denote it by $(M_{ij})_l^{p+1}$, where u_i^l,v_j^l are the dual variables associated with k sub problems for i=1,2,. . . ,m; j=1,2,. . .,n, n+1.

Step 7: Find $(A_{ij})_l^{p+1} = (M_{ij})_l^{p+1} \times (E_{ij})_l^{p+1}$ for all $i, j \notin \overline{B}$

where $(A_{ij})_l^{p+1}$ is the change in cost which occurs for introducing a non-basic cell (i, j) with value $(E_{ij})_l^{p+1}$ into the basis by making reallocation.

Step 8: Calculate $F_{ij}^{p+1}(NB)$ is the total fixed cost involved for introducing the variable x_{ij} with values $(E_{ij})_l^{p+1}$ for all $i, j \notin \overline{B}$ into the current basis to form a new basis.

$$F_{ij}^{p+1} \text{ (Difference)} = F_{ij}^{p+1} \text{ (NB)} - F^{p+1}\text{(Current)}$$

Step 9: Now add F_{ij}^{p+1} (Difference) and $(A_{ij})_l^{p+1}$ and denote it by $(\Delta_{ij})_l^{p+1}$ *i.e*

$$(\Delta_{ij})_l^{p+1} = F_{ij}^{p+1} \text{ (Difference)} + (A_{ij})_l^{p+1} \quad \text{for all } i, j \notin \overline{B}$$

Step 10: We check for adominated or non dominated solution as follows:

(i) $\overline{X}$ will dominate all other solutions X^* obtained by introducing the non-basic cell *(i, j)* for which $(\Delta_{ij})_l^{p+1} \geq 0$ for $l = 1,.. ,k$, with at least one inequality strictly positive, and $x_{ij}^* = \theta_{ij} > 0$. Therefore, such a non-basic cell possibly cannot enter the basis.

(ii) If for a non-basic cell (i, j), $(\Delta_{ij})_l^{p+1} \leq 0$ for $l = 1,...,k$, with at least one inequality, with a strictly negative sign, then the solution X^* obtained by introducing the cell *(i, j)* will dominate $\overline{X}$ if $x_{ij}^* = \theta_{ij} > 0$.

(iii) If there are non-basic cells *(i, j)* and *(h, k)* such that $\theta_{ij}(\Delta_{ij})_l^{p+1} \le \theta_{hk}(\Delta_{ij})_l^{p+1}$, $l = 1,2,...,k$ with strict inequality in at least one of the k inequalities, where $x_{cd}^{*} = \theta_{cd}$ if the cell (c,d) is introduced into the basis, then the solution obtained by introducing the cell (h, k) is dominated by the solution resulting from introducing the cell (i, j).

(iv) $\overline{X}$ is a non-dominated solution if for all non-basic cells *(i, j)*, $(\Delta_{ij})_l^{p+1} > 0$ for at least one l i.e. $\overline{X}$ is a unique optimal solution for at least one of the k sub-problems (P^l), l = 1,…, k. Thus the above observations (ii) and (iv) help one analyzing the character of any basic feasible solution, but still, they do not cover all the possibilities. We may have the possibility when for at least one non-basic cell, s $(l \le s < k)$ of the k quantities $(\Delta_{ij})_l^{p+1}$, $l = 1,...,k$ are negative, while the remaining k-s are positive, it being assumed that all the k quantities for the remaining non-basic cells are non-positive. For this case to find out whether $\overline{X}$ is dominated or non-dominated is given in Appendix 2.1.

Step 11: Let Z_{p+1} be the optimal cost of problem (P_{p+1}) and X_{p+1} be the optimal solution corresponding to Z_{p+1}.

Step 12: Compute

$$T_{p+1} = \max_{\substack{1 \le i \le m \\ 1 \le j \le n+1}} \left[t_{ij}, x_{ij} > 0 \text{ according to } X^{p+1} \right]$$

Then the trade-off pair (Z_{p+1}, T_{p+1}) is called the cost-time trade-off pair at the ($p+1$)-th iteration. Obviously, $Z_{p+1} > Z_p$, $T_{p+1} < T_p$.

Step 13: Set $p=p+1$, go to Step 3.

Step 14: Suppose after q^{th} iteration the solution is infeasible i.e. $Z_{q+1} \ge M$. Then identify the complete set of efficient trade-off pairs

$$(Z_1,T_1), (Z_2,T_2), \ldots, (Z_q,T_q)$$

$$\text{where } Z_1 < Z_2 < \ldots < Z_q \text{ and } T_1 > T_2 > \ldots > T_q$$

3.5 NUMERICAL EXAMPLE

The above algorithm is explained by considering the following 3×3 fixed-charge linear multi-objective transportation problem where the units of cost and time are taken in one standard scale.

$$\text{Minimize} \quad \{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{1} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{2} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{3} x_{ij} + \sum_{i=1}^{m} F_i\}$$

$$\text{Minimize} \quad T = \max[t_{ij} / x_{ij} > 0]$$

$$\text{Subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \; i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \; j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \;\; i = 1,2,3; \;\; j = 1,2,3.$$

In order to solve the above problem it is presented as two parts as shown below:

$$\text{Minimize the cost function} \quad \{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{1} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{2} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{3} x_{ij} + \sum_{i=1}^{m} F_i\}$$

$$\text{Subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \; i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \; j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \;\; i = 1,2,3; \;\; j = 1,2,3.$$

and

$$\text{Minimize the time function} \quad \max[t_{ij} / x_{ij} > 0]$$

$$\text{Subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \; i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \; j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \;\; i = 1,2,3; \;\; j = 1,2,3.$$

Table 3.1 gives the values of variable cost C_{ij}^{l} (i=1, 2, 3; j=1, 2, 3;l=*1,2,3*) and Table 3.2 gives the values of time t_{ij} (i=1, 2, 3; j=1, 2, 3). The fixed costs are:

F_{11}=100, F_{12}=50, F_{13} =50

F_{21}=150, F_{22}=50, F_{23}=50

F_{31}=200, F_{32}=50, F_{33}=50

TABLE 3.1

	c_{ij}^1			a_i
c_{ij}^1	5	9	10	9
	4	6	2	14
	4	2	3	17
b_j	5	12	8	

	c_{ij}^2		
c_{ij}^2	1	2	4
	3	7	4
	2	9	5

	c_{ij}^3		
c_{ij}^3	4	7	2
	1	5	8
	9	8	1

TABLE 3.2

Destination j→ *Origin i↓*	1	2	3	a_i
1	15	7	2	9
2	10	13	11	14
3	6	8	14	17
b_j	5	12	8	

For solution purpose the above data is written together and given in Table 3.3. The upper entries of each row represent the cost $c_{ij}^l,\ l = 1, 2, 3.$ and south west entry represent t_{ij}.

TABLE 3.3

Destination j→ *Origin i↓*	1	2	3	a_i
1	5 1 4 15	9 2 7 7	10 4 2 2	9
2	4 3 1 10	6 7 5 13	2 4 8 11	14
3	4 2 9 6	2 9 8 8	3 5 1 14	17
b_j	5	12	8	

The total cost which is to be minimized is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}{}^{l} x_{ij} + \sum_{i=1}^{3} F_i$$

$$F_i = \sum_{l=1}^{3} \delta_{il} F_{il} \ , \qquad i = 1,2,3$$

where

$$\delta_{i1} = 1, if \ \sum_{j=1}^{3} x_{ij} > 0, \ i = 1,2,3,$$
$$= 0, otherwise;$$

$$\delta_{i2} = 1, if \ \sum_{j=1}^{3} x_{ij} > 8, \ i = 1,2,3,$$
$$= 0, otherwise.$$

$$\delta_{i3} = 1, if \ \sum_{j=1}^{3} x_{ij} > 11, \ i = 1,2,3,$$
$$= 0, otherwise.$$

Since, the two objectives are in consideration, and first objective is the total cost of first k objectives of problem $P_1$of chapter 2 and can be considered as single objective for which the optimal solution is given in Table 3.4.

TABLE 3.4

Destination ***j***→ *Origin* ***i***↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	5 1 4 **5** 15	9 2 7 **3** 7	10 4 2 2	0 0 0 **1** 0	100	0 0 0	9
2	4 3 1 10	6 7 5 13	2 4 8 11	0 0 0 **14** 0	0	0 0 0	14
3	4 2 9 6	2 9 8 **9** 8	3 5 1 **8** 14	0 0 0 0	300	-7 7 1	17
v_j^1 v_j^2 v_j^3	5 1 4	9 2 7	10 -2 0	0 0 0	400		
b_j	5	12	8	15			

From Table 3.4 $X^5 = (0,8,0,1,0,0,0,14,5,4,8,0)$, for which $Z = 1547$ where Z is the total cost. Now, from Table 3.3, it can be seen that maximum time corresponding to their to this cost is 15. So the first trade off (Z_1, T_1) = (1547, 15). Now, applying Step 3, we modify the problem P_5 to problem P_6 and the basic feasible solution of P_6 is given below in Table 3.5

TABLE 3.5

Destinationj→ *Origin i*↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	M M M	9 2 7 **9**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1 **5**	6 7 5 **3**	2 4 8 **6**	0 0 0	250	-3 5 -2	14
3	4 2 9	2 9 8	3 5 1 **2**	0 0 0 **15**	200	-2 6 -9	17
v_j^1 v_j^2 v_j^3	7 -2 3	9 2 7	5 -1 10	2 -6 9	600		
b_j	5	12	8	15			

Applying Step 6, we get (C_{ij}^{l} -u_i^l-v_j^l) values, for all $i, j \notin \overline{B}$ which are given in Table 3.6.

TABLE 3.6

ij	11	13	14	24	31	32
$C_{ij}^1 - u_i^1 - v_j^1$	M-7	5	-2	1	-1	-5
$C_{ij}^2 - u_i^2 - v_j^2$	M+2	5	6	1	-2	1
$C_{ij}^3 - u_i^3 - v_j^3$	M-3	-8	-9	-7	15	10

Applying Step 7, we get the values of $(A_{ij})_l^2$, which are displayed in Table 3.7.

TABLE 3.7

ij	11	13	14	24	31	32
$(A_{ij})_1^2$	5(M-7)	30	-12	6	-2	-10
$(A_{ij})_2^2$	5(M+2)	30	36	6	-4	2
$(A_{ij})_3^2$	5(M-3)	-48	-54	-42	30	20

Applying Step 8, we get the following results which are displayed in Table 3.8.

TABLE 3.8

ij	11	13	14	24	31	32
1	150	150	100	150	150	150
2	250	250	250	150	250	250
3	200	200	200	200	200	200
$F_{ij}\,(NB)$	600	600	550	500	600	600
$F_{ij}\,(Difference)$	0	0	-50	-100	0	0

Applying Step 9, we get the values of $(\Delta_{ij})_l^2$, which are displayed in Table 3.9

TABLE 3.9

ij	11	13	14	24	31	32
$(\Delta_{ij})_1^2$	5(M-7)	30	-62	-94	-2	-10
$(\Delta_{ij})_2^2$	5(M+2)	30	-14	-94	-4	2
$(\Delta_{ij})_3^2$	5(M-3)	-48	-104	-142	30	20

From Table 3.5 $X^1(0,9,0,0,5,36,0,0,2,15), Z = 2151$.Now, applying step 10(ii), Since, $(\Delta_{ij})_l^2 \leq 0$, l=1,2,3, the cell (2,4) enter to the basis and the new solution is given in Table 3.10.

TABLE 3.10

Destinationj→ *Origin i*↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	M M M	9 2 7 **9**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1 **5**	6 7 5 **3**	2 4 8	0 0 0 **6**	150	-3 5 -2	14
3	4 2 9	2 9 8	3 5 1 **8**	0 0 0 **9**	200	-3 5 -2	17
v_j^1 v_j^2 v_j^3	7 -2 3	9 2 7	6 0 3	3 -5 2	500		
b_j	5	12	8	15			

Applying Step 6, we get the following results as shown in Table 3.11.

TABLE 3.11

ij	11	13	14	23	31	32
$C_{ij}^1 - u_i^1 - v_j^1$	M-7	4	-3	-1	0	-4
$C_{ij}^2 - u_i^2 - v_j^2$	M+2	4	5	-1	-1	2
$C_{ij}^3 - u_i^3 - v_j^3$	M-3	-1	-2	7	8	3

Applying Step 7, we get the values of $(A_{ij})_l^2$, which are given in Table 3.12.

TABLE 3.12

ij	11	13	14	23	31	32
$(A_{ij})_1^2$	5(M-7)	24	-18	-6	0	-12
$(A_{ij})_2^2$	5(M+2)	24	30	-6	-5	6
$(A_{ij})_3^2$	5 (M-3)	-6	-12	42	40	9

Applying Step 8, we get the following results which are displayed in Table 3.13.

TABLE 3.13

ij	11	13	14	23	31	32
1	150	150	100	150	150	150
2	150	250	250	250	150	150
3	200	200	200	200	300	250
$F_{ij}\,(NB)$	500	600	550	600	600	550
$F_{ij}\,(Difference)$	0	100	50	100	100	50

Applying Step 9, we get the values of $(\Delta_{ij})_l^2$, which are displayed in Table 3.14.

TABLE 3.14

ij	11	13	14	23	31	32
$(\Delta_{ij})_1^2$	5(M-7)	124	32	94	100	38
$(\Delta_{ij})_2^2$	5(M+2)	124	80	94	95	56
$(\Delta_{ij})_3^2$	5(M-3)	94	38	142	140	59

From Table 3.10 $X^2(0,9,0,0,5,3,0,6,0,0,8,9)$, for which $Z =$ 1821 where Z is the total cost. Now, applying step 10(i), Since $(\Delta_{ij})_l^2 \geq 0$, l=1,2,3, the no cell enter to the basis. Now, from Table 3.3, it can be seen that maximum time corresponding to their to this cost is 14. So the second trade off (Z_2, T_2) = (1821, 14). Now, applying Step 3, we modify the problem P_6 to problem P_7 and the basic feasible solution of P_7 is given below in Table 3.15.

TABLE 3.15

Destinationj→ *Origin i*↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	M M M	9 2 7 **9**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1	6 7 5 **3**	2 4 8 **8**	0 0 0 **3**	200	-3 5 -2	14
3	4 2 9 **5**	2 9 8	M M M	0 0 0 **12**	200	-3 5 -2	17
v_j^1 v_j^2 v_j^3	7 -3 11	9 2 7	5 -1 10	3 -5 2	550		
b_j	5	12	8	15			

Applying Step 6, we get (C_{ij}^{l} -u_i^l-v_j^l) values, for all $i, j \notin \overline{B}$ which are given in Table 3.16.

TABLE 3.16

ij	11	13	14	21	32	33
$C_{ij}^1 - u_i^1 - v_j^1$	M-7	5	-3	0	-4	M-2
$C_{ij}^2 - u_i^2 - v_j^2$	M+3	5	5	1	2	M-4
$C_{ij}^3 - u_i^3 - v_j^3$	M-11	-8	-2	-8	3	M-8

Applying Step 7, we get the values of $(A_{ij})_l^3$, which are displayed in Table 3.17.

TABLE 3.17

ij	11	13	14	21	32	33
$(A_{ij})_1^3$	3(M-7)	40	-9	0	-12	8(M-2)
$(A_{ij})_2^3$	3(M+3)	40	15	3	6	8(M-4)
$(A_{ij})_3^3$	3(M-11)	-64	-6	-24	9	8(M-8)

Applying Step 8, we get the following results which are displayed in Table 3.18.

TABLE 3.18

ij	11	13	14	21	32	33
1	150	150	100	150	150	150
2	250	200	250	250	150	150
3	200	200	200	200	200	300
$F_{ij}\,(NB)$	600	550	550	600	500	600
$F_{ij}\,(Difference)$	50	0	0	50	-50	50

Applying Step 9, we get the values of $(\Delta_{ij})_l^3$, which are displayed in Table3.19

TABLE 3.19

ij	11	13	14	21	32	33
$(\Delta_{ij})_1^3$	50+3(M-7)	40	-9	0	-62	50+8(M-2)
$(\Delta_{ij})_2^3$	50+3(M+3)	40	15	53	-44	50+8(M-4)
$(\Delta_{ij})_3^3$	50+3(M-11)	-64	-6	26	-41	50+8(M-8)

From Table 3.15 $X^1 = (0,9,0,0,0,3,8,3,5,0,0,12)$, for which $Z = 2053$ .Now, applying step10(ii), Since $(\Delta_{ij})_l^3 \leq 0$,l=1,2,3, the cell (3,2) enter to the basis and the new solution is given in Table 3.20.

TABLE 3.20

Destinationj→ *Origin i↓*	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	M M M	9 2 7 **9**	10 4 2	0 0 0	150	0 0 0	9
2	4 3 1	6 7 5	2 4 8 **8**	0 0 0 **6**	150	-7 7 1	14
3	4 2 9 **5**	2 9 8 **3**	M M M	0 0 0 **9**	200	-7 7 1	17
v_j^1 v_j^2 v_j^3	11 -5 8	9 2 7	9 -3 7	7 -7 -1	500		
b_j	5	12	8	15			

Applying Step 6, we get the following results as shown in Table 3.21.

TABLE 3.21

ij	11	13	14	21	22	33
$C_{ij}^1 - u_i^1 - v_j^1$	M-11	1	-7	0	4	M-2
$C_{ij}^2 - u_i^2 - v_j^2$	M+5	7	7	1	-2	M-4
$C_{ij}^3 - u_i^3 - v_j^3$	M-8	-5	1	-8	-3	M-8

Applying Step 7, we get the values of $(A_{ij})_l^3$, which are given in Table 3.22.

TABLE 3.22

ij	11	13	14	21	22	33
$(A_{ij})_1^3$	5(M-11)	8	-63	0	12	8(M-2)
$(A_{ij})_2^3$	5(M+5)	56	63	5	-6	8(M-4)
$(A_{ij})_3^3$	5 (M-8)	-40	9	-40	-9	8(M-8)

Applying Step 8, we get the following results which are displayed in Table 3.23.

TABLE 3.23

ij	11	13	14	21	22	33
1	150	150	0	150	150	150
2	150	0	150	250	200	0
3	200	300	300	200	200	300
$F_{ij}\,(NB)$	500	450	450	600	550	450
$F_{ij}\,(Difference)$	0	-50	-50	100	50	-50

Applying Step 9, we get the values of $(\Delta_{ij})_l^3$, which are displayed in Table 3.24.

TABLE 3.24

ij	11	13	14	21	22	33
$(\Delta_{ij})_1^3$	5(M-11)	-42	-113	100	62	-50+8(M-2)
$(\Delta_{ij})_2^3$	5(M+5)	6	13	105	44	-50+8(M-4)
$(\Delta_{ij})_3^3$	5(M-8)	-90	-41	60	41	-50+8(M-8)

From Table 3.20 $X^2 = (0,9,0,0,0,0,8,6,5,3,0,9)$, for which $Z = 1906$.

Since there is no cell for which $(\Delta_{ij})_l^1 \leq 0$ for l=1, 2, 3 with at least one inequality so it is must to check the character of this solution is dominate or non dominate .

Initially, $H^{-1} = I_3$, s_1 = -1, s_2 = -1, s_3 = -1. As $\sum_{l=1}^{k} s_l (\Delta_{ij})_l^2$

Is most positive for the cell (1,3), therefore for this cell

$$Y_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -42 \\ 6 \\ -90 \end{bmatrix} = \begin{bmatrix} -42 \\ 6 \\ -90 \end{bmatrix}$$

Since, the first component is most positive. So, q_2 leaves the basis, hence $E_H^T = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}$

Performing the simplex iteration.

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 42 & 0 \\ 0 & -6 & 0 \\ 0 & 90 & -1 \end{bmatrix}$$ we get, s_1 = -1, s_2 = -132, s_3 = -1

and $$H^{-1} = \begin{bmatrix} 1 & 7 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 15 & 1 \end{bmatrix}$$

Now, cell (1,4) is negative and Y_A is not negative. Therefore X^2 is a dominated solution.

Now R= {(1,3), (1,4)}

Applying step 10, the cell (1,3) enter to the basis and the new solution is given in Table 3.25.

TABLE 3.25

Destinationj→ *Origin i↓*	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	M M M	9 2 7 **1**	10 4 2 **8**	0 0 0	150	0 0 0	9
2	4 3 1	6 7 5	2 4 8	0 0 0 **14**	0	-7 7 1	14
3	4 2 9 **5**	2 9 8 **11**	M M M	0 0 0 **1**	0	-7 7 1	17
v_j^1 v_j^2 v_j^3	11 -5 8	9 2 7	10 4 2	7 -7 -1	450		
b_j	5	12	8	15			

Applying Step 6, we get the following results in Table 3.26.

TABLE 3.26

ij	11	14	21	22	23	33
$C_{ij}^1 - u_i^1 - v_j^1$	M-11	-7	0	4	-1	M-3
$C_{ij}^2 - u_i^2 - v_j^2$	M+5	7	1	-2	-7	M-11
$C_{ij}^3 - u_i^3 - v_j^3$	M-8	1	-8	-3	5	M-3

Applying Step 7, we get the values of $(A_{ij})_l^3$ which are tabulated in Table 3.27.

TABLE 3.27

ij	11	14	21	22	23	33
$(A_{ij})_1^3$	M-11	-7	0	44	-8	8(M-3)
$(A_{ij})_2^3$	M+5	7	5	-22	-56	8(M-11)
$(A_{ij})_3^3$	M-8	1	-40	-33	40	8(M-3)

Applying Step 8, we get the following results which are displayed in Table 3.28.

TABLE 3.28

ij	11	14	21	22	23	33
1	150	100	150	150	150	150
2	0	0	150	200	150	0
3	300	300	250	200	200	300
F_{ij} (*NB*)	450	400	550	550	500	450
F_{ij} (*Difference*)	0	-50	100	100	50	0

Applying Step 9, we get the following values of $(\Delta_{ij})_l^3$ which are tabulated in Table 3.29.

TABLE 3.29

ij	11	14	21	22	23	33
$(\Delta_{ij})_1^3$	M-11	-57	100	144	42	8(M-3)
$(\Delta_{ij})_2^3$	M+5	-43	105	78	-6	8(M-11)
$(\Delta_{ij})_3^3$	M-8	-49	60	67	90	8(M-3)

From Table 3.25 $X^3 = (0,1,8,0,0,0,0,14,5,11,0,1)$, for which $Z = 1780$.Now, applying step10(ii), Since $(\Delta_{ij})_l^2 \leq 0$, l=1,2,3, the cell (1,4) enter to the basis and the new solution is given in Table 3.30.

TABLE 3.30

Destinationj→ *Origin i*↓	1	2	3	4	F^1(current)	u_i^1 u_i^2 u_i^3	a_i
1	M M M	9 2 7 **0**	10 4 2 **8**	0 0 0 **1**	100	0 0 0	9
2	4 3 1	6 7 5	2 4 8	0 0 0 **14**	0	0 0 0	14
3	4 2 9 **5**	2 9 8 **12**	M M M	0 0 0	300	-7 7 1	17
v_j^1 v_j^2 v_j^3	11 -5 8	9 2 7	10 4 2	0 0 0	400		
b_j	5	12	8	15			

Applying Step 6, we get the following results in Table 3.31.

TABLE 3.31

ij	11	21	22	23	33	34
$C_{ij}^1 - u_i^1 - v_j^1$	M-11	-7	-3	-8	M-3	7
$C_{ij}^2 - u_i^2 - v_j^2$	M+5	8	5	0	M-11	-7
$C_{ij}^3 - u_i^3 - v_j^3$	M-8	-7	-2	6	M-3	-1

Applying Step 7, we get the values of $(A_{ij})_l^3$ which are tabulated in Table 3.32.

TABLE 3.32

ij	11	21	22	23	33	34
$(A_{ij})_1^3$	0	0	0	-64	8(M-3)	7
$(A_{ij})_2^3$	0	0	0	0	8(M-11)	-7
$(A_{ij})_3^3$	0	0	0	48	8(M-3)	-1

Applying Step 8, we get the following results which are displayed in Table 3.33

TABLE 3.33

ij	11	21	22	23	33	34
1	100	100	100	0	100	150
2	0	0	0	150	0	0
3	300	300	300	300	300	300
$F_{ij}\,(NB)$	400	400	400	450	400	450
$F_{ij}\,(Difference)$	0	0	0	50	0	50

Applying Step 9, we get the following values of $(\Delta_{ij})_l^3$ which are tabulated in Table 3.34.

TABLE 3.34

ij	11	21	22	23	33	34
$(\Delta_{ij})_1^3$	0	0	0	-14	8(M-3)	57
$(\Delta_{ij})_2^3$	0	0	0	0	8(M-11)	43
$(\Delta_{ij})_3^3$	0	0	0	90	8(M-3)	49

From Table 3.30 $X^4 = (0,0,8,1,0,0,0,14,5,12,0,0),$ for which $Z = 1631$.

Since there is no cell for which $(\Delta_{ij})_l^1 \leq 0$ for *l=1, 2, 3* with at least one inequality so it is must to check the character of this solution is dominate or non dominate .

Taking initially, $H^{-1} = I_3$, $s_1 = -1$, $s_2 = -1$, $s_3 = -1$. As for non basic cell $(2,3) \sum_{l=1}^{k} s_l\,(\Delta_{ij})_l^2$ is most negative and also fixed charge increase. So, (2,3) cannot enter the basis. Then all $(\Delta_{ij})_l^2 \geq 0$ (for all $i, j \notin B$).Now R=ϕ Hence, the third cost-time trade-off pair is

(Z_3,T_3) = (1631,8). After third iteration, we see that the solution is infeasible.

Then second cost-time trade–off pairs is dominated.Hence we get two cost-time trade-off pairs as (1547,15) and(1631,8).

The result shows that the minimum cost is 1547 which corresponds to the pair (1547,15) and the minimum time is 8 which corresponds to the pair (1631,8).

The fixed charge multi-objective transportation problem is considered with an algorithm which is a modification of algorithm proposed by Basu et al (1994) in conjunction with approach given by Gupta and Gupta (1982). Also this problem is extended by including one more nonlinear objective with is conflicting in nature and a cost-time trade off pairs has been obtained.

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