# METHODS FOR SOLVING FULLY FUZZY LINEAR PROGRAMMING PROBLEM

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Submitted by

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**INDIA**

**DEDICATED**

**TO**

**GOD, MY PARENTS**

**AND**

**SUPERVISOR**

*Certificate*

I hereby declare that the work which is being presented here in the dissertation entitled **"METHODS FOR SOLVING FULLY FUZZY LINEAR PROGRAMMING PROBLEM"** in partial fulfillment of the requirement for the award of degree of **Master of Science** in **Mathematics and Computing** submitted in School of Mathematics and Computer Applications, Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Vikas Sharma**, Assistant Professor, SMCA and refer other researcher's work which are duly listed in the reference section.

The matter presented in this thesis has not been submitted in any other University/Institute for the award of my degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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## *Abstract*


Linear programming problem is one of the important branch of Mathematical programming problem. Applications of LP exist in almost all the areas like military, industry, agriculture, transportation, economics, health systems and even behavioral and social sciences. But in real world all situations may not be deterministic. There may exists different kinds of uncertainties in social, industrial and economic systems, such as events may occurs randomly, unavailability of system data and due to the error in measurements, etc. For dealing with such vagueness and ambiguities fuzzy decision making methods are used and in this decision making process either coefficients or variables in the objective function or constraints or both are taken as fuzzy in nature. In this thesis, fully fuzzy linear programming problems is discussed in which all the parameters of LPP are fuzzy in nature.

In first chapter of the dissertation, fully fuzzy linear programming problem is introduced. The brief description of basic concepts, definitions, arithmetics operations that are used throughout work and detailed literature survey of fully fuzzy linear programming problem and summary of the thesis has also been discussed in this chapter. In Chapter 2, an algorithm for ordering the generalized and normal trapezoidal fuzzy numbers on the basis of their rank, mode, divergence, and spread are discussed. Numerical examples are also discussed which elaborate the concept of ordering of fuzzy numbers. Further, in Chapter 3 a method is presented for solving the fully fuzzy linear problem with equality constraints using the concept of ranking function in which all the parameters are triangular fuzzy numbers and the numerical examples are shown. In the last chapter, another method known as Mehar's method with Yager's ranking approach have been discussed for solving fully fuzzy linear programming problem, where all the parameters are unrestricted L-R fuzzy numbers or L-R flat fuzzy numbers, numerical examples are discussed to elaborate the algorithm.

# Contents

# Chapter 1

# Introduction

Mathematical programming is regarded as one of the important areas of applied mathematics with extensive applications in engineering, economics, and natural sciences. A mathematical programming problem has well defined objective function and set of constraints, the systematic determination of optimal solutions leads to the development of large family of methods and algorithms. The need for an efficient and systematic decision-making approach drives the need for optimization strategies and are used in almost every field like chemical, electrical, biomedical, agricultural, etc.

The general form of mathematical programming problem is given as below

$$\max / \min Z = f(x)$$

subject to

$$g_i(x) \leq 0, \ i = 1, 2, \ldots, m$$

$$x \geq 0$$

where $f(x)$ and $g_i(x), \ i = 1, 2, \ldots m$ are some functions of the vector $x \in R^n$. Thus, we find a vector $x$ which optimize (maximize or minimize) the objective function $f(x)$ subject to the constraints $g_i(x) \leqslant 0, \ i = 1, 2, \ldots m$.

Linear programming problem is a special case of mathematical programming program where, the objective functions and all the constraints are linear function. A solution which satisfies all the constraints is called as feasible solution and a feasible solution which gives the best value of objective function is termed as optimal feasible solution.

In the year 1947, Dantzig [14] a member of the US Air force, developed the simplex method for finding the optimal solution of LPP. It is the most common and widely used technique to solve LP problems. Since, the optimal value of the objective of an LPP is attained at one of the vertex of the feasible region, therefore the main idea behind simplex method is to move from one vertex from another vertex in the improving direction of the objective function, till one which gives optimal value of the objective is not found. As the feasible region of an LPP is closed and bounded and has finite number of vertices, the simplex method gives optimal solution in finite number of steps.

The general form of linear programming problem is stated as

$$\min\ z = c^T x$$

subject to

$$Ax = B$$

$$x \geq 0$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $c = (c_1, c_2, \ldots, c_n)^T$, $b = (b_1, b_2, \ldots, b_n)^T$ and $A = \{a_{ij}\}$ is an $(m \times n)$ matrix.

In many practical situations, the decision makers might not really want to actually maximize or minimize the objective function but may want to reach desirable level which is not defined precisely. For example, the actual purpose may be to improve the profit or cost situation or constraints can also be imprecise in nature. For example, consider a constraint x $\leq$ 4, where $x$ is any resource one may say that x is not exactly $\leq$ 4 and assumes that the solutions are acceptable even if $x$ is slightly $>$ 4 while solving LPP small violations make solutions infeasible so their is need to relax right hand side of the constraints to get flexible solutions. To deal with such situations Zadeh [44] in 1965, developed the theory of fuzzy sets which is more advantageous than crisp sets. In crisp set a element is either a member of set or not but fuzzy set allows the element to be partially in the set associated with membership function ranging from 0 to 1. The advantage of fuzzy set theory is that it provides an way to model vague, ambiguous and imprecise data by using fuzzy numbers which are generalization of regular real number each of which have its membership value lying in interval [0,1]. In the recent past, many fuzzy decision making methods have been developed to make decisions under fuzzy environment. Zimmermann [45] used this concept of fuzzy set theory into an general LPP and introduced Fuzzy linear programming.

Many methods are available for solving FLP [24, 40]. In this dissertation, the fully fuzzy linear programming problem is discussed, where all the parameters i.e. coefficients as well as variables in the objective function and constraints are taken as fuzzy numbers.

The general form of fully fuzzy linear programming problem is given as

$$\text{maximize } \widetilde{Z} = \widetilde{C}^T \otimes \widetilde{X}$$

subject to

$$\widetilde{A}\widetilde{X} = \widetilde{B}$$

$$\widetilde{X} \geqslant 0$$

where $\widetilde{C}^T = (\widetilde{c}_j)_{1\times n},\ \widetilde{X} = (\widetilde{x}_j)_{n\times 1},\ \widetilde{A} = (\widetilde{a}_{ij})_{m\times n},\ \widetilde{B} = (\widetilde{b}_i)_{m\times 1}$ and $(\widetilde{a}_{ij}),\ (\widetilde{c}_j),\ (\widetilde{x}_j),$ $(\widetilde{b}_i) \in F(R)$, where $F(R)$ is an set of fuzzy numbers defined on set of real numbers.

As in fuzzy decision making process fuzzy numbers are employed but since, fuzzy numbers have no natural order like real numbers therefore it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other. Thus, an efficient approach for ordering the fuzzy numbers is by use of ranking function and the idea behind this is ordering of the fuzzy numbers by converting them into real number. Ranking of fuzzy numbers play an important role in risk analysis, decision making, optimization, forecasting etc. Fuzzy numbers must be ranked before an action is taken by a decision maker. In this dissertation, the methods which are described for solving fully fuzzy linear programming problem are discussed by using ranking function. A Ranking function is defined as $\Re : F(R) \to R$ where $F(R)$ is a set of fuzzy numbers defined on real line, which maps each fuzzy number into the real line, where a natural order exists.

## 1.1 Definitions

In this section, some of the fundamental definitions, operations and concepts of fuzzy set theory initiated by Zadeh[44] are reviewed which are used throughout the work:

**Definition 1.1.1** *A fuzzy set $A$ in $X$ is characterized by its membership function, denoted by $\mu_{\widetilde{A}} : X \to [0,1]$ and the set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x));\ x \in X\}$ defined by $\mu_{\widetilde{A}}(x)$ for each $x \in X$ is called fuzzy set.*

**Definition 1.1.2** *A $\alpha$- level set of a fuzzy set $\widetilde{A}$ is defined as an ordinary set $\widetilde{A}_\alpha$ for which the degree of membership function exceeds the level $\alpha$.*

$$\widetilde{A}_\alpha = \{x \in X : \mu_{\widetilde{A}}(x) \geq \alpha\}$$

**Definition 1.1.3** *A fuzzy set is convex if and only if*

$$\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_{x_1}, \mu_{x_2}]$$

*for alll $x_1, x_2 \in X$ and $\lambda \in [0,1]$. Alternatively, a fuzzy set is convex if all $\alpha$-level sets are convex.*

**Definition 1.1.4** *A fuzzy set $\widetilde{A}$ is said to be normal if there exist at least one $x \in X$ such that $\mu_{\widetilde{A}}(x) = 1$.*

**Definition 1.1.5** *A fuzzy number is a convex normalized fuzzy set of the real line $R^1$ whose membership function is piecewise continuous.*

**Definition 1.1.6** *The support of fuzzy set $\widetilde{A}$ is the crisp subset of $X$ and is defined as*

$$supp(\widetilde{A}) = \{x \in X : \mu_{\widetilde{A}}(x) > 0\}$$

**Definition 1.1.7** *A fuzzy number $\widetilde{A} = (a, b, c)$ is said to be a triangular fuzzy number if its membership function is given by*

$$\mu_{\widetilde{A}}(x)=\begin{cases} \dfrac{x-a}{b-a} & a \leq x \leq b \\ \dfrac{x-c}{b-c} & b \leq x \leq c \\ 0, & otherwise \end{cases}$$

**Definition 1.1.8** *A triangular fuzzy number $(a, b, c)$ is said to be non negative fuzzy number if $a \geq 0$.*

**Definition 1.1.9** *A fuzzy number $\widetilde{A} = (a, b, c; w)$ is said to be a generalized triangular fuzzy number if its membership function is given by*

$$\mu_{\widetilde{A}}(x)=\begin{cases} w\left(\dfrac{x-a}{b-a}\right) & a \leq x \leq b \\ w\left(\dfrac{x-c}{b-c}\right) & b \leq x \leq c \\ 0, & otherwise \end{cases}$$

**Definition 1.1.10** *Two triangular fuzzy numbers $\widetilde{A} = (a, b, c)$ and $\widetilde{B} = (e, f, g)$ are said to be equal if and only if $a = e$, $b = f$, $c = g$.*

**Definition 1.1.11** *A ranking function is a function $\Re : F(R) \to R$ where $F(R)$ is a set of fuzzy numbers defined on a set of real numbers, maps each fuzzy number into the real number, where a natural order exist. For a triangular fuzzy number $A = (a, b, c)$, the ranking function is defined as $\Re(\widetilde{A}) = \dfrac{a + 2b + c}{4}$, see Kaufmann and Gupta [23].*

*Let $\widetilde{A} = (a, b, c)$ and $\widetilde{B} = (e, f, g)$ be two triangular fuzzy numbers then*

*(1) $\widetilde{A} \geq \widetilde{B}$ if and only if $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$*

*(1)* $\widetilde{A} \leq \widetilde{B}$ *if and only if* $\Re(\widetilde{A}) \leq \Re(\widetilde{B})$

*(1)* $\widetilde{A} \approx \widetilde{B}$ *if and only if* $\Re(\widetilde{A}) = \Re(\widetilde{B})$

**Definition 1.1.12** *Following arithmetic operations are defined for two triangular fuzzy numbers* $\widetilde{A} = (a, b, c)$ *and* $\widetilde{B} = (e, f, g)$.

*(1)* $\widetilde{A} \oplus \widetilde{B} = (a, b, c) \oplus (e, f, g) = (a + e, b + f, c + g)$

*(2)* -$\widetilde{A} = -(a, b, c) = (-c, -b, -a)$

*(3)* $\widetilde{A} \ominus \widetilde{B} = (a, b, c) \ominus (e, f, g) = (a - e, b - f, c - g)$

*(4) Let* $\widetilde{A} = (a, b, c)$ *be any triangular fuzzy number and* $\widetilde{B} = (e, f, g)$ *be non-negative triangular fuzzy number then their multiplication is represented as*

$$\widetilde{A} \times \widetilde{B} = \begin{cases} (ax, by, cz) & a \geq 0 \\ (az, by, cz) & a < 0, c \geq 0 \\ (az, by, cx) & c < 0 \end{cases}$$

**Definition 1.1.13** *A fuzzy number* $\widetilde{A} = (a, b, c, d)$ *is said to be a trapezoidal fuzzy number if its membership function is given by*

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \dfrac{x-d}{c-d} & c \leq x \leq d \\ 0, & otherwise \end{cases}$$

**Definition 1.1.14** *A trapezoidal fuzzy number* $(a, b, c, d)$ *is said to be non negative fuzzy number if* $a \geq 0$.

**Definition 1.1.15** *A fuzzy number* $\widetilde{A} = (a, b, c, d; w)$ *is said to be a generalized* *trapezoidal fuzzy number if its membership function is given by*

$$\mu_{\widetilde{A}}(x)=\begin{cases} w\left(\dfrac{x-a}{b-a}\right) & a \leq x \leq b \\ w & b \leq x \leq c \\ w\left(\dfrac{x-d}{c-d}\right), & c \leq x \leq d \\ 0 & otherwise \end{cases}$$

**Definition 1.1.16** *A ranking function is a function $\Re : F(R) \to R$ where $F(R)$ is a set of fuzzy numbers defined on a set of real numbers, which maps each fuzzy number into the real number, where a natural order exist. For a trapezoidal fuzzy number $A = (a, b, c, d)$, the ranking function is defined as $\Re(\widetilde{A}) = \dfrac{a+b+c+d}{4}$, see Kaufmann at.el [23].*

*Let $\widetilde{A} = (a, b, c, d)$ and $\widetilde{B} = (e, f, g, h)$ be two trapezoidal fuzzy numbers then*

*(i) $\widetilde{A} \geq \widetilde{B}$ if and only if $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$*

*(ii) $\widetilde{A} \leq \widetilde{B}$ if and only if $\Re(\widetilde{A}) \leq \Re(\widetilde{B})$*

*(iii) $\widetilde{A} \approx \widetilde{B}$ if and only if $\Re(\widetilde{A}) = \Re(\widetilde{B})$*

**Definition 1.1.17** *Mode of a generalized trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d; w)$ is defined as mode$(\widetilde{A})$=$w\dfrac{(b+c)}{2}$.*

**Definition 1.1.18** *Divergence of a generalized trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d; w)$ is defined as divergence$(\widetilde{A})$=$w(d-a)$.*

**Definition 1.1.19** *Left spread of a generalized trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d; w)$ is defined as left spread$(\widetilde{A})$=$w(b-a)$.*

**Definition 1.1.20** *Following arithmetic operations are defined for two generalized trapezoidal fuzzy numbers $\widetilde{A} = (a, b, c, d; w_1)$ and $\widetilde{B} = (e, f, g, h; w_2)$.*

*(i) $\widetilde{A} \oplus \widetilde{B} = (a, b, c, d; w_1) \oplus (e, f, g, h; w_2) = (a+e, b+f, c+g, d+h; minimum(w_1, w_2))$*

*(ii) $\widetilde{A} \ominus \widetilde{B} = (a, b, c, d; w_1) \ominus (e, f, g, h; w_2) = (a-e, b-f, c-g, d-h; minimum(w_1, w_2))$*

*(iii)* $\widetilde{A} \otimes \widetilde{B} = (a_1, b_1, c_1, d_1; minimum(w_1, w_2))$, *where* $a_1 = minimum(ae, ah, fd, dh), b_1 = minimum(bf, bg, cf, cg), c_1 = maximum(bf, bg, cf, cg), d_1 = maximum(ae, ah, fd, dh)$.

*(iv)Scalar multiplication of a generalized trapezoidal fuzzy number* $\widetilde{A} = (a, b, c, d; w_1)$ *is defined as*

$$\lambda\widetilde{A} = \begin{cases} (\lambda a, \lambda b, \lambda c, \lambda d; w_1), & \lambda > 0 \\ (\lambda d, \lambda c, \lambda b, \lambda a; w_1), & \lambda < 0 \end{cases}$$

**Definition 1.1.21** *A function* $L : [0, \infty) \to [0, 1]$ $(or R : [0, \infty) \to [0, 1])$ *is said to be reference function of fuzzy number if and only if*

*(i)* $L(0) = 1$ $(or\ R(0) = 1)$ .

*(ii)* $L(or R)$ *is non-increasing on* $[0, \infty)$.

**Definition 1.1.22** *A fuzzy number* $\widetilde{A}$*, which is defined on universal set of real numbers R, denoted as* $(m, n, \alpha, \beta)_{LR}$ *is said to be an LR flat fuzzy number if its membership function* $\mu_{\widetilde{A}}(x)$ *is given as*

$$\mu_{\widetilde{A}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & x \leq m,\ \alpha > 0, \\ R\left(\dfrac{x-n}{\beta}\right), & x \geq n,\ \beta \geq 0, \\ 1, & m \leq x \leq n \end{cases}$$

**Definition 1.1.23** *An L-R flat fuzzy number* $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ *is said to be non-negative L-R flat fuzzy number if* $m - \alpha \geq 0$ *and is said to be non-positive L-R flat fuzzy number if* $n + \beta \leq 0$.

**Definition 1.1.24** *An L-R flat fuzzy number* $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ *is said to be unrestricted L-R flat fuzzy number if* $m - \alpha$ *is unrestricted in sign.*

**Definition 1.1.25** *Let* $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ *be an L-R flat fuzzy number and* $\lambda$ *be a real number in the interval [0,1] then the crisp set,* $A = \{x \in X : \mu_{\widetilde{A}}(x) \geq \lambda\} = \{m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)\}$, *is said to be* $\lambda$*-cut of* $\widetilde{A}$.

**Definition 1.1.26** *let $\widetilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be any two L-R flat fuzzy numbers then $\widetilde{A}_1 = \widetilde{A}_2$ iff $m_1 = m_2,\ n_1 = n_2,\ \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

**Definition 1.1.27** *If $m = n$ then an L-R flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be an L-R fuzzy number is denoted as $(m, m, \alpha, \beta)_{LR}$ or $(n, n, \alpha, \beta)_{LR}$ or $(m, \alpha, \beta)_{LR}$ or $(n, \alpha, \beta)_{LR}$.*

**Definition 1.1.28** *The ranking function for an L-R flat fuzzy numbers is defined as*

*where* $\Re(m, n, \alpha, \beta) = \frac{1}{2}\left(\int_0^1 (m - \alpha L^{-1}(\lambda))d\lambda + \int_0^1 (n + \beta R^{-1}(\lambda))d\lambda\right),$

$0 \leqslant \lambda \leqslant 1.$

*Let $\widetilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}, \widetilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be any two L-R flat fuzzy numbers then*

*(i) $\widetilde{A} \preceq \widetilde{B}$ iff $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$*

*(ii) $\widetilde{A} \succeq \widetilde{B}$ iff $\Re(\widetilde{A}) \leq \Re(\widetilde{B})$*

*(iii) $\widetilde{A} \approx \widetilde{B}$ iff $\Re(\widetilde{A}) = \Re(\widetilde{B})$.*

## 1.2 Literature Survey

Linear programming is one of most frequently used operation research technique. Linear programming problems(LPP) was first developed by Kantorovich [22]. Then in 1947, Dantzig [14] developed a method known as simplex method to solve LPP. Neumann [38] introduced theory of duality and applied it in game theory since then it is being used to find the solutions for major problems in mathematics, economics, engineering and military applications. The first polynomial time algorithm to solve LPP is introduced by Khachiyan in 1979 for solving LPP which is based on the concept that a feasible solution is enclosed in an ellipse. Practically, this method is longer as compared to simplex method. Karmarkar in 1984 proposed an algorithm named as interior point method to solve large scale linear programming problems efiiciently. The beauty of the approach is that it gives polynomial time complexity of the solution which is an excellent improvement over the simplex method. Linear programming problem has been applied to large number of areas including transportation, production, inventory management, finance, scheduling, agriculture, and distribution, etc.

In the crisp environment, all the parameters in LPP are well defined and precise. But in the real world, Many times the value of all the parameters of LPP is estimated by the experts. Clearly, assumptions made by experts for all parameters may not be accurate.

In the year 1965, Bellmann and Zadeh [44] introduced the concept of decision making in fuzzy environment, in which the process of decision making is discussed when the goals and the constraints are fuzzy in nature. Many methods have been developed in which either objective function is considered fuzzy in nature or constraints or both are fuzzy in nature. The concept of fuzzy set theory into LPP was first introduced by Zimmermann [45], now number of algorithms are available in literature to solve fuzzy LPP [45, 40, 24, 32, 28, 31].

In this dissertation, we have discussed methods for solving Fully fuzzy linear programming (FLP) problems i.e. coefficients of objective function constraints and all the variables are fuzzy in nature. The solution methodology is based upon ranking the objective function of LPP and the first method for ranking was proposed by Jain [20] in 1976. Yagar [43] proposed four indices which are taken into consideration for ordering fuzzy quantities in [0,1]. Kaufmann and Gupta [24], presented an approach for the ranking of fuzzy numbers. Campos and Gonzalez [7] proposed a subjective approach for ranking fuzzy numbers. Wang and Chen [12] developed an method which is based on integral value index for ranking of fuzzy numbers. Cheng [8] presented distance method for ranking the fuzzy numbers. Kwang and Lee [34] proposed a ranking method by considering the complete possibility distributions of fuzzy numbers in their evaluations. Modarres et al. [37] proposed a ranking method which is based on preference function that measures the fuzzy numbers point by point and the most preferred number is identified at every point.

Chu and tsao [13] proposed a method for ranking the fuzzy numbers with the area between the centroid point and original point. Deng et al. [15] gives centroid-index method for ranking the fuzzy numbers. Chen and Chen [9] proposed an method for ranking of generalized trapezoidal fuzzy numbers. In the year 2008, Wang and Lee [42] for the development of ranking index of the fuzzy numbers uses the concept of centroid. Chen and Tang [11] for ranking the p -norm trapezoidal fuzzy numbers proposed an method. Chen and Wang [12] based on ranking of the fuzzy numbers study the fuzzy risk analysis. Abbasbandy and Hajarii [1] based on the left and right spreads at some $\alpha-$levels of trapezoidal fuzzy numbers proposed an new approach for ranking these trapezoidal fuzzy numbers. Chen and Chen [10] proposed an method for fuzzy risk analysis which is based on ranking the generalized fuzzy numbers with different heights and spreads. Ramli et al. [20] gives the detailed description of different methods for ranking of fuzzy numbers. Kumar et al. [25] for ranking the generalized trapezoidal fuzzy numbers introduced RM approach.

In the literature many methods are available for solving FFLP. Bukley and Feuring [6] developed an method for solving FFLP problem with inequality constraints. The objective function is firstly changed into a multi-objective fuzzy linear programming problem and then fuzzy flexible programming is used to evaluate the whole non dominated set to the multi-objective fuzzy linear program. Further an evolutionary algorithm is developed to solve the fuzzy flexible program.

Hashemi et al. [18] proposed two-phase approach to find the optimal solution of FFLP problems. It is based on mean and standard deviation of the fuzzy numbers. In this approach, the first phase maximizes the possibilistic mean value of the fuzzy objective function and obtains a set of feasible solutions. The second phase minimizes the standard deviation of the original fuzzy objective function, and consider all the basic feasible solutions which are obtained at the end of the first phase. They also generalized the concept of linear programming duality and extend the duality as well as the weak duality theory in fuzzy.

Dehghan et al. [16] proposed a fuzzy linear programming approach for finding the exact solution of fully fuzzy linear system of equations when all the elements of the coefficient matrix are non-negative fuzzy numbers.

Allahviranloo et al. [2] proposed an defuzzification method to solve the fully fuzzy linear programming problem. For an approximation of fuzzy numbers in objective function and coefficient matrix the concept of nearest symmetric triangular fuzzy number is applied.

Lotfi et al. [35] proposed lexicographic method to find the optimal solution of FFLP problem when the elements of the coefficient matrix are symmetric fuzzy numbers. The concept of the symmetric triangular fuzzy number and an approach to defuzzify a general fuzzy quantity is introduced. First, the fuzzy triangular number is approximated to its nearest symmetric triangular number, with the assumption that all decision variables are symmetric triangular then symmetric fuzzy solution is an optimal solution. Every FLP model converted into two crisp complex linear problems, first a problem is designed in which the center objective value will be calculated and since the center of a fuzzy number is preferred to its margin. With a special ranking on fuzzy numbers, the FFLP transform to multi objective linear programming (MOLP) where all variables and parameters are crisp.

Kumar et al. [27] proposed a method to find the fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints. They introduced a method for changing the inequality constraints into equality constraints by introducing slack and surplus fuzzy variables. then Kumar et al. [26] introduced a method for finding the solution of FFLP problem with equality constraints. In this FFLP problem is converted into crisp LPP and then obtained crisp LPP is solved to find the optimal solution. This method is applied to solve the FFLS of equations in which their is no restriction on the elements of the coefficient matrix which are not necessarily non-negative fuzzy numbers and also to solve the FFLP problems in which the elements of the coefficient matrix are not symmetric triangular fuzzy numbers.

Sharma and Dubey [41] proposed Gomorys method to find the fuzzy optimal solution of fully FLPP with inequality constraints by representing all the parameters as trapezoidal fuzzy numbers and Yagers general linear ranking function is used for solving Fuzzy number.

Jayalakshmi and Pandian [21] introduced bound and decomposition method to solve FFLP problems. In the proposed method, the given FFLP problem is decomposed into three crisp linear programming (CLP) problems with bounded variables constraints, the three CLP problems are solved separately and by using its optimal solutions.

Babbar et al. [3] discussed about some new numerical methods which are used to solve a fully fuzzy linear system with triangular fuzzy numbers of the type $(m, \alpha, \beta)$. The main idea behind the Proposed methods is to remove the restrictions that FFLS restricts the coefficient matrix and the solutions to be nonnegative fuzzy numbers.

Kumar et al. [32] introduced a new general form of FFLP problems as the existing general form of FFLP problems is valid only if the parameters which are represented by flat fuzzy numbers are all positive. The proposed general form is applicable if their is negative sign in any of the flat fuzzy parameter.

Further, Kumar et al. [29] introduced the product of unrestricted L-R flat fuzzy numbers and then by using the proposed product, a new method named as Mehars method is developed for solving FFLP problems. In the development of Mehar's method Yager's ranking approach is used and it has been found that all FFLP problems can be solved with the Mehar's method which may not be solvable by several existing methods.

Khan et al. [33] proposed a revised simplex method for solving FFLP without converting it into the crisp LPP. By using the concept of ranking function together with Gaussian elimination process the linear programming problems are solved in fully fuzzy environment and they have also shown that how the dual of FFLP is written.

Bhradwaj and Kumar [5] improves the existing technique which is proposed in Khan at.el[33] for solving FFLP problems without converting it into crisp problem to find the optimal solution of FFLP. Hence to find an efficient algorithm to solve FFLP without converting it into crisp problem is still an open area of research.

Hatami [19] adopted Mehars method which was developed by Kumar at.el [28] for solving FLP problems in which the elements of the coefficient matrix are real numbers and all the other parameters are fuzzy numbers. Here, the method is used for solving FFLP problems involving symmetric trapezoidal fuzzy numbers having both equality and inequality constraints.

## 1.3 Summary of the thesis

In this thesis, we have discussed LPP under fuzzy environment. Some methods for solving fully fuzzy linear programming problem using ranking function has been elaborated.

Since, fuzzy numbers can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than other. In Chapter 2, an efficient approach for ordering the fuzzy numbers based on rank, mode, divergence and spread has been discussed. The main advantage of the proposed approach is that the it provides the correct ordering of generalized and normal trapezoidal fuzzy numbers.

In Chapter 3, a method is presented which is used for finding an optimal solution of the fully fuzzy linear programming problem with equality constraints using the concept of ranking function where all the parameters of FFLP problem are triangular fuzzy numbers. The presented algorithm finds the exact solution of special type of fully fuzzy linear system of equations and FFLP problem in which all the elements of the coefficient matrix are non negative fuzzy numbers and symmetric fuzzy numbers, respectively. The problem is first converted into its crisp linear programming problem and then solution is found.

Mathematically, a fully fuzzy linear programming problem with m fuzzy equality constraints and n fuzzy variables may be formulated as follows:

$$\text{maximize } \widetilde{Z} = \widetilde{C}^T \otimes \widetilde{X}$$

subject to

$$\widetilde{A}\widetilde{X} = \widetilde{B}$$

$$\widetilde{X} \geqslant 0$$

where $\widetilde{C}^T = (\widetilde{c}_j)_{1\times n}$, $\widetilde{X} = (\widetilde{x}_j)_{n\times 1}$, $\widetilde{A} = (\widetilde{a}_{ij})_{m\times n}$, $\widetilde{B} = (\widetilde{b}_i)_{m\times 1}$ and $(\widetilde{a}_{ij})$, $(\widetilde{c}_j)$, $(\widetilde{x}_j)$, $(\widetilde{b}_i)$ $\in F(R)$, where $F(R)$ is an set of fuzzy numbers defined on set of real numbers.

In chapter 4, a modified product for unrestricted L-R fuzzy numbers and L-R flat fuzzy numbers has been discussed and using this product another method named as Mehar's method for solving the FFLP problems is discussed. Further, the merits of Mehar's method over other existing techniques for solving FFLP is also discussed. General form of FFLP problem in which all the parameters are unrestricted L-R fuzzy numbers is

$$\text{Maximize(or Minimize) } \widetilde{Z} = \sum_{j=1}^{n} (\widetilde{c}_j \odot \widetilde{x}_j)$$

subject to

$$\widetilde{a_{ij}} \odot \widetilde{x}_j = \widetilde{b}_i \quad \forall\ i = 1, 2, \ldots, m,$$

where $\widetilde{a_{ij}}, \widetilde{b}_i$ and $\widetilde{c}_j$ are non- negative and non- positive L-R fuzzy number and $\widetilde{x}_j$ is non negative L-R fuzzy number. By using the product defined for LR fuzzy numbers and using Yager's ranking approach [43], this FFLP is converted to a crips linear programming problem, which can be solved by using simplex method.

# Chapter 2

# Ranking of Generalized Trapezoidal Fuzzy Numbers Based on Rank, Mode, Divergence and Spread

In this chapter, a new approach is discussed for the ranking of generalized trapezoidal fuzzy numbers. The proposed approach is based on rank, mode, divergence and spread. The main advantage of the this approach is that it provides ordering of generalized and normal trapezoidal fuzzy numbers correctly and more efficiently. And few of examples are discussed.

## 2.1 Some important results

In this section, some important results have been discussed which are used in the proposed ranking approach

**Proposition 2.1.1** *Let* $\widetilde{A} = (a_1, b_1, c_1, d_1; w_1)$ *and* $\widetilde{B} = (a_2, b_2, c_2, d_2; w_2)$ *be two generalized trapezoidal fuzzy numbers such that*

*(1)* $\Re(\widetilde{A}) = \Re(\widetilde{B})$*, (2) mode(*$\widetilde{A}$*)=mode(*$\widetilde{B}$*), (3) divergence(*$\widetilde{A}$*)=divergence(*$\widetilde{B}$*) then*

*(i) Left spread(*$\widetilde{A}$*)* $>$ *Left spread(*$\widetilde{B}$*) iff* $w_1 b_1 > w_2 b_2$

*(ii) Left spread($\widetilde{A}$) $<$ Left spread($\widetilde{B}$) iff* $w_1b_1 < w_2b_2$

*(iii) Left spread($\widetilde{A}$)=Left spread($\widetilde{B}$) iff* $w_1b_1 = w_2b_2$

***Proof:***

*(1). Suppose the fuzzy numbers* $\widetilde{A}$ *and* $\widetilde{B}$ *has same rank i.e.* $\Re(\widetilde{A}) = \Re(\widetilde{B})$

$$\Rightarrow w_1\frac{(a_1+b_1+c_1+d_1)}{4} = w_2\frac{(a_2+b_2+c_2+d_2)}{4}$$

$$\Rightarrow w_1(a_1+b_1+c_1+d_1) = w_2(a_2+b_2+c_2+d_2) \tag{2.1.1}$$

*(2). Now, iff the fuzzy numbers* $\widetilde{A}$ *and* $\widetilde{B}$ *has same mode i.e. mode($\widetilde{A}$)=mode($\widetilde{B}$)*

$$\Rightarrow w_1\frac{(b_1+c_1)}{2} = w_2\frac{(b_2+c_2)}{2}$$

$$\Rightarrow w_1(b_1+c_1) = w_2(b_2+c_2) \tag{2.1.2}$$

*(3). If divergence of the fuzzy numbers* $\widetilde{A}$ *and* $\widetilde{B}$ *are same i.e.divergence($\widetilde{A}$)= divergence($\widetilde{B}$),we have*

$$w_1(d_1-a_1) = w_2(d_2-a_2) \tag{2.1.3}$$

*by using* (2.1.1) *we have*

$$w_1(a_1+d_1) + w_1(b_1+c_1) = w_2(a_2+d_2) + w_2(b_2+c_2)$$

*since, by using* (2.1.2) *we get,*

$$w_1(a_1+d_1) + w_1(b_1+c_1) = w_2(a_2+d_2) + w_2(b_1+c_1)$$

$$w_1(a_1+d_1) = w_2(a_2+d_2) \tag{2.1.4}$$

*adding* (2.1.3) *and* (2.1.4)

$$w_1(a_1+d_1) + w_1(d_1-a_1) = w_2(a_2+d_2) + w_2(d_2-a_2)$$

*we get*

$$w_1 d_1 = w_2 d_2 \tag{2.1.5}$$

*and by putting* (2.1.6) *in* (2.1.3) *we get,*

$$w_1 a_1 = w_2 a_2 \tag{2.1.6}$$

*(a) Left spread*($\widetilde{A}$) $>$ *Left spread*($\widetilde{B}$)

*iff* $w_1(b_1 - a_1) > w_2(b_2 - a_2)$
*iff* $w_1 b_1 > w_2 b_2$ $\quad (\because w_1 a_1 = w_2 a_2)$
*hence, Left spread*($\widetilde{A}$) $>$ *Left spread*($\widetilde{B}$) *iff* $w_1 b_1 > w_2 b_2$.

*(b) Left spread*($\widetilde{A}$) $<$ *Left spread*($\widetilde{B}$)

*iff* $w_1(b_1 - a_1) < w_2(b_2 - a_2)$
*iff* $w_1 b_1 < w_2 b_2$ $\quad (\because w_1 a_1 = w_2 a_2)$

*(c) Left spread*($\widetilde{A}$)=*Left spread*($\widetilde{B}$)

*iff* $w_1(b_1 - a_1) = w_2(b_2 - a_2)$
*iff* $w_1 b_1 = w_2 b_2$ $\quad (\because w_1 a_1 = w_2 a_2)$

**Proposition 2.1.2** *Let* $\widetilde{A} = (a_1, b_1, c_1, d_1, w_1)$ *and* $\widetilde{B} = (a_2, b_2, c_2, d_2, w_2)$ *be two generalized trapezoidal fuzzy numbers such that*

*(1)* $\Re(\widetilde{A}) = \Re(\widetilde{B})$, *(2) mode*($\widetilde{A}$)=*mode*($\widetilde{B}$), *(3) divergence*($\widetilde{A}$)=*divergence*($\widetilde{B}$) *then*
*(i) Left spread*($\widetilde{A}$) $>$ *Left spread*($\widetilde{B}$) *iff Right spread*($\widetilde{A}$) $>$ *Right spread*($\widetilde{B}$)

*(ii) Left spread*($\widetilde{A}$) $<$ *Left spread*($\widetilde{B}$) *iff Right spread*($\widetilde{A}$) $<$ *Right spread*($\widetilde{B}$)

*(iii) Left spread*($\widetilde{A}$)=*Left spread*($\widetilde{B}$) *iff Right spread*($\widetilde{A}$)=*Right spread*($\widetilde{B}$)

***Proof :*** *From preposition 1, we have*

$$w_1 a_1 = w_2 a_2$$

$$w_1 d_1 = w_2 d_2$$

$$w_1(b_1 + c_1) = w_2(b_2 + c_2)$$

*(a) Left spread*$(\widetilde{A}) >$ *Left spread*$(\widetilde{B})$

$\Leftrightarrow w_1b_1 > w_2b_2$ *(by preposition 1(i))*

$\Leftrightarrow w_1c_1 < w_2c_2$

$\Leftrightarrow -w_1c_1 > -w_2c_2$ *(adding $w_1d_1$ both sides)*

$\Leftrightarrow w_1d_1 - w_1c_1 > w_1d_1 - w_2c_2$

$\Leftrightarrow w_1d_1 - w_1c_1 > w_2d_2 - w_2c_2$ $(\because w_1d_1 = w_2d_2)$

$\Leftrightarrow w_1(d_1 - c_1) > w_2(d_2 - c_2)$

$\Leftrightarrow$ *Right spread*$(\widetilde{A}) >$ *Right spread*$(\widetilde{B})$

*(b) Left spread*$(\widetilde{A}) <$ *Left spread*$(\widetilde{B})$

$\Leftrightarrow w_1b_1 < w_2b_2$ *(by preposition 1(ii))*

$\Leftrightarrow w_1c_1 > w_2c_2$

$\Leftrightarrow -w_1c_1 < -w_2c_2$ *(adding $w_1d_1$ both sides)*

$\Leftrightarrow w_1d_1 - w_1c_1 < w_1d_1 - w_2c_2$

$\Leftrightarrow w_1d_1 - w_1c_1 < w_2d_2 - w_2c_2$ $(\because w_1d_1 = w_2d_2)$

$\Leftrightarrow w_1(d_1 - c_1) < w_2(d_2 - c_2)$

$\Leftrightarrow$ *Right spread*$(\widetilde{A}) <$ *Right spread*$(\widetilde{B})$

*(c) Left spread*$(\widetilde{A}) =$ *Left spread*$(\widetilde{B})$

$\Leftrightarrow w_1b_1 = w_2b_2$ *(by preposition 1(iii))*

$\Leftrightarrow w_1c_1 = w_2c_2$

$\Leftrightarrow -w_1c_1 = -w_2c_2$ *(adding $w_1d_1$ both sides)*

$\Leftrightarrow w_1d_1 - w_1c_1 = w_1d_1 - w_2c_2$

$\Leftrightarrow w_1d_1 - w_1c_1 = w_2d_2 - w_2c_2$ $(\because w_1d_1 = w_2d_2)$

$\Leftrightarrow w_1(d_1 - c_1) = w_2(d_2 - c_2)$

$\Leftrightarrow$ *Right spread*$(\widetilde{A})=$ *Right spread*$(\widetilde{B})$

## 2.2 Algorithm for ranking of generalized trapezoidal fuzzy numbers

Let $\widetilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers.

For comparing the two generalized trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ the steps are as follows:

**Step 1:** Find the rank of the two generalized trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ i.e. $\Re(\widetilde{A})$ and $\Re(\widetilde{B})$.

**Case(1):** If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then $\widetilde{A} > \widetilde{B}$
**Case(2):** If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then $\widetilde{A} < \widetilde{B}$
**Case(3):** If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then go to step 2,

**Step 2:** Find the mode of the two generalized trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ i.e. mode$(\widetilde{A})$ and mode$(\widetilde{B})$.

**Case(1):** If mode$(\widetilde{A}) >$ mode$(\widetilde{B})$ then $\widetilde{A} > \widetilde{B}$
**Case(2):** If mode$(\widetilde{A}) <$ mode$(\widetilde{B})$ then $\widetilde{A} < \widetilde{B}$
**Case(3):** If mode$(\widetilde{A})$=mode$(\widetilde{B})$ then go to step 3,

**Step 3:** Find the divergence of the two generalized trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ i.e. divergence$(\widetilde{A})$ and divergence$(\widetilde{B})$.

**Case(1):** If divergence$(\widetilde{A}) >$ divergence$(\widetilde{B})$ then $\widetilde{A} > \widetilde{B}$
**Case(2):** If divergence$(\widetilde{A}) <$ divergence$(\widetilde{B})$ then $\widetilde{A} < \widetilde{B}$
**Case(3):** If divergence$(\widetilde{A})$=divergence$(\widetilde{B})$ then go to step 4,

**Step 4:** Find the Left Spread of the two generalized trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ i.e. Left Spread$(\widetilde{A})$ and Left Spread$(\widetilde{B})$.

**Case(1):** Left spread$(\widetilde{A}) >$ Left spread$(\widetilde{B})$
i.e. $w_1 b_1 > w_2 b_2$ then $\widetilde{A} > \widetilde{B}$ from preposition 1(i)
**Case(2):** Left spread$(\widetilde{A}) <$ Left spread$(\widetilde{B})$
i.e $w_1 b_1 < w_2 b_2$ then $\widetilde{A} < \widetilde{B}$ from preposition 1(ii)

**Case(3):** Left spread($\widetilde{A}$)=Left spread($\widetilde{B}$)

i.e. $w_1 b_1 = w_2 b_2$ then go to step 5, from preposition 1(iii)

**Step 5:** Find $w_1$ and $w_2$

**Case(1):** If $w_A > w_B$ then $\widetilde{A} > \widetilde{B}$

**Case(2):** If $w_A < w_B$ then $\widetilde{A} < \widetilde{B}$

**Case(3):** If $w_A = w_B$ then $\widetilde{A} \sim \widetilde{B}$

## 2.3 Numerical examples

In this section, some numericals are explained to elaborate the concept of comparing the fuzzy numbers on the basis of their rank, mode, divergence and spreads.

**Example 1:** Let $\widetilde{A} = (0.1, 0.3, 0.4, 0.6; 1)$ and $\widetilde{B} = (1, 1, 1, 1; 1)$ are two generalized trapezoidal fuzzy numbers.

**Solution:**

**Step 1:**Find rank of $\widetilde{A}$ and $\widetilde{B}$.

$\Re(\widetilde{A}) = 1\left(\dfrac{0.1 + 0.3 + 0.4 + 0.6}{4}\right) = 0.35$ and $\Re(\widetilde{B}) = 0.7\left(\dfrac{1 + 1 + 1 + 1}{4}\right) = 1$

Since,

$$\Re(\widetilde{A}) = 0.35, \text{ and } \Re(\widetilde{B}) = 1$$

clearly the $\Re(\widetilde{A}) < \Re(\widetilde{B})$

$\Rightarrow \widetilde{A} < \widetilde{B}$.

as ranks of the fuzzy numbers are not same thus no need to follow the further steps of algorithm.

**Example 2:** Let $\widetilde{A} = (1, 1, 1, 1; 1)$ and $\widetilde{B} = (0.1, 0.3, 0.4, 0.7; 1)$ are two generalized trapezoidal fuzzy numbers.

**Solution:**

**Step 1:**Find rank of $\widetilde{A}$ and $\widetilde{B}$.

$\Re(\widetilde{A}) = 1\left(\dfrac{1+1+1+1}{4}\right) = 1$ and $\Re(\widetilde{B}) = 1\left(\dfrac{0.1+0.3+0.4+0.7}{4}\right) = 0.375$

Since,

$$\Re(\widetilde{A}) = 1 \text{ and } \Re(\widetilde{B}) = 0.375$$

clearly the $\Re(\widetilde{A}) > \Re(\widetilde{B})$

$\Rightarrow \widetilde{A} > \widetilde{B}$.

as ranks of the fuzzy numbers are not same thus no need to follow the further steps of algorithm.

when the rank of the fuzzy numbers became same then we compare the fuzzy numbers on the basis of mode here, an example is shown.

**Example 3:** Let $\widetilde{A} = (0.2, 0.4, 0.5, 0.9; 0.35)$ and $\widetilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$ are two generalized trapezoidal fuzzy numbers.

**Solution:**

**Step 1:**Find rank of $\widetilde{A}$ and $\widetilde{B}$.

$\Re(\widetilde{A}) = 0.35\left(\dfrac{0.2+0.4+0.6+0.8}{4}\right) = 0.175$ and $\Re(\widetilde{B}) = 0.7\left(\dfrac{0.1+0.2+0.3+0.4}{4}\right) =$ 0.175

clearly, $\Re(\widetilde{A}) = \Re(\widetilde{B}) = 0.175$, so go to step 2,

**Step 2:**Find mode of $\widetilde{A}$ and $\widetilde{B}$.

$\text{mode}(\widetilde{A}) = 0.35\left(\dfrac{0.4+0.5}{2}\right) = 0.1575$ and $\text{mode}(\widetilde{B}) = 0.7\left(\dfrac{0.2+0.3}{2}\right) = 0.175$

since,

$$\text{mode}(\widetilde{A}) = 0.1575 \text{ and } \ \text{mode}(\widetilde{A}) = 0.175$$

clearly, $\text{mode}(\widetilde{A}) < \text{mode}(\widetilde{B})$

$\Rightarrow \widetilde{A} < \widetilde{B}$.

if the mode also became same then we compare the generalized trapezoidal fuzzy numbers on the basis of their divergence and if it also became same then we compare them by their spreads.

**Example 4:** Let $\widetilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\widetilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ are two generalized fuzzy numbers.

**Solution:**

**Step 1:** Find rank of $\widetilde{A}$ and $\widetilde{B}$.

$\Re(\widetilde{A}) = 1\left(\dfrac{0.1 + 0.2 + 0.4 + 0.5}{4}\right) = 0.3$ and $\Re(\widetilde{B}) = 1\left(\dfrac{0.1 + 0.3 + 0.3 + 0.5}{4}\right) = 0.3$

Since, $\Re(\widetilde{A}) = \Re(\widetilde{B}) = 0.3$, so go to step 2,

**Step 2:** Find mode of $\widetilde{A}$ and $\widetilde{B}$.

$$\text{mode}(\widetilde{A}) = 1\left(\frac{0.2 + 0.4}{2}\right) = 0.3 \text{ and mode}(\widetilde{B}) = 1\left(\frac{0.3 + 0.3}{2}\right) = 0.3$$

Since, $\text{mode}(\widetilde{A}) = \text{mode}(\widetilde{B}) = 0.3$, so go to step 3,

**Step 3:**Find divergence of $\widetilde{A}$ and $\widetilde{B}$.

$$divergence(\widetilde{A}) = 1(0.5 - 0.1) = 0.4 \text{ and divergence}(\widetilde{B}) = 1(0.5 - 0.1) = 0.4$$

Since, $\text{divergence}(\widetilde{A}) = \text{divergence}(\widetilde{B}) = 0.4$, so go to step 4,

**Step 4:**Find left spread of $\widetilde{A}$ and $\widetilde{B}$.

$$\text{Left Spread}(\widetilde{A}) = 1(0.2) = 0.2 \text{ and Left Spread}(\widetilde{B}) = 1(0.3) = 0.3$$

Since, $\text{Left Spread}(\widetilde{A}) = 0.2$ and $\text{Left Spread}(\widetilde{B}) = 0.3$,

$\Rightarrow \widetilde{A} < \widetilde{B}$.

here, an example is shown to compare the generalized trapezoidal fuzzy numbers when the left spreads also became same.

**Example 5:** Let $\widetilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\widetilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$ are two generalized fuzzy numbers.

**Solution:**

**Step 1:** Find rank of $\widetilde{A}$ and $\widetilde{B}$. $\Re(\widetilde{A}) = 0.35\left(\dfrac{0.2+0.4+0.6+0.8}{4}\right) = 0.175$ and $\Re(\widetilde{B}) = 0.7\left(\dfrac{0.1+0.2+0.3+0.4}{4}\right) = 0.175$

Since, $\Re(\widetilde{A}) = \Re(\widetilde{B}) = 0.175$, so go to step 2,

**Step 2:**Find mode of $\widetilde{A}$ and $\widetilde{B}$.

$$\text{mode}(\widetilde{A}) = 0.35\left(\frac{0.4+0.6}{2}\right) = 0.175 \text{ and } \text{mode}(\widetilde{B}) = 0.7\left(\frac{0.2+0.3}{2}\right) = 0.175$$

Since, $\text{mode}(\widetilde{A}) = \text{mode}(\widetilde{B}) = 0.175$, so go to step 3,

**Step 3:**Find divergence of $\widetilde{A}$ and $\widetilde{B}$.

$$\text{divergence}(\widetilde{A}) = 0.35(0.8-0.2) = 0.21 \text{ and } \text{divergence}(\widetilde{B}) = 0.7(0.4-0.1) = 0.21$$

Since, $\text{divergence}(\widetilde{A}) = \text{divergence}(\widetilde{B}) = 0.21$, so go to step 4,

**Step 4:**Find left spread of $\widetilde{A}$ and $\widetilde{B}$.

$$\text{Left Spread}(\widetilde{A}) = 0.35(0.4) = 0.14 \text{ and Left Spread}(\widetilde{B}) = 0.7(0.2) = 0.14$$

Since, $\text{Left Spread}(\widetilde{A}) = \text{Left Spread}(\widetilde{B}) = 0.07$, so go to step 5,

**Step 5:**Find $w_1$ and $w_2$

$$w_1 = 0.35, w_2 = 0.7$$

clearly, $w_1 < w_2 \Rightarrow \widetilde{A} < \widetilde{B}$.

Example is shown to compare three generalized fuzzy numbers on the basis of their ranks.

**Example 6:** Let $\widetilde{A} = (0, 0.4, 0.6, 0.8; 1)$, $\widetilde{B} = (0.2, 0.5, 0.5, 0.9; 1)$ and $\widetilde{C} = (0.1, 0.6, 0.7, 0.8; 1)$ are three generalized trapezoidal fuzzy numbers.

**Solution:**

**Step 1:**Find rank of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$.

$\Re(\widetilde{A}) = 1\left(\dfrac{0+0.4+0.6+0.8}{4}\right) = 0.45,\ \Re(\widetilde{B}) = 1\left(\dfrac{(0.2+0.5+0.5+0.9)}{4}\right) = 0.525$

and $\Re(\widetilde{C}) = 1\left(\dfrac{(0.1+0.6+0.7+0.8)}{4}\right) = 0.55$

Since, $\Re(\widetilde{A}) = 0.45,\ \Re(\widetilde{B}) = 0.525$ and $\Re(\widetilde{C}) = 0.55$

clearly the $\Re(\widetilde{A}) < \Re(\widetilde{B})$ and $\Re(\widetilde{B}) < \Re(\widetilde{C})$

$\Rightarrow \widetilde{A} < \widetilde{B} < \widetilde{C}$.

## 2.4 Concluding remark

In this Chapter, for the ranking of generalized trapezoidal fuzzy numbers an simple approach is shown. The main advantage of the proposed approach is that it provides the correct ordering of generalized and normal trapezoidal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems.

# Chapter 3

# Fully Fuzzy Linear Programming Problem using Ranking Function

In this chapter, a method is presented for solving the fully fuzzy linear problem with equality constraints using the concept of ranking function in which all the parameters are triangular fuzzy numbers and the numerical examples are shown.

## 3.1 Fully Fuzzy Linear Programming Problem

In crisp Linear programming problem, all the parameters are well defined and precise. However, in real life problems there may exists uncertainty about the parameters. In such a situation the parameters of linear programming problem may be represented as fuzzy numbers. Fully Fuzzy Linear Programming Problems with m fuzzy equality constraints and n fuzzy variables may be formulated as follows:

$$\text{maximize } \widetilde{Z} = \widetilde{C}^T \otimes \widetilde{X}$$

subject to

$$\widetilde{A}\widetilde{X} = \widetilde{B}$$

$$\widetilde{X} \geqslant 0$$

where $\widetilde{C}^T = (\widetilde{c}_j)_{1\times n}$, $\widetilde{X} = (\widetilde{x}_j)_{n\times 1}$, $\widetilde{A} = (\widetilde{a}_{ij})_{m\times n}$, $\widetilde{B} = (\widetilde{b}_i)_{m\times 1}$ and $(\widetilde{a}_{ij})$, $(\widetilde{c}_j)$, $(\widetilde{x}_j)$, $(\widetilde{b}_i)$

$\in F(R)$, where $F(R)$ is an set of fuzzy numbers defined on set of real numbers.

## 3.2 Algorithm to find the fuzzy optimal solution of FFLP problems

In this section, various steps for solving fully fuzzy linear programming problem has been shown:

The formulation for FFLP is given as

$$(P1) \quad \text{Maximize(or Minimize) } \widetilde{Z} = \widetilde{C}^T \otimes \widetilde{X}$$

subject to

$$\widetilde{A}\widetilde{X} = \widetilde{B}$$

$$\widetilde{X} \geqslant 0$$

The steps are as follows:

**Step 1:** Substituting the values of $\widetilde{C}^T = (\widetilde{c}_j)_{1\times n},\ \widetilde{X} = (\widetilde{x}_j)_{n\times 1},\ \widetilde{A} = (\widetilde{a}_{ij})_{m\times n},\ \widetilde{B} = (\widetilde{b}_i)_{m\times 1}$, then the above FFLP problem is written as,

$$\text{Maximize (or Minimize) } (\sum_{j=1}^{n} \widetilde{c}_j \otimes \widetilde{x}_j)$$

subject to $\quad (\sum_{j=1}^{n} \widetilde{a}_{ij} \otimes \widetilde{x}_j = \widetilde{b}_j) \qquad \forall\ i = 1, 2, \ldots, m$

$\widetilde{x}_i$ is a non-negative triangular fuzzy number.

**Step 2:** If all the parameters $\widetilde{c}_j,\ \widetilde{x}_j,\ \widetilde{a}_{ij},$ and $\widetilde{b}_i$ in step 1 are represented by triangular fuzzy numbers $(p_j, q_j, r_j), (x_j, y_j, z_j), (a_{ij}, b_{ij}, c_{ij})$ and $(b_i, g_i, h_i)$ respectively then the problem will become

$$\text{Maximize (or Minimize) } (\sum_{j=1}^{n}(p_j, q_j, r_j) \otimes (x_j, y_j, z_j))$$

subject to
$$(\sum_{j=1}^{n}(a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i)) \quad \forall\ i = 1, 2, \ldots, m$$

$(x_j, y_j, z_j)$ is a non-negative triangular fuzzy number.

**Step 3:** By using the arithmetic operations defined in definition 1.1.12. Assume $(a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j) = (m_{ij}, n_{ij}, o_{ij})$ the FFLP problem which is obtained in step 2 is represented as

$$\text{Maximize (or Minimize) } (\sum_{j=1}^{n}(p_j, q_j, r_j) \otimes (x_j, y_j, z_j))$$

subject to
$$(\sum_{j=1}^{n}(m_{ij}, n_{ij}, o_{ij}) = (b_i, g_i, h_i)) \quad \forall\ i = 1, 2, \ldots, m$$

$(x_j, y_j, z_j)$ is a non-negative triangular fuzzy number.

**Step 4:** the fuzzy linear programming problem which is given in step 3 is converted into the following crisp linear programming problem:

$$\text{Maximize (or Minimize) } (\sum_{j=1}^{n}(p_j, q_j, r_j) \otimes (x_j, y_j, z_j))$$

subject to

$$\sum_{j=1}^{n} m_{ij} = b_i, \quad \forall\ i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} n_{ij} = g_i, \quad \forall\ i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} o_{ij} = h_i, \quad \forall\ i = 1, 2, \ldots, m$$

$$y_j - x_j \geq 0, z_j - y_j \geq 0, \quad \forall\ j = 1, 2, \ldots, n$$

**Step 5:** Find the optimal solution $x_j,\ y_j$ and $z_j$ by solving the crisp linear programming problem which is obtained in step 4.

**Step 6.** Find the fuzzy optimal solution by putting the values of $x_j,\ y_j$ and $z_j$ in $\widetilde{x_i} = (x_j, y_j, z_j)$.

**Step 7.** Find the fuzzy optimal value by putting $\widetilde{x_i}$ in $(\sum_{j=1}^{n} \widetilde{c_j} \otimes \widetilde{x_j})$.

**Remark 3.2.1** *The Fully Fuzzy Linear Programming Problem will have $\widetilde{X}$ as its fuzzy optimal solution if it satisfies the following characteristics:*

*(i) $\widetilde{X}$ is a non-negative fuzzy number,*

*(ii) It satisfies the constraints $\widetilde{A}\widetilde{X} = \widetilde{B}$,*

*(iii) If their exist any other non-negative fuzzy number $\widetilde{X'}$ such that it satisfies the set of constraints of the given problem i.e. $\widetilde{A}\widetilde{X'} = \widetilde{B}$ then*

** for maximization problem, $\Re(\widetilde{C}^T \otimes \widetilde{X}) > \Re(\widetilde{C}^T \otimes \widetilde{X'})$,*

** for minimization problem, $\Re(\widetilde{C}^T \otimes \widetilde{X}) < \Re(\widetilde{C}^T \otimes \widetilde{X'})$.*

**Remark 3.2.2** *If $\widetilde{X}$ is an optimal solution of the fully fuzzy linear programming problem and let their exist an fuzzy number $\widetilde{Y}$ such that it satisfies the following characteristics:*

*(i) $\widetilde{Y}$ is a non-negative fuzzy number,*

*(ii) $\widetilde{A}\widetilde{Y} = \widetilde{B}$,*

*(iii) and $\Re(\widetilde{C}^T \otimes \widetilde{X}) = \Re(\widetilde{C}^T \otimes \widetilde{Y})$,*

*then $\widetilde{Y}$ is said to be an alternative fuzzy optimal solution.*

## 3.3 Numerical examples

**Example 1.** Consider the following FFLS with arbitrary coefficients

$$\widetilde{3} \otimes \widetilde{x_1} + \widetilde{2} \otimes \widetilde{x_2} = \widetilde{21}$$

$$\widetilde{1} \otimes \widetilde{x_1} + \widetilde{3} \otimes \widetilde{x_2} = \widetilde{14}$$

where $\widetilde{x_1}, \widetilde{x_2}$ are non-negative triangular fuzzy numbers.

**solution:**

When all the elements of coefficient matrix are not non-negative triangular fuzzy numbers,

$$(2, 3, 4) \otimes \widetilde{x_1} + (1, 2, 3) \otimes \widetilde{x_1} = (5, 21, 43),$$

$$(-1, 1, 2) \otimes \widetilde{x_1} + (1, 3, 4) \otimes \widetilde{x_1} = (-6, 14, 34),$$

where $\widetilde{x_1},\ \widetilde{x_2}$ are non-negative triangular fuzzy numbers.

Putting $\widetilde{x_1} = (x_1, y_1, z_1)$ and $\widetilde{x_2} = (x_2, y_2, z_2)$ then FFLS will become

$$(2, 3, 4) \otimes (x_1, y_1, z_1) + (1, 2, 3) \otimes (x_2, y_2, z_2) = (5, 21, 43),$$

$$(-1, 1, 2) \otimes (x_1, y_1, z_1) + (1, 3, 4) \otimes (x_2, y_2, z_2) = (-6, 14, 34),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Now, with the help of Step 3 of an algorithm and by using the arithmetic operations which are defined in Definition 1.1.12 the above given FFLS problem will become,

$$\text{Maximize } (1x_1 + 2x_2, 6y_1 + 3y_2, 9z_1 + 8z_2)$$

subject to

$$(2x_1 + x_2, 3y_1 + 2y_2, 4z_1 + 3z_2) = (6, 16, 30),$$

$$(-1z_1 + x_2, y_1 + 3y_2, 2z_1 + 4z_2) = (1, 17, 30),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

by using step 4 of an algorithm the given FFLS is converted into crisp linear system of equations

$$2x_1 + x_2 = 5,$$

$$3y_1 + 2y_2 = -6,$$

$$4z_1 + 3z_2 = 21,$$

$$-z_1 + x_2 = 14,$$

$$y_1 + 3y_2 = 43,$$

$$2z_1 + 4z_2 = 34,$$

$y_1 - x_1 \geq 0,\ z_1 - y_1 \geq 0,\ y_2 - x_2 \geq 0,\ z_2 - y_2 \geq 0.$

On solving the above CLP using two phase method the optimal solution obtained is $x_1 = 2,\ y_1 = 5,\ z_1 = 7,\ x_2 = 1,\ y_2 = 3,\ z_2 = 5.$

Using Step 6, the fuzzy optimal solution is given by $\widetilde{x_1} = (2, 5, 7),\ \widetilde{x_2} = (1, 3, 5).$

**Example 2.** Consider the following FFLP problem

$$\text{Maximize } (\widetilde{6} \otimes \widetilde{x_1} + \widetilde{3} \otimes \widetilde{x_2})$$

subject to

$$\widetilde{3} \otimes \widetilde{x_1} + \widetilde{2} \otimes \widetilde{x_2} = \widetilde{16}$$

$$\widetilde{1} \otimes \widetilde{x_1} + \widetilde{3} \otimes \widetilde{x_2} = \widetilde{17}$$

where $\widetilde{x_1}, \widetilde{x_2}$ are non-negative triangular fuzzy numbers.

**solution:**

When all the elements of coefficient matrix are not non-negative triangular fuzzy numbers,

$$\text{Maximize } ((1, 6, 9) \otimes \widetilde{x_1} + (2, 3, 8) \otimes \widetilde{x_1})$$

. subject to

$$(2, 3, 4) \otimes \widetilde{x_1} + (1, 2, 3) \otimes \widetilde{x_1} = (6, 16, 30),$$

$$(-1, 1, 2) \otimes \widetilde{x_1} + (1, 3, 4) \otimes \widetilde{x_1} = (1, 17, 30),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Putting $\widetilde{x_1} = (x_1, y_1, z_1)$ and $\widetilde{x_2} = (x_2, y_2, z_2)$ then FFLP problem will become

$$\text{Maximize } ((1,6,9) \otimes (x_1, y_1, z_1) + (2,3,8) \otimes (x_2, y_2, z_2))$$

. subject to

$$(2,3,4) \otimes (x_1, y_1, z_1) + (1,2,3) \otimes (x_2, y_2, z_2) = (6,16,30),$$

$$(-1,1,2) \otimes (x_1, y_1, z_1) + (1,3,4) \otimes (x_2, y_2, z_2) = (1,17,30),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Now, with the help of Step 3 of an algorithm and by using the arithmetic operations which are defined in Definition 1.1.12 the above given FFLP problem will become,

$$\text{Maximize } (1x_1 + 2x_2, 6y_1 + 3y_2, 9z_1 + 8z_2)$$

subject to

$$(2x_1 + x_2, 3y_1 + 2y_2, 4z_1 + 3z_2) = (6,16,30),$$

$$(-1z_1 + x_2, y_1 + 3y_2, 2z_1 + 4z_2) = (1,17,30),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

using the concept of ranking function which is defined in definition 1.1.11, the above mentioned FFLP problem will become

$$\text{Maximize } \left(\frac{1}{4}(x_1 + 2x_2 + 12y_1 + 6y_2 + 9z_1 + 8z_2)\right)$$

subject to

$$(2x_1 + x_2, 3y_1 + 2y_2, 4z_1 + 3z_2) = (6,16,30),$$

$$(-1z_1 + x_2, y_1 + 3y_2, 2z_1 + 4z_2) = (1,17,30),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

by using step 4 of an algorithm the given FFLP is converted into crisp linear programming problem

$$\text{Maximize } \left(\frac{1}{4}(x_1 + 2x_2 + 12y_1 + 6y_2 + 9z_1 + 8z_2)\right)$$

subject to

$$2x_1 + x_2 = 6,$$

$$3y_1 + 2y_2 = 1,$$

$$4z_1 + 3z_2 = 30,$$

$$-z_1 + x_2 = 6,$$

$$y_1 + 3y_2 = 17,$$

$$2z_1 + 4z_2 = 30,$$

$y_1 - x_1 \geq 0,\ z_1 - y_1 \geq 0,\ y_2 - x_2 \geq 0,\ z_2 - y_2 \geq 0.$

The optimal solution of the above CLP problem is $x_1 = 1,\ y_1 = 1,\ z_1 = 2,\ x_2 = 4,\ y_2 = 5,\ z_2 = 6.$

Using Step 6, the fuzzy optimal solution is given by $\widetilde{x_1} = (1, 2, 3),\ \widetilde{x_2} = (4, 5, 6).$

putting the values $\widetilde{x_1}$ and $\widetilde{x_2}$ in the objective function

$(1, 6, 9) \otimes (1, 2, 3) + (2, 3, 8) \otimes (4, 5, 6),$ by using the arithmetic operations defined in the definition 1.1.11

= (1,12,27)+(8,15,48)=(9,27,75)

Thus, the optimal solution of the given FFLP problem is (9,27,75)

**Example 3.**Consider the following FLPP

$$\text{Maximize } (\widetilde{2} \otimes \widetilde{x_1} + \widetilde{3} \otimes \widetilde{x_2})$$

subject to

$$\widetilde{1} \otimes \widetilde{x_1} + \widetilde{2} \otimes \widetilde{x_2} = \widetilde{10}$$

$$\widetilde{2} \otimes \widetilde{x_1} + \widetilde{1} \otimes \widetilde{x_2} = \widetilde{8}$$

where $\widetilde{x_1}, \widetilde{x_2}$ are non-negative triangular fuzzy numbers.

**Solution:**

When all the elements of coefficient matrix are non-negative triangular fuzzy numbers

$$\text{Maximize } ((1, 2, 3) \otimes \widetilde{x_1} + (2, 3, 4) \otimes \widetilde{x_1}).$$

subject to

$$(0, 1, 2) \otimes \widetilde{x_1} + (1, 2, 3) \otimes \widetilde{x_1} = (2, 10, 24),$$

$$(1, 2, 3) \otimes \widetilde{x_1} + (0, 1, 2) \otimes \widetilde{x_2} = (1, 8, 21),$$

where $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Putting $\widetilde{x_1} = (x_1, y_1, z_1)$ and $\widetilde{x_2} = (x_2, y_2, z_2)$ then FFLP problem will become

$$\text{Maximize } ((1, 2, 3) \otimes (x_1, y_1, z_1) + (2, 3, 4) \otimes (x_2, y_2, z_2)).$$

subject to

$$(0, 1, 2) \otimes (x_1, y_1, z_1) + (1, 2, 3) \otimes (x_2, y_2, z_2) = (2, 10, 24),$$

$$(1, 2, 3) \otimes (x_1, y_1, z_1) + (0, 1, 2) \otimes (x_2, y_2, z_2) = (1, 8, 21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Now, with the help of Step 3 of an algorithm and by using the arithmetic operations which are defined in Definition 1.1.12 the above given FFLP problem will become,

$$\text{Maximize } (1x_1 + 2x_2, 2y_1 + 3y_2, 3z_1 + 4z_2).$$

subject to

$$(0x_1 + x_2, 1y_1 + 2y_2, 2z_1 + 3z_2) = (2, 10, 24),$$

$$(1z_1 + 0x_2, 2y_1 + y_2, 3z_1 + 2z_2) = (1, 8, 21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

using the concept of ranking function which is defined in definition 1.1.11, the above mentioned FFLP problem will become

$$\text{Maximize } \left(\frac{1}{4}(x_1 + 2x_2 + 4y_1 + 6y_2 + 3z_1 + 4z_2)\right).$$

subject to

$$(0x_1 + x_2, 1y_1 + 2y_2, 2z_1 + 3z_2) = (2, 10, 24),$$

$$(1z_1 + 0x_2, 2y_1 + y_2, 3z_1 + 2z_2) = (1, 8, 21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

by using step 4 of an algorithm the given FFLP is converted into crisp linear programming problem

$$\text{Maximize } \left(\frac{1}{4}(x_1 + 2x_2 + 4y_1 + 6y_2 + 3z_1 + 4z_2)\right)$$

subject to

$$0x_1 + x_2 = 2,$$

$$y_1 + 2y_2 = 10,$$

$$2z_1 + 3z_2 = 24,$$

$$x_1 + 0x_2 = 1,$$

$$2y_1 + 3y_2 = 24,$$

$$3z_1 + 2z_2 = 21,$$

$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0.$

The optimal solution of the above CLP problem is $x_1 = 1, y_1 = 1, z_1 = 2, x_2 = 4, y_2 = 5, z_2 = 6$.

Using Step 6, the fuzzy optimal solution is given by $\widetilde{x_1} = (1, 2, 3), \widetilde{x_2} = (2, 4, 6)$.

putting the values $\widetilde{x_1}$and$\widetilde{x_2}$ in the objective function

$(1, 2, 3) \otimes (1, 2, 3) + (2, 3, 4) \otimes (2, 4, 6)$, by using the arithmetic operations defined in the definition 1.1.11

= (1,2,9)+(4,6,12)=(5,16,33)

Thus, the optimal solution of the given FFLP problem is (5,16,33)

**Example 4.**Consider the following FLPP

$$\text{Maximize } (\widetilde{2} \otimes \widetilde{x_1} + \widetilde{3} \otimes \widetilde{x_2})$$

subject to

$$\widetilde{1} \otimes \widetilde{x_1} + \widetilde{2} \otimes \widetilde{x_2} = \widetilde{10}$$

$$\widetilde{2} \otimes \widetilde{x_1} + \widetilde{1} \otimes \widetilde{x_2} = \widetilde{8}$$

where $\widetilde{x_1}, \widetilde{x_2}$ are non-negative triangular fuzzy numbers.

**Solution:**

When all the elements of coefficient matrix are non-negative triangular fuzzy numbers

$$\text{Maximize } ((1, 2, 3) \otimes \widetilde{x_1} + (2, 3, 4) \otimes \widetilde{x_1}).$$

subject to

$$(0, 1, 2) \otimes \widetilde{x_1} + (0, 2, 3) \otimes \widetilde{x_1} = (2, 10, 24),$$

$$(1, 2, 3) \otimes \widetilde{x_1} + (0, 1, 2) \otimes \widetilde{x_1} = (1, 8, 21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Putting $\widetilde{x_1} = (x_1, y_1, z_1)$and$\widetilde{x_2} = (x_2, y_2, z_2)$ then FFLP problem will become

$$\text{Maximize } ((1,2,3) \otimes (x_1, y_1, z_1) + (2,3,4) \otimes (x_2, y_2, z_2)).$$

subject to

$$(0,1,2) \otimes (x_1, y_1, z_1) + (0,2,3) \otimes (x_2, y_2, z_2) = (2,10,24),$$

$$(1,2,3) \otimes (x_1, y_1, z_1) + (0,1,2) \otimes (x_2, y_2, z_2) = (1,8,21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

Now, with the help of Step 3 of an algorithm and by using the arithmetic operations which are defined in Definition 1.1.12 the above given FFLP problem will become,

$$\text{Maximize } (1x_1 + 2x_2, 2y_1 + 3y_2, 3z_1 + 4z_2),$$

subject to

$$(0x_1 + 0x_2, 1y_1 + 2y_2, 2z_1 + 3z_2) = (2,10,24),$$

$$(1z_1 + 0x_2, 2y_1 + y_2, 3z_1 + 2z_2) = (1,8,21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

using the concept of ranking function which is defined in definition 1.1.11, the above mentioned FFLP problem will become

$$\text{Maximize } \left(\frac{1}{4}(x_1 + 2x_2 + 4y_1 + 6y_2 + 3z_1 + 4z_2)\right)$$

subject to

$$(0x_1 + 0x_2, 1y_1 + 2y_2, 2z_1 + 3z_2) = (2,10,24),$$

$$(1z_1 + 0x_2, 2y_1 + y_2, 3z_1 + 2z_2) = (1,8,21),$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are non-negative triangular fuzzy numbers.

by using step 4 of an algorithm the given FFLP is converted into crisp linear programming problem

$$\text{Maximize } \left(\frac{1}{4}(x_1 + 2x_2 + 4y_1 + 6y_2 + 3z_1 + 4z_2)\right).$$

subject to

$$0x_1 + 0x_2 = 2,$$

$$y_1 + 2y_2 = 10,$$

$$2z_1 + 3z_2 = 24,$$

$$x_1 + 0x_2 = 1,$$

$$2y_1 + 3y_2 = 24,$$

$$3z_1 + 2z_2 = 21,$$

$y_1 - x_1 \geq 0, z_1 - y_1 \geq 0, y_2 - x_2 \geq 0, z_2 - y_2 \geq 0.$

This type of problem is not solvable since the constraint $0x_1 + 0x_2 = 2$, is not valid, therefore the given FFLP is not solvable.

**Remark 3.3.1** *Since the two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are equivalent if $\Re(\widetilde{A}) = \Re(\widetilde{B})$ thus if we replace the fuzzy number $(0, 2, 3)$ with the fuzzy number which is equivalent to this i.e. having same rank then the above FFLP is solvable.*

## 3.4 Concluding Remarks

In this chapter, it is shown that by using the proposed method in real life situations it is better to use for solving the FFLP problems than existing methods.

(i) Clearly in the Examples 1 and 2 which are shown above the values of $\widetilde{x_1}$ and $\widetilde{x_2}$ satisfies all the constraints. Also it is not possible to find the any non-negative triangular fuzzy number $\widetilde{x_1}$ and $\widetilde{x_2}$ which satisfies the following conditions:

(i) $\Re((1, 6, 9) \otimes \widetilde{x_1} + (2, 3, 8) \otimes \widetilde{x_2} > \Re(9, 27, 75)$.

(ii) $\Re((1, 2, 3) \otimes \widetilde{x_1} + (2, 3, 4) \otimes \widetilde{x_2} > \Re(5, 16, 33)$.

(ii) The method shown in [16] can be applied only to find the exact solution of a special type of FFLS of equations for which all the elements of the coefficient matrix are non-negative fuzzy numbers. The method is not applicable if elements of coefficient matrix are not non-negative.

(iii) The existing method [35] can be applied only for solving a special type of FFLP problems in which all the elements of the coefficient matrix are symmetric fuzzy numbers. To apply the existing method for solving a FFLP problem in which all the elements of coefficient matrix are not symmetric fuzzy numbers, firstly it is required to approximate the fuzzy numbers into nearest symmetric fuzzy numbers and due to this conversion the obtained solutions are not exact.

Thus, the method discussed in this chapter can be applied to solve these special type of FFLS and FFLP problems.

# Chapter 4

# Mehar's Method for solving FFLP problem

In this chapter, a method named as Mehar's method with Yager's ranking approach have been discussed for solving fully fuzzy linear programming problem having some or all the parameters as unrestricted L-R fuzzy numbers and L-R flat fuzzy numbers and the numerical example is shown.

## 4.1 Mathematical Formulation

General form of FFLP problem in which all the parameters are unrestricted L-R fuzzy numbers

(i)

$$\text{Maximize(or Minimize) } \widetilde{Z} = \sum_{j=1}^{n} (\widetilde{c_j} \odot \widetilde{x_j})$$

subject to

$$\widetilde{a_{ij}} \odot \widetilde{x_j} = \widetilde{b_i} \quad \forall\ i = 1, 2, \ldots, m,$$

where $\widetilde{a_{ij}}, \widetilde{b_i}$ and $\widetilde{c_j}$ are non- negative and non- positive L-R fuzzy number and $\widetilde{x_j}$ is non negative L-R fuzzy number.

(ii) Based on the product $\otimes$ i.e. when the spreads are smaller as compared to their means.

$$\text{Maximize(or Minimize) } \widetilde{Z} = \sum_{j=1}^{n} (\widetilde{c_j} \otimes \widetilde{x_j})$$

subject to

$$\widetilde{a_{ij}} \otimes \widetilde{x_j} = \widetilde{b_i} \quad \forall\ i = 1, 2, \ldots, m,$$

where $\widetilde{a_{ij}}, \widetilde{b_i}$ and $\widetilde{c_j}$ are non- negative or non- positive L-R fuzzy numbers and $\widetilde{x_j}$ is non negative L-R fuzzy number.

General form of FFLP problem in which some or all the parameters are unrestricted L-R flat fuzzy numbers

(iii)

$$\text{Maximize(or Minimize) } \widetilde{Z} = \sum_{j=1}^{n} (\widetilde{c_j} \odot \widetilde{x_j})$$

subject to

$$\widetilde{a_{ij}} \odot \widetilde{x_j} = \widetilde{b_i} \quad \forall\ i = 1, 2, \ldots, m,$$

where $\widetilde{a_{ij}}, \widetilde{b_i}$ and $\widetilde{c_j}$ and $\widetilde{x_j}$ are L-R flat fuzzy numbers.

(iv) Based on the product $\otimes$ i.e. when the spreads are smaller as compared to their means.

$$\text{Maximize(or Minimize) } \widetilde{Z} = \sum_{j=1}^{n} (\widetilde{c_j} \otimes \widetilde{x_j})$$

subject to

$$\widetilde{a_{ij}} \otimes \widetilde{x_j} = \widetilde{b_i} \quad \forall\ i = 1, 2, \ldots, m,$$

where $\widetilde{a_{ij}}, \widetilde{b_i}$ and $\widetilde{c_j}$ and $\widetilde{x_j}$ are L-R flat fuzzy numbers

## 4.2 Arithmetic operation of L-R flat fuzzy numbers

Following arithmetic operations are defined for two L-R flat fuzzy numbers $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$.

**1. Addition:** $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR} \oplus \widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR} = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$.

**2. Substraction:** Let $\widetilde{A_1} = (m_3, n_3, \alpha_3, \beta_3)_{RL}$ be any R-L flat fuzzy number.

$\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR} \ominus \widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR} = (m_1 - m_3, n_1 - n_3, \alpha_1 + \beta_3, \beta_1 + \alpha_3)_{LR}$.

**3. Product:** (i) If $\widetilde{A_1}$ and $\widetilde{A_2}$ both are non-negative, then

$\widetilde{A_1} \odot \widetilde{A_2} = (m_1 m_2, n_1 n_2, m_1\alpha_2 + \alpha_1 m_2 - \alpha_1\alpha_2, n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2)_{LR}$

(ii) If $\widetilde{A_1}$ is non-positive and $\widetilde{A_2}$ is non-negative, then

$\widetilde{A_1} \odot \widetilde{A_2} = (m_1 n_2, n_1 m_2, \alpha_1 n_2 - \beta_2 m_1 + \alpha_1\beta_2, \beta_1 m_2 - n_1\alpha_2 - \beta_1\alpha_2)_{LR}$

(iii) If $\widetilde{A_1}$ is non-negative and $\widetilde{A_2}$ is non-positive, then

$\widetilde{A_2} \odot \widetilde{A_2} = (n_1 m_2, m_1 n_2, n_1\alpha_2 - \beta_1 m_2 + \beta_1\alpha_2, m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2)_{LR}$

(iv) If $\widetilde{A_1}$ and $\widetilde{A_2}$ both are non-positive, then

$\widetilde{A_1} \odot \widetilde{A_2} = (n_1 n_2, m_1 m_2, -n_1\beta_2 - \beta_1 n_2 - \beta_1\beta_2, -m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2)_{LR}$

(v) Let $\lambda$ be any real number then

$$\lambda\widetilde{A} = \begin{cases} (\lambda m_1, \lambda n_1, \lambda\alpha_1, \lambda\beta_1)_{LR}, & \lambda \geq 0 \\ (\lambda n_1, \lambda m_1, -\lambda\beta_1, -\lambda\alpha_1)_{LR}, & \lambda < 0 \end{cases}$$

following is the product for L-R flat fuzzy numbers in which the spreads are smaller as compared to the mean values.

**4 Product:** (i) If $\widetilde{A_1}$ and $\widetilde{A_2}$ both are non-negative, then

$\widetilde{A_1} \otimes \widetilde{A_2} = (m_1 m_2, n_1 n_2, m_1\alpha_2 + \alpha_1 m_2, n_1\beta_2 + \beta_1 n_2)_{LR}$

(ii) If $\widetilde{A_1}$ is non-positive and $\widetilde{A_2}$ is non-negative, then

$\widetilde{A_1} \otimes \widetilde{A_2} = (m_1 n_2, n_1 m_2, \alpha_1 n_2 - \beta_2 m_1, \beta_1 m_2 - n_1\alpha_2)_{LR}$

(iii) If $\widetilde{A_1}$ is non-negative and $\widetilde{A_2}$ is non-positive, then

$\widetilde{A_2} \otimes \widetilde{A_2} = (n_1m_2, m_1n_2, n_1\alpha_2 - \beta_1m_2, m_1\beta_2 - \alpha_1n_2)_{LR}$

(iv) If $\widetilde{A_1}$ and $\widetilde{A_2}$ both are non-positive, then

$\widetilde{A_1} \otimes \widetilde{A_2} = (n_1n_2, m_1m_2, -n_1\beta_2 - \beta_1n_2, -m_1\alpha_2 - \alpha_1m_2)_{LR}$

with the above defined product following are the counter examples that are not solvable

**Example 1** Find the fuzzy optimal solution of the FFLP problem.

$$\text{Maximize } (4,4,0,0)_{LR} \odot \widetilde{x_1} \oplus (1,1,1,1)_{LR} \odot \widetilde{x_2}$$

subject to

$$(2,5,5,2)_{LR} \odot \widetilde{x_1} \oplus (-1,5,1,2)_{LR}\widetilde{x_2} \preceq (-17,45,25,46)_{LR}$$

$$(1,2,1,1)_{LR} \odot \widetilde{x_1} \oplus (3,5,2,2)_{LR}\widetilde{x_2} \preceq \left(11,39,\frac{193}{5},\frac{168}{5}\right)_{LR}$$

where, $\widetilde{x_1}, \widetilde{x_2}$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$.

**Example 2** Find the fuzzy optimal solution of the FFLP problem.

$$\text{Maximize}(1,1,1,1)_{LR} \otimes \widetilde{x_1} \oplus (4,4,0,0)_{LR} \otimes \widetilde{x_2}$$

subject to

$$(2,3,1,1)_{LR} \otimes \widetilde{x_1} \oplus (-3,-2,1,1)_{LR}\widetilde{x_2} \preceq (-27,-4,21,32)_{LR}$$

$$(1,2,1,1)_{LR} \otimes \widetilde{x_1} \oplus (3,5,2,2)_{LR}\widetilde{x_2} \preceq \left(11,39,\frac{193}{5},\frac{168}{5}\right)_{LR}$$

where, $\widetilde{x_1}, \widetilde{x_2}$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$

**Example 3**

The fuzzy optimal solution of the FFLP problem given below is solvable by proposed method.

$$\text{Maximize } (2,1,2)_{LR} \odot \widetilde{x_1} \oplus (-3,2,1)_{LR} \odot \widetilde{x_2}$$

subject to

$$(1,1,2)_{LR} \odot \widetilde{x_1} \oplus (2,1,3)_{LR}\widetilde{x_2} \preceq (20,10,5)_{LR}$$

$$(-2,1,1)_{LR} \odot \widetilde{x_1} \oplus (5,2,3)_{LR}\widetilde{x_2} \preceq (-1,3,2)_{LR}$$

where, $\widetilde{x_1}, \widetilde{x_2}$ are L-R fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$.

**Example 4**

The fuzzy optimal solution of the FFLP problem given below is solvable by proposed method.

$$\text{Maximize } (1,1,1)_{LR} \otimes \widetilde{x_1} \oplus (2,1,2)_{LR} \odot \widetilde{x_2}$$

subject to

$$(4,1,0)_{LR} \otimes \widetilde{x_1} \oplus (-2,1,1)_{LR}\widetilde{x_2} \preceq (5,2,3)_{LR}$$

$$(-3,1,2)_{LR} \otimes \widetilde{x_1} \oplus (4,1,2)_{LR}\widetilde{x_2} \preceq (4,1,1)_{LR}$$

where, $\widetilde{x_1}, \widetilde{x_2}$ are L-R fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$.

thus product has been modified and based on the proposed product all the above examples are solved in section 4.7

## 4.3 Modified Product for unrestricted L-R flat fuzzy numbers

In this section, results for modified product for unrestricted L-R flat fuzzy numbers are explained.

**Theorem 4.3.1** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 - \alpha_1 < 0$ *and* $m_1 > 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m_1', n_1', \alpha_1', \beta_1')_{LR}$ *where,* $m_1' = minimum\{m_1m_2, n_1m_2\}$, $n_1' = maximum\{m_1n_2, n_1n_2\}$, $\alpha_1' = minimum\{m_1m_2, n_1m_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 +$

$\beta_1 m_2 - \beta_1\alpha_2\}$, $\beta_1' = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\} - maximum\{m_1n_2, n_1n_2\}$.

***Proof:***

*Let* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *be two L-R flat fuzzy numbers such that* $m_1 - \alpha_1$ *and* $m_1 \geq 0$

*By using the definition* 1.1.22 *we get ,*

$A_{1\lambda} = [m_1 - \alpha_1 L^{-1}(\lambda), n_1 + \beta_1 R^{-1}(\lambda)]$ *and* $A_{2\lambda} = [m_2 - \alpha_2 L^{-1}(\lambda), n_2 + \beta_2 R^{-1}(\lambda)]$

*since,* $m_1 - \alpha_1 < 0$ *and* $m_1 \geq 0$ *so* $m_1 - \alpha_1 L^{-1}(\lambda) \geq 0$ *for* $\lambda \geq L\left(\dfrac{m_1}{\alpha_1}\right)$

*and* $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ *for* $\lambda \leq L\left(\dfrac{m_1}{\alpha_1}\right)$ *and* $n_1 + \beta_1 R^{-1}(\lambda) \geq 0$ *for all* $\lambda$

*the cases for finding the product of* $\widetilde{A_1}$ *and* $\widetilde{A_2}$ *:*

***Case 1:*** *If* $m_2 - \alpha_2 \geq 0$ *then* $m_2 - \alpha_2 L^{-1}(\lambda) \geq 0$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \geq 0 \forall \lambda$*, thus the following two subcases occur*

***(i)*** *If* $m_1 - \alpha_1 \geq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

*on putting* $\lambda = 1$ *we get,*

*since L(0) = 1 and R(0)=1*

$$A_{1\lambda}A_{2\lambda} = [m_1m_2, n_1n_2], \tag{4.3.1}$$

***(ii)*** *If* $m_2 - \alpha_2 \geq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

*on putting* $\lambda = 0$

*since* $L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(n_2 + \beta_2)]$ *then,*

$$A_{1\lambda}A_{2\lambda} = [m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2] \qquad (4.3.2)$$

*Thus by* (4.3.1) *and* (4.3.2) *we get,* $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_1'', n_1'', \alpha_1'', \beta_1'')_{LR}$

*where* $m_1'' = m_1m_2, n_1'' = n_1n_2, \alpha_1'' = m_1m_2 - m_1n_2 - m_1\beta_2 + \alpha_1n_2 + \alpha_1\beta_2, \beta_1'' = n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2 - n_1n_2$

***Case 2:***

*If* $m_2 - \alpha_2 < 0, m_2 \geq 0$ *then*

$m_2 - \alpha_2L^{-1}(\lambda) \geq 0$ *for* $\lambda \leq L\left(\frac{m_2}{\alpha_2}\right), m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *for* $\lambda \geq L\left(\frac{m_2}{\alpha_2}\right)$ *and* $n_2 + \beta_2R^{-1}(\lambda) \geq 0 \forall\lambda$

*thus four subcases may arise to find the product of* $A_{1\lambda}$ *and* $A_{2\lambda}$

*since we want to find the product of* $A_{1\lambda}$ *and* $A_{2\lambda}$ *corresponding to* $\lambda = 0$ *and* $\lambda = 1$, *so we consider only two subcases as follows*

***(i)****If* $m_1 - \alpha_1L^{-1}(\lambda) \leq 0$ *and* $m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [minimum\{[(m_1-\alpha_1L^{-1}(\lambda))(n_2+\beta_2R^{-1}(\lambda)), (n_1+\beta_1R^{-1}(\lambda))(n_2+\beta_2R^{-1}(\lambda))]\},$ $maximum\{(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))\}]$

*since on putting* $\lambda = 0, L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$$A_{1\lambda}A_{2\lambda} = [minimum\{(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(m_2 - \alpha_2)\}, maximum\{(m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2)\}] \qquad (4.3.3)$$

$A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\},$ $maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\}]$

***(ii)****If* $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ *and* $m_2 - \alpha_2L^{-1}(\lambda) \geq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))]$

*on putting* $\lambda = 1$ *we get,*

*since* $L(0) = 1$ *and* $R(0) = 1$

$$A_{1\lambda}A_{2\lambda} = [m_1m_2, n_1n_2] \tag{4.3.4}$$

*now, by using* (4.3.3) *and* (4.3.4) *we get* , $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_2'', n_2'', \alpha_2'', \beta_2'')_{LR}$

*where,* $m_2'' = m_1m_2, n_2'' = n_1n_2, \alpha_2'' = m_1m_2 - minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_2'' = maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\} - n_1n_2$.

***Case 3:***

*If* $m_2 < 0, n_2 \geq 0$ *then*

$m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2R^{-1}(\lambda) \geq 0 \forall\lambda$

*thus the following two cases occur:*

***(i)*** *If* $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ *and* $m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))]$

*on putting* $\lambda = 1$

*since* $L^{-1}(1) = 0$ *and* $R^{-1}(1) = 0$

$$A_{1\lambda}A_{2\lambda} = [n_1m_2, n_1n_2] \tag{4.3.5}$$

***(ii)*** *If* $m_1 - \alpha_1L^{-1}(\lambda) \leq 0$ *and* $m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [minimum\{[(m_1 - \alpha_1L^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda))]\}, maximum\{(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))\}]$

*since on putting* $\lambda = 0, L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [minimum\{(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(m_2 - \alpha_2)\}, maximum\{(m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2)\}]$

$A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\},$

$maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\}]$

$$(4.3.6)$$

*by using* (4.3.5) *and* (4.3.6) *we get,*

$\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_3'', n_3'', \alpha_3'', \beta_3'')_{LR}$

*where* $m_3'' = n_1m_2, n_3'' = n_1n_2, \alpha_3'' = n_1m_2 - minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_3'' = maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\} - n_1n_2$

***Case 4:***

*If* $n_2 < 0, n_2 + \beta_2 \geq 0$ *then*

$m_2 - \alpha_2L^{-1}(\lambda) \leq 0 \ \forall \ \lambda$ *and* $n_2 + \beta_2R^{-1}(\lambda) \leq 0$ *for* $\lambda \leq R\left(\frac{-n_2}{\beta_2}\right)$ *and* $n_2 + \beta_2R^{-1}(\lambda) \geq 0$ *for* $\lambda \geq R\left(\frac{-n_2}{\beta_2}\right)$ *so four cases occur for finding the product of* $A_{1\lambda}$ *and* $A_{2\lambda}$

*since we want to find the product of* $A_{1\lambda}$ *and* $A_{2\lambda}$ *corresponding to* $\lambda = 0$ *and* $\lambda = 1$ *so we consider only the following two cases:*

***(i)*** *If* $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ *and* $n_2 + \beta_2R^{-1}(\lambda) \leq 0$ *then,*

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (m_1 - \alpha_1L^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))]$

*on putting* $\lambda = 1$

$R^{-1}(1) = 0$ *and* $L^{-1}(1) = 0$

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1)(m_2 - \alpha_2), (m_1 - \alpha_1)(n_2 + \beta_2)]$

$$A_{1\lambda}A_{2\lambda} = [n_1m_2, m_1n_2] \tag{4.3.7}$$

***(ii)*** *If* $m_1 - \alpha_1L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2R^{-1}(\lambda) \geq 0$ *then,*

$A_{1\lambda}A_{2\lambda} = [minimum\{[(m_1 - \alpha_1L^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda))]\}, maximum\{(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 +$

$\beta_2 R^{-1}(\lambda))\}]$

*since on putting* $\lambda = 0, L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [minimum\{(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(m_2 - \alpha_2)\}, maximum\{(m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2)\}]$

$A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\},$
$maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\}]$

(4.3.8)

*by using* (4.3.7) *and* (4.3.8) *we get,*

$\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_4'', n_4'', \alpha_4'', \beta_4'')_{LR}$

*where* $m_4'' = n_1m_2, n_4'' = m_1n_2, \alpha_4'' = n_1m_2 - minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_4'' = maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\} - m_1n_2$

***Case 5:*** *If* $n_2 + \beta_2 < 0$ *then*

$m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2R^{-1}(\lambda) \leq 0$ *for all* $\lambda$ *thus the two subcases occur as follows:*

***(i)*** *If* $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (m_1 - \alpha_1L^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))]$

*on putting* $\lambda = 1$

$R^{-1}(1) = 0$ *and* $L^{-1}(1) = 0$

$$A_{1\lambda}A_{2\lambda} = [n_1m_2, m_1n_2] \tag{4.3.9}$$

***(ii)****If* $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda))]$

*on putting* $\lambda = 0$

$R^{-1}(0) = 1$ *and* $L^{-1}(0) = 1$ *by definition 1.1.13*

$A_{1\lambda}A_{2\lambda} = [(n_1+\beta_1)(m_2-\alpha_2), (m_1-\alpha_1)(m_2-\alpha_2)]$ $A_{1\lambda}A_{2\lambda} = [n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2, m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2]$

$$(4.3.10)$$

*therefore, by* (4.3.9) *and* (4.3.10) *we get,* $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_5'', n_5'', \alpha_5'', \beta_5'')_{LR}$

*where,* $m_5'' = n_1m_2, n_5'' = m_1n_2, \alpha_5'' = n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2, \beta_5'' = m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2$

*Thus Combining the results of all the above five cases we get,*

*If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 - \alpha_1 < 0$ *and* $m_1 > 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m_1', n_1', \alpha_1', \beta_1')_{LR}$ *where,* $m_1' = minimum\{m_1m_2, n_1m_2\}$, $n_1' = maximum\{m_1n_2, n_1n_2\}$, $\alpha_1' = minimum\{m_1m_2, n_1m_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}$, $\beta_1' = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - maximum\{m_1n_2, n_1n_2\}$.

**Theorem 4.3.2** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 < 0$ *and* $n_1 \geq 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m_2', n_2', \alpha_2', \beta_2')_{LR}$ *where,* $m_2' = minimum\{m_1n_2, n_1m_2\}$, $n_2' = maximum\{m_1m_2, n_1n_2\}$, $\alpha_2' = minimum\{m_1n_2, n_1m_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}$, $\beta_1' = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\} - maximum\{m_1m_2, n_1n_2\}$.

***proof:***

*Let* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *be two L-R flat fuzzy numbers such that* $m_1 - \alpha_1$ *and* $m_1 \geq 0$

*By using the definition* 1.1.22 *we get ,*
$A_{1\lambda} = [m_1 - \alpha_1L^{-1}(\lambda), n_1 + \beta_1R^{-1}(\lambda)]$ *and* $A_{2\lambda} = [m_2 - \alpha_2L^{-1}(\lambda), n_2 + \beta_2R^{-1}(\lambda)]$

*since,* $m_1 < 0$ *and* $n_1 \geq 0$ *so* $m_1 - \alpha_1L^{-1}(\lambda) \leq 0$ *for all* $\lambda$ *and* $n_1 + \beta_1R^{-1}(\lambda) \geq 0$ *for all* $\lambda$

*the cases for finding the product of* $\widetilde{A_1}$ *and* $\widetilde{A_2}$ *:*

***Case 1:*** *If* $m_2 - \alpha_2 \geq 0$ *then* $m_2 - \alpha_2 L^{-1}(\lambda) \geq 0$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \geq 0 \forall \lambda$, *thus the following two subcases occur*

***(i)*** *If* $m_1 - \alpha_1 \leq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

*on putting* $\lambda = 1$ *we get,*

*since* $L(0) = 1$ *and* $R(0) = 1$

$$A_{1\lambda}A_{2\lambda} = [m_1 n_2, n_1 n_2], \tag{4.3.11}$$

***(ii)*** $A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

*on putting* $\lambda = 0$

$R^{-1}(0) = 1$ *and* $L^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(n_2 + \beta_2)]$

$$A_{1\lambda}A_{2\lambda} = [m_1 n_2 - n_1\alpha_1 + \beta_2 m_1 - \beta_2\alpha_1, n_1 n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2] \tag{4.3.12}$$

*therefore, by* (4.3.11) *and* (4.3.12) *we get,* $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_1', n_1', \alpha_1', \beta_1')_{LR}$

*where,* $m_1' = m_1 n_2, n_1' = n_1 n_2, \alpha_1' = m_1 n_2 - n_2\alpha_1 + \beta_2 m_1 - \alpha_1\beta_2, \beta_1' = n_1 n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2$

***Case 2:***

*If* $m_2 - \alpha_2 < 0, m_2 \geq 0$ *then*

$m_2 - \alpha_2 L^{-1}(\lambda) \geq 0$ *for* $\lambda \leq L\left(\dfrac{m_2}{\alpha_2}\right), m_2 - \alpha_2 L^{-1}(\lambda) \leq 0$ *for* $\lambda \geq L\left(\dfrac{m_2}{\alpha_2}\right)$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \geq 0 \forall \lambda$

*thus two subcases may arise to find the product of* $A_{1\lambda}$ *and* $A_{2\lambda}$

*so we consider the two subcases as follows*

***(i)****If* $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ *and* $m_2 - \alpha_2 L^{-1}(\lambda) \leq 0$ *for* $\lambda \leq L\left(\dfrac{m_2}{\alpha_2}\right)$ *then*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))]$

*on putting* $\lambda = 1$

*since* $L^{-1}(1) = 0$ *and* $R^{-1}(1) = 0$

$$A_{1\lambda}A_{2\lambda} = [m_1(n_2, n_1 n_2] \tag{4.3.13}$$

***(ii)*** *If* $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ *and* $m_2 - \alpha_2 L^{-1}(\lambda) \geq 0$ *for* $\lambda \geq L\left(\frac{m_2}{\alpha_2}\right)$ *then*

$A_{1\lambda}A_{2\lambda} = [minimum\{[(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))]\}, maximum\{(m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))\}]$

*since on putting* $\lambda = 0, L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [minimum\{(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(m_2 - \alpha_2)\}, maximum\{(m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2)\}]$

$$A_{1\lambda}A_{2\lambda} = [minimum\{m_1 n_2 - \alpha_1 n_2 + \beta_2 m_1 - \alpha_1 \beta_2, n_1 m_2 - n_1 \alpha_2 + \beta_1 m_2 - \beta_1 \alpha_2\}, maximum\{m_1 n_2 + m_1 \beta_2 - \alpha_1 n_2 - \alpha_1 \beta_2, n_1 n_2 + \beta_1 n_2 + n_1 \beta_2 + \beta_1 \beta_2\}] \tag{4.3.14}$$

*now, by using* (4.3.13) *and* (4.3.14) *we get* , $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_2'', n_2'', \alpha_2'', \beta_2'')_{LR}$

*where,* $m_2'' = m_1 n_2, n_2'' = n_1 n_2, \alpha_2'' = m_1 n_2 - minimum\{m_1 n_2 - \alpha_1 n_2 + \beta_2 m_1 - \alpha_1 \beta_2, n_1 m_2 - n_1 \alpha_2 + \beta_1 m_2 - \beta_1 \alpha_2\}, \beta_2'' = maximum\{m_1 n_2 + m_1 \beta_2 - \alpha_1 n_2 - \alpha_1 \beta_2, n_1 n_2 + \beta_1 n_2 + n_1 \beta_2 + \beta_1 \beta_2\} - n_1 n_2$.

***Case 3:***

*If* $m_2 < 0, n_2 \geq 0$ *then*

$m_2 - \alpha_2 L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \geq 0 \forall \lambda$

*thus the following two cases occur:*

***(i)*** *If* $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ *and* $m_2 - \alpha_2 L^{-1}(\lambda) \leq 0$ *for all* $\lambda$ *then*

$A_{1\lambda}A_{2\lambda} = [minimum\{[(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))]\}, maximum\{(m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (n_1 + \beta_1 R^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda))\}]$

*since on putting* $\lambda = 0, L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [minimum\{(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(m_2 - \alpha_2)\}, maximum\{(m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2)\}]$

$$A_{1\lambda}A_{2\lambda} = [minimum\{m_1 n_2 - \alpha_1 n_2 + \beta_2 m_1 - \alpha_1 \beta_2, n_1 m_2 - n_1 \alpha_2 + \beta_1 m_2 - \beta_1 \alpha_2\}, maximum\{m_1 n_2 + m_1 \beta_2 - \alpha_1 n_2 - \alpha_1 \beta_2, n_1 n_2 + \beta_1 n_2 + n_1 \beta_2 + \beta_1 \beta_2\}] \tag{4.3.15}$$

***(ii)*** *for* $\lambda = 1, L^{-1}(1) = 0$ *and* $R^{-1}(1) = 0$

$$A_{1\lambda}A_{2\lambda} = [minimum\{m_1 n_2, n_1 m_2\}, maximum\{m_1 n_2, n_1 n_2\}] \tag{4.3.16}$$

*now, by using* (4.3.15) *and* (4.3.16) *we get* , $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m''_3, n''_3, \alpha''_3, \beta''_3)_{LR}$

*where,* $m''_3 = minimum\{m_1 n_2, n_1 m_2\}, n''_3 = maximum\{m_1 n_2, n_1 n_2\}$,

$\alpha''_3 = minimum\{m_1 n_2, n_1 m_2\} - minimum\{m_1 n_2 - \alpha_1 n_2 + \beta_2 m_1 - \alpha_1 \beta_2, n_1 m_2 - n_1 \alpha_2 + \beta_1 m_2 - \beta_1 \alpha_2\}, \beta''_3 = maximum\{m_1 n_2 + m_1 \beta_2 - \alpha_1 n_2 - \alpha_1 \beta_2, n_1 n_2 + \beta_1 n_2 + n_1 \beta_2 + \beta_1 \beta_2\} - maximum\{m_1 n_2, n_1 n_2\}$.

***Case 4:***

*If* $n_2 < 0, n_2 + \beta_2 \geq 0$ *then*

$m_2 - \alpha_2 L^{-1}(\lambda) \leq 0 \forall \lambda$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \leq 0$ *for* $\lambda \leq R\left(\frac{-n_2}{\beta_2}\right)$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \geq 0$ *for* $\lambda \geq R\left(\frac{-n_2}{\beta_2}\right)$ *so four cases occur for finding the product of* $A_{1\lambda}$ *and* $A_{2\lambda}$

***(i)*** *If* $m_1 - \alpha_1 L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2 R^{-1}(\lambda) \leq 0$ *then,*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1 L^{-1}(\lambda))(n_2 + \beta_2 R^{-1}(\lambda)), (m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))]$

*on putting* $\lambda = 1$

$R^{-1}(1) = 0$ *and* $L^{-1}(1) = 0$ *by definition 1.1.13*

$A_{1\lambda}A_{2\lambda} = [(m_1 - \alpha_1)(n_2 + \beta_2), (m_1 - \alpha_1)(m_2 - \alpha_2)]$

$$A_{1\lambda}A_{2\lambda} = [m_1n_2, m_1m_2] \tag{4.3.17}$$

***(ii)****If* $m_1 - \alpha_1L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2R^{-1}(\lambda) \geq 0$ *then,*

$A_{1\lambda}A_{2\lambda} = [minimum\{[(m_1 - \alpha_1L^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda))]\}, maximum\{(m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (n_1 + \beta_1R^{-1}(\lambda))(n_2 + \beta_2R^{-1}(\lambda))\}]$

*since on putting* $\lambda = 0, L^{-1}(0) = 1$ *and* $R^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [minimum\{(m_1 - \alpha_1)(n_2 + \beta_2), (n_1 + \beta_1)(m_2 - \alpha_2)\}, maximum\{(m_1 - \alpha_1)(m_2 - \alpha_2), (n_1 + \beta_1)(n_2 + \beta_2)\}]$

$$A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\}] \tag{4.3.18}$$

*by using* (4.3.17) *and* (4.3.18)*, we get*

$\widetilde{A_1} \odot \widetilde{A_2} \simeq (m_4'', n_4'', \alpha_4'', \beta_4'')_{LR}$

*where,* $m_4'' = m_1n_2, n_4'' = m_1m_2, \alpha_4'' = m_1n_2 - minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1 - \alpha_1\beta_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}, \beta_4'' = maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + \beta_1n_2 + n_1\beta_2 + \beta_1\beta_2\} - m_1m_2.$

***Case 5:*** *If* $n_2 + \beta_2 < 0$ *then*

$m_2 - \alpha_2L^{-1}(\lambda) \leq 0$ *and* $n_2 + \beta_2R^{-1}(\lambda) \leq 0$ *for all* $\lambda$ *thus the two subcases occur as follows:*

***(i)*** *If* $m_1 - \alpha_1L^{-1}(\lambda) \geq 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1R^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda)), (m_1 - \alpha_1L^{-1}(\lambda))(m_2 - \alpha_2L^{-1}(\lambda))]$

*on putting* $\lambda = 1$

$R^{-1}(1) = 0$ *and* $L^{-1}(1) = 0$

$$A_{1\lambda}A_{2\lambda} = [n_1 m_2, m_1 m_2] \tag{4.3.19}$$

***(ii)****Now, for* $\lambda = 0$ *then*

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1 R^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda)), (m_1 - \alpha_1 L^{-1}(\lambda))(m_2 - \alpha_2 L^{-1}(\lambda))]$

*on putting* $\lambda = 0$

$R^{-1}(0) = 1$ *and* $L^{-1}(0) = 1$

$A_{1\lambda}A_{2\lambda} = [(n_1 + \beta_1)(m_2 - \alpha_2), (m_1 - \alpha_1)(m_2 - \alpha_2)]$

$$A_{1\lambda}A_{2\lambda} = [n_1 m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2, m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2] \tag{4.3.20}$$

*therefore, by* (4.3.19) *and* (4.3.20) *we get,* $\widetilde{A_1} \odot \widetilde{A_2} \simeq (m''_5, n''_5, \alpha''_5, \beta''_5)_{LR}$

*where,* $m''_5 = n_1 m_2, n''_5 = m_1 m_2, \alpha''_5 = n_1 m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2, \beta''_5 = m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2 - m_1 m_2$

*Thus Combining the results of all the above five cases we get,*

$\widetilde{A_1} \odot \widetilde{A_2} \approx (m'_2, n'_2, \alpha'_2, \beta'_2)_{LR}$ *where,* $m'_2 = minimum\{m_1 n_2, n_1 m_2\}$,

$n'_2 = maximum\{m_1 m_2, n_1 n_2\}, \alpha'_2 = minimum\{m_1 n_2, n_1 m_2\} - minimum\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2, n_1 m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2\}, \beta'_1 = maximum\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\} - maximum\{m_1 m_2, n_1 n_2\}$.

**Theorem 4.3.3** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $n_1 < 0$ *and* $n_1 + \beta_1 \geq 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m'_3, n'_3, \alpha'_3, \beta'_3)_{LR}$ *where,* $m'_3 = minimum\{m_1 n_2, n_1 n_2\}$, $n'_3 = maximum\{n_1 m_2, m_1 m_2\}$, $\alpha'_3 = minimum\{m_1 n_2, n_1 n_2\} - minimum\{m_1 n_2 + m_1\beta_2 - \alpha_1 n_2 - \alpha_1\beta_2, n_1 m_2 - n_1\alpha_2 + \beta_1 m_2 - \beta_1\alpha_2\}$, $\beta'_3 = maximum\{m_1 m_2 - m_1\alpha_2 - \alpha_1 m_2 + \alpha_1\alpha_2, n_1 n_2 + n_1\beta_2 + \beta_1 n_2 + \beta_1\beta_2\} - maximum\{n_1 m_2, m_1 m_2\}$.

***proof:*** *is similar to that of Theorem* 4.3.1 *and Theorem* 4.3.2.

**Theorem 4.3.4** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $n_1 + \beta_1 < 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m'_4, n'_4, \alpha'_4, \beta'_4)_{LR}$ *where,* $m'_4 = minimum\{m_1n_2, n_1n_2\}$, $n'_4 = maximum\{m_1m_2, n_1m_2\}$, $\alpha'_4 = minimum\{m_1n_2, n_1n_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 + n_1\beta_2 + \beta_1n_2 + \beta_1\beta_2\}$, $\beta'_4 = maximum\{n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2, m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2\} - maximum\{m_1m_2, n_1m_2\}$.

***Proof:*** *is similar to that of Theorem* 4.3.1 *and Theorem* 4.3.2.

**Theorem 4.3.5** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 - \alpha_1 \geq 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m'_5, n'_5, \alpha'_5, \beta'_5)_{LR}$ *where,* $m'_5 = minimum\{m_1m_2, n_1m_2\}$, $n'_5 = maximum\{m_1n_2, n_1n_2\}$, $\alpha'_5 = minimum\{m_1m_2, n_1m_2\} - minimum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2 + \alpha_1\alpha_2,\ n_1m_2 - n_1\alpha_2 + \beta_1m_2 - \beta_1\alpha_2\}$, $\beta'_5 = maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2 - \alpha_1\beta_2, n_1n_2 - n_1\beta_2 - \beta_1n_2 + \beta_1\beta_2\} - maximum\{m_1n_2, n_1n_2\}$.

***Proof:*** *is similar to that of Theorem* 4.3.1 *and* 4.3.2.

**Product to find $\widetilde{A_1} \otimes \widetilde{A_2}$**

**Theorem 4.3.6** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 - \alpha_1 \geq 0$ *then* $\widetilde{A_1} \otimes \widetilde{A_2} \approx (m'_1, n'_1, \alpha'_1, \beta'_1)_{LR}$ *where,* $m'_1 = minimum\{m_1m_2, n_1m_2\}$, $n'_1 = maximum\{m_1n_2, n_1n_2\}$, $\alpha'_1 = minimum\{m_1m_2, n_1m_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}$, $\beta'_1 = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\} - maximum\{m_1n_2, n_1n_2\}$.

***Proof:***

***Case (i)*** *Neglecting the terms* $\alpha_1\beta_2$ *and* $\beta_1\beta_2$ *from the results obtained in case(1) of Theorem* 4.3.1, *we get* $\widetilde{A_{1\lambda}A_{2\lambda}} = [m_1, m_2, n_1, n_2]$ *for* $\lambda = 1$ *and* $\widetilde{A_{1\lambda}A_{2\lambda}} = [m_1n_1 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + n_1\beta_2 + \beta_1n_2]$ *for* $\lambda = 0$. *Combining the both, we get*

$A_{1\lambda} \otimes A_{2\lambda} \simeq (m''_1, n''_1, \alpha''_1, \beta''_1)_{LR}$

*where,* $m''_1 = m_1m_2, n''_1 = n_1n_2, \alpha''_1 = m_1m_2 - m_1n_2 + m_1\beta_2 - \alpha_1n_2, \beta''_1 = n_1n_2 + n_1\beta_2 + \beta_1n_2 - n_1n_2.$

***Case (ii)*** *Neglecting the terms* $\alpha_1\beta_2,\ \beta_1\alpha_2,\ \alpha_1\alpha_2$ *and* $\beta_1\beta_2$ *from the results obtained in case (ii) of Theorem* 4.3.1, *we get*

$A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + \beta_1n_2 + n_1\beta_2\}] for \lambda = 0$ *and* $A_{1\lambda}A_{2\lambda} = [m_1m_2, n_1n_2]$ *for* $\lambda = 1$.

*Combining the both,we get*

$\widetilde{A_1} \otimes \widetilde{A_2} \approx (m''_2, n''_2, \alpha''_2, \beta''_2)_{LR}$

*where,* $m''_2 = m_1m_2, n''_2 = n_1n_2, \alpha''_2 = m_1m_2 - minimum\{m_1n_2 - \alpha_1n_2 + \beta_2m_1, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}, \beta''_2 = maximize\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + \beta_1n_2 + n_1\beta_2\} - n_1n_2.$

***Case (iii)*** *Neglecting the terms* $\alpha_1\beta_2,\ \beta_1\alpha_2,\ \alpha_1\alpha_2$ *and* $\beta_1\beta_2$ *from the results obtained in case 3 of Theorem* 4.3.1, *we get*

$A_{1\lambda}A_{2\lambda} = [n_1m_2, n_1n_2]$ *for* $\lambda = 1$ *and* $A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 + \beta_1m_2 - n_1\alpha_2\}, maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta 1n_2\}$ *for* $\lambda = 0$, *Combining the both, we get*

$\widetilde{A_1} \otimes \widetilde{A_2} \approx (m''_3, n''_3, \alpha''_3, \beta''_3)_{LR}$

*where,* $m''_3 = n_1m_2, n''_3 = n_1n_2, \alpha''_3 = n_1m_2 - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 + \beta_1m_2 - n_1\alpha_2\}, \beta''_3 = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta 1n_2\} - n_1n_2$

***Case (iv)*** *Neglecting the terms* $\alpha_1\beta_2,\ \beta_1\alpha_2,\ \alpha_1\alpha_2$ *and* $\beta_1\beta_2$ *from the results obtained in case 4 of Theorem* 4.3.1, *we get*

$A_{1\lambda}A_{2\lambda} = [n_1m_2, m_1n_2]$ *for* $\lambda = 1$ *and* $A_{1\lambda}A_{2\lambda} = [minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 + \beta_1m_2 - n_1\alpha_2\}, maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1n_2 + n_1\beta_2 + \beta 1n_2\}$ *for* $\lambda = 0$, *Combining the both, we get*

$\widetilde{A_1} \otimes \widetilde{A_2} \approx (m''_4, n''_4, \alpha''_4, \beta''_4)_{LR}$

*where,* $m''_4 = n_1m_2, n''_4 = m_1n_2, \alpha''_4 = n_1m_2 - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1m_2 +$

$\beta_1 m_2 - n_1\alpha_2\}, \beta''_4 = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1n_2 + n_1\beta_2 + \beta 1 n_2\} - m_1n_2$

***case (v)*** *Neglecting the terms* $\beta_1\alpha_2$ *and* $\alpha_1\alpha_2$ *from the results obtained in case 4 of Theorem* 4.3.1, *we get*

$A_{1\lambda}A_{2\lambda} = [n_1m_2, m_1n_2]$ *for* $\lambda = 1$ *and* $A_{1\lambda}A_{2\lambda} = [n_1m_2 - n_1\alpha_2 + \beta_1 m_2, m_1m_2 - m_1\alpha_2 - \alpha_1 m_2]$ *for* $\lambda = 0$. *Combining the both, we get*

$\widetilde{A_1} \otimes \widetilde{A_2} \approx (m''_5, n''_5, \alpha''_5, \beta''_5)_{LR}$

*where,*$m''_5 = n_1m_2, n''_5 = m_1n_2, \alpha''_5 = n_1m_2 - n_1\alpha_2 + \beta_1 m_2, \beta 5'' = m_1m_2 - m_1\alpha_2 - \alpha_1 m_2$.

*Combining the results of all five cases the following result is obtained:*

*where,* $m'_1 = minimum\{m_1m_2, n_1m_2\}$, $n'_1 = maximum\{m_1n_2, n_1n_2\}$, $\alpha'_1 = minimum\{m_1m_2, n_1m_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1 n_2, n_1m_2 - n_1\alpha_2 + \beta_1 m_2\}$, $\beta'_1 = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1n_2 + n_1\beta_2 + \beta_1 n_2\} - maximum\{m_1n_2, n_1n_2\}$.

**Theorem 4.3.7** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 < 0$ *and* $n_1 \geq 0$ *then* $\widetilde{A_1} \otimes \widetilde{A_2} \approx (m'_2, n'_2, \alpha'_2, \beta'_2)_{LR}$ *where,* $m'_2 = minimum\{m_1n_2, n_1m_2\}$, $n'_2 = maximum\{m_1m_2, n_1n_2\}$, $\alpha'_2 = minimum\{m_1n_2, n_1m_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1 n_2, n_1m_2 - n_1\alpha_2 + \beta_1 m_2\}$, $\beta'_1 = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1n_2 + n_1\beta_2 + \beta_1 n_2\} - maximum\{m_1m_2, n_1n_2\}$.

***Proof:*** *is similar to that of Theorem* 4.3.6.

**Theorem 4.3.8** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $n_1 < 0$ *and* $n_1 + \beta_1 \geq 0$ *then* $\widetilde{A_1} \otimes \widetilde{A_2} \approx (m'_3, n'_3, \alpha'_3, \beta'_3)_{LR}$ *where,* $m'_3 = minimum\{m_1n_2, n_1n_2\}$, $n'_3 = maximum\{n_1m_2, m_1m_2\}$, $\alpha'_3 = minimum\{m_1n_2, n_1n_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1 n_2, n_1m_2 - n_1\alpha_2 + \beta_1 m_2\}$ ,$\beta'_3 = maximum\{m_1m_2 - m_1\alpha_2 - \alpha_1 m_2, n_1n_2 + n_1\beta_2 + \beta_1 n_2\} - maximum\{n_1m_2, m_1m_2\}$.

***Proof:*** *is similar to that of Theorem* 4.3.6.

**Theorem 4.3.9** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $n_1 + \beta_1 < 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m'_4, n'_4, \alpha'_4, \beta'_4)_{LR}$ *where,* $m'_4 = minimum\{m_1n_2, n_1n_2\}$, $n'_4 = maximum\{m_1m_2, n_1m_2\}$, $\alpha'_4 =$

$minimum\{m_1n_2, n_1n_2\} - minimum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 + n_1\beta_2 + \beta_1n_2\}$, $\beta'_4 = maximum\{n_1m_2 - n_1\alpha_2 + \beta_1m_2, m_1m_2 - m_1\alpha_2 - \alpha_1m_2\} - maximum\{m_1m_2, n_1m_2\}$.

***Proof:*** *is similar to that of Theorem* 4.3.6.

**Theorem 4.3.10** *If* $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *are two L-R flat fuzzy numbers such that* $m_1 - \alpha_1 \geq 0$ *then* $\widetilde{A_1} \odot \widetilde{A_2} \approx (m'_5, n'_5, \alpha'_5, \beta'_5)_{LR}$ *where,* $m'_5 = minimum\{m_1m_2, n_1m_2\}$, $n'_5 = maximum\{m_1n_2, n_1n_2\}$, $\alpha'_5 = minimum\{m_1m_2, n_1m_2\} - minimum\{m_1m_2 - m_1\alpha_2 - \alpha_1m_2, n_1m_2 - n_1\alpha_2 + \beta_1m_2\}$, $\beta'_5 = maximum\{m_1n_2 + m_1\beta_2 - \alpha_1n_2, n_1n_2 - n_1\beta_2 - \beta_1n_2\} - maximum\{m_1n_2, n_1n_2\}$.

***Proof:*** *is similar to that of Theorem* 4.3.6.

The product for L-R fuzzy numbers $\widetilde{A_1} = (m_1, \alpha_1, \beta_1)$ and $\widetilde{A_2} = (m_1, \alpha_1, \beta_1)$ is obtained by putting $m_1 = n_1, m_2 = n_2$ in all the above theorems.

**Remark 4.3.11** *If* $m = n$ *and* $L(x) = R(x) = maximize(0, 1-x)$ *then an L-R flat fuzzy number* $(m, n, \alpha, \beta)_{LR}$ *is said to be a triangular fuzzy number and is denoted as* $(m, \alpha, \beta)$.

**Remark 4.3.12** *If* $m \neq n$ *and* $L(x) = R(x) = maximize(0, 1-x)$ *then an L-R flat fuzzy number* $(m, n, \alpha, \beta)_{LR}$ *is said to be a trapezoidal fuzzy number and is denoted as* $(m, n, \alpha, \beta)$.

Therefore, the product of triangular and trapezoidal fuzzy numbers can be found by putting $L(x) = R(x) = maximize(0, 1-x)$ in the product which is given for L-R fuzzy numbers or L-R flat fuzzy numbers.

## 4.4 Yager's Ranking Approach

For comparing the fuzzy numbers their are many ranking approaches have been proposed. A relatively simple computational and easily understandable ranking approach is proposed by Yager[43] and in this chapter, yager's Ranking is used in the

Mehar's method for solving FFLP problems.Yager[43] proposed a procedure for ordering fuzzy sets in which a ranking index $\Re(\widetilde{A})$ is calculated for an LR flat fuzzy number. Let $\widetilde{A_1} = (m, n, \alpha, \beta)_{LR}$ be any L-R flat fuzzy number then

$$\Re(m, n, \alpha, \beta) = \frac{1}{2}\left(\int_0^1 (m - \alpha L^{-1}(\lambda))d\lambda + \int_0^1 (n + \beta R^{-1}(\lambda))d\lambda\right), \quad 0 \leqslant \lambda \leqslant 1. \tag{4.4.1}$$

**Lemma 4.4.1** *If* $L(x) = R(x) = maximize(0, 1-x)$ *then* $\Re(\widetilde{A}) = \left(m + n - \frac{\alpha}{2} + \frac{\beta}{2}\right)$

***Proof:*** *Since* $L(x) = R(x) = maximize(0, 1 - x)$, *thus* $L^{-1}(\lambda) = R^{-1}(\lambda) = 1 - \lambda$ *where* $\lambda \in [0, 1]$.

*on substituting the values of* $L^{-1}(\lambda)$ *and* $R^{-1}(\lambda)$ *in* (4.3.1) *we get*

$\Re(m, n, \alpha, \beta) = \frac{1}{2}\left(\int_0^1 (m - \alpha(1 - \lambda))d\lambda + \int_0^1 (n + \beta(1 - \lambda))d\lambda\right) = \frac{1}{2}\left(m + n - \frac{\alpha}{2} + \frac{\beta}{2}\right).$

## 4.5 Algorithm for Mehar's Method

The following steps for finding the optimal solution fully fuzzy number:

**Step 1:** Let us consider $\widetilde{a_{ij}} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}, \widetilde{x_j} = (x_j, y_j, \alpha''_j, \beta''_j)_{LR}, \widetilde{b_i} = (b_i, g_i, \gamma_i, \delta_i)_{LR}$ and $\widetilde{c_j} = (p_j, q_j, \alpha'_j, \beta'_j)_{LR}$ then the fully fuzzy programming problem is written as

$$\text{Maximize(or Minimize)} \sum_{j=1}^{n} ((p_j, q_j, \alpha'_j, \beta'_j)_{LR} \odot (x_j, y_j, \alpha''_j, \beta''_j)_{LR})$$

subject to

$$\sum_{j=1}^{n} ((a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}) \odot (x_j, y_j, \alpha''_j, \beta''_j)_{LR}) \preceq, \approx, \succeq (b_i, g_i, \gamma_i, \delta_i)_{LR},$$

$i = 1, 2, \ldots, m$ and where, $(x_j, y_j, \alpha''_j, \beta''_j)_{LR}$ is a L-R flat fuzzy number.

**Step 2:** now, let us consider $(p_j, q_j, \alpha'_j, \beta'_j)_{LR} \odot (x_j, y_j, \alpha''_j, \beta''_j)_{LR} \simeq (s_j, t_j, \alpha''''_j, \beta''''_j)_{LR}$ and $(a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR} \odot (x_j, y_j, \alpha''_j, \beta''_j)_{LR} \simeq (m_{ij}, n_{ij}, \gamma'_{ij}, \delta'_{ij})_{LR}$ then the fully fuzzy linear programming problem which is obtained in step 1 is written as:

$$\text{Maximize(or Minimize)} \sum_{j=1}^{n}((s_j, t_j, \alpha''''_j, \beta''''_j)_{LR})$$

subject to

$$\sum_{j=1}^{n}((m_{ij}, n_{ij}, \gamma'_{ij}, \delta_{ij})'_{LR}) \preceq, \approx, \succeq (b_i, g_i, \gamma_i, \delta_i)_{LR}$$

, $i = 1, 2, \ldots, m$ and where, $(x_j, y_j, \alpha''_j, \beta''_j)_{LR}$ is a L-R flat fuzzy number.

**Step 3:** By using yager's ranking approach , the fully fuzzy linear programming problem which is obtained in step 2 is written as:

$$\text{Maximize(or Minimize) } \Re(\sum_{j=1}^{n}(s_j, t_j, \alpha''''_j, \beta''''_j)_{LR})$$

subject to

$$\Re(\sum_{j=1}^{n}((m_{ij}, n_{ij}, \gamma'_{ij}, \delta_{ij})'_{LR})) \preceq, \approx, \succeq \Re((b_i, g_i, \gamma_i, \delta_i)_{LR}),$$

$i = 1, 2, \ldots, m$ and where, $x_j \leq y_j,\ \alpha''_j \geq 0,\ \beta''_j \geq 0.$

**Step 4:** Now, in the step 3 the crisp linear programming problem which is obtained solve it and find the optimal solution $x_j, y_j, \alpha''_j, \beta''_j$ and put these values in $\widetilde{x}_j = (x_j, y_j, \alpha''_j, \beta''_j)_{LR}$ and find the optimal solution.

**Step 5:** By putting $\widetilde{x}_j$ in $\sum_{j=1}^{n}(\widetilde{c}_j \odot \widetilde{x}_j)$ find the fuzzy optimal value of fully fuzzy linear programming problem.

## 4.6 Numerical examples

In this section, it has been shown that Mehar's method can be used for solving FFLP problems in which some or with all the parameters are represented as unrestricted L-R fuzzy or L-R flat fuzzy numbers and is shown through examples

**Example 1**

Find the fuzzy optimal solution of the FFLP problem.

$$\text{Maximize } (4,4,0,0)_{LR} \odot \widetilde{x_1} \oplus (1,1,1,1)_{LR} \odot \widetilde{x_2}$$

subject to

$$(2,5,5,2)_{LR} \odot \widetilde{x_1} \oplus (-1,5,1,2)_{LR}\widetilde{x_2} \preceq (-17,45,25,46)_{LR}$$

$$(1,2,1,1)_{LR} \odot \widetilde{x_1} \oplus (3,5,2,2)_{LR}\widetilde{x_2} \preceq \left(11,39,\frac{193}{5},\frac{168}{5}\right)_{LR}$$

where, $\widetilde{x_1}, \widetilde{x_2}$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$.

**Solution:**Now, solving this FFLP problem by using the algorithm which is defined in section 4.5.

**Step 1:** Putting $\widetilde{x_1} = (x_1, y_1, \alpha_1, \beta_1)$ and $\widetilde{x_2} = (x_2, y_2, \alpha_2, \beta_2)$ in the given problem then the FFLP problem is written as

$$\text{Maximize } (4,4,0,0)_{LR} \odot (x_1,y_1,\alpha_1,\beta_1) \oplus (1,1,1,1)_{LR} \odot (x_2,y_2,\alpha_2,\beta_2)$$

subject to

$$(2,5,5,2)_{LR} \odot (x_1,y_1,\alpha_1,\beta_1) \oplus (-1,5,1,2)_{LR} \odot (x_2,y_2,\alpha_2,\beta_2) \preceq (-17,45,25,46)_{LR}$$

$$(1,2,1,1)_{LR} \odot (x_1,y_1,\alpha_1,\beta_1) \oplus (3,5,2,2)_{LR} \odot (x_2,y_2,\alpha_2,\beta_2) \preceq \left(11,39,\frac{193}{5},\frac{168}{5}\right)_{LR}$$

where, $(x_1, y_1, \alpha_1, \beta_1), (x_2, y_2, \alpha_2, \beta_2)$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$

**Step 2:** By using the arithmetic operations for L-R flat fuzzy numbers which are defined in the section 4.3 the fully fuzzy linear programming problem in step 2 will become.

Maximize $(minimum\{4x_1, 4x_1\}, maximum\{4y_1, 4y_1\}, minimum\{4x_1, 4x_1\} - minimum\{4x_1 - 4\alpha_2 - 0 - 0, 4x_1 - 4\alpha_1 + 0 - 0\}, maximum\{4y_1 + 4\beta_1 - 0 - 0, 4y_1 + 4\beta_1 + 0 + 0\} - maximum\{4y_1, 4y_1\})_{LR} \oplus (minimum\{x_2, x_2\}, maximum\{y_2, y_2\}, minimum\{x_2, x_2\} - minimum\{y_2 - \alpha_2 - y_2 + \alpha_2, y_2 - \alpha_2 + y_2 - \alpha_2\}, maximum\{y_2 + \beta_2 - y_2 - \beta_2, y_2 + \beta_2 + y_2 + \beta_2\} - maximum\{x_2, x_2\})_{LR}$.

subject to

$(minimum\{2x_1, 5x_1\}, maximum\{2y_1, 5y_1\}, minimum\{2x_1, 5x_1\} - minimum\{2y_1 + 2\beta_1 - 5y_1 - 5\beta_1, 5x_1 + 5\alpha_1 + 2x_1 - 2\alpha_1\}, maximum\{2x_1 - 2\alpha_1 - 5x_1 - 5\alpha_1, 5y_1 + 5\beta_1 + 2y_1 + 2\beta_1\} - maximum\{2y_1, 5y_1\})_{LR} \oplus (minimum\{-x_2, 5x_2\}, maximum\{-y_2, 5y_2\}, minimum\{-x_2, 5x_2\} - minimum\{-y_2 - \beta_2 - y_2 - \beta_2, 5x_2 - 5\alpha_2 + 2x_2 - 2\alpha_2\}, maximum\{-x_2 + \alpha_2 - x_2 + \alpha_2, 5y_2 + 5\beta_2 + 2y_2 + 2\beta_2\} - maximum\{-y_2, 5y_2\})_{LR} \preceq (-17, 45, 25, 46)_{LR}$

$(minimum\{x_1, 2x_1\}, maximum\{y_1, 2y_1\}, minimum\{x_1, 2x_1\} - minimum\{y_1 + \beta_1 - y_1 - \beta_1, x_1 - 2\alpha_1 + x_1 - \alpha_1\}, maximum\{y_1 + \beta_1 - y_1 - \beta_1, 2y_1 + 2\beta_1 + y_1 + \beta_1\} - maximum\{y_1, 2y_1\})_{LR} \oplus (minimum\{3x_2, 5x_2\}, maximum\{3y_2, 5y_2\}, minimum\{3x_2, 5x_2\} - minimum\{3x_2 + \alpha_2 - 2x_2 - 2\alpha_2, 5x_2 - 5\alpha_2 + 2x_2 - 2\alpha_2\}, maximum\{3y_2 + 3\beta_2 - 2y_2 - \beta_2, 5y_2 + 5\beta_2 + 2y_2 + 2\beta_2\} - maximum\{3y_2, 5y_2\})_{LR} \approx \left(11, 39, \frac{193}{5}, \frac{168}{5}\right)_{LR}$

where, $(x_1, y_1, \alpha_1, \beta_1), (x_2, y_2, \alpha_2, \beta_2)$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1 - x\}$.

on simplifying, the FFLP problem will become

Maximize $(minimum\{4x_1, 4x_1\}, maximum\{4y_1, 4y_1\}, minimum\{4x_1, 4x_1\} - minimum\{4x_1 - 4\alpha_2, 4x_1 - 4\alpha_1\}, maximum\{4y_1 + 4\beta_1, 4y_1 + 4\beta_1\} - maximum\{4y_1, 4y_1\})_{LR} \oplus (minimum\{x_2, x_2\}, maximum\{y_2, y_2\}, minimum\{x_2, x_2\} - minimum\{0, 2y_2 -$

$2\alpha_2\}$, $maximum\{0, 2y_2 + 2\beta_2\} - maximum\{x_2, x_2\})_{LR}$.

subject to

$(minimum\{2x_1, 5x_1\}, maximum\{2y_1, 5y_1\}, minimum\{2x_1, 5x_1\} - minimum\{-3y_1 - 3\beta_1, 7x_1 + 7\alpha_1\},\ maximum\{-3x_1 + 3\alpha_1, 7y_1 + 7\beta_1\} - maximum\{2y_1, 5y_1\})_{LR} \oplus (minimum\{-y_2, 5x_2\}, maximum\{-y_2, 5y_2\}, minimum\{-y_2, 5x_2\} - minimum\{-2y_2 - 2\beta_2, 7x_2 - 7\alpha_2\},\ maximum\{-2x_2 - 2\alpha_2, 7y_2 + 7\beta_2\} - maximum\{-y_2, 5y_2\})_{LR} \preceq (-17, 45, 25, 46)_{LR}$

$(minimum\{x_1, 2x_1\},\ maximum\{y_1, 2y_1\},\ minimum\{x_1, 2x_1\} - minimum\{0, 3x_1 - 3\alpha_1\},\ maximum\{0, 3y_1 + 3\beta_1\} - maximum\{y_1, 2y_1\})_{LR} \oplus\ (minimum\{3x_2, 5x_2\}, maximum\{3y_2, 5y_2\},\ minimum\{3x_2, 5x_2\} - minimum\{x_2 - \alpha_2, 7x_2 - 7\alpha_2\}, maximum\{y_2 + \beta_2, 7y_2 + 7\beta_2\} - maximum\{3y_2, 5y_2\})_{LR} \approx \left(11, 39, \frac{193}{5}, \frac{168}{5}\right)_{LR}$

where, $(x_1, y_1, \alpha_1, \beta_1), (x_2, y_2, \alpha_2, \beta_2)$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1 - x\}$

**Step 3:** By using yager's ranking function which is given as $minimum(a, b) = \frac{a+b}{2} - \left|\frac{a-b}{2}\right|$ and $maximum(a, b) = \frac{a+b}{2} + \left|\frac{a-b}{2}\right|$ and by using the step 3 of Mehar's method the FFLP problem which is obtained in Step 2 is written as

Maximize $(2x_x - \alpha_1 + 2y_1 + \beta_1 + \frac{1}{2}x_2 - \frac{1}{4}\alpha_2 - \frac{1}{4}|x_2 - \alpha_2| + \frac{1}{2}|y_2 + \beta_2| + \frac{1}{2}y_2 + \frac{1}{4}\beta_2)$

subject to

$\frac{11}{2}y_1 + 2\beta_1 + \frac{11}{2}x_1 - 2\alpha_1 - \frac{1}{2}|3y_1 + 3\beta_1 + 7x_1 - 7\alpha_1| + \frac{9}{2}y_2 + \frac{5}{2}\beta_2 + \frac{9}{2}x_2 - \frac{5}{2}\alpha_2 - |y_2 + \beta_2 + \frac{7}{2}x_2 - \frac{7}{2}\alpha_2| - \frac{3}{2}|x_1| - \frac{1}{2}|y_2 + 5x_2| + \frac{3}{2}|y_1| + \frac{1}{2}|x_2 + 5y_2| - \frac{1}{2}|7y_1 + 7\beta_1 +3x_1 - 3\alpha_1| + |x_2 - \alpha_2 + \frac{7}{2}y_2 + \frac{7}{2}\beta_2| \leqslant 77.$

$3\alpha_1 - \frac{3}{2}\alpha_1 - \frac{3}{2}|x_1 - \alpha_1| + 8x_2 - 4\alpha_2 - 3|x_2 - \alpha_2| - \frac{1}{2}|x_1| - |x_2| + 3y_1 + \frac{1}{2}|y_1| + 8y_2 +|y_2|\frac{3}{2}\beta_1 + \frac{3}{2}|y_1 + \beta_1| + 4\beta_2 + 3|y_2 + \beta_2| = 95.$

$x_1 \leqslant y_1,\ x_2 \leqslant y_2,\ \alpha_2 \geqslant 0,\ \alpha_2 \geqslant 0,\ \beta_1 \geqslant 0,\ \beta_2 \geqslant 0.$

**Step 4:** Solving the crisp Non-linear programming problem which is obtained in step 3 the optimal solution is $x_1 = \frac{377}{122}, x_2 = \frac{583}{122}, y_1 = \frac{377}{122}, y_2 = \frac{583}{122}, \alpha_1 = 0, \alpha_2 = 0, \beta_1 = 0, \beta_1 = 0$. Putting these values in $\widetilde{x_1} = (x_1, y_1, \alpha_1, \beta_1)_{LR}, \widetilde{x_1} = (x_1, y_1, \alpha_1, \beta_1)_{LR}$ the fuzzy optimal solution is $\widetilde{x_1} = \left(\frac{377}{122}, \frac{377}{122}, 0, 0\right)_{LR}$,and

$\widetilde{x_1} = \left(\frac{583}{122}, \frac{583}{122}, 0, 0\right)_{LR}$

**Step 5:** Putting the values of $\widetilde{x_1} and \widetilde{x_2}$ obtained from Step 4 in $(4,4,0,0)_{LR} \odot \widetilde{x_1} \oplus (1,1,1,1)_{LR} \odot \widetilde{x_2}$

$(4,4,0,0)_{LR} \odot \left(\frac{377}{122}, \frac{377}{122}, 0, 0\right)_{LR} \oplus (1,1,1,1)_{LR} \odot \left(\frac{583}{122}, \frac{583}{122}, 0, 0\right)_{LR}$ the fuzzy optimal value is $\left(\frac{2091}{122}, \frac{2091}{122}, \frac{583}{122}, \frac{583}{122}\right)$.

**Example 2** Find the fuzzy optimal solution of the FFLP problem

$$\text{Maximize}(1,1,1,1)_{LR} \otimes \widetilde{x_1} \oplus (4,4,0,0)_{LR} \otimes \widetilde{x_2}$$

subject to

$$(2,3,1,1)_{LR} \otimes \widetilde{x_1} \oplus (-3,-2,1,1)_{LR} \widetilde{x_2} \preceq (-27,-4,21,32)_{LR}$$

$$(\ 1,2,1,1)_{LR} \otimes \widetilde{x_1} \oplus (3,5,2,2)_{LR} \widetilde{x_2} \preceq \left(11, 39, \frac{193}{5}, \frac{168}{5}\right)_{LR}$$

where, $\widetilde{x_1}, \widetilde{x_2}$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$

**Solution:**By using above defined algorithm for solving FFLP problem i.e. Mehar's method

**Step 1:** Putting $\widetilde{x_1} = (x_1, y_1, \alpha_1, \beta_1)$ and $\widetilde{x_2} = (x_2, y_2, \alpha_2, \beta_2)$ in given problem then the FFLP problem is written as

$$\text{Maximize } (1,1,1,1)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1) \oplus (4,4,0,0)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2)$$

subject to

$$(2,3,1,1)_{LR} \otimes (x_1, y_1, \alpha_1, \beta_1) \oplus (-3,-2,1,1)_{LR} \otimes (x_2, y_2, \alpha_2, \beta_2) \preceq (-27,-4,21,32)_{LR}$$

$(1,2,1,1)_{LR}\otimes(x_1,y_1,\alpha_1,\beta_1)\oplus(3,5,2,2)_{LR}\otimes(x_2,y_2,\alpha_2,\beta_2)\preceq\left(11,39,\frac{193}{5},\frac{168}{5}\right)_{LR}$

where, $(x_1,y_1,\alpha_1,\beta_1),(x_2,y_2,\alpha_2,\beta_2)$ are L-R flat fuzzy numbers and $L(x)=R(x)=maximum\{0,1-x\}$

**Step 2:** By using the arithmetic operations for L-R flat fuzzy numbers which are defined in the section 4.3 the fully fuzzy linear programming problem in step 2 will become.

Maximize $(minimum\{4x_1,4x_1\}, maximum\{4y_1,4y_1\}, minimum\{4x_1,4x_1\} -minimum\{4x_1-4\alpha_2-0,4x_1-4\alpha_1+0\}, maximum\{4y_1+4\beta_1-0,4y_1+4\beta_1+0\}- maximum\{4y_1,4y_1\})_{LR}\ \oplus\ (minimum\{x_2,x_2\}, maximum\{y_2,y_2\}, minimum\{x_2, x_2\}-minimum\{y_2-\alpha_2-y_2,y_2-\alpha_2+y_2\},\ maximum\{y_2+\beta_2-y_2,y_2+\beta_2+y_2\}- maximum\{x_2,x_2\})_{LR}$.

subject to

$(minimum\{2x_1,3x_1\}, maximum\{2y_1,3y_1\}, minimum\{2x_1,3x_1\}-minimum\{2x_1-2\alpha_1-x_1,3x_1-3\alpha_1+x_1\}, maximum\{2y_1+2\beta_1-y_1,3y_1+3\beta_1+y_1\}-maximum\{2y_1,3y_1\})_{LR} \oplus (minimum\{-3y_2,-2y_2\}, maximum\{-3x_2,-2x_2\}, minimum\{-3y_2,-2y_2\}- minimum\{-3y_2-3\beta_2-y_2,-2y_2-2\beta_2+y_2\}, maximum\{-2x_2-2\alpha_2+x_2,-3x_2- 3\alpha_2-x_2\}-maximum\{-3x_2,-2x_2\})_{LR}\preceq(-27,-4,21,32)_{LR}$

$(minimum\{x_1,2x_1\}, maximum\{y_1,2y_1\}, minimum\{x_1,2x_1\}-minimum\{y_1+\beta_1- y_1,x_1-2\alpha_1+x_1\}, maximum\{y_1+\beta_1-y_1,2y_1+2\beta_1+y_1\}-maximum\{y_1,2y_1\})_{LR}\oplus (minimum\{3x_2,5x_2\},\ maximum\{3y_2,5y_2\}, minimum\{3x_2,5x_2\}-minimum\{3x_2+ \alpha_2-2x_2,5x_2-5\alpha_2+2x_2\}, maximum\{3y_2+3\beta_2-2y_2,5y_2+5\beta_2+2y_2\}- maximum\{3y_2,5y_2\})_{LR}\approx\left(11,39,\frac{193}{5},\frac{168}{5}\right)_{LR}$

where, $(x_1,y_1,\alpha_1,\beta_1),(x_2,y_2,\alpha_2,\beta_2)$ are L-R flat fuzzy numbers and $L(x)=R(x)=maximum\{0,1-x\}$

on simplifying the FFLP problem will become,

Maximize $(minimum\{4x_1,4x_1\}, maximum\{4y_1,4y_1\}, minimum\{4x_1,4x_1\}-minimum$

$\{4x_1-4\alpha_1, 4x_1-4\alpha_1\}$, $maximum\{4y_1+4\beta_1, 4y_1+4\beta_1\}-maximum\{4y_1, 4y_1\})_{LR} \oplus$ $(minimum\{x_2, x_2\}$, $maximum\{y_2, y_2\}$, $minimum\{x_2, x_2\} - minimum\{-\alpha_2, 2y_2 - \alpha_2\}$, $maximum\{\beta_2, 2y_2 + 2\beta_2\} - maximum\{x_2, x_2\})_{LR}$.

subject to

$(minimum\{2x_1, 3x_1\}$, $maximum\{2y_1, 3y_1\}$, $minimum\{2x_1, 3x_1\} - minimum\{x_1 - 2\alpha_1, 4x_1 - 3\alpha_1\}$, $maximum\{y_1 + 2\beta_1, 4y_1 + 3\beta_1\} - maximum\{2y_1, 3y_1\})_{LR} \oplus$ ( $minimum\{-3y_2, -2_2\}$, $maximum\{-3x_2, -2x_2\}$, $minimum\{-3y_2, -2y_2\}-$ $minimum\{-4y_2 - 3\beta_2, -y_2 - 2\beta_2\}$, $maximum\{-x_2 - 2\alpha_2, -4x_2 - 3\alpha_2\}-$ $maximum\{-3x_2, -2x_2\})_{LR} \preceq (-27, -4, 21, 32)_{LR}$

$(minimum\{x_1, 2x_1\}$, $maximum\{y_1, 2y_1\}$, $minimum\{x_1, 2x_1\}-minimum\{\beta_1, -2\alpha_1\}$, $maximum\{\beta_1, 4y_1 + 2\beta_1\} - maximum\{y_1, 2y_1\})_{LR} \oplus$ $(minimum\{3x_2, 5x_2\}$, $maximum\{3y_2, 5y_2\}$, $minimum\{3x_2, 5x_2\} - minimum\{x_2 + \alpha_2, 7x_2 - 5\alpha_2\}$, $maximum\{y_2 + 3\beta_2, 7y_2 + 5\beta_2\} - maximum\{3y_2, 5y_2\})_{LR} \approx \left(11, 39, \frac{193}{5}, \frac{168}{5}\right)_{LR}$

where, $(x_1, y_1, \alpha_1, \beta_1), (x_2, y_2, \alpha_2, \beta_2)$ are L-R flat fuzzy numbers and $L(x) = R(x) = maximum\{0, 1 - x\}$

**Step 3:** By using yager's ranking function which is given as $minimum(a, b) = \frac{a+b}{2} - \left|\frac{a-b}{2}\right|$ and $maximum(a, b) = \frac{a+b}{2} + \left|\frac{a-b}{2}\right|$ and by using the step 3 of Mehar's method the FFLP problem which is obtained in Step 2 is written as

Maximize $\left(2x_1 - \frac{1}{2}\alpha_1 + 2y_1 + \beta_1 - \frac{1}{4}\alpha_2 + \frac{1}{8}|-2y_2 - \beta_2| - \frac{1}{4}|-y_2| + \frac{1}{2}y_2 + \frac{3}{8}\beta_2\right)$

subject to

$$\frac{5}{2}x_1 - \frac{5}{2}y_2 + \frac{5}{2}y_1 - \frac{5}{2}x_2 - \frac{1}{2}|-x_1| - \frac{5}{2}|-y_2| + \frac{5}{2}|y_1| + \frac{5}{2}|-x_2| + \frac{1}{4}|-x_1| - \frac{5}{2}\alpha_1$$
$$-\frac{3}{4}|\alpha_1 - x_1| + \frac{5}{2}|-y_2| - \frac{5}{2}|-y_2| - \frac{5}{2}\beta_2 - \frac{3}{4}|y_2 - \beta_2| + \frac{5}{8}\beta_1 - \frac{5}{4}|y_1| - \frac{5}{4}\alpha_2 + \frac{5}{4}|x_2 - \alpha_2|$$
$$-\frac{5}{4}|x_2| = -\frac{9}{2}\ .$$

$$3\alpha_1 - \frac{3}{2}\alpha_1 - \frac{3}{2}|x_1 - \alpha_1| + 8x_2 - 4\alpha_2 - 3|x_2 - \alpha_2| - \frac{1}{2}|x_1| - |x_2| + 3y_1 + \frac{1}{2}|y_1| + 8y_2+$$
$$|y_2|\frac{3}{2}\beta_1 + \frac{3}{2}|y_1 + \beta_1| + 4\beta_2 + 3|y_2 + \beta_2| = 95.$$

$x_1 \leqslant y_1,\ x_2 \leqslant y_2,\ \alpha_2 \geqslant 0,\ \alpha_2 \geqslant 0,\ \beta_1 \geqslant 0,\ \beta_2 \geqslant 0.$

**Step 4:** Solving the crisp Non-linear programming problem which is obtained in step 3 the optimal solution is $x_1 = \frac{67}{110},\ x_2 = \frac{628}{110},\ y_1 = \frac{67}{110},\ y_2 = \frac{628}{122},\ \alpha_1 = 0,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_1 = 0$. Putting these values in $\widetilde{x_1} = (x_1, y_1, \alpha_1, \beta_1)_{LR}$, $\widetilde{x_2} = (x_2, y_2, \alpha_2, \beta_2)_{LR}$ the fuzzy optimal solution is $\widetilde{x_1} = \left(\frac{67}{110}, \frac{67}{110}, 0, 0\right)_{LR}$,
$\widetilde{x_2} = \left(\frac{628}{110}, \frac{628}{110}, 0, 0\right)_{LR}$

**Step 5:** Putting the values of $\widetilde{x_1}$ and $\widetilde{x_2}$ obtained from Step 4 in $(4,4,0,0)_{LR} \otimes \widetilde{x_1} \oplus (1,1,1,1)_{LR} \otimes \widetilde{x_2}$ .
$(1,1,1,1)_{LR} \otimes \left(\frac{67}{110}, \frac{67}{110}, 0, 0\right)_{LR} \oplus (4,4,0,0)_{LR} \otimes \left(\frac{628}{110}, \frac{628}{110}, 0, 0\right)_{LR}$ the fuzzy optimal value is $\left(\frac{2579}{110}, \frac{2579}{110}, \frac{67}{110}, \frac{67}{110}\right)_{LR}$.
Mehar's method can also be used to solve the FFLP problems in which all the parameters are represented by L-R fuzzy numbers.

The fuzzy optimal solution of the FFLP problem given below in example 3 and 4 is solvable by proposed method.

**Example 3**

$$\text{Maximize } (2,1,2)_{LR} \odot \widetilde{x_1} \oplus (-3,2,1)_{LR} \odot \widetilde{x_2}$$

subject to

$$(1,1,2)_{LR} \odot \widetilde{x_1} \oplus (2,1,3)_{LR}\widetilde{x_2} \preceq (20,10,5)_{LR}$$

$$(-2,1,1)_{LR} \odot \widetilde{x_1} \oplus (5,2,3)_{LR}\widetilde{x_2} \preceq (-1,3,2)_{LR}$$

where, $\widetilde{x}_1, \widetilde{x}_2$ are L-R fuzzy numbers and $L(x) = R(x) = maximum\{0, 1 - x\}$.

**Example 4**

$$\text{Maximize } (1,1,1)_{LR} \otimes \widetilde{x_1} \oplus (2,1,2)_{LR} \odot \widetilde{x_2}$$

subject to

$$(4,1,0)_{LR} \otimes \widetilde{x_1} \oplus (-2,1,1)_{LR}\widetilde{x_2} \preceq (5,2,3)_{LR}$$

$$(-3,1,2)_{LR} \otimes \widetilde{x}_1 \oplus (4,1,2)_{LR}\widetilde{x}_2 \preceq (4,1,1)_{LR}$$

where, $\widetilde{x}_1, \widetilde{x}_2$ are L-R fuzzy numbers and $L(x) = R(x) = maximum\{0, 1-x\}$.

the solutions to the above given examples 3 and 4 can be similarly calculated by following the steps of the algorithm defined for Mehar's method. Thus the optimal solution for example 3 is $\left(-\frac{192}{31}, \frac{128}{31}, \frac{6932}{93}\right)_{LR}$ and to that of example 4 is $\left(\frac{1427}{167}, \frac{956}{167}, \frac{1427}{167}\right)_{LR}$

## 4.7 Concluding Remarks

(i) Due to non-existence of the product of unrestricted L-R fuzzy numbers or L-R flat fuzzy the it has been observed that the existing method [2] can be used for solving fully FLP problems of the type (i) and (ii) as shown in example 1 and 2 in which all the coefficients are represented by either non-negative L-R fuzzy numbers or non-positive L-R fuzzy numbers and all the decision variables are represented by non-negative L-R fuzzy numbers. However, the existing method [2] can not be used to find the fuzzy optimal solution of fully FLP problems (iii) and (iv) that the problems dissused in example 2 and 3 in which some or all the parameters are represented by unrestricted L-R fuzzy numbers or unrestricted L-R flat fuzzy numbers.

(i) The existing methods [6, 18, 27, 31] can be used only for solving such fully FLP problems in which some or all the parameters are represented by triangular or trapezoidal fuzzy numbers. Since, in the fully FLP problems, chosen in examples mentioned in section 4.6, all the parameters are represented by L-R fuzzy numbers or L-R flat fuzzy numbers so none of the chosen problems can be solved by the existing methods [6, 18, 27, 31].

(iii) The proposed Mehar's method can be used for solving such fully FLP problems in which some or all the parameters are represented by unrestricted L-R fuzzy numbers or L-R flat fuzzy numbers. So, all the chosen problems in example 1,2,3 and 4 can be solved by the Mehar's method.

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