Inverse Problems in Ordinary Differential Equations

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CERTIFICATE

*I hereby certify that the work which is being presented in the thesis entitled "***Inverse Problems in Ordinary Differential Equations***" in partial fulfillment of the requirements for the award of degree of Master of Science at School of Mathematics, Thapar Institute of Engineering and Technology, Patiala is an authentic record of my own work carried out under the supervision of Dr. Kavita.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

This thesis report includes solving the inverse problem in ordinary differential equation which is helpful in various fields. For example, we have considered a second order ordinary differential equation. In this second order ordinary differential equation, there are two types of equation. The first one is linear equation and other one is non-linear equation. In beginning, we solve the second order ordinary differential equation, then we implement the Least Square Fit Approach on linear differential equation and Newton method on non-linear differential equation to find the inverse problem. From the results, we have conclude that these methods are efficient for solving inverse problems associated with differential equations.

Chapter 1 of this report describes about the inverse problem and it's various use in many applications in differential equation. Chapter 2 describes the regularization techniques for solving the inverse problems. Chapter 3 describes the various methods for solving the inverse problem in ordinary differential equation. In the end, we presented the conclusion.

Chapter 1

Inverse Problem

1.1 Introduction

We will discuss about the inverse problems and it's various important applications in this introductory section. In the previous years, the quickly developing area in applied mathematics is the inverse problem. This growth has large applications in the field of sciences and industry. So, we can say that inverse problem is a dynamic and widening branch of Mathematics. When we are using the term ”inverse problem”, it is generally ask that ”inverse to what”, then we can say that two problems are opposite to each other if the origination of one problem involves the other problem. The one problem is the forward problem and another problem is the inverse problem. However, if we will research that there is a physical world problem at the back of the mathematical problem. There is a dissimilarity between direct(forward) problem and the inverse problem. We can say that direct problem is that problem in which we know about the future actions of physical system from the knowledge of existing state and real parameters of that system. The inverse problem is the observation of existing state of the system from the future knowledge or parameter identification.

Inverse problems are an analysis area which deals with inversion of the system. There are many examples in our daily life in which inverse problems are widely used like in medical computerized tomography (CT) scan, geophysics, astronomy, Image Formation, Elasticity Imaging, Bio-mechanical imaging, Physics, Economics, Groundwater flow modeling, Navigation and many other fields. So, in all these fields our aim is to determine some hidden factors from a set of predictions that produced them. The various applications are given below:-

A. Medical Computerized Tomography (CT) Scan

It is the powerful method for analysis of human organs and biological system. The inverse problems helps us to determine image structure from inside the body through the measurement of X-rays. The X-rays are very high speed radiations that have passed through the body and creates the photographic film.

B. Geophysics

In area of geophysics, the inverse problems have a crucial participation. We cannot determine the interior structure of the earth but it is possible with the help of the inverse problem. In inverse problem, there is transmission of seismic waves to find the position of an earthquake's point. Also, inverse problems helps to find the degree of compactness of earth from measurements of the earth's gravity field. So, this type of condition in which waves are transmit to examine the material are possible through inverse problem.

C. Image Formation

To calculate about the making of an image, there is mapping in the direct problem of an object from it's image to it's quantities. We know about the details of the direct problem. Thus, the inverse problem is to calculating the real picture from the given input and the correct information of the direct problem. In the case of explanation of deblurring pictures, we have an image which is a fine picture, the given input is the blurred image, and direct problem is the whole blurring process. The ill-posed problem is to determine the fine image from the blurred picture and to provide the information of the complete blurring process.

D. Elasticity Imaging

It is a modern and optimistic method in medical imaging. It tells us about the distinction of tissues of elastic modules. We know the characteristics of an object and it's boundary input and determine displacement field in the forward problem. In inverse problem, the whole condition is opposite, we know about the displacement field and determine the boundary knowledge and the material properties of the elastic modules.

E. Bio-mechanical imaging

It is feasible to see the image of physical characteristics of tissues in biomechanical imaging. It includes the characteristics like viscoelastic, Young's modulus, non-linear elastic properties. At the same time, it has been set up that the physical characterstics of tissue which are changed if the tissues are defected.

The maximum techniques in biomechanical imaging consists observing the distort tissue by using a imaging treatment like ultrasound or Magnet Resonance Imaging and then using the pre-deformation and post-deformation pictures to observe the internal problem of the tissue. Then, this input is used to observe the spatial distribution of the tissue along with an appropriate mechanical model.

F. Physics

There are lot of ill-posed problems which are appear in the branch of physics. These are given below:

(a) Calculate the shape of a mountain from the travel time:- Let us consider that we have a particle and we slide this particle in upward direction with starting energy on a frictionless mountain and estimate that how much time for a particle is neccessary to return it's starting position. If we alters the initial energy and calculate the time need to return back, then we can observe the shape of the mountain.

(b) Calculation of a potential from the period of oscillation:- Let we suppose that the motion of a particle. Initially, we know the time of oscillations of the particle with how much energy is required, then calculate the potential of the particle.

G. Economics

The demand theory of economics says about the forward problem. There is specified that in an exchange economy consists of N customers, consists the mutual demand function, which is the total of individual demand functions. So, there is a mapping of the cost system S to a item's quantity $x(S)$. There is given a mapping in the inverse problem : $S \to x(S)$, it is commonly to question that whether it is the total demand function of the market economy.

H. Groundwater flow modeling

We want to calculates the object boundary of an aquifer from calculations of pressure of a fluid that dips the aquifer in groundwater flow modeling. But unluckily, very small amount of noise in the data can creates the large amount of mistakes in the estimation. This uncertainity event is termed as ill-posedness. So, our aim is to calculates the parameters of an aquifer and reduce the errors in the estimation.

I. Navigation

When a sailor travels in a boat or submarine, then he finds the actual position in near to the actual time. It is possible by taking a lot of observations. For example:- by using the position of satellite, and also by taking the last location and using the knowledge such as keeping the data of speed. When all the separate pieces of data collect with each other, then it gives the detail of the movement of vessel.

The area of ill-posed theory is very vast. There are large number of applications of inverse problem which are found in many diverse areas. The mathematical approaches called as regularization that have been enlarged to handle with the inverse problems.

1.2 What is an inverse problem?

In the inverse problem, we have only knowledge of the effects and we are finding the causes, which consists the idea of solving the inverse problems. However, if there is a minor changes in the results, then there is a huge distiction in the causes. As a result, sometimes it is not possible to calculate the real cause uniquely by observing only results. So, inverse problems starts with the results and then determine the causes. It is the inverse of forward problem. We want to find an effect from a cause in direct problem but in an inverse problem the effect is known and we want to recover the causes.

Inverse problems are of two types:

(a) Well-posed problem

(b) Ill-posed problem

Well-posed problem:- According to Hadamard," a problem is called well-posed if it fulfills the subsequent given requirements:-

(a) the problem must have a solution (Existence),
(b) there is at most one solution to the problem (Uniqueness), and
(c) it must depend continuously on data and parameters (Stability).

If the problem violates any of these requirements, then it is called **ill-posed problem.**

In mathematical way, we can say that let X is a map defined from set A to set B i.e., $X : A \rightarrow B$, where A and B are Banach spaces or normed spaces, and $X(s) = p$ where X is a non-linear operator (generally). It becomes well-posed if X is a homeomorphism of A onto B. Also, we can say that the solution exists for any $p \in B$ is unique, and depends on p continuously, so that X^{-1} is the continuous mapping. If some of these conditions do not happened, then the problem becomes ill-posed.

Now, the question arises that why inverse problems are difficult than forward problem?

Actually, inverse problems are more harder than the forward problem because they tell us about the parameters that we cannot directly observe. The one reason is that presence of various models that suitably holds the data. The necessary things must be include the solution occurrence, solution uniqueness, and instability of the solution process.

1. Existence:- Actually, there may not be any model which fits absolutely in the data. This can occur in practice due to input data contains so much errors.

2. Uniqueness:- If there exists proper solutions, then that solutions may not be unique, even for an large amount of data points (i.e., there exists infinite answers).

3. Instability:- The procedure of evaluating an ill-posed solution is not stable because little conversion in calculation can occurs the large change in the approximated model. In the inverse problems, where this type of situation arises are termed to as ill-posed in the case of continuous systems, or ill-conditioned in the case of discrete linear systems. It is generally possible to stabilize the inversion process by adding extra constraints that bias the solution, a process i.e., generally termed as regularization.

An example

Let us take Fredholm first kind integral equation of convolution type in one dimension space:

$$y(t) = \int_0^1 p(t-t')x(t')dt' = (\rho x)(t); \;\; 0 < t < 1, \tag{1.2.1}$$

This equation is a one-dimensional kind of a representation which occurs in two-dimensional optical imaging. Here, $x, \ y, \ p$ denotes following things:-

$x =$ light source intensity,

$y =$ pixel intensity,

$p =$ kernel which shows the blurring effects which occurs during picture formation.

It's one-dimensional model is

$$p(t) = cexp\left(\frac{-t^2}{2\lambda^2}\right), \tag{1.2.2}$$

where $c = 1/\lambda\sqrt{2\pi}$ and $\lambda = 0.05$ are positive values.

The forward problem is related with equation (1.2.1), given the source x and the kernel p which determines the blurred picture y corresponding to a piecewise smooth source as shown in following Figure 1. Therefore, p is a smooth function, the estimation of $y = \lambda x$ by applying standard numerical quadrature is straightforward.

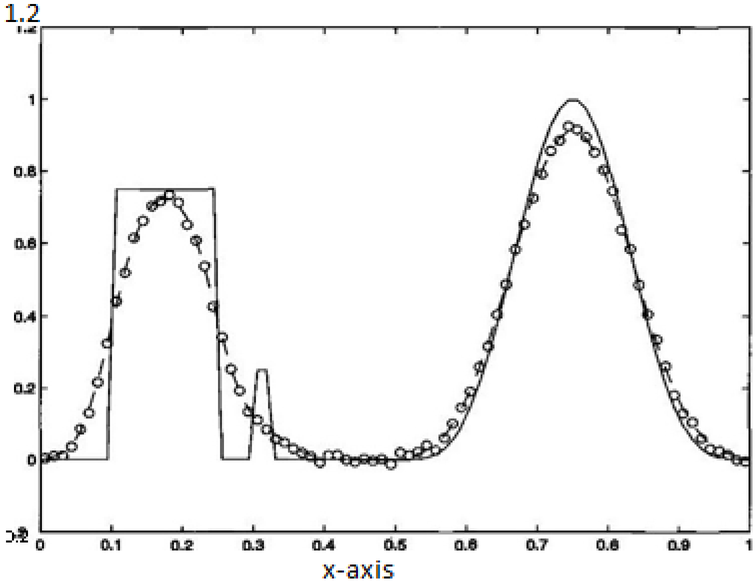

Figure 1:One dimensional image data

In Figure 1, the source function x is represented by solid line, the blurred image $y = \lambda x$ is represented by the dashed line, and the discrete noisy data d is represented by the circles. To calculate the approximate integrals we are using the midpoint quadrature.

An inverse problem is associated with it as given below:

We know the kernel p and blurred image y, find the source x. In beginning, the exact solution for this ill-posed problem looks straightforward. We can easily discretize the above equation (1.2.1) to get a discrete linear system $Px = d$. If we are applying the midpoint quadrature, then we get the values of P :

$$[P]_{ij} = kcexp\left(\frac{((i-j)k)^2}{2\lambda^2}\right) \; ; \; 1 \leq i \; ; \; j \leq n, \tag{1.2.3}$$

where $k = 1/n$. If the matrix P is non-singular, so we can evaluate the discrete approximation $p^{-1}d$ to x. To find the exact quadrature value, n should have big value. But unluckily, matrix P forms enlargely ill-conditioned as n becomes big, so noise in d may be largely amplified. Certain noise because of quadrature, can be controlled but noise in the picture making device cannot be controlled. As a result, straightforward solution approach looks not to success.
Therefore, we use the regularization techniques to control the errors which occurs due to noise.

1.3 How inverse problems are used for differential equations?

The two mathematical problems are differentiation and integration which are inverse to each other. It is not clear that which one is direct problem and which one of these problems is an inverse problem. However, the differential equation has the property of ill-posedness. In most of the calculus courses, differentiation might be viewed as ”inverse problem”. The various operations like physical and social gives a new idea for the modeling of the inverse problems. There are too many differential and integrable equations which are used in the formation of inverse problems. If we know about the given essential data in these type of models, then we can solve the inverse problem and get the approximate solution. Also, the behaviour of the model can be calculated under many conditions. The important input data consists of knowledge of domain’s shape and it’s size, initial input data and boundary material in the form of differential equations. If these type of data which explains the working of the system satisfies, then we can use the mathematical model possibly for learning the real model.
The differential equations are used for explaining the various processes whether they are natural or the social. If we know the input data of the differential equations, then we can easily solve that equations and also this type of differential equations fulfill the boundary conditions or initial input data. Also, there occurs so many difficulties in the field of science and engineering very oftenly. The less coefficient of input in the equation are not known, we have to find the conditions, if we know about some output values of the differential equation. So, these type of

difficulties are the inverse problems for the differential equation.

Inverse problem in linear order Ordinary Differential Equations.:- Suppose we take an example which describes that the inverse problems are used for linear differential equation of the quartic form:-

$$x^{iv}(t)+p_2(t)x^{''}(t)+p_0(t)x(t)=0. \tag{1.3.1}$$

The above coefficients of the equation $p_3(t)=p_1(t)=0$, $f(t)=0$ and $p_2(x)$, $p_0(x)$ are not known. Suppose $x_1(t)=sint$, $x_2(t)=cost$ be the two outputs of the equation (1.3.1). Now put $x_1(t)$, $x_2(t)$ in equation (1.3.1) which provides only one equation for the not known coefficients $p_2(t)p_0(t)=1$. Therefore, coming output value is not single of this type of ill-posedness problem. If there is stated another answers $x_1(t)=e^t$, $x_2(t)=e^{2t}$ of the equation (1.3.1), then putting the values provide two equations for not-known coefficients $p_2(t)+p_0(t)=1$ and $4p_2(t)+p_0(t)=16$. This type of model has a unique solution of equations $p_2(t)=5$, $p_0(t)=4$. This example shows that the necessary of analysis of knowledge in which we are using the inverse problem for solving it.

Inverse Problems for Differential Equations of Elastodynamics:- Let us consider that the differential equations of elastodynamics an let $x=(x_1,x_2,x_3)\in R^3$ be a three dimensional space variable and let $t\in R$ be a one-dimensional time variable and $s(x,t)=(s_1(x,t),s_2(x,t),s_3(x,t))$ is the displacement vector function of non-homogeneous anisotropic elastic material representing by density ρ and the elasticmoduli E_{ijkl}. The density ρ and the elasticmoduli E_{ijkl} are the varying functions of position $x=(x_1,x_2,x_3)$. By adding the properties of strain-energy function with Hooke law we search that E_{ijkl} satisfy the following property and strong convexity.

$$E_{ijkl}=E_{klij}=E_{lkij}=E_{ijlk},\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}>0. \tag{1.3.2}$$

The 3×3 real symmetric nonzero matrix i.e., $(\varepsilon_{ij})_{3\times3}$. The equations in the non-homogeneous anisotropic elastic materials for the motion are given below:-

$$\rho\frac{\partial^2 s_i}{\partial t^2}=\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\partial}{\partial x_j}(E_{ijkl}\frac{\partial s_k}{\partial x_l})+f_i\ ;\ i=1,\ 2,\ 3. \tag{1.3.3}$$

where f_i are the elements of body forces i.e., $f(x,t)=(f_1(x,t)f_2(x,t),f_3(x,t))$ acting per unit volume on the particle exactly at location x at some time t. Now, we determine the approximate solution of equation (1.3.3) for $f = 0$:-

$$s(x,t) = S(t - T(x))B(x). \tag{1.3.4}$$

Near a wavefront $t = T(x)$, we suppose that the components of $S = (S_1, S_2, S_3)$ are changing most quickly than $B(x)$ or E_{ijkl}. Therefore, sucessive derivatives $\partial S/\partial t$ and $\partial^2 S/\partial t^2$ are also changing still more fastly. Now, we substitute the equation (1.3.4) into (1.3.3), we get:

$$\left(\rho\delta_{ik} - \sum_{j=1}^{3}\sum_{l=1}^{3}\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_l}\right)\frac{\partial^2 S_k}{\partial t^2}B = D_i(SB), \tag{1.3.5}$$

where $D_i(SB)$ includes gradients, S is the elastic moduli, first-order derivatives of S and amplitude function $B(x)$.

Therefore, L.H.S. of (1.3.5) should be much smaller than $\partial^2 S/\partial t^2$.We find that the matrix coefficients of $\partial^2 S(t - T(x))/\partial t^2 B(x)$ must be singular:

$$det\left(\rho\delta_{ik} - \sum_{j=1}^{3}\sum_{l=1}^{3}\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_l}\right) = 0. \tag{1.3.6}$$

The above equation finds the certain wavefronts in an elastic medium, because this provides a constraint on the function $T(x)$. Also, in the non-homogeneous isotropic medium,

$$E_{ijkl} = \nu\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.3.7}$$

where,

$$\delta_{ij} = \begin{cases} 0; if i \neq j \\ 1; if i = j \end{cases}$$

Mostly, $\nu = \nu(x)$, $\mu = \mu(x)$ are termed as Lame functions.In an inhomogeneous isotropic medium, the special form equation (1.3.7) E_{ijkl} makes it possible to get following equation:-

$$\left(\nabla T.\nabla T - \frac{\rho}{\nu + 2\mu}\right)\left(\nabla T.\nabla T - \frac{\rho}{\mu}\right)^2 = 0. \tag{1.3.8}$$

$T(x)$ satisfies eikonal equation (1.3.8)

$$|\nabla T|^2 = \frac{1}{E_P^2(x)}, \tag{1.3.9}$$

or we can write this eikonal equation as follows:-

$$|\nabla T|^2 = \frac{1}{E_S^2(x)}, \tag{1.3.10}$$

where $E_P^2 = \sqrt{(\nu + 2\mu)/\rho}$ denotes local P-wave speed and $E_S^2 = \sqrt{\mu/\rho}$ denotes local S-wave speed.

Inverse Kinematic Problem of Seismic Waves:- Let us consider that a point's original location $x^0 \in R^3$ at a time, $t = 0$ happens energetic. In homogeneous isotropic medium, wavefronts flow from the origin, with radius E_Pt (for P-waves) and E_St for S-waves, reaching at generallocation x at time $t = |x - x^0|/E_P$ and $t = |x - x^0|/E_S$. We introduce the function $T(x, x^0)$ as the travel time need for wavefront to arrive x from x_0. The function$T(x, x^0)$ satisfies equation (1.3.9) for P-waves and equation (1.3.10) for S-waves.

The first inverse problem was the inverse kinematic problem which was studied in geophysics. Suppose that our Earth is an isotropic inhomogeneous elastic medium and the estimations of the seismic waves, starting from a point location x^0 and travelling in the interior of Earth, are provided for the points on the earth's surface. So, these estimations of input involve the propagate time $T(x, x^0)$ of seismic waves between the origin point x^0 and any point on earths surface. By using the estimation of the data, inverse kinematic problem calculates the speed of the seismic waves inside the Earth. We can explain mathematically inverse kinematic problem as given below:

Let D is the domain which is bounded by surface of the earth S, and suppose $T(x, x^0)$ is the function of the moving time which is need by a signal with no estimation of speed $E(x) > 0$ to arrive x from x^0. We have to find $C(x)$ for all x from D if function $T(x, x^0)$ is known for all points $x^0 \in S_1$ and $x \in S_2$, where $S_1 \subseteq S$ and $S_2 \subseteq S$ are subsets of S.

Herglotz and Wiechert and Zoeppritz were the earliest mathematicians who discovered that the inverse kinematic problem in considerations i.e.,

$$E(x)\frac{1}{n(r)}; r = |x| \; ; \; \frac{d}{dr}(rn(r)) > 0, \tag{1.3.11}$$

x_0 is the fixed point from S, if $T(x, x^0)$ is given for any x from S.Gerver and Markushevich have displayed that the state $\frac{d}{dr}(rn(r)) > 0$ can be removed but in this case there are many solutions for the inverse kinematic problems.

1.4 Literature Review

Till so far, many number of research papers have been published on inverse problem in ordinary differential equation. Some of them are listed below which are related to our thesis work:

1. **Using inverse problems as projects in an ordinary differential equations class**

   Paper: J. Graham, Using inverse problems as projects in an ordinary differential equations class, (1996)1-8. [4]

   In this research paper, author tells about the inverse problem, mathematical features of inverse problem. Solving the inverse problem is more creativity than solving the direct problems. There is discussion of various applications of inverse problems in many fields like medical, geophysical, non-destructive testing, etc. The second order linear and non-linear differential equation is solved by the data fitting approach and non-linear equation method.

2. **Introduction of Inverse Problem and Its Applications to Science and Technology**

   Paper: A. Garg, K. Goyal, Kshama Goyal, Introduction of Inverse Problem and it's applications to Science and Technology, 2(2013)30-34. [11]

   In this research paper, researchers describes the idea of ill-posed problem. By the side of carefully describing the problem, they also gives some mathematical examples like Fredholm integral equation, inhomogeneous Helmholtz equation etc. for recognized the article and the problems which are connected with the study of inverse problems. They also discuss about the various applications of ill-posed difficulties in the area of science and technology.

3. **Computational Methods for Inverse Problem**

   Book: Curtis R. Vogel, Computational Methods for Inverse Problem, (2002). [1]

   In this book, there is study of inverse problems and various applications of inverse problem, regularization of inverse problem by various techniques, numerical solution to discretize problem, numerical optimzation methods and identification of differential equation. The important topic is selecting the regularization parameter from the statistical perspective and also provides motivation for adjoint and higher order derivatives.

4. **Numerical algorithms for inverse and ill-posed Problems**

   Book: A.. Denisov, Numerical algorithms for inverse and ill-posed Problems, (1987). [6]

This book describes about inverse problem and ill-posed problems and numerical algorithms for the solution of the problem. Also discussion of inverse and ill-posed problems begins in science, ecology, engineering, medicine, etc. For the research of different processes there are certain mathematical techniques and models. The author tells that the inverse problem starts when some mathematical model parameters are not well-defined. It describes the inverse problems in ordinary and partialy differential equations and many numerical algorithms for solving the inverse problems .

5. **Introduction to Inverse Problems for Differential Equations**
   Book: A. H. Hasanoglu, V. G. Romanov, Introduction to inverse problems for differential equations, (2017). [3]
   The textbook describes the idea of inverse problems arising in PDE's. They tells the nature of inverse problem in differential and integral equations which are depends on well-defined mathematical models. The most unique properties of any inverse problem are invisible in the characterstics of corelating direct problems solutions related to PDE's which are arising in mathematical models. In this book, the author deals with inverse problems related to the second-order hyperbolic differential equations and for the electrodynamic differential equations and elliptic differential equations.

6. **A Survey on Inverse Problems for Applied Sciences**
   Paper: F. Yaman, V. G. Yakhno, and R. Potthast, A Survey on Inverse Problems for Applied Sciences, (2013). [5]
   In this research paper, the researcher tells that inverse problem has engineering applications and we can solve them easily by mathematically methods. It decribes the ill-posed problems for Time-Harmonic Acoustic and Electromagnetic Waves, Boundary Reconstruction Problems of Acoustic Waves. It solves the inverse problems for differential equations of elastodynamics and gives the idea regarding inversion based applications.

Chapter 2

Regularization Techniques for Solving the Inverse Problem

We got important knowledge from the discrete linear system $Px = d$ except ill-condition. Now, let us consider a discrete linear model i.e.,

$$d = Px_{true} + \eta, \tag{2.0.1}$$

with $\delta = ||\eta|| > 0$.
where,

$||.||$ = standard euclidean norm,
x_{true} = true discretized source,
η = noise in the data,
(δ)= noise level.

Next, let us suppose that P is an invertible, real-valued matrix. Then, it has a Singular Value Decomposition (SVD):

$$P = Udiag(s_i)V^T, \tag{2.0.2}$$

with decreasing singular values values $s_i > 0$.Also,

$$P^{-1} = P^T = Vdiag(s_i^{-1})U^T. \tag{2.0.3}$$

Now, we need some below factors:-
column vectors v_i of V termed as right singular vectors, and

column vectors u_i of U, which are the left singular vectors. These vectors satisfy:

$$u_i^T u_j = \delta_{ij}; v_i^T v_j = \delta_{ij}, \tag{2.0.4}$$

$$Pv_i = s_i u_i; P^T u_i = s_i v_i, \tag{2.0.5}$$

here, δ_{ij} is the Kronecker delta and $U^T = U^{-1}$ and $V^T = V^{-1}$.
If P is symmetric and positive definite matrix, then unique values of s_i are the eigenvalues of P and $U = V$ has columns which involves the orthonormalized eigenvectors. Now, post-multiply P^{-1} with equation (2.0.1),we get:

$$\Rightarrow P^{-1}d = P^{-1}Px_{true} + P^{-1}\eta$$

$$\Rightarrow P^{-1}d = x_{true} + P^{-1}\eta$$

$$\Rightarrow Vdiag(s_i^{-1}U^T)d = x_{true} + \sum_{i=1}^{n} s_i^{-1}(u_i^T \eta)v_i$$

$$\Rightarrow P^{-1}d = Vdiag(s_i^{-1})U^T = x_{true} + \sum_{i=1}^{n} s_i^{-1}(u_i^T \eta)v_i. \tag{2.0.6}$$

When we divides the small singular values s_i^{-1}, then there is some unstability. So, there are three methods to remove the error. These are given below:-
2.1) Truncated Singular Value Decomposition (i.e., TSVD)
2.2) Tikhonov(-Phillips) Regularization
2.3) Least Square Fit method

2.1 Truncated Singular Value Decomposition

To overcome this error in equation (2.0.6), we can change the s_i^{-1} i.e., multiplying equation (2.0.6) by a regularizing filter function $w_\alpha(s_i^2)$ and the product $w_\alpha(s_i^2)s^{-1} \to 0$ as $s \to 0$. This filters out the unique elements of $K^{-1}d$ relating to small singular values and gets an approximation to x_{true} with the following representation:

$$x_\alpha = Vdiag(w_\alpha(s_i^2)s_i^{-1})U^T d = \sum_{i=1}^{n} w_\alpha(s_i^2)s_i^{-1}(u_i^T d)v_i. \tag{2.1.1}$$

We should keep the unique elements relating to large singular values for getting some degree of accuracy. It is possible by considering $w_\alpha(s^2) \approx 1$ for large values of s^2. For example, of such a filter function:

$$w_\alpha(s^2) = \begin{cases} 1; if s^2 > \alpha \\ 0; if s^2 \leq \alpha \end{cases} \tag{2.1.2}$$

The approximation of equation (2.1.1) becomes:

$$x_\alpha = \sum_{s_i^2 > \alpha} s_i^{-1} (u_i^T d) v_i, \tag{2.1.3}$$

and this is called as the **Truncated Singular Value Decomposition (i.e., TSVD)** solution to $px = d$.

2.2 Tikhonov Regularization

Let us consider the Tikhonov Filter Function:

$$w_\alpha(s^2) = \frac{s^2}{s^2+\alpha}, \tag{2.2.1}$$

Put $w_\alpha(s^2)$ in equation (2.1.1), we get:

$$x_\alpha = \sum_{i=1}^{n} \frac{s_i(u_i^T d)}{s_i^2+\alpha} v_i = (P^T P + \alpha I)^{-1} P^T d. \tag{2.2.2}$$

The equation (2.2.2) is a result of equation (2.0.2) and equation (2.0.5). This technique is known as **Tikhonov Regularization.**

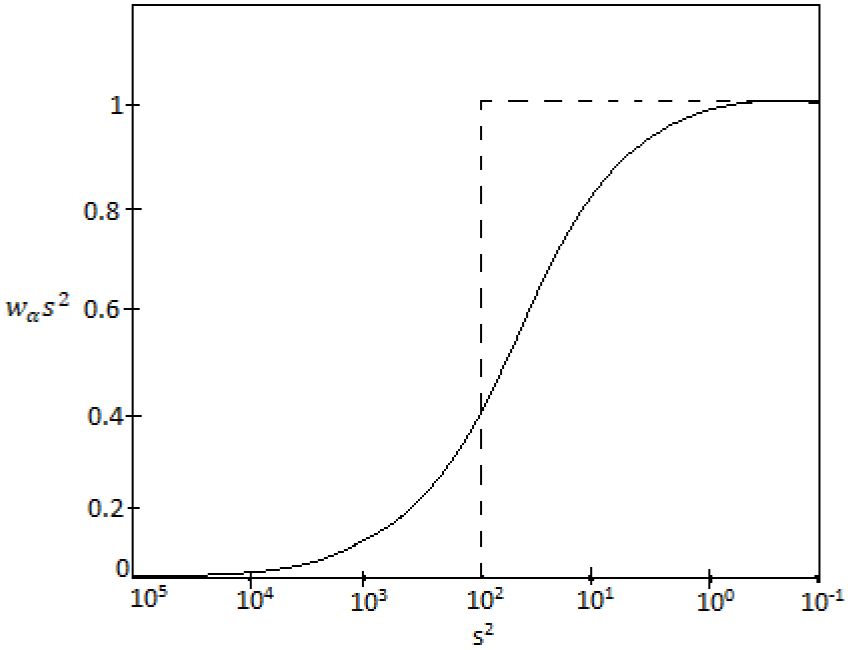

Figure 2:Plotting of Tikhonov's Filter Function

In Figure 2, dotted line represents the filter function w_α corresponding to TSVD regularization and solid line represents the squared singular values s^2 corresponding to Tikhonovs Regularization. The input data for the regularization constant is $\alpha = 10^{-2}$.

Problem:- Numerically implement standard Tikhonov regularization for the test problem. The true solution is given by:

$$f_{true} = \begin{cases} 0.75,\ 0.1 < x < 0.25, \\ 0.25,\ 0.3 < x < 0.32, \\ sin^4(2\pi x),\ 0.5 < x < 1, \\ 0,\ otherwise \end{cases}$$

Solution:- We solve this problem by using this formula $d = Px_{true} + \eta$, where $||\eta|| > 0$ i.e.,$\eta = 0.01(say)$.

$$\Rightarrow d = Px_{true} + 0.01,$$

From the equation (1.2.3), we know that:

$$[P]_{ij} = kcexp\left(\frac{((i-j)k)^2}{2\lambda^2}\right),\ 1 \le i,\ j \le n$$

$$\Rightarrow [P]_{ij} = \frac{1}{n}\frac{1}{\lambda\sqrt{2\pi}}exp\left(\frac{(i-j)^2}{2 \times \lambda^2 \times n^2}\right).$$

Also,we know that there are four inputs so the value of $n = 4, \lambda = 0.05$. Now put these values in the above equation,we get:

$$\Rightarrow [P]_{ij} = \frac{1}{4 \times 0.05\sqrt{2\pi}}exp\left(\frac{(i-j)^2}{2 \times (0.05)^2 \times (4)^2}\right)$$

$$\Rightarrow [P]_{ij} = \frac{1}{4 \times 0.05\sqrt{2\pi}}exp\left(\frac{(i-j)^2}{2 \times 0.0025 \times 16}\right)$$

$$\Rightarrow [P]_{ij} = \frac{1}{0.50142}exp\left(\frac{(i-j)^2}{0.08}\right)$$

$$\Rightarrow [P]_{ij} = 1.9943 \times exp\left(\frac{(i-j)^2}{0.08}\right).$$

So, we get the values of matrix $[P]_{ij}$ by putting the values of i and j, where $1 \le i \le 4$ and $1 \le j \le 4$. Also, we find the $[P]_{ij}$. Then, substituting the values of $[P]_{ij}$, x_{true} and $\eta = 0.01$ in equation (1.2.3) and we get the values of d. Then, we substitute all values in the equation (2.2.2) i.e., $x_\alpha = (P^TP + \alpha I)^{-1}P^Td$ for finding the values of x_α. In this way , we can find the solution of the problem.

The Matlab Code For Tikhonov Regularization is given below:-

```
clear variables
close all
clc
h=0.25;
gamma=0.05;
pi=22/7;
c=1/(gamma*sqrt(2*pi));
for i=1:4
for j=1:4
K(i,j)=h*c*(exp(-(((i-j)*h)^2)/(2*(gamma)^2)));
end
end
K
itta=0.01;
ftrue=[0.75;0.25;sin(2*pi*0.8)^4;0];
d=(K*ftrue)+itta;
d
alpha=10^(-2);
I=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1]
P=transpose(K);
P
falpha=(((P*K)+ (alpha*I))^-1)*P*d
Output:


K=
         1.9943 0.0000 0.0000 0.0000
         0.0000 1.9943 0.0000 0.0000
         0.0000 0.0000 1.9943 0.0000
         0.0000 0.0000 0.0000 1.9943
```

```
d =
         1.5057
         0.5086
         1.6373
         0.0100


 I =
        1  0  0  0
        0  1  0  0
        0  0  1  0
        0  0  0  1

P =
        1.9943 0.0000 0.0000 0.0000
        0.0000 1.9943 0.0000 0.0000
        0.0000 0.0000 1.9943 0.0000
        0.0000 0.0000 0.0000 1.9943

falpha =
                 0.7531
                 0.2544
                 0.8189
                 0.0050
```

2.3 Regularized Least Square Fit Method

Least squares are famous for solving the inverse problems because it helps in the easiest calculations. The drawback of least square in a data set is that only a strong sensitivity to a small number of large errors. First of all, we should know about the least square fit method, it is design of mathematical regression analysis that calculates best fit the line of a data set, by giving a clear signification of the connection between the data points. Every data point is a entitled of the connection between given self-contained variable and not well-known dependent variable. We consider a linear discrete system $Px = q$, where P is $m*n$ matrix with $m \leq n$. Suppose that components of P and q are added by some error. In this case, error in the data is the total least square :

$$min||(P,q)-(\bar{P},\bar{q})||_F, \tag{2.3.1}$$

subject to $\bar{q} = \bar{P}x$. If components of P are exact and only q consists some error. Then, the composition of least square problem is given below:

$$min_x||Px-q||_2. \tag{2.3.2}$$

Our main target is on ill-posed problem where unique value of P decompose slowly to zero. This type of problem arise in the discretization of ill posedness. In these problems, contains noise to the small singular values of P. Due to this, it is compulsory to evaluate a regularized solution in which error effect is filtered out. For least square problems, the filtering is done by Tikhonov's regularization method. For the least squares problems the general method of Tikhonov's regularization for Least Square Method is given below:-

$$min||Px-q||_2^2+\mu||Nx||_2^2, \tag{2.3.3}$$

where $\mu > 0$ that we select to control the size of output vector and L is a matrix which describes the semi-norm $||Nx||_2$ of output vector i.e., L represents the first or second degree differential operator. If N is identity matrix, then tikhonov problem is termed to be in it's standard form:

$$(P^TP+\mu N^TN)x = P^Tq. \tag{2.3.4}$$

When μ increases, then semi-norm $||Nx||_2$ of output vector decreases monotonically while residual i.e., $min||Px-q||_2$ increases monotonically.

Chapter 3

Inverse Problem in Ordinary Differential Equations

When we are solving the inverse problem in differential equations, it consists more innovation than finding standard direct problems which makes the most challenge to knowledge. By solving inverse problems in ordinary differential equation enable us to superior recognized the direct problem on which inverse problems are depend. Now, we consider the example problem of inverse problem which is given in the form of ordinary differential equation.

3.1 An Example Problem

The inverse problem which is based on second order ordinary differential equation which is used to prototype the harmonic motion. The problem posed will not be solved in full generality but we want to generate the concept that what is an inverse problem which is consisted in the model of harmonic motion. The problem becomes critically damped or overdamped in place of the assumption. In case of underdamped requires of alterations to the approaches which are given in the problem.

Formulate a method of recovering the coefficients in second order ordinary differential equation:

$$mx''(t) + bx'(t) + kx(t) = f(t), \tag{3.1.1}$$

from the measurements of $x(t)$. Here we are free to select initial conditions and function(s) in any manner which fits properly. We assume that the solution to homogeneous equation approaches zero over time.

3.2 Two Approaches for the Solution of Inverse Problems

We are applying two approaches to the solution to get proper solution. There are two methods to solve the second order ordinary differential equation:-
1.Data Fitting Method
2.Non-linear Equation Method

3.2.1 Data Fitting Method

We take $f(t) = t^2$ for the right hand side since this choice leads to a particular solution .

$$mx"(t) + bx'(t) + kx(t) = t^2, \tag{3.2.1}$$

where $m = 1, b = 10, k = 49$.
Now, we are solving this differential Equation by putting the values of $m,\ b,\ k$ in equation(3.2.1). Therefore it becomes:

$$x''(t) + 10x'(t) + 49x(t) = t^2. \tag{3.2.2}$$

Complementary function:-To calculate the value of this function, we are using D instead of derivative. So that it becomes in the following form:-

$$\begin{aligned} D^2 + 10D + 49 &= 0 \\ \Rightarrow D &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \Rightarrow D &= \frac{-10 \pm \sqrt{(10)^2 - 4 * 1 * 49}}{2 * 1} \\ \Rightarrow D &= \frac{-10 \pm \sqrt{100 - 196}}{2} \\ \Rightarrow D &= \frac{-10 \pm \sqrt{-96}}{2} \\ \Rightarrow D &= -5 \pm 2\sqrt{6}i. \end{aligned}$$

Therefore, $C.F. = e^{(-5t)}(c_1 cos(2\sqrt{6}t) + c_2 sin(2\sqrt{6}t))$.
Here, $e^{(-5t)}$ is decreasing function and $(c_1 cos(2\sqrt{6}t) + c_2 sin(2\sqrt{6}t))$ is bounded function. So, we neglect it due to very small term.

Particular Solution:-Now, we are solving the particular solution of equation (3.2.2).
We know that,

$$f(t) = at^2 + bt + c. \tag{3.2.3}$$

Differentiate equation (3.2.3) w.r.t. to t, we get $f'(t)$:-

$$\Rightarrow f'(t) = 2at + b.$$

Also differentiate $f'(t)$ w.r.t. to t, we get $f''(t)$:

$$\Rightarrow f''(t) = 2a.$$

Now, substitute the values of $f(t)$, $f'(t)$ and $f''(t)$ in the following equation (3.2.4), and we get:-

$$f''(t) + 10f'(t) + 49f(t) = t^2 \tag{3.2.4}$$

$$\Rightarrow 2a + 10(2at + b) + 49(at^2 + bt + c) = t^2$$

$$\Rightarrow 2a + 20at + 10b + 49at^2 + 49bt + 49c = t^2$$

$$\Rightarrow 49at^2 + (20a + 49b)t + 2a + 10b + 49c = t^2.$$

Now, we are comparing the coefficients of t on both sides, so that we calculate the values of a, b, and c respectively:-

$$49a = 1$$

$$\Rightarrow a = \frac{1}{49},$$

$$20a + 49b = 0$$

$$\Rightarrow 49b = -20a$$

Now, put the value of $a = 1/49$ in the above equation to find the value of b, we get:-

$$\Rightarrow b = \frac{-20}{49} \times \frac{1}{49}$$

$$\Rightarrow b = -\frac{20}{(49)^2}$$

$$\Rightarrow b = -\frac{2 \times 10}{(49)^2},$$

$$2a+10b+49c=0.$$

We substitute the values of $a=1/49$ and $b=-20/(49)^2$ in the above equation and we get:-

$$\begin{aligned}
&\Rightarrow 2\times\left(\frac{1}{49}\right)+10\times\left(-\frac{20}{(49)^2}\right)+49c=0\\
&\Rightarrow 49c=-\frac{2}{49}+\frac{200}{(49)^2}\\
&\Rightarrow c=-\frac{2}{(49)^2}+\frac{200}{(49)^3}\\
&\Rightarrow c=-\frac{2}{(49)^2}+\frac{2\times(10)^2}{(49)^3}.
\end{aligned}$$

Now, put all the above calculated values of a, b and c in the equation (3.2.3) and we get the following equation (3.2.5):-

$$f(t)=\frac{1}{49}t^2-\frac{2\times 10}{(49)^2}t-\frac{2}{(49)^2}+\frac{2\times(10)^2}{(49)^3}. \tag{3.2.5}$$

Therefore, we get the following particular solution $x_p(t)$ (in general) :-

$$x_p(t)=\frac{1}{k}t^2-\frac{2b}{k^2}t-\frac{2m}{k^3}+\frac{2b^2}{k^3} \tag{3.2.6}$$

$$\Rightarrow f(t)=a_2t^2-a_1t+a_0. \tag{3.2.7}$$

where,

$$a_2=\frac{1}{k},\ a_1=-\frac{2b}{k^2},\ a_0=-\frac{2m}{k^2}+\frac{2b^2}{k^3} \tag{3.2.8}$$

If we suppose that the system runs up till $x_p(t)$ is superior, then we can apply the least squares fit in the form of $a_2t^2+a_1t+a_0$ to input and evaluate the system for m, b, k (inverse values) from the calculated values of a_2, a_1, a_0. Clearly,by using this method the basic conditions are not required specially. When we perform least square fit on equation (3.2.7),we get:

$$\sum_{i=1}^{n} f(t)=a_2\sum_{i=1}^{n} t^2-a_1\sum_{i=1}^{n} t+a_0n. \tag{3.2.9}$$

Now,we are multiplying the equation(3.2.9) with t,we get:-

$$\sum_{i=1}^{n} tf(t) = a_2\sum_{i=1}^{n} t^3 - a_1\sum_{i=1}^{n} t^2 + a_0\sum_{i=1}^{n} t. \tag{3.2.10}$$

Also,multiply the equation(3.2.10) with t,we get the following equation(3.2.11) :-

$$\sum_{i=1}^{n} t^2 f(t) = a_2\sum_{i=1}^{n} t^4 - a_1\sum_{i=1}^{n} t^3 + a_0\sum_{i=1}^{n} t^2. \tag{3.2.11}$$

Now, we calculate the values of t, $f(t)$, $tf(t)$, t^2, t^3, t^4 and $t^2f(t)$ in equations (3.2.9), (3.2.10) and (3.2.11) from the below the table:

Table 1

t	$f(t)$	$tf(t)$	t^2	t^3	t^4	$t^2f(t)$
0	0.0008	0	0	0	0	0
1	0.0129	0.0129	1	1	1	0.0129
2	0.0658	0.1316	4	8	16	0.2633
$\sum t = 3$	$\sum f(t) =$ 0.0796	$\sum tf(t) =$ 0.1446	$\sum t^2 =$ 5	$\sum t^3 =$ 9	$\sum t^4 =$ 17	$\sum t^2f(t) =$ 0.2763

Put these all calculated values of t, $f(t)$, $tf(t)$, t^2, t^3, t^4 and $t^2f(t)$ in equations (3.2.9), (3.2.10) and (3.2.11) respectively, we get:

$$3a_0 - 3a_1 + 5a_2 = 0.08, \tag{3.2.12}$$

$$3a_0 - 5a_1 + 9a_2 = 0.14, \tag{3.2.13}$$

$$5a_0 - 9a_1 + 17a_2 = 0.27. \tag{3.2.14}$$

After solving these three above equations ,we get the values of a_0, a_1, a_2 respectively

$$a_0 = 0.02167;\ a_1 = 0.12;\ a_2 = 0.075 \tag{3.2.15}$$

Now put the values of equation (3.2.15) in equation (3.2.8),we get:

$$m = 6.605986;\ b = 10.677336;\ k = 13.34 \tag{3.2.16}$$

3.2.2 Non-linear Equation Method

We take $f(t) = F_0 cos(\omega t)$ on the right side for finding the particular solution in the following non-linear equation.

$$mx''(t) + bx'(t) + kx(t) = F_0 cos(\omega t), \tag{3.2.17}$$

where $m = 1,\ b = 10,\ k = 49$. Now, we are solving the differential Equation and it becomes:

$$x''(t) + 10x'(t) + 49x(t) = F_0 cos(\omega t). \tag{3.2.18}$$

Complementary function:-To calculate the value of this function, we are using D instead of derivative. So, it becomes in the following form:-

$$\begin{aligned}
D^2 + 10D + 49 &= 0 \\
\Rightarrow D &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
\Rightarrow D &= \frac{-10 \pm \sqrt{(10)^2 - 4*1*49}}{2*1} \\
\Rightarrow D &= \frac{-10 \pm \sqrt{100 - 196}}{2} \\
\Rightarrow D &= \frac{-10 \pm \sqrt{-96}}{2} \\
\Rightarrow D &= -5 \pm 2\sqrt{6}i.
\end{aligned}$$

Therefore, $C.F. = e^{(-5t)}(c_1 cos(2\sqrt{6}t) + c_2 sin(2\sqrt{6}t))$.
Here, $e^{(-5t)}$ is decreasing function and $(c_1 cos(2\sqrt{6}t) + c_2 sin(2\sqrt{6}t))$ is bounded function. So, we neglect it due to very small term.

Analytic Solution:-We are using the three different forcing functions in the form of $F_0 \cos(\omega t)$ and the amplitude is calculated . The analytic solution must have an amplitude which is given below equation:

$$A(m,k,b) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}. \tag{3.2.19}$$

We can get the system of non-linear equations from the three different values of $\omega = 1,\ 2,\ 3$, a 3×3 system of non-linear equation. Now, we are using the Newton's method for solving the system of equations for $m,\ b$, and k. The main advantage of this approach is that it can be easily used in terms of RLC circuit. So, in this case the actual data can be collected. Also, the need of initial conditions are not important in the analytic solution.

Also, we know that the values of $m,\ b$, and k respectively(i.e., $m = 1;\ k = 49;\ b = 10$). So, we can perform the newton method $(x_{n+1} = x_n - [J(x_n)]^{-1} * f(x_n))$ easily on equation (3.2.19) :-
At initial point,

$$x_0 = \begin{bmatrix} 1 \\ 49 \\ 10 \end{bmatrix};f(x_0) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

and

$$J(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial m} & \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial m} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial b} \\ \frac{\partial f_3}{\partial m} & \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial b} \\ \end{bmatrix}$$

where,
For $\omega = 1$,

$$f_1(m,k,b) = \frac{1}{\sqrt{(k-m)^2 + b^2}};$$

For $\omega = 2$

$$f_2(m,k,b) = \frac{1}{\sqrt{(k-4m)^2 + 4b^2}};$$

For $\omega = 3$

$$f_3(m,k,b) = \frac{1}{\sqrt{(k-9m)^2 + 9b^2}}.$$

Now,

$$\frac{\partial f_1}{\partial m} = \frac{k-m}{((k-m)^2 + b^2)^{\frac{3}{2}}};$$
$$\frac{\partial f_1}{\partial k} = \frac{-(k-m)}{((k-m)^2 + b^2)^{\frac{3}{2}}};$$
$$\frac{\partial f_1}{\partial b} = \frac{-b}{((k-m)^2 + b^2)^{\frac{3}{2}}};$$

$$\frac{\partial f_2}{\partial m} = \frac{4(k-4m)}{((k-4m)^2+4b^2)^{\frac{3}{2}}};$$

$$\frac{\partial f_2}{\partial k} = \frac{-(k-4m)}{((k-4m)^2+4b^2)^{\frac{3}{2}}};$$

$$\frac{\partial f_2}{\partial b} = \frac{-4b}{((k-4m)^2+4b^2)^{\frac{3}{2}}};$$

$$\frac{\partial f_3}{\partial m} = \frac{9(k-9m)}{((k-9m)^2+9b^2)^{\frac{3}{2}}};$$

$$\frac{\partial f_3}{\partial k} = \frac{-(k-9m)}{((k-9m)^2+9b^2)^{\frac{3}{2}}};$$

$$\frac{\partial f_3}{\partial b} = \frac{-9b}{((k-9m)^2+9b^2)^{\frac{3}{2}}}$$

, and

$$J(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial m} & \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial m} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial b} \\ \frac{\partial f_3}{\partial m} & \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial b} \end{bmatrix}.$$

Now, put the values of $m = 1;\ k = 49;\ b = 10$ in the above jacobian,we get:

$$J(x_0) = \begin{bmatrix} 0.00040 & -0.00040 & -0.000084 \\ 0.0015 & -0.00037 & -0.00033 \\ 0.00288 & -0.00032 & -0.00072 \end{bmatrix}$$

$$[J(x_0)]^{-1} = \begin{bmatrix} -4380.23012552301 & 7112.97071129706 & -2749.08472803347 \\ -3530.33472803347 & 1255.23012552301 & -163.441422594142 \\ -15951.8828451883 & 27894.0027894002 & -12312.5871687587 \end{bmatrix}$$

Now,we are using the Newton's Method :- $x_{n+1} = x_n - [J(x_n)]^{-1} * f(x_n)$

For $n = 0$:- $x_1 = x_0 - [J(x_0)]^{-1} * f(x_0))$. i.e.,

$$x_1 = \begin{bmatrix} 1 \\ 49 \\ 10 \end{bmatrix} - \begin{bmatrix} -4380.23012552301 & 7112.97071129706 & -2749.08472803347 \\ -3530.33472803347 & 1255.23012552301 & -163.441422594142 \\ -15951.8828451883 & 27894.0027894002 & -12312.5871687587 \end{bmatrix} * \begin{bmatrix} 0.02039 \\ 0.14250 \\ 0.02 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 49 \\ 10 \end{bmatrix} - \begin{bmatrix} 865.7 \\ 3389.4 \\ 1030 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} -864.7 \\ -3379.4 \\ -981 \end{bmatrix}$$

3.3 MATLAB Coding

3.3.1 Matlab coding for Data Fitting Method

The matlab code of the sample problem for the linear equation and we are performing the least square fit by using the MATLAB command ”polyfit”. The second matlab code using the MATLAB function ”ode45”.

```
datafit.m
tol=10e-6; t0=0; tf=5;
f='yp'; y0=[1;0];
[T,U]=ode45(f,t0,tf,y0,tol,1)
q=length(T);
x=T(q-20:q); y=U(q-20:q,1);
p=polyfit(x,y,2);
```

```
prime.m
function yprime = yp(t,y)
yprime(1)=y(2);
yprime(2)=t*t-10*y(2)-49*y(1);
```

3.3.2 Matlab coding for Non-linear Equation Method

The first matlab code is used to evaluate the Jacobian of the function in the non-linear equation method . The third matlab code computes the amplitudes of the analytic solution. The final matlab code computes the solution via a simplified version of Newton’s method.

```
jacobian.m
function y = jac(m,b,k,w)
y=zeros(3);
for j=1:3
denom = ((k-m*w(j)*w(j))^2+b*b*w(j)*w(j))^(3/2);
y(j,1) = (k-m*w(j)^2)*w(j)^2/denom; % derivative wrt m
y(j,2) = -b*w(j)*w(j)/denom; % derivative wrt b
y(j,3) = (m*w(j)^2-k)/denom;
end
```

```
Final.m
function y=F(m,b,k,w,amp)
y=zeros(3,1);
for j=1:3
y(j) = 1/((k-m*w(j)^2)^2+b^2*w(j)^2)^(1/2) - amp(j);
end
```

```
amplitude.m
function y=camp(m,b,k,w)
y=1/((k-m*w^2)^2 + b^2*w^2)^(1/2);
```

```
nonlinear.m
for j=1:3
amp(j) = camp(10,3,49,j);
end
w=1:3;
x=ones(3,1);
noise=2*rand(3,1)-1;
amp2=amp’+.001*noise*max(amp);
```

```
for i=1:10
x=x-inv(jac(x(1),x(2),x(3),w))*F(x(1),x(2),x(3),w,amp2)
end
```

Conclusion

In this thesis report, we have studied that inverse problems which are used in differential equation in many fields like medical imaging, geophysics, image formation, physics, ground-water flow model, navigation and so many other fields. We have solved the ordinary differential equation and then find the inverse problem of linear differential equation by using the method of Least Square Fit and of non-linear differential equation by using the Newton method.

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