GENERALIZED PARABOLIC FUZZY NUMBERS AND IT'S APPLICATION

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Under the guidance of
Dr. Harish Garg

School of Mathematics and Computer Applications
Thapar University
Patiala – 147004 (Punjab)
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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Generalized Parabolic Fuzzy numbers and it's application**" in partial fulfillment of the requirement for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own carried out under the supervision of Dr. Harish Garg, Assistant Professor, SMCA, Thapar University Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

Ansha
(ANSHA)
Reg. No. 301203001

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Date: July 16 2014

(Dr. Harish Garg)
Supervisor

Countersigned by:

Dr. Rajesh Kumar
Head SMCA
Thapar University
Patiala

Dr. S. K. Mohapatra
Dean of Academic Affairs
Thapar University
Patiala

Abstract

Probability theory is being used extensively in engineering and management for various analyses. As the conventional theory is based on the probabilities and binary state structure for analyzing the performance of the structure and therefore result based on it do not always provide useful results to the practitioners due to its limitation of being able to handle only quantitative information. So the result obtained are therefore not of much practical value. This is primarily due to the fact that there is significant impact on subjective information in relation to the available quantitative information. One way to handle the subjective information is the use of fuzzy set theory. Research in last two decades, however, has shown that probability theory is not the only possible way to represent imprecision and uncertainty. Fuzzy set theory provides a significant alternative to the probabilistic approach to find the various arithmetic operations for evaluating the performance of the system. In recent times, the use of fuzzy sets has been gaining popularity and is playing an important role in the areas of engineering and management disciplines. As compared to other research domains, the fuzzy arithmetic gained great interest in scientific areas such as decision problems, reliability analysis, optimization etc. In order to perform operations on fuzzy observations, fuzzy numbers came into existence.

The objective of this work is to carry out the analyze of the various arithmetic operations using generalized parabolic fuzzy numbers. For this various arithmetic operations has been studied by taking parabolic fuzzy numbers. As most of the data collected from the various resources are generally imprecise, vague and uncertain. So to handle these types of data, fuzzy set theory has been used and then analyzed the system in the form of fuzzy membership functions by using the concept of distribution and complementary distribution functions. The advantage of the proposed generalized parabolic fuzzy number is that it gives compressed range of prediction. Another advantages of the proposed approach are that they do not need the computation of $\alpha-$ cut of the fuzzy number. Results obtained by using fuzzy numbers are practically much better than those obtained by classical methods.

The present thesis is organized into five chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of arithmetic operations, membership functions etc., by using conventional, fuzzy and optimization techniques is presented in the first chapter. In **Chapter 2**, the basic and preliminaries related to the arithmetic operation and to be used in the subsequent chapters are given.

Chapter 3 presents a concept of trapezoidal fuzzy number is extended to parabolic fuzzy number. A generalized parabolic fuzzy number has been introduced here. Also, the definition of trapezoidal parabolic fuzzy number and arithmetic operations between two trapezoidal parabolic fuzzy numbers are introduced. Various arithmetic operations, such as addition, subtraction, multiplication, inverse, division etc are studied by using the concept of the distribution and complementary distribution functions. The major advantages of the technique are that they do not need the computation of $\alpha-$ cut of the fuzzy number and hence it becomes more powerful where the standard method fails. The operations have been validated through some elementary applications and results are compared with that of $\alpha-$ cut method and shows the supremacy of the result.

In **Chapter 4**, the expression of the defuzzification by the various methods such as center of area, the bisection of area, largest of maxima, smallest of maxima, regular weighted point etc., are computed by using trapezoidal fuzzy numbers and trapezoidal parabolic fuzzy numbers. Based on their results, it has been concluded that the defuzzified expression for triangular fuzzy numbers and triangular parabolic fuzzy numbers are special cases for our computed results.

Chapter 5 deals with the overall concluding observations of this study and a brief discussion on the scope for future work.

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Ansha

(Ansha)

Patiala

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Chapter 1

Introduction

Under the growing complexities of the system, problems in the real world quite often turn out to be complex owing to an element of uncertainty either in the parameters which define the problem or in the situations in which the problem occurs. The conventional approaches for uncertainty analysis are very important as it help for making effective decision-making. The conventional approach for uncertainty analysis assume a probability distribution for the failure probability of system components or basic events to find the uncertainty in the overall system failure probability. However, it is very difficult to make statistical interference in case of systems where available data is insufficient. As the probability approach has been applied successfully for many real world engineering problems but still there are some limitations to the probabilistic method. For instance, probabilistic methods are based on mass collection of data, which is random in nature, to achieve the requisite confidence level. But in large scale the complicated system has the massive fuzzy uncertainty due to which it is difficult to get the exact probability of the events. Thus, results based on probability theory do not always provide useful information to the practitioners due to the limitation of being able to handle only quantitative information. Moreover, in real world applications, sometimes there is insufficient data to accurately handle the statistics of parameters. Due to these limitations, the results based on probability theory do not always provide useful information to the practitioners and hence probabilistic approach is inadequate to account for such built-in uncertainties in the data.

To overcome these difficulties, methodologies based on fuzzy set theory [21] are being used in the risk analysis for propagating the basic event uncertainty. The probabilistic approaches deal with uncertainty, which is random in nature, while the fuzzy approach deals with the uncertainty, which is due to imprecision associated with the complexity of the system as well as vagueness of human judgement difficulties, Fuzzy set theory [21] has been viewed as a useful tool, especially for dealing with the complex systems, in which the interactions of the system's variables may be too complex to be precisely specified. However, we could find that fuzzy logic may obtain different simulated efficiency and performance while adopting various forms of fuzzy arithmetic, and the fuzzy arithmetic operations have necessary condition which operations have to use triangular fuzzy numbers [7]. In the framework of fuzzy arithmetic [12] various operations as, e.g., addition, subtraction, etc., are realized. These operations are made with the use of Zadeh's possibilistic extension principle [8] or its new, improved, and also possibilistic version proposed by Klir [14], which takes into account the so-called requisite constraints. Arithmetic operations are also performed under the assumption which was introduced by Zadeh [23] that the membership function of a fuzzy set is of a possibilistic character and that each element of the universal set, with a non-zero membership grade, belongs to a fuzzy set.

For the past few years, some people have worked on arithmetic operations on fuzzy numbers. Piegat [17] presented a definition of fuzzy which allows for a considerable fuzziness decrease in the number of arithmetic operations. He also introduced new notions such as qualifier, qualification algorithm and property function of set. Stefanini and Guerra [18] analyzed decomposition of fuzzy numbers in order to study some properties of fuzzy arithmetic operations and compared the proposed approximation with the results of standard fuzzy mathematics. Gao et al. [9] worked on four methods for solving multiplication operation of two fuzzy numbers. These are non-linear programming method, analytical method, computer method and computer simulation method. Akther and Ahmad [1] presented a way of computing arithmetic operations of fuzzy numbers as well as an analytic form of resultant membership functions. Mahanta et al. [15] gave method that can be utilized in cases where the method of α-cuts fails. Taleshian and Rezvani [19] gave methods for solving multiplication operation of two trapezoidal fuzzy numbers. Chutia et al. [4] developed a method of finding membership function for functions of triangular fuzzy variable from the concept of credibility theory and a method for computation of basic arithmetical operations of fuzzy variables is forwarded. Bansal [3] explored the arithmetic properties of an arbitrary trapezoidal fuzzy number. Oussalah [16] addressed theoretical results about some invariance properties concerning the relationships between the defuzzification outcomes and the arithmetic of fuzzy numbers. Gao et al. [9] worked on four methods for solving multiplication operation of two fuzzy numbers. These are non-linear programming method, analytical method, computer method and computer simulation method. Deschrijver [5] analyzed the arithmetic operations in both interval and intuitionistic fuzzy set theory. Banerjee and Roy [2] studied defuzzification method for generalized trapezoidal fuzzy numbers based on the Zadeh's extension principle method, interval method and vertex method. For more details about the applications and fuzzy arithmetic operation, we may refer to [2, 5, 6, 10, 11, 13, 17, 20] and their corresponding references.

1.1 Objective of the Thesis

The arithmetic operations on fuzzy quantities are widely used in the literature. There are two well-known additions of fuzzy quantities in vector space that have been adopted in the literature. One is based on the extension principle by directly considering the membership functions, and another one uses the $\alpha-$ level sets without considering the membership functions. Here, in this thesis, instead of using these methods for computing the membership functions, we compute the membership functions by using distribution and complementary distribution functions. Thus, the objective of the presented work is to compute the various arithmetic operations of fuzzy set theory by utilizing the available, imprecise and vague data. Fuzzy set theory has been used for handling the uncertainties in the data and then analyze the various arithmetic operations in the form of fuzzy membership functions using generalized parabolic fuzzy numbers. The concept of distribution and complementary distribution functions have been used for computing these membership functions. The advantage of this approach is that it gives a compressed range of prediction and it does not involve the computation of $\alpha-$ cuts for finding the membership functions.

Apart from their computation of the membership functions, the computation of the defuzzified values/expressions for the generalized parabolic fuzzy numbers has been established. Various types of defuzzification methods such as center of area, the bisection of area, mean of maxima, largest of maxima, regular weighted point etc., are computed for the fuzzy numbers. Based on their defuzzified values, the defuzzified values of the triangular, and trapezoidal parabolic and linear fuzzy numbers are extracted from it by considering it as a special case.

1.2 Structure of the Thesis

The present thesis is organized into five chapters including the present one that contains mainly the literature review. The rest of the chapters are described below:

In **Chapter 2**, the basic and preliminaries related to the arithmetic operation and to be used in the subsequent chapters are given.

Chapter 3 presents a concept of trapezoidal fuzzy number is extended to parabolic fuzzy number. A generalized parabolic fuzzy number has been introduced here. Also, the definition of trapezoidal parabolic fuzzy number and arithmetic operations between two trapezoidal parabolic fuzzy numbers are introduced. Various arithmetic operations, such as addition, subtraction, multiplication, inverse, division etc are studied by using the concept of the distribution and complementary distribution functions. The major advantages of the technique are that they do not need the computation of $\alpha-$ cut of the fuzzy number and hence it becomes more powerful where the standard method fails. The operations have been validated through some elementary applications and results are compared with that of $\alpha-$ cut method and shows the supremacy of the result.

In **Chapter 4**, the expression of the defuzzification by the various methods such as center of area, the bisection of area, largest of maxima, smallest of maxima, regular weighted point etc., are computed by using trapezoidal fuzzy numbers and trapezoidal parabolic fuzzy numbers. Based on their results, it has been concluded that the defuzzified expression for triangular fuzzy numbers and triangular parabolic fuzzy numbers are special cases for our computed results.

Chapter 5 deals with the overall concluding observations of this study and a brief discussion on the scope for future work.

Chapter 2

Preliminaries

This chapter presents some of the fundamental definitions and mathematical theory for fuzzy set theory. The focus is on defining the fuzzy set, $\alpha-$ cuts, convex and normal fuzzy set, fuzzy numbers and Parabolic fuzzy numbers.

2.1 Fuzzy set

Basically, a set is defined as a collection of objects, which share certain characteristics. A classical set is a collection of distinct objects. The classical set is defined in such a way that the universe of discourse is split into two groups: members and nonmembers. Consider an object x in a crisp set A. This object x is either a member or a nonmember of the given set A. In case of crisp sets, no partial membership exists. This binary issue of membership can be represented mathematically by the indicator function,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.1.1}$$

where χ_A is the membership in set A for element x in the universe. The membership concept represents mapping from an element x in universe X to one of the two elements in universe Y (either to element 0 or 1). There exist a function-theoretic set called value set for any set A defined on universe X, based on the mapping of characteristic function, The whole set is assigned a membership value 1, and the null set is assigned a membership value 0.

In other words, fuzzy sets [21] may be viewed as an extension and generalization of the basic concepts of crisp sets. An important property of fuzzy set is that it allows partial membership i.e. between 0 and 1. Zadeh extended the notion of valuation set $\{1, 0\}$ (definitely in / definitely out) to the interval of real values (degree of membership) between 1 and 0, denoted as [0, 1], where 0.0 represents the absolutely false and 1.0 represents absolutely truth. The fuzzy set $\widetilde{A}$ in the universe of discourse U is defined as a set of ordered pairs $(x, \mu_{\widetilde{A}}(x))$, i.e.

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x) \mid x \in U\} \tag{2.1.2}$$

where $\mu_{\widetilde{A}}(x)$ is the degree of membership of x in fuzzy set $\widetilde{A}$ and it indicates the degree that x belongs to $\widetilde{A}$. Clearly $\mu_{\widetilde{A}}(x) \in [0, 1]$.

2.2 Membership functions

Membership function defines the fuzziness in a fuzzy set irrespective of the elements in the set, which are discrete or continuous. The membership functions are generally represented in graphical form. There exist certain limitations for the shapes used to represent graphical form of membership function. The rule that describe fuzziness graphically are also fuzzy. But standard shapes of the membership functions are maintained over the years. The membership function defines all the information contained in a fuzzy set; hence it is important to discuss the various features of the membership functions. For a fuzzy set $\widetilde{A}$ a membership function, denoted by $\mu_{\widetilde{A}}(\cdot)$ maps U to the membership space M, i.e. $\mu_{\widetilde{A}} : U \to M$. The membership value ranges in the interval [0, 1] i.e. the range of the membership function is a subset of the non-negative real numbers whose supremum is finite.

The three main basic features involved in characterizing membership function are the following.

(i) **Core:** The core of a membership function for some fuzzy set $\widetilde{A}$ is defined as that region of universe that is characterized by complete membership in the set $\widetilde{A}$. The core has elements x of the universe such that

$$\mu_{\widetilde{A}}(x) = 1 \tag{2.2.1}$$

The core of a fuzzy set may be an empty set.

(ii) **Support:** The support of a membership function for a fuzzy set $\widetilde{A}$ is defined as that region of universe that is characterized by a nonzero membership in the set $\widetilde{A}$. The support comprises elements x of the universe such that

$$\mu_{\widetilde{A}}(x) > 0 \tag{2.2.2}$$

(iii) **Boundary:** The boundary of a membership function for a fuzzy set $\widetilde{A}$ is defined as the region of universe that contains a nonzero but not a complete membership. In other words the boundary comprises those elements x of the universe such that

$$0 < \mu_{\widetilde{A}}(x) < 1 \tag{2.2.3}$$

The boundary elements are those which possess partial membership in the fuzzy set $\widetilde{A}$.

2.3 $\alpha-$ cuts

α-cut is one of the most significant and extensively used concept in fuzzy set theory which was introduced by Zadeh [22]. When we want to exhibit an element $x \in X$ that typically belongs to a fuzzy set $\widetilde{A}$, we may demand that its membership value be greater than some threshold $\alpha \in [0, 1]$.

For a fuzzy set $\widetilde{A}$,

$$\begin{aligned} A_\alpha &= \{x \mid \mu_{\widetilde{A}}(x) > \alpha\}; \qquad \alpha \in [0,1) \\ A_\alpha &= \{x \mid \mu_{\widetilde{A}}(x) \geq \alpha\}; \qquad \alpha \in [0,1) \end{aligned}$$

are called strong $\alpha-$ cut and weak $\alpha-$ cut respectively.

2.4 Convex and Normal Set

A fuzzy set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)\} \subseteq U$ is called convex fuzzy set if the following relation between the elements x_1, x_2 and x_3 of $\widetilde{A}$ holds

$$\mu_{\widetilde{A}}(x_2) \geq \min[\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_3)] \tag{2.4.1}$$

Otherwise the fuzzy set is called non-convex fuzzy set.

A fuzzy set whose membership function has at least one element x in the universe whose membership value is unity is called normal fuzzy set.

2.5 Fuzzy number

A fuzzy number is an extension of a regular number in which the the value corresponding to element has its own weight between 0 and 1, called membership functions, instead of single values. Thus, a fuzzy number is a convex and normal fuzzy set of the real line $\mathbb{R}$ such that

(i) $\exists$ exactly one $x_0 \in \mathbb{R}$ with $\mu_{\widetilde{A}}(x_0) = 1$.

(ii) $\mu_{\widetilde{A}}$ is piecewise continuous.

and its membership function is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} f_A(x) & ; \quad \text{if} \quad a_1 \leq x \leq a_2 \\ 1 & ; \quad \text{if} \quad x = a_2 \\ g_A(x) & ; \quad \text{if} \quad a_2 \leq x \leq a_3 \\ 0 & ; \quad \text{if} \quad \text{otherwise} \end{cases} \tag{2.5.1}$$

where $0 \leq \mu_{\widetilde{A}}(x) \leq 1$ and $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 \leq a_2 \leq a_3$ and the two functions $f_A, g_A : \mathbb{R} \to [0, 1]$ are called the sides of the fuzzy numbers. The function f_A and g_A are nondecreasing and nonincreasing continuous functions respectively. Dubois and Prade named $f_A(x)$ as left reference function and $g_A(x)$ as right or complementary reference function of concerned fuzzy number. We denote this fuzzy number as $\widetilde{A} = (a_1, a_2, a_3)$ where $\widetilde{A}$ represents the fuzzy set of A.

The α-cuts of the triangular fuzzy set is defined in the closed interval form as below

$$A_\alpha = [A_1^{(\alpha)}, A_3^{(\alpha)}] \tag{2.5.2}$$

The basic arithmetic operations, i.e., addition, subtraction, multiplication and division, of fuzzy numbers depends upon the arithmetic of the interval of confidence. The four main arithmetic operation on two triangular fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ described by the $\alpha-$ cuts are given below for the following intervals:

$$A^{(\alpha)} = [A_1^{(\alpha)}, A_3^{(\alpha)}] \text{ and } B^{(\alpha)} = [B_1^{(\alpha)}, B_3^{(\alpha)}], \quad \alpha \in [0,\ 1]$$

(i) Addition : $\tilde{A} + \tilde{B} = [A_1^{(\alpha)} + B_1^{(\alpha)}, A_3^{(\alpha)} + B_3^{(\alpha)}]$

(ii) Subtraction : $\tilde{A} - \tilde{B} = [A_1^{(\alpha)} - B_3^{(\alpha)}, A_3^{(\alpha)} - B_1^{(\alpha)}]$

(iii) Multiplication : $\tilde{A} \cdot \tilde{B} = [P^{(\alpha)}, Q^{(\alpha)}]$
where $P^{(\alpha)} = \min(A_1^{(\alpha)} \cdot B_1^{(\alpha)}, A_1^{(\alpha)} \cdot B_3^{(\alpha)}, A_3^{(\alpha)} \cdot B_1^{(\alpha)}, A_3^{(\alpha)} \cdot B_3^{(\alpha)})$
and $Q^{(\alpha)} = \max(A_1^{(\alpha)} \cdot B_1^{(\alpha)}, A_1^{(\alpha)} \cdot B_3^{(\alpha)}, A_3^{(\alpha)} \cdot B_1^{(\alpha)}, A_3^{(\alpha)} \cdot B_3^{(\alpha)})$

(iv) Division : $\tilde{A} \div \tilde{B} = \tilde{A} \cdot \dfrac{1}{\tilde{B}}$, if $0 \notin \tilde{B}$

It is clear that the multiplication and division of two TFNs is not again a TFN with linear sides but it is a new fuzzy number with parabolic sides.

2.6 Parabolic Fuzzy Number

A fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$ is said to be a parabolic fuzzy number (PFN) if its membership function is defined as below

$$\mu_{\widetilde{A}}(x) = \begin{cases} \left(\dfrac{x - a_1}{a_2 - a_1}\right)^2, & ; \text{ if } a_1 \le x \le a_2 \\ 1 & ; \text{ if } x = a_2 \\ \left(\dfrac{a_3 - x}{a_3 - a_2}\right)^2, & ; \text{ if } a_2 \le x \le a_3 \\ 0, & ; \text{ if otherwise} \end{cases} \tag{2.6.1}$$

2.7 Generalized parabolic fuzzy number

A fuzzy number $\widetilde{A} = (a_1, a_2, a_3; \omega)$, defined on the universal set of real numbers $\mathbb{R}$, is said to be generalized fuzzy number it its membership function has the following characteristics

(i) $\mu_{\widetilde{A}}(x) : \mathbb{R} \to [0, 1]$ is continuous.

(ii) $\mu_{\widetilde{A}}(x) = 0$, for all $x \in (-\infty, a_1] \cup [a_3, \infty)$.

(iii) $\mu_{\widetilde{A}}(x)$ is strictly increasing on $[a_1, a_2]$ and strictly decreasing on $[a_2, a_3]$.

(iv) $\mu_{\widetilde{A}}(x) = \omega$ for all $x = a_2$ where $0 < \omega \leq 1$.

Thus, the membership function of the generalized parabolic fuzzy number is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \omega\left(\dfrac{x-a_1}{a_2-a_1}\right)^2, & ; \text{ if } a_1 \leq x \leq a_2 \\ \omega, & ; \text{ if } x = a_2 \\ \omega\left(\dfrac{a_3-x}{a_3-a_2}\right)^2, & ; \text{ if } a_2 \leq x \leq a_3 \\ 0, & ; \text{ if otherwise} \end{cases} \tag{2.7.1}$$

or $\quad \mu_{\widetilde{A}}(x) = \max\left(\min\left(\omega(\frac{x-a_1}{a_2-a_1})^2, \omega, \omega(\frac{a_3-x}{a_3-a_2})^2 \right), 0 \right)$

A generalized fuzzy number is said to be positive(negative) i.e. $\widetilde{A} \geq 0(\widetilde{A} \leq 0)$ if and only if $a_1 \geq 0(a_3 \leq 0)$.

Chapter 3

Arithmetic operations on the Generalized Parabolic fuzzy numbers

In this chapter, we have studied the basic arithmetic operations and defuzzification method for two generalized positive parabolic fuzzy numbers by using the concept of the distribution and complementary distribution functions. The major advantage of these operations is that they do not need the computation of $\alpha-$cut of the fuzzy number and hence it becomes more powerful where the standard method i.e., $\alpha-$ cuts method fails. Based on these operations, some elementary applications on mensuration have been illustrated and compared their results with generalized triangular fuzzy numbers.

3.1 Membership function for function of a fuzzy variable

Consider a fuzzy variable $\widetilde{X} = (a_1, a_2, a_3; \omega)$ with height of the variable is ω and membership functions is defined as below

$$\mu_X(x) = \begin{cases} \omega L_1(x) & \text{if } \; a_1 \le x \le a_2 \\ \omega & \text{if } \; x = a_2 \\ \omega R_1(x) & \text{if } \; a_2 \le x \le a_3 \end{cases}$$

where $L_1(x)$ and $R_1(x)$ are the nondecreasing and nonincreasing functions of x respectively.

Let $F(X) = [F(a_1), F(a_2), F(a_3); F(\omega)]$ be the fuzzy variable of the function $F(X)$. In order to find the fuzzy membership of $F(X)$. Let $z = F(x), x \in X$ or $x = \psi(z)$. Hence the density functions for the distribution functions $L_1(x)$ and $R_1(x)$ are obtained as

$$\begin{aligned} f_1(x) &= \frac{d}{dx}(L_1(x)) = \eta_1(z) \ \text{ at } \ x = \psi_1(z) \\ g_1(x) &= \frac{d}{dx}(R_1(x)) = \eta_2(z) \ \text{ at } \ x = \psi_2(z) \end{aligned}$$

Now, let,

$$\frac{dx}{dz} = \frac{d}{dz}(\psi_1(z)) = m_1(z) \quad ; \quad \frac{dx}{dz} = \frac{d}{dz}(\psi_2(z)) = m_2(z)$$

Then the distribution function for $F(x)$ would be given as

$$\int\limits_{F(a_1)}^{x} \eta_1(z)m_1(z)dz \quad ; \quad F(a_1) \leq x \leq F(a_2)$$

while their complementary distribution function would be given as

$$\int\limits_{F(a_3)}^{x} \eta_2(z)m_2(z)dz \quad ; \quad F(a_2) \leq x \leq F(a_3)$$

Hence, the membership function for the fuzzy variable function $F(x)$ is given by

$$\mu_{F(x)}(x) = \begin{cases} F(\omega) \int\limits_{F(a_1)}^{x} \eta_1(z)m_1(z)dz & ; \quad F(a_1) \leq x \leq F(a_2) \\ F(\omega) & ; \quad x = F(a_2) \\ F(\omega) \int\limits_{F(a_3)}^{x} \eta_2(z)m_2(z)dz & ; \quad F(a_2) \leq x \leq F(a_3) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

3.2 Arithmetic of fuzzy variables

In order to evaluate the fuzzy arithmetic for the parabolic fuzzy numbers, consider the two parabolic fuzzy numbers $X = [a_1, a_2, a_3; \omega_1]$ and $Y = [b_1, b_2, b_3, \omega_2]$ where ω_1, ω_2 represents the degree of their membership functions in crisp environment. Their corresponding membership functions are defined as

$$\mu_X(x) = \begin{cases} \omega_1 L_1(x) & ; \quad \text{if } \ a_1 \leq x \leq a_2 \\ \omega_1 & ; \quad \text{if } \ x = a_2 \\ \omega_1 R_1(x) & ; \quad \text{if } \ a_2 \leq x \leq a_3 \end{cases} \tag{3.2.1}$$

and

$$\mu_Y(y) = \begin{cases} \omega_2 L_1(y) & ; \quad \text{if} \;\; b_1 \le y \le b_2 \\ \omega_2 & ; \quad \text{if} \;\; y = b_2 \\ \omega_2 R_1(y) & ; \quad \text{if} \;\; b_2 \le y \le b_3 \end{cases} \tag{3.2.2}$$

where $L_1(x) = \left(\dfrac{x - a_1}{a_2 - a_1}\right)^2$, $L_1(y) = \left(\dfrac{y - b_1}{b_2 - b_1}\right)^2$ are the left distribution functions and $R_1(x) = \left(\dfrac{a_3 - x}{a_3 - a_2}\right)^2$, $R_1(y) = \left(\dfrac{b_3 - y}{b_3 - b_2}\right)^2$ are the right distribution functions of X and Y respectively. In order to find the distribution functions of their corresponding arithmetic operations, we start with equating $L_1(x)$ with $L_1(y)$ and $R_1(x)$ with $R_1(y)$ and obtain $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively, where $\phi_1(x) = b_1 \pm \left(\frac{(x-a_1)(b_2-b_1)}{(a_2-a_1)}\right)$, $\phi_2(x) = b_3 \mp \left(\frac{(a_3-x)(b_3-b_2)}{(a_3-a_2)}\right)$. Let Z be the resultant of the arithmetic operations of X and Y. Then at $y = \phi_1(x)$ and $y = \phi_2(x)$ we get $x = \psi_1(z)$ and $x = \psi_2(z)$ respectively. Based on these functions, we get the density function corresponding to the distribution and complementary distribution functions as

$$f_1(x) = \frac{d}{dx}(L_1(x)) = \eta_1(z) \;\; \text{at} \;\; x = \psi_1(z)$$
$$g_1(x) = \frac{d}{dx}(R_1(x)) = \eta_2(z) \;\; \text{at} \;\; x = \psi_2(z)$$

Also,

$$\frac{dx}{dz} = \frac{d}{dz}(\psi_1(z)) = m_1(z) \quad ; \quad \frac{dx}{dz} = \frac{d}{dz}(\psi_2(z)) = m_2(z)$$

Hence, the distribution function for fuzzy variable $F(z)$ where $F(z) = [z_1, z_2, z_3, \omega]$, $\omega = \min(\omega_1, \omega_2)$ are

$$\mu_{F(z)}(x) = \begin{cases} \omega \int\limits_{z_1}^{x} \eta_1(z) m_1(z) dz & ; \quad \text{if} \;\; z_1 \le x \le z_2 \\ \omega & ; \quad \text{if} \;\; x = z_2 \\ \omega \int\limits_{z_3}^{x} \eta_2(z) m_2(z) dz & ; \quad \text{if} \;\; z_2 \le x \le z_3 \end{cases}$$

Based on these functions, we obtain the membership functions of functions, such as addition, subtraction, multiplication, inverse etc.

3.2.1 Addition of fuzzy numbers

For addition of the fuzzy numbers X and Y, the fuzzy number $Z = X + Y = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$ be the resultant fuzzy number of X and Y. Now let $z = x + y$ we get $z = x + \phi_1(x)$ and $z = x + \phi_2(x)$ which implies that $x = \psi_1(z)$ and $x = \psi_2(z)$ where

$$x = \psi_1(z) = \frac{z - \frac{a_2 b_1}{a_2 - a_1} + \frac{a_1 b_2}{a_2 - a_1}}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}$$

Hence, $\eta_1(z) = \left(\frac{2}{(a_2 - a_1)^2}\right)\left(\frac{z - a_1 - b_1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right), \quad m_1(z) = \frac{1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}$

Thus, left sided distribution function for the fuzzy variable $Z = X + Y$ is

$$\begin{aligned}
\int\limits_{a_1+b_1}^{x} \eta_1(z)\ m_1(z)dz &= \int\limits_{a_1+b_1}^{x} \left(\frac{2}{(a_2 - a_1)^2}\right)\left(\frac{z - a_1 - b_1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)\left(\frac{1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)dz \\
&= \left(\frac{2}{(a_2 - a_1)^2}\right)\left(\frac{1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)^2 \int\limits_{a_1+b_1}^{x} (z - a_1 - b_1)dx \\
&= \left(\frac{1}{a_2 - a_1}\right)^2 \left(\frac{x - (a_1 + b_1)}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)^2 \\
&= \left(\frac{x - (a_1 + b_1)}{a_2 + b_2 - a_1 - b_1}\right)^2 \qquad ; \quad a_1 + b_1 \le x \le a_2 + b_2
\end{aligned}$$

Similarly, if $y = \phi_2(x)$ then $z = x + y$ becomes $x = \psi_2(z)$ where

$$\psi_2(z) = \frac{z + \frac{-a_2 b_3}{a_3 - a_2} + \frac{a_3 b_2}{a_3 - a_2}}{1 + \frac{b_3 - b_2}{a_3 - a_2}}$$

Here, in this case

$$\eta_2(z) = \left(\frac{-2}{(a_3 - a_2)^2}\right)\left(\frac{a_3 + b_3 - z}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right), \quad m_2(z) = \frac{1}{1 + \frac{b_3 - b_2}{a_3 - a_2}}$$

Thus, right sided distribution function for the fuzzy variable $Z = X + Y$ is

$$\begin{aligned}
\int\limits_{a_3+b_3}^{x} \eta_2(z)\ m_2(z)dz &= \int\limits_{a_3+b_3}^{x} \left(\frac{-2}{(a_3 - a_2)^2}\right)\left(\frac{a_3 + b_3 - z}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right)\left(\frac{1}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right)dz \\
&= \left(\frac{-2}{(a_3 - a_2)^2}\right)\left(\frac{1}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right)^2 \int\limits_{a_3+b_3}^{x} (a_3 + b_3 - z)dx \\
&= \left(\frac{-2}{(a_3 - a_2)^2}\right)\left(\frac{1}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right)^2 \frac{(a_3 + b_3 - x)^2}{-2} \\
&= \left(\frac{1}{a_3 - a_2}\right)^2 \left(\frac{(a_3 + b_3) - x}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right)^2
\end{aligned}$$

$$= \left(\frac{(a_3+b_3)-x}{a_3-a_2+b_3-b_2}\right)^2 \qquad ; \quad a_2+b_2 \le x \le a_3+b_3$$

Therefore, the membership functions of the fuzzy variable $Z = X + Y$ is given by

$$\mu_Z(x) = \begin{cases} \omega\left(\dfrac{x-(a_1+b_1)}{a_2-a_1+b_2-b_1}\right)^2 & ; \quad a_1+b_1 \le x \le a_2+b_2 \\ \omega & ; \quad x = a_2+b_2 \\ \omega\left(\dfrac{(a_3+b_3)-x}{a_3-a_2+b_3-b_2}\right)^2 & ; \quad a_2+b_2 \le x \le a_3+b_3 \end{cases}$$

where $\omega = \min(\omega_1, \omega_2)$.

3.2.2 Scalar multiplication of fuzzy variable

If $\widetilde{X}$ be the TFN and $y = kx$ be the transformation then $k\widetilde{X}$ is TFN is given by

$$k\widetilde{X} = \begin{cases} (ka_1, ka_2, ka_3; \omega_1) & \text{if } \ k > 0 \\ (ka_3, ka_2, ka_1; \omega_1) & \text{if } \ k < 0 \end{cases}$$

Using the transformation $z = kx$, we get $x = z/k$ and hence $\psi(z) = z/k$. Thus $\mid \frac{d}{dz}x \mid = \frac{1}{k} = m(z)$. Therefore,

$$\begin{aligned} \int_{ka_1}^{x} \eta_1(z)m(z)dz &= \int_{ka_1}^{x} \left(\frac{2(z-ka_1)}{k(a_2-a_1)^2}\right)\left(\frac{1}{k}\right)dz = \left(\frac{x-ka_1}{ka_2-ka_1}\right)^2 \\ \int_{ka_3}^{x} \eta_2(z)m(z)dz &= \int_{ka_3}^{x} \left(\frac{-2(ka_3-z)}{k(a_3-a_2)^2}\right)\left(\frac{1}{k}\right)dz = \left(\frac{ka_3-x}{ka_3-ka_2}\right)^2 \end{aligned}$$

Therefore, the membership functions of the fuzzy variable $k\widetilde{X}$, $k > 0$ is given by

$$\mu_{k\widetilde{X}}(x) = \begin{cases} \omega_1\left(\dfrac{x-ka_1}{ka_2-ka_1}\right)^2 & ; \quad ka_1 \le x \le ka_2 \\ \omega_1 & ; \quad x = ka_2 \\ \omega_1\left(\dfrac{ka_3-x}{ka_3-ka_2}\right)^2 & ; \quad ka_2 \le x \le ka_3 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $k < 0$, the membership functions of the fuzzy variable $k\widetilde{X}$, is given by

$$\mu_{k\widetilde{X}}(x) = \begin{cases} \omega_1\left(\dfrac{x-ka_3}{ka_2-ka_3}\right)^2 & \text{if } \ ka_3 \le x \le ka_2 \\ \omega_1 & \text{if } \ x = ka_2 \\ \omega_1\left(\dfrac{ka_1-x}{ka_1-ka_2}\right)^2 & \text{if } \ ka_2 \le x \le ka_1 \\ 0 & \text{otherwise} \end{cases}$$

3.2.3 Subtraction of fuzzy variable

The fuzzy membership function corresponding to fuzzy variable $Z = X - Y$ are defined, by using the property of addition and scalar multiplication of the fuzzy variable, as

$$\mu_Z(x) = \begin{cases} \omega\left(\dfrac{x-(a_1-b_3)}{a_2-a_1+b_3-b_2}\right)^2 & ; \quad a_1-b_3 \le x \le a_2-b_2 \\ \omega & ; \quad x = a_2-b_2 \\ \omega\left(\dfrac{(a_3-b_1)-x}{a_3-a_2+b_2-b_1}\right)^2 & ; \quad a_2-b_2 \le x \le a_3-b_1 \end{cases}$$

3.2.4 Multiplication of a fuzzy variables

In order to find the membership functions of the fuzzy variable $Z = XY$ where the distribution functions of X and Y are defined in (3.2.1) and (3.2.2) respectively. Thus at $y = \phi_1(x)$, $z = xy$ becomes $x = \frac{(a_1b_2-a_2b_1)\pm\sqrt{(a_1b_2-a_2b_1)^2+4(b_2-b_1)(a_2-a_1)z}}{2(b_2-b_1)} = \psi_1(z), (say)$. Take,

$$A_1 = (a_2-a_1)(b_2-b_1) \quad ; \quad B_1 = a_1(b_2-b_1)+b_1(a_2-a_1) \quad ; \quad C_1 = a_1b_1$$

Hence,

$$\begin{aligned} \eta_1(z) &= \frac{2}{(a_2-a_1)^2}\left[\frac{-a_1b_2-a_2b_1+2a_1b_1+\sqrt{B_1^2-4A_1(C_1-z)}}{2(b_2-b_1)}\right] \\ &= \frac{1}{a_2-a_1}\left[\frac{-B_1+\sqrt{B_1^2-4A_1(C_1-z)}}{A_1}\right] \\ \text{and} \quad m(z) &= \left|\frac{dx}{dz}\right| = \frac{a_2-a_1}{\sqrt{B_1^2-4A_1(C_1-z)}}. \end{aligned}$$

$$\begin{aligned} \therefore \int_{a_1b_1}^{x} \eta_1(z)m_1(z)dz &= \int_{a_1b_1}^{x} \frac{1}{a_2-a_1}\left[\frac{-B_1+\sqrt{B_1^2-4A_1(C_1-z)}}{A_1}\right]\frac{a_2-a_1}{\sqrt{B_1^2-4A_1(C_1-z)}}dz \\ &= \int_{a_1b_1}^{x} \frac{1}{A_1}\left[\frac{-B_1+\sqrt{B_1^2-4A_1(C_1-z)}}{\sqrt{B_1^2-4A_1(C_1-z)}}\right]dz \\ &= \int_{a_1b_1}^{x} \frac{1}{A_1}\left[\frac{-B_1}{\sqrt{B_1^2-4A_1(C_1-z)}}+1\right]dz \\ &= \frac{1}{A_1}\left[\frac{(-B_1)^2-B_1\sqrt{B_1^2-4A_1(C_1-z)}+2A_1x-2A_1C_1}{2A_1}\right] \\ &= \left[\frac{-B_1+\sqrt{B_1^2-4A_1(C_1-x)}}{2A_1}\right]^2 \quad ; \quad a_1b_1 \le x \le a_2b_2 \end{aligned}$$

Similarly, by taking

$$A_2 = (a_3 - a_2)(b_3 - b_2) \quad ; \quad B_2 = -a_3(b_3 - b_2) - b_3(a_3 - a_2) \quad ; \quad C_2 = a_3b_3$$

we get, the membership function for the complementary distribution functions as

$$\int_{a_3b_3}^{x} \eta_1(z)m_1(z)dz \;=\; \left[\frac{-B_2 + \sqrt{B_2^2 - 4A_2(C_2 - x)}}{2A_2}\right]^2 \qquad ; \quad a_2b_2 \le x \le a_3b_3$$

Hence, the membership function of the fuzzy variable $Z = XY$ is given by

$$\mu_{XY}(x) = \begin{cases} \omega\left(\dfrac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - x)}}{2A_1}\right)^2 & ; \quad a_1b_1 \le x \le a_2b_2 \\ \omega & ; \quad x = a_2b_2 \\ \omega\left(\dfrac{-B_2 + \sqrt{B_2^2 - 4A_2(C_2 - x)}}{2A_2}\right)^2 & ; \quad a_2b_2 \le x \le a_3b_3 \end{cases}$$

3.2.5 Inverse of a fuzzy variable

Consider a fuzzy variable $X = [a_1, a_2, a_3, \omega_1]$ with membership function given in eq. (3.2.1). Let $X^{-1} = [a_3^{-1}, a_2^{-1}, a_1^{-1}; \omega_1]$. Let $z = \frac{1}{x}$ so that $\mid \frac{dx}{dz} \mid = \frac{1}{z^2}$. Therefore for X^{-1} we have

$$\int_{x}^{a_1^{-1}} \eta_1(z)m(z)dz = \int_{x}^{a_1^{-1}} \left(\frac{2}{(a_2 - a_1)^2}(\frac{1}{z} - a_1)\right)\left(\frac{1}{z^2}\right)dz = \left(\frac{1 - a_1x}{x(a_2 - a_1)}\right)^2$$

$$\int_{a_3^{-1}}^{x} \eta_2(z)m(z)dz = \int_{a_3^{-1}}^{x} \left(\frac{2}{(a_3 - a_2)^2}(\frac{1}{z} - a_3)\right)\left(\frac{1}{z^2}\right)dz = \left(\frac{xa_3 - 1}{x(a_3 - a_2)}\right)^2$$

Thus, based on these distribution functions, fuzzy membership function of X^{-1} are

$$\mu_{X^{-1}}(x) = \begin{cases} \omega_1\left(\dfrac{xa_3 - 1}{x(a_4 - a_3)}\right)^2 & \text{if } \; a_3^{-1} \le x \le a_2^{-1} \\ \omega_1 & \text{if } \; x = a_2^{-1} \\ \omega_1\left(\dfrac{1 - a_1x}{x(a_2 - a_1)}\right)^2 & \text{if } \; a_2^{-1} \le x \le a_1^{-1} \\ 0 & \text{otherwise} \end{cases}$$

3.2.6 Division of fuzzy variables

Consider the two TFN X and Y defined their membership functions as in equations (3.2.1) and (3.2.2) respectively. Suppose $Z = \frac{X}{Y}$ then membership function of Z is given by $Z = X.Y^{-1}$.

3.3 Illustrative Examples

The above methodology for computing the membership functions of various arithmetic operation has been illustrated through a numerical examples as given below.

3.3.1 Example 1: Addition of two numbers

Let $X = [1, 2, 4; 1]$ and $Y = [3, 5, 6; 1]$ be two parabolic fuzzy numbers with membership functions as

$$\mu_{\tilde{X}}(x) = \begin{cases} (x-1)^2, & ; \text{ if } \quad 1 \le x \le 2 \\ \left(\dfrac{4-x}{2}\right)^2, & ; \text{ if } \quad 2 \le x \le 4 \\ 0, & ; \text{ otherwise} \end{cases} \quad \text{and} \quad \mu_{\tilde{Y}}(y) = \begin{cases} \left(\dfrac{y-3}{2}\right)^2, & ; \text{ if } \quad 3 \le y \le 5 \\ (6-y)^2, & ; \text{ if } \quad 5 \le y \le 6 \\ 0, & ; \text{ otherwise} \end{cases}$$

In order to evaluate the degree of membership of $X+Y$, we start with the equating of the distribution and complementary distribution functions and hence we get $y = 2x+1 = \phi_1(x)$ and $y = \frac{8-x}{2} = \phi_2(x)$. Now for $Z = X + Y$, we get $x = \psi_1(z) = \frac{z-1}{3}$, $x = \psi_2(z) = \frac{2z-8}{3}$, $\eta_1(z) = 2(\frac{z-4}{3})$, $\eta_2(z) = \frac{10-z}{3}$, $m_1(z) = \frac{1}{3}$ and $m_2(z) = \frac{2}{3}$.

Therefore, the distribution function of the fuzzy variable $Z = X + Y$ would now be given as

$$\int_4^x \eta_1(z)\ m_1(z) dz = \left(\frac{x-4}{3}\right)^2 \quad ; \quad 4 \le x \le 7$$

and the complementary distribution function is

$$\int_{10}^x \eta_2(z)\ m_2(z) dz = \left(\frac{10-x}{3}\right)^2 \quad ; \quad 7 \le x \le 10$$

Then the fuzzy membership function of X+Y is

$$\therefore \quad \mu_{X+Y}(x) = \begin{cases} \left(\dfrac{x-4}{3}\right)^2 & ; \text{ if } \quad 4 \le x \le 7 \\ \left(\dfrac{10-x}{3}\right)^2 & ; \text{ if } \quad 7 \le x \le 10 \\ 0 & ; \text{ otherwise} \end{cases}$$

The obtained results are depicted graphically in Fig. 3.1 along with the other existing results, linear and crisp, and are explained as below.

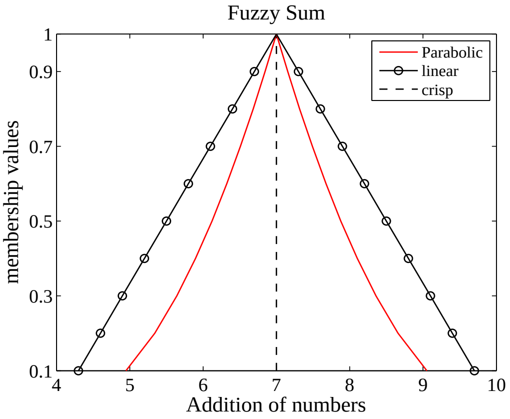

Figure 3.1: Membership function of Addition of two numbers

(i) The results computed by the crisp or traditional methodology are independent of the uncertainty level and hence it remain constant for all membership values. Therefore, their results are suitable only for a system whose data are precise.

(ii) The results computed by taking the linear membership functions are shown in Fig. 3.1 with linear legend. From the figure it is concluded that it contains a wide range of spread in the form of support and hence results are not so much practical as it contains a large amount of uncertainties.

(iii) On the other hand, the results computed by taking parabolic fuzzy numbers have reduced region and a smaller spread than the other results at any level of satisfaction. This means that uncertainties existing during the analysis are reduced up to the desired degree and hence decision makers/system analyst may use these results for further analysis which leads to a more sound and effective decision for future course of actions in lesser time.

Also, it has been concluded that the value of their resultant number is increasing from 4cm to 7cm at a nonlinear increasing rate $\frac{2}{9}(x-4)$ and then decreases from 7cm to 10cm at a nonlinear decreasing rate $\frac{2}{9}(10-x)$. The corresponding defuzzified value obtained by using COG method is 7cm.

3.3.2 Example 2: Length of Rod

Let length of a rod is a parabolic fuzzy number $\widetilde{A} = (12cm, 13.5cm, 15cm; 0.8)$. If the length $\widetilde{B} = (5cm, 6.5cm, 8cm; 0.7)$, a parabolic fuzzy number, is cut off from this rod then the remaining length of the rod $\widetilde{C}$ is $\widetilde{A}(-)\widetilde{B}$.

The parabolic membership function corresponding to fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are defined as below

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0.8\left(\dfrac{x-12}{1.5}\right)^2 & ; \quad \text{if} \quad 12 \leq x \leq 13.5 \\ 0.8\left(\dfrac{15-x}{1.5}\right)^2 & ; \quad \text{if} \quad 13.5 \leq x \leq 15 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\widetilde{B}}(y) = \begin{cases} 0.7\left(\dfrac{y-5}{1.5}\right)^2 & ; \quad \text{if} \quad 5 \leq y \leq 6.5 \\ 0.7\left(\dfrac{8-y}{1.5}\right)^2 & ; \quad \text{if} \quad 6.5 \leq y \leq 8 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Now $\widetilde{B} = (-8cm, -6.5cm, -5cm; 0.7)$ be the negative of the fuzzy number $\widetilde{B}$, then their corresponding membership functions is given as

$$\mu_{-\widetilde{B}}(x) = \begin{cases} 0.7\left(\dfrac{x+8}{1.5}\right)^2 & ; \quad \text{if} \quad -8 \leq x \leq -6.5 \\ 0.7 & ; \quad \text{if} \quad x = -6.5 \\ 0.7\left(\dfrac{x+5}{1.5}\right)^2 & ; \quad \text{if} \quad -6.5 \leq x \leq -5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Hence, using the property of the addition of the two parabolic fuzzy numbers, the membership functions of the remaining length of the rod is a parabolic fuzzy number $\widetilde{C}$ and is given as:

$$\therefore \ \mu_{\widetilde{C}}(x) = \begin{cases} 0.7\left(\dfrac{x-4}{3}\right)^2 & ; \quad \text{if} \quad 4 \leq x \leq 7 \\ 0.7 & ; \quad \text{if} \quad x = 7 \\ 0.7\left(\dfrac{10-x}{3}\right)^2 & ; \quad \text{if} \quad 7 \leq x \leq 10 \\ 0, & ; \quad \text{otherwise} \end{cases}$$

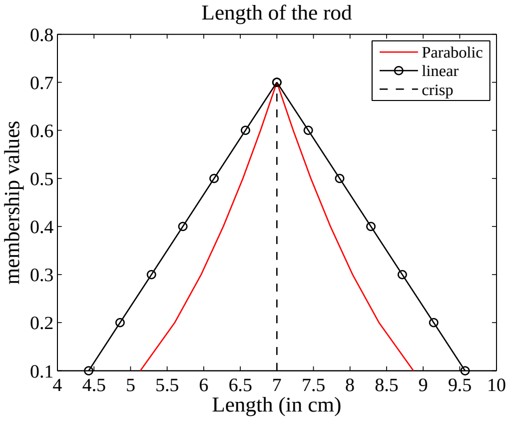

Figure 3.2: Membership functions of Length of rod

From above, we conclude that the remaining length of the rod lies between 4cm and 10 cm. Moreover, the value of this length is increased from 4cm to 7cm at a nonlinear increasing rate $\frac{1.4}{9}(x-4)$ and then decreases from 7cm to 10cm at a nonlinear decreasing rate $\frac{1.4}{9}(10-x)$. Also, there are 70% possibilities that the length takes the value 7cm. The corresponding membership values are plotted in Fig. 3.2 at different level of significance and concluded that the proposed one have less range of uncertainties than others. The defuzzified value of the remaining length of the rod is 7cm.

3.3.3 Example 3: Area of a rectangle

Let length and breadth of a rectangle are two parabolic fuzzy numbers given by $\widetilde{A} = (1cm, 2cm, 4cm; 0.75)$ and $\widetilde{B} = (3cm, 5cm, 6cm; 0.85)$. Then the area $\widetilde{C}$ of the rectangle is $\widetilde{A}(\cdot)\widetilde{B}$.

In order to evaluate the membership functions of $\widetilde{C}$ we equate the distribution and complementary distribution functions respectively of $\widetilde{A}$ and $\widetilde{B}$ and hence we get $\phi_1(x) = 2x+1$ and $\phi_2(x) = \dfrac{x+8}{2}$. Now for $Z = A.B$ we get $x = \psi_1(z) = \dfrac{-1 \pm \sqrt{1+8z}}{4}$, $x = \psi_2(z) = -4 \pm \sqrt{16+2z}$, $\eta_1(z) = \frac{-5+\sqrt{1+8z}}{2}$ and $\eta_2(z) = \frac{4-x}{2} = \frac{8-\sqrt{16+2z}}{2}$, $m_1(z) = \frac{1}{\sqrt{1+8z}}$ and $m_2(z) = \frac{1}{\sqrt{16+2z}}$.

Therefore, the distribution function of the fuzzy variable $\widetilde{C}$ is given by

$$\begin{aligned}
\int_{3}^{x} \eta_1(z)m_1(z)dz &= \int_{3}^{x} \left(\frac{-5+\sqrt{1+8z}}{2}\right)\left(\frac{1}{\sqrt{1+8z}}\right)dz \\
&= \frac{1}{2}\int_{3}^{x} \left(\frac{-5+\sqrt{1+8x}}{\sqrt{1+8x}}\right)dz \\
&= \left(\frac{\sqrt{1+8x}-5}{4}\right)^2
\end{aligned}$$

and the complimentary distribution function is given by

$$\begin{aligned}
\int_{10}^{x} \eta_2(z)m_2(z)dz &= \int_{10}^{x} \left(\frac{8-\sqrt{16+2z}}{2}\right)\left(\frac{1}{\sqrt{16+2z}}\right) dz \\
&= \left(\frac{8-\sqrt{16+2x}}{2}\right)^2
\end{aligned}$$

Hence, the membership functions of the area of the rectangle is given as

$$\therefore \quad \mu_{\widetilde{C}}(x) = \begin{cases} 0.75\left(\dfrac{\sqrt{1+8x}-5}{4}\right)^2 & ; \quad \text{if} \quad 3 \le x \le 10 \\ 0.75 & ; \quad \text{if} \quad x = 10 \\ 0.75\left(\dfrac{8-\sqrt{16+2x}}{2}\right)^2 & ; \quad \text{if} \quad 10 \le x \le 24 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The variation of their membership functions corresponding to linear and parabolic functions are summarized in Fig. 3.3 at different value of significance level. From this figure,

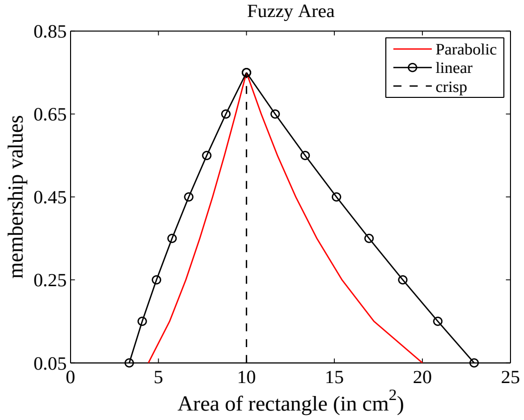

Figure 3.3: Membership functions of Area of a rectangle

it is concluded that resultant fuzzy number is a convex-concave type with a nonlinear increasing rate $\frac{\sqrt{1+8x}-5}{2\sqrt{1+8x}}$ from 3 to 10cm^2 and then decreases from 10 to 24cm^2 with nonlinear decreasing rate $\frac{8-\sqrt{16+2x}}{2\sqrt{16+2x}}$. Also, there is a probability of 75% that the area of a rectangle is 10cm^2. Hence, the area of a rectangle lies between 4 and 10 cm^2. The defuzzified values corresponding to linear and parabolic fuzzy numbers are 11.9768cm^2 and 11.2806cm^2 respectively. Thus, there is less variation in their defuzzified values in case of parabolic numbers as compared to linear numbers when compared with their crisp value 10cm^2.

3.3.4 Example 4: Length of the rectangle

Let area and breadth of the rectangle be given as a parabolic fuzzy numbers $\widetilde{A}$=(1cm^2,2cm^2,4cm^2; 0.75) and $\widetilde{B} = (3cm, 5cm, 6cm; 0.85)$ respectively, then the length of the rectangle is given by $\widetilde{A}(\div)\widetilde{B}$ or $\widetilde{A}(\cdot)\widetilde{B}^{-1}$.

Now based on the membership function of $\widetilde{B}$ we obtain the membership functions of $\widetilde{B}^{-1} = (6^{-1}, 5^{-1}, 3^{-1}; 0.85)$ as

$$\mu_{\widetilde{B}^{-1}}(y) = \begin{cases} 0.85\left(6-\frac{1}{y}\right)^2 & ; \text{ if } \ 6^{-1} \le y \le 5^{-1} \\ 0.85 & ; \text{ if } \ y = 5^{-1} \\ 0.85\left(\dfrac{\frac{1}{y}-3}{2}\right)^2 & ; \text{ if } \ 5^{-1} \le y \le 3^{-1} \\ 0 & ; \text{ otherwise} \end{cases}$$

Hence the membership function of the length of the rectangle is obtained by multiplying the two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}^{-1}$ as

$$\mu_{\widetilde{A}\cdot\widetilde{B}^{-1}}(x) = \begin{cases} 0.75\left(\dfrac{6x-1}{x+1}\right)^2 & ; \text{ if } \ \frac{1}{6} \le y \le \frac{2}{5} \\ 0.75 & ; \text{ if } \ x = \frac{2}{5} \\ 0.75\left(\dfrac{4-3x}{2(x+1)}\right)^2 & ; \text{ if } \ \frac{2}{5} \le y \le \frac{4}{3} \\ 0 & ; \text{ otherwise} \end{cases}$$

From this membership function, it has been concluded that there is a 75% probability that the length of the rectangle is 0.4cm and the range of the length of the rectangle is

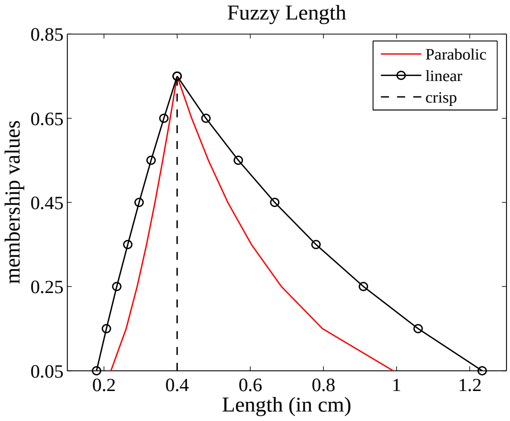

Figure 3.4: Membership functions of length of the rectangle

$[\frac{1}{6}, \frac{4}{3}]$. The variation of their membership values at different level of membership values are plotted in Fig. 3.4 which shows that its value is increased from $\frac{1}{6}$ to $\frac{2}{5}$ with a nonlinear increasing rate $10.5(\frac{6x-1}{(x+1)^3})$ while decreases from $\frac{2}{5}$ to $\frac{4}{3}$ with nonlinear rate $(\frac{10.5}{8})(\frac{4-3x}{(x+1)^3})$. Thus membership functions are a concave-convex type instead of linear one as in the case of linear membership functions. The corresponding values of their defuzzified values are 0.52028cm and 0.59294cm for parabolic and linear membership functions while their crisp value is 0.4cm. Hence there is 23.11% and 32.54% decrease in the defuzzified values of crisp and linear membership functions when parabolic membership functions have been used.

3.3.5 Example 5: Perimeter of Rectangle

Let the length and breadth of a rectangle are two parabolic fuzzy numbers $\widetilde{A}$=(12cm, 13.5cm,14cm; 0.9) and $\widetilde{B} = (6cm, 7.5cm, 9cm; 0.8)$, then perimeter $\widetilde{C}$ of rectangle is $2[\widetilde{A}(+)\widetilde{B}]$.

The parabolic membership functions of $\widetilde{A}$ and $\widetilde{B}$ are given as

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0.9\left(\dfrac{x-12}{1.5}\right)^2 & ; \quad \text{if} \quad 12 \le x \le 13.5 \\ 0.9 & ; \quad \text{if} \quad x = 13.5 \\ 0.9\left(\dfrac{14-x}{0.5}\right)^2 & ; \quad \text{if} \quad 13.5 \le x \le 14 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\mu_{\widetilde{B}}(x) = \begin{cases} 0.8\left(\dfrac{y-6}{1.5}\right)^2 & ; \quad \text{if} \quad 6 \le y \le 7.5 \\ 0.8 & ; \quad \text{if} \quad y = 7.5 \\ 0.8\left(\dfrac{9-y}{1.5}\right)^2 & ; \quad \text{if} \quad 7.5 \le y \le 9 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Now, by property of the addition of the two fuzzy numbers, we get

$$\mu_{\widetilde{C}}(x) = \begin{cases} 0.8\left(\dfrac{x-36}{6}\right)^2 & ; \quad \text{if} \quad 36 \le x \le 42 \\ 0.8 & ; \quad \text{if} \quad x = 42 \\ 0.8\left(\dfrac{46-x}{6}\right)^2 & ; \quad \text{if} \quad 42 \le x \le 46 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

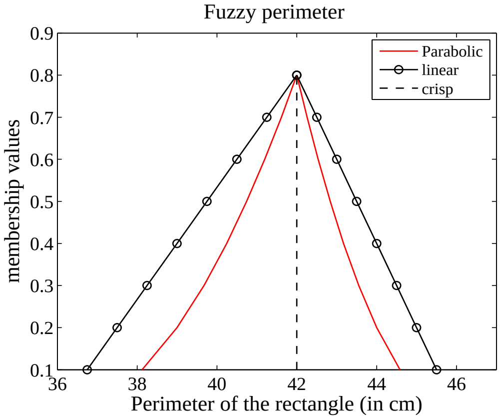

Figure 3.5: Membership functions of Perimeter of rectangle

The membership values corresponding to the perimeter of the rectangle are summarized graphically in Fig. 3.5 which shows that the level of uncertainties in the form of support are less as compared to the linear membership functions. Thus the results corresponding to parabolic membership functions are beneficial for system analyst for making more sound decision based on these results. Also, it has been concluded from the figure that there is a 80% probability of getting the perimeter of rectangular 42cm. On the other hand, there is an increase in their perimeter with a nonlinear increasing rate $(\frac{0.8}{18})(x-36)$ when $x \in [36, 42]$ while decreasing with a rate of $(\frac{0.8}{18})(48-x)$ when $x \in [42, 46]$. Their corresponding defuzzified values are 41.3518cm and 41.5433cm respectively, for linear and parabolic membership functions.

3.4 Conclusion

In this chapter, we have worked on the GPFNs and introduced their corresponding fuzzy arithmetic operations based on their distribution and their complementary distribution functions. This method is an alternative and useful for finding the membership functions because the standard method, $\alpha-$ cut, does not always yield results. The variations in the membership functions has been plotted and compared with the crisp methodology and by taking linear membership functions and we conclude that there is less range of uncertainties in the form of support during the analysis and hence proposed one is beneficial for system analyst. The validity of the method has been evaluated by solving some problems of mensuration using GPFNs and compared their results with the triangular membership functions with the existing method.

Chapter 4

Computation of defuzzified expressions for Parabolic fuzzy numbers

This chapter discusses the various defuzzification method for finding the defuzzified values from the computed fuzzy results. The generalized parabolic trapezoidal fuzzy numbers (GPFNs) have been taken for computing the formulae of the various defuzzification methods.

4.1 Defuzzification

Aggregating two or more fuzzy output sets (or membership functions) yield a new fuzzy set (or a new membership function) in the basic fuzzy inference algorithm. In most cases, a result in the form of a fuzzy set is converted into a crisp result by the *defuzzification* process. Defuzzification is necessary for hardware applications, because conventional systems' operations are based on crisp data exchange. Among the several methods which are suggested in the literature, the most widely used methods are listed in Table 4.1. As there

Table 4.1: Defuzzification Methods

Centroid Methods:	Maxima Methods:
Center of Gravity	Mean of Maximums
Center of Weights	Left-Right Maxima
Center of Largest Area	Maximum Probability
Center of Mass	

are two basic mechanisms: centroid and maxima. The centroid methods are based on finding a balance point of property that can be the total geometric figure, the weight (area) of each fuzzy set, the area of the largest fuzzy set, or the area of highest intersection. The maximum possibility method searches for the highest peak whereas the left-right maxima method searches for the peak in a selected direction. The mean of maxima method may also be considered as one of the centroid techniques since mean and center practically refer to the same property. In this work, we have studied the following defuzzification methods which are defined as below for the generalized fuzzy number $\widetilde{A} = (a, b, c, d; w)$:

4.1.1 Center of Area (COA)

This is one of the most commonly used technique for finding the defuzzified values and can be expressed as

$$x_{COA} = \frac{\int_x x \cdot \mu_{\widetilde{A}}(x)\ dx}{\int_x \mu_{\widetilde{A}}(x)\ dx}$$

where x_{COA} is the crisp output, $\mu_{\widetilde{A}}(x)$ is the aggregated membership function and x is the output variable.

4.1.2 Bisection of Area (BOA)

The bisection of area is the vertical line that divides the region into two sub-regions of equal area. The formula for x_{BOA} is given by

$$\int_a^{x_{BOA}} \mu_{\widetilde{A}}(x)\ dx = \int_{x_{BOA}}^d \mu_{\widetilde{A}}(x)\ dx$$

4.1.3 Smallest of Maxima (SOM)

The smallest output with the maximum membership function is taken as the crisp value and it is denoted by x_{SOM}.

4.1.4 Largest of Maxima (LOM)

The maximum x_{LOM} takes the largest amongst all x thats belongs to $[b, c]$.

4.1.5 Mean of Maxima (MOM)

In this method, defuzzification is calculated by taking the average of the defuzzification of LOM and SOM and can be represented as

$$x_{MOM} = \frac{x_{LOM} + x_{SOM}}{2}$$

4.1.6 Regular Weighted Point (RWP)

For the fuzzy number $\widetilde{A} = (a, b, c, d)$, the $\alpha-$cut is $A_\alpha = [L_A(\alpha), R_A(\alpha)]$ and the regular weighted point (RWP) for A is given by

$$\begin{aligned} RWP(A) &= \frac{\int_0^1 \frac{(L_A(\alpha)+R_A(\alpha))}{2} f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha} \\ &= \int_0^1 (L_A(\alpha) + R_A(\alpha))f(\alpha)d\alpha \\ \text{where} \quad f(\alpha) &= \begin{cases} 1 - 2\alpha & ; \quad \alpha \in [0, \frac{1}{2}] \\ 2\alpha - 1 & ; \quad \alpha \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

4.2 Expression of defuzzified value for GPFNs

In this section, we have computed the defuzzified expressions for GPFNs by using the above methods. For this we take the trapezoidal fuzzy numbers (TrFN) and trapezoidal parabolic fuzzy numbers (TrPFN) whose endpoints are represented by $\widetilde{A} = (a, b, c, d; w)$. Based on their expression, the defuzzified values for the triangular fuzzy number (TFN) and parabolic fuzzy number (PFN) have been computed by considering as a special case. The expression of the membership functions corresponding to TrFNs and TrPFNs are given in Eq. (4.2.1) and (4.2.2) respectively.

$$\mu_{\widetilde{A}}(x) = \begin{cases} w\left(\dfrac{x-a}{b-a}\right) & ; \quad a \le x \le b \\ w & ; \quad b \le x \le c \\ w\left(\dfrac{d-x}{d-c}\right) & ; \quad c \le x \le d \end{cases} \tag{4.2.1}$$

$$\text{and} \quad \mu_{\widetilde{A}}(x) = \begin{cases} w\left(\dfrac{x-a}{b-a}\right)^2 & ; \quad a \le x \le b \\ w & ; \quad b \le x \le c \\ w\left(\dfrac{d-x}{d-c}\right)^2 & ; \quad c \le x \le d \end{cases} \tag{4.2.2}$$

4.2.1 Centroid of area method

Case I: When TrFN has been taken

$$\begin{aligned}
\int \mu_{\widetilde{A}}(x)dx &= \int_a^b w\left(\frac{x-a}{b-a}\right)dx + \int_b^c wdx + \int_c^d w\left(\frac{d-x}{d-c}\right)dx \\
&= \frac{w}{(b-a)}\left(\frac{(x-a)^2}{2}\right)_a^b + w(c-b) + \frac{w}{(d-c)}\left(\frac{(d-x)^2}{-2}\right)_c^d \\
&= \frac{w}{2}(d+c-b-a) \\
\int x \cdot \mu_{\widetilde{A}}(x)dx &= \int_a^b xw\left(\frac{x-a}{b-a}\right)dx + \int_b^c xwdx + \int_c^d xw\left(\frac{d-x}{d-c}\right)dx \\
&= \frac{w}{(b-a)}\left[\frac{(x-a)^3}{3} + a\frac{(x-a)^2}{2}\right]_a^b + \frac{w(c^2-b^2)}{2} + \\
&\quad \frac{w}{(d-c)}\left[\frac{(d-x)^3}{3} - d\frac{(d-x)^2}{2}\right]_c^d \\
&= \frac{w(b-a)}{6}(2b+a) + \frac{w(c^2-b^2)}{2} + \frac{w(d-c)}{6}(d+2c) \\
&= \frac{w}{6}(d^2+c^2-b^2-a^2-ab+cd) \\
\therefore \quad \text{Defuzzification} &= \frac{\int x \cdot \mu_{\widetilde{A}}(x)dx}{\int \mu_{\widetilde{A}}(x)dx} \\
&= \frac{1}{3}\frac{d^2+c^2-b^2-a^2-ab+cd}{d+c-b-a} \\
&= \frac{1}{3}\left[d+c+b+a-\frac{cd-ab}{d+c-b-a}\right] \qquad (4.2.3)
\end{aligned}$$

Case II: When TrPFN has been taken

$$\begin{aligned}
\int \mu_{\widetilde{A}}(x)dx &= \int_a^b w\left(\frac{x-a}{b-a}\right)^2 dx + \int_b^c wdx + \int_c^d w\left(\frac{d-x}{d-c}\right)^2 dx \\
&= \frac{w}{(b-a)^2}\left(\frac{(x-a)^3}{3}\right)_a^b + w(c-b) + \frac{w}{(d-c)^2}\left(\frac{(d-x)^3}{-3}\right)_c^d \\
&= \frac{w}{3}(d+2c-2b-a)
\end{aligned}$$

$$\begin{aligned}
\int x \cdot \mu_{\widetilde{A}}(x)dx &= \int_a^b xw\left(\frac{x-a}{b-a}\right)^2 dx + \int_b^c xwdx + \int_c^d xw\left(\frac{d-x}{d-c}\right)^2 dx \\
&= \frac{w}{(b-a)^2}\left[\frac{(x-a)^4}{4} + a\frac{(x-a)^3}{3}\right]_a^b + \frac{w(c^2-b^2)}{2} \\
&\quad + \frac{w}{(d-c)^2}\left[\frac{(d-x)^4}{4} - d\frac{(d-x)^3}{3}\right]_c^d \\
&= \frac{w(b-a)}{12}(3b+a) + \frac{w(c^2-b^2)}{2} + \frac{w(d-c)}{12}(d+3c) \\
&= \frac{w}{12}(d^2+3c^2-3b^2-a^2-2ab+2cd) \\
\therefore \quad \text{Defuzzification} &= \frac{1}{4}\left[d+2c+2b+a-\frac{c^2-b^2+2cd-2ab}{d+2c-2b-a}\right] \qquad (4.2.4)
\end{aligned}$$

Special cases:

Case 1: For TFNs; $b = c$ then from Eq. (4.2.3), we have $x_{COA} = \frac{a+b+d}{3}$

Case 2: For TPFNs; $b = c$ then from Eq. (4.2.4), we have $x_{COA} = \frac{3a+2b+3d}{8}$

Case 3: For crisp number; $a = b = c = d$ then we have $x_{COA} = b$

4.2.2 Bisection of area method

Case I: When TrFN has been taken

By definition of bisection of area,

$$\begin{aligned}
& \int_a^{x_{BOA}} \mu_{\widetilde{A}}(x)dx = \int_{x_{BOA}}^d \mu_{\widetilde{A}}(x)dx \\
\Rightarrow \quad & \int_a^b \mu_{\widetilde{A}}(x)dx + \int_b^{x_{BOA}} \mu_{\widetilde{A}}(x)dx = \int_{x_{BOA}}^c \mu_{\widetilde{A}}(x)dx + \int_c^d \mu_{\widetilde{A}}(x)dx \\
\Rightarrow \quad & \int_a^b w\frac{x-a}{b-a}dx + \int_b^{x_{BOA}} wdx = \int_{x_{BOA}}^c wdx + \int_c^d w\frac{d-x}{d-c}dx \\
\Rightarrow \quad & \int_a^b \frac{x-a}{b-a}dx + \int_b^{x_{BOA}} dx = \int_{x_{BOA}}^c dx + \int_c^d \frac{d-x}{d-c}dx \\
\Rightarrow \quad & \frac{b-a}{2} + x_{BOA} - b = c - x_{BOA} + \frac{d-c}{2} \\
\Rightarrow \quad & x_{BOA} = \frac{a+b+c+d}{4} \qquad (4.2.5)
\end{aligned}$$

Case II: When TrPFN has been taken

$$
\begin{aligned}
& \int_a^{x_{BOA}} \mu_{\widetilde{A}}(x)dx = \int_{x_{BOA}}^{d} \mu_{\widetilde{A}}(x)dx \\
\Rightarrow \quad & \int_a^b \mu_{\widetilde{A}}(x)dx + \int_b^{x_{BOA}} \mu_{\widetilde{A}}(x)dx = \int_{x_{BOA}}^{c} \mu_{\widetilde{A}}(x)dx + \int_c^d \mu_{\widetilde{A}}(x)dx \\
\Rightarrow \quad & \int_a^b w\left(\frac{x-a}{b-a}\right)^2 dx + \int_b^{x_{BOA}} w dx = \int_{x_{BOA}}^{c} w dx + \int_c^d w\left(\frac{d-x}{d-c}\right)^2 dx \\
\Rightarrow \quad & \frac{b-a}{3} + x_{BOA} - b = c - x_{BOA} + \frac{d-c}{3} \\
\Rightarrow \quad & x_{BOA} = \frac{a+2b+2c+d}{6} \qquad (4.2.6)
\end{aligned}
$$

Special cases:

Case 1: For TFNs; $b = c$ then from Eq. (4.2.5), we have $x_{BOA} = \frac{a+2b+d}{4}$

Case 2: For TPFNs; $b = c$ then from Eq. (4.2.6), we have $x_{BOA} = \frac{a+b+d}{3}$

Case 3: For crisp number; we have $a = b = c = d$ then $x_{BOA} = b$

4.2.3 Regular Weighted Point

Case I: When TrFN has been taken

By definition of regular weighted point,

$$
\begin{aligned}
\text{RWP} &= \int_0^{1/2} (L_A(\alpha) + R_A(\alpha)) f(\alpha) d\alpha + \int_{1/2}^{1} (L_A(\alpha) + R_A(\alpha)) f(\alpha) d\alpha \\
&= \int_0^{1/2} [a + d + \alpha(b - a - d + c)](1 - 2\alpha) d\alpha \\
&\qquad + \int_{1/2}^{1} [a + d + \alpha(b - a - d + c)](2\alpha - 1) d\alpha \\
&= \left\{ \frac{a+d}{2} - \frac{a+d}{4} + \frac{b-a-d+c}{8} - \frac{(b-a-d+c)}{12} \right\} \\
&\qquad + \left\{ \frac{3(a+d)}{4} - \frac{a+d}{2} + \frac{7(b-a-d+c)}{12} - \frac{3(b-a-d+c)}{8} \right\} \\
&= \frac{a+d}{2} + \frac{b-a-d+c}{4} \\
&= \frac{a+b+c+d}{4} \qquad (4.2.7)
\end{aligned}
$$

Case II: When TrPFN has been taken

$$\begin{aligned}
\text{RWP} &= \int_0^{1/2}(L_A(\alpha)+R_A(\alpha))f(\alpha)d\alpha + \int_{1/2}^{1}(L_A(\alpha)+R_A(\alpha))f(\alpha)d\alpha \\
&= \int_0^{1/2}[a+d+\sqrt{\alpha}(b-a-d+c)](1-2\alpha)d\alpha \\
&\qquad + \int_{1/2}^{1}[a+d+\sqrt{\alpha}(b-a-d+c)](2\alpha-1)d\alpha \\
&= \left\{\frac{a+d}{2}-\frac{a+d}{4}+\frac{b-a-d+c}{3\sqrt{2}}-\frac{(b-a-d+c)}{5\sqrt{2}}\right\}+\left\{\frac{3(a+d)}{4}-\frac{a+d}{2}\right. \\
&\quad \left.+\frac{2(1-\frac{1}{4\sqrt{2}})(b-a-d+c)}{5/2}-\frac{(1-\frac{1}{2\sqrt{2}})(b-a-d+c)}{3/2}\right\} \\
&= \frac{a+d}{2}+\frac{2(\sqrt{2}+1)}{15}(b-a-d+c) \qquad (4.2.8)
\end{aligned}$$

Special cases:

Case 1: For TFNs; $b = c$ then from Eq. (4.2.7), we have $x_{RWP} = \frac{a+2b+d}{4}$

Case 2: For TPFNs; $b = c$ then from Eq. (4.2.8), we have $x_{RWP} = \frac{a+d}{2} + \frac{2(\sqrt{2}+1)}{15}(2b-a-d)$

Case 3: For crisp number; $a = b = c = d$ then we have $x_{RWP} = b$.

The complete summary of these defuzzified expressions is summarized in Table 4.2.

4.3 Conclusion

The objective of this chapter is to compute the defuzzified expressions for the GPFNs. For this various kinds of defuzzification methods, namely center of area, bisection of area, largest of maxima, smallest of maxima, regular weighted point etc., are used for computation. Trapezoidal fuzzy numbers and trapezoidal Parabolic fuzzy numbers have been used for computing their corresponding defuzzified expressions. Based on these expressions, it has been concluded that the defuzzified values of the triangular fuzzy numbers, triangular parabolic fuzzy numbers, crisp numbers etc., are taken to be as a special case. The summary of these expressions is also summarized in tabular form.

Table 4.2: Defuzzified formulas for different fuzzy numbers

Defuzzifier	TFN	TPFN	TrFN	TrPFN
COA	$\frac{a+b+d}{3}$	$\frac{3a+2b+3d}{8}$	$\frac{1}{3}\left[a+b+c+d-\frac{cd-ab}{d+c-b-a}\right]$	$\frac{1}{4}\left[a+2b+2c+d-\frac{c^2-b^2+2cd-2ab}{d+2c-2b-a}\right]$
BOA	$\frac{a+2b+d}{4}$	$\frac{a+b+d}{3}$	$\frac{a+b+c+d}{4}$	$\frac{a+2b+2c+d}{6}$
SOM	b	b	b	b
LOM	b	b	c	c
MOM	b	b	$\frac{b+c}{2}$	$\frac{b+c}{2}$
RWP	$\frac{a+2b+d}{4}$	$\frac{a+d}{2}+\frac{2(\sqrt{2}+1)}{15}(2b-a-d)$	$\frac{a+b+c+d}{4}$	$\frac{a+d}{2}+\frac{2(\sqrt{2}+1)}{15}(b-a-d+c)$

Chapter 5

Summary and Future Scope

The chapter presents a comprehensive summary of the research contributions made during the period of this thesis. It also outlines the managerial implications for the implementation of recommendations. Finally the scope for future work has been outlined.

5.1 Summary of the work

The conclusion made from the work presented in this thesis are summarized below:

(i) The research work presented in this thesis is an attempt to give an alternative approach for computing the membership functions of the various arithmetic operations in fuzzy environment. Here, in the literature, there are two well-known additions of fuzzy quantities. One is based on the extension principle by directly considering the membership functions, and another one uses the $\alpha-$ level sets without considering the membership functions. In this study, instead of using these methods for computing the membership functions, we compute the membership functions by using distribution and complementary distribution functions. Fuzzy set theory has been used for handling the uncertain, vague and imprecise data. A generalized trapezoidal parabolic fuzzy number has been used for computing their membership functions. The advantage of this approach is that it gives a compressed range of prediction and it does not involve the computation of $\alpha-$ cuts for finding the membership functions. Thus, this method is beneficial for those where computation of $\alpha-$ cut is difficult or even not possible.

(ii) Apart from their computation of the membership functions, the computation of the defuzzified values/expressions for the generalized parabolic fuzzy numbers have been established. Various types of defuzzification methods such as center of area, the bisection of area, mean of maxima, largest of maxima, regular weighted point etc., are computed for the fuzzy numbers. Based on their defuzzified expressions, the expressions of the triangular and parabolic fuzzy numbers are extracted from it by considering it as a special case.

5.2 Future scope of the work

The method for computing the various arithmetic operations using various fuzzy numbers can be extended in the following directions:

(i) The presented methodology will be further extended and improved using some optimization technique such as Genetic algorithm, Particle swarm optimization, Artificial bee colony etc.

(ii) The computation of the fuzzy membership functions can be computed by formulating a nonlinear optimization model instead of using fuzzy arithmetic operations.

(iii) The presented work done may be extended for a nonlinear fuzzy number such as exponential, hyperbolic etc.

(iv) In our study, we have taken the constant data i.e. it follows the exponential distribution. In the future, we may try to extend the proposed approach for a time varying distribution function.

(v) The study, based on these arithmetic operations may be extended for the applications part in reliability optimization, resource allocation, facility planning and management, inventory control, network analysis and job shop scheduling.

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