FOURTH ORDER COMPACT SCHEME FOR NAVIER-STOKES AND CONVECTION-DIFFUSION EQUATIONS

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BY

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CERTIFICATE

I hereby certify that the dissertation entitled ***"FOURTH ORDER COMPACT SCHEME FOR NAVIER-STOKES AND CONVECTION-DIFFUSION EQUATIONS",*** *which is being submitted by* ***Ms. Ramandeep Kaur (Roll no. 301103016),*** *in the partial fulfillment of the requirements for the award of degree of* ***MASTER OF SCIENCE*** *in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's authentic record of work studied under the supervision of* ***Dr. Vivek Sangwan,*** *Assistant Professor, SMCA, Thapar University, Patiala, during the period from January 2013 to June 2013.*

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other university for the award of any degree by the author.

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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TO

GOD, MY TEACHERS AND MY PARENTS

Contents

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Above all, I pay my reverence to the almighty of

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ABSTRACT

The present dissertation entitled "FOURTH ORDER COMPACT SCHEME FOR NAVIER-STOKES AND CONVECTION-DIFFUSION EQUATIONS" comprises the study of fourth order compact finite difference schemes under the supervision of Dr. Vivek Sangwan, Assistant Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

Differential equations arise in the mathematical modelling of many physical, chemical and biological phenomena and many more areas of science and engineering such as fluid dynamics, electromagnetism, material science, astrophysics, economy etc. Therefore, it becomes a necessity to develop methods for solving differential equations. There are two principle approaches for finding the solutions of differential equations. One is the asymptotic approach and the other one is numerical approach. In this regard, numerical approaches have many benefits over asymptotic approaches as these techniques are much easier to use as compared to asymptotic approaches. Also these require the problem solver very less information about the problem to be solved. In numerical approaches, one finds a solution of a differential equation numerically.

The aim of this work is to study the higher order compact finite difference schemes for solving the differential equations. These schemes enjoys the benefits of higher order accuracy, high stability, easy to implement and many more. Also, these schemes uses a compact $9-point$ stencil which has the much added advantage in terms of less boundary conditions requirement while solving the problem numerically and retaining higher order accuracy. The schemes provide very powerful tool for solving very complicated differential equations like convection-dominated problems and Navier-Stokes equations with high Reynolds numbers.

The work presented in this dissertation has been divided into three chapters. The

first chapter is introductory. In this chapter, we present a brief introduction of differential equations, solution methodology, finite difference schemes and some of the results which will be used in the later chapters. Towards the end of this chapter, compact fourth-order finite difference scheme has been discussed in brief.

In the second chapter, we consider the steady two-dimensional Navier-Stokes (N-S) equation

$$\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} = -\zeta.$$

Compact scheme has been derived for this N-S equation with a local truncation error of fourth order. The scheme allows the use of standard iterative schemes to solve the resulting systems of numerical equations. The computed solution, obtained using the compact scheme, of the Navier-Stokes equation has been compared with the exact solution. Towards the end of this chapter, the Extension of the scheme has been discussed in detail for the rectangular grids.

In the third chapter, the convection-diffusion equation

$$u_{xx} + u_{yy} + p(x,y)u_x + q(x,y)u_y = f(x,y)$$

has been considered. A different fourth order compact finite difference scheme has been presented for the above convection-diffusion equation with variable coefficients. The scheme has been defined on a single square cell of size $2h$ over a $9 - point$ stencil and has a truncation error of order h^4. Few numerical examples have been solved using the proposed scheme and the computed solutions have been compared with those available in the literature.

References of different publications have been cited at the end of the dissertation.

Chapter 1

INTRODUCTION

1.1 Differential Equation

In simple words, a differential equation can be described as a relation between dependent variables, independent variables and the derivatives of the dependent variables with respect to the independent variables. Differential equations arise in many areas of science and engineering. Many laws governing natural phenomena are relations (equations) involving rates at which things happen. For example: models which describes heat transfer, chemical process etc.

Differential equations can be categorized in two categories:

(1) Ordinary differential equations

(2) Partial differential equations

An ordinary differential equation (ODE) is a differential equation in which the unknown function is a function of a single independent variable. A partial differential equation (PDE) is a differential equation in which the unknown function is a function of more than one independent variables and their partial derivatives.

1.2 Solution Methodology

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions, functions that make the equation hold true. For most of the differential equations with complex or transcendental variable coefficients, non-linear differential equations, it is very much difficult to find the exact solution. Many properties of solutions of these differential equations may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated. Thus we can say, there are two principle approaches for finding the solutions of differential equations. One is the asymptotic approach and the other one is numerical approach.

In the asymptotic approach, one finds a solution of a differential equation by using the properties and the nature of the problem. These methods require the problem solver to have some apriori knowledge of the solution expected. Therefore, for nonlinear or differential equations having complex coefficient or differential equations governing the real life phenomenon or problems having complex domains, it is very much difficult to find the solution using asymptotic approach. In this regard, numerical approaches have benefits over asymptotic approaches. In numerical approach, one finds a solution of a differential equation numerically. These methods does not require the problem solver much information of the solution expected. Also, these methods are comparatively very easy to implement as against to asymptotic methods. Some of the numerical techniques include Finite Difference Methods, Finite Element Method, Finite Volume Method etc.

1.3 Finite Difference Methods

Finite difference methods are widely used for solving ordinary and partial differential equations. In finite difference methods, we replace the derivatives appearing in the differential equation by finite differences that approximate them. Applications of the finite difference methods can easily be seen in computational science and engineering disciplines, such as thermal engineering, fluid mechanics, etc.

In the present work, a finite difference approach has been used for solving differential equations.

Let $u(x, y)$ is a function with step size h in x and k in y, then finite difference approximations for its first and second derivatives by forward difference (FD), backward difference (BD) and central difference (CD) are as given below:

Using Taylor series expansion, we get

$$\begin{aligned} u(x+h, y) =& u(x, y) + h u_x(x, y) + \frac{h^2}{2!} u_{xx}(x, y) + ... && (1.3.1) \\ u_x(x, y) =& \frac{u(x+h, y) - u(x, y)}{h} + O(h) && (FD) \\ u(x-h, y) =& u(x, y) - h u_x(x, y) + \frac{h^2}{2!} u_{xx}(x, y) + ... && (1.3.2) \\ u_x(x, y) =& \frac{u(x, y) - u(x-h, y)}{h} + O(h) && (BD) \end{aligned}$$

Subtracting (1.3.1) from (1.3.2), we get

$$\begin{aligned} u_x(x, y) =& \frac{u(x+h, y) - u(x-h, y)}{2h} + O(h^2) && (CD) \\ u(x, y+k) =& u(x, y) + k u_y(x, y) + \frac{k^2}{2!} u_{yy}(x, y) + ... && (1.3.3) \end{aligned}$$

$$\begin{aligned} u_y(x, y) =& \frac{u(x, y+k) - u(x, y)}{k} + O(k) && (FD) \\ u(x, y-k) =& u(x, y) - k u_y(x, y) + \frac{k^2}{2!} u_{yy}(x, y) + ... && (1.3.4) \\ u_y(x, y) =& \frac{u(x, y) - u(x, y-k)}{k} + O(k). && (BD) \end{aligned}$$

Subtracting (1.3.3) from (1.3.4), we get

$$u_y(x, y) = \frac{u(x, y+k) - u(x, y-k)}{2k} + O(k^2). \qquad (CD)$$

For second order derivatives, using Taylor series expansion, we get

$$\begin{aligned}
u(x+2h,y) =& u(x,y) + 2hu_x(x,y) + \frac{4h^2}{2!}u_{xx}(x,y) + ...\\
u_{xx}(x,y) =& \frac{u(x+2h,y) - 2hu_x(x,y) - u(x,y)}{2h^2}\\
u_{xx}(x,y) =& \frac{u(x+2h,y) - 2u(x+h,y) + u(x,y)}{2h^2} + O(h) && (FD)\\
u(x-2h,y) =& u(x,y) - 2hu_x(x,y) + \frac{4h^2}{2!}u_{xx}(x,y) + ...\\
u_{xx}(x,y) =& \frac{u(x-2h,y) + 2hu_x(x,y) - u(x,y)}{2h^2}\\
u_{xx}(x,y) =& \frac{u(x-2h,y) - 2u(x-h,y) + u(x,y)}{2h^2} + O(h) && (BD)\\
u_{xx}(x,y) =& \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2) && (CD)\\
u(x,y+2k) =& u(x,y) + 2ku_y(x,y) + \frac{4k^2}{2!}u_{yy}(x,y) + ...\\
u_{yy}(x,y) =& \frac{u(x,y+2k) - 2ku_y(x,y) - u(x,y)}{2k^2}\\
u_{yy}(x,y) =& \frac{u(x,y+2k) - 2u(x,y+k) + u(x,y)}{2k^2} + O(k) && (FD)\\
u(x,y-2k) =& u(x,y) - 2ku_y(x,y) + \frac{4k^2}{2!}u_{yy}(x,y) + ...\\
u_{yy}(x,y) =& \frac{u(x,y-2k) + 2ku'(x,y) + u(x,y)}{2k^2}\\
u_{yy}(x,y) =& \frac{u(x,y-2k) - 2u(x,y-k) + u(x,y)}{2k^2} + O(k) && (BD)\\
u_{yy}(x,y) =& \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2} + O(k^2) && (CD)
\end{aligned}$$

Now we will explain through example how these differences are used for solving partial differential equations.

Example: Find the numerical solution of the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad 0 \le x \le 1, \quad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 1$$

and initial condition

$$u(x,0) = \begin{cases} 1+2x, & 0 \le x \le \frac{1}{2}, \\ 3-2x, & \frac{1}{2} \le x \le 1 \end{cases}$$

using explicit finite difference method taking $\Delta x = 0.2$, $\Delta t = 0.02$ and compute up to $t = 0.24$ up to six decimal places.

Solution: We note that the problem is symmetric about $x = 0.5$ since the initial temperature $u(x, 0)$ is symmetric and the boundary conditions at $x = 0$ and $x = 1.0$ are also same. Therefore, the temperature at the subsequent times will remain also symmetric about $x = 0.5$.

The domain $0 \leq x \leq 1.0$ is subdivided into five sub-intervals each of width $\Delta x = 0.2$.

We have $x_i = 0.2 \times i$, $i = 0(1)5$

Due to symmetry at $x = 0.5$, for all j

$$u_{0,j} = u_{5,j}, \quad u_{1,j} = u_{4,j}, \quad u_{2,j} = u_{3,j}$$
$$r = \frac{\Delta t}{\Delta x^2} = 0.02/0.04 = 0.5.$$

Putting $r = 0.5$ in the explicit formulae

$$\begin{aligned} u_{i,j+1} =& r u_{i-1,j} + (1 - 2r)u_{i,j} + r u_{i+1,j} \\ =& \frac{1}{2}(u_{i-1,j} + u_{i+1,j}), \quad i = 1, 2 \end{aligned}$$

The values for $i = 3$ and $i = 4$ can be written by symmetry. Computed values are:

Put $i = 1, j = 0$

$$\begin{aligned} u_{1,1} =& \frac{1}{2}(u_{0,0} + u_{2,0}) \\ =& 1.4 \end{aligned}$$

Put $i = 2, j = 0$

$$\begin{aligned} u_{2,1} =& \frac{1}{2}(u_{1,0} + u_{3,0}) \\ =& 1.6 \end{aligned}$$

Put $i = 3, j = 0$

$$\begin{aligned} u_{3,1} =& \frac{1}{2}(u_{2,0} + u_{4,0}) \\ =& 1.6 \end{aligned}$$

Put $i = 4, j = 0$

$$\begin{aligned} u_{4,1} =& \frac{1}{2}(u_{3,0} + u_{5,0}) \\ =& 1.4 \end{aligned}$$

Put $i = 1, j = 1$

$$\begin{aligned} u_{1,2} =& \frac{1}{2}(u_{0,1} + u_{2,1}) \\ =& 1.3 \end{aligned}$$

Put $i = 2, j = 1$

$$\begin{aligned} u_{2,2} =& \frac{1}{2}(u_{1,1} + u_{3,1}) \\ =& 1.5 \end{aligned}$$

Put $i = 3, j = 1$

$$\begin{aligned} u_{3,2} =& \frac{1}{2}(u_{2,1} + u_{4,1}) \\ =& 1.5 \end{aligned}$$

Put $i = 4, j = 1$

$$\begin{aligned} u_{4,2} =& \frac{1}{2}(u_{3,1} + u_{5,1}) \\ =& 1.3 \end{aligned}$$

1.4 Compact Finite Difference Scheme

The Navier-Stokes equations have been used to model fluid dynamics phenomena describing flows of an incompressible viscous fluid. These equations are highly nonlinear and are

$x \to$	0	0.2	0.4	0.6	0.8	1.0
$t = 0.00$	1.0	1.4	1.8	1.8	1.4	1.0
0.02	1.0	1.4	1.6	1.6	1.4	1.0
0.04	1.0	1.3	1.5	1.5	1.3	1.0
0.06	1.0	1.25	1.4	1.4	1.25	1.0
0.08	1.0	1.20	1.325	1.325	1.20	1.0
0.10	1.0	1.1625	1.2625	1.2625	1.1625	1.0
0.12	1.0	1.13125	1.21250	1.21250	1.13125	1.0
0.14	1.0	1.10625	1.171875	1.171875	1.10625	1.0
0.16	1.0	1.085938	1.139062	1.139062	1.085938	1.0
0.18	1.0	1.069531	1.112500	1.112500	1.069531	1.0
0.20	1.0	1.056250	1.091016	1.091016	1.056250	1.0
0.22	1.0	1.045508	1.073633	1.73633	1.045508	1.0
0.24	1.0	1.036816	1.059570	1.059570	1.036816	1.0

very difficult to solve, especially when the approximate solutions are required to have a high accuracy. A Navier-Stokes equation may be linearized in stream function and vorticity formulation [4, 5]. Computing an accurate solution of a convection diffusion equation thus becomes an issue. To begin with some stable and accurate numerical methods have been discussed for solving the $2D$ convection diffusion equation with high Reynolds numbers.

The general convection diffusion equation is of the form

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} + p(x,y)\frac{\partial U(x,y)}{\partial x} + q(x,y)\frac{\partial U(x,y)}{\partial y} = f(x,y) \qquad (1.4.1)$$

where $p(x,y)$ and $q(x,y)$ are continuous functions of variables x and y. The magnitude of p and q simulates the Reynolds numbers (denoted by Re) which determines the convection strength of a flow. Ω is convex $2D$ domain, and $\partial\Omega$ is the boundary of ω. This equation appears in many transport problems. Also the equation can be considered as a particular case of the steady state incompressible Navier-Stokes equations[4].

Suppose equation (1.4.1) is discretized by some finite difference scheme, and a linear system

$$A^h u^h = f^h \qquad (1.4.2)$$

is obtained, where h is the uniform grid spacing of the discretized space Ω^h and u^h is the numerical solution over Ω^h. The linear system (1.4.2) is usually of very large dimension. For such large sparse linear systems, the use of direct solution methods based on Gaussian elimination results in engaging a very large memory and CPU cost, and iterative methods such as *Jacobi* and SOR are sensitive to the meshsize h, the type of boundary conditions and other factors. The coefficient matrix A^h is nonsymmetric and far from diagonally dominant if *Re* is large[15]. This property adds further difficulty to classical iterative methods.

Equation (1.4.1) may be discretized by the central difference scheme (CDS); the resulting linear system (1.4.2) is a five-point formula with a truncation error of order h^2. In the case of CDS, classical iterative methods for solving the resulting linear system may not converge when the convective terms dominate and the cell Reynolds number (*Re*) is greater than a certain constant. For this reason, the upwind difference scheme (UPS) has been used for many years despite that it is only first-order accurate.

Due to the importance in practical applications, various attempts have been made to solve the convection diffusion equation and the incompressible Navier-Stokes equations with iterative (especially multigrid) methods [6, 12, 5]. For example, de Zeeuw [26] developed a black-box multigrid solver with some matrix-dependent prolongations and restrictions for solving convection dominated problems. This and other methods have demonstrated the efficiency of multigrid techiques in solving convection-dominated problems discretized by UPS.

Recently, there has been increasing interest in developing fourth-order compact schemes for solving equation (1.4.1) and the incompressible Navier-Stokes equations with large Reynolds numbers [4, 17]. There has been numerous work on the construction of compact schemes for the incompressible Navier-Stokes equations(see example, [5, 15, 31]). The most noted ones include the work of Gupta [15] and Dennis et al. [5]. Almost all of these schemes are geared towards steady flow calculations even though the ideas may, in principle, be applied to unsteady flows as well. In the case of steady flows, these schemes have shown a great deal of potential [31].

The main purpose of the present work is to introduce an efficient fourth-order scheme which overcomes all the above mentioned difficulties and which is easy to implement. For this purpose, a fourth-order compact finite difference scheme has been presented for a set of second-order partial differential equations. Compact finite differencing is a mean of achieving high order discretisations of differential equations without an enlargement of the bandwidth of the resulting set of discrete equations. For example, for second-order problems in one space dimension, tridiagonal systems having fourth-order accuracy are produced. High accuracy coupled with easily solved systems are clearly most desirable properties of a numerical method.

Ideally these schemes offer two attractive features: **high order accuracy** and **small stencil**. Consequently the number of numerical boundary conditions needed is considerably reduced, compared with standard high order schemes. This is of great importance for the computation of viscous incompressible flows for which numerical boundary conditions have always been an issue.

The scheme which has been presented in the next two chapters is essentially compact and retains all the nice features of compact schemes. The numerical results reported by the compact schemes have the following advantages over the other above mentioned difference schemes:

(1) **Unconditional stability**: Although the coefficient matrices are no longer diagonally dominant for large Re, the schemes have been shown numerically stable for large Reynolds numbers.

(2) **High accuracy**: It has been shown that these schemes do produce numerical

solution of fourth-order accuracy for the convection diffusion equation [7, 9] and of high accuracy for the Navier-Stokes equations with small to medium *Re*.

(3) **Works well for Convection-dominated problems**: Since the simultaneous presence of the convection and viscous terms makes it difficult to construct simple and efficient compact schemes which fit nicely the structure of both the momentum equation and the kinematic constraint, it is natural to relax slightly the requirement of compactness(for the convection terms) so long as it does not complicate the treatment of the boundary conditions.

(4) **Easy to implement**: The scheme is very simple and easy to implement. The complexity of the scheme is essentially the same as that of a standard second-order scheme. The simplicity of this scheme enables us to prove convergence with fourth-order accuracy.

(5) **Highly suitable for parallel computation**: The simplicity of this method also makes it very attractive for implementation on parallel machines. The compactness of the stencil enables one to pass very little information between different processors.

(6) **Easy boundary treatment**: Since the computational stencil involves only the nearest nine grid points, the schemes are of compact type and no special formula is needed for approximating grid points near the boundaries of a regularly structured domain.

However, until recently, computational advantages of these compact schemes have not been fully investigated. For example, it is not clear if these schemes can be used to solve incompressible Navier-Stokes equations of very large *Re* because of the limitations of available computer power and of the difficulty of solving the resulting linear system (lack of diagonal dominance) with traditional *SOR* type iterative methods.

To fully investigate the potential of using the fourth-order compact schemes for solving Navier-Stokes equations with large *Re*, nontraditional iterative methods seem necessary. One class of promising methods are multigrid techniques which has been successfully used with the first and second-order discretization schemes for solving problems in computational fluid dynamics (including the driven cavity problem)(see for example [6, 12]). A preliminary investigation on combining the fourth-order compact schemes with multigrid techniques was made by Altas and Burrage [27] for diffusion-dominated flow problems. However, multigrid solution and accelerated multigrid solution methods with the fourth-order compact schemes for solving convection-dominated problems are relatively new [1, 18].

The papers presented here uses multigrid techniques to evaluate the fourth-order compact schemes in solving steady state incompressible Navier-Stokes equations for large Re, exemplified by solving the square driven cavity problem. This follows from Gupta's work [15] using SOR iterative method with these schemes to solve a driven cavity problem for $Re \leq 2000$ and Zhang's work [7] employing the fourth-order compact scheme and multigrid techniques to compute a high accuracy solution of the convection diffusion equation with very large Reynolds numbers.

The scheme presented also compares well with spectral methods.
It has the advantage of being simple, robust, efficient and much more stable. It is well known that if the viscous term is treated explicitly, therefore the compact schemes resolve boundary layers better than other finite difference methods on uniform grids. On the other hand, for high Reynolds number flows, it is clear that boundary layer is the most difficult part to resolve. Aside from that, while fourth-order schemes are in theory less accurate than spectral methods, the difference can only be seen at a very high level of accuracy [20].

An exception has been found in the high order finite difference schemes of compact type that are computationally efficient and stable and yield highly accurate numerical solutions at least for the linear and quasilinear partial differential equations. Simplest version of such compact schemes for the poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{1.4.3}$$

which can be discretized at a grid point (x, y) using a nine-point finite difference approximation is given by:

$$4[u_1 + u_2 + u_3 + u_4] + u_5 + u_6 + u_7 + u_8 - 20u_0 = \frac{1}{2}h^2[f_1 + f_2 + f_3 + f_4 + 8f_0]. \tag{1.4.4}$$

(See Fig.1 for the computational stencil in the next chaphter.) This approximation was named Mehrstellenverfahren by Collatz [3]. It has a local truncation error of order h^4 and is an approximation of compact type as it involves only the eight nearest neighbors of the point (x, y). This type of approximations have been obtained for other elliptic equations by many researchers like the Hodie schemes of Lynch and Rice [23], the OCI schemes of Berger et al. [1], and the SCHOS schemes of Gupta et al. All reduce to the above difference approximation when applied to the Poisson equation. Similar compact schemes of order h^6 have also been obtained [24].
The compact schemes of Gupta et al. (called SCHOS) which were applied to the convection-diffusion equations in particular and were found to yield high accuracy, have been applied to a large number of test problems including problems of convection-dominated flows. In the papers presented below, these finite difference schemes have been extended to the Navier-Stokes equations. As a test of this method, the model problem of a lid driven cavity for small to moderate values of the Reynolds number has been solved and the numerical solutions have been compared with the highly accurate benchmark solutions available in the literature.

In the chapters to follow, in the second Chapter, nine-point compact finite difference discretization scheme has been discussed in detail for solving equation (1.4.1). The benchmark lid driven square driven cavity problem has been solved using the proposed forth-order compact scheme. The numerical results for the square driven cavity problem have been compared with those obtained by other researchers using other methods. In the third Chapter, the streamfunction and vorticity formulation of the incompressible Navier-Stokes equations has been presented. It is almost explicit. For $2D$ problems, only two Poisson or Helmholtz equations are solved at each time step. No iteration is required between the boundary values of vorticity and the interior field variables. The multigrid solvers for the convection diffusion(and the Poisson) equation and for the incompressible Navier-Stokes equations are also outlined. Furthermore, the Poisson and Helmholtz equations can be solved using standard fast Poisson solvers designed for second-order schemes.

Chapter 2

A COMPACT FOURTH-ORDER FINITE DIFFERENCE SCHEME FOR THE STEADY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

2.1 Introduction

In this chapter, the steady two-dimensional Navier-stokes (N-S) equations have been solved by fourth-order compact finite difference scheme. As stated in the first Chapter, that finite difference methods of obtaining approximate numerical solutions of the steady incompressible Navier-stokes equations can vary considerably in terms of accuracy and efficiency. In the area of finite difference methods it has been discovered that although central difference approximates are locally second-order-accurate, they often suffer from computational instability and the resulting solutions exhibit non-physical oscillations. The upwind difference approximations are computationally stable, although they are only first-order-accurate, and the resulting solution exhibits the effects of artificial viscosity. The second-order upwind methods are no better than the first-order upwind difference ones for large values of *Re*. The higher-order finite difference methods of conventional type do not allow direct iterative techniques. An exception has been found in the high-order finite difference schemes of compact type, which are computationally efficient, highly stable and yield highly accurate numerical solutions([4, 5, 15]).

The approximation

$$\frac{1}{6h^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix}\psi = -\frac{1}{12}\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix}\zeta$$

to the equation

$$\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} = -\zeta \tag{2.1.1}$$

is fourth-order accurate when applied to equation (2.1.1). Gupta et. al. [14], Dennis and Hudson [5] and Gupta [15] noted that this technique can be generalized also to provide a fourth-order-accurate nine-point scheme for solutions to the convection-diffusion equation

$$\frac{\partial^2\zeta}{\partial^2 x} + \frac{\partial^2\zeta}{\partial^2 y} - Re((p(x,y)\frac{\partial\zeta}{\partial x} + q(x,y)\frac{\partial\zeta}{\partial y}) = f(x,y). \tag{2.1.2}$$

With the choices $p(x,y) = \psi_y$, $q(x,y){=}{-}\psi_x$ and $f(x,y) = 0$, the pair of equations (2.1.1) and (2.1.2) forms the steady two-dimensional Navier-Stokes equations. However, in this case a problem arises in that the approximations needed to obtain the velocities $p\ (x,y)$ and $q(x,y)$ to fourth-order accuracy will extend outside the (3×3)-point domain ([5, 15]). In the present work, a compact fourth-order finite difference scheme for the time-independent $N-S$ equations with the novelty of 'genuine compactness', i.e. the compact scheme is strictly within the nine-point stencil, has been derived. It is shown that the new scheme yields highly accurate numerical solutions while still allowing SOR-type iterations for low-to-medium Reynolds numbers.

The organization of the chapter is as follows[32]. In the next section, the compact fourth-order finite difference scheme for the Navier-Stokes equations has been introduced. In Section 3, the new fourth-order scheme for the Navier-Stokes equations which possess an exact solution has been presented. The model problem of the lid-driven cavity is described

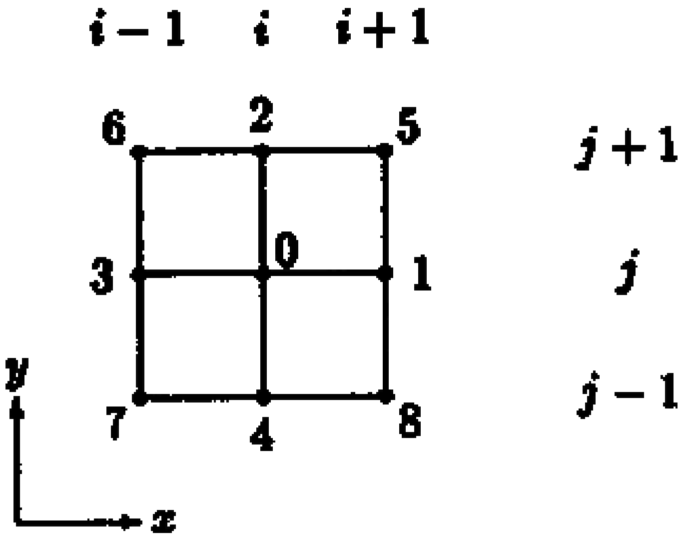

Figure 1. Computational stencil

in Section 4 with detailed comparisons of the solutions obtained using the proposed compact scheme with the existing solutions in the literature. In Section 5, we have derived in detail the possible extensions of the present method for non-uniform mesh.

2.2 Numerical Methods

The Navier-Stokes equations representing the two-dimensional steady flow of an incompressible viscous fluid are given in streamfunction-vorticity form as

$$\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} = -\zeta, \tag{2.2.1}$$

$$\frac{\partial^2\zeta}{\partial^2 x} + \frac{\partial^2\zeta}{\partial^2 y} = Re(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}). \tag{2.2.2}$$

Here ψ is the streamfunction, ζ is the vorticity and Re is the non-dimensional Reynolds number. Assuming a uniform grid in both $x-$ and $y-$directions, we number the grid points $(x, y), (x + h, y), (x, y + h)$, $(x - h, y), (x, y - h), (x + h, y + h), (x - h, y + h), (x - h, y - h)$ and $(x + h, y - h)$ as $0, 1, 2, 3, 4, 5, 6, 7$ and 8 respectively (see Figure 1) been discretized uniformly with grid size h. In writing the finite difference approximations, a single subscript j denotes the corresponding function value at the grid point numbered j. For completeness we first reiterate the derivation of the fourth-order compact scheme for

$$u_{xx} + u_{yy} = f(x, y). \tag{2.2.3}$$

By using Taylor expansion, we have

$$u(x+h, y) = u(x, y) + hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) + \frac{h^3}{3!}u_{xxx}(x, y) + \frac{h^4}{4!}u_{xxxx(x,y)} \tag{2.2.4}$$

$$u(x+(-h), y) = u(x, y) - hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) - \frac{h^3}{3!}u_{xxx}(x, y) + \frac{h^4}{4!}u_{xxxx(x,y)} \tag{2.2.5}$$

Adding equations (2.2.4) and (2.2.5):

$$\begin{aligned}
u(x+h, y) + u(x-h, y) &= 2u(x, y) + 2\frac{h^2}{2!}u_{xx}(x, y) + 2\frac{h^4}{4!}u_{xxxx}(x, y) + O(h^4) \\
u_1 + u_3 &= 2u_0 + 2\frac{h^2}{2!}u_{xx}(x, y) + 2\frac{h^4}{4!}u_{xxxx}(x, y) + O(h^4) \\
\frac{u_1 + u_3 - 2u_0}{h^2} &= u_{xx}(x, y) + \frac{h^2}{12}u_{xxxx}(x, y) + O(h^4) \\
\frac{u_1 + u_3 - 2u_0}{h^2} &= \delta_x^2 \\
\delta_x^2 &= u_{xx}(x, y) + \frac{h^2}{12}u_{xxxx}(x, y) + O(h^4) \\
\delta_x^2 &= u_{xx}(1 + \frac{h^2}{12}u_{xx}) + O(h^4 \\
u_{xx} &= \delta_x^2(1 + \frac{h^2}{12}u_{xx})^{-1} + O(h^4) \\
\frac{\partial^2}{\partial x^2} &= \delta_x^2(1 + \frac{h^2}{12}\delta_x^2)^{-1} + O(h^4)
\end{aligned}$$

Proceeding in the similar manner as above, its counterpart in the y-direction is given by

$$\frac{\partial^2}{\partial y^2} = \delta_y^2(1 + \frac{h^2}{12}\delta_y^2)^{-1} + O(h^4)$$

Using these values in equation (2.2.3), we get

$$\begin{aligned}
(1 + \frac{h^2}{12}\delta_x^2)^{-1}\delta_x^2 u + (1 + \frac{h^2}{12}\delta_y^2)^{-1}\delta_y^2 u &= f(x, y) + O(h^6) \\
(1 + \frac{h^2}{12}\delta_y^2)\delta_x^2 u + (1 + \frac{h^2}{12}\delta_x^2)\delta_y^2 u &= (1 + \frac{h^2}{12}\delta_x^2)(1 + \frac{h^2}{12}\delta_y^2)f(x, y) + O(h^6) \\
(1 + \frac{h^2}{12}\delta_y^2)\delta_x^2 u + (1 + \frac{h^2}{12}\delta_x^2)\delta_y^2 u &= (1 + \frac{h^2}{12}(\delta_x^2 + \delta_y^2))f(x, y) + O(h^6)
\end{aligned}$$

Now,

$$\begin{aligned}
(1+\frac{h^2}{12}\delta_y^2)\delta_x^2 u &= (1+\frac{h^2}{12}\delta_y^2)\frac{u_1+u_3-2u_0}{h^2} \\
&= \frac{u_1+u_3-2u_0}{h^2}+\frac{1}{12}(\frac{u_5+u_8-2u_1}{h^2}+\frac{u_6+u_7-2u_3}{h^2}-2\frac{u_2+u_4-2u_0}{h^2}) \\
&= \frac{1}{12h^2}(12u_1+12u_3-24u_0+u_5+u_8-2u_1+u_6+u_7-2u_3-2u_2-2 \\
&\quad u_4+4u_0) \\
&= \frac{1}{12h^2}(10u_1-2u_2+10u_3-2u_4+u_5+u_6+u_7+u_8-20u_0) \qquad (2.2.6) \\
(1+\frac{h^2}{12}\delta_x^2)\delta_y^2 u &= (1+\frac{h^2}{12}\delta_x^2)\frac{u_2+u_4-2u_0}{h^2} \\
&= \frac{u_2+u_4-2u_0}{h^2}\frac{1}{12}(\frac{u_5+u_6-2u_2}{h^2}+\frac{u_8+u_7-2u_4}{h^2}-2\frac{u_3+u_1-2u_0}{h^2}) \\
&= \frac{1}{12h^2}(12u_2+12u_4-24u_0+u_6+u_5-2u_2+u_7+u_8-2u_4-2u_3 \\
&\quad -2u_1+4u_0) \\
&= \frac{1}{12h^2}(10u_2-2u_1-2u_3-+10u_4+u_5+u_6+u_7+u_8-20u_0) \qquad (2.2.7)
\end{aligned}$$

Adding equations (2.2.6) and (2.2.7):

$$\begin{aligned}
(1+\frac{h^2}{12}\delta_y^2)\delta_x^2 u+&(1+\frac{h^2}{12}\delta_x^2)\delta_y^2 u \\
&= \frac{1}{12h^2}(8u_1+8u_2+8u_3+8u_4+2u_5+2u_6+2u_7+2u_8-40u_0) \\
&= \frac{1}{6h^2}(4(u_1+u_2+u_3+u_4)+u_5+u_6+u_7+u_8-20u_0) \qquad (2.2.8)
\end{aligned}$$

Now,

$$\begin{aligned}
(1+\frac{h^2}{12}(\delta_x^2+\delta_y^2))f_0 &= f_0+\frac{h^2}{12}\delta_x^2 f_0+\frac{h^2}{12}\delta_y^2 f_0 \\
&= f_0+\frac{h^2}{12}\frac{f_1+f_3-2f_0}{h^2}+\frac{h^2}{12}\frac{f_2+f_4-2f_0}{h^2} \\
&= \frac{1}{12}(12f_0+f_1+f_3-2f_0+f_2+f_4-2f_0) \\
&= \frac{1}{12}(8f_0+f_1+f_2+f_3+f_4) \qquad (2.2.9)
\end{aligned}$$

Using (2.2.8) and (2.2.9), we get

$$
\begin{aligned}
&(1+\frac{h^2}{12}\delta_y^2)\delta_x^2 u+(1+\frac{h^2}{12}\delta_x^2)\delta_y^2 u=(1+\frac{h^2}{12}(\delta_x^2+\delta_y^2))f_0\\
&\frac{1}{6h^2}(4(u_1+u_2+u_3+u_4)+u_5+u_6+u_7+u_8-20u_0)=\frac{1}{12}(8f_0+f_1+f_2+f_3+f_4)\\
&4(u_1+u_2+u_3+u_4)+u_5+u_6+u_7+u_8-20u_0=\frac{h^2}{2}(8f_0+f_1+f_2+f_3+f_4) \quad (2.2.10)
\end{aligned}
$$

Substituting (2.2.10) in equation (2.2.1):

$$4(\psi_1+\psi_2+\psi_3+\psi_4)+\psi_5+\psi_6+\psi_7+\psi_8-20\psi_0=-\frac{h^2}{2}(8\zeta_0+\zeta_1+\zeta_2+\zeta_3+\zeta_4) \quad (2.2.11)$$

Now, for solving the equation (2.2.2), assuming that

$$g(x,y)=\psi_y\zeta_x-\psi_x\zeta_y,$$

we can rewrite equation (2.2.2) as

$$\zeta_{xx}+\zeta_{yy}=Re(g(x,y)). \quad (2.2.12)$$

Now using the equation (2.2.3):

$$
\begin{aligned}
&(1+\frac{h^2}{12}\delta_y^2)\delta_x^2\zeta+(1+\frac{h^2}{12}\delta_x^2)\delta_y^2\zeta=(1+\frac{h^2}{12}(\delta_x^2+\delta_y^2))g(x,y)+O(h^6)\\
&8(\zeta_1+\zeta_2+\zeta_3+\zeta_4)+2(\zeta_5+\zeta_6+\zeta_7+\zeta_8)-40\zeta_0=12h^2Re\ g(x,y)+h^4Re(\delta_x^2 g(x,y)+\\
&\qquad\delta_y^2 g(x,y))+O(h^6)\\
&\qquad=12h^2Re\ g(x,y)+h^4Re(g_{xx}+g_{yy})+O(h^6)\\
&\qquad=I_1+I_2+O(h^6) \quad (2.2.13)
\end{aligned}
$$

Differentiating $g(x,y)$ with respect to x and y, we get

$$
\begin{aligned}
g_x&=\psi_{yx}\zeta_x+\psi_y\zeta_{xx}-\psi_{xx}\zeta_y-\psi_x\zeta_{yx}\\
g_y&=\psi_{yy}\zeta_x+\psi_y\zeta_{xy}-\psi_{xy}\zeta_y-\psi_x\zeta_{yy}\\
g_{xx}&=\psi_{yxx}\zeta_x+2\psi_{yx}\zeta_{xx}+\psi_y\zeta_{xxx}-\psi_{xxx}\zeta_y-2\psi_{xx}\zeta_{yx}-\psi_x\zeta_{yxx}\\
g_{yy}&=\psi_{yyy}\zeta_x+2\psi_{yy}\zeta_{xy}+\psi_y\zeta_{xyy}-\psi_{xyy}\zeta_y-2\psi_{xy}\zeta_{yy}-\psi_x\zeta_{yyy}
\end{aligned}
$$

Using,

$$\zeta_{xx} + \zeta_{yy} = Re\ g(x, y) \tag{2.2.14}$$

We get

$$\begin{aligned} g_{xx} + g_{yy} &= \zeta_x(\psi_{xx} + \psi_{yy})_y + \psi_y(\zeta_{xx} + \zeta_{yy})_x - \zeta_y(\psi_{xx} + \psi_{yy}) - \psi_x \\ &\quad (\zeta_{xx}\zeta_{yy})_y + 2\psi_{xy}(\zeta_{xx} - \zeta_{yy}) + 2\zeta_{yx}(\psi_{yy} - \psi_{xx}) \\ &= Re\psi_y g_x - Re\psi_x g_y + 2\psi_{xy}(\zeta_{xx} - \zeta_{yy}) + 2\zeta_{yx}(\psi_{yy} - \psi_{xx}) \end{aligned} \tag{2.2.15}$$

This result implies that $g_{xx} + g_{yy}$ is a combination of first and second derivatives of ψ and ζ, which can be approximated to truncation error of order $O(h^2)$ by the 3×3 grid points. That is, I_2 in (2.2.13) can be approximated by ψ_j, ζ_j, $0 \leqslant \text{j} \leqslant 8$, giving a truncation error of order $O(h^6)$. We now consider the term I_1. First note that

$$\begin{aligned} g(x, y) &= \psi_y\zeta_x - \psi_x\zeta_y \\ \psi_y = \psi_{24} &= \psi_2 - \psi_4 = 2h\psi_x + \frac{2h^3}{3!}\psi_{yyy} + O(h^5) \\ \psi_x = \psi_{13} &= \psi_1 - \psi_3 = 2h\psi_x + \frac{2h^3}{3!}\psi_{xxx} + O(h^5) \end{aligned}$$

The above results, together with similar ones for ζ, yield

$$\begin{aligned} \psi_{24}\zeta_{13} - \psi_{13}\zeta_{24} &= (2h\psi_y + \frac{2h^3}{3!}\psi_{yyy})(2h\zeta_x + \frac{2h^3}{3!}\zeta_{xxx}) - (2h\psi_x + \frac{2h^3}{3!}\psi_{xxx})(2h\zeta_y \\ &\quad + \frac{2h^3}{3!}\zeta_{yyy}) \\ &= 4h^2(\psi_y\zeta_x - \psi_x\zeta_y + \frac{2h^4}{3}(\psi_y\zeta_{xxx} + \psi_{yyy}\zeta_x - \psi_x\zeta_{yyy} - \psi_{xxx}\zeta_y \\ &\quad + O(h^6) \\ &= 4h^2 g + \frac{2h^4}{3}(\psi_y\zeta_{xxx} + \psi_{yyy}\zeta_x - \psi_x\zeta_{yyy} - \psi_{xxx}\zeta_y + O(h^6) \end{aligned} \tag{2.2.16}$$

This result implies that approximations for $h^2 g$ (or, equivalently, for I_1) involve the use of ψ_{xxx}, ψ_{yyy}, ζ_{xxx}, ζ_{yyy}. However, in order to approximate these third derivatives to $O(h^2)$, extra points outside the (3×3) -point domain are required. To avoid this, we observe that

$$\begin{aligned}
\psi_y\zeta_{xxx} + \psi_{yyy}\zeta_x - \psi_x\zeta_{yyy} - \psi_{xxx}\zeta_y & \\
&= \psi_y(\zeta_{xx} + \zeta_{yy})_x - \psi_y\zeta_{yyx} + \zeta_x(\psi_{xx} + \psi_{yy})_y - \zeta_x\psi_{xxy} \\
&\quad - \psi_x(\zeta_{xx} + \zeta_{yy})_y + \psi_x\zeta_{xxy} - \zeta_y(\psi_{xx} + \psi_{yy})_x + \zeta_y\psi_{yyx} \\
&= Re\psi_y g_x - Re\psi_x g_y - \psi_y\zeta_{yyx} - \zeta_x\psi_{xxy} + \psi_x\zeta_{xxy} + \zeta_y\psi_{yyx}
\end{aligned} \tag{2.2.17}$$

Combining (2.2.16) and (2.2.17) gives:

$$\begin{aligned}
12h^2 g &= 3(\psi_{24}\zeta_{13} - \psi_{13}\zeta_{24}) - 2h^4 Re\psi_y g_x - Re\psi_x g_y - \psi_y\zeta_{yyx} - \zeta_x\psi_{xxy} \\
&\quad + \psi_x\zeta_{xxy} + \zeta_y\psi_{yyx} + 0(h^6) \\
12h^2 g + h^4(g_{xx} + g_{yy}) & \\
&= 3(\psi_{24}\zeta_{13} - \psi_{13}\zeta_{24}) - 2h^4 Re\psi_y g_x - Re\psi_x g_y - \psi_y\zeta_{yyx} - \zeta_x\psi_{xxy} + \psi_x \\
&\quad \zeta_{xxy} + \zeta_y\psi_{yyx} + h^4(Re\psi_y g_x - Re\psi_x g_y + 2\psi_{xy}(\zeta_{xx} - \zeta_{yy}) + 2\zeta_{xy}(\psi_{yy} \\
&\quad - \psi_{xx}) \\
&= 3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} + h^4(Re\psi_x g_y - Re\psi_y g_x) + 2(\psi_y\zeta_{yyx} + \zeta_x\psi_{xxy} - \\
&\quad \psi_x\zeta_{xxy} - \zeta_y\psi_{yyx} + 2\psi_{xy}(\zeta_{xx} - \zeta_{yy}) + 2\zeta_{xy}(\psi_{yy} - \psi_{xx}) \\
&= 3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} + h^4[T_1 + T_2 + T_3] + O(h^6),
\end{aligned} \tag{2.2.18}$$

where

$$\begin{aligned}
T_1 &= Re\psi_x g_y - Re\psi_y g_x \\
&= Re(\psi_x g_y - \psi_y g_x) \\
&= Re(\psi_x(\psi_{yy}\zeta_x + \psi_y\zeta_{xy} - \psi_{xy}\zeta_y - \psi_x\zeta_{yy})[-\psi_y(\psi_{yx}\zeta_x + \psi_y\zeta_{xx} - \psi_{xx}\zeta_y - \psi_x\zeta_{yx}) \\
&= Re(\psi_x\psi_{yy}\zeta_x + \psi_x\psi_y\zeta_{xy} + \psi_y\zeta_y\psi_{xx} + 2\psi_x\psi_y\zeta_{xy} - \psi_x\zeta_y\psi_{xy} - \psi_y\zeta_x\psi_{xy} - \psi_x^2\zeta_{yy} \\
&\quad \psi_x^2\zeta_{xx}) \\
T_2 &= 2\psi_{xy}(\zeta_{xx} - \zeta_{yy}) - 2\zeta_{xy}(\psi_{yy} - \psi_{xx}) \\
T_3 &= \psi_y\zeta_{yyx} - \zeta_x\psi_{xxy} + \psi_x\zeta_{xxy} + \zeta_y\psi_{yyx}
\end{aligned}$$

It is clear that each term in T_1, T_2 and T_3 can be approximated up to $O(h^2)$ within the nine-point stencil. Using the difference formulae:

$$
\begin{aligned}
u_{xx} &= \frac{u_1 + u_3 - 2u_0}{h^2} + O(h^2), \\
u_x &= \frac{u_1 - u_3}{2h} + O(h^2), \\
u_{xy} &= \frac{u_5 - u_6 + u_7 - u_8}{4h^2} + O(h^2), \\
u_{xxy} &= \frac{u_5 + u_6 - u_7 - u_8 - 2(u_2 - u_4)}{2h^3} + O(h^2),
\end{aligned}
$$

we obtain,

$$
\begin{aligned}
T_1 = {} & \frac{Re}{4h^4}[\psi_{13}\zeta_{13}\psi_{204} + \psi_{24}\zeta_{24}\psi_{103} + \frac{1}{2}\psi_{13}\psi_{24}(\zeta_{56} + \zeta_{78}) - \frac{1}{4}(\psi_{13}\zeta_{24} + \psi_{24}\zeta_{13})(\psi_{56} \\
& + \psi_{78}) - \psi_{13}^2\zeta_{204} - \psi_{24}^2\zeta_{103}] + O(h^2), \\
T_2 = {} & \frac{1}{2h^4}(\psi_{56} + \psi_{78})(\zeta_{12} + \zeta_{34}) - \frac{1}{2h^4}(\zeta_{56} + \zeta_{78})(\psi_{12} + \psi_{34}) + O(h^2), \\
T_3 = {} & -\frac{2}{h^4}\zeta_{13}\psi_{24} + \frac{2}{h^4}\psi_{13}\zeta_{24} + \frac{1}{2h^4}\psi_{24}(\zeta_{56} - \zeta_{78}) + \frac{1}{2h^4}\zeta_{13}(\psi_{57} + \psi_{68}) - \frac{1}{2h^4}\zeta_{24} \\
& (\psi_{56} - \psi_{78}) - \frac{1}{2h^4}\psi_{13}(\zeta_{57} + \zeta_{68}) + O(h^2).
\end{aligned}
$$

Substituting the above results into (2.2.18) and using (2.2.13), we obtain the fourth-order compact scheme for (2.2.2):

$$
\begin{aligned}
& 8(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4) + 2(\zeta_5 + \zeta_6 + \zeta_7 + \zeta_8) - 40\zeta_0 \\
& \qquad = \frac{Re}{4h^4}[\psi_{13}\zeta_{13}\psi_{204} + \psi_{24}\zeta_{24}\psi_{103} + \frac{1}{2}\psi_{13}\psi_{24}(\zeta_{56} + \zeta_{78}) - \frac{1}{4}(\psi_{13}\zeta_{24} + \psi_{24} \\
& \qquad \zeta_{13})(\psi_{56} + \psi_{78}) - \psi_{13}^2\zeta_{204} - \psi_{24}^2\zeta_{103}] + \frac{1}{2h^4}(\psi_{56} + \psi_{78})(\zeta_{12} + \zeta_{34}) - \\
& \qquad \frac{1}{2h^4}(\zeta_{56} + \zeta_{78})(\psi_{12} + \psi_{34}) - \frac{2}{h^4}\zeta_{13}\psi_{24} + \frac{2}{h^4}\psi_{13}\zeta_{24} + \frac{1}{2h^4}\psi_{24}(\zeta_{56} - \\
& \qquad \zeta_{78}) + \frac{1}{2h^4}\zeta_{13}(\psi_{57} + \psi_{68}) - \frac{1}{2h^4}\zeta_{24}(\psi_{56} - \psi_{78}) - \frac{1}{2h^4}\psi_{13}(\zeta_{57} + \zeta_{68}) \\
& \qquad + O(h^2), \qquad (2.2.19)
\end{aligned}
$$

where $f_{ij} := f_i - f_j$ and $f_{ikj} := f_i - 2f_k + f_j$. The fourth-order compact scheme for the$N-S$ equations (2.2.1) and (2.2.2) is given by (2.2.11) and (2.2.19).

The fourth-order compact scheme (2.2.11) and (2.2.19) may be solved by pointwise iteration methods as described in [8] or by Newton's method with direct solvers at each stage as described in [6].

2.3 Navier-Stokes Equations with Exact Solution

In this section, numerical solutions of (2.2.1) and (2.2.2) using the new fourth-order compact scheme (2.2.11) and (2.2.19) has been obtained. The test problem used in this section is chosen such that the analytical solution is available, so that a rigorous comparison can be made. For test problem, consider the Navier-Stokes equations (2.2.1) and (2.2.2) in $\Omega = (0,1) \times (0,1)$. The boundary conditions on the stream function are given by

$$\begin{aligned} \psi &= \frac{y-x}{Re} - e^{x+y} \\ \psi(x,0) &= \frac{-x}{Re} - e^{x} \\ \psi(x,1) &= \frac{1-x}{Re} - e^{x+1} \\ \psi(0,y) &= \frac{y}{Re} - e^{y} \\ \psi(1,y) &= \frac{y-1}{Re} - e^{1+y} \end{aligned}$$

and the boundary conditions on the vorticity function are given by

$$\begin{aligned} \zeta &= 2e^{x+y} \\ \zeta(x,0) &= 2e^{x} \\ \zeta(x,1) &= 2e^{x+1} \end{aligned}$$

$$\begin{aligned} \zeta(0,y) &= 2e^{y} \\ \zeta(1,y) &= 2e^{1+y} \end{aligned}$$

The exact solution for the above problem is given by[26]:

$$\psi = \frac{y-x}{Re} - e^{x+y}, \quad \zeta = 2e^{x+y}.$$

Table 2.1: RMS errors in Ω for the streamfunction and vorticity at $Re = 1000$

	$\psi - error,$ $\zeta - error$	$\psi - error,$ $\zeta - error$	$\psi - error,$ $\zeta - error$	$\psi - error,$ $\zeta - error$
$Grid$	11×11	21×21	41×41	81×81
$h^2 scheme$	1.41(-4),2.71(-4)	3.35(-5),6.63(-5)	8.17(-6),1.63(-5)	2.02(-6),4.01(-6)
$h^4 scheme$	4.72(-8),9.45(-8)	2.80(-9),5.59(-9)	1.70(-10),3.40(-10)	1.05(-11),2.10(-11)

We notice that the above solution is smooth in $\bar{\Omega}$:=$[0, 1] \times [0, 1]$.

The test problem has been solved for various Reynolds numbers ranging from $Re =\ 5\ to\ 1000$, but since the results appear to be Re-independent, only those for $Re = 1000$ are shown. For the sake of comparison the results using a second-order central difference scheme are also presented. The RMS errors in Ω for the streamfunction and vorticity are given in $Table 1$. It is observed that the results for the h^2 scheme, the central difference scheme, are in good agreement with those obtained by Brameley and Sloan.

It is also seen that the convergence orders for the h^2 scheme and the h^4 scheme, (2.2.11) and (2.2.19) is of fourth-order accuracy when the solutions of (2.2.1) and (2.2.2) are smooth.

2.4 Extensions

2.4.1 Extension to rectangular grids

In this Section, the schemes introduced in Section 2 has been extended to rectangular grids, i.e. the mesh sizes Δx in the x-direction and Δy in the y-direction are different. We use Taylor expansion in the x- and y- directions separately when discretizing the differential equations. In invoking equations (2.2.1) and (2.2.2) to eliminate third order directional derivatives (as ψ in the equations (2.2.15) and (2.2.17), we use

$$\begin{aligned}
-\psi_{xxx} =&\psi_{yyx} + \zeta_x \\
\zeta_{xxx} =&Reg_x - \zeta_{yyx} \\
\psi_{yyy} =& - \zeta_y - \psi_{xxy} \\
-\zeta_{yyy} =& - Reg_y + \zeta_{xxy}
\end{aligned} \tag{2.4.1}$$

Again consider the differential equation:

$$\begin{aligned}
u_{xx} + u_{yy} =&f(x,y) \\
(1+\frac{(\Delta y)^2}{12}\delta_y^2)\delta_x^2 u + (1+\frac{(\Delta x)^2}{12}\delta_x^2)\delta_y^2 u =& (1+\frac{(\Delta x)^2}{12}\delta_x^2)(1+\frac{(\Delta y)^2}{12}\delta_y^2)f + O(h^4)
\end{aligned}$$

Now,

$$\begin{aligned}
(1+\frac{(\Delta y)^2}{12}\delta_y^2)\delta_x^2 u &= (1+\frac{(\Delta y)^2}{12}\delta_y^2)(\frac{u_1+u_3-2u_0}{(\Delta x)^2}) \\
&= \frac{1}{12(\Delta x)^2}[10u_1 - 2u_2 + 10u_3 - 2u_4 + u_5 + u_6 + u_7 + u_8 - 20u_0]
\end{aligned} \tag{2.4.2}$$

$$\begin{aligned}
(1+\frac{(\Delta x)^2}{12}\delta_x^2)\delta_y^2 u &= (1+\frac{(\Delta x)^2}{12}\delta_x^2)(\frac{u_2+u_4-2u_0}{(\Delta y)^2}) \\
&= \frac{1}{12(\Delta y)^2}[10u_2 - 2u_1 + 10u_4 - 2u_3 + u_5 + u_6 + u_7 + u_8 - 20u_0]
\end{aligned} \tag{2.4.3}$$

Adding equations (2.4.2) and (2.4.3), we get

$$\begin{aligned}
(1+\frac{(\Delta y)^2}{12}\delta_y^2)\delta_x^2 u+&(1+\frac{(\Delta x)^2}{12}\delta_x^2)\delta_y^2 u \\
=&\frac{1}{12(\Delta x)^2}[10u_1 - 2u_2 + 10u_3 - 2u_4 + u_5 + u_6 + u_7 + u_8 - 20u_0]+ \\
&\frac{1}{12(\Delta y)^2}[10u_2 - 2u_1 + 10u_4 - 2u_3 + u_5 + u_6 + u_7 + u_8 - 20u_0] \\
=&\frac{1}{12(\Delta x)^2}[10(u_1+u_3) - 2(u_2+u_4) + u_5 + u_6 + u_7 + u_8 - 20u_0]+ \\
&\frac{1}{12(\Delta y)^2}[10(u_2+u_4) - 2(u_1+u_3) + u_5 + u_6 + u_7 + u_8 - 20u_0] \\
=&\frac{1}{12(\Delta x)(\Delta y)}[10\lambda(u_1+u_3) - 2\lambda(u_2+u_4) + \lambda(u_5 + u_6 + u_7 + u_8
\end{aligned}$$

$$\begin{aligned}
&-20u_0)] + \frac{1}{12(\Delta y)(\Delta x)}[10\gamma(u_2+u_4) - 2\gamma(u_1+u_3) + \gamma(u_5+u_6\\
&+u_7+u_8-20u_0)]\\
=&\frac{1}{12(\Delta x)(\Delta y)}[(10\lambda-2\gamma)(u_1+u_3) + (10\gamma-2\lambda)(u_2+u_4) + (\lambda+\gamma)\\
&(u_5+u_6+u_7+u_8-20u_0)]
\end{aligned} \tag{2.4.4}$$

$$\begin{aligned}
&(1+\frac{(\Delta x)^2}{12}\delta_x^2)(1+\frac{(\Delta y)^2}{12}\delta_y^2)f(x,y)\\
&\qquad =f_0 + \frac{(\Delta x)^2}{12}(\delta_x^2 f_0) + \frac{(\Delta y)^2}{12}(\delta_y^2 f_0) + \frac{(\Delta x)^2}{12}\frac{(\Delta y)^2}{12}\delta_x^2\delta_y^2 f_0\\
&\qquad =f_0 + \frac{(\Delta x)^2}{12}(\frac{f_1+f_3-2f_0}{(\Delta x)^2} + \frac{(\Delta y)^2}{12}(\frac{f_2+f_4-2f_0}{(\Delta y)^2}\\
&\qquad =12\frac{\Delta x\Delta y}{12}[12f_0+f_1+f_3-2f_0+f_2+f_4-2f_0]\\
&\qquad =\Delta x\Delta y[8f_0+f_1+f_2+f_3+f_4]
\end{aligned} \tag{2.4.5}$$

Finally, we can obtain the fourth-order compact scheme on a non-uniform grid for (2.2.1) :

$$\begin{aligned}
&\psi_{xx}+\psi_{yy} = -\zeta\\
\Rightarrow\ &(10\lambda-2\gamma)(\psi_1+\psi_3) + (10\gamma-2\lambda)(\psi_2+\psi_4) + (\lambda+\gamma)(\psi_5+\psi_6+\psi_7+\psi_8-20\psi_0)\\
&= -\Delta x\Delta y[8\zeta_0+\zeta_1+\zeta_2+\zeta_3+\zeta_4]
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2\zeta}{\partial^2 x} + \frac{\partial^2\zeta}{\partial^2 y} =Re(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y})\\
\Rightarrow\ &(10\lambda-2\gamma)(\zeta_1+\zeta_3) + (10\lambda-2\gamma)(\zeta_3+\zeta_4) + (\lambda+\gamma)(\zeta_5+\zeta_6+\zeta_7+\zeta_8-20\zeta_0\\
&=12\Delta x\Delta y(1+\frac{(\Delta x)^2}{12}\delta_x^2)(1+\frac{(\Delta y)^2}{12}\delta_y^2)Reg(x,y)\\
&=12\Delta x\Delta y[Reg(x,y) + \frac{(\Delta x)^2}{12}\delta_x^2 Reg(x,y)) + \frac{(\Delta y)^2}{12}\delta_y^2)Reg(x,y)]\\
&=12\Delta x\Delta y Reg(x,y) + (\Delta x)^3\Delta y Reg_{xx}(x,y)) + (\Delta x)(\Delta y)^3)Reg_{yy}(x,y)]
\end{aligned}$$

Now,

$$
\begin{aligned}
Reg(x,y) &= \psi_y\zeta_x - \psi_x\zeta y\\
\psi_x = \psi_{13} = \psi_1 - \psi_3 &= 2\Delta x\psi_x + \frac{2(\Delta x)^3}{3!}\psi_{xxx} + O(h^5)\\
\psi_y = \psi_{24} = \psi_2 - \psi_4 &= 2\Delta y\psi_y + \frac{2(\Delta y)^3}{3!}\psi_{yyy} + O(h^5)
\end{aligned}
$$

$$
\begin{aligned}
&\psi_{24}\zeta_{13} - \psi_{13}\zeta_{24}\\
&\quad=(2\Delta y\psi_y + \frac{2(\Delta y)^3}{3!}\psi_{yyy})(2\Delta x\zeta_x + \frac{2(\Delta x)^3}{3!}\zeta_{xxx}) - (2\Delta x\psi_x + \frac{2(\Delta x)^3}{3!}\\
&\quad\psi_{xxx})(2\Delta y\zeta_y + \frac{2(\Delta)^3}{3!}\psi_{yyy}\\
&\quad=4\Delta x\Delta y\psi_y\zeta_x + \frac{(2\Delta y(\Delta x)^3}{3}\psi_y\zeta_{xxx} + \frac{2\Delta x(\Delta y)^3}{3}\psi_{yyy}\zeta_x) + \frac{4(\Delta x)^3(\Delta y)^3}{3!3!}\\
&\quad\psi_{yyy}\zeta_{xxx} - 4\Delta x\Delta y\psi_x\zeta_y - \frac{(2\Delta y(\Delta x)^3}{3}\psi_{xxx}\zeta_y + \frac{2\Delta x(\Delta y)^3}{3}\psi_x\zeta_{yyy})+\\
&\quad\frac{4(\Delta y)^3(\Delta x)^3}{3!3!}\psi_{yyy}\zeta_{xxx}
\end{aligned}
$$

$$
\begin{aligned}
&=4\Delta x\Delta y(\psi_y\zeta_x - \psi_x\zeta_y) + \frac{(2\Delta y(\Delta x)^3}{3}(\psi_y\zeta_{xxx} - \psi_{xxx}\zeta_y + \frac{2\Delta x(\Delta y)^3}{3}\\
&(\psi_{yyy}\zeta_x - \psi_x\zeta_{yyy}) + O(h^6)\\
&=4\Delta x\Delta yReg(x,y) + \frac{(2\Delta y(\Delta x)^3}{3}[\psi_y(Reg_x - \zeta_{yyx}) + \zeta y(\psi_{yyx} + \zeta_x)]\\
&\quad+\frac{2\Delta x(\Delta y)^3}{3}(-\zeta_y\zeta x - \psi_{xxy}\zeta_x - Reg_y\psi_x + \psi_x\zeta_{xxy}]
\end{aligned}
$$

Using the notations $\lambda= \frac{\Delta y}{\Delta x}$, $\gamma= \frac{\Delta x}{\Delta y}$ and $f_{ij} := f_i - f_j$, f_{ikj}:=$f_i - 2f_k + f_j$, we get

$$
\begin{aligned}
&\Rightarrow 12\Delta x\Delta yRe\ g(x,y)\\
&\qquad=3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} - 2\Delta y(\Delta x)^3[\psi_yReg_x - \psi_y\zeta_{yyx}) + \zeta y\psi_{yyx} + \zeta_x\zeta_y]\\
&\qquad\quad - 2\Delta x(\Delta y)^3[-\zeta_y\zeta x - \psi_{xxy}\zeta_x - Reg_y\psi_x + \psi_x\zeta_{xxy}
\end{aligned}
$$

Now,

$$\begin{aligned}
&12\Delta x\Delta y Reg(x,y) + \Delta y(\Delta x)^3 Reg_{xx} + \Delta x(\Delta y)^3 Reg_{yy}\\
&\quad =3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} + \Delta y(\Delta x)^3[(-2\psi_y Reg_x + 2\zeta_{yyx}\psi_y - 2\psi_{yyx}\zeta_y - 2\zeta_x\zeta_y)\\
&\quad + Re(\psi_{yxx}\zeta_x + 2\psi_{yx}\zeta_{xx} + \zeta_{xxx}\psi_y - \psi_{xxx}\zeta_y - 2\psi_{xx}\zeta_{yx} - \psi_x\zeta_{yxx}) + \Delta x\\
&\quad (\Delta y)^3[2\zeta_y\zeta_x + 2\psi_{xxy}\zeta_x + 2Reg_y\psi_x - 2\psi_x\zeta_{xxy}) + Re(\zeta_{yyy}\zeta_x + 2\psi_{yy}\zeta_{xy}+\\
&\quad \zeta_{xxy}\psi_y - \psi_{xyy}\zeta_y - 2\psi_{xy}\zeta_{yy} - \psi_x\zeta_{yyy})
\end{aligned}$$

$$\begin{aligned}
&=3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} + \Delta y(\Delta x)^3[(-2\psi_y Reg_x + 2\zeta_{yyx}\psi_y - 2\psi_{yyx}\zeta_y - 2\zeta_x\zeta_y+\\
&\quad Re\psi_{yxx}\zeta_x + 2Re\psi_{yx}\zeta_{xx} + Re(Reg_x\psi_y - \psi_y\zeta_{yyx}) + Re(\psi_{yyx}\zeta_y + Re\psi_x\zeta_y\\
&\quad - 2Re\psi_{xx}\zeta_{yx} - Re\psi_x\zeta_{yxx}) + \Delta x(\Delta y)^3[2\zeta_y\zeta x + 2\psi_{xxy}\zeta_x + 2Reg_y\psi_x - 2\\
&\quad \psi_x\zeta_{xxy}) - Re\zeta_y\zeta_x - Re\psi_{xxy}\zeta x + 2Re\zeta_{xy}\psi_{yy} + Re\zeta_{xyy}\psi_y - Re\psi_{xyy}\zeta_y - 2\\
&\quad Re\psi_{xy}\zeta_{yy} - Re(Reg_y\psi_x + \zeta_{xxy}\psi_x)\\
&=3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} + \Delta y(\Delta x)^3[(-\psi_y Reg_x + \zeta_{yyx}\psi_y - \psi_{yyx}\zeta_y - Re\ \zeta_x\ \zeta_y+\\
&\quad Re\psi_{yxx}\zeta_x + 2Re\psi_{yx}\zeta_{xx} - 2Re\psi_{xx}\zeta_{yx}) - Re(\psi_x\zeta_{xxy} + \Delta x(\Delta y)^3[-\zeta_y\zeta_x+\\
&\quad \psi_{xxy}\zeta_x + Reg_y\psi_x - \psi_x\zeta_{xxy} + Re2\psi_{yy}\zeta_{xy} + Re\psi_{xyy}\zeta y - Re\zeta_{xyy}\psi_{yy} - 2Re\\
&\quad \zeta_{yy}\psi_{xy}
\end{aligned}$$

$$\begin{aligned}
&12\Delta x\Delta y Reg(x,y) + \Delta y(\Delta x)^3 Reg_{xx} + \Delta x(\Delta y)^3 Reg_{yy}\\
&\quad =3\psi_{24}\zeta_{13} - 3\psi_{13}\zeta_{24} + \Delta y(\Delta x)^3[-Re(\psi_y\zeta_x\psi_{yx} + \psi_y^2\zeta_{xx} - \psi_y\psi_{xx}\zeta_y - \psi_x\zeta_{yx}\\
&\quad \psi_y) + \zeta_{yyx}\psi_y - \psi_{yyx}\zeta_y - Re\zeta_x\zeta_y + Re\psi_{yxx}\zeta_x + 2Re\psi_{yx}\zeta_{xx} - 2Re\psi_{xx}\zeta_{yx})\\
&\quad - Re(\psi_x\zeta_{xxy}\frac{(\zeta_{56}+\zeta_{78})}{4\Delta x\Delta y} + \frac{\psi_{24}^2}{4(\Delta y)^2}\frac{\zeta_{103}}{(\Delta x)^2} - \frac{\psi_{24}\zeta_{24}}{2\Delta y 2\Delta x}\frac{\psi_{103}}{(\Delta x)^2} - \frac{\psi_{13}\psi_{24}}{2\Delta x 2\Delta y}\\
&\quad \frac{(\zeta_{56}+\zeta_{78})}{4\Delta x\Delta y} + \frac{\psi_{13}\psi_{24}}{2\Delta x 2\Delta y}\frac{(\zeta_{56}+\zeta_{78})}{4\Delta x\Delta y} + \frac{\psi_{24}}{2\Delta y}\frac{(\zeta_{56}-\zeta_{78})}{2(\Delta y)^2\Delta x} - \frac{2\psi_{24}\zeta_{13}}{4(\Delta y)^3\Delta x}+\\
&\quad \frac{2\zeta_{24}\psi_{13}}{4(\Delta y)^3\Delta x} - \frac{\zeta_{13}\zeta_{24}}{4\Delta x\Delta y} + \frac{\zeta_{13}(\psi_{56}-\psi_{78})}{4(\Delta x)^3\Delta y} - \frac{\zeta_{13}\psi_{24}}{2(\Delta x)^3\Delta y} + \frac{\psi_{56}+\psi_{78}\zeta_{103}}{4(\Delta x)^3\Delta y}\\
&\quad - \frac{2\psi_{103}(\zeta_{56}+\zeta_{78})}{4(\Delta x)^3\Delta y} - \frac{\psi_{13}(\zeta_{56}-\zeta_{78})}{4(\Delta x)^3\Delta y} + \frac{\zeta_{24}\psi_{13}}{(\Delta x)^3\Delta y} + \Delta x(\Delta y)^3[Re(\frac{\psi_{13}\psi_{24}}{2\Delta x 2\Delta y}
\end{aligned}$$

$$\frac{(\zeta_{56}+\zeta_{78})}{4\Delta x\Delta y}-\frac{\psi_{13}^2}{4(\Delta x)^2}\frac{\zeta_{204}}{(\Delta y)^2}+Re\frac{\psi_{13}\zeta_{13}}{2\Delta y2\Delta x}\frac{\psi_{103}}{(\Delta x)^2}-\frac{\psi_{13}\zeta_{24}}{2\Delta x2\Delta y}\frac{(\psi_{56}+\psi_{78})}{4\Delta x\Delta y}$$
$$+\frac{\psi_{13}}{2\Delta x}\frac{(\zeta_{56}-\zeta_{78})}{2(\Delta x)^2\Delta y}+\frac{\psi_{24}}{2\Delta y}\frac{(\zeta_{56}-\zeta_{78})}{2(\Delta y)^2\Delta x}-\frac{2\psi_{24}\zeta_{13}}{4(\Delta y)^3\Delta x}+\frac{2\zeta_{24}\psi_{13}}{4(\Delta y)^3\Delta x}+\frac{\zeta_{13}\zeta_{24}}{4\Delta x\Delta y}$$
$$+\frac{\zeta_{13}(\psi_{56}-\psi_{78})}{4(\Delta x)^3\Delta y}-\frac{\zeta_{13}\psi_{24}}{2(\Delta x)^3\Delta y}-\frac{(\psi_{56}+\psi_{78})\zeta_{204}}{2(\Delta y)^3\Delta x}-\frac{2\psi_{204}(\zeta_{56}+\zeta_{78})}{4(\Delta y)^3\Delta y}-$$
$$\frac{\zeta_{24}(\zeta_{56}-\zeta_{78})}{4(\Delta y)^3\Delta x}+\frac{\zeta_{24}\psi_{13}}{2(\Delta x)^3\Delta y}]$$

$$12\Delta x\Delta y Reg(x,y)+\Delta y(\Delta x)^3 Reg_{xx}+\Delta x(\Delta y)^3 Reg_{yy}$$
$$=3\psi_{24}\zeta_{13}-3\psi_{13}\zeta_{24}-\frac{Re}{4}\gamma\psi_{13}\psi_{24}(\zeta 56+\zeta 78)-\frac{Re}{4}\gamma\psi_{24}^2\zeta_{103}+\frac{Re}{4}\gamma(\psi_{56}+$$
$$\psi_{78})+\frac{Re}{4}\gamma\psi_{13}\psi_{24}(\zeta_{56}+\zeta_{78})+\frac{(\gamma)^2}{4}(\zeta_{56}+\zeta_{78})\psi_{24}-\frac{(\gamma)^2}{2}\psi_{24}\zeta_{13}-\frac{(\gamma)^2}{4}($$
$$\psi_{56}+\psi_{78}\zeta_{24}+\frac{(\gamma)^2}{2}\zeta_{24}\psi_{13}-\frac{(\Delta x)^2}{4}\zeta_{13}\zeta_{24}+\frac{1}{4}\zeta_{13}(\psi_{56}-\psi_{78})-\frac{1}{2}\zeta_{13}\psi_{24}+\frac{1}{4}$$
$$(\psi_{56}+\psi_{78})\zeta_{103}-\frac{1}{2}\psi_{103}(\zeta_{56}+\zeta_{78})-\frac{1}{4}\psi_{13}(\zeta_{56}-\zeta_{78})+\frac{\zeta_{24}\psi_{13}}{2}+\frac{(\Delta y)^2}{2}\zeta_{24}$$
$$\zeta_{13}+\frac{(\lambda)^2}{4}\zeta_{13}(\psi_{56}-\psi_{78})-\frac{Re}{4}\lambda\psi_{13}\zeta_{24}(\psi_{56}+\psi_{78})-\frac{Re}{4}\lambda\psi_{13}^2\zeta_{204}+\frac{Re}{4}\lambda\psi_{13}$$
$$\psi_{24}(\zeta_{56}+\zeta_{78})-\frac{(\lambda)^2}{4}(\zeta_{56}-\zeta_{78})\psi_{13}-\frac{(\lambda)^2}{2}\psi_{13}\zeta_{24}-\frac{(\lambda)^2}{2}\zeta_{13}\psi_{24}-\frac{1}{4}\zeta_{24}(\psi_{56}-\psi_{78})$$
$$+\frac{1}{2}\zeta_{24}\psi_{13}-\frac{1}{4}(\psi_{56}+\psi_{78})\zeta_{204}+\frac{1}{2}\psi_{204}(\zeta_{56}+\zeta_{78})+\frac{1}{4}\psi_{24}(\zeta_{56}-\zeta_{78})-\frac{\zeta_{13}\psi_{24}}{2}+$$
$$\frac{Re}{4}(\lambda)^2\psi_{13}\zeta_{13}\psi_{103}$$

$$=3\psi_{24}\zeta_{13}-3\psi_{13}\zeta_{24}-\frac{1}{2}(\psi_{24}\zeta_{13}-\psi_{13}\zeta_{24})-\frac{1}{2}(\psi_{24}\zeta_{13}-\psi_{13}\zeta_{24})-\frac{(\gamma)^2}{2}(\psi_{24}\zeta_{13}-\psi_{13}$$
$$\zeta_{24})-\frac{(\lambda)^2}{2}(\psi_{24}\zeta_{13}-\psi_{13}\zeta_{24})+\frac{Re}{4}((\Delta y)^2-(\Delta x)^2)\zeta_{13}\zeta_{24}\frac{Re}{4}[\psi_1(\zeta_{67}+3\zeta_{85})+\psi_2$$
$$(\zeta_{78}+3\zeta_{56})+\psi_3(\zeta_{85}+3\zeta_{67})+\psi_4(\zeta_{56}+3\zeta_{78})+\psi_5(\zeta_{34}+3\zeta_{12})+\psi_6(\zeta_{41}+3\zeta_{23})+$$
$$\psi_7(\zeta_{12}+3\zeta_{34})+\psi_8(\zeta_{23}+3\zeta_{41})]+\lambda_2\frac{Re}{4}[\zeta_{13}(\psi_{57}+\psi_{68})-\psi_{13}(\zeta_{57}+\zeta_{68})]+\gamma^2\frac{Re}{4}$$
$$[\psi_{24}(\zeta_{56}-\zeta_{78})-\zeta_{24}(\psi_{56}-\psi_{78})]+\frac{Re^2}{4}(\lambda\psi_{13}(\zeta_{13}\psi_{204}-\psi_{13}\zeta_{204})+\gamma\psi_{24}(\zeta_{24}\psi_{103}$$
$$-\psi_{24}\zeta_{103})+\frac{\lambda+\gamma}{4}\psi_{13}\psi_{24}(\zeta_{56}+\zeta_{78})-\frac{1}{4}(\gamma\psi_{24}\zeta_{13}+\lambda\psi_{13}\zeta_{24})(\psi_{56}+\psi_{78})$$

Finally, the fourth-order compact scheme for equation (2.2.2) over the non-uniform grid is given by:

$$
\begin{aligned}
&(10\lambda - 2\gamma)(\zeta_1 + \zeta_3) + (10\lambda - 2\gamma)(\zeta_3 + \zeta_4) + (\lambda + \gamma)(\zeta_5 + \zeta_6 + \zeta_7 + \zeta_8 - 20\zeta_0) \\
&\quad = \frac{Re}{2}(4 - \lambda^2 - \gamma^2)(\psi_{24}\zeta_{13} - \psi_{13}\zeta_{24}) + \frac{Re}{4}(\Delta y^2 - \Delta x^2)\zeta_{13}\zeta_{24} + \frac{Re}{4}[\psi_1 \\
&\quad (\zeta_{67} + 3\zeta_{85}) + \psi_2(\zeta_{78} + 3\zeta_{56}) + \psi_3(\zeta_{85} + 3\zeta_{67}) + \psi_4(\zeta_{56} + 3\zeta_{78}) + \psi_5 \\
&\quad (\zeta_{34} + 3\zeta_{12}) + \psi_6(\zeta_{41} + 3\zeta_{23}) + \psi_7(\zeta_{12} + 3\zeta_{34}) + \psi_8(\zeta_{23} + 3\zeta_{41})] + \lambda_2 \\
&\quad \frac{Re}{4}[\zeta_{13}(\psi_{57} + \psi_{68}) - \psi_{13}(\zeta_{57} + \zeta_{68})] + \gamma^2\frac{Re}{4}[\psi_{24}(\zeta_{56} - \zeta_{78}) - \zeta_{24}(\psi_{56} \\
&\quad - \psi_{78})] + \frac{Re^2}{4}(\lambda\psi_{13}(\zeta_{13}\psi_{204} - \psi_{13}\zeta_{204}) + \gamma\psi_{24}(\zeta_{24}\psi_{103} - \psi_{24}\zeta_{103}) \\
&\quad + \frac{\lambda + \gamma}{4}\psi_{13}\psi_{24}(\zeta_{56} + \zeta_{78}) - \frac{1}{4}(\gamma\psi_{24}\zeta_{13} + \lambda\psi_{13}\zeta_{24})(\psi_{56} + \psi_{78})
\end{aligned}
$$

where $\lambda = \frac{\Delta y}{\Delta x}$, $\gamma = \frac{\Delta x}{\Delta y}$ and again $f_{ij} := f_i - f_j$, $f_{ikj} := f_i - 2f_k + f_j$. It is easy to verify that when $\Delta x = \Delta y$, the above scheme is in coincidence with the one given in Section 2. In order for the above scheme to allow SOR-type iteration, the mesh ratio γ is required to satisfy $\gamma \ \varepsilon \ (\frac{1}{\sqrt{5}, \sqrt{5}})$.

2.4.2 Extension to more general domains

If a domain can be transformed into a rectangular one by conformal mappings, then the present compact fourth-order methods can be extended to solve the transformed equations in the rectangular domain. By using a conformal mapping $x = x(\sigma, \eta)$ and (σ, η), the resulting Navier-Stokes equations can be written as

$$\psi_{\sigma\sigma} + \psi_{\eta\eta} = -\beta(\sigma, \eta)\zeta, \tag{2.4.6}$$

$$\zeta_{\sigma\sigma} + \zeta_{\eta\eta} = Re(\psi_\eta \zeta_\sigma - \psi_\sigma \zeta_\eta) \tag{2.4.7}$$

where $\beta \ (\sigma, \eta)$ is a known function. A fourth-order compact scheme for (2.4.6) and (2.4.7) can be constructed in similar way as discussed above. The only extra steps are add the term $\zeta \ (\beta_\sigma \ \zeta_\eta$-$\beta_\eta \ \zeta_\sigma)$ to the right-hand sides of (2.2.15) and (2.2.17). Since β_σ and β_η

are known functions, the extra term can be readily approximated up to $O(h^2)$ with the nine-point stencil.

2.5 Remarks and Conclusions

2.5.1 Non-compact fourth-order schemes

Hou and Wetton [19] employed fourth -order streamfunction methods for the time - dependent, incompressible N-S equations. Wide schemes which are built using standard fourth-order difference operators are employed instead of compact ones and the boundary terms are handled by extrapolating the streamfunction values. Evidence is given this approach is preferable to using compact differencing for high-Reynolds-number flows. This property of compact schemes has been well documented in [19]. Indeed, it has been found that for the driven cavity problem the convergence becomes slow and SOR pointwise iteration does not work when $Re \geq 9000$. One of the reasons for this is that the truncation errors for all the compact schemes are of order $O(h^4 Re)$, while the truncation errors for convectional fourth-order schemes are of order $O(h^4 Re)$.

2.5.2 Conclusions

In this work, a compact fourth-order scheme for the time-independent Navier-Stokes equations with the novelty of ‘genuine compactness’ has been developed. In deriving compact fourth-order schemes, the main difference between the method proposed and previous ones is the following. To obtain a compact fourth-order scheme for (2.2.2), previous researches employ Taylor expansion for (2.2.2) but do not use (2.2.1) which gives the relation between ψ and ζ. However, this procedure also employs equation (2.2.1), so that the compact scheme for (2.2.2) is strictly with in the nine-point stencil. The key point with the present scheme is that it allows direct iteration for low-to-medium Re.

Chapter 3

A SINGLE CELL HIGH ORDER SCHEME FOR THE CONVECTION-DIFFUSION EQUATION WITH VARIABLE COEFFICIENTS

3.1 Introduction

In this chapter, we consider the convection-diffusion equation

$$LU \equiv u_{xx} + u_{yy} + p(x,y)u_x + q(x,y)u_y = f(x,y) \tag{3.1.1}$$

This equation often appears in the description of transport phenomena. The magnitudes of $p(x,y)$ *and* $q(x,y)$ determine the ratio of the convection to diffusion. In many problems of practical interest the convective terms dominate the diffusion. Numerical simulation of (3.1.1) becomes increasingly difficult as the ratio of the convection to diffusion increases. When the equation (3.1.1) is discretized using central differences(CDS), the resulting scheme has a truncation error of order h^2. In the case of CDS, iterative methods for solving the resulting system of linear equations do not converge when the convective terms dominate and the cell Reynolds number is greater than a certain constant. In addition, direct methods for solving the system of linear equations may give erroneous results. If the convective terms are approximated by suitable forward or backward differences and the diffusion terms by central differences, the resulting scheme is called the upwind or the upstream scheme or the UDS. There are several variations of UDS and also combinations of the CDS and the UDS schemes. The UDS introduces artificial viscosity and hence the results are in error when the convection dominates. Both the truncation and the discretization error of UDS are of order h and hence a very fine mesh is needed if accurate results are required. Such a refinement of the mesh is often uneconomical .

Recently, a new finite difference scheme for the special case of equation (3.1.1) where p and q are constants has been proposed [12]. In this work, a generalization of the scheme to the case of variable coefficients $p(x, y)$ and $q(x, y)$ has been presented. The new scheme has a truncation error of order h^4 and the resulting system of linear equations can be solved by iterative methods even for large absolute values of $p(x, y)$ and $q(x, y)$.

The coefficients of the non-diagonal terms in the difference equation do not have the same sign for all values of $p(x, y)$ and $q(x, y)$, therefore it is not possible to predict the order of the discretization error from that of the truncation error using the theoretical results known so far. There are some difference schemes for which the order of the discretization error is reduced by 1 as the transport number becomes large. However, from the numerical results of several test problems, it appears that this is not the case for the method proposed in this chapter[13]. The order of discretization error is defined for the asymptotic case when the mesh size $h \to 0$. The numerical estimates of the order determined from the errors calculated by using two different mesh sizes may not reach its asymptotic value as long as the derivatives appearing in the expression for the truncation error vary with the change in the mesh. This is indeed the case when the transport number is large. A better estimate of the order of the discretization error is obtained in such cases by refining the mesh. Several test problems have been solved using the proposed scheme and also using the UDS and the CDS. In almost all cases the proposed scheme produced better results for a given mesh size. Only those schemes which are designed on the basis of the exact solution of a particular problem give better results when the mesh is crude. Such schemes are a lot more complicated and difficult to implement. The rate at which the error decreases as the mesh is refined is not as fast as that of the scheme presented here, and hence the presented scheme gives better results as the mesh is refined. In the scheme proposed here, the coefficients can be computed easily when the grid is uniform. the procedure presented in the chapter can be extended to irregular meshes. In such cases, the difference scheme is not obtained explicitly for each mesh point but is computed as the difference equations are assembled. The procedure has been generalized to the case of the diffusion convection equation when the diffusion coefficients are variable, and the resulting scheme has been applied to some problems of flows in porous media. The preliminary results of these extensions are quite promising and will be reported in the future. In this chapter[13], only the equation (3.1.1) on regular meshes has been considered. In the derivation of the difference scheme, the solution $u(x, y)$ is first expressed locally on a mesh element in terms of a linear combination of the basis functions which are chosen to be polynomials in the present case. The functions $p(x, y), q(x, y)$ and $f(x, y)$ are expanded in a similar manner. A set of linear equations for the unknown coefficients in the expansion of $u(x, y)$ are obtained by demanding that the differential equation (3.1.1) be satisfied locally. Additional equations are obtained by interpolating the solution over a set of mesh points which lie on the cell. This technique has been used to obtain single cell high order schemes for the Poisson, the Helmholtz, the biharmonic and other linear equations. The difference scheme derived here is a 9-point scheme. Only those mesh points which lie on a single square cell of side $2h$ are involved, thereby keeping the bandwidth as small as possible for the order of the truncation error achieved. No special formulae are needed for points near the boundary. The new finite difference scheme for equation (3.1.1) is presented in the next section. The results of numerical experiments with this scheme are given in Section 3.

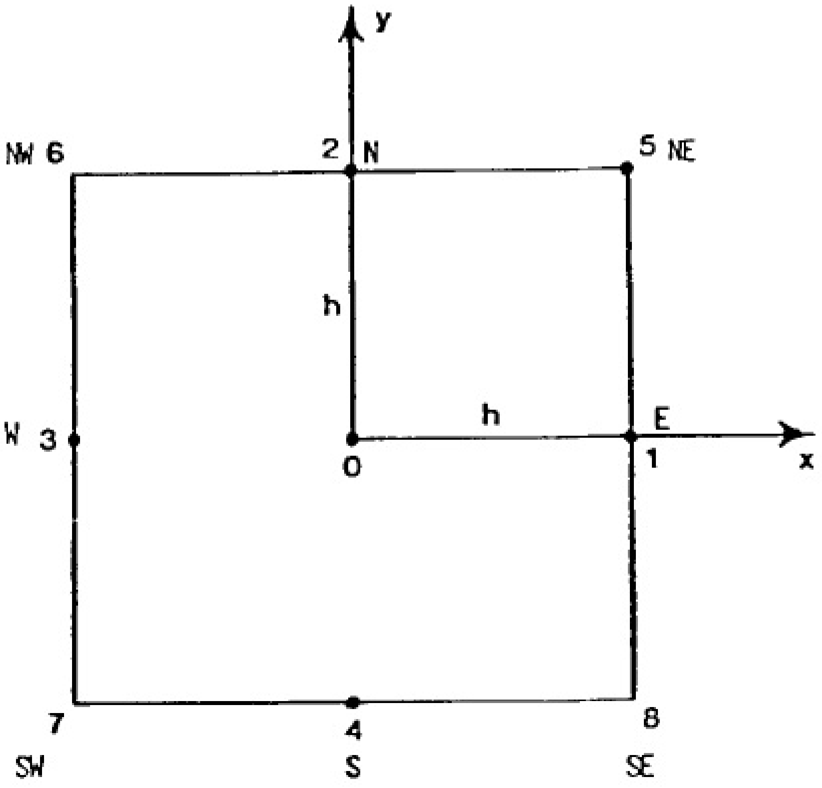

Figure 1. Labelling of grid points

3.2 The Finite Difference Scheme

The finite difference formula for a mesh point (x, y) which is denoted by '$0'$' in *Figure* 1 involves the other eight mesh points at $(x \pm h, y), (x, y \pm h)$. These points are denoted either by numbers $1 - 8$ or by letters showing their directions with respect to the point '$0'$' as in *Figure* 1. The difference formula involves the coefficients $\lambda_{i,j}$, $\mu_{i,j}$ and $c_{i,j}$ which appear in the expansions of $p(x, y)$, $q(x, y)$ *and* $f(x, y)$ along with the nodal values $u_k = u(x_k, y_k)$ for $k = 0, 1, 2, ...$ Alternatively, the coefficients in the expansion of the known functions can be expressed in terms of their partial derivatives. In practice it is more convenient to use the nodal values of the known functions rather than their derivatives. Therefore, an alternative formulation of the difference formula involving the nodal values of $p(x, y)$, $q(x, y)$ *and* $f(x, y)$ is also given.

The differential equation is given by

$$u_{xx} + u_{yy} + p(x,y)u_x + q(x,y)u_y = f(x,y). \tag{3.2.1}$$

We assume that, locally, the solution $u(x,y)$ and the functions p, q, f can be expressed by two-dimensional power series:

$$u(x,y) = \sum a_{i,j}x^i y^j, \qquad f(x,y) = \sum c_{i,j}x^i y^j \tag{3.2.2}$$

$$p(x,y) = \sum \lambda_{i,j}x^i y^j, \qquad q(x,y) = \sum \mu_{i,j}x^i y^j \tag{3.2.3}$$

Substituting equations (3.2.2) and (3.2.3) into equation (3.2.1) and comparing the coefficients of $x^i y^j$, we get

$$\begin{aligned}
x^i y^j c_{i,j} =& \sum_{i=2}^{\infty} a_{i,j} i(i-1)x^{i-2}y^j + \sum_{j=2}^{\infty} a_{i,j} j(j-1)x^i y^{j-2} + \sum_{i,j} \lambda_{i,j}x^i y^j \sum_{i=1}^{\infty} i a_{i,j} \\
& x^{i-1}y^j + \sum_{i,j} \mu_{i,j}x^i y^j \sum_{j=1}^{\infty} (j) a_{i,j} x^i y^{j-1} \\
x^i y^j c_{i,j} =& \sum_{i=0}^{\infty} (i+2)(i+1)a_{i+2,j}x^i y^j + \sum_{i=0}^{\infty} (j+2)(j+1)a_{i,j+2}x^i y^j + \sum_{i,j} \lambda_{i,j} \\
& x^i y^j \sum_{i=0}^{\infty} (i+1)a_{i+1,j}x^{i-1}y^j + \sum_{i,j} \mu_{ij}x^i y^j \sum_{j=0}^{\infty} (j+1)a_{i,j+1}x^i y^j \\
c_{i,j} =& (i+1)(i+2)a_{i+2,j} + (j+1)(j+2)a_{i,j+2} + \lambda_{0,0}(i+1)a_{i+1,j} + i\lambda_{1,0} \\
& a_{i,j} + \lambda_{2,0}(i-1)a_{i-1,j} + ... + \lambda_{1,1} i a_{i,j-1} + \mu_{0,0}(j+1)a_{i,j+1} + \mu_{0,1} j \\
& a_{i,j} + \mu_{0,2}(j-1)a_{i,j-1} + ... + \mu_{1,1} j a_{i-1,j} ...
\end{aligned}$$

$$\begin{aligned}
c_{i,j} =& (i+1)(i+2)a_{i+2,j} + (j+1)(j+2)a_{i,j+2} + \sum_{i \le r, s \le j} [(i-r+1)\lambda_{r,\ s} \\
& a_{i-r+1,j-s} + (j-s+1)\mu_{r,s}a_{i-r,j+1-s}]
\end{aligned} \tag{3.2.4}$$

The equations (3.2.4) constitute the constraints imposed by the differential equation (3.2.1) on the coefficients a_{ij}, c_{ij}, λ_{ij} and μ_{ij} of the expansions (3.2.2) and (3.2.3).

In particular,

$$
\begin{aligned}
c_{0,0} &= 2(a_{2,0} + a_{0,2}) + \lambda_{0,0}a_{1,0} + \mu_{0,0}a_{0,1} \qquad (3.2.5)\\
c_{1,0} &= 6a_{3,0} + 2a_{1,2} + 2\lambda_{0,0}a_{2,0} + \lambda_{1,0}a_{1,0} + \mu_{0,0}a_{1,1} + \mu_{1,0}a_{0,1}\\
c_{0,1} &= 2a_{2,1} + 6a_{0,3} + \lambda_{0,0}a_{1,1} + \lambda_{0,1}a_{1,0} + 2\mu_{0,0}a_{0,2} + \mu_{0,1}a_{0,1}\\
c_{2,0} &= 12a_{4,0} + 2a_{22} + 3\lambda_{0,0}a_{3,0} + 2\lambda_{1,0}a_{2,0} + \lambda_{2,0}a_{1,0} + \mu_{0,0}a_{2,1}+\\
&\quad \mu_{1,0}a_{2,1} + \mu_{1,0}a_{1,1} + \mu_{2,0}a_{0,1}\\
c_{0,2} &= 2a_{2,2} + 12a_{0,4} + \lambda_{0,0}a_{1,2} + \lambda_{0,1}a_{1,1} + \lambda_{0,2}a_{1,0} + 3\mu_{0,0}a_{0,3}+\\
&\quad 2\mu_{0,1}a_{0,2} + \mu_{0,2}a_{0,1}\\
c_{1,1} &= 6a_{3,1} + 6a_{1,3} + 2\lambda_{0,0}a_{2,1} + \lambda_{1,0}a_{1,1} + \lambda_{1,1}a_{1,0} + 2\mu_{0,0}a_{1,2}+\\
&\quad \mu_{0,1}a_{1,1} + \mu_{1,1}a_{0,1}
\end{aligned}
$$

The above six constraints ensure the satisfaction of the differential equation (3.1.1) for $u = x^i y^j$ for $i + j \le 4$. The constraints involve 15 unknown values of $a_{i,j},\ 0 \le i + j \le 4$. The remaining nine equations, relating the values of $a_{i,j}$, are obtained by collocation on the nine points $0 - 8$ of the single cell of side $2h$ ($see\ Figure$ 1). In particular,

$$
\begin{aligned}
u_N = u_2 &= u(x, y + h)\\
&= u_{x,y} + hu_y(x, y) + \frac{h^2}{2!}u_{yy}(x, y) + \frac{h^3}{3!}u_{yyy}(x, y) \qquad (3.2.6)
\end{aligned}
$$

Now,

$$
\begin{aligned}
u &= \sum_{j=0} a_{i,j}x_i y_j\\
u_y &= \sum_{j=1} j a_{i,j}x_i y_{j-1}\\
&= \sum_{j=0} (j + 1)a_{i,j+1}x_i y_j
\end{aligned}
$$

$$
\begin{aligned}
u_y(0,0) &= a_{0,1} \\
u_{yy} &= \sum_{j=2} j(j-1)a_{i,j}x_i y_{j-2} \\
&= \sum_{j=0} (j+2)(j+1)a_{i,j+2}x_i y_j \\
u_{yy}(0,0) &= 2a_{0,2}
\end{aligned}
$$

Putting these values of u_y and u_{yy} in equation (3.2.6), we get

$$
\begin{aligned}
u_N = u_2 &= a_{0,0} + a_{0,1}h + a_{0,2}h^2 + a_{0,3}h^3 + a_{0,4}h^4 + ... \\
u_{NE} = u_5 &= u(x+h, y+h) \\
&= u(x+h,y) + hu_y(x+h,y) + \frac{h^2}{2!}u_{yy}(x+h,y) + \frac{h^3}{3!}u_{yyy}(x+h,y) + ... \\
&= u(x,y) + hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \frac{h^3}{3!}u_{xxx}(x,y) + \frac{h^4}{4!}u_{xxxx}(x,y) + ...+ \\
&\quad h[u_y(x,y) + hu_{x,y}(x,y) + \frac{h^2}{2!}u_{xxy}(x,y) + \frac{h^3}{3!}u_{xxxy}(x,y) + ...] + \frac{h^2}{2!}[u_{yy} \\
&\quad (x,y) + hu_{xyy} + \frac{h^2}{2!}u_{yy}u_{xx}(x,y) + ...] \\
&= u(x,y) + h(u_x + u_y) + \frac{h^2}{2!}(u_{yy} + u_{xx} + 2u_{xy}) + \frac{h^3}{3!}(u_{xxx} + 3u_{xxy} + 3u_{xyy}) \\
&\quad + \frac{h^4}{4!}(u_{xxxx} + 4u_{xxxy} + 4u_{xx}u_{yy}) \\
&= a_{0,0} + h(a_{l,0} + a_{0,1} + h^2(a_{2,0} + a_{0,2} + a_{1,1}) + h^3(a_{3,0} + a_{2,1} + a_{1,2}) + h^4 \\
&\quad (a_{4,0} + a_{3,1} + a_{2,2}) \\
u_S = u_4 &= u(x, y-h) \\
&= u_{x,y} - hu_y(x,y) + \frac{h^2}{2!}u_{yy}(x,y) - \frac{h^3}{3!}u_{yyy}(x,y) + ... \\
&= a_{0,0} - a_{0,1}h + a_{0,2}h^2 - a_{0,3}h^3 + a_{0,4}h^4 + ...
\end{aligned}
$$

Now, adding u_N and u_S , we get

$$
\begin{aligned}
u_2 + u_4 &= 2a_{0,0} + 2a_{0,2}h^2 + 2a_{0,4}h^4 + ... \\
u_2 + u_4 - 2u_0 &= 2a_{0,2}h^2 + 2a_{0,4}h^4 + ... \\
u_E = u_1 &= u(x+h, y) \\
&= u_{x,y} + hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \frac{h^3}{3!}u_{xxx}(x,y) + ... \qquad (3.2.7)
\end{aligned}
$$

Again

$$
\begin{aligned}
u &= \sum_{i=0} a_{i,j} x_i y_j \\
u_x &= \sum_{i=1} i a_{i,j} x_{i-1} y_j \\
&= \sum_{i=0} (i+1) a_{i+1,j} x_i y_j \\
u_x(0,0) &= a_{1,0} \\
u_{xx} &= \sum_{i=2} i(i-1) a_{i,j} x_{i-2} y_j \\
&= \sum_{i=0} (i+2)(i+1) a_{i+2,j} x_i y_j
\end{aligned}
$$

$$u_{xx}(0,0) = 2a_{2,0}$$

Putting the values u_x and u_{xx} in equation (3.2.7), we have

$$
\begin{aligned}
u_E = u_1 &= a_{0,0} + a_{1,0}h + a_{2,0}h^2 + a_{3,0}h^3 + a_{4,0}h^4 + ... \\
u_1 - u_0 &= ha_{1,0} + h^2 a_{2,0} + ... \\
u_{NW} = u_6 &= u(x-h, y+h) \\
&= u(x-h,y) + hu_y(x-h,y) + \frac{h^2}{2!}u_{yy}(x-h,y) + \frac{h^3}{3!}u_{yyy}(x-h,y) + ... \\
&= u(x,y) - hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) - \frac{h^3}{3!}u_{xxx}(x,y) + \frac{h^4}{4!}u_{xxxx}(x,y) + .. \\
&\quad + h[u_y(x,y) - hu_{x,y}(x,y) + \frac{h^2}{2!}u_{xxy}(x,y) - \frac{h^3}{3!}u_{xxxy}(x,y) + ...] + \frac{h^2}{2!} \\
&\quad [u_{yy}(x,y) - hu_{xyy} + \frac{h^2}{2!}u_{yy}u_{xx}(x,y) + ...] \\
&= u(x,y) - h(u_x + u_y) + \frac{h^2}{2!}(u_{yy} + u_{xx} - 2u_{xy}) - \frac{h^3}{3!}(u_{xxx} - 3u_{xxy} + \\
&\quad 3u_{xyy}) + \frac{h^4}{4!}(u_{xxxx} - 4u_{xxxy} + 4u_{xx}u_{yy}) \\
&= a_{0,0} - h(a_{l,0} + a_{0,1} + h^2(a_{2,0} + a_{0,2} - a_{1,1}) - h^3(a_{3,0} - a_{2,1} + a_{1,2}) \\
&\quad + h^4(a_{4,0} - a_{3,1} + a_{2,2})
\end{aligned}
$$

Now, adding u_{NE} and u_{NW},

$$u_5 + u_6 = 2a_{0,0} + 2h^2(a_{2,0} + a_{0,2}) + 2h^3 a_{2,1} + 2h^4(a_{4,0} + a_{2,2}) + ... \qquad (3.2.8)$$

Subtracting u_{NE} from u_{NW}:

$$u_5 - u_6 = 2h(a_{1,0} + a_{0,1} + 2h^2 a_{1,1} + 2h^3(a_{3,0} + a_{1,2} + ... \tag{3.2.9}$$

$$\begin{aligned} u_W = u_3 &= u(x-h, y) \\ &= u_{x,y} - hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) - \frac{h^3}{3!}u_{xxx}(x,y) + ... \\ u_3 &= a_{0,0} - a_{1,0}h + a_{2,0}h^2 - a_{3,0}h^3 + a_{4,0}h^4 + ... \\ u_3 - u_0 &= -a_{1,0}h + a_{2,0}h^2 - a_{3,0}h^3 + a_{4,0}h^4 + ... \end{aligned}$$

Again, adding u_E and u_W:

$$u_1 + u_3 - 2u_0 = 2h^2 a_{2,0} + 2h^4 a_{4,0} + ...$$

Subtracting u_E from u_W:

$$\begin{aligned} u_1 - u_3 &= 2ha_{1,0} + 2h^3 a_{3,0} + ... \\ u_{SW} = u_7 &= u(x-h, y-h) \\ &= u(x-h,y) - hu_y(x-h,y) + \frac{h^2}{2!}u_{yy}(x-h,y) - \frac{h^3}{3!}u_{yyy}(x-h,y) + ... \\ &= u(x,y) - hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) - \frac{h^3}{3!}u_{xxx}(x,y) + \frac{h^4}{4!}u_{xxxx}(x,y) + ... - \\ &\quad h[u_y(x,y) - hu_{x,y}(x,y) + \frac{h^2}{2!}u_{xxy}(x,y) - \frac{h^3}{3!}u_{xxxy}(x,y) + ...] + \frac{h^2}{2!}[u_{yy} \\ &\quad (x,y) - hu_{xyy} + \frac{h^2}{2!}u_{yy}u_{xx}(x,y) + ...] \\ &= u(x,y) - h(u_x + u_y) + \frac{h^2}{2!}(u_{yy} + u_{xx} + 2u_{xy}) - \frac{h^3}{3!}(u_{xxx} + 3u_{xxy} + 3u_{xyy}) \\ &\quad + \frac{h^4}{4!}(u_{xxxx} + 4u_{xxxy} + 4u_{xx}u_{yy}) \\ u_7 &= a_{0,0} - h(a_{l,0} + a_{0,1} + h^2(a_{2,0} + a_{0,2} + a_{1,1}) - h^3(a_{3,0} + a_{2,1} + a_{1,2}) + h^4 \\ &\quad (a_{4,0} - a_{3,1} + a_{2,2}) \end{aligned}$$

$$\begin{aligned}
u_{SE} = u_8 &= u(x+h, y-h)\\
&= u(x+h,y) - hu_y(x+h,y) + \frac{h^2}{2!}u_{yy}(x+h,y) - \frac{h^3}{3!}u_{yyy}(x+h,y) + ...\\
&= u(x,y) + hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \frac{h^3}{3!}u_{xxx}(x,y) + \frac{h^4}{4!}u_{xxxx}(x,y) + ... -\\
&\quad h[u_y(x,y) + hu_{x,y}(x,y) + \frac{h^2}{2!}u_{xxy}(x,y) + \frac{h^3}{3!}u_{xxxy}(x,y) + ...] + \frac{h^2}{2!}[u_{yy}\\
&\quad (x,y) + hu_{xyy} + \frac{h^2}{2!}u_{yy}u_{xx}(x,y) + ...]\\
&= u(x,y) + h(u_x - u_y) + \frac{h^2}{2!}(u_{yy} + u_{xx} - 2u_{xy}) - \frac{h^3}{3!}(u_{xxx} - 3u_{xxy} + 3u_{xyy})\\
&\quad + \frac{h^4}{4!}(u_{xxxx} - 4u_{xxxy} + 4u_{xx}u_{yy})\\
u_8 &= a_{0,0} + h(a_{l,0} + a_{0,1}) + h^2(a_{2,0} + a_{0,2} - a_{1,1}) + h^3(a_{3,0} - a_{2,1} + a_{1,2}) + h^4\\
&\quad (a_{4,0} - a_{3,1} + a_{2,2})
\end{aligned}$$

Adding u_{SW} and u_{SE}:

$$u_7 + u_8 = 2a_{0,0} + 2h^2(a_{2,0} + a_{0,2}) - 2h^3a_{2,1} + 2h^4(a_{4,0} + a_{2,2}) + ... \tag{3.2.10}$$

Subtracting u_{SW} from u_{SE}:

$$u_7 - u_8 = -2h(a_{1,0} + a_{0,1} + 2h^2a_{1,1} - 2h^3(a_{3,0} + a_{1,2} + ... \tag{3.2.11}$$

Adding (3.2.9) and (3.2.11), we obtain relations of the form

$$u_5 - u_6 + u_7 - u_8 = 4h^2a_{1,1} + ...$$

Adding (3.2.8) and (3.2.10), we obtain relations of the form

$$u_5 + u_6 + u_7 + u_8 = 4a_{0,0} + 4h^2(a_{2,0} + a_{0,2}) + 4h^4(a_{4,0} + a_{2,2})$$

$$u_5 + u_6 + u_7 + u_8 - 4a_{0,0} = 4h^2(a_{2,0} + a_{0,2}) + 4h^4(a_{4,0} + a_{2,2})$$

$$u_5 + u_6 + u_7 + u_8 - 4u_0 = 4h^2(a_{2,0} + a_{0,2}) + 4h^4(a_{4,0} + a_{0,4})$$

Proceeding in the similar manner, we get

$$u_1 + u_2 + u_3 + u_4 - 4u_0 = 2h^2(a_{2,0} + a_{0,2}) + 2h^4(a_{4,0} + a_{0,4})$$

We use the notation

$$\Diamond u_0 = u_1 + u_2 + u_3 + u_4$$

$$\Box u_0 = u_5 + u_6 + u_7 + u_8$$

Now, from (3.2.5) we have,

$$\begin{aligned}
h^2 c_{0,0} &= 2h^2(a_{2,0} + a_{0,2}) + \lambda_{0,0} h(a_{1,0}h) + \mu_{0,0} h(a_{0,1}h) \\
&= 2h_2(a_{2,0} + a_{0,2}) + \lambda_{0,0} h(u_1 - u_0 - h^2 a_{2,0} - h^3 a_{3,0}) + \mu_{0,0} h(u_2 - u_0 - a_{0,1}h \\
&\quad - a_{0,2}h^2 - a_{0,3}h^3 - a_{0,4}h^4) \\
&= u_1 + u_2 + u_3 + u_4 - 4u_0 - 2h^4(a_{4,0} + a_{0,4}) + \lambda_{0,0} h(u_1 - u_0 - h^2 a_{2,0} - h^3 \\
&\quad a_{3,0}...) + \mu_{0,0} h(u_2 - u_0 - a_{0,1}h - a_{0,2}h^2 - a_{0,3}h^3 - a_{0,4}h^4...) \\
&= \Diamond u_0 - 4u_0 - 2h^4(a_{4,0} + a_{0,4}) + \lambda_{0,0} h(u_1 - u_0 - h^2 a_{2,0} - h^3 a_{3,0}...) + \mu_{0,0} h \\
&\quad (u_2 - u_0 - a_{0,1}h - a_{0,2}h^2 - a_{0,3}h^3 - a_{0,4}h^4...) + O(h^4) \qquad (3.2.12)
\end{aligned}$$

Neglecting terms of order h^3, we obtain the upwind difference scheme of order h.

$$h^2 c_{0,0} = \Diamond u_0 - 4u_0 + \lambda_{0,0} h(u_1 - u_0) + \mu_{0,0} h(u_2 - u_0) \qquad (3.2.13)$$

If the terms containing h^3 are eliminated in (3.2.12), we obtain

$$\begin{aligned}
h^2 c_{0,0} &= \Diamond u_0 - 4u_0 + \lambda_{0,0} h(u_1 - u_0) + \mu_{0,0} h(u_2 - u_0) - \lambda_{0,0} h(h^2 a_{2,0}) \\
&\quad - \mu_{0,0} h(h^2 a_{0,2}) \\
&= \Diamond u_0 - 4u_0 + \lambda_{0,0} h(u_1 - u_0) + \mu_{0,0} h(u_2 - u_0) - \frac{\lambda_{0,0} h}{2}(u_1 + \\
&\quad u_3 - 2u_0) - \frac{\mu_{0,0} h}{2}(u_2 + u_4 - 2u_0) \\
&= \Diamond u_0 - 4u_0 + \frac{\lambda_{0,0} h}{2}(u_1 - u_3) + \frac{\mu_{0,0} h}{2}(u_2 - u_4) + O(h^4) \qquad (3.2.14)
\end{aligned}$$

The central difference scheme results when $O(h^4)$ terms are neglected in (3.2.14). This result cannot be improved any further without using more constraints from (3.2.4), (3.2.5). The two constraints involving $c_{1,0}$, $c_{0,1}$ contain $a_{3,0}$, $a_{0,3}$ which also appear in the next set of constraints for $c_{2,0}$, $c_{0,2}$. Using these constraints and relations in (3.2.6), (3.2.7), we obtain,

$$
\begin{aligned}
&6h^2c_{0,0}+\frac{h^4}{2}(c_{1,0}\lambda_{0,0}+c_{0,1}\mu_{0,0})+h^4(c_{2,0}+c_{0,2})\\
&\quad=6[\Diamond u_0-4u_0+\frac{\lambda_{0,0}h}{2}(u_1-u_3)+\frac{\mu_{0,0}h}{2}(u_2-u_4)]+\frac{h^4}{2}[(6a_{3,0}+2a_{1,2}\\
&\qquad+2\lambda_{0,0}a_{2,0}+\lambda_{1,0}a_{1,0}+\mu_{0,0}a_{1,1}+\mu_{1,0}a_{0,1})\lambda_{0,0}+(2a_{2,1}+6a_{0,3}+\lambda_{0,0}\\
&\qquad a_{1,1}+\lambda_{0,1}a_{1,0}+2\mu_{0,0}a_{0,2}+\mu_{0,1}a_{0,1})\mu_{0,0}]+h^4[(12a_{4,0}+2a_{22}+3\lambda_{0,0}\\
&\qquad a_{3,0}+2\lambda_{1,0}a_{2,0}+\lambda_{2,0}a_{1,0}+\mu_{0,0}a_{2,1}+\mu_{1,0}a_{2,1}+\mu_{1,0}a_{1,1}+\mu_{2,0}a_{0,1})+\\
&\qquad(2a_{2,2}+12a_{0,4}+\lambda_{0,0}a_{1,2}+\lambda_{0,1}a_{1,1}+\lambda_{0,2}a_{1,0}+3\mu_{0,0}a_{0,3}+2\mu_{0,1}a_{0,2}\\
&\qquad+\mu_{0,2}a_{0,1})]\\
&\quad=6\Diamond u_0-24u_0+3\lambda_{0,0}h(u_1-u_3)+3\mu_{0,0}h(u_2-u_4)+2h^4(\lambda_{0,0}a_{1,2}+\mu_{0,0}\\
&\qquad a_{2,1})+a_{2,0}h^4(\lambda_{0,0}^2+2\lambda_{1,0})+a_{0,2}h^4(\mu_{0,0}^2+2\mu_{0,1})+h^4a_{1,1}(\lambda_{0,0}\mu_{0,0}+\\
&\qquad\lambda_{0,1}+\mu_{1,0})+h^4a_{1,0}(\frac{\lambda_{1,0}\lambda_{0,0}}{2}+\frac{\mu_{0,0}\lambda_{0,1}}{2}+\lambda_{0,2}+\lambda_{2,0})+h^4(\mu_{2,0}+\mu_{0,2}\\
&\qquad+\frac{\lambda_{0,0}\mu_{1,0}}{2}+\frac{\mu_{0,0}\mu_{0,1}}{2})+4h^4a_{2,2}+O(h^6)
\end{aligned}
$$

$$
\begin{aligned}
&6h^2c_{0,0} + \frac{h^4}{2}(c_{1,0}\lambda_{0,0} + c_{0,1}\mu_{0,0}) + h^4(c_{2,0} + c_{0,2})\\
&\quad =6\Diamond u_0 - 24u_0 + 3\lambda_{0,0}h(u_1 - u_3) + 3\mu_{0,0}h(u_2 - u_4) + \frac{h\lambda_{0,0}}{2}(u_5 - u_6\\
&\qquad - u_7 + u_8) + \frac{h\mu_{0,0}}{2}(u_5 + u_6 - u_7 - u_8) - h[\lambda_{0,0}(u_1 - u_3)) + \mu_{0,0}(u_2 -\\
&\qquad u_4)] + \frac{h^2}{2}(\lambda_{0,0}^2 + 2\lambda_{1,0})(u_1 + u_3) + \frac{h^2}{2}(\mu_{0,0}^2 + 2\mu_{0,1})(u_2 + u_4) - h^2\\
&\qquad (\lambda_{0,0}^2 + \mu_{0,0}^2 + 2\lambda_{1,0} + 2\mu_{0,1})u_0 + \frac{h^2}{4}(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0})(u_5 - u_6\\
&\qquad + u_7 - u_8) + \frac{h^3}{2}(\mu_{2,0} + \mu_{0,2} + \frac{\lambda_{0,0}\mu_{1,0}}{2} + \frac{\mu_{0,0}\mu_{0,1}}{2}(u_1 - u_3) + \frac{h^3}{2}(\\
&\qquad \mu_{2,0} + \mu_{0,2} + \frac{\lambda_{0,0}\mu_{1,0}}{2} + \frac{\mu_{0,0}\mu_{0,1}}{2}(u_2 - u_4 + (u_5 + u_6 + u_7 + u_8\\
&\qquad - 4u_0) - 2(u_1 + u_2 + u_3 + u_4 - 4u_0)\\
&\quad =4\Diamond u_0 + \Box u_0 - 20u_0 + 2h\lambda_{0,0}(u_1 - u_3) + 2h\mu_{0,0}(u_2 - u_4) + \frac{h\lambda_{0,0}}{2}((u_5\\
&\qquad - u_6 - u_7 + u_8) + \frac{h\mu_{0,0}}{2}(u_5 + u_6 - u_7 - u_8) + \frac{h^2}{2}(\lambda_{0,0}^2 + 2\lambda_{1,0})(u_1 +\\
&\qquad u_3) + \frac{h^2}{2}(\mu_{0,0}^2 + 2\mu_{0,1})(u_2 + u_4) - h^2(\lambda_{0,0}^2 + \mu_{0,0}^2 + 2\lambda_{1,0} + 2\mu_{0,1})u_0 +\\
&\qquad \frac{h^2}{4}(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0})(u_5 - u_6 + u_7 - u_8) + \frac{h^3}{2}(\lambda_{2,0} + \lambda_{0,2} + \frac{\lambda_{0,0}\lambda_{1,0}}{2}\\
&\qquad + \frac{\mu_{0,0}\lambda_{0,1}}{2}(u_1 - u_3) + \frac{h^3}{2}(\mu_{2,0} + \mu_{0,2} + \frac{\lambda_{0,0}\mu_{1,0}}{2} + \frac{\mu_{0,0}\mu_{0,1}}{2}(u_2 - u_4)\\
&\qquad + O(h^6)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=o}^{8}\alpha_k u_k =& u_1[4 + 2h\lambda_{0,0} + \frac{h^2}{2}(\lambda_{0,0}^2 + 2\lambda_{1,0}) + \frac{h^3}{2}(\lambda_{2,0} + \lambda_{0,2} + \frac{1}{2}\lambda_{0,0}\lambda_{1,0} + \frac{1}{2}\mu_{0,0}\lambda_{0,1})]\\
&+ u_2[4 + 2h\mu_{0,0} + \frac{h^2}{2}(\mu_{0,0}^2 + 2\mu_{0,1}) + \frac{h^3}{2}(\mu_{2,0} + \mu_{0,2} + \frac{1}{2}\lambda_{0,0}\mu_{1,0} + \frac{1}{2}\mu_{0,0}\mu_{0,1})]\\
&+ u_3[4 - 2h\lambda_{0,0} + \frac{h^2}{2}(\lambda_{0,0}^2 + 2\lambda_{1,0}) - \frac{h^3}{2}(\lambda_{2,0} + \lambda_{0,2} + \frac{1}{2}\lambda_{0,0}\lambda_{1,0} + \frac{1}{2}\mu_{0,0}\lambda_{0,1})]\\
&+ u_4[4 - 2h\mu_{0,0} + \frac{h^2}{2}(\mu_{0,0}^2 + 2\mu_{0,1}) - \frac{h^3}{2}(\mu_{2,0} + \mu_{0,2} + \frac{1}{2}\lambda_{0,0}\mu_{1,0} + \frac{1}{2}\mu_{0,0}\mu_{0,1})]\\
&+ u_5[1 + \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) + \frac{h^2}{4}(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0})] + u_6[1 - \frac{h}{2}(\lambda_{0,0} - \mu_{0,0})\\
&- \frac{h^2}{4}(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0})] + u_7[[1 - \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) + \frac{h^2}{4}(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} +\\
&\mu_{1,0})] + u_8[1 - \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) - \frac{h^2}{4}(\lambda_{0,0}\mu_{0,0} + \lambda_{0,1} + \mu_{1,0})] + u_0[-20 - h^2(\lambda_{0,0}^2\\
&+ \mu_{0,0}^2 + 2\lambda_{1,0} + 2\mu_{0,1})]
\end{aligned}
$$

Neglecting the terms of order h^6, we get the fourth order difference scheme given by equation (3.2.14).

The fourth order difference approximation of equation (3.1.1) is given by

$$\sum_{k=o}^{8} \alpha_k u_k = 6c_{0,0}h^2 + (c_{2,0} + c_{0,2})h^4 + (\lambda_{0,0}c_{1,0} + \mu_{0,0}c_{O,1})\frac{h^4}{2}, \tag{3.2.15}$$

where

$$\begin{aligned}
\alpha_1 \equiv \alpha_E =& 4 + 2h\lambda_{0,0} + h^2R_3 + \frac{h^3}{2}R_6 \\
\alpha_2 \equiv \alpha_N =& 4 + 2h\mu_{0,0} + h^2R_4 + \frac{h^3}{2}R_5 \\
\alpha_3 \equiv \alpha_W =& 4 - 2h\lambda_{0,0} + h^2R_3 - \frac{h^3}{2}R_6 \\
\alpha_4 \equiv \alpha_S =& 4 - 2h\mu_{0,0} + h^2R_4 - \frac{h^3}{2}R_5 \\
\alpha_5 \equiv \alpha_{NE} =& 1 + \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) + \frac{h^2}{4}R_2 \\
\alpha_6 \equiv \alpha_{NW} =& 1 - \frac{h}{2}(\lambda_{0,0} - \mu_{0,0}) - \frac{h^2}{4}R_2 \\
\alpha_7 \equiv \alpha_{SW} =& 1 - \frac{h}{2}(\lambda_{0,0} + \mu_{0,0}) + \frac{h^2}{4}R_2 \\
\alpha_8 \equiv \alpha_{SE} =& 1 + \frac{h}{2}(\lambda_{0,0} - \mu_{0,0}) - \frac{h^2}{4}R_2 \\
\alpha_0 \equiv& -20 - h^2R_1.
\end{aligned} \tag{3.2.16}$$

Here, the following notations have been used:

$$\begin{aligned}
R_1 =& \lambda_{0,0}^2 + \mu_{0,0}^2 + 2\lambda_{1,0} + 2\mu_{0,1} \\
R_2 =& \mu_{1,0} + \lambda_{0,1} + \lambda_{0,0}\mu_{0,0} \\
R_3 =& \frac{1}{2}\lambda_{0,0}^2 + \lambda_{1,0} \\
R_4 =& \frac{1}{2}\mu_{0,0}^2 + \mu_{0,1} \\
R_5 =& \mu_{2,0} + \mu_{0,2} + \frac{1}{2}\lambda_{0,0}\mu_{1,0} + \frac{1}{2}\mu_{0,0}\mu_{0,1} \\
R_6 =& \lambda_{2,0} + \lambda_{0,2} + \frac{1}{2}\lambda_{0,0}\lambda_{1,0} + \frac{1}{2}\mu_{0,0}\lambda_{0,1}.
\end{aligned} \tag{3.2.17}$$

Replacing $\lambda_{i,j}$, $\mu_{i,j}$ and $c_{i,j}$ in terms of the nodal values of the functions $p(x,y)$, $q(x,y)$ and $f(x,y)$, we get an alternative finite difference scheme given by

$$\begin{aligned}\sum_{j=0}^{8}\alpha_j u_j =& 6f_0h^2 + h^4\frac{(f_E+f_W-2f_0)}{2h^2} + \frac{f_N+f_S-2f_0}{2h^2} + \frac{h^4}{2}[\frac{p_0(f_E-f_W)}{2h} \\ &+ \frac{q_0(f_N-f_S)}{2h}] \\ =&\frac{h^2}{2}[f_N+f_S+f_E+f_W+8f_0] + \frac{h^3}{4}[p_0(f_E-f_W)+q_0(f_N-f_S)]\end{aligned} \tag{3.2.18}$$

where

$$\begin{aligned}\alpha_1 \equiv \alpha_E =& 4+2hp_0+\frac{h^2}{2}[p_0^2+2\frac{(p_E-p_W)}{2h}]+\frac{h^3}{2}[\frac{(p_E+p_W-2p_0)}{2h^2}+\frac{p_N+p_S-2p_0}{2h^2} \\ &+\frac{p_0}{2}\frac{(p_E-p_W)}{2h}+\frac{q_0}{2}\frac{(p_N-p_S)}{2h}] \\ =&4+2hp_0+\frac{h^2}{2}p_0^2+\frac{h}{2}(p_E-p_W)+\frac{h}{2}(p_E+p_W-4p_0+p_N+p_S)+\frac{h^2}{8} \\ &p_0(p_E-p_W)+\frac{h^2}{8}q_0(f_N-f_S) \\ =&4+\frac{h}{4}[4p_0+3p_E-p_W+p_N+p_S]+\frac{h^2}{8}[4p_0^2+p_0(p_E-p_W)+q_0(p_N-p_S) \\ \alpha_2 \equiv \alpha_N =& 4+2hq_0+\frac{h^2}{2}[q_0^2+2\frac{(q_N-q_S)}{2h}]+\frac{h^3}{2}[\frac{(q_E+q_W-2q_0)}{2h^2}+\frac{q_N+q_S-2q_0}{2h^2} \\ &+\frac{q_0}{2}\frac{(q_E-q_W)}{2h}+\frac{q_0}{2}\frac{(q_N-q_S)}{2h}] \\ =&4+2hq_0+\frac{h^2}{2}q_0^2+\frac{h}{2}(q_N-q_S)+\frac{h}{4}(q_E+q_W-4q_0+q_N+q_S) \\ &+\frac{h^2}{8}p_0(q_E-q_W)+\frac{h^2}{8}q_0(q_N-q_S) \\ =&4+\frac{h}{4}[4q_0+3q_N-q_S+q_E+q_W]+\frac{h^2}{8}[4q_0^2+p_0(q_E-q_W)+q_0(q_N-q_S)\end{aligned}$$

$$
\begin{aligned}
\alpha_3 \equiv \alpha_W =& 4 - 2hp_0 + \frac{h^2}{2}[p_0^2 + 2\frac{(p_E - p_W)}{2h}] - \frac{h^3}{2}[\frac{(p_E + p_W - 2p_0)}{2h^2} + \frac{p_N + p_S - 2p_0}{2h^2} \\
& + \frac{p_0}{2}\frac{(p_E - p_W)}{2h} + \frac{q_0}{2}\frac{(p_N - p_S)}{2h}] \\
=& 4 - 2hp_0 + \frac{h^2}{2}p_0^2 + \frac{h}{2}(p_E - p_W) - \frac{h}{2}(p_E + p_W - 4p_0 + p_N + p_S) - \frac{h^2}{8}p_0 \\
& (p_E - p_W) - \frac{h^2}{8}q_0(p_N - p_S) \\
=& 4 - \frac{h}{4}[4p_0 - p_E + 3p_W + p_N + p_S] + \frac{h^2}{8}[4p_0^2 + p_0(p_E - p_W) - q_0(p_N - p_S) \\
\alpha_4 \equiv \alpha_S =& 4 - 2hq_0 + \frac{h^2}{2}[q_0^2 + 2\frac{(q_N - q_S)}{2h}] - \frac{h^3}{2}[\frac{(q_E + q_W - 2q_0)}{2h^2} + \frac{q_N + q_S - 2q_0}{2h^2} \\
& + \frac{q_0}{2}\frac{(q_E - q_W)}{2h} + \frac{q_0}{2}\frac{(q_N - q_S)}{2h}] \\
=& 4 - 2hq_0 + \frac{h^2}{2}q_0^2 + \frac{h}{2}(q_N - q_S) - \frac{h}{4}(q_E + q_W - 4q_0 + q_N + q_S) - \frac{h^2}{8}p_0(q_E \\
& - q_W) - \frac{h^2}{8}q_0(q_N - q_S) \\
=& 4 - \frac{h}{4}[4q_0 - q_N + 3q_S + q_E + q_W] + \frac{h^2}{8}[4q_0^2 + p_0(q_E - q_W) - q_0(q_N - q_S)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_5 \equiv \alpha_{NE} =& 1 + \frac{h}{2}p_0 + \frac{h}{2}q_0 + \frac{h^2}{4}[p_0q_0 + \frac{(p_N - p_S}{2h} + \frac{q_E - q_W}{2h}\frac{p_0}{2}\frac{(p_E - p_W)}{2h} + \frac{q_0}{2} \\
& \frac{(p_N - p_S)}{2h}] \\
=& 1 + \frac{h}{2}(p_0 + q_0) + \frac{h}{8}(p_N - p_S + q_E - q_W) - \frac{h^2}{4}(p_0q_0) \\
\alpha_6 \equiv \alpha_{NW} =& 1 - \frac{h}{2}p_0 + \frac{h}{2}q_0 - \frac{h^2}{4}[p_0q_0 + \frac{(p_N - p_S}{2h} + \frac{q_E - q_W}{2h}\frac{p_0}{2}\frac{(p_E - p_W)}{2h} + \frac{q_0}{2} \\
& \frac{(p_N - p_S)}{2h}] \\
=& 1 - \frac{h}{2}(p_0 - q_0) - \frac{h}{8}(p_N - p_S + q_E - q_W) - \frac{h^2}{4}(p_0q_0) \\
\alpha_7 \equiv \alpha_{SW} =& 1 - \frac{h}{2}p_0 - \frac{h}{2}q_0 + \frac{h}{8}(p_N - p_S + q_E - q_W) + \frac{h^2}{4}p_0q_0 \\
\alpha_8 \equiv \alpha_{SE} =& 1 + \frac{h}{2}p_0 - \frac{h}{2}q_0 - \frac{h}{8}(p_N - p_S + q_E - q_W) + \frac{h^2}{4}p_0q_0 \\
\alpha_0 =& -[20 + h^2[p_0^2 + q_0^2 + 2\frac{2(p_E - p_W)}{2h} + 2(\frac{q_N - q_S}{2h}] \\
=& -[20 + h^2[(p_0^2 + q_0^2) + h(p_E - p_W) + hq_N - q_S]
\end{aligned}
$$

Note that both the difference schemes (3.2.14) and (3.2.17) reduce to the scheme given in [12] when $p(x,y)$ and $q(x,y)$ are constants.

3.3 Numerical Results

Numerical results for several test problems have been obtained by using the two $9-point$ schemes given by (3.2.14) and (3.2.17). The scheme (3.2.14) involves the derivatives of the functions $p(x,y)$, $q(x,y)$ and $f(x,y)$ whereas the scheme (3.2.17) involves only the nodal values of these functions. The results obtained by these two schemes do not differ significantly as far as the order of the maximum errors are concerned. Hence the results obtained by using the scheme (3.2.17), which is more useful in practice, are reported here. This scheme is called the single cell high order scheme or $SCHOS$ in the sequel. All test problems given here are solved on a unit square $[0,1]\times[0,1]$ using a uniform mesh h. Boundary values of the solutions are assumed to be known. The system of linear equations is solved by using the successive over-relaxation (SOR) iterative method. Sometimes it is necessary to use a relaxation parameter less than 1 with the CDS. The convergence criterion for the iteration was chosen to be 10^{-6}.

3.3.1 Test problem 1

Consider the boundary value problem

$$\begin{aligned} &-\epsilon(u_{xx}+u_{yy}) + u_x = 0, && && 0 \le x,y \le 1 \\ &u(x,0) = 0, && u(x,1) = 0, && 0 \le x \le 1 \\ &u(0,y) = sin\pi y, && u(1,y) = 2sin\pi y, && 0 \le y \le 1 \end{aligned} \tag{3.3.1}$$

Comparison of (3.3.1) and (3.1.1) shows that $-p(x,y) = \frac{1}{\epsilon} = P$ (say), $q(x,y) = 0$ and $f(x,y) = 0$. The exact solution

$$u = e^{\frac{Px}{2}} sin\pi y \frac{[2e^{\frac{-P}{2}} sinh\sigma x + sinh\sigma(1-x)]}{sinh\sigma}$$

where $\sigma^2 = \pi^2 + \frac{P^2}{4}$, shows the presence of a boundary layer near $x = 1$ whose thickness is of order $\frac{1}{P}$ for large P. The boundary layer is expected to affect the numerical results adversely as P increases.

This problem has been studied by Gartland [10], and a special five point stencil involving modified Bessel functions was proposed. In Tables $1-3$, some results have been presented for this problem.

Table 3.1: Maximum relative errors for problem 1, $h = \frac{1}{32}$:

P	UDS	CDS	$SCHOS$
10	0.9166(-1)	0.4537(-2)	0.6011(-4)
20	0.1262	0.1576(-1)	0.1399(-3)
40	0.1686	0.5925(-1)	0.1511(-2)
100	0.2264	0.3002	0.3517(-1)

In $Table 1$, the maximum relative errors for $P = 10, 20, 40$ and 100 for $h = 1/32$ show that the errors due to UDS range between 9 and 23 percent, whereas the CDS gives acceptable results for $P \leq 40$ but not for $P = 100$. The errors for $SCHOS$ remain ≤ 4 percent. As expected, the maximum relative errors occur near the corners at $x = 1$ in all cases.

In $Table 2$, the maximum absolute errors for different values of h are given. Numerical estimates for the order of the discretization errors which are obtained by considering the errors due to mesh sizes h and $2h$ are also given. It is clear from the table that the order of $SCHOS$ is about twice that of CDS.

In order to compare the obtained results with those given by Gartland, the average relative

Table 3.2: Maximum absolute errors and the estimated orders for problem 1 :

P	h^{-1}	CDS	$Order$	$SCHOS$	$Order$
40	8	0.5122		0.1256	
	16	1.5	0.2009(-1)	0.1399(-3)	2.65
	32	0.6723(-1)	1.78	0.1712(-2	3.55
100	8	0.9060		0.4249	
	16	0.5618	0.68	0.1670	1.35
	32	0.2872	0.97	0.3365(-1)	2.31
	64	0.9493(-1)	1.60	0.3151(-2)	3.41

Table 3.3: Average relative errors for *problem*1, $P = 100$:

h^{-1}	*CDS*	*SCHOS*	*Gartland*
8	1.18(1)	8.13(-2)	6.81(-3)
16	1.46(-1)	1.25(-2)	4.94(-3)
32	1.61(-2)	1.26(-3)	1.90(-3)
64	2.23(-3)	1.79(-4)	5.71(-4)

errors for $P = 100$ in $Table3$ has been presented. Clearly, the special method proposed by Gartland gives better results when the mesh is crude. However, as the mesh is refined, the errors due to $SCHOS$ decrease rapidly and for $h \leq 1/32$ the errors due to $SCHOS$ are consistently smaller. The order of the Gartland method is not even as high as that of the CDS.

3.3.2 Test problem 2

Consider the boundary value problem

$$\begin{aligned} \phi_{xx} + \phi_{yy} =& P\cos\theta\phi_x + P\sin\theta\phi_y, && 0 \leq x, y \leq 1 \\ \phi(x,0) = 0&, \phi(x,1) = 0, && 0 \leq x \leq 1 \\ \phi(0,y) =& 4y(1-y), \phi(1,y) = 0, && 0 \leq y \leq 1 \end{aligned} \tag{3.3.2}$$

Comparison of (3.1.1) and (3.3.2) shows that $p(x,y)= -P\cos\theta$ and $q(x,y)$ = -$P\sin\theta$ and $f(x,y) = 0$. The exact solution is given by

$$\phi = e^{\frac{-P(x\cos\theta + y\sin\theta)}{2}} \sum_{n=1}^{\infty} B_n \sin h[\sigma_n(1-x)] \sin n\pi y$$

where

$$\sigma_n^2 = n^2\pi^2 + \frac{P^2}{4}$$

Table 3.4: Maximum absolute errors and the estimated orders for *problem*1:

	h^{-1}	UDS	$Order$	CDS	$Order$	$SCHOS$	$Order$
$\theta = 0$	8	0.1227		0.3420		0.8280(-1)	
	16	0.1604		0.1532	1.15	0.1323(-1)	2.65
	32	0.1256	0.35	0.4449(-1)	1.78	0.1123(-2)	3.55
$\theta = \frac{\pi}{8}$	8	0.2886		0.3688		0.6931(-1)	
	16	0.2268	0.34	0.1286	1.52	0.1019(-1)	2.77
	32	0.1394	0.70	0.3484(-1)	1.88	0.8128(-3)	3.64
$\theta = \frac{\pi}{4}$	8	0.2467		0.2833		0.4932(-1)	
	16	0.2035	0.28	0.8031(-1)	1.82	0.5977(-2)	3.04
	32	0.1218	0.74	0.1950(-1)	2.04	0.4066(-3)	3.88

and

$$B_n = \frac{8}{\sin h\sigma_n} \int_0^1 y(1-y)e^{-\frac{P \sin\theta y}{2}} \sin n\pi y dy$$

This problem represents the convection of θ (temperature or concentration) in a fluid moving with a uniform velocity at an angle θ towards the x-axis. For $\theta = 0$ a boundary layer develops on $x = 1$ as in the case of problem 1, whereas for $\theta \neq 0$, boundary layers develop on $x = 1$ and also on $y = 1$ for P large. Numerical results obtained by using UDS are known to be affected adversely when the direction of the flow is not aligned with the direction of the finite difference grid. This is known as the grid orientation problem. This problem has been chosen to study whether the numerical solutions obtained by $SCHOS$ are affected by the grid orientation. This problem has been studied by Stubley et. al [28] in which they proposed a special method called QIS which uses a $9 - point$ stencil.

In Table 4, the absolute maximum errors for $P = 40$, $h = 1/8, 1/16$ and $1/32$ for $\theta = 0$, $\pi/8$ and $\pi/4$ have been presented. Estimates of the order of the method obtained from the numerical results are also given. Most of the observations made in the case of problem 1 remain valid for this problem as well. It is clearly seen that the results of UDS deteriorate as θ increases, whereas this is not the case for both the CDS and the $SCHOS$. The results for $\theta = \pi/4$ appear to be better than those for $\theta = 0$, however, this is due to a decrease in the effective transport number for $\theta = \pi/4$. In any case the $SCHOS$ is not

Table 3.5: Maximum absolute errors and the estimated orders for *problem*3, equation (3.3.3) :

P	h^{-1}	UDS	$Order$	CDS	$Order$	$SCHOS$	$Order$
100	8	0.1678		0.6174(-2)		0.3081(-2)	
	16	0.9894(-1)	0.76	0.1633(-2)	1.92	0.2615(-3)	3.56
	32	0.5426(-1)	0.86	0.4154(-3)	1.97	0.1775(-4)	3.88
1000	8	0.2017		-		0.9448(-2)	
	16	0.1238	0.70	-		0.1845(-2)	2.36
	32	0.6819(-1)	0.86	-		0.1864(-3)	3.31

affected by the grid orientation. As in the case of problem 1, the results obtained by using special methods such as QIS for this problem are more accurate than those obtained by $SCHOS$ when the mesh is crude. However, as the mesh is refined, the results obtained by $SCHOS$ improve rapidly. The maximum error over a coarse 7×7 mesh ($h = 1/8$) for $P = 80$ are given in [28]. For $h = 1/32$ the results given by $SCHOS$ and QIS are comparable, whereas for $h = 1/8, 1/16$, QIS results are better.

3.3.3 Test problem 3

In problems 1 and 2 the coefficient functions were constants. In this problem, the coefficients are considered to be variables; thus in (3.1.1) let

$$p(x, y) = Px, \qquad q(x, y) = -Py$$

with the exact solution

$$u = xy(1 - x)(l - y)e^{x+y} \tag{3.3.3}$$

Numerical solutions for$P = 100$ and 1000are given in Table 5. For $P = 1000$, the CDS failed to converge with $S.O.R.$ Instead of considering $p(x, y)$ and $q(x, y)$ as first degree polynomials, we considered another problem with the exact solution same as in equation

(3.3.3) but $p(x, y) = exp(x + y)$ and $q(x, y) = l/exp(x + y)$. Here, again the results were similar to those given for (3.3.3). The CDS did not converge for $P > 100$.

3.4 Conclusion

Several test problems have been solved using the single cell high order method proposed here. From the numerical experiments it appears that the scheme gives good results. Further testing over a wider range of the values of the parameters is necessary before its usefulness is established. The scheme is simple, easy to implement and the resulting system of linear equations can be solved by iterative methods. The rate of convergence of the proposed scheme is twice as that of the central difference scheme and about 3 to 4 times that of the upwind difference scheme. The proposed scheme is not affected by the grid orientation nor does it introduce artificial viscosity. The method of derivation carried over to irregular meshes with some modifications.

The difference scheme has been arrived at by approximating the solution locally by means of polynomials. Therefore, the accuracy of the numerical solutions will be affected for mesh sizes for which such an approximation is not sufficiently accurate. In particular, when the convection is large as compared with the diffusion and a boundary layer exists in which the solution varies exponentially, one could expect a deterioration of the numerical results if a crude mesh is used. This can be seen in problems 1 and 2 for the values of $Ph > 6$.

In the derivation of the scheme, it is assumed that the solution of the problem is sufficiently smooth. If the smoothness condition is not satisfied, the order of the scheme drops and it may give results no better than the lower order schemes.

The proposed scheme uses a 9-point stencil and hence requires additional computations as compared to the 5-point schemes. In order to compare methods for a given order of the error it is necessary to obtain work estimates in terms of arithmetic operations. Of course, the rate of convergence of the iterative method also plays a decisive role in such estimates.

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