APPROXIMATION OF FUNCTIONS

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Certificate

I hereby certify that the work which is being presented in this thesis entitled **"APPROXIMATION OF FUNCTIONS"** in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics and Computing to the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala is a record of my own work studied under the supervision of Dr. Meenu Rani.
The matter embodied in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

In this report, we review the different types of continuity of functions and relation between them. We also describe the convergence of sequence and series and various definitions related to approximation theory, which are important for the convergence point of view. We discuss the basic theme of approximating functions by polynomial functions. Then, we study some properties and convergence behaviour of the Bernstein polynomials. The Bèzier curve and its properties have also been introduced. Lastly, we obtain the Bèzier variant of the certain Genuine hybrid operators and discuss their approximation behaviour.

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Navdeep kaur

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Introduction

To study the approximation of functions, we recall some basic definitions i.e. continuity, convergence etc.

0.1 Continuity

Pointwise continuity

Definition 1. A function f is said to be pointwise continuous at a point a in its domain if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever x is in the domain of f and $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

Continuity

Definition 2. A function f is said to be continuous on S iff $\forall\ a \in S\ \forall\ \epsilon > 0$ there is $\delta > 0$ for all $x \in S$ and $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

Uniformly continuity

Definition 3. A function f is said to be uniformly continuous on S iff for all $\epsilon > 0$ there is $\delta > 0$ and for all $a \in S$ and for all $x \in S$, we have $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

Note:

- It is obvious that uniform continuous function is always continuous.
- A continuous function on a closed and bounded set is uniformly continuous.

Example 1. The function $f(x) = x^3$ is continuous but not uniformly continuous on $S = (0, \infty)$.

Example 2. The function $f(x) = \dfrac{1}{x}$ is continuous but not uniformly continuous in an open interval $(0, 1)$.

Modulus of continuity

Definition 4. If f is uniformly continuous on a closed and bounded interval $I = [a, b]$, then for any $\delta > 0$, the modulus of continuity $\omega(f, \delta)$ as:

$$\omega(f, \delta) = \sup_{\substack{x,y\in I \\ |x-y|<\delta}} \{|f(x) - f(y)|\}.$$

Properties:

- The set of all co-ordinates (x, y) which satisfy the condition $|x - y| < \delta_2$ is more comprehending than the set of co-ordinates for which $|x - y| < \delta_1$. Then we obtain
$$\omega(f, \delta_1) \leqq \omega(f, \delta_2).$$
- For a function f to be uniformly continuous on the intercept $\langle a, b\rangle$ is necessary and sufficient that
$$\lim_{\delta\to 0} \omega(f, \delta) = 0.$$
- For the natural number n, we have
$$\omega(f, n\delta) \leqq n\omega(f, \delta).$$
- For each positive value of λ we have
$$\omega(f, \lambda\delta) \leqq (\lambda + 1)\omega(f, \delta).$$

Absolute continuity

Definition 5. Let $[a, b] \subseteq \mathbb{R}$. A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous (AC) on $[a, b]$ if for every $\epsilon > 0$ there is $\delta > 0$ such that whenever a finite sequence of collection of disjoint sub-intervals (x_k, y_k) of $[a, b]$ with $x_k, y_k \in [a, b]$ satisfies

$$\sum_k (y_k - x_k) < \delta$$

then

$$\sum_k (f(y_k) - f(x_k)) < \epsilon.$$

Properties:

- Let $f(x)$ be absolutely continuous function on closed interval $[a,b]$. Then $f(x)$ is of bounded variation on $[a,b]$. Consequently, $f'(x)$ exists for almost every $x \in [a,b]$.
- If $f(x)$ is absolutely continuous on $[a,b]$ and $f'(x) = 0$ for almost every $x \in [a,b]$, then $f(x)$ is constant.

Lipschitz continuity

Definition 6. A function $f(x)$ from $S \subset \mathbb{R}$ into $\mathbb{R}$ is Lipschitz continuous at $x \in S$ if there is a constant M such that

$$|f(y) - f(x)| \leqq M\,|y - x|$$

for all $y \in S$ sufficiently near x. Thus we write

$$f(x) \in Lip_M 1.$$

Properties:

- If the derivative $f'(x)$ with $|f'(x)| \leqq M$ exists everywhere on the interval (a,b), then
$$f(x) \in Lip_M 1.$$
- The two statements
$$f(x) \in Lip_M \alpha \quad \text{and} \quad \omega(f,\delta) \leqq M\delta^{\alpha}$$
are fully equivalent.
- A Lipschitz function need not be differentiable.

Bounded variation

Definition 7. The total variation of a real-valued (or more generally complex-valued) function $f : [a,b] \to \mathbb{R}$ is given by

$$\bigvee_a^b (f) = \sup_{Q \in P} \sum_{i=0}^{nQ-1} \left| f(x_{i+1}) - f(x_i) \right|,$$

where supremum is taken over the set

$P = \{Q = \{x_0, ..., x_{nQ}\} \,|Q \text{ is partition of } [a,b] \text{ satisfying } x_i \leq x_{i+1} \text{ for } 0 \leq i \leq nQ - 1\}$

of all partitions of the interval considered.

A real-valued function f is said to be of bounded variation (BV function) on $[a,b] \subset \mathbb{R}$ if

$$f \in BV(a,b) \Leftrightarrow \bigvee_a^b (f) < +\infty.$$

Properties:

- $AC[a,b] \subset C[a,b] \cap BV[a,b]$.
- If $f : [a,b] \to \mathbb{R}$, then $V[f;a,b] \geq |f(b) - f(a)|$.
- A monotonic function defined on a closed interval $[a,b]$ is a function of bounded variation.
- An integral is a function of bounded variation.
- A continuous function may or may not be a function of bounded variation.
- A function of bounded variation is necessarily bounded.
- If f' exists and is bounded on $[a,b]$, then f is of bounded variation on $[a,b]$.

Example 3. Define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Here function f is continuous, but has unbounded variation on $[-1,1]$.

0.2 Convergence

Pointwise convergence of a sequence:

Definition 8. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function f i.e. $\lim\limits_{n\to\infty} f_n = f$ pointwise iff $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for every x in the domain.

Uniform convergence of sequence:

Definition 9. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to be uniformly converges to f on a set E if for each $\epsilon > 0$, an integer $\mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for $n \geq \mathbb{N}$ and all $x \in E$.

Pointwise convergence of series:

Definition 10. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions with common domain X. The corresponding series $\sum_{n=1}^{\infty} f_n(x)$ is said to be pointwise convergent to the sum function $f(x)$ if the corresponding sequence of partial sums $(S_n)_{n=1}^{\infty}$ $\left(\text{where } S_n(x) = \sum_{k=1}^{n} f_n(x)\right)$ is pointwise convergent to $f(x)$.

Uniform convergence of a series:

Definition 11. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions with common domain X. The corresponding series $\sum_{n=1}^{\infty} f_n(x)$ is said to be uniformly Convergent to the sum function $f(x)$ if the corresponding sequence of partial sums is uniformly convergent to $f(x)$.

Example 4. $f_n(x) = x^n$ on the interval $[0, 1]$. Here, for $0 \leq x < 1$, we have $x^n \to 0$. On the other hand, for $x = 1$, $x^n \to 1$. So here is a case of a sequence of continuous functions converging pointwise but not uniformly on a closed interval to a limit function which is not continuous.

Relation: We have the following relation for functions over a closed and bounded (i.e. compact) subset of the $\mathbb{R}$:

Continuously differentiable $\subseteq$ Lipschitz continuous $\subseteq$ Uniformly continuous $\subseteq$ Continuous.

Lipschitz continuous $\subseteq$ absolutely continuous $\subseteq$ bounded variation.

Chapter 1

Bernstein Polynomials

Approximation: An approximation is a value or quantity is nearly but not exactly correct or a thing that is simillar to something else. In mathematics, approximation is concerned with how functions can be best approximated with simpler functions, and with quantitative characterizing the error introduced near by. Here, we discuss the basic theme of approximation of functions by polynomial functions and it has become possible due to Weierstrass.

1.1 Weierstrass Approximation Theorem

The Weierstrass approximation theorem [19] is the key moment in expansion of approximation theory which was given by Weierstrass in 1885. This theorem characterizes that there exists a sequence of polynomials which is dense in space of all continuous functions on a closed interval. In mathematical analysis, the Weierstrass approximation theorem states that every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated arbitrarily well by polynomials.
In other words, Assume that $f(x)$ is a continuous function defined on $[a, b]$ and for a given $\epsilon > 0$, there exists a polynomial $p(x)$ on $[a, b]$ such that $|f(x) - p(x)| < \epsilon$, for all $x \in [a, b]$.
There are many proofs available of this important theorem. The early ones are due to Runge(1885), Picard (1891), Lerch (1892 and 1903), Volterra (1897), Lebesgue (1898), Mittag-Leffler (1900), Fejér (1900 and 1916), Landau (1908), De la Valle

Poussin(1908), Jackson (1911), Sierpinski (1911), Bernstein (1912), and Montel (1918). Here we mention the proof by Korovkin [12] which is based on the Bernstein polynomials.

1.2 Bernstein Polynomials

Definition 12. For $f \in B[0,1]$, Bernstein polynomials are defined as

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $B[0,1]$ is the space of all bounded functions in $[0,1]$.

1.2.1 Properties

- B_n is a monotone operator i.e. if $f(x) \geq g(x), x \in [0,1]$, then $B_n(f;x) \geq B_n(g;x)$.
- Bernstein polynomials of degree n are non-negative.
- The Bernstein operator is linear.
- $\sum_{k=0}^{n} p_{n,k}(x) = 1$.
- At the interval endpoints 0 and 1, only the first and the last Bernstein polynomials are nonzero.
- $p_{n,k}(1-x) = p_{n,n-k}(x)$ (Symmetry).

1.2.2 Convergence

We can approximate a continuous function on a closed interval uniformly by the sequence of the Bernstein polynomials which gives a constructive proof of Weierstrasss Theorem. Although Taylor's polynomial is also a good idea to approximate a function but, they are not suitable; for even setting aside questions of convergence, they are applicable only to the functions that are infinite times differentiable, but not to all continuous functions.

To approximate the function by Bernstein polynomial, we need the following:

1. f only needs to be continuous.

2. It is essential that the domain $[a, b]$ be closed and bounded.

1.3 Linear Positive Operators

Let S be the linear space of all real valued functions acting on a non-empty set X. The operator $T : S \to S$ is called linear positive if for $\alpha, \beta \in \mathbb{R}$ and $f, g \in S$;

$(i)\, T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$,

$(ii)\, T(f) \geq 0$ for any non-negative function $f \in S$.

The foundation of the theory of approximation by general sequences of linear positive operators was introduced by T. Popoviciu [16], H. Bohman [2] and P. P. Korovkin [12].

1.4 Korovkin Theorem

If $(L_n)_{n\geq 1}$ is a sequence of linear positive operators from C[a,b] into C[a,b] such that for every $g \in (e_0, e_1, e_2)$, where $e_i(t) = t^i$, i = 0,1,2, we have

$$\lim_{n\to\infty} L_n(g) = g$$

uniformly in [a,b], then, for every $f \in [a, b]$ there holds

$$\lim_{n\to\infty} L_n(f) = f$$

uniformly in [a,b].

Uniform convergence of a continuous function by Bernstein polynomial :-

Theorem 1.4.1. *If $f \in C[0, 1]$, then $\lim_{n\to\infty} B_n(f; x) = f(x)$, uniformly for $x \in [0, 1]$.*

Proof. For a continuous function f defined on $[0, 1]$, consider the $nth-$degree Bernstein polynomial:

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Let $f(x) = 1$,

$$B_n(1, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = [x + (1-x)]^n = 1.$$

The first-order moment is given by

$$\begin{aligned} B_n(t, x) &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left(\frac{k}{n}\right) \\ &= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} x^{k+1} (1-x)^{n-k-1} \\ &= x B_{n-1}(1, x) = x. \end{aligned}$$

The second-order moment is given by

$$
\begin{aligned}
B_n(t^2,x) &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k^2}{n^2}\right) \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k(k-1)+k}{n^2}\right) \\
&= \frac{1}{n^2} \left[\sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} x^k (1-x)^{n-k} \right. \\
&\quad \left. + \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \right] \\
&= \frac{1}{n^2} [\, n(n-1)x^2 B_{n-2}(1,x) + nx B_{n-1}(1,x)] \\
&= \frac{1}{n^2} \left[n(n-1)x^2 + nx \right] = x^2 + \frac{x(1-x)}{n}.
\end{aligned}
$$

Thus, $B_n(e_i(t),x)$ converges uniformly to $e_i(x)$ in the interval $[0,1]$, where $e_i(t) = t^i$, i=0,1,2.

Since, Bernstein polynomials are linear positive operators, therefore all the conditions of Korovkin theorem are satisfied. Thus, $B_n(f,x)$ converges uniformly to the continuous function $f(x)$ in the interval $[0,1]$.

Now, we shall generalize this concept for interval $[a,b]$. Let the function $f(x)$ be continuous in $[a,b]$, then we define

$$\phi(y) = f(a + y(b-a)).$$

Since the function $\phi(y)$ is continuous in $[0,1]$, there exists a polynomial $p(y)$ such that

$$|p(y) - \phi(y)| < \epsilon, \;\; y \in [0,1].$$

$$y = \frac{x-a}{b-a}, \;\; Q(x) = p\left(\frac{x-a}{b-a}\right),$$

and

$$\phi\left(\frac{x-a}{b-a}\right) = f\left(a + (b-a)\frac{x-a}{b-a}\right) = f(x),$$

$$\epsilon > \left|p\left(\frac{x-a}{b-a}\right) - \phi\left(\frac{x-a}{b-a}\right)\right| = |Q(x) - f(x)|\,,\ x \in [a,b].$$

Thus the proof is completed. $\square$

After Bernstein, many researches have introduced different sequence of the linear positive operators and discussed their approximation properties. Now, we list some of the expert mathematicians who involved in significant activities in development of theory of approximation and made many efforts to improve the degree of approximation of these linear positive operators: Kantorovich [11], Phillips [15], Baskakov [7], Durrmeyer [5], D. D. Stancu [18], Lupas and Lupas [13], Srivastava and Gupta [17], Bèzier [4] etc.

Chapter 2

Bèzier Curves

2.1 Bèzier Curves

Bèzier curve is an approximation curve. The curve was first proposed in 60's by P. Bèzier. Initially it was used to define scultptured surface of automobile bodies. A parametric nth order Bèzier curve is defined by $(n + 1)$ control points.

Definition 13. A Bèzier curve is a parametric curve frequently used in computer graphics and related fields. Generalizations of Bèzier curves to higher dimensions are called Bèzier surfaces, of which the Bèzier triangle is a special case. A Bèzier curve is a mathematically defined curve used in two-dimensional graphic applications. The curve is defined by $(n + 1)$ points: the initial position and the terminating position (which are called "anchors") and $(n - 1)$ separate middle points (which are called "handles"). The shape of a Bèzier curve can be altered by moving the handles. These points are control points which defines the shape of curve. If we change the control points then it will change the shape of curve.

It uses the co-ordinates of the other points in order to define the curve but it doesn't pass through, that's why it is called approximation curve. A control polygon is nothing but a polygon formed by joining the control points. Another name given for control polygon is characteristic polygon.
Now, we define 3rd order Bèzier curve (Cubic Bèzier curve) by using four control points.

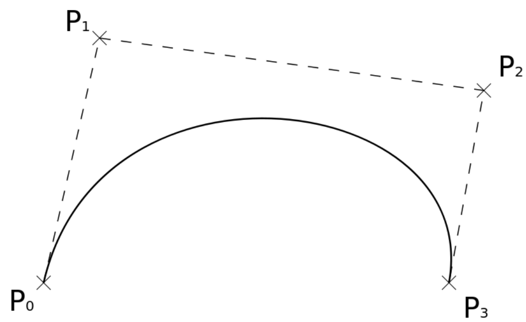


Figure 2.1: Cubic Bèzier curve

Input to Cubic Bèzier curve

First point, $P_0 = (x_0, y_0, z_0)$

Second point, $P_1 = (x_1, y_1, z_1)$

Third point, $P_2 = (x_2, y_2, z_2)$

Fourth point, $P_3 = (x_3, y_3, z_3)$

- First point P_0 is starting point
- Fourth point P_3 is end point
- Order of point is important.

Mathematically, a cubic Bèzier curve is defined as

$$x(u) = (1-u)^3 x_0 + 3(1-u)^2 u x_1 + 3(1-u)u^2 x_2 + u^3 x_3,\ 0 \leq u \leq 1.$$

Any curve can be defined as a function of a single parameter. The parameter shown here is u. Here x is a function of u. We can also defined it as

$$x(u) \quad = \quad \binom{3}{0}(1-u)^3 x_0 + \binom{3}{1}(1-u)^2 u x_1 + \binom{3}{2}(1-u)u^2 x_2 + \binom{3}{3}u^3 x_3$$

or

$$x(u) = \sum_{i=0}^{3} \binom{3}{i}(1-u)^{3-i} u^i x_i.$$

The basis function on the range u in $[0,1]$ for the cubic Bèzier curve:

blue: $x = (1-u)^3$, green: $x = 3(1-u)^2 x$, red: $x = 3(1-u)u^2$, cyan: $x = u^3$.

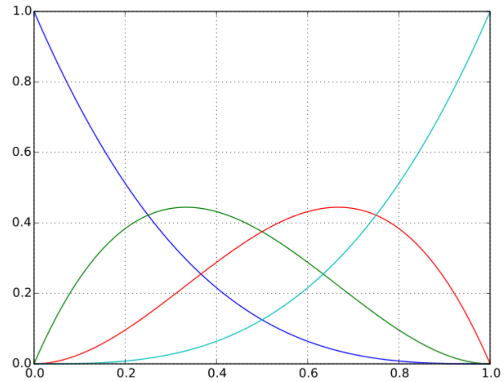


Figure 2.2: Basis function

Quartic Bèzier curve is defined as

$$x(u) = \sum_{i=0}^{4} \binom{4}{i} (1-u)^{4-i} u^i x_i.$$

Quintic Bèzier curve is defined as

$$x(u) = \sum_{i=0}^{5} \binom{5}{i} (1-u)^{5-i} u^i x_i.$$

Generic Bèzier curve is defined as

$$x(u) = \sum_{i=0}^{n} \binom{n}{i} (1-u)^{n-i} u^i x_i.$$

Properties of Bèzier curve

- The curve begins at P_0 and ends at P_n; this is the so-called endpoint interpolation property.

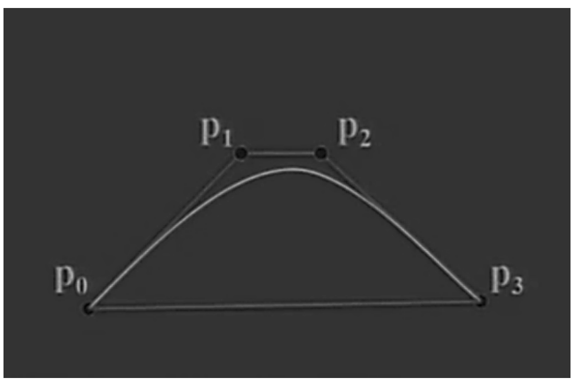
- The curve is a straight line if and only if all the control points are collinear.

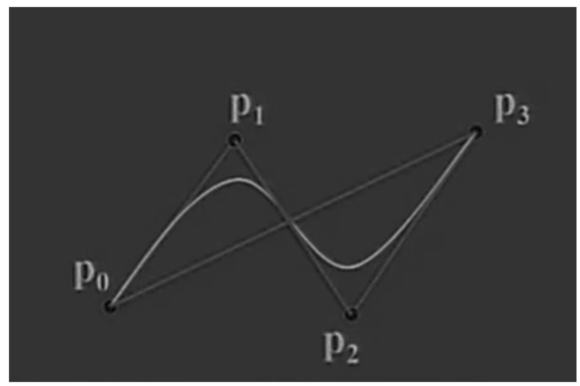
- The start and end of the curve is tangent to the first and last point of the Bèzier polygon, respectively.

- Curvature at end pionts are defined by end points and their two adjacent pionts

$$\begin{aligned}\kappa_0 &= \frac{2|(p_1 - p_0) \times (p_2 - p_1)|}{3|p_1 - p_0|^3}\\ \kappa_1 &= \frac{2|(p_{n-1} - p_{n-2}) \times (p_n - p_{n-1})|}{3|p_n - p_{n-1}|^3}.\end{aligned}$$

- Reversing the sequence of control points doesnot change the shape of curve.
- Convex hull property.





- Partition of unity

$$x(u) = \sum_{i=0}^{n} \binom{3}{i} (1-u)^{n-i} u^i = 1.$$

For n=3,

$$(1-u)^3 + 3(1-u)^2 u + 3(1-u)u^2 + u^3 = 1.$$

- A curve can be split at any point into two subcurves, or into arbitrarily many subcurves, each of which is also a Bèzier curve.
- Closed Bèzier curve:

Applications:

- Bèzier curves are widely used in computer graphics to model smooth curves. As the curve is completely contained in the convex hull of its control points,

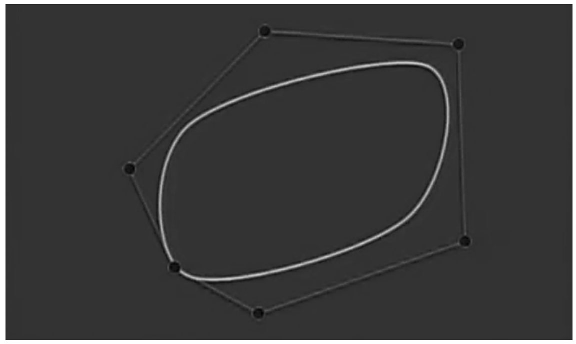


Figure 2.3: Closed Bèzier curve

the points can be graphically displayed and used to manipulate the curve intuitively. Affine transformations such as translation and rotation can be applied on the curve by applying the respective transform on the control points of the curve.

- In animation applications, such as Adobe Flash and Syn fig, Bèzier curves are used to outline, for example, movement. Users outline the wanted path in Bèzier curves, and the application creates the needed frames for the object to move along the path.

- True Type fonts use composite Bèzier curves composed of quadratic Bèzier curves. Other languages and imaging tools (such as Post Script, Asymptote, Meta font, and SVG) use composite Bèzier composed of cubic Bèzier curves for drawing curved shapes. Open Type fonts can use either kind, depending on the flavor of the font.

Chapter 3

Bèzier variant of the Certain Genuine Hybrid Operators

A Bèzier curve is a parametric curve frequently used in computer graphics and related field. It is well known that Bèzier basis functions play an important role in computer aided design. Zeng and Chen [20] were the first to estimate the rate of convergence for the Bèzier variant of Bernstein Durrmeyer operators. Bojanic and Cheng [4], [3] who estimated the rate of convergence with derivatives of bounded variation for Bernstein and Hermite-Fejer polynomials by using different methods. Guo [6] studied it for the Bernstein-Durrmeyer polynomials by using Berry Esseen theorem. Zeng and Tao [21] also introduced the Bèzier type Baskaov-Durrmeyer operators and estimated the rate of convergence for $\theta \geq 1$. Abel and Gupta [1] introduced the Bezier variant of the Baskakov operators and then Gupta [9] estimated the convergence of Bèzier type Baskakov-Kantorovich operators and studied the rate of convergence for $0 < \theta < 1$. Several other Bèzier variants of some summation-integral type operators were studied in [8] and [9].

In [10] Gupta et al. proposed a general hybrid family of summation-integral type operators based on the parameters $\rho > 0$ and $c \in \{0, 1\}$ in the following way:

$$B_{\alpha}^{\rho}(f; x, c) = \sum_{k=1}^{\infty} p_{\alpha,k}(x, c) \int_{0}^{\infty} \theta_{\alpha,k}^{\rho}(t) f(t) dt + p_{\alpha,0}(x, c) f(0) \tag{3.0.1}$$

where

$$p_{\alpha,k}(x, c) = \frac{(-x)^k}{k!} \phi_{\alpha,c}^{k}(x)$$

$$\theta^{\rho}_{\alpha,k}(t) = \frac{\alpha\rho}{\Gamma(k\rho)} e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1}.$$

For the space defined as:

$$C_{\gamma}[0,\infty) = \left\{ f \in [0,\infty) : |f(t)| \le Ce^{\gamma t} \,\text{for some}\, \gamma \ge 0, t \in [0,\infty) \right\}$$

It is observed that the operators $B^{\rho}_{\alpha}(f;x,c)$ are well defined for $\alpha\rho > \gamma$, and preserve the linear functions.

In this section we attempt to continue the study of Bèzier variants of different sequences of operators. We propose a Bèzier variant of the operator (3.0.1) after applying parameter $\beta \ge 1$ which is defined as:

(3.0.2)

$$B^{\rho}_{\alpha,\beta}(f;x,c) = \sum_{k=1}^{\infty} \left[J^{\beta}_{\alpha,k}(x,c) - J^{\beta}_{\alpha,k+1}(x,c) \right] \int_0^{\infty} \theta^{\rho}_{\alpha,k}(t) f(t) dt + Q^{\beta}_{\alpha,0}(x,c) f(0),$$

where

$$Q^{\beta}_{\alpha,k}(x,c) = J^{\beta}_{\alpha,k}(x,c) - J^{\beta}_{\alpha,k+1}(x,c) \text{ and } J_{\alpha,k}(x,c) = \sum_{j=k}^{\infty} p_{\alpha,j}(x,c)$$

when $k < \infty$ and 0 otherwise. Clearly, $B^{\rho}_{\alpha,\beta}(f;x,c)$ is a linear positive operator. If $\beta = 1$, then the operators $B^{\rho}_{\alpha,\beta}(f;x,c)$ reduce to the operators $B^{\rho}_{\alpha}(f;x,c)$. The aim of this chapter is to investigate a direct approximation result and the rate of convergence for functions having a derivative equivalent with a function of bounded variation on every finite subinterval of $[0,\infty)$ for the operators (3.0.2).

3.1 Auxiliary Results

Lemma 3.1.1. *For $B^{\rho}_{\alpha}(t^m;x,c)$, $m = 0,1,2$ one has*

- $B^{\rho}_{\alpha}(1;x,c) = 1;$
- $B^{\rho}_{\alpha}(t;x,c) = x;$
- $B^{\rho}_{\alpha}(t^2;x,c) = \frac{x}{\alpha}(1+\rho) + x^2\left(\frac{c}{\alpha}+1\right);$
- *If $\alpha \to \infty$, then $B^{\rho}_{\alpha}(t^2;x,c) = x^2$.*

Consequently,

$$B_\alpha^\rho((t-x)^2;x,c)=\frac{x(1+\rho(1+cx))}{\alpha\rho},$$

and

$$B_{\alpha,\beta}^\rho((t-x)^2;x,c)\leq\frac{\beta\lambda x(1+cx)}{\alpha}$$

for sufficiently large α and $\lambda>1$. From [14], one has

$$B_\alpha^\rho((t-x)^{2r};x,c)=O(\alpha^{-r}). \tag{3.1.1}$$

Remark 3.1.2. *We have*

$$\begin{aligned}B_{\alpha,\beta}^\rho(1,x,c) &= \sum_{k=0}^{\infty}Q_{\alpha,k}^\rho(x,c)\\ &= [J_{\alpha,0}(x,c)]^\beta\\ &= \left[\sum_{j=0}^{\infty}p_{\alpha,j}(x,c)\right]^\beta=1.\end{aligned}$$

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $[0,\infty)$ endowed with the norm

$$||f||=\sup_{[0,\infty)}|f(x)|.$$

Lemma 3.1.3. *For every $f\in C_B[0,\infty)$, we have*

$$||B_{\alpha,\beta}^\rho(f;x,c)||\leq||f||.$$

Proof. Applying Remark (3.1.2), the proof of this lemma easily follows. Hence the details are omitted.

□

Remark 3.1.4. *For $0\leq a,\ b\leq 1,\ \alpha\geq 1,$ using the inequality*

$$|a^\alpha-b^\alpha|\leq\alpha|a-b|$$

and from the definition of $B_{\alpha,\beta}^\rho(x,c)$, for all $k=0,1,2,...,$ we have

$$0<[J_{n,k}(x,c)]^\beta-[J_{n,k+1}(x,c)]^\beta\leq\beta\left[J_{n,k}(x,c)-J_{n,k+1}(x,c)\right]=\beta p_{n,k}(x,c).$$

Hence from the definition of $B_{\alpha,\beta}^\rho(f;x,c)$, we get

$$|B_{\alpha,\beta}^\rho(f;x,c)|\leq\beta B_\alpha^\rho(|f|;x,c).$$

3.2 Main Results

To obtain our first result, we first define the Ditzian-Totik first order modulus of smoothness and the K-functional. Let $\phi(x) = \sqrt{x(1+cx)}$, $f \in C[0,\infty)$. The first order modulus of smoothness is given by

$$w_\phi(f,t) = \sup_{0<h\leq t} \left\{ \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right) \right|, x \pm \frac{h\phi(x)}{2} \in [0,\infty) \right\}$$

and the Petrees $K-functional$ is defined by

$$\overline{K}_\phi(f,t) = \inf_{g\in W_\phi} \left\{ ||f-g|| + t\,||\phi g'|| + t^2\,||g'||\,, t>0 \right\}$$

Where $W_\phi = \{g : g \in AC_{loc}, ||\phi g'|| < \infty, ||g'|| < \infty\}$ and $g \in AC_{loc}$ on every $[a,b] \subset [0,\infty)$. It is well known that $\bar{K}_\phi(f,t)$ and $w_\phi(f,t)$ are equivalent which means that there exists a constant $C > 0$ such that

$$C^{-1} w_\phi(f,t) \leq \overline{K}_\phi(f,t) \leq C w_\phi(f,t) \tag{3.2.1}$$

Now, we introduce a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness.

Theorem 3.2.1. *Let f is a continuous function on $[0,\infty)$ and $\phi(x) = \sqrt{x(1+cx)}$, then for every $x \in [0,\infty)$, we have*

$$\left| B^\rho_{\alpha,\beta}(f;x,c) - f(x) \right| \leq C w_\phi\left(f; \frac{1}{\sqrt{\alpha}}\right), \tag{3.2.2}$$

where C is a constant independent of α and x.

Proof. For fixed α and x, choosing $g = g_{\alpha,x} \in W_\phi$ and using the representation

$$g(t) = g(x) + \int_x^t g'(u)du.$$

Appling the operators to both sides, we get

$$\left| B^\rho_{\alpha,\beta}(g;x,c) - g(x) \right| = \left| B^\rho_{\alpha,\beta}\left(\int_x^t g'(u)du; x, c \right) \right|. \tag{3.2.3}$$

Now to find the estimate we split the domain into two parts: $F^c_\alpha = [0, \frac{1}{\alpha}]$ and $F_\alpha = (\frac{1}{\alpha}, \infty)$.

First, if $x \in (\frac{1}{\alpha}, \infty)$ then $B^{\rho}_{\alpha,\beta}((t-x)^2; x, c) \sim \frac{\lambda\beta}{\alpha}\phi^2(x)$.

Now,

$$\left|\int_x^t g'(u)du\right| \leq ||\phi g'|| \left|\int_x^t \frac{1}{\phi(u)}du\right|. \tag{3.2.4}$$

For any $x, t \in (0, \infty)$, we find that

$$\begin{aligned}
\left|\int_x^t \frac{1}{\phi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1+cu)}}du\right| \\
&\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{(1+cu)}}\right) du\right| \\
&\leq 2\left(\sqrt{t} - \sqrt{x} + \frac{\sqrt{1+ct} - \sqrt{1+cx}}{c}\right) \\
&= 2\left|t-x\right|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1+ct}+\sqrt{1+cx}}\right) \\
&< 2\left|t-x\right|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1+cx}}\right) \\
&\leq \frac{2(1+c)}{\sqrt{c(c-1)}}\frac{|t-x|}{\phi(x)}. \qquad (3.2.5)
\end{aligned}$$

Combining (3.2.3)-(3.2.5) and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|B^{\rho}_{\alpha,\beta}(g; x, c) - g(x)\right| &< \frac{2(1+c)}{\sqrt{c(c-1)}}\,||\phi g'||\,\phi^{-1}(x)B^{\rho}_{\alpha,\beta}(|t-x|; x) \\
&\leq \frac{2(1+c)}{\sqrt{c(c-1)}}\,||\phi g'||\,\phi^{-1}(x)\left(B^{\rho}_{\alpha,\beta}\left((t-x)^2; x\right)\right)^{\frac{1}{2}} \\
&\leq \frac{2(1+c)}{\sqrt{c(c-1)}}\,||\phi g'||\left(\frac{\beta\lambda}{\alpha}\right)^{\frac{1}{2}} \qquad (3.2.6) \\
&\leq C\,||\phi g'||\,\frac{1}{\sqrt{\alpha}}.
\end{aligned}$$

For $x \in F^c_{\alpha} = [0, \frac{1}{\alpha}]$, $B^{\rho}_{\alpha,\beta}((t-x)^2; x) \sim \frac{\beta\lambda}{\alpha^2}$ and

$$\left|\int_x^t g'(u)du\right| \leq ||g'||\,|t-x|\,.$$

Therefore, using the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\left|B^{\rho}_{\alpha,\beta}(g;x,c)-g(x)\right| &\leq ||g'||\, B^{\rho}_{\alpha,\beta}\left(|t-x|\,;x,c\right)\\
&\leq ||g'||\left(B^{\rho}_{\alpha,\beta}(t-x)^2;x,c\right)^{\frac{1}{2}}\\
&\leq ||g'||\,\frac{\sqrt{\lambda\beta}}{\alpha}\\
&\leq C\,||g'||\,\frac{1}{\alpha}.
\end{aligned}
\tag{3.2.7}
$$

From (3.2.6) $-$ (3.2.7), we obtain

$$
\left|B^{\rho}_{\alpha,\beta}(g;x,c)-g(x)\right| \leq C\left(||\phi g'||\,\frac{1}{\sqrt{\alpha}}+||g'||\,\frac{1}{\alpha}\right)
\tag{3.2.8}
$$

Using Lemma (3.1.3) and equation (3.2.8), we can write

$$
\begin{aligned}
\left|B^{\rho}_{\alpha,\beta}(f;x,c)-f(x)\right| &\leq \left|B^{\rho}_{\alpha,\beta}(f-g;x)\right|+|f(x)-g(x)|+\left|B^{\rho}_{\alpha,\beta}(g;x,c)-g(x)\right|\\
&\leq C\left(||f-g||+||\phi g'||\,\frac{1}{\sqrt{\alpha}}+||g'||\,\frac{1}{\alpha}\right).
\end{aligned}
$$

Taking the infimum on the right hand side of the above inequality over all $g\in W_{\phi}$, we get

$$
\left|B^{\rho}_{\alpha,\beta}(f;x,c)-f(x)\right| = C\overline{K}_{\phi}\left(f;\frac{1}{\sqrt{\alpha}}\right)
$$

Using $\overline{K}_{\phi}(f,t)\sim w_{\phi}(f,t)$ and equation (3.2.1), we get the desired relation (3.2.2). This completes the proof of the theorem. □

Now, we shall discuss the rate of approximation of functions with a derivative of bounded variation on $[0,\infty)$. Let $DBV_{\gamma}[0,\infty)$, $\gamma\geq 0$, denote the class of all absolutely continuous functions f defined on $[0,\infty)$, having a derivative f' equivalent with a function of bounded variation on every finite subinterval of $[0,\infty)$ and $|f(t)|\leq Mt^{\gamma}$.

We observe that the functions $f\in DBV_{\gamma}[0,\infty)$ possess a representation

$$
f(x)=\int_0^x g(t)dt+f(0),
$$

where $g\in BV[0,\infty)$, i.e. g is a function of bounded variation on every finite subinterval of $[0,\infty)$.

In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators $B^{\rho}_{\alpha,\beta}$ in an integral form as follows:

$$B^{\rho}_{\alpha,\beta}(f;x,c) = \int_0^{\infty} K^{\rho}_{\alpha,\beta}(x,t,c) f(t)\,dt \tag{3.2.9}$$

where the kernel $K^{\rho}_{\alpha,\beta,c}(x,t)$ is given as

$$K^{\rho}_{\alpha,\beta}(x,t,c) = \sum_{k=1}^{\infty} Q^{\beta}_{\alpha,k}(x,c)\theta^{\rho}_{\alpha,k}(t) + Q^{\beta}_{\alpha,0}(x,c)\delta(t)$$

$\delta(u)$ is the Dirac delta function.

Lemma 3.2.2. *For a fixed $x \in (0,\infty)$ and sufficiently large α, we have*
1) $\xi^{\rho}_{\alpha,\beta}(x,y,c) = \int_0^y K^{\rho}_{\alpha,\beta}(x,t,c)dt \leq \beta \frac{\lambda x(1+cx)}{\alpha} \frac{1}{(x-y)^2}$, $0 \leq y < x$,
2) $1 - \xi^{\rho}_{\alpha,\beta}(x,z,c) = \int_z^{\infty} K^{\rho}_{\alpha,\beta}(x,t,c)dt \leq \beta \frac{\lambda x(1+cx)}{\alpha} \frac{1}{(z-x)^2}$, $x < z < \infty$.

Proof. 1) Using Lemma(3.1.1) and Remark (3.1.4), we get

$$\begin{aligned}
\xi^{\rho}_{\alpha,\beta}(x,y,c) &= \int_0^y K^{\rho}_{\alpha,\beta}(x,t,c)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
&\leq B^{\rho}_{\alpha,\beta}\left((t-x)^2;x,c\right)(x-y)^{-2} \\
&\leq \beta B^{\rho}_{\alpha}\left((t-x)^2;x,c\right)(x-y)^{-2} \\
&\leq \frac{\beta\lambda x(1+cx)}{\alpha}\frac{1}{(x-y)^2}.
\end{aligned}$$

2) The proof of second part is same. □

Theorem 3.2.3. *Let $f \in DBV_{\gamma}[0,\infty)$. Then, for every $x \in (0,\infty)$ sufficiently large α, we have*

$$\begin{aligned}
\left|B^{\rho}_{\alpha,\beta}(f;x,c) - f(x)\right| \leq\ & \frac{1}{\beta+1}\left|f'(x+) + \beta f'(x-)\right|\sqrt{\frac{2\beta x(1+cx)}{\alpha}} \\
&+\frac{\beta}{\beta+1}\left|f'(x+) + f'(x-)\right|\sqrt{\frac{2\beta x(1+cx)}{\alpha}} \\
&+\frac{\lambda\beta(1+cx)}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\bigvee_{x-x/k}^{x+x/k} f'_x + \frac{x}{\sqrt{\alpha}}\bigvee_{x-x/\sqrt{\alpha}}^{x+x/\sqrt{\alpha}} f'_x \\
&+\frac{\lambda\beta(1+cx)}{\alpha x}\left|f(2x) - f(x) - xf'(x+)\right| \\
&+\frac{\beta C(\alpha,c,r,x)}{\alpha^r} + \frac{|f(x)|}{x}\frac{\lambda\beta(1+cx)}{\alpha},
\end{aligned}$$

where $\bigvee_a^b f(x)$ *denotes the total variation of* $f(x)$ *on* $[a,b]$ *and* f'_x *is defined by*

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \le t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t < \infty \end{cases} . \tag{3.2.10}$$

Proof. Since $B^{\rho}_{\alpha,\beta}(1, x, c) = 1$, using (3.2.9), for every $x \in (0, \infty)$ we obtain

$$\begin{aligned} B^{\rho}_{\alpha,\beta}(t; x, c) - f(x) &= \int_0^{\infty} K^{\rho}_{\alpha,\beta}(x, t, c)\,(f(t) - f(x))\,dt \\ &= \int_0^{\infty} K^{\rho}_{\alpha,\beta}(x, t, c) \int_x^t f'(u)du dt. \end{aligned} \tag{3.2.11}$$

For any $f \in DBV_{\gamma}[0, \infty)$, using equation (3.2.10) we get

$$\begin{aligned} f'(u) &= f'_x(u) + \frac{1}{\beta+1}\left(f'(x+) + \beta f'(x-)\right) \\ &+\frac{1}{2}\left(f'(x+) - f'(x-)\right)\left(sgn(u-x) + \frac{\beta-1}{\beta+1}\right) \\ &\times\delta_x(u)\left[f'(u) - \frac{1}{2}\left(f'(x+) - f'(x-)\right)\right], \end{aligned} \tag{3.2.12}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x \end{cases} .$$

From equations (3.2.11) and (3.2.12), we have

$$\begin{aligned} B^{\rho}_{\alpha,\beta}(t; x, c) - f(x) &= \int_0^{\infty} K^{\rho}_{\alpha,\beta}(x, t, c) \int_x^t \left[f'_x(u) + \frac{1}{\beta+1}\left(f'(x+) + \beta f'(x-)\right)\right. \\ &+\frac{1}{2}\left(f'(x+) - f'(x-)\right)\left(sgn(u-x) + \frac{\alpha-1}{\alpha+1}\right) \\ &\left.+\ \delta_x(u)\left[f'(u) - \frac{1}{2}\left(f'(x+) - f'(x-)\right)\right] du\right] dt \\ &= A_1 + A_2 + A_3 + A^{\rho}_{\alpha,\beta}(f'_x, x, c) + D^{\rho}_{\alpha,\beta}(f'_x, x, c). \end{aligned}$$

where

$$
\begin{aligned}
A_1 &= \int_0^\infty \left(\int_x^t \frac{1}{\beta+1} \left(f'(x+) + \beta f'(x-) \right) du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt, \\
A_2 &= \int_0^\infty K^{\rho}_{\alpha,\beta}(x,t,c) \int_x^t \left(\frac{1}{2} \left(f'(x+) - f'(x-) \right) \left(sgn(u-x) + \frac{\beta-1}{\beta+1} \right) du \right) dt, \\
A_3 &= \int_0^\infty \left(\int_x^t \delta_x(u) \left(f'(u) - \frac{1}{2} \left(f'(x+) - f'(x-) \right) \right) du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
A^{\rho}_{\alpha,\beta}(f'_x, x, c) &= \int_0^x \left(\int_x^t f'_x(u)du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt, \\
D^{\rho}_{\alpha,\beta}(f'_x, x, c) &= \int_x^\infty \left(\int_x^t f'_x(u)du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt.
\end{aligned}
$$

By applying the definition of $\delta_x(u)$, we obtain

$$
\begin{aligned}
A_3 &= \int_0^\infty \left(\int_x^t \delta_x(u) \left(f'(u) - \frac{1}{2} \left(f'(x+) - f'(x-) \right) \right) du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
(3.2.13) \quad &= 0
\end{aligned}
$$

also

$$
\begin{aligned}
A_1 &= \int_0^\infty \left(\int_x^t \frac{1}{\beta+1} \left(f'(x+) + \beta f'(x-) \right) du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
(3.2.14) \quad &= \frac{1}{\beta+1} \left(f'(x+) + \beta f'(x-) \right) \int_0^\infty (t-x)\, K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
&= \frac{1}{\beta+1} \left(f'(x+) + \beta f'(x-) \right) B^{\rho}_{\alpha,\beta}\left((t-x); x, c \right)
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 &= \int_0^\infty K^{\rho}_{\alpha,\beta}(x,t,c) \int_x^t \left(\frac{1}{2} \left(f'(x+) - f'(x-) \right) \left(sgn(u-x) + \frac{\beta-1}{\beta+1} \right) du \right) dt, \\
&= \frac{1}{2} \left(f'(x+) - f'(x-) \right) \left[- \int_0^x \left(\int_t^x \left(sgn(u-x) + \frac{\beta-1}{\beta+1} \right) du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt \right] \\
(3.2.15) &+ \int_x^\infty \left(\int_x^t \left(sgn(u-x) + \frac{\beta-1}{\beta+1} \right) du \right) K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
&\leq \frac{\beta}{\beta+1} \left(f'(x+) + f'(x-) \right) \int_0^\infty |t-x|\, K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
&= \frac{\beta}{\beta+1} \left(f'(x+) + f'(x-) \right) B^{\rho}_{\alpha,\beta}\left(|t-x|; x, c \right).
\end{aligned}
$$

Using Lemma (3.1.3) and equations (3.2.11)-(3.2.15) and applying the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\left|B^{\rho}_{\alpha,\beta}(t;x,c)-f(x)\right| &\leq \frac{1}{\beta+1}\left|f'(x+)+\beta f'(x-)\right|\left(\beta B^{\rho}_{\alpha}\left((t-x)^2;x,c\right)\right)^{\frac{1}{2}}\\
&\quad+\frac{\beta}{\beta+1}\left|f'(x+)-f'(x-)\right|\left(\beta B^{\rho}_{\alpha}\left((t-x)^2;x,c\right)\right)^{\frac{1}{2}}\\
&\quad+\left|A^{\rho}_{\alpha,\beta}(f'_x,x,c)\right|+\left|D^{\rho}_{\alpha,\beta}(f'_x,x,c)\right|\\
&\leq \frac{1}{\beta+1}\left|f'(x+)+\beta f'(x-)\right|\sqrt{\frac{\beta\lambda x(1+cx)}{\alpha}}\\
&\quad+\frac{\beta}{\beta+1}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{\beta\lambda x(1+cx)}{\alpha}}\\
&\quad+\left|A^{\rho}_{\alpha,\beta}(f'_x,x,c)\right|+\left|D^{\rho}_{\alpha,\beta}(f'_x,x,c)\right|.
\end{aligned} \tag{3.2.16}$$

Thus our problem is reduced to calculate the estimates of the terms $A^{\rho}_{\alpha,\beta}(f'_x,x,c)$ and $D^{\rho}_{\alpha,\beta}(f'_x,x,c)$. Since $\int_a^b d_t\xi^{\rho}_{\alpha,\beta}(x,t,c)\leq 1$ for all $[a,b]\subseteq[0,\infty)$, using integration by parts and applying Lemma (3.2.2) with $y=x-x/\sqrt{\alpha}$, we have

$$\begin{aligned}
\left|A^{\rho}_{\alpha,\beta}(f'_x,x,c)\right| &= \left|\int_0^x\left(\int_x^t f'_x(u)du\right)d_t\xi^{\rho}_{\alpha,\beta}(x,t,c)\right|\\
&= \left|\int_0^x \xi^{\rho}_{\alpha,\beta}(x,t,c)f'_x dt\right|\\
&\leq \int_0^y\left|\xi^{\rho}_{\alpha,\beta}(x,t,c)\right|\left|f'_x dt\right|+\int_y^x\left|\xi^{\rho}_{\alpha,\beta}(x,t,c)\right|\left|f'_x dt\right|\\
&\leq \frac{\beta\lambda x(1+cx)}{\alpha}\int_0^y\bigvee_t^x\left(f'_x\right)(x-t)^{-2}dt+\int_y^x\bigvee_t^x\left(f'_x\right)dt\\
&\leq \frac{\beta\lambda x(1+cx)}{\alpha}\int_0^y\bigvee_t^x\left(f'_x\right)(x-t)^{-2}dt+\frac{x}{\sqrt{\alpha}}\bigvee_{x-x/\sqrt{\alpha}}^x\left(f'_x\right)\\
&= \frac{\beta\lambda x(1+cx)}{\alpha}\int_0^{x-x/\sqrt{\alpha}}\bigvee_t^x\left(f'_x\right)(x-t)^{-2}dt+\frac{x}{\sqrt{\alpha}}\bigvee_{x-x/\sqrt{\alpha}}^x f'_x.
\end{aligned}$$

Substituting $u=x/(x-t)$, we have

$$\begin{aligned}
\beta\frac{\lambda x(1+cx)}{\alpha}\int_0^{x-x/\sqrt{\alpha}}\bigvee_t^x f'_x(x-t)^{-2}dt &= \beta\frac{\lambda x(1+cx)}{\alpha}x^{-1}\int_1^{\sqrt{\alpha}}\bigvee_{x-x/u}^x\left(f'_x\right)du\\
&\leq \beta\frac{\lambda(1+cx)}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\int_k^{k+1}\bigvee_{x-x/k}^x\left(f'_x\right)du\\
&\leq \frac{\lambda(1+cx)}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\bigvee_{x-x/k}^x\left(f'_x\right).
\end{aligned}$$

Thus,

$$\left|A^{\rho}_{\alpha,\beta}(f'_x,x,c)\right| \leq \frac{\lambda(1+cx)}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\bigvee_{x-x/k}^{t}(f'_x)+\frac{x}{\sqrt{\alpha}}\bigvee_{x-x/\sqrt{\alpha}}^{x}f'_x. \tag{3.2.17}$$

Again, using integration by parts in $D^{\rho}_{\alpha,\beta}(f'_x,x,c)$ and applying Lemma (3.2.2) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|D^{\rho}_{\alpha,\beta}(f'_x,x,c)\right| \\
&\leq \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)du\right)d_tK^{\rho}_{\alpha,\beta}(x,t,c)\right|+\left|\int_x^{2x}\left(\int_x^t f'_x(u)du\right)d_t\left(1-\xi^{\rho}_{\alpha,\beta}(x,t,c)\right)\right| \\
&\leq \left|\int_{2x}^{\infty}\left(f(t)-f(x)\right)K^{\rho}_{\alpha,\beta}(x,t,c)\right|+|f'(x-)|\left|\int_{2x}^{\infty}(t-x)\,K^{\rho}_{\alpha,\beta}(x;t,c)dt\right| \\
&\quad+\left|\int_x^{2x}f'_x(u)du\right|\left|1-\xi^{\rho}_{\alpha,\beta}(x,2x,c)\right|+\left|\int_x^{2x}f'_x(t)\left(1-\xi^{\rho}_{\alpha,\beta}(x,t,c)\right)dt\right| \\
&\leq \left|\int_{2x}^{\infty}f(t)K^{\rho}_{\alpha,\beta}(x,t,c)\right|+|f(x)|\left|\int_{2x}^{\infty}K^{\rho}_{\alpha,\beta}(x,t,c)\right| \\
&\quad+|f'(x+)|\left(\int_{2x}^{\infty}(t-x)^2K^{\rho}_{\alpha,\beta}(x,t,c)dt\right)^{\frac{1}{2}}+\frac{\beta\lambda(1+cx)}{\alpha x}\left|\int_x^{2x}\left(f'(u)-f'(x+)\right)du\right| \\
&\quad+\left|\int_x^{x+x/\sqrt{\alpha}}f'_x(t)dt\right|+\frac{\beta\lambda x(1+cx)}{\alpha}\left|\int_{x+x/\sqrt{\alpha}}^{2x}(t-x)^{-2}f'_x(t)dt\right|.
\end{aligned}$$

We see that there exists an integer $r(2r\geq\gamma)$, such that $f(t)=O(t^{2r})$, as $t\to\infty$. Now proceeding in a manner similar to the estimate of $A^{\rho}_{\alpha,\beta}(f'_x,x,c)$, on substituting $t=x+\frac{x}{u}$ we get

$$\begin{aligned}
\left|D^{\rho}_{\alpha,\beta}(f'_x,x,c)\right| &\leq M\int_{2x}^{\infty}t^{2r}K^{\rho}_{\alpha,\beta}(x,t,c)dt+|f(x)|\int_{2x}^{\infty}K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
&\quad+|f'(x+)|\sqrt{\frac{\lambda\beta x(1+cx)}{\alpha}}+\frac{\lambda\beta(1+cx)}{\alpha x}\left|f(2x)-f(x)-xf'(x+)\right| \\
&\quad+\frac{x}{\sqrt{\alpha}}\bigvee_{x}^{x+x/\sqrt{\alpha}}(f'_x)+\frac{\lambda\beta x(1+cx)}{\alpha}\left|\int_{x+x/\sqrt{\alpha}}^{2x}(t-x)^{-2}f'_x(t)dt\right| \\
&\leq M\int_{2x}^{\infty}t^{2r}K^{\rho}_{\alpha,\beta}(x,t,c)dt+|f(x)|\int_{2x}^{\infty}K^{\rho}_{\alpha,\beta}(x,t,c)dt \\
&\quad+|f'(x+)|\sqrt{\frac{\lambda\beta x(1+cx)}{\alpha}}+\frac{\lambda\beta(1+cx)}{\alpha x}\left|f(2x)-f(x)-xf'(x+)\right| \\
&\quad+\frac{x}{\sqrt{\alpha}}\bigvee_{x}^{x+x/\sqrt{\alpha}}(f'_x)+\frac{\lambda\beta(1+cx)}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\bigvee_{x}^{x+x/\sqrt{\alpha}}f'_x. \qquad (3.2.18)
\end{aligned}$$

For $t \geq 2x$, we get $t \leq 2(t-x)$ and $x \leq t-x$. Now using equation (3.1.1) and Lemma (3.2.2), we obtain

$$\begin{aligned}\int_{2x}^{\infty} t^{2r} K^{\rho}_{\alpha,\beta}(x,t,c)dt + |f(x)| \int_{2x}^{\infty} K^{\rho}_{\alpha,\beta}(x,t,c)dt &\leq 2^{2r}\int_{2x}^{\infty}(t-x)^{2r} K^{\rho}_{\alpha,\beta}(x,t,c)dt \\ &\quad + \frac{|f(x)|}{x^2}\int_{2x}^{\infty}(t-x)^{2} K^{\rho}_{\alpha,\beta}(x,t,c)dt \\ &\leq \frac{\beta C(\alpha,c,r,x)}{\alpha^r} + \frac{|f(x)|}{x^2}\frac{\lambda\beta x(1+cx)}{\alpha}. \qquad (3.2.19)\end{aligned}$$

Collecting the estimates (3.2.16)-(3.2.19), we get the required result. □

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