Numerical Solutions of Some Parabolic Partial Differential Equations Using Finite Difference Methods

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under

the guidance of

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to the

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INDIA

DEDICATED

TO

GOD, MY PARENTS AND MY TEACHERS

CONTENTS

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Numerical Solutions of Some Parabolic Partial Differential Equations Using Finite Difference Methods**" which is being submitted for the award of degree of master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Ram Jiwari.

The matter presented in the thesis has not been submitted for the award of any other degree of this or any other university.

Rishu Singla

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

In this thesis an attempt has been made to solve some parabolic partial differential equations by using finite differences methods. The chapter wise summary of the thesis is as follows

In chapter 2, we consider one-dimensional convection-diffusion parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=D\frac{\partial^2 u}{\partial x^2}\ , \qquad 0<x<L,\ 0<t<T$$

The convection–diffusion equation is a parabolic partial differential equation, which describes physical phenomena where energy is transformed inside a physical system due to two processes: convection and diffusion. The term convection means the movement of molecules within fluids, whereas, diffusion describes the spread of particles through random motion from regions of higher concentration to regions of lower concentration. In this chapter we have developed some finite difference schemes based on weighted average for solving the one dimensional advection–diffusion equation with constant coefficients. These techniques are based on the two-level finite difference approximation. By changing the values of weighed parameter θ, we obtained the Forward Time Cantered Space (FTSC) , Upwind scheme, Lax-Wendroff and Crank-Nicolson schemes. In order to check the accuracy of proposed methods three test examples are considered with analytical solution available in literature. The examples are solved by all four schemes and compared each other. It has been concluded that the Lax-Wendroff scheme is in good agreement with the analytical solution as compare to the other schemes.

In chapter 3, we consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}, \qquad (x,t)\in\Omega$$

The nonlinear partial differential equation is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical

modelling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman [14] in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger [23, 24] use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc

In this chapter, we present a combined numerical scheme based on Hopf-Cole transformation and Crank-Nicolson finite difference method for the numerical solutions of one dimensional Burgers' equation. The scheme has shown to be unconditionally stable and is second order accurate in space and time. The advantage of the proposed method is that there is no restriction in choosing mesh sizes. In support of the predicted theory, the two test examples have been considered and solved numerically by the proposed scheme and compared with the analytical solutions obtained by using Hopf-Cole transformation. The Figures are plotted to show the physical phenomenon of the given problem.

Chapter 1

Introduction

1.1 Partial Differential Equations

The mathematical formulation of most problems in science involving rates of change with respect to two or more independent variables, usually representing time, length or angle, leads either to a partial differential equation or to set of such equation.

The general form of linear second-order partial differential equation is

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial t}+c\frac{\partial^2 u}{\partial t^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial t}+fu=g \tag{1.1}$$

Here $u=u(x,t)$ and a, b, c, d, e, f *and* g are functions of x *and* t only-they do not depend of u.

The first three terms containing the second derivatives are called the Principal part of the partial differential equation. They determine the nature of the general solution to the equation. In fact, the coefficients of the Principal part can be used to classify the PDE as follows.

The PDE is said to be elliptic if $b^2-4ac<0$. The Laplace equation has $a=1, b=0$ *and* $c=1$ is therefore an elliptic PDE.

The PDE is said to be hyperbolic if $b^2-4ac>0$ .The wave equation has $a=1, b=0$ *and* $c=-1$ is therefore a hyperbolic PDE.

The PDE is said to be parabolic if $b^2-4ac=0$ .The heat equation has $a=1, b=0$ *and* $c=0$ is therefore a parabolic PDE.

The simple examples for the parabolic, elliptic and hyperbolic equation are as follows

1.1.1 Parabolic Equation

A parabolic equation in one dimension space may be written as

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2} \tag{1.2}$$

where k is a constant. The equation (1.2) represents conduction of heat in the x-direction with u denoting temperature at a point x in a homogeneous medium at time t. Besides heat conduction, equation (1.2) also represents several other physical processes, like diffusion of gas, fluid flow, etc.

A parabolic equation in two space dimensions can be written as

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = k\nabla^2 u$$

where $\nabla^2 u = \nabla.\nabla u \; and \; \nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$. The operator ∇^2 is known as Laplacian operator or simply Laplacian. In case of heat conduction (diffusion) the parameter k is called coefficient of heat conduction (diffusion) and is equal to $k = \frac{k}{pc}$ where k is conductivity, p the density and c is the specific heat of the medium.

As parabolic equation is time-dependent, they are known as 'transient problems'.

1.1.2 Elliptic Equation

With time increasing, all transient problems tend to reach steady state, i.e. when there is no more change in the value of u in spite of increase in time, which mathematically means $\frac{\partial u}{\partial t} = 0$. The elliptic equation describes steady state processes and can be represented as

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{(Laplace equation)} \tag{1.3}$$

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad \text{(Poisson equation)} \tag{1.4}$$

Since u may also represent voltage in a conductor or velocity potential of fluids or gravitational pontential in space, the elliptic equation are generally referred to as potential problems.

1.1.3 Hyperbolic Equation

The most common example of a hyperbolic equation in one-space dimension is the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

where c is constant.

Equation (1.5) represents the motion of a vibrating string stretched between two points where u denotes the displacement of a point on the string at a distance x, at any instance t while the string vibrates in the u-x plane. The string is assumed to be uniform and elastic and that $c^2 = \frac{T}{m}$, where T is the tension in the string and m is its mass per unit length. The equation may also represent the displacement of a longitudinally vibrating bar or of sound waves in a pipe. In two-space dimension it may represent deflection of a membrane.

1.2 Numerical Solutions of Partial Differential Equations

Partial differential equations (PDEs) form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. The vast majority of PDEs model cannot be solved analytically. So, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solution numerically. In most cases, the approximate solution is represented by functional values at certain discrete points (grid points or mesh points). There seems a bridge between the derivatives in the PDE and the functional values at the grid points. The numerical technique is such a bridge, and the corresponding approximate solution is termed the numerical solution. Currently, there are many numerical techniques available in the literature. Among them, the finite difference (FD), finite element (FE), and finite volume (FV) methods fall under the category of low order methods, whereas spectral and pseudo spectral methods are considered global methods. Sometimes the latter two methods are considered as subsets of the method of weighted.

1.2.1 Finite Element Method

Finite element method (FEM) represents a powerful and general class of techniques for the approximate solution of partial differential equations. The basic idea in the FEM is to find the solution of a complicated problem by replacing it by a simpler one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution. This method is mostly used for the accurate solution of complex engineering problems with abundant software available commercially. FEM was first developed in 1956 for the analysis of aircraft structural problems. Thereafter, within a decade, the potentialities of the method for the solution of different types of applied science and engineering problems were recognized.

Over the years, the FEM technique has been so well established that today it is considered to be one of the best methods for solving a wide variety of practical problems efficiently. In fact, the method has become one of the active research areas for applied mathematicians. Based on the variational principle, basic procedures of the FEM include: obtaining functional (variational expressions) from corresponding differential equations, dividing interested region into small elements, constructing interpolation model for each element, assembling all elements' contributions to the global system, and finally solving the global-matrix problems. The systematic generality of FEM makes it possible to construct a general-purposed computer program for a wide range of problems. In this method, the region is divided into subregions (elements), which could be different shapes i.e. triangular, rectangular, curvilinear, ring, or infinite.

Moreover, non uniform unstructured meshes and adaptive meshing procedures can be employed to significantly improve the accuracy and efficiency of FEM programs. Furthermore, FEM scheme can be established not only by the variational method but also by the Galerkin method or the least squares method, so FEM can still be used even though a variational principle does not exit or cannot be identified. Boundary conditions can be easily applied once the mesh generation is done.However, the pre-and post-processes of the computed set up always play an important role for a good FEM program. Many researchers have been using finite element method for the solutions of PDEs since 1956. Gerisch *et al.* [14,15] have used high-order linearly implicit two-step peer - finite element methods for time dependent PDEs successfully.

1.2.2 Finite Volume Method

Finite volume methods (FVMs) form a relatively general class of discretizations for certain types of partial differential equations. These methods start from balance equations over local control volumes, e.g., the conservation of mass in diffusion problems. When these conservation equations are integrated by parts over each control volume, certain terms yield integrals over the boundary of the control volume. For example, mass conservation can be written as a combination of source terms inside the control volume and fluxes across its boundary. Of course the fluxes between neighbouring control volumes are coupled. If this natural coupling of boundary fluxes is included in the discretization, then the local conservation laws satisfied by the continuous problem are guaranteed to hold locally also for the discrete problem. This is an important aspect of FVMs that makes them suitable for the numerical treatment of, e.g., problems in fluid dynamics. Another valuable property is that when FVMs are applied to elliptic problems that satisfy a boundary maximum principle, they yield discretizations that satisfy a discrete boundary maximum principle even on fairly general grids. FVMs were proposed originally as a means of generating finite difference methods on general grids.

Today, however, while FVMs can be interpreted as finite difference schemes, their convergence analysis are usually facilitates by the construction of a related finite element method and a study of its convergence properties. The fundamental idea of the finite volume method can be implemented in various ways in the construction of the control volumes, in the localization of the degree of freedom, and in the discretization of the fluxes through the boundaries of the control volumes. There are two basically two classes of FVM. First, in *cell-centred* methods each control volume that surrounds a grid point has no vertices of the original triangulation lying on its boundary. The second approach, *vertex-centred* methods, uses vertices of the underlying triangulation as vertices of control volumes.

1.2.3 Method of Weighted Residuals

The methods of weighted residuals are the approximate methods which determine the solution of the differential equation in the form of functions which are closed in some sense to the exact solution. Consider a differential equation

$$\ell(u)=0 \tag{1.6}$$

with initial condition, $I(u)=0$, and boundary condition, $S(u)=0$. The solution of differential equation $U(x)$ is approximated by a finite series of functions $\phi_k(x)$ as follows:

$$U(x)=U_0(x)+\sum_{k=1}^{N} a_k\,\phi_k(x) \tag{1.7}$$

where $\phi_k(x)$ are the basis or trial functions, a_k are the coefficients to be determined that satisfy the differential equation, and *N* are the number of functions. The form of $U_0(x)$ is chosen to satisfy the boundary and the initial conditions exactly. There is another approach in which exact solutions of the differential equation are known and these are added together to satisfy the boundary conditions approximately. It is also possible to formulate a method in which the differential equation and the boundary conditions are satisfied approximately.

In general, the approximate solution does not satisfy the partial differential equation exactly, and substituting its value results in a residual, *R*,

$$R(x,a_1,a_2,...,a_N)=\ell(U(x)) \tag{1.8}$$

which in turn is minimized in some sense. For a given *N* the $a_k's$ are chosen by requiring that an integration of the weighted residual over the domain is zero. Thus

$$\langle W_k(x),R\rangle=0. \tag{1.9}$$

By letting $k=1,2,...N$ a system of equations involving only $a_k's$ is obtained. For unsteady partial differential equation this would be a system of ordinary differential equations, for steady problems a system of algebraic equations obtained. Different choices of $W_k(x)$ give rise to the different methods within the class.

1.3 Differential Quadrature Method

The differential quadrature method (DQM) is a higher order numerical technique for solving partial differential equations. In the nineteen century, most of the numerical simulations of engineering problems can be carried out by the low order FD, FE, and FV

methods using a large number of grid points. In some practical applications, however, numerical solutions of PDEs are required at only a few specified points in the physical domain. To achieve an acceptable degree of accuracy, low order methods still require the use of a large number of grid points to obtain accurate solutions at these specified points. In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Richard Bellman and his associates [5] introduced the method of differential quadrature in the early 1970s. The DQM, akin to the conventional integral quadrature method, approximates the partial derivative of a function at any location by a linear summation of all the function values along a mesh line. The key procedure in the differential quadrature application lies in the determination of the weighting coefficients. Initially, Bellman and his associates proposed two methods to compute the weighting coefficients for the first order derivative. The first method is based on an ill-conditioned algebraic equation system. The second method uses a simple algebraic formulation, but the coordinates of the grid points are fixed by the roots of the shifted Legendre polynomial. In earlier applications of the DQM, Bellman's first method was usually used because it allows the use of an arbitrary grid point distribution. However, since the algebraic equation system of this method is ill-conditioned, the number of the grid points usually used is less than 13. This drawback limits the application of the DQM.

The DQM and its applications were rapidly developed after the late 1980s, thanks to the innovative work in the computation of the weighting coefficients by researchers [6,7,8,30,31 and 48]. As a result, the DQM has emerged as a powerful numerical discretization tool in the past decade. As compared to the conventional low order finite difference and finite element methods, the DQM can obtain very accurate numerical results using a considerably smaller number of grid points and hence requiring relatively little computational effort.

1.4 Finite Difference Methods

The finite difference techniques are based upon the approximations that permit replacing differential equation by finite difference equation. There finite difference approximations are algebraic in form, and the solutions are related to grid points. Thus, a finite difference solution basically involves three steps:-

1) Dividing the solution into grids of notes.
2) Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3) Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

Forward-difference formula

$$f'(x_0) \cong \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Backward-difference formula

$$f'(x_0) \cong \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}$$

Central-difference formula

$$f'(x_0) \cong \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$$

The approach used for obtaining above finite difference equations is Taylor's series:-

$$f(x_o + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) + \frac{1}{3!}(\Delta x)^3 f'''(x_0) + o(\Delta x)^4 \quad (1.10)$$

$$f(x_0 - \Delta x) = f(x_o) - \Delta x f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) - \frac{1}{3!}(\Delta x)^3 f'''(x_0) + o(\Delta x)^4 \quad (1.11)$$

where $o(\Delta x)^4$ is the error introduced by truncating the series.

Subtracting (1.10) from (1.11), we obtain

$$f(x_0 + \Delta x) - f(x_0 - \Delta x) = 2\Delta x f'(x_0) + o(\Delta x)^3,$$

which can be re-written as

$$f'(x_0) \cong \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} + o(\Delta x)^2$$

i.e. the central-difference formula. Note that the $o(\Delta x)^2$ means the truncation error is the order of $(\Delta x)^2$ for the central-difference.

The forward-difference and backward-difference formula could be obtained by re-arranging (1.10) and (1.11) respectively, and we have

$$f'(x_0) \cong \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + o(\Delta x), \qquad \text{for forward difference,}$$

and

$$f'(x_0) \cong \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} + o(\Delta x),$$

for backward difference. We can find the truncation errors of their two formulas are of order Δx . On adding (1.10) and (1.11), we have

$$f(x_0 + \Delta x) + f(x_0 - \Delta x) = 2f(x_0) + (\Delta x)^2 f''(x_0) + o(\Delta x)^4,$$

and we have

$$f''(x_0) \cong \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2} + o(\Delta x)^2$$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

Our main aim of the present study is to discuss some finite difference schemes for the numerical solutions of some parabolic equations. So, we are giving some finite difference schemes for parabolic equations in the following chapter.

1.5 Finite Difference Methods for Solving Parabolic Equations

We will discuss a few finite difference methods, their merits and demerits, for solving one-dimensional parabolic equation (1.2). Generally the equation is divided by the parameter k throughout which is absorbed in t. Hence the resulting equation is written in the normalised form as,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\ t \ge 0 \tag{1.12}$$

As can be seen, equation (1.12) is defined in the space domain $0 \le x \le L$. This domain can also be normalised varying from 0 to 1 by change of variable, if required. Let us suppose Dirichlet conditions are prescribed at both the ends x=0 and x=L, (values of u are given) and an initial condition is prescribed at time t=0 as given below:

$$u(0,t) = u_0 \qquad t > 0 \qquad (1.12(a))$$

$$u(L,t) = u_L \qquad t > 0 \qquad (1.12(b))$$

$$u(x,0) = f(x) \qquad t = 0 \qquad (1.12(c))$$

The domain of integration of the partial differential equation (1.12) or its solution domain is $D = [0 \le x \le L] \times [t \ge 0]$

Let us consider *x-t* plane such that x is represented by horizontal axis and time t by vertical axis. A rectangular mesh is formed in domain *D* by drawing lines parallel to the axes. We subdivide the interval $0 \le x \le L$ into, say N subintervals each of width Δx such that $N\Delta x = L$ and mark the points on the x-axis as x_i, $i = 0,1,...,N$ where $x_0 = 0$, $x_N = L$ *and* $x_i = i\Delta x$.

We draw lines through these points parallel to t-axis and also draw lines parallel to x-axis at distances $t = \Delta t = t_1$, $2\Delta t = t_2$, $j\Delta t = t_j$, etc.

In this way the domain D is subdivided into rectangular meshes. The points of intersection of these lines are called mesh points, grid points and pivotal or nodal points. We find the solution at these mesh points in a step by step manner in t-direction. That is, if $u_{i,j}$ denotes the value of u at the mesh point (i, j), then to start with we compute at $t = t_1 = \Delta t$ the values of u_{i1}, $i = 1,2,....,N-1$.

Once the values are known at $t = t_1$, the process may be repeated to get the values at $t = t_2 = 2\Delta t$, *i.e.* u_{i2}, $i = 1,2,....,N-1$.In general when the values of $u_{i,j}$ have been computed up to $t = t_j = j\Delta t$, *i.e. jth* time level, the values at the next time level $(j+1)th$, are computed to give $u_{i,j+1}$, $i = 1,2,....,N-1$. It may be noted that the values of u at boundaries $x_0 = 0$ and,

$x_N = L$ are known as $u_0 \, and \, u_L$, by virtue of prescribed boundary conditions (1.12(a)) and (1.12(b)) respectively and values of $u_{i,0}$ on account of initial condition (1.12(c)) as $u_{i0} = f(x_i), i = 0(1)N, t = 0$.

1.5.1 Explicit Scheme for Parabolic Partial Differential Equation

Consider a simple example of a parabolic (or diffusion) partial differential equation with one spatial independent variable

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.13}$$

The equivalent finite difference approximation is

$$\frac{u(i, j+1) - u(i, j)}{\Delta t} = \frac{u(i-1, j) - 2u(i, j) + u(i+1, j)}{(\Delta x)^2}.$$

where $x = i\Delta x, \; i = 1,2,3....,n, \; t = j\Delta t, \; j = 1,2....,N-1$, we use the forward difference formula for the derivative with respective to t and central difference formula for the with respect to x. If we let

$$r = \frac{\Delta t}{k(\Delta x)^2} \tag{1.14}$$

equation (1.13) could be written as

$$u(i, j+1) = ru(i+1, j) + (1-2r)u(i, j) + ru(i-1, j) \tag{1.15}$$

This explicit formula can be used to compute $u(x, t+\Delta t)$ explicitly in terms of $u(x, t)$. Thus the values of u along the first time row, t= Δt, can be calculated in terms of the boundary and initial conditions, then the values of u along the second row, $t = 2\Delta t$, are calculated in terms of the first time row, and so on.

1.5.2 Crank-Nicolson (C-N) Scheme for Parabolic Partial Differential Equation (Implicit Scheme)

The C-N scheme is also an implicit scheme. Instead of a mesh point, the discretization is made at the midpoint of jth and $(j+1)th$ levels, i.e. at $(x_i, t_{j+1/2})$. The time derivative as well as the space derivative in (1.12), both is approximated by central differences, i.e.

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=\frac{1}{2}\left[\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{\Delta x^2}+\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{\Delta x^2}\right]+o(\Delta t^2)+o(\Delta x^2) \qquad (1.16)$$

It may be noted that since the values of u are not available at the midpoint, the second derivative at the point $\left(i, j+\frac{1}{2}\right)$ is replaced by the average values at the (i, j) and $(i, j+1)$ points. It must also be noted that the truncation error in the C-N scheme is $o(\Delta t^2)+o(\Delta x^2)$.

while it is $o(\Delta t)+o(\Delta x^2)$ in both the methods, namely Explicit and Implicit, discussed earlier. Rearranging various terms in (1.11) and putting $r=\frac{\Delta t}{\Delta x^2}$, we get

$$-ru_{i+1,j+1}-ru_{i-1,j+1}+2(1+r)u_{i,j+1}=ru_{i+1,j}+ru_{i-1,j}+2(1-r)u_{i,j}+\Delta t[o(\Delta t^2)+o(\Delta x^2)]$$

Neglecting the error terms, the C-N scheme becomes

$$-ru_{i+1,j+1}-ru_{i-1,j+1}+2(1+r)u_{i,j+1}=ru_{i+1,j}+ru_{i-1,j}+2(1-r)u_{i,j} \qquad i=1,2,....,N-1 \qquad (1.17)$$

As may be seen that C-N scheme (1.15) is an implicit scheme. In order to find the value of u at the $(j+1)th$ level, we have to invoke it for i=1(1)N-1 which will result in N-1 equation in N-1 unknowns and their solution will give the value of u at the $(j+1)th$ time level. It may also be mentioned that the resulting system of equations is tridiagonal and can be very easily solved by Gaussian Elimination or some other standard method.

1.6 Stability of Explicit Scheme

In the Explicit scheme the values of u at the level of time $j+1$ are given explicitly in terms of the values of u at the previous level j, i.e.

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j} \quad , \quad i = 1,2,....,N-1 \tag{1.18}$$

Let us suppose that the values of u at the jth level are not the true values and have certain error in them. We have computed u at the $(j+1)th$ level using these values which will also have error in it. Let us denote the true value by u, approximate (actually computed) value by u^* and the associated error in u^* by e, so that the computed value can be expressed as,

$$u^*_{i,j+1} = ru^*_{i-1,j} + (1-2r)u^*_{i,j} + ru^*_{i+1,j} \tag{1.19}$$

Subtracting (1.19) from (1.18), we get

$$e_{i,j+1} = re_{i-1,j} + (1-2r)e_{i,j} + re_{i+1,j} \ , \ i = 1,2,...,N-1 \tag{1.20}$$

where $e_{i,j}$ denotes the error in the exact value u and the computed value u^* at the (i,j) mesh points. It may be remembered that $u_{0,j+1} = u_0$ and $u_{N,j+1} = u_L$ on account of the boundary conditions and their value are not being computed at any time. Hence there is no question of error in the values of u at the endpoints so that $e_{0,j} = e_{N,j} = 0$. We can now write (1.20) in matrix form as follows:

$$\begin{bmatrix} e_{1,j+1} \\ e_{2,j+1} \\ e_{3,j+1} \\ ..... \\ ..... \\ e_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & 0 & ..... & 0 \\ r & 1-2r & r & 0 & ..... & 0 \\ 0 & r & 1-2r & r & ..... & 0 \\ & & ..... & & & \\ & & & ..... & & \\ & & & & r & 1-2r \end{bmatrix} \begin{bmatrix} e_{1,j} \\ e_{2,j} \\ e_{3,j} \\ ..... \\ ..... \\ e_{N-1,j} \end{bmatrix} \tag{1.21}$$

or $\quad e_{j+1} = Ae_j \tag{1.22}$

where

$$e^T{}_{j+1} = \begin{pmatrix} e_{1,j+1} & e_{2,j+1} & ..... & e_{N-1,j+1} \end{pmatrix} \tag{1.22a}$$

$$e^T{}_j = \begin{pmatrix} e_{1,j} & e_{2,j} & ..... & e_{N-1,j} \end{pmatrix} \tag{1.22b}$$

and

$$A=\begin{bmatrix} 1-2r & r & 0 & 0 & ..... & 0 \\ r & 1-2r & r & 0 & ..... & 0 \\ 0 & r & 1-2r & r & ..... & 0 \\ & & ..... & & & \\ & & & ..... & & \\ & & & & r & 1-2r \end{bmatrix} \tag{1.22c}$$

Then according to formula (1.22) these error will become Ae_0 at the next step ignoring the computing error introduced at that step, so that $e_1 = Ae_0$. Then at the following step the contribution of e_1 will be $e_2 = Ae_1 = A^2 e_0$. Proceeding in the same manner, after k step, the error e_0 will contribute to overall error

$$e_k = A^k e_0, \quad k = 1,2,..... \tag{1.23}$$

It should be noted that local computational error has been neglected at each time level. We observe from (1.23) that e_k is dependent of A^k. If elements of matrix A become smaller and smaller tending to zero, or remain bounded, as k becomes infinitely larger than the scheme will be stable. On the other hand, if they increase, then the scheme will be unstable.

Let us suppose that the matrix $A\{(N-1)\times(N-1)\}$ possesses eigen value λ_s with corresponding eigen vectors $v_s, s=1,2....,N-1$ and that they are all distinct. They satisfy the equation,

$$Av = \lambda v \qquad or \qquad Av_s = \lambda_s v_s, \qquad s = 1,2,....,N-1 \tag{1.24}$$

We can express the error vector e_0 by a linear combination of these vectors, say

$$e_0 = \sum_{s=1}^{N-1} c_s v_s \tag{1.25}$$

where $v^T{}_s = (v_{1s} \quad v_{2s} \quad ..... \quad v_{N-1,s})$ (1.26)

It may be noted that the $(N-1)$ coefficient, c_s can be uniquely determined from the system of equation (1.25). From (1.25) we have,

$$A^k e_0 = A^{k-1} \sum_{s=1}^{N-1} c_s A v_s \text{ , from (1.25)}$$

$$= A^{k-1} \sum c_s \lambda_s v_s \text{ , from (1.24)}$$

$$= \sum_{s=1}^{N-1} c_s \lambda_s^k v_s \tag{1.27}$$

The terms $A^k e_0$ on the left side (1.27) denotes the propagation of error e_0 at the kth step as shown in (1.23). if this error is not to increase indefinitely with k, then we must have

$$|\lambda_s| \le 1,\ s = 1,2,...,N-1$$

0r $$|\lambda_{\max}| \le 1$$

Thus for the Explicit scheme to be stable, the modulus of the largest eigen value of matrix A, given by (1.22c), should not exceed unity.

1.7 Stability of Implicit Scheme

We see that Crank-Nicolson Scheme (1.17) is linear in u values defined at the mesh points. Hence the error will also follow the same formula. Changing subscripts i and j by p and q respectively, the error formula may be written as

$$-re_{p-1,q+1} + 2(1+r)e_{p,q+1} - re_{p+1,q+1} = re_{p-1,q} + 2(1-r)e_{p,q} + re_{p+1,q} \qquad (1.28)$$

Writing in matrix form remembering that there is no error at $x=0$ and $x=L$

$$Pe_{q+1} = Qe_q$$

or $$e_{q+1} = p^{-1}Qe_q \qquad (1.29)$$

where

$$P = \begin{bmatrix} 2(1+r) & -r & 0 & ..... & 0 \\ -r & 2(1+r) & -r & ..... & 0 \\ & & ..... & & \\ & & ..... & & \\ 0 & 0 & -r & & 2(1+r) \end{bmatrix}, \quad Q = \begin{bmatrix} 2(1-r) & r & 0 & ..... & 0 \\ r & 2(1-r) & r & ..... & 0 \\ & & ..... & & \\ & & ..... & & \\ 0 & 0 & r & & 2(1-r) \end{bmatrix}$$

$$e_{q+1}^T = \begin{pmatrix} e_{1,q+1} & e_{2,q+1} & ..... & e_{N-1,q+1} \end{pmatrix}$$

$$e_q^T = \begin{pmatrix} e_{1,q} & e_{2,q} & ..... & e_{N-1,q} \end{pmatrix}$$

For stability the eigen values of $p^{-1}Q$ should be less than or equal to one in modulus.

Let us define a tridiagonal matrix T as

$$T=\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdots\cdots & & & \\ & & \cdots\cdots & & \\ & & & -1 & 2 \end{bmatrix}$$

Then we can write (1.29) as

$$e_{q+1}=(2I+rT)^{-1}(2I-rT)e_q$$

$$= Se_q \quad \text{where matrix} \quad S=(2I+rT)^{-1}(2I-rT)$$

If μ is an eigen value of T then the eigen value λ of matrix S is given by $\lambda=\frac{2-\mu r}{2+\mu r}$. For stability $|\lambda|\le 1$, so that $-1\le\frac{2-\mu r}{2+\mu r}\le 1$ which implies that $\mu\ge 0$. Using Brauer's theorem on matrix T, we see that $-2\le\mu-2\le 2$ *or* $0\le\mu\le 4$. Hence, C-N scheme is stable for all values of r, i.e. unconditionally stable.

Alternatively, we can also use the fact that the eigen values of matrix T is given by

$$\mu_s=4\sin^2\left(\frac{s\pi}{2N}\right),\quad s=1,2,...,N-1$$

Hence, $$\lambda_s=\frac{1-2r\sin^2\left(s\pi/2N\right)}{1+2r\sin^2\left(s\pi/2N\right)}$$

Obviously $|\lambda_s|$ will always be less than 1 .

1.8 Organisation of Thesis

In this thesis an attempt has been made to solve some parabolic equations by using some finite differences methods. The chapter wise summary of the thesis is as follows.

In chapter 2, we consider one-dimensional convection-diffusion parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=D\frac{\partial^2 u}{\partial x^2}\ ,\qquad 0<x<L,\ 0<t<T$$

The convection–diffusion equation is a parabolic partial differential equation, which describes physical phenomena where energy is transformed inside a physical system due to two processes: convection and diffusion. The term convection means the movement of molecules within fluids, whereas, diffusion describes the spread of particles through random motion from regions of higher concentration to regions of lower concentration. In this chapter we have developed some finite difference schemes based on weighted average for solving the one dimensional advection–diffusion equation with constant coefficients. These techniques are based on the two-level finite difference approximation. By changing the values of weighed parameter θ, we obtained the Forward Time Cantered Space (FTSC) , Upwind scheme, Lax-Wendroff and Crank-Nicolson schemes. In order to check the accuracy of proposed methods three test examples are considered with analytical solution available in literature. The examples are solved by all four schemes and compared each other. It has been concluded that the Lax-Wendroff scheme is in good agreement with the analytical solution as compare to the other schemes.

In chapter 3, we consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}, \qquad (x,t) \in \Omega$$

The nonlinear partial differential equation is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modelling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman [21] in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger [23,24] use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc

In this chapter, we present a combined numerical scheme based on Hopf-Cole transformation and Crank-Nicolson finite difference method for the numerical solutions of one dimensional Burgers' equation. The scheme has shown to be unconditionally stable and is second order accurate in space and time. The advantage of the proposed method is that there is no restriction in choosing mesh sizes. In support of the predicted theory, the two test examples have been considered and solved analytically by using Hopf-Cole transformation. The figures are plotted to show the physical phenomenon of the given problem.

Chapter 2

Weighted Finite Difference Techniques for the Numerical Solutions of Advection-Diffusion Equation

2.1 Introduction

Problems of environmental pollution (for rivers, coasts, groundwater, and the atmosphere) can be reduced to the solution of a mathematical model of diffusion-dispersion. The mathematical model describing the transport and diffusion processes is the one-dimensional advection-diffusion equation (ADE). Mathematical modeling of heat transport, pollutants, and suspended matter in water and soil involves the numerical solution of a convection-diffusion equation. The convection-diffusion equation is a parabolic partial differential equation, which describes physical phenomena where energy is transformed inside a physical system due to two processes: convection and diffusion. The term convection means the movement of molecules within fluids, whereas, diffusion describes the spread of particles through random motion from regions of higher concentration to regions of lower concentration. It is necessary to calculate the transport of fluid properties or trace constituent concentrations within a fluid for applications such as water quality modelling, air pollution, meteorology, oceanography and other physical sciences. When velocity field is complex, changing in time and transport process cannot be analytically calculated, and then numerical approximations to the convection equation are indispensable. Various numerical techniques have been developed and compared for solving the one dimensional convection-diffusion equation with constant coefficient [1, 9, 16, 17 and 18]. Most of these techniques are based on the two-level finite difference approximations. In [32] several different numerical techniques will be developed and compared for solving the three-dimensional advection-diffusion equation with constant coefficient. These techniques are based on the two-level fully explicit and fully implicit finite difference approximations.

In [33] new classes of high-order accurate methods have developed for solving the two-dimensional unsteady convection-diffusion equation based on the method of lines approach.

In [34] a new practical scheme designing approach has presented whose application is based on the modified equivalent partial differential equation (MEPDE). These methods are second-order accurate and techniques that are third order or fourth order accurate. In [35] a variety of explicit and implicit algorithms has been studied dealing with the solution of the one dimensional advection equation. These schemes are based on the weighted finite difference approximations. In [36] several finite difference schemes are discussed for solving the two-dimensional Schrodinger equation with Dirichlet's boundary conditions. In [37] several different computational LOD procedures were developed and discussed for solving the two-dimensional transport equation. These schemes are based on the time-splitting finite difference approximations in [38] the solution of Cauchy reaction-diffusion problem is presented by means of variational iteration method. The main object of this study is to develop a user friendly, economical and stable method which can work for higher values of peclet number for convection-diffusion equation by using redefined cubic B-splines collocation method.

One of the best tools for solving the ADE is spreadsheet. There are many advantages of spreadsheets such as having numerical and visual feedback, fast calculating capabilities. One of the most advantages of spreadsheet is its graphical interface. The solution obtained through the spreadsheet can easily be plotted at the same worksheet. Any changes in the input parameters of the solution domain will be directly reflected to the graphical representation of the solutions. Spreadsheets are user-friendly easy to programmer Spreadsheets have an increasing popularity in engineering problems. Several studies have been carried out using spreadsheets for the last 10 years. The application of them is carried out in different fields of engineering problems such as in the solutions of partial differential equations [39], one-dimensional transient heat-conduction problems [54], free-surface seepage problems [22], steady-state groundwater applications [50,40], transient groundwater applications [19], and the groundwater parameter estimation[20].

In this chapter, we have proposed a numerical scheme based on weighted finite difference techniques for the numerical solutions of advection-diffusion equation. By changing only the weighting parameter in the proposed scheme, we obtained four numerical schemes explicit, Crank-Nicholson, implicit and Lax-Wendroff. In order to test the accuracy of the schemes three test problems have been considered and a comparison a made between the proposed schemes.

2.2 Mathematical Model of Advection-Diffusion Equation

The mathematical model is given as:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < L, 0 < t \le T \tag{2.1}$$

With initial conditions

$$u(x,0) = f(x) \quad , \quad 0 \le x \le L \tag{2.2}$$

and boundary conditions

$$u(0,t) = g(t) \ , \ 0 < t \le T \tag{2.3}$$

$$u(L,t) = h(t) \ , \ 0 < t \le T \tag{2.4}$$

where f, g and h are known functions, while the function c is unknown, *c* is the velocity in *x* direction and *D* is the dispersion coefficient. Note that c and D are considered to the positive constant values.

Little progress has been made so far to solve the one-dimensional ADE with analytical methods when c and D constant. So much effort has been put into developing stable and accurate numerical solutions of equation (2.1). In the present study, various weighted finite difference techniques have been simulated using spreadsheets.

2.3 Numerical Solution of Advection-Diffusion Equation by Finite Difference Method

The solution of the problem is covered by a mesh of grid-lines

$$x_i = i\Delta x, \qquad i = 0,1,...,M$$

$$t_n = n\Delta t, \qquad n = 0,1,...,N$$

and they are parallel to the space and time coordinate axes. The constant spatial and temporal grid-spacing are $\Delta x = \frac{L}{M}$ and $\Delta t = \frac{T}{N}$

Consider the following approximations of the derivatives in the ADE which incorporate a weight θ as follows:

$$\frac{\partial u}{\partial t} = \frac{u(i,n+1) - u(i,n)}{\Delta t} \tag{2.5}$$

$$c\frac{\partial u}{\partial x} = c\left\{\theta\left[\frac{u(i,n) - u(i-1,n)}{\Delta x}\right]\right\} + (1-\theta)\left[\frac{u(i+1,n) - u(i-1,n)}{2\Delta x}\right] \tag{2.6}$$

$$D\frac{\partial^2 u}{\partial x^2} = D\frac{u(i-1,n) - 2u(i,n) + u(i+1,n)}{(\Delta x)^2} \tag{2.7}$$

Put the value in equation (2.1)

$$\frac{u(i,n+1) - u(i,n)}{\Delta t} + c\left\{\theta\left[\frac{u(i,n) - u(i-1,n)}{\Delta x}\right]\right\} + (1-\theta)\left[\frac{u(i+1,n) - u(i-1,n)}{2\Delta x}\right] = D\frac{u(i-1,n) - 2u(i,n) + u(i+1,n)}{(\Delta x)^2}$$

$$u(i,n+1) = -c\Delta t\left\{\theta\left[\frac{u(i,n) - u(i-1,n)}{\Delta x}\right] + (1-\theta)\left[\frac{u(i+1,n) - u(i-1,n)}{2\Delta x}\right]\right\} + D\Delta t\frac{u(i-1,n) - 2u(i,n) + u(i+1,n)}{(\Delta x)^2} + u(i,n)$$

$$u(i,n+1) = \frac{c\Delta t\theta u(i-1,n)}{\Delta x} + \frac{c(1-\theta)\Delta t u(i-1,n)}{2\Delta x} + \frac{D\Delta t u(i-1,n)}{(\Delta x)^2} - \frac{c\Delta t\theta u(i,n)}{\Delta x} - 2\frac{D\Delta t u(i,n)}{(\Delta x)^2} + u(i,n) - \frac{c(1-\theta)\Delta t u(i+1,n)}{2\Delta x} + \frac{D\Delta t u(i+1,n)}{(\Delta x)^2}$$

$$u(i,n+1) = \frac{\Delta t}{\Delta x}u(i-1,n)\left[c\theta + \frac{c(1-\theta)}{2} + \frac{D}{(\Delta x)}\right] + u(i,n)\left[1 - \frac{2D\Delta t}{(\Delta x)^2} - \frac{c\Delta t\theta}{\Delta x}\right] u(i+1,n)\left[\frac{-c(1-\theta)\Delta t}{\Delta x} + \frac{D\Delta t}{(\Delta x)^2}\right]$$

$$u(i,n+1)=\left[\frac{(1+\theta)c\Delta t}{2\Delta x}+\frac{D\Delta t}{(\Delta x)^2}\right]u(i-1,n)+\left[1-\frac{2D\Delta t}{(\Delta x)^2}-\frac{c\Delta t\theta}{\Delta x}\right]u(i,n)$$
$$+\left[-\frac{c(1-\theta)\Delta t}{\Delta x}+\frac{D\Delta t}{(\Delta x)^2}\right]u(i+1,n)$$

where $cr=\frac{c\Delta t}{\Delta x}$ is the courant number and $pe=\frac{c\Delta x}{D}$ is the peclet number.

$$u(i,n+1)=\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)+(1+\theta)cr\right]u(i-1,n)+\left[1-2\left(\frac{cr}{pe}\right)-cr\theta\right]u(i,n)+\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)-cr(1-\theta)\right]u(i+1,n) \quad (2.8)$$

2.3.1 Upwind Explicit Scheme

Assume that $\theta=0$ in Equation (2.8) and it may be written as the following upwind explicit-type finite difference formula in solving the ADE.

$$u(i,n+1)=\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)+cr\right]u(i-1,n)+\left[1-2\left(\frac{cr}{pe}\right)\right]u(i,n)+\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)-cr\right]u(i+1,n)$$

2.3.2 The FTCS (Forward Time Cantered Space)-Type Scheme

If assuming $\theta=1$ in Equation (2.8) may be written the following FTCS-type finite difference formula to solve the ADE.

$$u(i,n+1)=\left[\left(\frac{cr}{pe}\right)+cr\right]u(i-1,n)+\left[1-2\left(\frac{cr}{pe}\right)-cr\right]u(i,n)+\left[\left(\frac{cr}{pe}\right)\right]u(i+1,n)$$

2.3.3 Crank Nicholson Scheme

Assuming $\theta=\frac{1}{2}$ in Equation (2.8) yields the following

$$u(i,n+1)=\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)+\frac{3}{2}cr\right]u(i-1,n)+\left[1-2\left(\frac{cr}{pe}\right)-\frac{1}{2}cr\right]u(i,n)+\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)-\frac{1}{2}cr\right]u(i+1,n)$$

2.3.4 The Lax-Wendroff Scheme

Putting $\theta = cr$ in Equation (2.8) yields the following

$$u(i,n+1)=\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)+cr+(cr)^2\right]u(i-1,n)+\left[1-2\left(\frac{cr}{pe}\right)-(cr)^2\right]u(i,n)$$
$$+\frac{1}{2}\left[2\left(\frac{cr}{pe}\right)-cr+(cr)^2\right]u(i+1,n)$$

2.4 Numerical Experiments

In order to check the accuracy of the proposed schemes three test examples have been considered. The examples are chosen such that their exact solutions are known and already discussed in literature. In each example, absolute errors are computed by the following formulas

$$Absolute\ Eroor=\left|u_{ij}-\bar{u}_{ij}\right|$$

where u_{ij} and $\bar{u}_{ij}$ denote the exact and numerical solution of the problem.

Example 1:

A problem for which the exact solution is known is considered to test the accuracy of the proposed methods described in above subsections for solving the advection-diffusion equation. These techniques are applied to solve (2.1)-(2.4) with *g(t), h(t),* and *f(x)* known and *u* unknown. In this example, we have considered the advection-diffusion equation (2.1)-(2.4) with initial and boundary conditons

$$f(x)=\exp\left(-\frac{(x+0.5)^2}{0.00125}\right)$$

$$g(0,t)=\frac{0.025}{\sqrt{0.000625+0.02t}}\exp\left(-\frac{(0.5-t)^2}{(0.00125+0.04t)}\right)$$

$$g(1,t)=\frac{0.025}{\sqrt{0.000625+0.02t}}\exp\left(-\frac{(1.5-t)^2}{(0.00125+0.04t)}\right)$$

D=0.01, C=1.0,

The analytical solution to the one-dimensional advection-diffusion in a region bounded by $0 \le x \le 1$ is taken from Ref. [41] and given as:

$$u(x,t)=\frac{0.025}{\sqrt{0.000625+0.02t}}\exp\left(-\frac{(x+0.5-t)^2}{(0.00125+0.04t)}\right)$$

In this example the values of the various parameters are chosen D=0.01, m^2/s ,C=1 m/s with time step Δt = 0.0001. The problem model has been solved for the different θ values and the results have been shown in Table 1 and Figures 1-8. The Table show the maximum absolute error of the problem. The Figures compare the exact and numerical solutions at different time and θ. As can be seen in Table 1, the Lax-Wendroff Scheme gives the best result. It can be seen from the Table that the maximum absolute errors for the scheme Lax-Wendroff Scheme are smaller than the other.

Example 2:

A problem for which the exact solution is known is considered to test the accuracy of the proposed methods described for solving the advection-diffusion equation. These techniques are applied to solve (2.1)-(2.4) with *g(t), h(t),* and *f(x)* known and *u* unknown. The following initial and boundary conditions are considered

$$f(x)=\exp\left(-\frac{(x-b)^2}{D}\right)$$

$$g(0,t)=\frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(-b-ct)^2}{D(4t+1)}\right)$$

$$g(9,t)=\frac{1}{\sqrt{4t+1}}\exp\left(-\frac{(9-b-ct)^2}{D(4t+1)}\right)$$

The analytical solution to the one-dimensional advection-diffusion of a Gaussian pulse of unit height, centred at $x = 1$ in a region bounded by $0 \le x \le 9$ as given Noye and Tan [1] is

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x-b-ct)^2}{D(4t+1)} \right)$$

where c is the velocity in the x direction, b is the center of the initial Gaussian pulse, D is the diffusion coefficient in the x direction, and t is the time coordinate. The values of the various parameters used are D=0.01, m^2/s , c =1 m/s and the Gaussian pulse of unit height centered at x =1. The numerical results of the example are given in Table 2 and Figures 9-16. The Table show the maximum absolute error of the problem. The Figures compare the exact and numerical solutions at different time and θ.

However, when the problem is solved with Lax-Wendrof scheme the numerical results are good in agreement with the analytical solution.

Example 3:

In this example, we have considered the advection-diffusion equation (2.1) – (2.4) with the following initial and boundary

$$f(x) = \frac{1}{\sqrt{s}} \exp\left(-50 \frac{x^2}{s} \right)$$

$$g(0,t) = \frac{1}{\sqrt{s}} \exp\left(-50 \frac{(-t)^2}{s} \right)$$

$$g(1,t) = \frac{1}{\sqrt{s}} \exp\left(-50 \frac{(1-t)^2}{s} \right)$$

The exact solution of the problem is given by

$$u(x,t) = \frac{1}{\sqrt{s}} \exp\left(-50 \frac{(x-t)^2}{s} \right)$$

The values of the various parameters are D=0.01, m^2/s, C=1 m/s. The grid space and time step are taken to be Δx = 0.0101 and Δt = 0.00001 respectively. The numerical results have

been shown in Table 3 in form of maximum absolute errors and in Figures 17-24. The Figures compare the exact and numerical solutions at different time and θ.

However, when the problem is solved with Lax-Wendrof scheme the numerical results are good in agreement with the analytical solution.

Table 1: Maximum absolute error of Example 1 for different scheme at different time *t.*

Maximum Absolute error					
Δt	t	FTCS	Upwind Explicit	Crank Nicholson	Lax-Wendroff
0.0001	0 .5	4.4606×10^{-4}	8.1000×10^{-3}	4.4000×10^{-3}	4.333×10^{-4}
0.0001	0.75	5.7627×10^{-4}	1.3100×10^{-3}	7.1000×10^{-3}	5.6719×10^{-4}
0.0001	1.0	6.1156×10^{-4}	1.7000×10^{-3}	9.3000×10^{-3}	6.0400×10^{-4}
0.0001	2.0	1.5217×10^{-4}	2.700×10^{-3}	1.4000×10^{-3}	1.4912×10^{-4}

Table 2: Maximum absolute error of Example 2 for different scheme at different time *t.*

Maximum Absolute error					
Δt	t	FTCS	Upwind explicit	Lax Wendroff	Crank Nicholson
0.0001	1	1.489×10^{-2}	2.397×10^{-2}	1.488×10^{-2}	1.870×10^{-2}
0.0001	2	1.067×10^{-2}	1.849×10^{-2}	1.067×10^{-2}	1.442×10^{-2}
0.0001	4	6.600×10^{-2}	1.370×10^{-2}	6.600×10^{-2}	1.067×10^{-2}
0.0001	6	4.770×10^{-2}	1.137×10^{-2}	4.770×10^{-2}	8.850×10^{-2}

Table 3: Maximum absolute error of Example 3 for different scheme at different time *t.*

Maximum Absolute error					
Δt	t	FTCS	Upwind explicit	Lax Wendroff	Crank Nicholson
0.0001	0 .5	2.85000×10^{-3}	4.03000×10^{-3}	2.86000×10^{-3}	3.46000×10^{-3}
0.0001	0.75	5.04000×10^{-3}	6.71000×10^{-3}	5.04000×10^{-3}	5.91000×10^{-3}
0.0001	1.00	4.13000×10^{-3}	5.75000×10^{-3}	4.13000×10^{-3}	4.97000×10^{-3}
0.0001	2.00	5.0874×10^{-4}	1.0000×10^{-3}	5.0914×10^{-4}	7.5138×10^{-4}

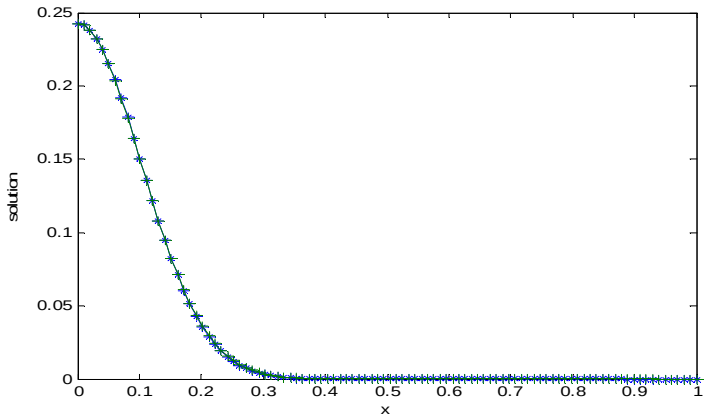

Figure 1: Comparison of exact and numerical solutions of Example 1 for $\theta = 0$ (explicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t=0.5.

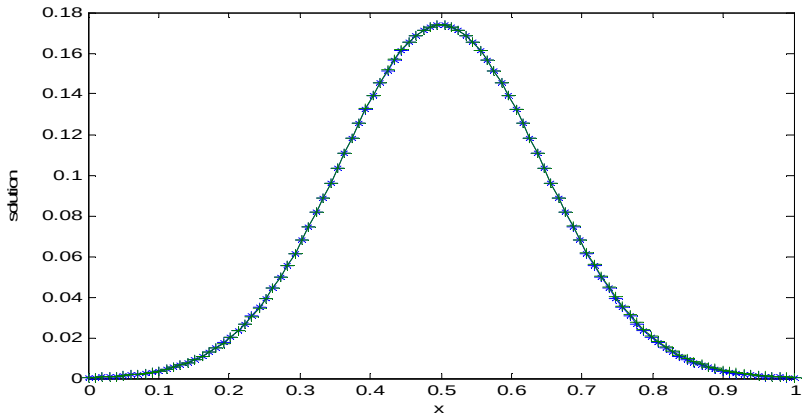

Figure 2: Comparison of exact and numerical solutions of Example 1 for $\theta = 0$ (explicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t=1.0

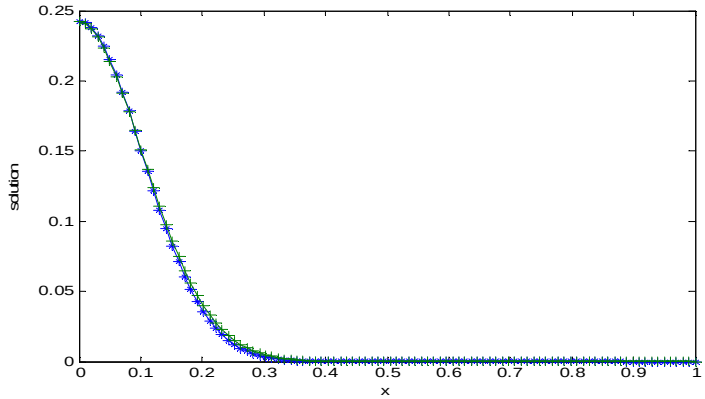

Figure 3: Comparison of exact and numerical solutions of Example 1 for $\theta = 1/2$ (Crank-Nicholson), N=100, D=0.01, C=1 Δt =0.0001 at t=.5

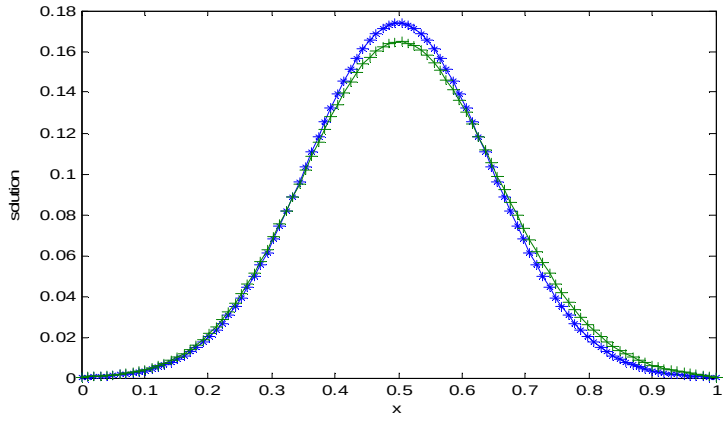

Figure 4: Comparison of exact and numerical solutions of Example 1 for $\theta = 1/2$ (Crank-Nicholson), N=100, D=0.01, C=1 Δt =0.0001 at t=1

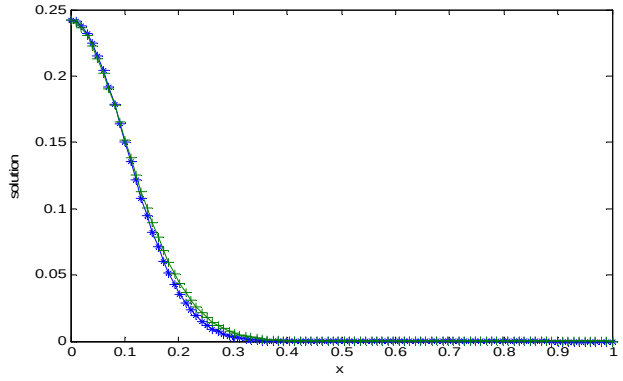

Figure 5: Comparison of exact and numerical solutions of Example 1 for $\theta = 1$(Implicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t=.5

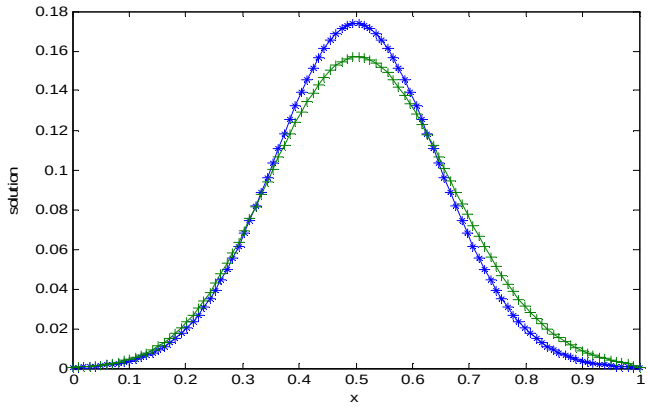

Figure 6: Comparison of exact and numerical solutions of Example 1 for $\theta = 1$(Implicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t=1

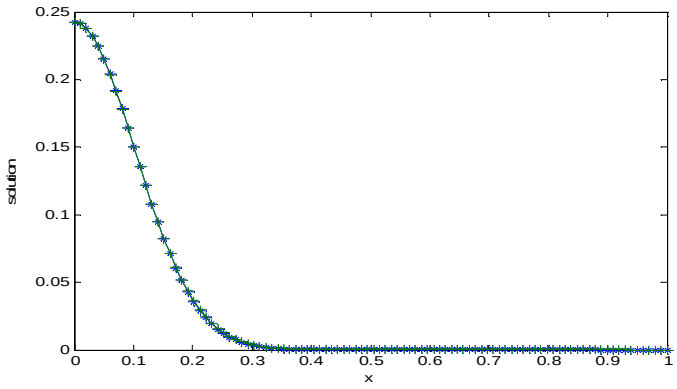

Figure 7: Comparison of exact and numerical solutions of Example 1 for $\theta = cr$ (Lax-Wendroff), N=100, D=0.01, C=1 Δt =0.0001 at t=.5

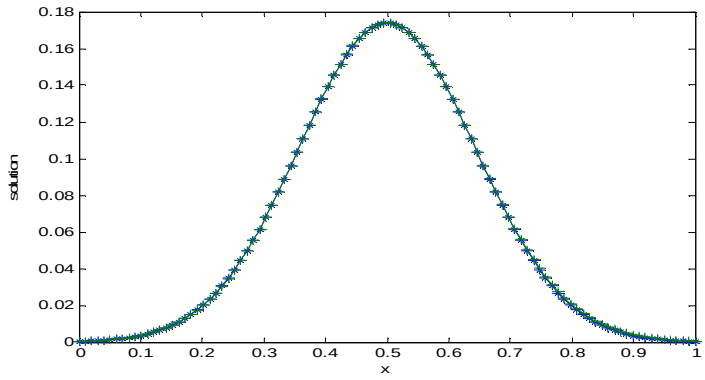

Figure 8: Comparison of exact and numerical solutions of Example 1 for $\theta = cr$ (Lax – Wendroff), N=100, D=0.01, C=1, Δt =0.0001 at t=1

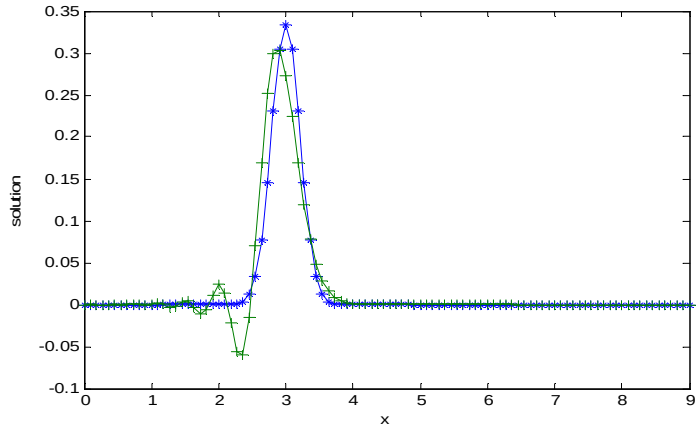

Figure 9: Comparison of exact and numerical solutions of Example 2 for $\theta = 0$ (Explicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t=2

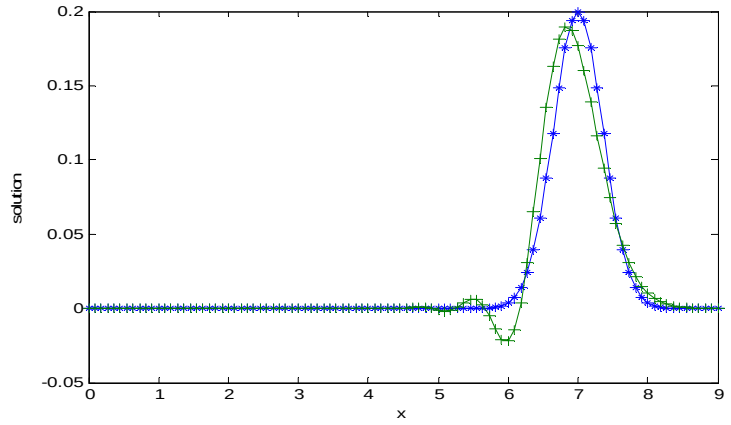

Figure 10: Comparison of exact and numerical solutions of Example 2 for $\theta = 0$ (Explicit scheme), N=100, D=0.01, C=1, Δt =0.0001 at t=6

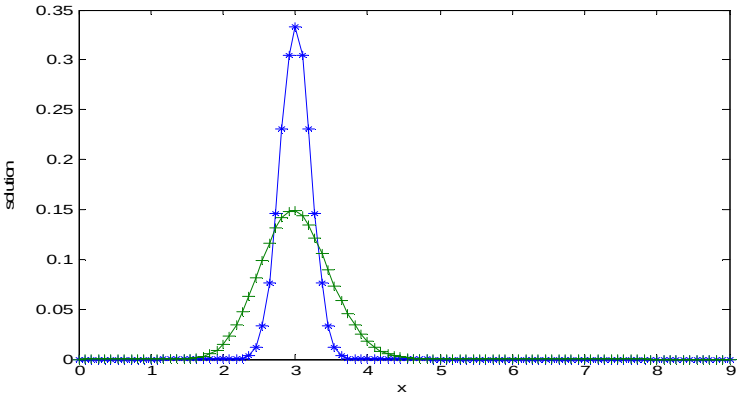

Figure 11: Comparison of exact and numerical solutions of Example 2 for $\theta = 1$ (Implicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t=2.

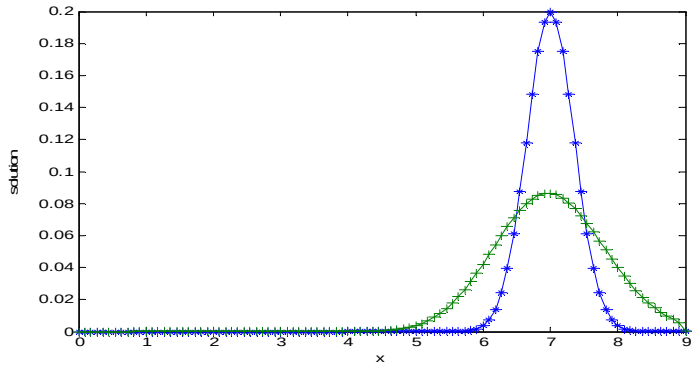

Figure 12: Comparison of exact and numerical solutions of Example 2 for $\theta = 1$ (Implicit scheme), N=100, D=0.01, C=1 Δt =0.0001 at t =6.

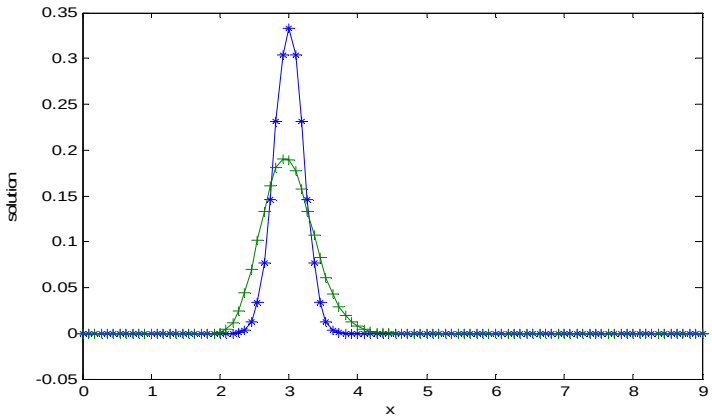

Figure 13: Comparison of exact and numerical solutions of Example 2 for $\theta = 1/2$ (Crank-Nicholson), N=100, D=0.01, C=1 Δt =0.0001 at t =2.

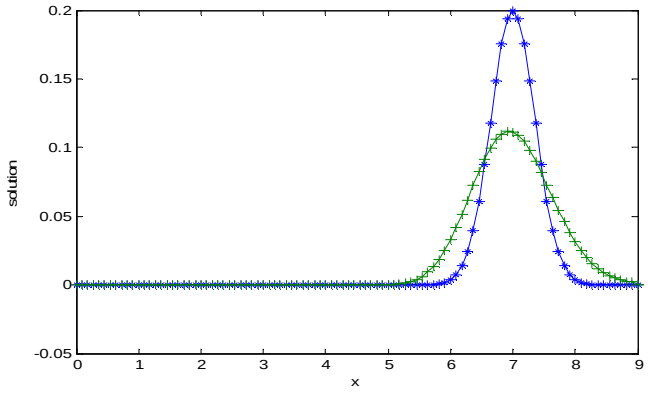

Figure 14: Comparison of exact and numerical solutions of Example 2 for $\theta = 1/2$ (Crank-Nicholson), N=100, D=0.01, C=1 Δt =0.0001 at t =6.

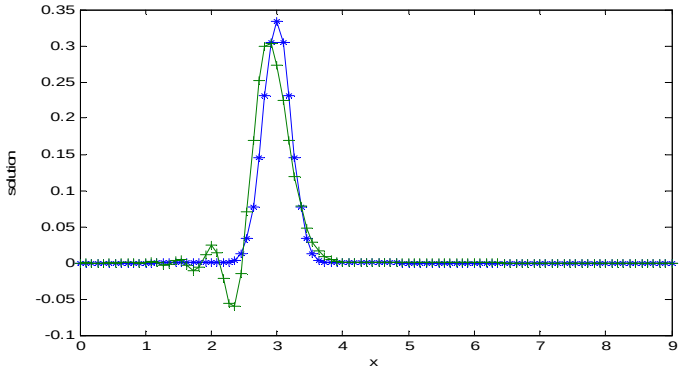

Figure 15: Comparison of exact and numerical solutions of Example 2 for $\theta = cr$ (Lax-Wendroff), N=100, D=0.01, C=1 Δt =0.0001 at t =2.

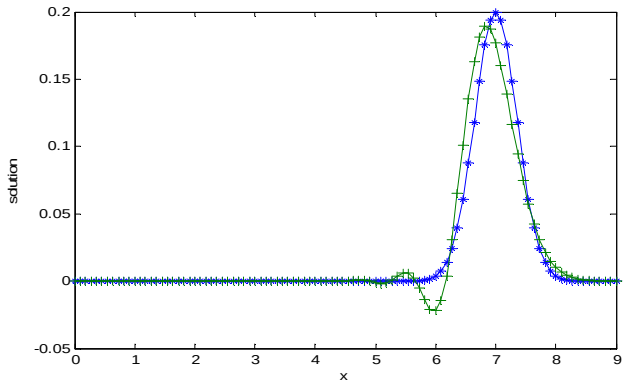

Figure 16: Comparison of exact and numerical solutions of Example 2 for $\theta = cr$ (Lax-Wendroff), N=100, D=0.01, C=1 Δt =0.0001 at t =6

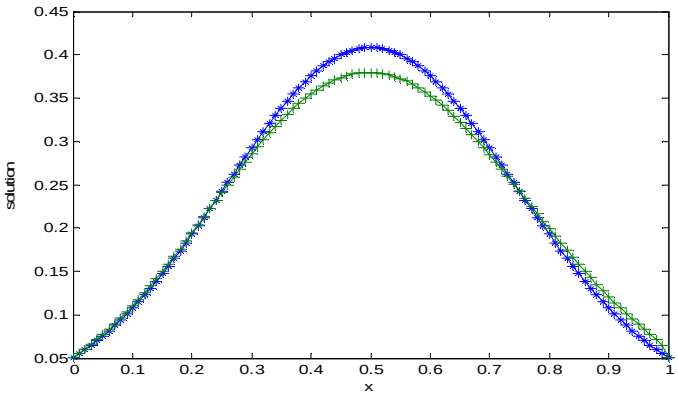

Figure 17: Comparison of exact and numerical solutions of Example 3 for $\theta = 0$ (Implicit scheme), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t = .5

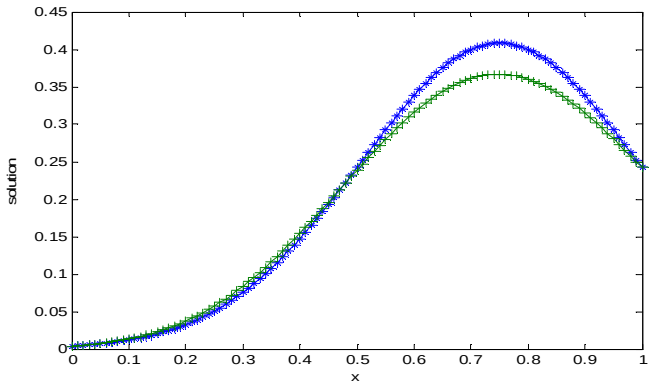

Figure18: Comparison of exact and numerical solutions of Example 3 for $\theta = 0$ (Implicit scheme), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t = .75.

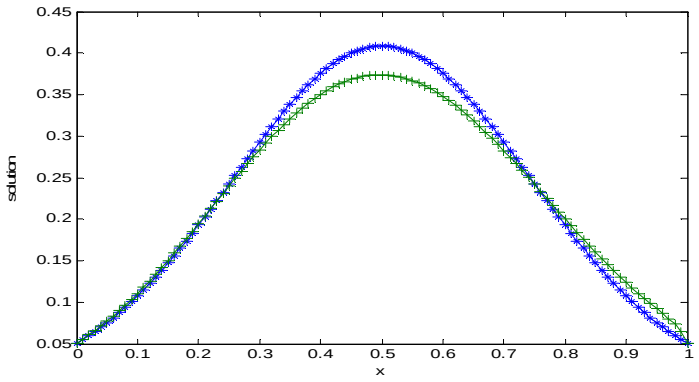

Figure 19: Comparison of exact and numerical solutions of Example 3 for $\theta = 1/2$ (Crank-Nicholson), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t =0.5.

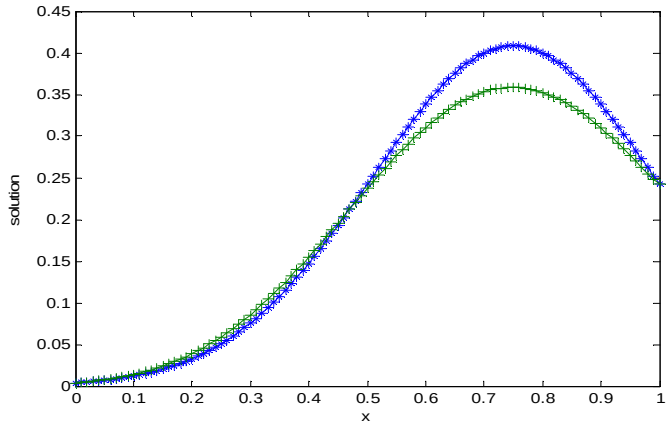

Figure 20: Comparison of exact and numerical solutions of Example 3 for $\theta = 1/2$ (Crank-Nicholson), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t =.75

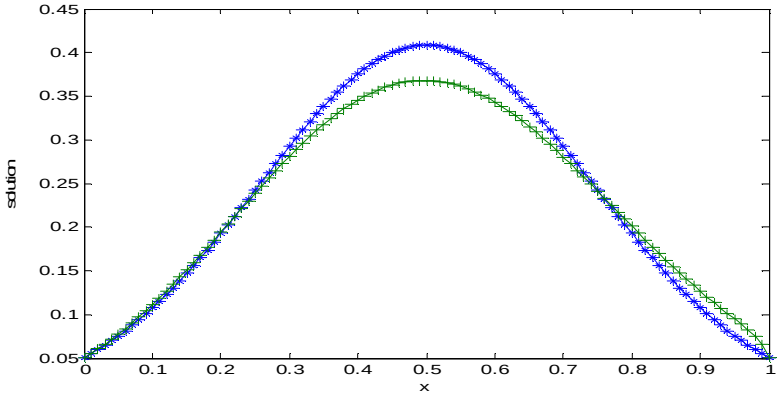

Figure 21: Comparison of exact and numerical solutions of Example 3 for $\theta = 1$(Explicit scheme), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t =.5.

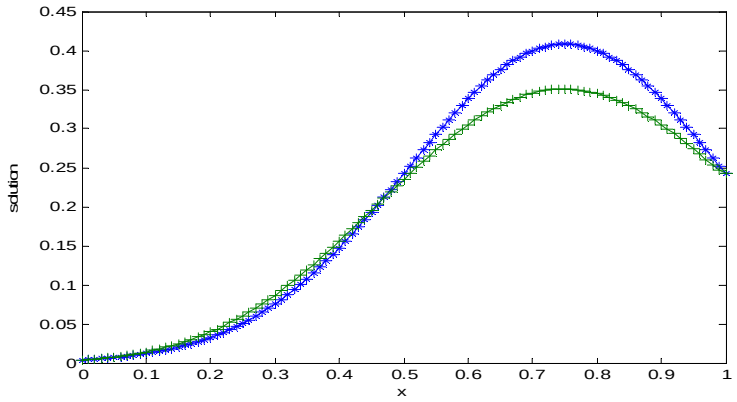

Figure 22: Comparison of exact and numerical solutions of Example 3 for $\theta = 1$ (Explicit scheme), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t =.75.

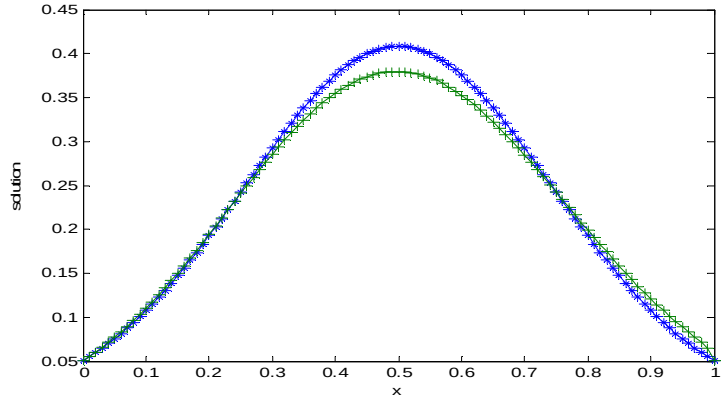

Figure 23: Comparison of exact and numerical solutions of Example 3 for $\theta = cr$ (Lax-Wendroff), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t =.5

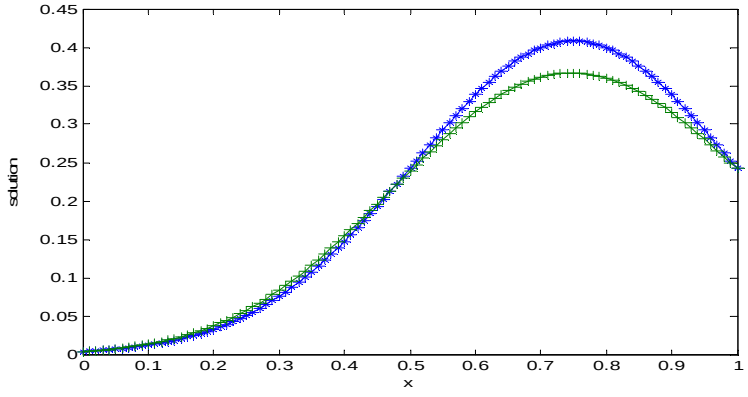

Figure 24: Comparison of exact and numerical solutions of Example 3 for $\theta = cr$ (Lax-Wendroff), N=101, D=0.01, C=1 , Δt =0.0001,s=6 at t =.75

Chapter 3

A Numerical Method Based on Crank-Nicoloson Scheme for Burgers' Equation

3.1 Introduction:

Consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} \qquad (x,t) \in \Omega \times [0,T) \tag{3.1}$$

where

$$\Omega = (0,1)\times(0,t]$$

with initial condition

$$u(x,0) = f(x) \qquad 0 < x < 1 \tag{3.2}$$

and boundary conditions

$$u(0,t) = g_1(t) \qquad 0 \le t \le T \tag{3.3}$$

$$u(1,t) = g_2(t) \qquad 0 \le t \le T \tag{3.4}$$

where Re is the Reynolds number and $f, g_1 \text{ and } g_2$ are the sufficiently smooth given functions.

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*This Chapter is the part of the research paper: M. K. Kadalbajoo and A. Awasthi, A numerical method based on Crank-Nicolson scheme for Burgers' equation, Applied Mathematics and Computation 182 (2006) 1430-1442.

The nonlinear partial differential equation (3.1) is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modelling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman [21] in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger [23, 24] use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc.

The Burgers equation (3.1) is one of the very few nonlinear partial differential equations which can be solved exactly for restricted set of initial functions f(x) only. The transformation

$$u = -\left(\frac{2}{\mathrm{Re}}\right)\frac{\phi_x}{\phi} \tag{3.5}$$

relates $u(x,t)\, and\, \phi(x,t)$ and if ϕ is a solution of the linear diffusion equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} \tag{3.6}$$

then u is a solution of the quasilinear Burgers equation (3.1). Conversely, u is a solution of (3.1) then ϕ from equation (3.5) is a solution of equation (3.6), apart from an arbitrary time-dependent multiplicative factor which is irrelevant in equation (3.5).

The transformation (3.5) appeared first in technical report by Lagerstrom et al. [43], and was published by Cole [25]. At about the same time it was given independently by Hopf [12]. Therefore, the transformation (3.5) is known as Hopf-Cole transformation [52]. Benton and Platzman [13] have been given 35 distinct analytic solutions of Burgers equation with different initial conditions.

Numerically Burgers' equation has been treated by many researchers. Event [10] introduced the group-Explicit method in order to solve Burgers equation. These methods are

semi-explicit and shown to be unconditionally stable and accurate of order $o\left(\Delta t+(\Delta x)^2+\frac{\Delta t}{\Delta x}\right)$. It is observed that his methods having sever consistency condition, i.e. methods are consistent if and only if $\frac{\Delta t}{\Delta x}\to 0, when\, \Delta t\to 0, \Delta x\to 0$.

Mittal and Singhal [44,45] have been given a spectral method using the finitely reproducing property of the non-linear operator in order to solve Burgers' equation with different boundary conditions. Kutluay et al. [49] has transformed this equation to linear heat equation with Neumann's boundary conditions by using Hopf-Cole transformation. The explicit finite difference and exact explicit finite difference methods see [49] are used to solve the transformed linear heat equation with Neumann boundary conditions. The explicit method and exact explicit methods have the severe stability condition and there by small size of time step to be taken. The study of the solution of Burgers' equation has been carried out for last half Century and still it is an active area of research to develop better numerical schemes to approximate its solution. In 1965, Hopf and Cole [53] propose a transformation known as Hopf-Cole transformation to solve the Burger' equation. Caldwell and Smith [26] use finite difference and cubic spline finite element methods to solve Burgers' equation. Evans et al. [11] introduce the group-explicit method and Kakuda et al. [27] propose a generalized boundary element approach to solve Burgers' equation. Ali et al. [2] use a cubic B-spline finite element method based on a collocation formulation to solve Burgers' equation. Gardner et al. [28] apply Petrov-Galerkin method with quadratic B-spline spatial finite elements and use a least squares technique using linear space-time finite elements [29]. In [51], Ozis and Ozdes generate a sequence of approximate solutions based on variational approach which converges to the exact solution. Recently, various numerical algorithms such as automatic differentiation method [3], novel numerical scheme [42], Quartic B-spline differential quadrature Method [4], modified cubic B-splines collocation method [46], numerical scheme based on differential quadrature method [47] have used for the numerical solution of the Burgers' equation.

In the present chapter, we use Crank-Nicoloson finite difference method on the transformed linear heat equation with Neumann boundary conditions. The stability of the method is analyzed and found to be unconditionally stable. The present method has the accuracy of second order in space and time. Several numerical experiments have been carried out and their results are presented to demonstrate the efficiency of the proposed method.

3.2 Hopf-Cole Transformation

Independently, Hopf [25] and Cole [12] noted remarkable result that (3.1) may be reduced to linear heat equation by the non-linear transformation given by the following Theorem.

Theorem 1. In the context with initial and boundary conditions of (3.1). If $\phi(x,t)$ is any solution to the heat equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{\mathrm{Re}} \frac{\partial^2 \phi}{\partial x^2} \tag{3.7}$$

then the nonlinear transformation[Hopf-Cole]

$$u = -2 \frac{1}{\mathrm{Re}} \frac{\phi_x}{\phi} \tag{3.8}$$

is a solution to (3.1)

proof: let $u = \dfrac{\partial \psi}{\partial x}, \qquad \psi = \psi(x,t)$

put these values in equation (3.1)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{\mathrm{Re}} \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial x}\right) + \left(\frac{\partial \psi}{\partial x}\right) \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{1}{\mathrm{Re}} \frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial x}\right)$$

and integrating w.r.t x

$$\int \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial x}\right) dx + \int \left(\frac{\partial \psi}{\partial x}\right) \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) dx = \frac{1}{\mathrm{Re}} \int \frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial x}\right) dx$$

$$\int \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial x}\right) dx + \frac{2}{2} \int \left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2}\right) dx = \frac{1}{\mathrm{Re}} \int \frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial x}\right) dx$$

$$\frac{\partial \psi}{\partial x} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 = \frac{1}{\mathrm{Re}} \frac{\partial^2 \psi}{\partial x^2}$$

Introduce $\psi = -2\frac{1}{\mathrm{Re}}\log\phi$, we have

$$\exp\left(-\frac{\mathrm{Re}}{2}\psi\right) = \phi$$

Let $u = \frac{\partial\psi}{\partial x}$ & integrating $\int_0^x u = \psi$

with initial condition

$$\phi(x,0) = \exp\left(-\frac{\mathrm{Re}}{2}\int_0^x u_0(\xi)d\xi\right) \quad , \quad 0 < x < 1$$

and boundary condition

$$\phi_x(0,t) = 0 = \phi_x(1,t), \qquad t \geq 0 \tag{3.9}$$

The above theorem shows that Burgers equation (3.1) can be reduced to a linear heat equation with Neumann boundary conditions (3.7).

3.3 Difference Scheme

The solution domain Ω of equation (3.1) is discretized with uniform mesh described as $\Omega = \{(x_j, t_j) : i = 0(1)N,\ j = 0(1)M\}$.Divide the interval [0,1] into N equal subintervals. Divide the interval [0,T] into M equal subintervals. Let $h = \frac{1}{N}$ be the mesh width in space and set $x_i = ih \ for \ i = 0,1,...,N.$ let $k = \frac{T}{M}$ be the mesh width in time and set $t_j = jk \ for \ K = 0,1,...,M.$

3.3.1 The Crank-Nicolson Finite Difference Method

The Crank-Nicolson finite difference discretization to linearized heat equation with Neumann boundary conditions is given by

$$\frac{\partial\phi}{\partial t} = \frac{1}{\mathrm{Re}}\frac{\partial^2\phi}{\partial x^2}$$

$$\left.\frac{\partial \phi}{\partial t}\right|_{i,j+1/2} = \frac{1}{\mathrm{Re}} \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j+1/2}$$

$$\frac{\partial}{\partial t}\phi(x_i, t_{j+1/2}) = c^2 \frac{\partial^2}{\partial x^2}\phi(x_i, t_{j+1/2})$$

using Central Difference formulae on both sides , we get..

$$\frac{\phi(x_i, t_{j+1}) - \phi(x_i, t_j)}{K} = \frac{1}{2\,\mathrm{Re}}\left(\frac{\partial^2}{\partial x^2}\phi(x_i, t_j) + \frac{\partial^2}{\partial x^2}\phi(x_i, t_{j+1})\right)$$

$$\phi(x_i, t_{j+1}) - \phi(x_i, t_j) = \frac{K}{2\,\mathrm{Re}}\left(\frac{\phi(x_{i+1}, t_j) + \phi(x_{i-1}, t_j) - 2\phi(x_i, t_j)}{h^2} + \frac{\phi(x_{i+1}, t_{j+1}) + \phi(x_{i-1}, t_{j+1}) - 2\phi(x_i, t_{j+1})}{h^2}\right)$$

where $r = \dfrac{K}{\mathrm{Re}h^2}$

$$2(\phi_{i,j+1} - \phi_{i,j}) = r(\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j+1} + \phi_{i-1,j+1} - 2\phi_{i,j+1})$$

$$-r\phi_{i+1,j+1} - r\phi_{i-1,j+1} + 2(1+r)\phi_{i,j+1} = r\phi_{i+1,j} + r\phi_{i-1,j} + 2(1-r)\phi_{i,j}$$

Dividing by 2 on both sides

$$\frac{-r}{2}\phi_{i+1,j+1} + (1+r)\phi_{i,j+1} - \frac{r}{2}\phi_{i-1,j+1} = \frac{r}{2}\phi_{i+1,j} + \frac{r}{2}\phi_{i-1,j} + (1-r)\phi_{i,j} \qquad i = 1(1)N-1 \qquad (3.10(a))$$

$$-r\phi_{i+1,j+1} + (1+r)\phi_{i,j+1} = r\phi_{i+1,j} + (1-r)\phi_{i,j} \qquad i = 0 \qquad (3.10(b))$$

$$-r\phi_{i-1,j+1} + (1+r)\phi_{i.j+1} = r\phi_{i-1,j} + (1-r)\phi_{i,j} \qquad i = N \qquad (3.10(c))$$

where $r = \dfrac{K}{\mathrm{Re}h^2}$, $j = 0(1)M$ *and* $\phi_{i,j}$ is the discrete approximation to $\phi(x_i, t_j)$ at the point (i,j).

The approximate solution of Burgers' equation (3.1) in terms of the approximate solution of heat (3.10), by using the Hopf-Cole transformation equation (3.8) is given by

$$u_{i,j} = -\left(\frac{2}{\mathrm{Re}}\right)\left\{\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2h\phi_{i,j}}\right\} = -\left(\frac{1}{\mathrm{Re}}\right)\left\{\frac{\phi_{i+1,j}-\phi_{i-1,j}}{h\phi_{i,j}}\right\}$$

3.4 Stability Analysis

We analyse the stability of linearized heat equation with Neumann boundary conditions by using matrix method. Writing equation (3.10) in matrix form, we have

$$\left(I-\frac{r}{2}A\right)\phi_{j+1} = \left(I+\frac{r}{2}A\right)\phi_j \tag{3.11}$$

where A is the matrix of order N+1, given by

$$\mathrm{A} = \begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \cdots\cdots\cdots & & & \\ & \cdots\cdots\cdots & & & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}$$

and $\phi_k (k = j, j+1)$ is a component vector given by

$$\phi_k = \begin{pmatrix} \phi_{0,k} \\ \phi_{1,k} \\ \cdots\cdots \\ \cdots\cdots \\ \phi_{N-1,k} \\ \phi_{N,k} \end{pmatrix}$$

with I as the identity matrix of order N+1 and $r = K/\mathrm{Re}h^2$, is the mesh ratio. write equation (3.11) in the form

$$\phi_{j+1} = \left(I-\frac{r}{2}A\right)^{-1}\left(I+\frac{r}{2}A\right)\phi_j \tag{3.12}$$

we introduce a diagonal matrix

$$D=\begin{pmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & ..... & & \\ & & ..... & & \\ & & & 1 & \\ & & & & \sqrt{2} \end{pmatrix}$$

In such a way so that A is similar to a symmetric matrix

$$\tilde{A}=D^{-1}AD$$

Let $B=\left(I-\frac{r}{2}A\right)^{-1}\left(I+\frac{r}{2}A\right)$ and (3.13)

$$\tilde{B}=D^{-1}BD=D^{-1}\left(I-\frac{r}{2}A\right)^{-1}\left(I+\frac{r}{2}A\right)D=\left[D^{-1}\left(I-\frac{r}{2}A\right)^{-1}D\right]\left[D^{-1}\left(I+\frac{r}{2}A\right)D\right]$$

$$=\left[D^{-1}\left(I-\frac{r}{2}A\right)D\right]^{-1}\left[D^{-1}\left(I+\frac{r}{2}A\right)D\right]=\left[\left(I-\frac{r}{2}\tilde{A}\right)\right]^{-1}\left[\left(I+\frac{r}{2}\tilde{A}\right)\right]$$

but matrices $\left(I-\frac{r}{2}\tilde{A}\right)^{-1}$ *and* $\left(I+\frac{r}{2}\tilde{A}\right)$ are symmetric and commute, and so B is symmetric. Therefore , B is similar to a symmetric matrix $\tilde{B}$. Hence , we have

$$p(B)\equiv p(\tilde{B})$$

where $p(B)$ is the spectral radius of B. The necessary and sufficient condition for stability is given by

$$p(B)\le 1 \qquad (3.14)$$

The spectral radius of the matrix B is given by

$$p(B)=\max|\mu_i|$$

where $\mu_i(i=0,1,...,N)$ are the eigen values of the matrix $\left(I-\frac{r}{2}A\right)^{-1}\left(I+\frac{r}{2}A\right)$

The values of $\mu_i's$ are given by

$$\mu_i = \frac{1+\frac{r}{2}\lambda_i}{1-\frac{r}{2}\lambda_i}, \qquad i = 0,1,2,....,N, \tag{3.15}$$

where λ_i $(i = 0,1,....,N)$ are the eigen values of the matrix A.

The condition of stability of the scheme (3.11) with the help of equation (3.14) and (3.15) is

$$\lambda_i \leq 0, \quad \forall i$$

The eigen values λ_j of the matrix A are given by

$$Det(A - \lambda I) = 0 \tag{3.16}$$

and simplification of equation (3.16) , we get

$$\left((\lambda+2)^2 - 4\right)Q_{N-1}(\lambda) = 0 \tag{3.17}$$

where

$$Q_N(\lambda) = \begin{vmatrix} -(2+\lambda) & 1 & & & \\ 1 & -(2+\lambda) & 1 & & \\ & ..... & & & \\ & & ..... & & \\ & & 1 & -(2+\lambda) & 1 \\ & & & 1 & -(2+\lambda) \end{vmatrix} \tag{3.18}$$

Using equation (3.18) in equation (3.17), we have

$$(-1)^{N-1}\left((\lambda+2)^2 - 4\right)\prod_i^{N-1}\left(4\cos^2\frac{\pi i}{2N} + \lambda\right) = 0$$

$$\lambda(\lambda+2)\prod_i^{N-1}\left(4\cos^2\frac{\pi i}{2N} + \lambda\right) = 0$$

which gives

$$\lambda = 0, \ or \ \lambda = -2 \ \ or$$

$$\lambda = -4\cos^2\frac{\pi i}{2N}, \quad i = 0,1,2,....,N.$$

Therefore , the method is unconditionally stable.

By using Taylor's series one can easily see that the proposed method has accuracy of order $0(h^2 + K^2)$ and consistent with the given problem.

3.5 Numerical Experiments

In order to demonstrate the adaptability and the accuracy of the present method, we consider some test examples. The exact solution of these examples is obtained by Hopf-Cole transformation. The numerical solutions generated by proposed method are compared with exact solution at the different nodal points.

Example1: Consider Burger's equation (3.1) with initial condition

$$u(x,0)=\sin \pi x \quad 0 < x < 1 \tag{3.19}$$

and homogeneous boundary conditions

$$u(0,t)=u(1,t)=0 \qquad 0 \le t \le T$$

By Hopf-Cole transformation

$$u(x,t)=-\frac{2}{\mathrm{Re}}\frac{\phi_x}{\phi} \tag{3.20}$$

Burgers' equation is transformed to the following linear heat equation

$$\frac{\partial \phi}{\partial t}=\frac{1}{\mathrm{Re}}\frac{\partial^2 \phi}{\partial x^2} \qquad 0 < x < 1,\ t > 0 \tag{3.21}$$

with initial condition

$$\phi(x,0)=\exp\left\{-\frac{\mathrm{Re}}{2\pi}[1-\cos(\pi x)]\right\} \qquad 0 < x < 1 \tag{3.22}$$

with boundary conditions[Neumann type]

$$\phi_x(0,t)=\phi_x(1,t)=0 \qquad 0 \le t \le T \tag{3.23}$$

The numerical results of the example are presented in Tables 1 and Figure 1-3 for different values of *Re* at different time and *x*. The Table shows that the results are good in agreement to the exact solutions. The Figures show the physical behaviour of the problem.

Example 2: Consider Burger's equation (3.1) with initial condition

$$u(x,0)= 4x(1-x) \qquad 0 < x < 1 \tag{3.24}$$

and boundary condition

$$u(0,t)= 0=u(1,t) \qquad 0 \le t \le T \tag{3.25}$$

By Hopf-Cole transformation

$$u(x,t)= -\frac{2}{\mathrm{Re}}\frac{\phi_x}{\phi}$$

Burgers' equation is transformed to the following linear heat equation

$$\frac{\partial \phi}{\partial t}= \frac{1}{\mathrm{Re}}\frac{\partial^2 \phi}{\partial x^2} \qquad 0 < x < 1,\ t > 0$$

with initial condition

$$\phi(x,0)= \exp\left\{-\frac{\mathrm{Re}}{2\pi}[1-\cos(\pi x)]\right\} \qquad 0 < x < 1$$

with boundary conditions[Neumann type]

$$\phi_x(0,t)= \phi_x(1,t)= 0 \qquad 0 \le t \le T$$

The numerical results of the example are presented in Tables 2 and Figure 4-6 for different values of *Re* at different time and *x*. The Table shows that the results are good in agreement to the exact solutions. The Figures show the physical behaviour of the problem

Table1: Comparison of the numerical solution with the exact solution at different times of Example 1 for $\Delta t = 0.01$ and $\Delta x = 0.0125$.

x	*T*	*Re=10*		*Re=100*	
		Computed solution	Exact Solution	Computed Solution	Exact Solution
0.25	0.4	0.30881	0.30889	0.34229	0.34191
	0.6	0.24069	0.24074	0.26902	0.26896
	1.0	0.16254	0.16256	0.18817	0.18819
0.50	0.4	0.56955	0.56963	0.66797	0.66071
	0.6	0.44714	0.44721	0.53211	0.52942
	1.0	0.29188	0.29192	0.37500	0.37442
0.75	0.4	0.62540	0.62544	0.93680	0.91026
	0.6	0.48715	0.48721	0.77724	0.76724
	1.0	0.28744	0.28747	0.55833	0.55605

Table 2: Comparison of the numerical solution with the exact solution at different times of Example 2 for $\Delta t = 0.01$ and $\Delta x = 0.0125$

x	T	Re=10		Re=100	
		Computed solution	Exact Solution	Computed Solution	Exact Solution
0.25	0.4	0.31743	0.31752	0.36273	0.36226
	0.6	0.24609	0.24614	0.28212	0.28204
	1.0	0.16558	0.16560	0.19467	0.19469
0.50	0.4	0.58446	0.58454	0.69186	0.68368
	0.6	0.45791	0.45798	0.55125	0.54832
	1.0	0.29831	0.29834	0.38627	0.38568
0.75	0.4	0.64558	0.64562	0.94940	0.92050
	0.6	0.50261	0.50268	0.79399	0.78299
	1.0	0.29582	0.29586	0.57170	0.56932

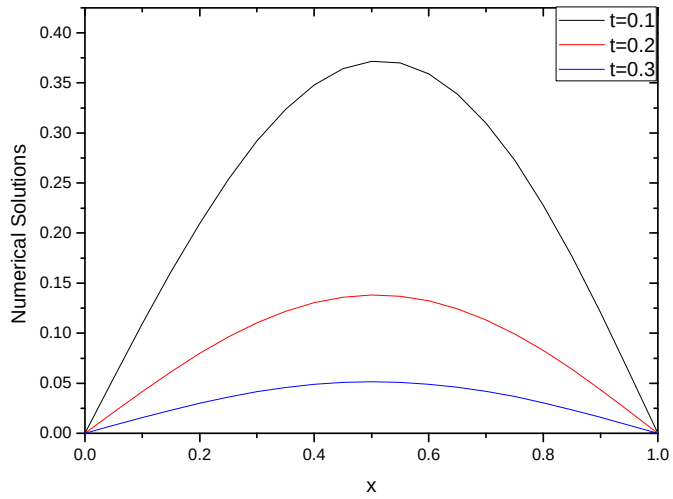

Figure 1: Numerical Solution of Example 1 at different times t and values of *Re=1* and $\Delta t = 0.0001$.

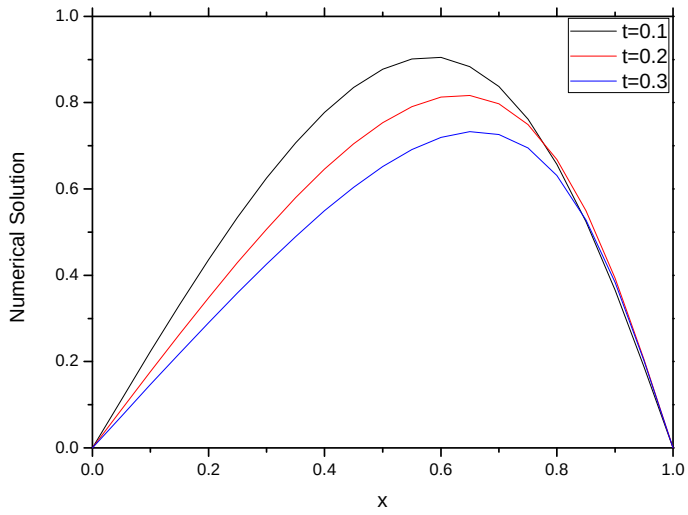

Figure 2: Numerical Solution of Example 1 at different times t and values of Re=*10* and $\Delta t = 0.0001$.

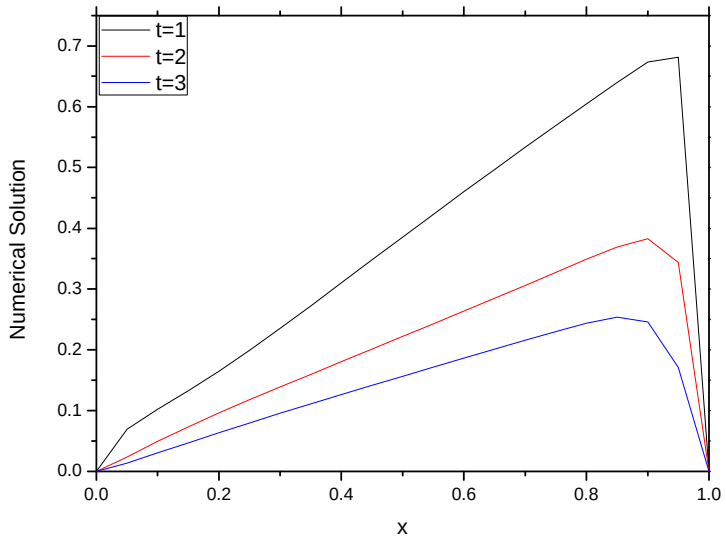

Figure 3: Numerical Solution of Example 1 at different times *t* and values of *Re=100* and $\Delta t = 0.0001$.

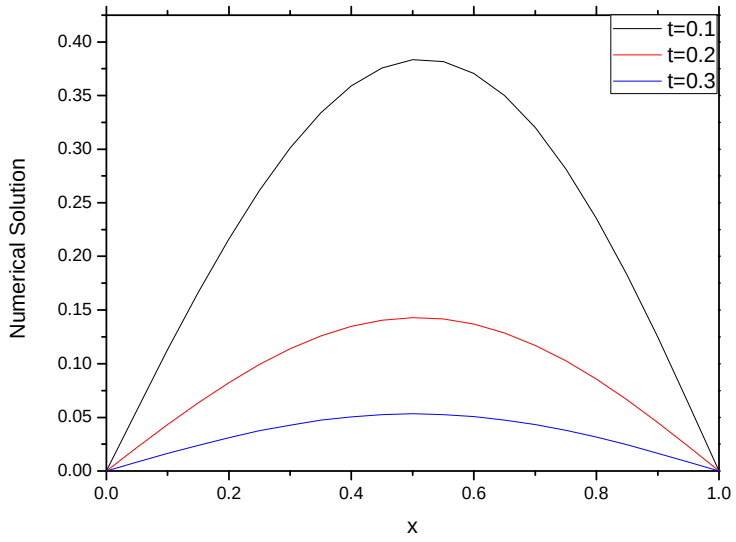

Figure 4: Numerical Solution of Example 2 at different times t and values of Re=1 and $\Delta t = 0.0001$

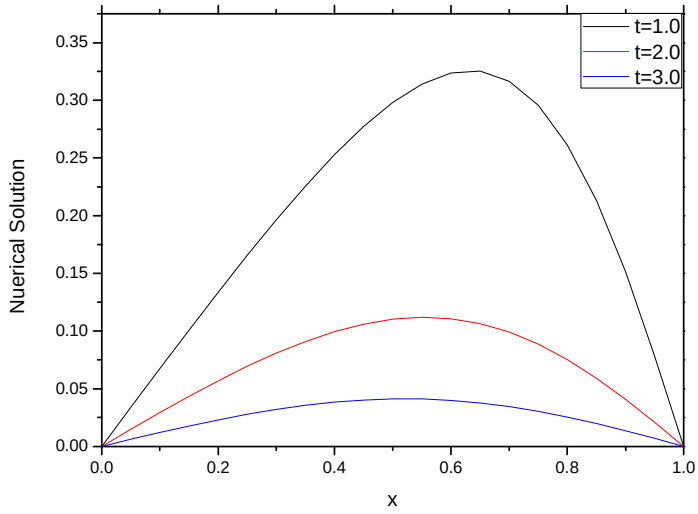

Figure 5: Numerical Solution of Example 2 at different times t and values of Re=10 and $\Delta t = 0.0001$

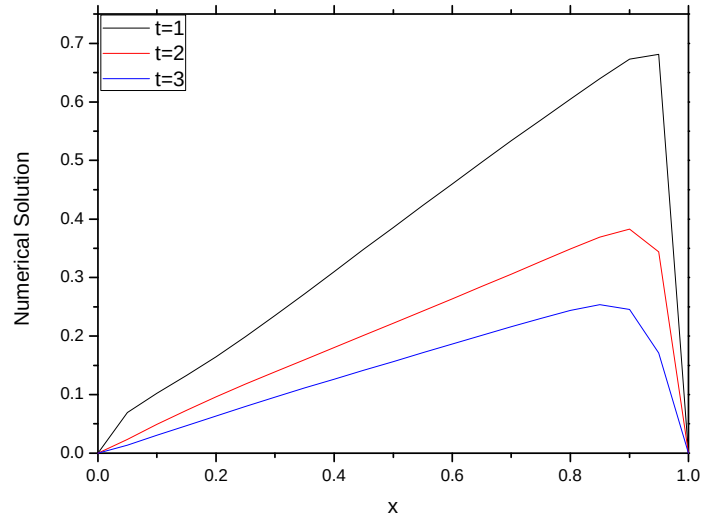

Figure 6: Numerical Solution of Example 2 at different times t and values of Re=10 and $\Delta t = 0.0001$

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