CLASSICAL METHOD FOR SOME NONLINEAR SYSTEMS

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DEDICATED

TO

GOD, MY PARENTS AND MY TEACHERS

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CERTIFICATE

*I hereby certify that the work which is being presented in this thesis entitled "**Classical method for some nonlinear systems**" which is being submitted for the award of degree of Master of Sciences, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Rajesh Kumar Gupta.***

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

The objective of this thesis entitled, "***Classical method for some nonlinear systems*** ", is to obtain the Lie symmetries and the exact solutions of nonlinear partial differential equations (PDEs) or their systems, which represent some of the important physical phenomenon. The nonlinear phenomena are encountered in a variety of situations in physics as well as in other natural applied sciences. Most of these phenomena are governed by nonlinear partial differential equations. The study of these systems of differential equations is often regarded as a difficult and confusing endeavour due to various limitations posed by the intrinsic nonlinearity.

The exact solutions of these differential equations which not only play a central role in the theories of these physical phenomena but have also become more and more sought after during last few years. The mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The group theoretic techniques can be categorized in this class and generally a variety of exact solutions may be furnished in a systematic manner.

This thesis comprises six chapters. The brief outline of the research work presented chapter wise in this thesis is as follows:

In chapter 1, we have described the nonlinear partial differential equations and the exact solutions. It contains the various definition of Lie groups. It also includes the preliminary material and literature review.

Chapter 2 is a brief description of Lie classical method based on the application to some nonlinear partial differential equations.

In chapter 3, firstly we study the classical Lie symmetries of the modified Korteweg-de Vries equation which is obtained through the Lie group method of the infinitesimal transformations. Secondly, using the classical symmetries of the equation, similarity reductions are obtained.

In Chapter 4, we investigate symmetries and reductions of the Burger's Korteweg-de Vries equation. Corresponding to each basic vector field, the reductions of the system to ordinary differential equations is obtained. These reduced systems are further studied for exact solutions.

Chapter 5 presents symmetry reductions and exact solutions of Gardner's equation. This chapter deals with the classical Lie method to obtain symmetries and reductions. Reduced systems are studied further to generate some exact solutions.

In Chapter 6, we study a form of extended form of modified Korteweg- de Vries equation. For each generator in the optimal system, the equation is reduced to a system of ordinary differential equations, which is further studied with the aim of deriving certain

exact solution.

CHAPTER 1

INTRODUCTION

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines. Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions —the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

The study of differential equations is a wide field in pure and applied mathematics, physics, meteorology, and engineering. All of these disciplines are concerned with the properties of differential equations of various types [3,18]. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable. Instead, solutions can be approximated using numerical methods. The theory of differential equations is well developed and the methods used to study them vary significantly with the type of the equation.

Differential equations can be further classified as ordinary differential equations and partial differential equations [3]. An ordinary differential equation (ODE) is a differential equation in which the unknown function (also known as the dependent variable) is a function of a single independent variable. In the simplest form, the unknown function is a real or complex valued function. An ordinary differential equation is a relation that contains the functions of only one independent variable, and one or more of its derivatives with respect to that variable. The order of differential equation is given by the maximum number of times the supposed unknown function in it has been differentiated. Ordinary differential equation arises in many differential contexts including geometry etc. Much study has been devoted to the solutions of ordinary differential equations. In the case where the equation is linear, it can be solved with analytical methods. Unfortunately, most of the interesting differential equations are nonlinear and, with a few exceptions, cannot be solved exactly [4, 24]. Approximate solutions can be obtained by using computer approximations.

A partial differential equation (PDE) is a differential equation in which the unknown function is a function of multiple independent variables and the equation involves its partial derivatives. The order is defined similarly to the case of ordinary differential equations, but further classification into elliptic, hyperbolic, and parabolic equations, especially for second-order linear equations, is of utmost importance. The study of partial differential equations is a fundamental subject area of mathematics which links important strands of pure mathematics to applied and computational mathematics. A solution (or a particular solution) to a partial differential equation is a function that solves the equation or, in other words turns into identity when substituted into the equation.

A solution is called general if it contains all particular solutions of the equations concerned. The term exact solution is often used for second and higher order nonlinear partial differential equations to denote a particular solution. The exact solutions are also helpful in designing and testing of numerical algorithms. The proposed work will be devoted to obtaining the exact solutions of nonlinear partial differential equations or their systems, which represent some of the important physical systems. Partial differential equations and their solutions exhibit rich and complex structures. Unfortunately, closed analytical expressions for their solutions can be found only in very special circumstances and these are mostly of limited theoretical and practical interest. Thus scientists and mathematicians have naturally lead to seeking techniques for the approximation of solutions [18, 24].

Both ordinary and partial differential equations are broadly classified as linear and nonlinear. A differential equation is linear if the unknown function and its derivatives appear to the power 1 (products are not allowed) and nonlinear otherwise. Linear differential equations are a further subclass for which the space of solutions is a linear subspace i.e. the sum of any set of solutions or multiples of solutions is also a solution. The coefficients of the unknown function and its derivatives in a linear differential equation are allowed to be (known) functions of the independent variable or variables if these coefficients are constants then one speaks of a constant coefficient linear differential equation. Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. With the development in physics and other sciences nonlinear phenomenon have really caught up with us. Virtually all fundamental equations of physics are nonlinear and are difficult to solve.

Various standard strategies adopted to derive exact solutions to nonlinear partial differential equations. But these solutions do not provide much information about the system. Strategies adopted to derive exact solutions of nonlinear partial differential equations are being avoided due to cumbersome and complicated calculations. But the strong desire of exact and more general solutions to nonlinear partial differential equations governing nonlinear phenomenon in technological enhancement has made tremendous growth in research interest. . Nonlinear equations are difficult to solve and the linear approximations used to describe them are often a tacit admission that the underlined equations cannot be solved. In fact, the study of nonlinear systems of differential equations is regarded as a difficult and a confusing endeavour. Whilst linear analysis as a mathematical discipline began in nineteenth century and in the intervening years achieved many spectacular successes throughout sciences, on the other hand the nonlinear equations by virtue of their inherent complexity remained much harder to understand because of their lack of simple superposed solutions.

Consequently, numerical methods are being applied to obtain approximate solutions of these equations. But these solutions, however fail to provide much information about the system. The study of nonlinear differential equations has not only provided information about the phenomenon, but has, in fact, helped in making more precise some of the concepts and theories developed in the last century mathematics.

Exact solutions for nonlinear equations are rare, and the methods, which can generates families of them, are not only increasingly popular, but increasingly sought. So far, number of methods have been proposed to construct the exact solutions, the most effective methods include the classical Lie approach [1,5,17,21], the non classical approach [2,11,25], Steinberg's symmetry reduction method [6,7], the direct method [15,26], the hyperbolic function expansion methods [15], the elliptic functions expansion methods [16,17]. The main tool in our study will be Lie classical method for the reduction of partial differential equations (PDEs) and other method including hyperbolic secant-tangent functions expansion method for some exact solutions of ordinary differential equations (ODEs). Software like Maple can be utilized during the research to derive and to test the authenticity of solutions and for other related purposes.

1.1 Literary Review

From 1870, Sophus Lie's work put the theory of differential equations on a more satisfactory foundation. He showed that the integration theories of the older mathematicians can, by the introduction of what are now called Lie groups [7] be referred to a common source, and that ordinary differential equations which admit the same infinitesimal transformations [7] present comparable difficulties of integration. He also emphasized the subject of transformations of contact. Lie's work systematically related a miscellany of topics in ordinary differential equations (ODEs) including separable equation, homogenous equations, and the use of laplace equation. Lie (1881) also indicated that for linear partial differential equations (PDEs), invariance under a Lie group leads directly to superpositions of solutions in terms of transforms.

A Lie group of transformations [7] admitted by a differential equations corresponds to a mapping of each of its solutions to another solutions of the same differential equations. There are an infinite number of ways of representing such a mapping by allowing an arbitrary change of independent variables. This representation is unique if the independent variables are kept fixed. Towards the end of the nineteenth century, Sophus Lie introduced the notion of Lie group in order to study the solutions of ordinary differential equations (ODEs). He showed the following main property: the order of an ordinary differential equation can be reduced by one if it is invariant under one-parameter Lie group of point transformations. This observation unified and extended the available integration techniques. Lie devoted the remainder of his mathematical career to developing these continuous groups that have now an impact on many areas of mathematically-based sciences. The applications of Lie groups to differential systems were mainly established by Lie and Emmy Noether, and then advocated by Élie Cartan. Roughly speaking, a Lie point symmetry of a system is a local group of transformations that maps every solution of the system to another solution of the same system. In other words, it maps the solution set of the system to itself. Elementary examples of Lie groups are translations, rotations and scalings. Application of Lie group theory presents the solutions of some nonlinear partial differential equations which are considered in this thesis.

CHAPTER 2

METHODOLOGY

(2.1) Preliminaries:

Now we will present some basic definitions from Bluman and Anco[7] that are useful for the thesis.

Group: A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

i) Closure property. For any elements a and b of G, $\phi(a,b)$ is an element of G.

ii) Associative property. For any elements a,b,c of G:

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c).$$

iii) Identity element. There exists a unique identity element e of G such that for any element a of G:

$$\phi(a,e) = \phi(e,a) = a$$

iv) Inverse element. For any element a of G, there exists a unique inverse element a^{-1} in G such that

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e$$

One-parameter group of transformations: Let $\bar{x} = (x_1, x_2, ..., x_n)$ lie in region $D \subset \Re^n$. The set of transformations

$$\bar{x}^* = \overline{X}(\bar{x};\varepsilon),$$

defined for each $\bar{x}$ in D and parameter ε in set $T \subset \Re$, with $\phi(\varepsilon,\delta)$ defining a law of composition of parameters ε and δ in T, forms a *one-parameter group of transformations* on D, if the following hold:

i) For each ε in T, the transformations are one-to-one onto D. [Hence, $\bar{x}^*$ lies in D.]

ii) T with the law of composition ϕ forms a group G.

iii) For each $\bar{x}$ in D, $\bar{x}^* = \bar{x}$ when $\varepsilon = \varepsilon_0$ corresponds to the identity e, i.e.,

$$\overline{X}(\bar{x};\varepsilon_0) = \bar{x}.$$

iv) If $\bar{x}^* = \overline{X}(\bar{x};\varepsilon)$, $\bar{x}^{**} = \overline{X}(\bar{x}^*;\delta)$, then

$$\bar{x}^{**} = \overline{X}(\bar{x};\phi(\varepsilon,\delta)).$$

One-parameter Lie group of transformations: A one-parameter group of transformations defines a *one-parameter Lie group of transformations* if, in addition to satisfying axioms (i)-(iv) of the above definition, the following holds:

v) ε is a continuous parameter, i.e., T is an interval in $\Re$. Without loss of generality, $\varepsilon = 0$ corresponds to the identity element e.

vi) $\overline{X}$ is infinitely differentiable with respect to $\overline{x}$ in D and an analytic function of ε in T.

vii) $\phi(\varepsilon,\delta)$ is an analytical function of ε and δ, $\varepsilon,\delta \in T$.

Infinitesimal Transformations: Consider a one-parameter (ε) Lie group of transformations

$$\overline{x}^* = \overline{X}(\overline{x};\, \varepsilon)$$

with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding about $\varepsilon = 0$, in some neighborhood of $\varepsilon = 0$, we get

$$\begin{aligned}\overline{x}^* &= \overline{x} + \varepsilon\left(\left.\frac{\partial \overline{X}(\overline{x};\, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right) + \frac{1}{2}\varepsilon^2\left(\left.\frac{\partial^2 \overline{X}(\overline{x};\, \varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0}\right) + \dots \\ &= \overline{x} + \varepsilon\left(\left.\frac{\partial \overline{X}(\overline{x};\, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right) + O(\varepsilon^2).\end{aligned}$$

Let $\overline{\xi}(\overline{x}) = \left.\dfrac{\partial \overline{X}(\overline{x};\, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}$. Then the transformation $\overline{x} + \varepsilon\overline{\xi}(\overline{x})$ is called the *infinitesimal transformation* of the Lie group of transformations . The components of $\overline{\xi}(\overline{x})$ are called the infinitesimals.

Infinitesimal Generators: The *infinitesimal generator* of the one-parameter Lie group of transformations is the operator

$$\mathrm{X} = \mathrm{X}(\overline{x}) = \overline{\xi}(\overline{x}).\nabla = \sum_{i=1}^{n} \xi_i(\overline{x})\frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right).$$

For any differentiable function $F(\overline{x}) = F(x_1, x_2, \dots, x_n)$, one has

$$\mathrm{X}F(\overline{x}) = \overline{\xi}(\overline{x}).\nabla F(\overline{x}) = \sum \xi_i(\overline{x})\frac{\partial F(\overline{x})}{\partial x_i}.$$

Invariant Functions: An infinitely differentiable function $F(\bar{x})$ is *invariant function* of the Lie group of transformations if and only if, for any group transformation,

$$F(\bar{x}^*) = F(\bar{x}).$$

If $F(\bar{x})$ is an invariant function then $F(\bar{x})$ is called an invariant and $F(\bar{x})$ is said to be invariant.

Theorem 1: $F(\bar{x})$ *is invariant under a Lie group of transformations if and only if* $\mathrm{X}F(\bar{x}) \equiv 0.$

Theorem 2: (Infinitesimal Criterion for the Invariance of a System of PDEs). *Let*

$$\mathrm{X} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\nu(x,u)\frac{\partial}{\partial u^\nu}$$

be the infinitesimal generator of the Lie group of point transformations. Let

$$\mathrm{X}^{(k)} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\nu(x,u)\frac{\partial}{\partial u^\nu} + \eta_i^{(1)\nu}(x,u,\partial u)\frac{\partial}{\partial u_i^\nu} + \dots$$
$$+\eta_{i_1 i_2 \dots i_k}^{(k)\nu}(x,u,\partial u,\dots,\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\nu}$$

be the kth-extended infinitesimal generator, where $\eta_i^{(1)\nu}$ *is given by and* $\eta_{i_1 i_2 \dots i_j}^{(j)\mu}$*),* $\nu = 1,2,\dots,m$ *and* $i_j = 1,2,\dots,n,$ *for* $j = 1,2,\dots,k,$ *in terms of* $\xi(x,u) = (\xi_1(x,u),\xi_2(x,u),\dots,\xi_n(x,u)),\ \eta(x,u) = (\eta^1(x,u),\eta^2(x,u),\dots,\eta^m(x,u)).$ *Then the one-parameter Lie group of point transformations is admitted by the system of PDEs if and only if*

$$\mathrm{X}^{(k)}F^\sigma(x,u,\partial u,\partial^2 u,\dots,\partial^k u) = 0,$$

when u *satisfies for each* $\sigma = 1,2,\dots N$.

Commutator : For an r-parameter Lie group of transformations with infinitesimal generators $\mathrm{X}_\alpha, \alpha = 1,2,\dots,r$, the *commutator* (Lie Bracket) of X_α and X_β is a first order operator defined by

$$[\mathrm{X}_\alpha,\mathrm{X}_\beta] = \mathrm{X}_\alpha\mathrm{X}_\beta - \mathrm{X}_\beta\mathrm{X}_\alpha = \sum_{i,j=1}^{n}\left[\left(\xi_{\alpha i}(\bar{x})\frac{\partial}{\partial x_i}\right)\left(\xi_{\beta j}(\bar{x})\frac{\partial}{\partial x_j}\right) - \left(\xi_{\beta i}(\bar{x})\frac{\partial}{\partial x_i}\right)\left(\xi_{\alpha j}(\bar{x})\frac{\partial}{\partial x_j}\right)\right]$$
$$= \sum_{j=1}^{n}\eta_j(\bar{x})\frac{\partial}{\partial x_j},$$

where $\eta_j(\bar{x}) = \sum_{i=1}^{n}\left(\xi_{\alpha i}(\bar{x})\frac{\partial \xi_{\beta j}(\bar{x})}{\partial x_i} - \xi_{\beta i}(\bar{x})\frac{\partial \xi_{\alpha j}(\bar{x})}{\partial x_i}\right).$

It immediately follows that

$$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha].$$

Adjoint : *Let G be a Lie group with Lie algebra $\mathbf{L}$. For each vector $v \in \mathbf{L}$, the adjoint vector ad v at $w \in \mathbf{L}$ is*

$$ad\ v\big|_w = [w, v] = -[v, w].$$

The adjoint representation $\mathrm{Ad}\,G$ of the underlying Lie group can be reconstructed either by integrating the system of linear ordinary differential equations

$$\frac{dw}{d\varepsilon} = \text{ad } v\big|_w, \quad w(0) = w_0,$$

with solution

$$w(\varepsilon) = Ad(\exp(\varepsilon v))w_0,$$

or, more simply, by summing the Lie series

$$Ad(\exp(\varepsilon v))w_0 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\text{ad } v)^n (w_0)$$

$$= w_0 - \varepsilon[v, w_0] + \frac{\varepsilon^2}{2}[v,[v, w_0]] - \cdots.$$

Optimal system : We need at present only one solution from each equivalence classes, as the rest may be found by applying appropriate group symmetries, a complete set of such solutions is referred to as an "optimal solution" of group invariant solution.The problem of deriving an optimal system of group invariant solutions is equivalent to an optimal system of Lie symmetries. The method used here is given by Olver [26] which basically consists of taking linear combinations of the generator and reducing them into their simplest equivalent form by applying a carefully chosen adjoint transformations.

$$adj[\exp(\varepsilon v_i)]v_j = v_j - \varepsilon[v_i, v_j] + \frac{\varepsilon^2}{2}[v_i,[v_i, v_j]] + \ldots\ldots\ldots$$

where $[v_i, v_j]$ is the usual commutator.

In this thesis, we deal with the methods of group invariant solutions, based on the theory of continuous group of transformations, better known as "Lie groups", acting on the space of independent and dependent variables of the system. The method is due originally to Sophus Lie who unified and extended the bewildered special methods of integration of differential equations. Through the constructive procedures Lie established that, in the case of ordinary differential equations (ODEs), invariance under one-parameter symmetry group implies that the order of the equation can be reduced by one.

Lie's work [7] for ordinary differential equations examines in a systematic and comprehensive way a wide spectrum of topics such as integrating factors, separable and homogenous equations, reduction of order, methods of undetermined coefficients and variation of parameters, Euler equation and homogenous equations with constant coefficients. Further, for linear partial differential equation, Lie has established that the invariance under continuous group of transformations leads directly to superposition of solution in terms of transformations.

The work put up by this thesis is being primarily based been based on certain concepts of group symmetry through the applications of the last two techniques mentioned above.

By symmetry group of a single or a system of partial differential equations, we mean a continuous group of transformations acting on the space independent and dependent variables which leaves the equation invariant. The solutions of partial differential equation(s) are all found by solving a reduced system of differential equations involving a fewer independent variables. Thus, in particular, the solutions of partial differential equation in two independent variables which is invariant under one parameter symmetry group can be found by solving a 'reduced' ordinary differential equations.

From a view point of deriving explicit solutions to a system of differential equations, ordinary or partial, group theoretic methods carry a lot of potential yet these methods could not be very popular as the algebraic calculations involved were two complex and cumbersome. However, with the advent of software such as Maple and some others, the task has been greatly simplified.

As mentioned earlier the work comprising this thesis is based primarily on the applications of Lie Classical method to some nonlinear partial differential equations. The problems are dealt-with in two phases-in the first, the symmetries of the system under investigation are derived using the classical Lie method and then in second phase, other successful deduction of the reduced systems of ordinary differential equations, the efforts are confined to furnish the exact solutions.

2.1.1 Invariance for a System of PDEs

In Lie classical theory, we work with the applications of nonlinear partial differential equations. Now, we will follow a procedure to derive infinitesimals corresponding to each derivative as considered by Bluman and Anco [7], we obtain the following.

Consider a kth order partial differential equation

$$u_{i_1,i_2.....i_k} = f\left(x,u,\partial u,\partial^2 u,............,\partial^k u\right),$$

where $f\left(x,u,\partial u,\partial^2 u,............,\partial^k u\right)$, does not depend explicitly on $u_{i_1,i_2.....i_k}$.

We first show how to derive the Lie group of infinitesimal transformations with infinitesimal

$$u^* = U^*(x,t,u;\varepsilon) = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2)$$
$$x^* = X^*(x,t,u;\varepsilon) = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2)$$
$$t^* = T^*(x,t,u;\varepsilon) = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2)$$

where η, ξ and τ be the infinitesimal corresponding to u, x and t.

On invoking the invariance criterion the following relation from the coefficients of the first order of ε is deduced. Then the symmetry determining is in the form

$$\eta^k_{i_1,i_2.....i_k} = \xi_j \frac{\partial f}{\partial x_j} + \eta\frac{\partial f}{\partial u} + \eta^1_j \frac{\partial f}{\partial u_j} + ...+ \eta^k_{j_1,j_2,....,j_k} \frac{\partial f}{\partial u_{j_1,j_2,...j_k}},$$

where u satisfies (2.1).

Now we find the values of $\eta^x, \eta^t, \eta^{xx}, \eta^{xt}, \eta^{xxx}, \eta^{xxt}$ etc.

First we need to calculate the auxiliary functions $\frac{\partial x}{\partial x^*}, \frac{\partial x}{\partial t^*}, \frac{\partial t}{\partial t^*}, \frac{\partial t}{\partial x^*}$

By $\frac{\partial x}{\partial x^*}$ we understand that $u = u(x,t)$ and that only t^* is held fixed.

Hence

$$x = x^* - \varepsilon\xi(x,t,u) + O(\varepsilon^2)$$
$$\frac{\partial x}{\partial x^*} = 1 - \varepsilon\left(\frac{\partial \xi}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial x}{\partial x^*}\right) + O\left(\varepsilon^2\right),$$
$$\frac{\partial x}{\partial x^*}\left(1 + \varepsilon\left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\right)\right) = 1 + O(\varepsilon^2),$$
$$\frac{\partial x}{\partial x^*} = \left(1 - \varepsilon\left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\right)\right) + O(\varepsilon^2).$$

Similarly

$$\frac{\partial x}{\partial t^*} = -\varepsilon\left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial t}\right) + O(\varepsilon^2),$$
$$\frac{\partial t}{\partial t^*} = 1 - \varepsilon\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial u}\frac{\partial u}{\partial t}\right) + O(\varepsilon^2),$$
$$\frac{\partial t}{\partial x^*} = -\varepsilon\left(\frac{\partial \tau}{\partial x} + \frac{\partial \tau}{\partial u}\frac{\partial u}{\partial x}\right) + O(\varepsilon^2). \qquad (2.1.1)$$

Hence, we find the following extensions:

First extensions

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial}{\partial x^*}\left(u^* + \varepsilon\eta(x,t,u)\right) + O(\varepsilon^2),$$

$$= \frac{\partial\left(u^* + \varepsilon\eta(x,t,u)\right)}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial t}{\partial x^*}\frac{\partial u}{\partial t} + O(\varepsilon^2). \qquad (2.1.2)$$

Substituting (2.2.1) into (2.2.2) we lead to

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial u}{\partial x} + \varepsilon\left(\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right)\frac{\partial u}{\partial x} - \frac{\partial \tau}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial \xi}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial \tau}{\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) + O(\varepsilon^2). \qquad (2.1.3)$$

Let η^x and η^t denote infinitesimals of $\frac{\partial u^*}{\partial x^*}$ and $\frac{\partial u^*}{\partial t^*}$ respectively. Then

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t. \qquad (2.1.4)$$

and similarly

$$\eta^t = \eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t. \qquad (2.1.5)$$

Second extensions

$$\eta^{xx} = \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3$$
$$+ (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{tx} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx} - \tau_{uu}u_x^2 u_t.$$

$$\eta^{xt} = \eta_{xt} + \left(\eta_{xu} - \tau_{tx}\right)u_t + \left(\eta_{tu} - \xi_{xt}\right)u_x - \tau_{xu}u_t^2 + \left(\eta_{uu} - \xi_{xu} - \tau_{tu}\right)u_x u_t$$
$$- \xi_{tu}u_x^2 - \tau_{uu}u_t^2 u_x + \xi_{uu}u_x^2 u_t - \tau_x u_{tt} + \left(\eta_u - \xi_x - \tau_t\right)u_{xt} - \xi_t u_{xx} - 2\tau_u u_t u_{xt}$$
$$- 2\xi_u u_{xt}u_x - \tau_u u_x u_{tt} - \xi_u u_{xx}u_t.$$

$$\eta^{tt} = \eta_{tt} + \left(2\eta_{tu} - \tau_{tt}\right)u_t - \xi_{tt}u_x + \left(\eta_{uu} - 2\tau_{tu}\right)u_t^2 - 2\xi_{tu}u_x u_t - \tau_{uu}u_t^3 - \xi_{uu}u_t^2 u_x +$$
$$\left(\eta_u - 2\tau_t\right)u_{tt} - 2\xi_t u_{xt} - 3\tau_u u_{tt}u_x - 2\xi_u u_{xt}u_t.$$

Third extensions

$$\eta^{txx} = \eta_{txx} + (\eta_{xxu} - \tau_{xxt})u_t + (2\eta_{xtu} - \xi_{xxt})u_x - \xi_u u_t u_{xxx} - 2\tau_u u_t u_{txx} - 2\tau_u u_x u_{ttx}$$
$$+ (\eta_{uut} - 2\xi_{xtu})u_x^2 + (2\eta_{xuu} - \xi_{xxu} - 2\tau_{xtu})u_x u_t - \tau_{xxu}u_t^2 - \tau_{xuu}u_x u_t^2 - \xi_{uut}u_x^3$$
$$+ (\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu})u_x^2 u_t - \xi_{uuu}u_x^3 u_t - \tau_{uuu}u_x^2 u_t^2 + (2\eta_{uu} - \xi_{xx} - 2\tau_{tx})u_{tx}$$
$$+ (\eta_{tu} - 2\xi_{tx})u_{xx} - \tau_{xx}u_{tt} - 4\tau_{xu}u_t u_{xt} - 2\tau_{xu}u_{tt}u_x + 2(\eta_{uu} - 2\xi_{xu} - \tau_{tu})u_x u_{tx}$$
$$+ (\eta_{uu} - 2\xi_{xu} - \tau_{tu})u_{xx}u_t - 3\xi_{tu}u_x u_{xx} - 3\xi_{uu}u_t u_x u_{xx} - \tau_{uu}u_{xx}u_t^2 - 3\xi_u u_{tx}u_{xx}$$
$$- 2\tau_u u_{xt}^2 + (\eta_u - 2\xi_x - \tau_t)u_{txx} - \tau_u u_{xx}u_{tt} - 2\tau_x u_{ttx} - \xi_t u_{xxx} - 3\xi_u u_x u_{txx}.$$

$$\eta^{xxx} = \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}u_t - 3\tau_{xxu}u_x u_t + 3(\eta_{xuu} - \xi_{xxu})u_x^2$$
$$- 3\tau_{xuu}u_t u_x^2 + (\eta_{uuu} - 3\xi_{xuu})u_x^3 - \tau_{uuu}u_x^3 u_t - \xi_{uuu}u_x^4 - 3\tau_{xx}u_{tx} + 3(\eta_{xu} - \xi_{xx})u_{xx}$$
$$+ 3(\eta_{uu} - 3\xi_{xu})u_x u_{xx} - 3\tau_{xu}u_t u_{xx} - 6\tau_{xu}u_x u_{tx} - 6\xi_{uu}u_x^2 u_{xx} - 3\tau_{uu}u_{xt}u_x^2$$

$$-3\tau_{uu}u_{xx}u_x u_t - 3\xi_u u_{xx}^2 - 3\tau_u u_{xx}u_{tx} - 3\tau_x u_{txx} + (\eta_u - 3\xi_x)u_{xxx} - 4\xi_u u_x u_{xxx}$$
$$-\tau_u u_{xxx}u_t - 3\tau_u u_{xxt}u_x .$$

2.2. Lie classical approach: An Algorithmic Overview

This algorithmic overview has been considered from Bluman and Anco [7]. When a given system of PDEs is subjected to invariance under one-parameter Lie group of transformations, one arrives at an over determined linear system of equations for the group infinitesimals. The classical method essentially consists of finding symmetry reductions of PDEs with the help of determining equations obtained under the condition of invariance of the system of PDEs These infinitesimals of the transformations help us obtain the reductions of the system. The symmetries and reductions reported in chapters are based on the application of this method. Consider a system of N PDEs with m dependent variables $u=(u^1,u^2,\ldots,u^m)$ and n independent variables $x=(x_1,x_2,\ldots,x_n)$, given by

$$F^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0,\ \ \mu=1,2,\ldots,N.$$

Let the one-parameter Lie group of point transformations leaves invariant the system of PDEs. Then, apply the prolonged operator $\mathrm{X}^{(k)}$ given by to each equation of the system and require that

$$\mathrm{X}^{(k)}F^{\mu}\Big|_{F^{\nu}=0}=0 \qquad \mu,\nu=1,2,\ldots,N .$$

The meaning of this condition is that $\mathrm{X}^{(k)}$ vanishes on the solution set of the originally given system. Precisely, this condition assures that $u(x)$ is solution of whenever $u^*(x^*)$ is one. Following the procedure as mentioned in section, a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator X admitted by the given system of PDEs is obtained. The solutions of the determining equations will lead to the explicit forms of ξ and η. Construct the corresponding characteristics equations and obtain u in terms of $n-1$ new independent variables. Rewrite the system in these new coordinates to get the reduced form of the system.

CHAPTER 3

MODIFIED KORTEWEG-dE VRIES EQUATION

The Korteweg-de Vries equation [17] is a mathematical model of waves on shallow water surfaces. It is particularly known as nonlinear partial differential equation whose solutions can be exactly and precisely specified. Korteweg-de Vries equation is related to modified Korteweg-de Vries equation using Mirura transformation.The mKdV equation is one of the most important nonlinear wave equation in physics and mechanics whose general form is

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0.$$

The equation was studied in [9,31] and some exact solutions were found. This equation is used as a mathematical model to study physical phenomenon arising in certain areas of interest. For example, in the coastal waves in ocean and liquid drops and bubbles, in the issues of atmospheric blocking phenomenon and dipole blocking [5,25,28,29,30]. The dimensionless form of mKdV equation can also be studied with time dependent coefficient. Here, the first term represents the evolution term while the second term is the nonlinear term.

The Korteweg de Vries equation was originally suggested in connection with certain regime of surface water waves [22]. It has been extensively studied and derived model for individual propagation of small amplitude long waves in a number of physical system. For example, Korteweg-de Vries equation can describe the evolution of shallow water waves, ion acoustic waves, long waves in sheer flows.The Korteweg-de Vries equation’s cousin is modified Korteweg-de Vries equation.

3.1 One-parameter Lie group of transformation and infinitesimal transformation

Consider the modified Korteweg-de Vries equation

$$u_t - \beta u^2 u_x + u_{xxx} = 0. \tag{3.1.1}$$

Let the group of infinitesimal transformations be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$
$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2). \tag{3.1.2}$$

The invariance under (3.1.2) means that if u is the solution of (3.1.1), then u^* is also a solution of it.

On invoking the invariance criterion as explained earlier, the following relation from the coefficients of the first order of ε is deduced:

$$\eta^t - \beta u^2\eta^x - 2\beta u u_x\eta + \eta^{xxx} = 0, \tag{3.1.3}$$

where η^t, η^x and η^{xxx} are prolonged infinitesimals acting on the enlarged space corresponding to u_t, u_x and u_{xxx} respectively.

The method of determining symmetry group mainly consists of finding the infinitesimals τ, ξ and η, which are functions of x, t and u.

The general solution of equation (3.1.3) provides infinitesimal elements τ, ξ and η for which the equation poses Lie symmetry.

Using the expressions for η^t, η^x and η^{xxx} we obtain the following

$$(\eta_t + \beta u^2 u_x \eta_u - \eta_u u_{xxx} - \beta\tau_t u^2 u_x + \tau_t u_{xxx} - \xi_t u_x - \tau_u \beta^2 u^4 u_x^2 - \tau_u u_{xxx}^2 + 2\beta u^2 u_x u_{xxx}\tau_u$$
$$- \beta\xi_u u^2 u_x^2 + \xi_u u_x u_{xxx}) + (-\beta u^2\eta_x - \beta u^2\eta_x u_x + \beta u^2\xi_x u_x + \beta^2 u^4\tau_t u_x - \beta u^2\tau_t u_{xxx} +$$
$$\beta u^2\xi_u u_x^2 + \beta^2 u^4\tau_u u_x^2 - \beta u^2\tau_u u_x u_{xxx}) - 2\beta u u_x\eta + (\eta_{xxx} + 3\eta_{xxu}u_x - \xi_{xxx}u_x - \beta u^2 u_x\tau_{xxx}$$
$$+ \tau_{xxx}u_{xxx} - 3\beta u^2\tau_{xxu}u_x^2 + 3\tau_{xxu}u_x u_{xxx} + 3\eta_{xuu}u_x^2 - 3\xi_{xxu}u_x^2 - 3\beta\tau_{xuu}u^2 u_x^3 + 3\tau_{xuu}u_x^2 u_{xxx} +$$
$$\eta_{uuu}u_x^3 - 3\xi_{xuu}u_x^3 - \beta\tau_{uuu}u^2 u_x^4 + \tau_{uuu}u_x^3 u_{xxx} - \xi_{uuu}u_x^4 - 3\tau_{xx}u_{tx} + 3\eta_{xu}u_{xx} - 3\xi_{xx}u_{xx} +$$
$$3\eta_{uu}u_x u_{xx} - 9\xi_{xu}u_x u_{xx} - 3\beta u^2\tau_{xu}u_x u_{xx} + 3\tau_{xu}u_{xx}u_{xxx} - 6\tau_{xu}u_x u_{tx} - 6\xi_{uu}u_x^2 u_{xx} - 3\tau_{uu}u_{xt}u_x^2$$
$$- 3\beta\tau_{uu}u^2 u_x^2 u_{xx} + 3\tau_{uu}u_x u_{xx}u_{xxx} - 3\xi_u u_{xx}^2 - 3\tau_u u_{xx}u_{tx} - 3\tau_x u_{txx} + \eta_u u_{xxx} - 3\xi_x u_{xxx} - 4\xi_u u_x u_{xxx}$$
$$- \beta\tau_u u^2 u_x u_{xxx} + \tau_u u_{xxx}^2 - 3\tau_u u_x u_{xxt}) = 0.$$

where u_t must be replaced by equation $u_t = \beta u^2 u_x - u_{xxx}$ On substituting the coefficients of different differentials equal to zero lead to number of PDEs in τ, ξ and η that need to be satisfied. The set of determining equations for the group of infinitesimals τ, ξ and η, which is obtained from above equation, after equating the coefficients of various derivative terms to zero, is as follows:

$$\tau_u = 0,$$
$$\tau_x = 0,$$
$$\xi_u = 0,$$
$$\eta_{uu} = 0,$$
$$\eta_{xu} - \xi_{xx} = 0,$$
$$\tau_t - \beta u^2 \tau_t - 3\xi_x = 0,$$
$$\beta u^2 \eta_u - \beta u^2 \tau_t - \xi_t - 2\beta u \eta - \beta u^2 \eta_x + \beta u^2 \xi_x + \beta^2 u^4 \tau_t + 3\eta_{xxu} - \xi_{xxx} = 0,$$
$$\eta_t - \beta u^2 \eta_x + \eta_{xxx} = 0.$$

The set of equations helps us to obtain infinitesimals τ, ξ and η as follows

$$\xi(x,t,u) = a + \frac{cx}{3},$$
$$\tau(x,t,u) = b + ct,$$
$$\eta(x,t,u) = \frac{-cu}{3}.$$

where a, b, c are arbitrary constants.

The Lie algebra associated with above equation consists of the following three vector fields

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t},$$
$$V_3 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{3}\frac{\partial}{\partial u}. \qquad (3.1.4)$$

3.2 Optimal system of Vector fields

The commutator table and the adjoint table for Lie algebra(3.1.4) is as follows:
The commutator (Lie Bracket) of V_α and V_β is first order operator defined by
$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha$.
The commutator table is given as follows:

COMMUTATOR TABLE

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$a_1^{'}V_2$
V_3	0	$-a_1^{'}V_2$	0

Formulae for adjoint table is

$$adj[\exp(\varepsilon V_i)V_j] = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i,[V_{i,}V_j]] + \dots\dots\dots\dots$$

ADJOINT TABLE

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 - \varepsilon a_1^{'}V_2$
V_3	V_1	$V_2 + \varepsilon a_1^{'}V_2$	V_3

We deduce an optimal system of sub algebras with their corresponding generators

$V_3 + a_1 V_1,$

$V_2 + a_1 V_1,$

$V_1.$

3.3 Reductions and Invariant Solutions

Generator (1)

The generator (1) is V_1.

The corresponding characteristic equation is given by,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

The generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$t = \zeta,$

$u(x,t) = F(t) = F(\zeta).$

Using the similarity variable and the forms of the similarity solution, the nonlinear partial differential equation (PDE) (3.1) reduces to the following nonlinear ordinary differential equation (ODE),

$F' = 0.$

It gives constant solution.

Generator (2)

The generator (2) is $V_2 + a_1 V_1$

The characteristic equation is given by

$$\frac{dt}{1} = \frac{dx}{a_1} = \frac{du}{0}.$$

The generator (2) in the optimal system defines the similarity variable and similarity solution as follows

$$x = a_1 t + \zeta,$$

$$u(x,t) = F(\zeta) = F(x - a_1 t).$$

We get the following ODE,

$$-a_1 F'(\zeta) - bF^2(\zeta)F'(\zeta) + F'''(\zeta) = 0.$$

Using maple its solution is obtained as follows:

$$F(\xi)=c_3,$$

$$F(\zeta)=\frac{\sqrt{-3ba_1}\,\tanh\left(-c_1+\frac{1}{2}\sqrt{-2a_1\zeta}\right)}{b},$$

$$F(\zeta)=\frac{\sqrt{-3ba_1}\,\tanh\left(c_1+\frac{1}{2}\sqrt{-2a_1\zeta}\right)}{b},$$

$$F(\zeta)=-\frac{\sqrt{-3ba_1}\,\tanh\left(-c_1+\frac{1}{2}\sqrt{-2a_1\zeta}\right)}{b}$$

$$F(\zeta)=-\frac{\sqrt{-3ba_1}\,\tanh\left(c_1+\frac{1}{2}\sqrt{-2a_1\zeta}\right)}{b},$$

$$F(\zeta)=\frac{\sqrt{-3b\left(-c_3^2+a_1\right)}JacobiCN\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2+2a_1}}{b}\right)}{b},$$

$$F(\zeta)=-\frac{\sqrt{-3b\left(-c_3^2+a_1\right)}JacobiCN\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2+2a_1}}{b}\right)}{b},$$

$$F(\zeta)=\frac{6c_3JacobiDN\left(c_2+c_3x,\frac{\sqrt{2c_3^2-a_1}}{c_3}\right)}{-6b},$$

$$F(\zeta)=-\frac{6c_3JacobiDN\left(c_2+c_3x,\frac{\sqrt{2c_3^2-a_1}}{c_3}\right)}{-6b},$$

$$F(\zeta)=\frac{\sqrt{3}\sqrt{b\left(c_3^2-a_1\right)}JacobiNC\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2-a_1}}{c_3}\right),}{b}$$

$$F(\zeta) = -\frac{\sqrt{3}\sqrt{b\left(c_3^2 - a_1\right)} JacobiNC\left(c_2 + c_3 x, \frac{1}{2}\frac{\sqrt{2c_3^2 - a_1}}{c_3}\right)}{b},$$

$$F(\zeta) = \frac{\sqrt{6}\sqrt{b\left(c_3^2 - a_1\right)} JacobiNC\left(c_2 + c_3 x, \frac{1}{2}\frac{\sqrt{2c_3^2 - a_1}}{c_3}\right)}{b},$$

$$F(\zeta) = -\frac{\sqrt{6}\sqrt{b\left(c_3^2 - a_1\right)} JacobiNC\left(c_2 + c_3 x, \frac{1}{2}\frac{\sqrt{2c_3^2 - a_1}}{c_3}\right)}{b}.$$

where JacobiNC,JacobiCN,JacobiDN are inverse of elliptic integrals and doubly periodic elliptic functions.Thus, the following are solutions of equation (3.1),

$$u(x,t) = c_3,$$

$$u(x,t) = \frac{\sqrt{-3ba_1}\tanh\left(-c_1 + \frac{1}{2}\sqrt{-2a_1(x - a_1 t)}\right)}{b}, \qquad (3.3.1)$$

$$u(x,t) = -\frac{\sqrt{-3ba_1}\tanh\left(-c_1 + \frac{1}{2}\sqrt{-2a_1(x - a_1 t)}\right)}{b}, \qquad (3.3.1')$$

$$u(x,t) = \frac{\sqrt{-3ba_1}\tanh\left(c_1 + \frac{1}{2}\sqrt{-2a_1(x - a_1 t)}\right)}{b}, \qquad (3.3.1'')$$

$$u(x,t) = -\frac{\sqrt{-3ba_1}\tanh\left(c_1 + \frac{1}{2}\sqrt{-2a_1(x - a_1 t)}\right)}{b}, \qquad (3.3.1''')$$

$$u(x,t) = \frac{\sqrt{-3b\left(-c_3^2 + a_1\right)} JacobiCN\left(c_2 + c_3 x, \frac{1}{2}\frac{\sqrt{2c_3^2 + 2a_1}}{c_3}\right)}{b},$$

$$u(x,t) = -\frac{\sqrt{-3b\left(-c_3^2 + a_1\right)} JacobiCN\left(c_2 + c_3 x, \frac{1}{2}\frac{\sqrt{2c_3^2 + 2a_1}}{c_3}\right)}{b},$$

$$u(x,t)=\frac{6\,JacobiDN\left(c_2+c_3x,\frac{\sqrt{2c_3^2-2a_1}}{c_3}\right)}{\sqrt{-6b}},$$

$$u(x,t)=-\frac{6\,JacobiDN\left(c_2+c_3x,\frac{\sqrt{2c_3^2-2a_1}}{c_3}\right)}{\sqrt{-6b}},$$

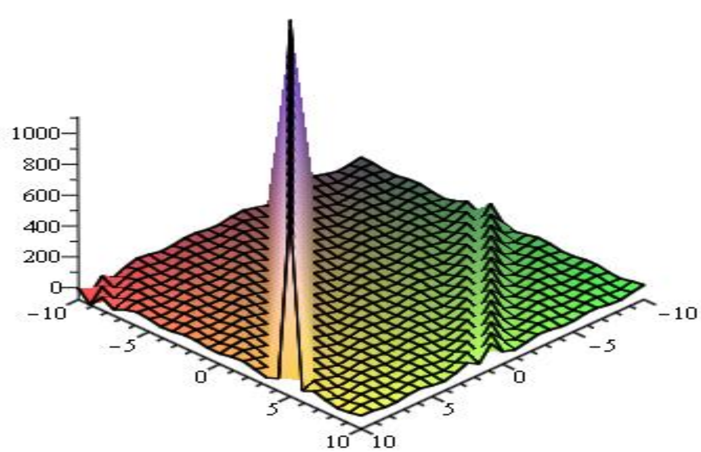
$$u(x,t)=\frac{\sqrt{3}\sqrt{b\left(c_3^2-a_1\right)}JacobiNC\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2-2a_1}}{c_3}\right)}{b},$$

$$u(x,t)=-\frac{\sqrt{3}\sqrt{b\left(c_3^2-a_1\right)}JacobiNC\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2-2a_1}}{c_3}\right)}{b},$$

$$u(x,t)=\frac{\sqrt{6}\sqrt{b\left(c_3^2-a_1\right)}JacobiNC\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2-2a_1}}{c_3}\right)}{b},$$

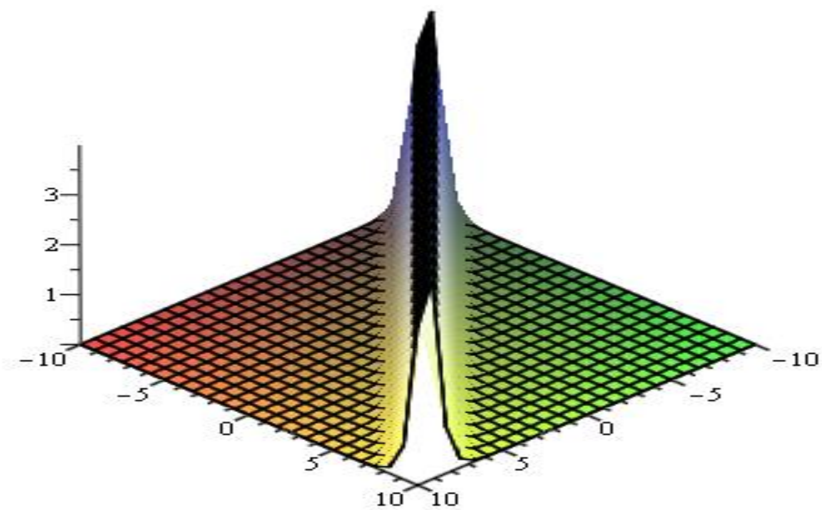
$$u(x,t)=-\frac{\sqrt{6}\sqrt{b\left(c_3^2-a_1\right)}JacobiNC\left(c_2+c_3x,\frac{1}{2}\frac{\sqrt{2c_3^2-2a_1}}{c_3}\right)}{b}.$$

The graphs of the solutions are as follows:
FIGURE 1



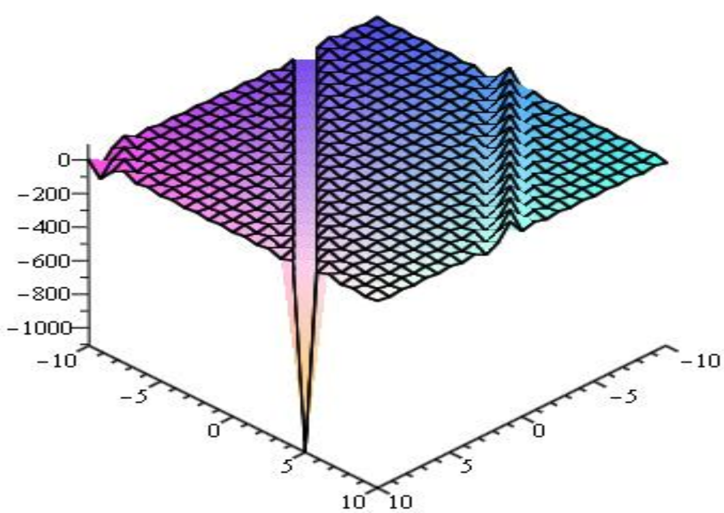
Section (3.3) for the solution (3.3.1) with
$a_1=-1, b=1, c_1=-1$

FIGURE 2



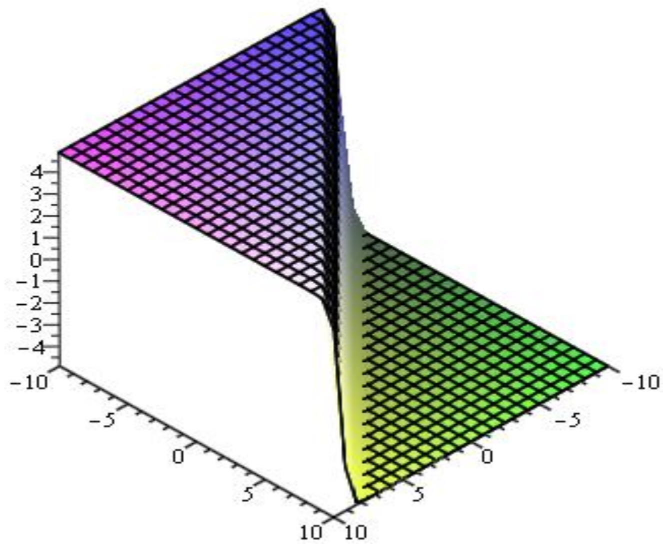
Section (3.3) for the solution (3.3.1) with $a_1 = 8, b = -1, c_1 = -1$

FIGURE 3



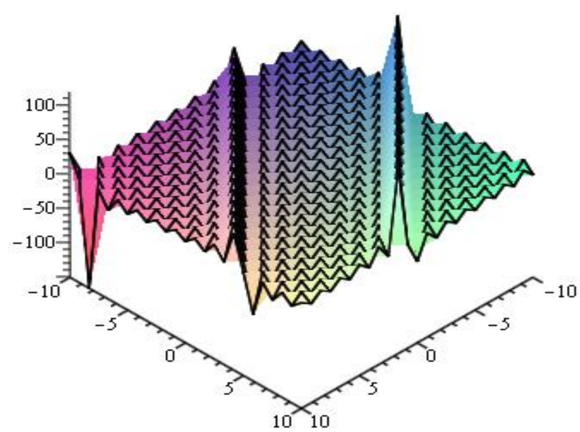
Section (3.3) for the solution (3.3.1') with $a_1 = -2, b = 4, c_1 = 1$

FIGURE 4



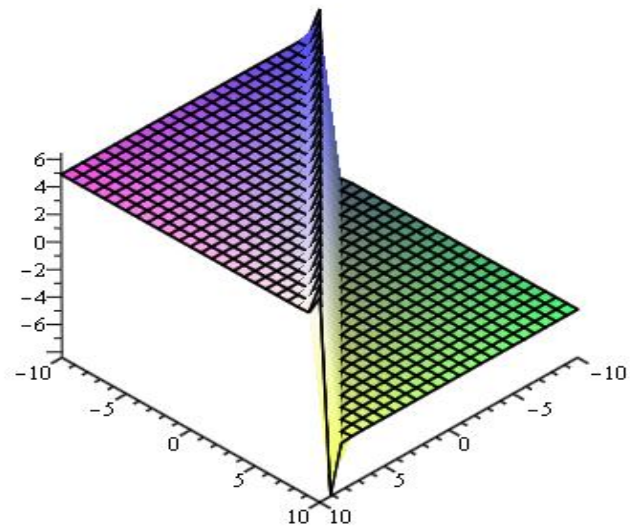
Section (3.3) for the solution (3.3.1’’) with $a_1 = 4, b = -2, c_1 = 1$

FIGURE 5



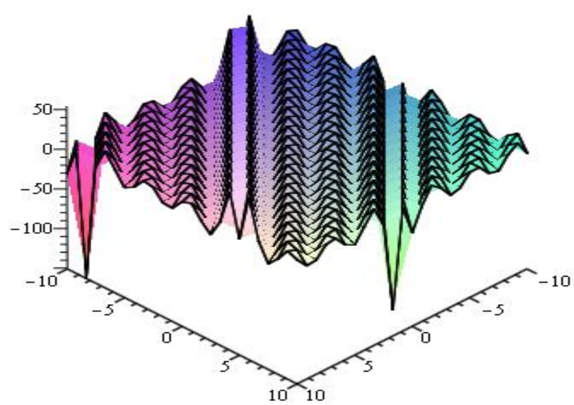
Section (3.3) for the solution (3.3.1’’)with $a_1 = 1, b = -8, c_1 = -1$

FIGURE 6



Section (3.3) for the solution (3.3.1''') with $a_1 = -8, b = 1, c_1 = -1$

FIGURE 7



Section (3.3) for the solution (3.3.1''') with $a_1 = 2, b = -4, c_1 = 1$

Generator (3)

The generator (3) is $V_3 + a_1 V_1$

The corresponding characteristic equation is given by

$$\frac{dx}{\frac{x}{3} + a_1} = \frac{dt}{t} = \frac{du}{-\frac{u}{3}}.$$

The generator (3) in the optimal system defines the similarity variable and similarity solution as follows

$$\zeta = \frac{(x+3a_1)^3}{t},$$

$$u = t^{1/3}\left(F\left(\frac{(x+3a_1)^3}{3}\right)\right)^{1/3}.$$

Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (3.1.1) reduces to the following nonlinear ordinary differential equation (ODE).

$$F + \zeta F' + F' + 3bF^2F' - 3F''' = 0.$$

This ODE cannot be further solved.

CHAPTER 4

BURGER'S KORTEWEG-dE VRIES EQUATION

Burger's equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics such as modelling of such as gas dynamics and traffic flow. It is named for Johannes Martinus Burgers (1895-1981). It is well known that many physical phenomenon can be described by the Korteweg-de Vries Burger's equation (KdVB equation). It arises in various contexts as a model equation incorporating the effects of dispersion, dissipation and non linearity.

KdV-Burger's equation has various physical applications, for instance, it can serve as a nonlinear wave model for a fluid in an elastic tube [16], of a liquid with small bubbles and turbulence [15]. A number of theoretical issues related to KdVB equation have received considerable attention. In particular, the travelling wave solution has been studied extensively.

Typical examples are provided by the behaviour of long waves in shallow water and waves in plasmas. We note that KdVB equation is non integrable. The KdV-Burger's equation is one dimension generalisation of model description of density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. Several studies employing a large variety of methods have been conducted to derive explicit solutions for KdV-Burger's equation [27,33]. With the aid of mathematica, new explicit and exact travelling wave and solitary solutions for KdV-Burger's equation are obtained by using improved sine cosine methods and the Wu elimination method [12, 13, 14, 35].

We can also study extended tanh-method [27] to obtain some exact solutions of KdV-Burgers equation. This standard Tanh-method was first developed in 1996 by Malfliet and Hereman [21]. Lie's method to continuous transformation groups is applied to deduce a class of invariant(similarity) solutions.

4.1 Lie symmetries of optimal system

Consider the Burger's Korteweg-de Vries equation [15,33]

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = 0. \qquad (4.1.1)$$

Let the group of infinitesimal transformations be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$
$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2). \qquad (4.1.2)$$

The invariance under (4.1.2) means that if u is the solution of (4.1.1), then u^* is also a solution of it.

On invoking the invariance criterion as explained earlier, the following relation from the coefficients of the first order of ε is deduced:

$$\eta^t + \alpha u\eta^x + \alpha\eta u_x + \beta\eta^{xx} + \gamma\eta^{xxx} = 0, \qquad (4.1.3)$$

where η^t, η^x and η^{xxx} are prolonged infinitesimals acting on the enlarged space corresponding to u_t, u_x and u_{xxx} respectively.

The method of determining symmetry group mainly consists of finding the infinitesimals τ, ξ and η, which are functions of x, t and u.

The general solution of equation (4.1.3) provides infinitesimal elements τ, ξ and η for which the equation poses Lie symmetry. Using the expressions for $\eta^t, \eta^x, \eta^{xx}$ and η^{xxx}

$$(\eta_t - \alpha\eta_u uu_x - \beta u_{xx}\eta_u - \gamma\eta_u u_{xxx} + \alpha\tau_t uu_x + \beta\tau_t u_{xx} + \gamma\tau_t u_{xxx} - \xi_t u_x - \tau_u\alpha^2 u^2 u_x^2 -$$
$$\tau_u\beta^2 u_{xx}^2 - \tau_u\gamma^2 u_{xxx}^2 - 2\tau_u\alpha\beta uu_x - 2\tau_u\beta\gamma u_{xx}u_{xxx} - 2\tau_u\alpha\gamma uu_x u_{xxx} + \xi_u\alpha uu_x^2 + \xi_u\beta u_x u_{xx} +$$
$$\xi_u\gamma u_x u_{xxx}) + \alpha\eta u_x + (\alpha u\eta_x + \alpha\eta_u uu_x - \alpha\xi_x uu_x + \alpha^2\tau_x u^2 u_x + \alpha\beta uu_{xx}\tau_x + \alpha\gamma\tau_x uu_{xxx} -$$
$$\alpha\xi_u uu_x^2 + \alpha^2\tau_u u^2 u_x^2 + \alpha\beta\tau_u uu_x u_{xx} + \alpha\gamma\tau_u uu_x u_{xxx}) + (\beta\eta_{xx} + 2\beta\eta_{xu}u_x - \beta\xi_{xx}u_x +$$
$$\beta\alpha\tau_{xx}uu_x + \beta^2\tau_{xx}u_{xx} + \beta\gamma\tau_{xx}u_{xxx} + \beta\eta_{uu}u_x^2 - 2\beta\xi_{xu}u_x^2 + 2\beta\alpha\tau_{xu}uu_x^2 + 2\beta^2\tau_{xu}u_x u_{xx}$$
$$+ 2\beta\gamma\tau_{xu}u_x u_{xxx} - \beta\xi_{uu}u_x^3 + \beta\eta_u u_{xx} - 2\beta\xi_x u_{xx} - 2\beta\tau_x u_{tx} - 3\beta\xi_u u_x u_{xx} + \alpha\beta\tau_u uu_x u_{xx} +$$
$$\beta^2\tau_u u_{xx}^2 + \beta\gamma\tau_u u_{xx}u_{xxx} - 2\beta\tau_u u_x u_{tx} + \alpha\beta\tau_{uu}uu_x^3 + \beta^2\tau_{uu}u_{xx}u_x^2 + \gamma\beta\tau_{uu}u_x^2 u_{xxx}) + \gamma\eta_{xxx} +$$
$$3\gamma\eta_{xxu}u_x - \gamma\xi_{xxx}u_x + \gamma\tau_{xxx}(\alpha uu_x + \beta u_{xx} + \gamma u_{xxx}) + 3\gamma\tau_{xxu}u_x(\alpha uu_x + \beta u_{xx} + \gamma u_{xxx}) + 3\gamma\eta_{xuu}u_x^2$$
$$- 3\gamma\xi_{xxu}u_x^2 + \gamma\eta_{uuu}u_x^3 + 3\gamma\tau_{xuu}u_x^2(\alpha uu_x + \beta u_{xx} + \gamma u_{xxx}) - 3\gamma\xi_{xuu}u_x^3 + \gamma\tau_{uuu}u_x^3(\alpha uu_x + \beta u_{xx}$$
$$+ \gamma u_{xxx}) - \gamma\xi_{uuu}u_x^4 - 3\gamma\tau_{xx}u_{tx} + 3\gamma\eta_{xu}u_{xx} - 3\gamma\xi_{xx}u_{xx} + 3\gamma\eta_{uu}u_x u_{xx} - 9\gamma\xi_{xu}u_x u_{xx} + 3\gamma\tau_{xu}u_{xx}($$
$$\alpha uu_x + \beta u_{xx} + \gamma u_{xxx}) - 6\gamma\tau_{xu}u_x u_{tx} - 6\gamma\xi_{uu}u_x^2 u_{xx} - 3\gamma\tau_{uu}u_{xt}u_x^2 + 3\gamma\tau_{uu}u_{xx}u_x(\alpha uu_x + \beta u_{xx} +$$
$$\gamma u_{xxx}) - 3\gamma\xi_u u_{xx}^2 - 3\gamma\tau_u u_{xx}u_{tx} - 3\gamma\tau_x u_{txx} + \gamma\eta_u u_{xxx} - 3\gamma\xi_x u_{xxx} - 4\gamma\xi_u u_x u_{xxx} + \gamma\tau_u u_{xxx}(\alpha uu_x +$$
$$\beta u_{xx} + \gamma u_{xxx}) - 3\gamma\tau_u u_{xxt}u_x = 0$$

where u_t must be replaced by equation $u_t = -\alpha uu_x - \beta u_{xx} - \gamma u_{xxx}$. On substituting the coefficients of different differentials equal to zero lead to number of PDEs in τ, ξ and η that need to be satisfied. The set of determining equations for the group of infinitesimals τ, ξ and η, which is obtained from above equation, after equating the coefficients of various derivative terms to zero, is as follows:

$\tau_u = 0,$
$\tau_x = 0,$
$\xi_u = 0,$
$\eta_{uu} = 0,$
$\tau_t = 3\xi_x,$
$-\beta\eta_u + \beta\tau_t + \beta\eta_u - 2\beta\xi_x + 3\gamma\eta_{xu} - 3\gamma\xi_{xx} = 0,$
$\alpha u\tau_t - \xi_t - \alpha u\xi_x + \alpha\eta + 2\beta\eta_{xu} - \beta\xi_{xx} + \alpha\beta u\tau_{xx} + 3\gamma\eta_{xuu} - \gamma\xi_{xxx} = 0,$
$\eta_t + \alpha u\eta_x + \beta\eta_{xx} + \gamma\eta_{xxx} = 0.$

The set of equations helps us to obtain infinitesimals τ, ξ and η as follows

$$\xi(x,t,u) = a + ct,$$
$$\tau(x,t,u) = b,$$
$$\eta(x,t,u) = \frac{c}{\alpha}.$$

where a, b, c are arbitrary constants. The Lie algebra associated with above equation consists of the following three vector fields,

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t},$$
$$V_3 = t\frac{\partial}{\partial x} + \frac{1}{\alpha}\frac{\partial}{\partial u}. \qquad (4.1.4)$$

The commutator table and the adjoint table for Lie algebra (4.1.1) is as follows.
The commutator (Lie Bracket) of V_α and V_β is first order operator defined by

$$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha.$$

COMMUTATOR TABLE

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	V_1
V_3	0	$-V_1$	0

Formulae for adjoint table is

$$adj[\exp(\varepsilon V_i)V_j] = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i,[V_{i,}V_j]] + ...$$

ADJOINT TABLE

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 - \varepsilon V_1$
V_3	V_1	$V_2 + \varepsilon V_1$	V_3

We deduce an optimal system of sub algebras with their corresponding generators:

$V = V_1,$

$V = V_2 + a_1 V_1,$

$V = V_3 + a_2^{'} V_2.$

4.2 Symmetry Reductions and Exact Solutions of Burger's KdV equation

Generator (1)

The generator (1) is V_1.

The corresponding characteristic equation is given by,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

The generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$t = \zeta.$

$u(x,t) = F(t) = F(\zeta).$

Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (4.1) reduces to the following nonlinear ordinary differential equation (ODE)

$F^{'} = 0.$

It gives a constant solution. (4.2.1)

Generator (2)

The generator (2) is $V_2 + a_1 V_1$

The characteristic equation is given by

$$\frac{dt}{1}=\frac{dx}{a_1}=\frac{du}{0}.$$

The generator (2) in the optimal system defines the similarity variable and similarity solution as follows

$$x-a_1t=\zeta,$$

$$u(x,t)=F(\zeta).$$

Using the similarity variable and the forms of the similarity solution, the nonlinear partial differential equation (PDE) (4.1) reduces to the following nonlinear ordinary differential equation (ODE).

$$-a_1F^{'}+\alpha FF^{'}+\beta F^{''}+\gamma F^{'''}=0.$$

Its solution in maple is

$$f(\zeta)=c_3,$$

$$f(\zeta)=\frac{1}{25}\left(\frac{3\beta^2+25a_1\gamma}{\alpha\gamma}\right)+\frac{6}{25}\beta^2\tanh\left(\frac{-c_1+\frac{1}{10}\frac{\beta\zeta}{\gamma}}{\alpha\gamma}\right)-\frac{3}{25}\beta^2\tanh\left(\frac{-c_1+\frac{1}{10}\frac{\beta\zeta}{\gamma}}{\alpha\gamma}\right)^2,$$

$$f(\zeta)=\frac{1}{25}\left(\frac{3\beta^2+25a_1\gamma}{\alpha\gamma}\right)+\frac{6}{25}\beta^2\tanh\left(\frac{c_1+\frac{1}{10}\frac{\beta\zeta}{\gamma}}{\alpha\gamma}\right)-\frac{3}{25}\beta^2\tanh\left(\frac{c_1+\frac{1}{10}\frac{\beta\zeta}{\gamma}}{\alpha\gamma}\right)^2.$$

Thus, the following are solutions of equation (4.1), then equations are obtained

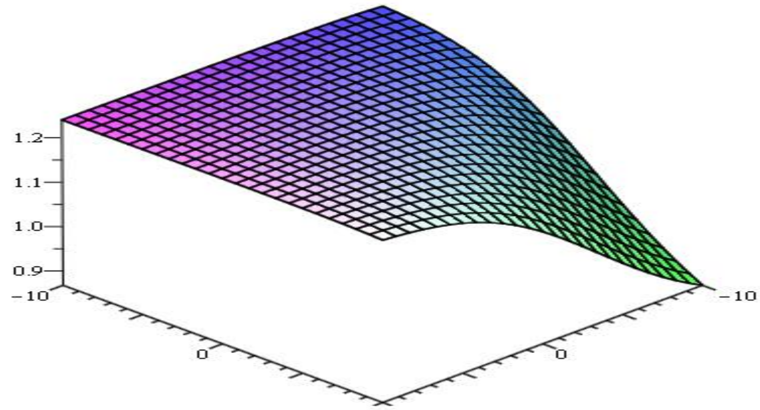
$$u(x,t)=c_3,$$

$$u(x,t)=\frac{1}{25}\left(\frac{3\beta^2+25a_1\gamma}{\alpha\gamma}\right)+\frac{6}{25}\beta^2\tanh\left(\frac{-c_1+\frac{1}{10}\frac{\beta(x-a_1t)}{\gamma}}{\alpha\gamma}\right)-\frac{3}{25}\beta^2\tanh\left(\frac{-c_1+\frac{1}{10}\frac{\beta(x-a_1t)}{\gamma}}{\alpha\gamma}\right)^2, \quad (4.2.2)$$

$$u(x,t)=\frac{1}{25}\left(\frac{3\beta^2+25a_1\gamma}{\alpha\gamma}\right)+\frac{6}{25}\beta^2\tanh\left(\frac{c_1+\frac{1}{10}\frac{\beta(x-a_1t)}{\gamma}}{\alpha\gamma}\right)-\frac{3}{25}\beta^2\tanh\left(\frac{c_1+\frac{1}{10}\frac{\beta(x-a_1t)}{\gamma}}{\alpha\gamma}\right)^2, \quad (4.2.2^{'})$$

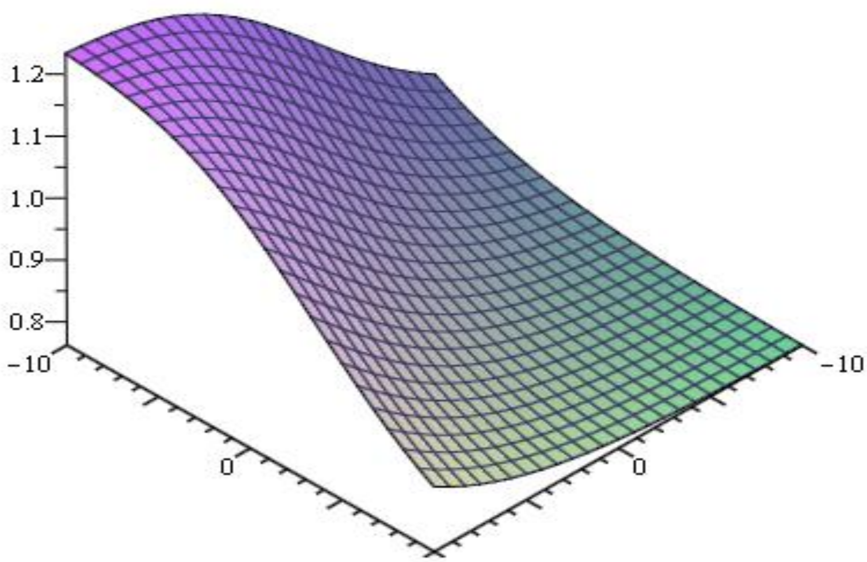
The graphs are as follows

FIGURE 1



Section (4.2) for the solution (4.2.2) with $\alpha = 1, \beta = 1, \gamma = 1, c_1 = 1, a_1 = 1$

FIGURE 2



Section (4.2) for the solution (4.2.2') with $\alpha = 1, \beta = 1, \gamma = 1, c_1 = 1, a_1 = 1$

Generator (3)

The generator (3) is $V_3 + a_2^{'} V_2$

The corresponding characteristic equation is given by

$$\frac{dx}{t} = \frac{dt}{a_2^{'}} = \frac{du}{1/\alpha}.$$

The generator (3) in the optimal system defines the similarity variable and similarity solution as follows

$$\zeta = a_2^{'} x - \frac{t^2}{2},$$

$$u(x,t) = \frac{1}{\alpha a_2^{'}}\left(t + F\left(a_2^{'} x - \frac{t^2}{2}\right)\right).$$

Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (4.1) reduces to the following nonlinear ordinary differential equation (ODE).

$$1 + \frac{FF^{'}}{\alpha a_2^{'}} + \frac{\beta a_2^{'} F^{'''}}{\alpha} + \frac{\gamma a_2^{'2} F^{''''}}{\alpha} = 0.$$

In this case we are able to find only reduction.

CHAPTER 5

GARDNER EQUATION

The Gardner equation is well known in the mathematical literature since the late sixties of 20th century. Initially it appeared in the context of construction of local conversation laws admitted by the KdV equation. Later on, the Gardner equation was generalized and found to be applicable in various branches of physics (solid state and plasma physics, fluid dynamics and quantum field theory).

The mathematical theory of the non linear evolution equations, starting from the Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation is an area of research for the past few decades [25, 28, 29, 32, 36]. The Korteweg-de Vries equation (KdV) is well known model for description of non linear long internal waves in the ocean. Its coefficients are defined by vertical density and currents stratisfaction. In a shelf zone the coefficient of quadratic nonlinearity may tend to zero, and the nonlinear model should be modified. In particular, the Gardner's Equation, that is also known as mixed KdV-mKdV equation. The Gardner's Equation which differs from the KdV by presence of an additional term of cubic nonlinearity may be used as the generalized model [10, 23, 34].

The Gardner equation shows up, particularly, in the context of internal gravity waves in a density-stratified ocean. This is commonly described by the KdV equations and its versions with small nonlinearity. However, there are situations when waves with strong nonlinearity is experienced, as in the case of Coastal Ocean Probe Experiment during 1995 in the Oregon Bay, the problem of creating an adequate theoretical model was deemed necessary. This lead to the study of Gardner's Equation [11].

5.1 Invariance Analysis

Consider the Gardner equation [23,34]

$$u_t + uu_x + u^2u_x + u_x + u + u_{xxx} = 0. \tag{5.1.1}$$

Let the group of infinitesimal transformation be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2).$$

The invariance means that if u is the solution of Gardner's equation, then u* is also a solution of it. On invoking the invariance criterion the following relation from these coefficients of the first order of ε is deduced

$$\eta^{t}+u\eta^{x}+\eta u_{x}+u^{2}\eta^{x}+2uu_{x}\eta+\eta^{x}+\eta+\eta^{xxx}=0. \tag{5.1.2}$$

where $\eta^{t},\eta^{x},\eta^{xxx}$ are infinitesimal generators corresponding to u_{t},u_{x},u_{xxx}. The method of obtaining the symmetry group of the given equation mainly consists of finding infinitesimals τ,ξ and η, which are functions of x, t and u. the general solution of equation (5.1.2) provides the infinitesimal element τ,ξ and η.

Substitute the values of $\eta^{t},\eta^{x},\eta^{xxx}$ in (5.1.2), we get

$$\begin{aligned}
&(\eta_{t}+u\eta_{x}+u^{2}\eta_{x}+\eta_{x}+\eta+\eta_{xxx})+u_{xxx}(-\eta_{u}+\tau_{t}+u\tau_{u}+u^{2}\tau_{x}+\tau_{x}+\tau_{xxx}+\eta_{u}-3\xi_{x})+\\
&u_{x}(-\eta_{u}-u\eta_{u}-u^{2}\eta_{u}+\tau_{t}+u\tau_{t}+u^{2}\tau_{t}-\xi_{t}+u\eta_{u}-u\xi_{x}+u\tau_{x}+u^{2}\tau_{x}+u^{3}\tau_{x}+u^{2}\eta_{u}\\
&-u^{2}\xi_{x}+u^{2}\tau_{x}+u^{3}\tau_{x}+u^{4}\tau_{x}+\eta_{u}-\xi_{x}+\tau_{x}+u^{2}\tau_{x}+u\tau_{x}+\eta+2u\eta+3\eta_{xxu}-\xi_{xxx}+\tau_{xxx}\\
&+u\tau_{xxx}+u^{2}\tau_{xxx})+u_{x}^{2}(-\tau_{u}-u^{2}\tau_{u}-u^{4}\tau_{u}-2\tau_{u}u^{3}-2u\tau_{u}-2u^{2}\tau_{u}+\xi_{u}+u^{2}\xi_{u}-u\xi_{u}\\
&+u\tau_{u}+u^{2}\tau_{u}+u^{3}\tau_{u}-u^{2}\xi_{u}+u^{2}\tau_{u}+u^{3}\tau_{u}+u^{4}\tau_{u}-\xi_{u}+\tau_{u}+u\tau_{u}+u^{2}\tau_{u}+3\tau_{xxu}+\\
&3u\tau_{xxu}+3u^{2}\tau_{xxu}+3\eta_{xuu}-3\xi_{xxu}+u\xi_{u})+u_{x}u_{xxx}(-2\tau_{u}-2u\tau_{u}-2u^{2}\tau_{u}+\xi_{u}+u\tau_{u}+\\
&u^{2}\tau_{u}+\tau_{u}+3\tau_{xxu}-4\xi_{u}+\tau_{u}+u\tau_{u}+u^{2}\tau_{u})+u_{x}^{3}(3\tau_{xuu}+3u\tau_{xuu}+3u^{2}\tau_{xuu}+\eta_{uuu}\\
&-3\xi_{xuu})+u_{x}^{2}u_{xxx}(3\tau_{xuu})+u_{x}^{3}u_{xxx}(\tau_{uuu})+u_{x}^{4}(\tau_{uuu}+u\tau_{uuu}+u^{2}\tau_{uuu}-\xi_{uuu})+u_{xx}(3\eta_{xu}-\\
&3\xi_{xx})+u_{x}u_{xx}(3\eta_{uu}-9\xi_{xu}+3\tau_{xu}+3u\tau_{xu}+3u^{2}\tau_{xx})+u_{xxx}^{xx}(3\tau_{xu})+u_{x}u_{tx}(-6\tau_{xu})+\\
&u_{x}^{2}u_{xx}(-6\xi_{uu}+3\tau_{uu}+3u\tau_{uu}+3u^{2}\tau_{uu})+u_{x}^{2}u_{xt}(-3\tau_{uu})+u_{x}u_{xx}u_{xxx}(3\tau_{uu})+u_{xx}^{2}(-3\xi_{u})\\
&+u_{xx}u_{tx}(-3\tau_{u})+u_{txx}(-3\tau_{x})+u_{xxt}u_{x}(-3\tau_{u})+\eta u_{x}+2uu_{x}\eta+\eta=0.
\end{aligned}$$

where u_{t} must be replaced by substituting its value from Gardner's equation where $u_{t}=-uu_{x}-u^{2}u_{x}-u_{x}-u-u_{xxx}$ Equating coefficients of $u_{xx}^{2},u_{xt},u_{xxt},u_{x}$, constant etc. equal to zero and we obtain the following equations.

$$\begin{aligned}
&\tau_{u}=0,\\
&\tau_{x}=0,\\
&\xi_{u}=0,\\
&\eta_{uu}=0,\\
&\tau_{t}=3\xi_{x},\\
&\eta_{xu}-\xi_{xx}=0,\\
&\eta_{t}+u\eta_{x}+u^{2}\eta_{x}+\eta+\eta_{xxx}=0,\\
&-\eta_{u}-u\eta_{u}-u^{2}\eta_{u}+\tau_{t}+u\tau_{t}+u^{2}\tau_{t}-\xi_{t}+u\eta_{u}-u\xi_{x}+u^{2}\eta_{u}-u^{2}\xi_{x}+\\
&\eta_{u}-\xi_{x}+\eta+2u\eta+3\eta_{xxu}-\xi_{xxx}=0.
\end{aligned}$$

The solutions of determining equations is given by

$\xi(x,t,u) = a,$

$\tau(x,t,u) = b,$

$\eta(x,t,u) = 0.$

where a, b, c are three arbitrary parameters. Hence, the point symmetry generators admitted by the Gardner's equation (5.1.1) are

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t}. \qquad (5.1.3)$$

The commutator table and adjoint table for Lie algebra (5.1.3) can easily be constructed as follows:

The commutator (Lie Bracket) of V_α and V_β is first order operator defined by

$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha.$

COMMUTATOR TABLE

	V_1	V_2
V_1	0	0
V_2	0	0

Formulae for adjoint table is

$$adj[\exp(\varepsilon V_i)V_j] = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i,[V_{i,}V_j]] + ...$$

ADJOINT TABLE

	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

5.2 Exact Solutions for basic Vector fields of Optimal system

In this section, corresponding to each generator in the optimal system of sub algebra, the reductions of PDEs (5.1.1) into ODEs in terms of similarity variable ξ and the new dependent variables – F, are obtained using the auxiliary equations. Some exact solutions of each of reduced system are then attempted.

Generator (1)

The generator (1) is V_1.

The corresponding characteristic equation is given by,

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{0}.$$

The generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$t=\zeta,$

$u(x,t)=F(t)=F(\zeta).$

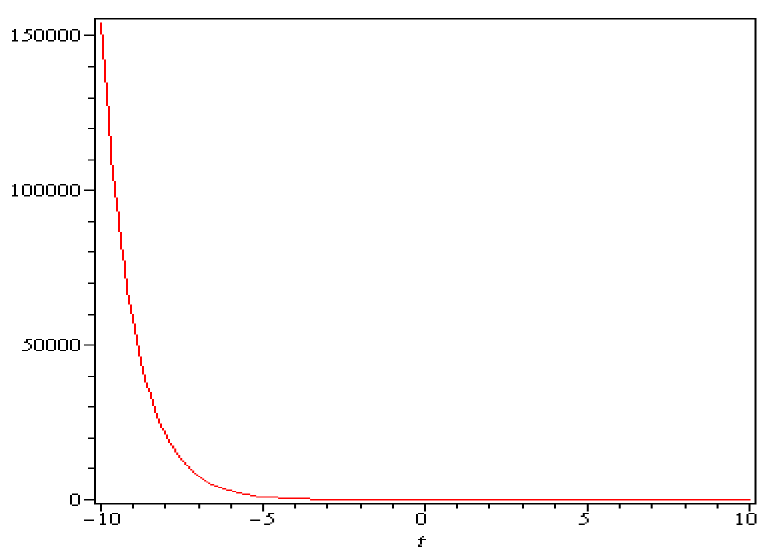
Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (5.1.1) reduces to the following nonlinear ordinary differential equation (ODE)

$F^{'}+F=0.$

Using maple we obtain its solution as

$$F(t)=c_1e^{-t}. \qquad (5.2.1)$$

and its graph is as follows



Section (5.2) for solution (5.2.1) with $c_1=7, t=-10..10$

Generator (2)

The generator (2) is V_2

The corresponding characteristic equation is given by,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Thus, the generator (2) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = x,$$

$$u(x,t) = F(x).$$

Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (5.1.1) reduces to the following nonlinear ordinary differential equation (ODE)

$$FF^{'} + F^{2}F^{'} + F^{'} + F + F^{'''} = 0.$$

In this case we can only find the reduction.

CHAPTER 6

EXTENDED FORM OF mKdV EQUATION

The Korteweg-de Vries equation arises as an approximate equation governing weakly nonlinear long waves in a shallow channel. When higher order effects are included, extended form of mKdV equation [20] is obtained. In recent paper [19,20] Lou and Lin studied nonlinear evolution equation in the form

$$u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0.$$

We study extended Korteweg-de Vries equation, that is the usual Korteweg-de Vries equation but with inclusion of extra nonlinear term, for the case when the coefficient of the cubic nonlinear term has an opposite polarity to that of coefficient of linear dispersive term. As this equation is integrable, the number and type of solutions formed can be determined from an appropriate spectral problem. For initial disturbances of small amplitude, the number and type of solitons generated is similar to the well known situation for Korteweg-de Vries equation.

Here we consider the analogous problem for the extended Korteweg-de Vries equation [19,20], which has an additional cubic term whose coefficient has the opposite polarity to the coefficient of linear dispersive term. This equation supports a family of solitons ranging from small amplitude ones to similar to those of KdV equation. Like KdV equation [28], this extended equation is also integrable and has an associated scattering problem whose discrete eigen values again determine the solitons which can emerge from a localised initial condition.

However unlike KdV equation, the solutions of extended KdV equation also depend critically on the shape of initial disturbances. The extended KdV can be transformed into higher order member of integrable equations. We are solving the extended mKdV equation using Lie classical method [7].

6.1 Lie Symmetries of Extended KdV equation

Consider the extended Korteweg-de Vries equation

$$u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0. \tag{6.1.1}$$

Let the group of infinitesimal transformation be defined as

$$\begin{aligned} u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \\ x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2). \end{aligned} \tag{6.1.2}$$

The invariance under (6.1.2) means that if u is the solution of (6.1.1), then u^* is also a solution of (6.1.1).

On invoking the invariance criterion as explained earlier, the following relation from the coefficients of the first order of ε is deduced:

$$\eta^t + a_1\eta^{xxx} + a_2\eta^x + a_3u\eta^x + a_3\eta u_x + a_4u^2\eta^x + 2a_4uu_x\eta = 0, \qquad (6.1.3)$$

where $\eta^t, \eta^{xxx}, \eta^x$ are prolonged infinitesimals acting on enlarged space corresponding to u_t, u_{xxx} and u_x respectively.

The method of determining symmetry group mainly consists of finding the infinitesimals τ, ξ and η, which are functions of x, t and u.

The general solution of equation (6.1.3) provides infinitesimal elements τ, ξ and η for which the equation poses Lie symmetry.

Using the expressions for η^t, η^x and η^{xxx} we obtain the following and solving the equation (6.1.3) by arranging the substituted values, we get

$$\begin{aligned}
&\eta_t - a_1\eta_u u_{xxx} - a_2\eta_u u_x - a_3uu_x\eta_u - a_4u^2u_x\eta_u + a_1\tau_t u_{xxx} + a_2\tau_t u_x + a_3\tau_t uu_x + \\
&a_4\tau_t u^2u_x - \xi_t u_x - a_1^2\tau_u u_{xxx}^2 - a_2^2\tau_u u_x^2 - 2a_1a_2\tau_u u_x u_{xxx} - a_3^2\tau_u u^2u_x^2 - a_4^2\tau_u u^4u_x^2 - \\
&2a_3a_4\tau_u u^3u_x^2 - 2a_1a_3\tau_u uu_x u_{xxx} - 2a_1a_4\tau_u u^2u_x u_{xxx} - 2a_2a_3\tau_u uu_x^2 - 2a_2a_4\tau_u u^2u_x^2 \\
&+ a_1\xi_u u_x u_{xxx} + a_2\xi_u u_x^2 + a_3\xi_u uu_x^2 + a_4\xi_u u^2u_x^2 + a_3\eta u_x + 2a_4uu_x\eta + (a_2\eta_x + a_2\eta_u u_x \\
&- a_2\xi_x u_x + a_1a_2\tau_x u_{xxx} + a_2^2\tau_x u_x + a_2a_3\tau_x uu_x + a_2a_4u^2u_x\tau_x - a_2\xi_u u_x^2 + a_1a_2\tau_u u_x u_{xxx} \\
&+ a_2^2\tau_u u_x^2 + a_2a_3\tau_u uu_x^2 + a_2a_4\tau_u u^2u_x^2) + (a_3u\eta_x + a_3\eta_u uu_x - a_3uu_x\xi_x + a_1a_3u\tau_x u_{xxx} \\
&+ a_2a_3uu_x\tau_x + a_3^2u^2\tau_x u_x + a_3a_4u^3u_x\tau_x - a_3uu_x^2\xi_u 1 + a_1a_3uu_x u_{xxx}\tau_u + a_2a_3u\tau_u u_x^2 + \\
&a_3^2u^2\tau_u u_x^2 + a_3a_4u^3\tau_u u_x^2) + a_4u^2\eta_x + a_4u^2\eta_u u_x - a_4u^2u_x\xi_x + a_4u^2\tau_x(a_1u_{xxx} + a_2u_x + \\
&a_3uu_x + a_4u^2u_x) - a_4u^2u_x^2\xi_u + a_4u^2u_x\tau_u(a_1u_{xxx} + a_2u_x + a_3uu_x + a_4u^2u_x) + a_1\eta_{xxx} + \\
&3a_1\eta_{xxu}u_x - a_1u_x\xi_{xxx} + a_1^2\tau_{xxx}uu_{xxx} + a_1a_2\tau_{xxx}u_x + a_1a_3\tau_{xxx}uu_x + a_1a_4\tau_{xxx}u^2u_x + 3a_1^2\tau_{xxu}u_x u_{xxx} \\
&+ 3a_1a_2\tau_{xxu}u_x^2 + 3a_1a_3u\tau_{xxu}u_x^2 + 3a_1a_4u^2\tau_{xxu}u_x^2 + 3a_1\eta_{xuu}u_x^2 - 3a_1\xi_{xxu}u_x^2 + 3a_1^2\tau_{xuu}u_x^2u_{xxx} \\
&+ 3a_1a_2\tau_{xuu}u_x^3 + 3a_1a_3u\tau_{xuu}u_x^3 + 3a_1a_4u^2\tau_{xuu}u_x^3 + a_1\eta_{uuu}u_x^3 - 3a_1\xi_{xuu}u_x^3 + a_1^2\tau_{uuu}u_x^3u_{xxx} \\
&+ a_1a_2\tau_{uuu}u_x^4 + a_1a_3u\tau_{uuu}u_x^4 + a_1a_4u^2\tau_{uuu}u_x^4 - a_1\xi_{uuu}u_x^4 - 3a_1\tau_{xx}u_{tx} + 3a_1\eta_{xu}u_{xx} - 3a_1\xi_{xx}u_{xx} \\
&+ 3\eta_{uu}a_1u_x u_{xx} - 9a_1\xi_{xu}u_x u_{xx} + 3a_1^2\tau_{xu}u_{xx}u_{xxx} + 3a_1a_2\tau_{xu}u_{xx}u_x + 3a_1a_3u\tau_{xu}u_x u_{xx} + \\
&3a_1a_4u^2\tau_{xx}u_{xx}u_x - 6a_1\tau_{xu}u_x u_{tx} - 6a_1\xi_{uu}u_x^2u_{xx} - 3a_1\tau_{uu}u_{xt}u_x^2 + 3a_1^2\tau_{uu}u_x u_{xx}u_{xxx} + 3a_1a_2\tau_{uu}u_x^2u_{xx} \\
&+ 3a_1a_3\tau_{uu}uu_x^2u_{xx} + 3a_1a_4u^2\tau_{uu}u_{xx}u_x^2 - 3a_1\xi_u u_{xx}^2 - 3a_1\tau_u u_{xx}u_{tx} - 3a_1\tau_x u_{txx} + a_1\eta_u u_{xxx} \\
&- 3a_1\xi_x u_{xxx} - 4a_1\xi_u u_x u_{xxx} + a_1^2\tau_u u_{xxx}^2 + a_1a_2\tau_u u_x u_{xxx} + a_1a_3u\tau_u u_x u_{xxx} + a_1a_4u^2\tau_u u_x u_{xxx} - 3a_1\tau_u u_{xxt}u_x = 0
\end{aligned}$$

where u_t must be replaced by equation $u_t = -a_1u_{xxx} - a_2u_x - a_3uu_x - a_4u^2u_x$. On substituting the coefficients of different differentials equal to zero lead to number of

PDEs in τ,ξ and η, that need to be satisfied. The set of determining equations for the group of infinitesimals τ,ξ and η, which is obtained from above equation, after equating the coefficients of various derivative terms to zero, is as follows:

$$\tau_u = 0,$$
$$\tau_x = 0,$$
$$\xi_u = 0,$$
$$\eta_{uu} = 0,$$
$$\eta_{xu} - \xi_{xx} = 0,$$
$$\tau_t = 3\xi_x,$$
$$\eta_t + a_2\eta_x + a_3u\eta_x + a_4u^2\eta_x + a_1\eta_{xxx} = 0,$$

$$-a_2\eta_u - a_3u\eta_u - a_4u^2\eta_u + a_2\tau_t + a_3\tau_t u + a_4\tau_t u^2 - \xi_t - a_2\xi_x + a_2\eta_u + a_3u\eta_u - a_3u\xi_x + 2a_4u\eta + a_4u^2\eta_u$$
$$-a_4u^2\xi_x + 3a_1\eta_{xxu} - a_1\xi_{xxx} + a_3\eta = 0.$$

The set of equations helps us to obtain infinitesimals τ,ξ and η as follows

$$\xi(x,t,u) = a + c\left(\frac{x}{3} - \frac{ta_3^2}{6a_4} + \frac{2}{3}ta_2\right),$$
$$\tau(x,t,u) = b + ct,$$
$$\eta(x,t,u) = c\left(\frac{-a_3 - 2a_4u}{6a_4}\right).$$

where a, b, c are arbitrary constants. The Lie algebra associated with above equation consists of the following three vector fields,

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t}, \qquad (6.1.4)$$
$$V_3 = \left(\frac{x}{3} - \frac{ta_3^2}{6a_4} + \frac{2}{3}ta_2\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(\frac{-a_3 - 2a_4u}{6a_4}\right)\frac{\partial}{\partial u}.$$

The commutator table and the adjoint table for Lie algebra (6.1.4) can be constructed. The commutator (Lie Bracket) of V_α and V_β is first order operator defined by

$$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha.$$

COMMUTATOR TABLE

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$a_1^{'}V_1 + a_2^{'}V_2$
V_3	0	$-(a_1^{'}V_1 + a_2^{'}V_2)$	0

Formulae for adjoint table is

$$adj[\exp(\varepsilon V_i)V_j] = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_{i,} V_j]] + \ldots\ldots\ldots\ldots$$

ADJOINT TABLE

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 - \varepsilon(a_1^{'}V_1 + a_2^{'}V_2)$
V_3	V_1	$V_2 + \varepsilon(a_1^{'}V_1 + a_2^{'}V_2)$	V_3

We deduce an optimal system of sub algebras with their corresponding generators

$V = V_3,$

$V = V_2 + a_1 V_1,$

$V = V_1.$

6.2 Group invariant solutions of Extended KdV equation

Generator (1)

The generator (1) is V_1.

The corresponding characteristic equation is given by,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

The generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$t = \zeta,$

$u(x,t) = F(t) = F(\zeta).$

Using the similarity variable and the forms of the similarity solution, the non linear partial differential equation (PDE) (6.1) reduces to the following nonlinear ordinary differential equation (ODE).

$F^{'} = 0$

This equation gives us constant solution.

Generator (2)

The generator (2) is $V_2 + a_1 V_1$.

The characteristic equation is given by

$$\frac{dt}{1} = \frac{dx}{a_1} = \frac{du}{0}.$$

The generator (2) in the optimal system defines the similarity variable and similarity solution as follows

$x = a_1 t + \zeta,$

$u(x,t) = F(\zeta) = F(x - a_1 t).$

Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (6.1) reduces to the following nonlinear ordinary differential equation (ODE).

$-a_1 F^{'} + a_1 F^{''''} + a_2 F^{'} + a_3 F F^{'} + a_4 F^2 F^{'} = 0.$

Using maple its solution obtained are as follows

$$F(\zeta) = \frac{-1}{2}\frac{a_3}{a_4} - \frac{1}{2}\frac{\sqrt{12a_4a_1 - 12a_4a_2 + 3a_3^2}\tanh\left(-c_1 + \frac{1}{4}\sqrt{\frac{-2a_4a_1\left(4a_4a_1 - 4a_4a_2 + a_3^2\right)}{a_4a_1}}\zeta\right)}{a_4},$$

$$F(\zeta) = \frac{-1}{2}\frac{a_3}{a_4} + \frac{1}{2}\frac{\sqrt{12a_4a_1 - 12a_4a_2 + 3a_3^2}\tanh\left(-c_1 + \frac{1}{4}\sqrt{\frac{-2a_4a_1\left(4a_4a_1 - 4a_4a_2 + a_3^2\right)}{a_4a_1}}\zeta\right)}{a_4},$$

$$F(\zeta) = \frac{-1}{2}\frac{a_3}{a_4} + \frac{1}{2}\frac{\sqrt{12a_4a_1 - 12a_4a_2 + 3a_3^2}\tanh\left(c_1 + \frac{1}{4}\sqrt{\frac{-2a_4a_1\left(4a_4a_1 - 4a_4a_2 + a_3^2\right)}{a_4a_1}}\zeta\right)}{a_4}.$$

Thus, the following solution of the equation (6.1) is obtained

$$u(x,t)=\frac{-1}{2}\frac{a_3}{a_4}+\frac{1}{2}\frac{\sqrt{12a_4a_1-12a_4a_2+3a_3^2}\tanh\left(-c_1+\frac{1}{4}\sqrt{\frac{-2a_4a_1\left(4a_4a_1-4a_4a_2+a_3^2\right)}{a_4a_1}}(x-a_1t)\right)}{a_4},$$

$$u(x,t)=\frac{-1}{2}\frac{a_3}{a_4}-\frac{1}{2}\frac{\sqrt{12a_4a_1-12a_4a_2+3a_3^2}\tanh\left(c_1+\frac{1}{4}\sqrt{\frac{-2a_4a_1\left(4a_4a_1-4a_4a_2+a_3^2\right)}{a_4a_1}}(x-a_1t)\right)}{a_4},$$

$$u(x,t)=\frac{-1}{2}\frac{a_3}{a_4}+\frac{1}{2}\frac{\sqrt{12a_4a_1-12a_4a_2+3a_3^2}\tanh\left(c_1+\frac{1}{4}\sqrt{\frac{-2a_4a_1\left(4a_4a_1-4a_4a_2+a_3^2\right)}{a_4a_1}}(x-a_1t)\right)}{a_4}.$$

Generator (3)

The generator (3) is V_3.

The corresponding characteristic equation is given by

$$\frac{dx}{\frac{x}{3}-\frac{ta_3^2}{6a_4}+\frac{2}{3}ta_2}=\frac{dt}{t}=\frac{du}{\frac{(-a_3-2a_4u)}{6a_4}}.$$

Using the similarity variable, the forms of the similarity solution, the nonlinear partial differential equation (PDE) (6.1) reduces to the following nonlinear ordinary differential equation (ODE).

$$\zeta F'+4F+12F'''+3F'F^2=0.$$

In this case we can only find reduction.

DISCUSSION

This thesis deals with the Lie Classical Method and its application to some nonlinear partial differential equations representing some interesting physical systems via Modified Korteweg de Vries equation, Burger's Korteweg de Vries equation, Extended Modified Korteweg de Vries equation, Gardner's equation. To determine the admissible symmetries, we refer to as the classical Lie approach. After obtaining the point symmetries admitted by a system under investigation, a formal approach of identifying an optimal system of Lie sub algebras has been opted with the help of the adjoint action of the Lie algebra. The basic generators contained in the optimal system have then been exploited to achieve the desired reductions of Partial Differential Equations to Ordinary Differential Equations. Corresponding to each generator in the optimal system of sub algebra, the reductions of Partial Differential Equations into Ordinary Differential Equations in terms of similarity variable and the new dependent variables are obtained using the auxiliary equations. Some exact solution of each reduced systems are then attempted. The resulting Ordinary Differential Equations have been examined subsequently for various types of exact solutions. The various exact solutions are in hyperbolic functions. Some plots to have an idea about the nature of solutions are obtained in the thesis.

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