Symmetries and Exact Solutions of Some Systems of Nonlinear Partial Differential Equations by Lie Classical Method

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Under

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled " **Symmetries and Exact Solutions of Some Systems of Nonlinear Partial Differential Equations by Lie Classical Method** *" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Rajesh Kumar Gupta.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

The objective of this thesis entitled, **" Symmetries and Exact Solutions of Some Systems of Nonlinear Partial Differential Equations by Lie Classical Methods"**, is to obtain the Lie symmetries and the exact solutions of nonlinear partial differential equations or their systems, which represent some of the important physical phenomenon. The nonlinear phenomena are encountered in a variety of situations in physics as well as in other natural applied sciences. Most of these phenomena are governed by nonlinear partial differential equations. The study of these systems of differential equations is often regarded as a difficult and confusing endeavour due to various limitations posed by the intrinsic nonlinearity.

This thesis comprises of five chapters. The brief outline of the research work presented chapter wise in this thesis is as follows:

In chapter 1, We have described the nonlinear partial differential equations and presented literature review.

In chapter 2, It contains the various definitions of Lie groups. It also includes the preliminary material.

In chapter 3, It presents symmetry reductions and exact solutions of Gardner equation. This chapter deals with the classical Lie method to obtain symmetries and reductions.

In chapter 4, We investigated symmetries and reductions of the Buckmaster equation. Corresponding to each basic vector field, the reductions of the system to ordinary equations are obtained. These reduced systems are further studied for exact solutions.

In chapter 5, We studied the classical Lie symmetries of the Degasperis-Procesi equation which is obtained through the Lie group method of the infinitesimal transformations. Secondly using the classical symmetries of the equation, similarity reduction are obtained.

Contents

Chapter 1

INTRODUCTION

1.1 Introduction

Nonlinear partial differential equations are partial differential equations with nonlinear terms. They describe many different physical systems, ranging from gravitation to fluid dynamics. They are difficult to study: there are almost no general techniques that work for all such equations, and usually each individual equation has to be studied as a separate problem. Various standard strategies are adopted to get approximate solutions of nonlinear partial differential equations. But these solutions do not provide much information about the system.

Strategies adopted to drive exact solutions to nonlinear equations are being avoided due to complicated and cumbersome calculations. Modern approaches seek methods applicable to non-linear partial differential equations as well as linear ones. In this context existence and uniqueness results, and theorems concerning the regularity of solutions, are more difficult. Since it is unlikely that explicit solutions can be obtained for any but the most special of problems, methods of solving the partial differential equation involve analysis within the appropriate function space - for example, seeking convergence of a sequence of functions which can be shown to approximately solve the partial differential equation, or describing the sought-for function as a fixed point under a self-map on the function space, or as the point at which some real-valued function is minimized.

Differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable. Instead, solutions can be approximated using numerical methods.

A partial differential equation is a differential equation that contains unknown multivariable functions and their partial derivatives. Partial differential equations are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. Partial differential equation can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. The term exact solution is often used for second and higher order nonlinear partial differential equations to denote a particular solution. The exact solutions are also helpful in designing and testing of numerical algorithms. The proposed work will be devoted to obtaining the exact solutions of nonlinear partial differential equations or their systems, which represent some of the important physical systems. Exact solutions for nonlinear equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but increasingly sought. So far, a number of methods have been proposed to construct the exact solutions; the group theoretic methods like Lie's classical method [1, 5], nonclassical method [2, 3], Steinberg's symmetry reduction method [6]; direct method [11], modified direct method [24]; truncated Painleve approach [9]; transformation methods [21]; anstze-based methods [10]; hyperbolic functions expansion methods [15]; elliptic functions expansion methods [14] and sine-cosine method [35] etc.

In this thesis applications of Lie Classical method is one of the useful methods in group theoretic techniques for solving partial differential equation.

In this thesis, three nonlinear partial differential equations are considered for exact solutions which are as follows:-

1-Gardner Equation-

$$u_t = 6(u+\varepsilon^2 u^2)u_x - u_{xxx}$$

2-Buckmaster Equation-

$$u_t = u_{xx}^4 + u_x^3$$

3-Degasperis-Procesi Equation-

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}.$$

All the solutions obtained for these nonlinear partial differential equation are exact analytic solutions. To check the correctness of these solutions, software MAPLE has been used.

1.2 Literature Review

In 1870, Sophus Lie introduced the notion of a continuous group of transformations. He was motivated by the lectures of his fellow Norwegian, Sylow, on the works of Abel and Galosis on solving algebraic equations. He developed the theory of “finite and continous groups”. Lie devoted his mathematical career to the development and application of his monument theory of continous groups. These groups are now known as “Lie groups”.

Lie groups of transformation are characterized by infinitesimal generators [7]. Lie gave an algorithm to find all infinitesimals generators of point transformations [7] and, more generally, contact transformations admitted by a given differential equation. Significantly, for a given differential equation, the basic applications of Lie groups of transformations only require knowledge of the admitted infinitesimal generators [7].

The entire subject lay dormant for nearly half a century until G. Birkhoff (1950) [8] called attention to the unexploited application of the Lie group of transformation to differential equations. Ovsiannikov [29] and his coworkers began a systematic program of successfully applying these methods to wide range of physically important problem.

This was followed by the work of Bluman and Cole [3, 4]. Since then, the theory has witness a veritable explosion of research both in the application to physical systems and its development Olver [28].

Lie’s continuous group theoretic ideas have been classified as direct methods and group theoretic methods. The direct method consists of separation of variables devised by Miller [26], and dimensional analysis due to Sedov [30]. Group theoretic methods are divided into two categories namely inspectional methods and deductive methods. Inspectional methods are two fold in the sense that the first one is due to Birkhoff [8] and the other is due to Hellums and Churchill [19]. In the class of deductive procedures, there are the following techniques proposed by different authors :Non-classical method (Bluman and Cole [3, 4]), Classical Lie method (Olver [28]) etc.

Chapter 2

METHODOLOGY

2.1 Preliminaries

Some basis definitions from Bluman and Anco [7] that are used in this thesis:-

GROUP

A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:
(1) **Closure property**- For any elements a and b of G, $\phi(a,b)$ is a element of G.
(2) **Associative property**- For any elements a,b,c of G:

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c).$$

(3) **Identity element**- There exists a unique identity element e of G such that for any element a of G:

$$\phi(a,e) = \phi(e,a) = a.$$

(4) **Inverse element**- For any element a of G there exists a unique inverse element a^{-1} in G such that:

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e.$$

ONE-PARAMETER GROUP OF TRANSFORMATION

Let $x = (x_1, x_2, x_3, ..., x_n)$ lie in a region $D \subset R^n$.The set of transformations

$$x^* = X(x;\varepsilon),$$

defined for each x in D and parameter ε in set $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters ε and δ in S, forms a one-parameter group of transformations on D if the following hold:

(1) For each ε in S the transformations are one-to-one onto D.

(2) S with the law of composition ϕ forms a group G.

(3) For each x in D, $x^* = x$ when $\varepsilon = \varepsilon_0$ corresponds to the identity e, i.e

$$X(x; \varepsilon_0) = x.$$

(4) If $x^* = X(x; \varepsilon)$, $x^{**} = X(x^*; \delta)$, then

$$x^{**} = X(x; \phi(\varepsilon, \delta)).$$

ONE-PARAMETER LIE GROUP OF TRANSFORMATION

A one-parameter group of transformations defines a one-parameter Lie group of transformations if, in addition to satisfying the above property (1) to (4), the following hold:

(5) ε is a continuous parameter, i.e., S is an interval in R. Without loss of generality, $\varepsilon = 0$ corresponds to the identity element e.

(6) X is infinitely differentiable with respect to x in D and an analytic function of ε in S.

(7) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, ε in S, δ in S.

Consider an example

$$x^* = x + \varepsilon,$$

$$y^* = y, \varepsilon \in R$$

And

$$\phi(\varepsilon, \delta) = \varepsilon + \delta.$$

This forms a one-parameter lie group of transformations.

INFINITESIMAL TRANSFORMATIONS

Consider a one-parameter (ε) Lie group of transformations

$$x^* = X(x; \varepsilon) \tag{2.1.1}$$

with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding (2.1.1) about ε=0, in some neighborhood of $\varepsilon = 0$, we get

$$x^* = x + \varepsilon\left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}\right) + \frac{1}{2}\varepsilon^2\left(\frac{\partial^2 X(x;\varepsilon)}{\partial \varepsilon^2}|_{\varepsilon=0}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x^* = x + \varepsilon(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}) + \bigcirc(\varepsilon^2)$$

Let

$$\xi(x) = \frac{\partial X(x;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}$$

The transformation $x + \varepsilon\xi(x)$ is called the infinitesimal transformation of the Lie group of transformations (2.1.1). The components of $\xi(x)$ are called the infinitesimals of (2.1.1).

INFINITESIMAL GENERATOR

The infinitesimal generator of the one-parameter Lie group of transformations (2.1.1) is the operator

$$X = X(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}$$

where ∇ is the gradient operator

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})$$

For any differentiable function $F(X) = F(x_1, x_2, ..., x_n)$, one has

$$XF(x) = \xi(x).\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial x_i}$$

INVARIANT FUNCTION

An infinitely differentiable function $F(x)$ is an invariant function of Lie group of transformations (2.1.1) if and only if, for any group of transformation (2.1.1),

$$F(x^*) = F(x)$$

If $F(x)$ is invariant function of (2.1.1), then $F(x)$ is called an invariant of (2.1.1) and $F(x)$ is said to be invariant under (2.1.1).

THEOREM

$F(x)$ is invariant under a lie group of transformations (2.1.1) if and only if

$$XF(x) = 0.$$

COMMUTATOR

For an r-parameter Lie group of transformation with infinitesimal generators X_α, $\alpha = 1, 2, ..., r$, the commutator (Lie Bracket) of X_α and X_β is a first order operator defined by

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = \sum_{i,j=1}^{n} \left[\left(\xi_{\alpha i}(x) \frac{\partial}{\partial x_i} \right) \left(\xi_{\beta j}(x) \frac{\partial}{\partial x_j} \right) - \left(\xi_{\beta i}(x) \frac{\partial}{\partial x_i} \right) \left(\xi_{\alpha j}(x) \frac{\partial}{\partial x_j} \right) \right]$$

$$[X_\alpha, X_\beta] = \sum_{j=1}^{n} \eta_j(x) \frac{\partial}{\partial x_j},$$

where

$$\eta_j(x) = \sum_{i=1}^{n} \left(\xi_{\alpha i}(x) \frac{\partial \xi_{\beta j}(x)}{\partial x_i} - \xi_{\beta i}(x) \frac{\partial \xi_{\alpha j}(x)}{\partial x_i} \right).$$

It immediately follows that

$$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]$$

ADJOINT VECTOR

Let G be a Lie group with Lie algebra L. For each vector $v \in L$, the adjoint vector ad v at $w \in L$ is ad $v|_w = [w, v] = -[v, w]$

The adjoint representation Ad G of the underlying Lie group can be reconstructed either by integrating the system of linear ordinary differential equations

$\frac{dw}{d\varepsilon}$=ad $v|_w$, $w(0) = w_0$,

with solution

$w(\varepsilon) = Ad(exp(\varepsilon v))w_0$,

Or, more simply, by summing the Lie series

$$Ad(exp(\varepsilon v))w_0 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adv)^n(w_0) = w_0 - \varepsilon[v, w_0] + \frac{\varepsilon^2}{2}[v, [v, w_0]] - ....$$

OPTIMAL SYSTEM

We need at present only one solution from each equivalence classes, as the rest may be found by applying appropriate group symmetries, a complete set of such solutions is referred to as an "optimal solution" of group invariant solution. The problem of deriving an optimal system of group invariant solutions is equivalent to an optimal system of Lie symmetries. The method used here is given by Olver [28] which basically consists of taking linear combinations of the generator and reducing them into their simplest equivalent form by applying a carefully chosen adjoint transformation.

$$adj[exp(\varepsilon v_i)] = v_j - \varepsilon[v_i, v_j] + \frac{\varepsilon^2}{2}[v_i[v_i, v_j]] + ...$$

where $[v_i, v_j]$ is the usual commutator.

In this thesis, we deal with the methods of group invariant solutions, based on the theory of continuous group of transformations, better known as “Lie groups”, acting on the space of independent and dependent variables of the system. The method is due to originally to Sophus Lie who unified and extended the special methods of integrating the differential equations. Through the constructive procedures Lie established that, in the case of ordinary differential equations, invariance under one-parameter symmetry group implies that the order of the equation can be reduced by one.

2.2 Invariance for the System of Partial Differential Equations

In Lie classical theory, we work with the applications of one parameter Lie group of transformations to nonlinear partial differential equations. Now, we will follow a procedure to derive infinitesimals corresponding to each derivative as considered by Bluman and Anco [7], we obtain the following.

Consider a kth order partial differential equation

$$u_{i_1,i_2,......,i_k} = f(x,u,\partial u,\partial^2 u,....,\partial^k u), \tag{2.2.1}$$

where $f(x,u,\partial u,\partial^2 u,....,\partial^k u)$, does not depend explicitly on $u_{i_1,i_2,......,i_k}$.

We first show how to derive the Lie group of infinitesimal transformations with infinitesimal

$$u^* = U^*(x,t,u;\varepsilon) = u+\varepsilon\eta(x,t,u)+\bigcirc(\varepsilon^2)$$
$$x^* = X^*(x,t,u;\varepsilon) = u+\varepsilon\xi(x,t,u)+\bigcirc(\varepsilon^2)$$
$$t^* = T^*(x,t,u;\varepsilon) = u+\varepsilon\tau(x,t,u)+\bigcirc(\varepsilon^2)$$

Where η, ξ and τ be the infinitesimal corresponding to u, x and t.

On invoking the invariance criterion the following relation from the coefficients of the first order of ε is deduced. Then the symmetry determining is in the form

$$u^k_{i_1,i_2,......,i_k} = \xi_j\frac{\partial f}{\partial x_j}+\eta\frac{\partial f}{\partial u}+\eta^1_j\frac{\partial f}{\partial u_j}+...........+\eta^k_{j_1,j_2,.....,j_k}\frac{\partial f}{\partial u_{j_1,j_2,......,j_k}},$$

where u satisfies (2.2.1)

Now we find the values of η^x, η^t, η^{xx}, η^{xt}, η^{xxx}, η^{xxt} etc.

First we need to calculate the auxiliary functions $\frac{\partial x}{\partial x^*}, \frac{\partial x}{\partial t^*}, \frac{\partial t}{\partial t^*}, \frac{\partial t}{\partial x^*}$.

By $\frac{\partial x}{\partial x^*}$ we understand that $u = u(x,t)$ and that only t^* is held fixed.

Hence

$$x = x^* - \varepsilon\xi(x,t,u)+\bigcirc(\varepsilon^2)$$

$$\frac{\partial x}{\partial x^*} = 1-\varepsilon\left(\frac{\partial \xi}{\partial x}\frac{\partial x}{\partial x^*}+\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial x}{\partial x^*}\right)+\bigcirc(\varepsilon^2),$$

$$\frac{\partial x}{\partial x^*}\left(1+\varepsilon\left(\frac{\partial \xi}{\partial x}+\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\right)\right)=1+\bigcirc(\varepsilon^2),$$

$$\frac{\partial x}{\partial x^*}=\left(1-\varepsilon\left(\frac{\partial \xi}{\partial x}+\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\right)\right)+\bigcirc(\varepsilon^2).$$

Similarly

$$\frac{\partial x}{\partial t^*}=-\varepsilon\left(\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial t}\right)+\bigcirc(\varepsilon^2),$$

$$\frac{\partial t}{\partial t^*}=1-\varepsilon\left(\frac{\partial \tau}{\partial t}+\frac{\partial \tau}{\partial u}\frac{\partial u}{\partial t}\right)+\bigcirc(\varepsilon^2),$$

$$\frac{\partial t}{\partial x^*}=-\varepsilon\left(\frac{\partial \tau}{\partial x}+\frac{\partial \tau}{\partial u}\frac{\partial u}{\partial x}\right)+\bigcirc(\varepsilon^2), \tag{2.2.2}$$

Hence, we find the following extensions:

First extensions

$$\frac{\partial u^*}{\partial x^*}=\frac{\partial}{\partial x^*}(u^*+\varepsilon\eta(x,t,u))+\bigcirc(\varepsilon^2),$$

$$=\frac{\partial(u^*+\varepsilon\eta(x,t,u))}{\partial x}\frac{\partial x}{\partial x^*}+\frac{\partial t}{\partial x^*}\frac{\partial u}{\partial t}+\bigcirc(\varepsilon^2). \tag{2.2.3}$$

Substituting (2.2.1) into (2.2.2) we lead to

$$\frac{\partial u^*}{\partial x^*}=\frac{\partial u}{\partial x}+\varepsilon\left(\frac{\partial \eta}{\partial x}+\left(\frac{\partial \eta}{\partial u}-\frac{\partial \xi}{\partial x}\right)\frac{\partial u}{\partial x}-\frac{\partial \tau}{\partial x}\frac{\partial u}{\partial t}-\frac{\partial \xi}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial \tau}{\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)+\bigcirc(\varepsilon^2). \tag{2.2.4}$$

Let η^x and η^t denote infinitesimals of $\frac{\partial u^*}{\partial x^*}$ and $\frac{\partial u^*}{\partial t^*}$ respectively. Then

$$\eta^x=\eta_x+(\eta_u-\xi_x)u_x-\tau_x u_t-\xi_u u_x^2-\tau_u u_x u_t. \tag{2.2.5}$$

And similarly

$$\eta^t=\eta_t+(\eta_u-\tau_t)u_t-\xi_t u_x-\tau_u u_t^2-\xi_u u_x u_t. \tag{2.2.6}$$

Second extensions

$$\begin{aligned}\eta^{xx}=&\eta_{xx}+(2\eta_{xu}-\xi_{xx})u_x-\tau_{xx}u_t+(\eta_{uu}-2\xi_{xu})u_x^2-2\tau_{xu}u_x u_t-\xi_{uu}u_x^3+(\eta_u-2\xi_x)u_{xx}\\&-2\tau_x u_{xt}-3\xi_u u_x u_{xx}-\tau_u u_t u_{xx}-2\tau_u u_t u_{tx}-\tau_{uu}u_x^2 u_t.\end{aligned}$$

$$\begin{aligned}\eta^{xt}=&\eta_{xt}+(\eta_{xu}-\tau_{tx})u_t+(\eta_{tu}-\xi_{xt})u_x-\tau_{xu}u_t^2+(\eta_{uu}-\xi_{xu}-\tau_{tu})u_x u_t-\xi_{tu}u_x^2-\tau_{uu}u_t^2 u_x\\&+\xi_{uu}u_x^2 u_t-\tau_x u_{tt}+(\eta_u-\xi_x-\tau_t)u_{xt}-\xi_t u_{xx}-2\tau_u u_t u_{xt}-2\xi_u u_x u_{xt}-\tau_u u_x u_{tt}-\xi_u u_{xx}u_t.\end{aligned}$$

$$\begin{aligned}\eta^{tt}=&\eta_{tt}+(2\eta_{tu}-\tau_{tt})u_t-\xi_{tt}u_x+(\eta_{uu}-2\tau_{tu})u_t^2-2\xi_{tu}u_x u_t-\tau_{uu}u_t^3-\xi_{uu}u_t^2 u_x+(\eta_u\\&-2\tau_t)u_{tt}-2\xi_t u_{xt}-3\tau_u u_x u_{tt}-2\xi_u u_x u_{xt}u_t.\end{aligned}$$

Third extensions

$$\begin{aligned}\eta^{txx} =&\eta_{txx}+(\eta_{xxu}-\tau_{xxt})u_t+(2\eta_{xtu}-\xi_{xxt})u_x-\xi_u u_t u_{xxx}-2\tau_u u_t u_{txx}-2\tau_u u_x u_{ttx}+\\ &(\eta_{uut}-2\xi_{xtu})u_x^2+(2\eta_{xuu}-\xi_{xxu}-2\tau_{xtu})u_x u_t-\tau_{xxu}u_t^2-2\tau_{xxu}u_x u_t^2-\xi_{uut}u_x^3+(\eta_{uuu}\\ &-2\xi_{xuu}-\tau_{tuu})u_x^2 u_t-\xi_{uuu}u_x^3 u_t-\tau_{uuu}u_x^2 u_t^2+(2\eta_{uu}-\xi_{xx}-2\tau_{tx})u_{tx}+(\eta_{tu}-2\xi_{tx})u_{xx}\\ &-\tau_{xx}u_{tt}-4\tau_{xu}u_t u_{xt}-2\tau_{xu}u_{tt}u_x+2(\eta_{uu}-2\xi_{xu}-\tau_{tu})u_x u_{tx}+(\eta_{uu}-2\xi_{xu}-\tau_{tu})u_{xx}u_t\\ &-3\xi_{tu}u_x u_{xx}-3\xi_{uu}u_t u_x u_{xx}-\tau_{uu}u_{xx}u_t^2-3\xi_u u_{tx}u_{xx}-2\tau_u u_{xt}^2+(\eta_u-2\xi_x-\tau_t)u_{txx}\\ &-\tau_u u_{xx}u_{tt}-2\tau_x u_{ttx}-\xi_t u_{xxx}-3\xi_u u_x u_{txx}-3\xi uuu_{xt}u_x^2-4\tau_{uu}u_{xt}u_x u_t-\tau_{uu}u_{tt}u_x^2.\end{aligned}$$

$$\begin{aligned}\eta^{xxx} =&\eta_{xxx}+(3\eta_{xxu}-\xi_{xxx})u_x-\tau_{xxx}u_t-3\tau_{xxu}u_x u_t+3(\eta_{xuu}-\xi_{xxu})u_x^2-3\tau_{xuu}u_t u_x^2+(\eta_{uuu}\\ &-3\xi_{xuu})u_x^3-\tau_{uuu}u_x^3 u_t-\xi_{uuu}u_x^4-3\tau_{xx}u_{tx}+3(\eta_{xu}-\xi_{xx})u_{xx}+3(\eta_{uu}-3\xi_{xu})u_x u_{xx}\\ &-3\tau_{xu}u_t u_{xx}-6\tau_{xu}u_x u_{tx}-6\xi_{uu}u_x^2 u_{xx}-3\tau_{uu}u_{xt}u_x^2-3\tau_{uu}u_{xx}u_x u_t-3\xi_u u_{xx}^2-3\tau_u u_{xx}u_{tx}\\ &-3\tau_x u_{txx}+(\eta_u-3\xi_x)u_{xxx}-4\xi_u u_x u_{xxx}-\tau_u u_{xxx}u_t-3\tau_u u_{xxt}u_x.\end{aligned}$$

2.3 Lie Classical Approach: An Algorithmic Overview

This algorithmic overview has been considered from Bluman and Anco [7]. The classical method essentially consists of finding symmetry reductions of PDEs with the help of determining equations obtained under the condition of invariance of the system of PDEs. When a given system of PDEs is subjected to invariance under one-parameter Lie group of transformations, one arrives at an over determined linear system of equations for the group infinitesimals. These infinitesimals of the transformations help us to obtain the reductions of the system. The symmetries and reductions reported in chapters 3, 4 and 5 are based on the applications of this method.

Consider a system of N PDEs with m dependent variables $u=(u^1,u^2,...,u^m)$ and n independent variables $x=(x_1,x_2,....,x_n)$, given by

$$F^\mu(x,u,\partial u,\partial^2 u,...,\partial^k u)=0, \mu=1,2,,...,N.$$

The point wise procedure is as follows:-

1) Let the one-parameter Lie group of point transformations leaves invariant the system of partial differential equations.

2) Then, apply the prolonged operator $X^{(k)}$ given to each equation of the system and require that

$$X^{(k)}F^\mu|_{F^\nu=0}=0 \qquad \mu,\nu=1,2,....,N.$$

The meaning of this condition is that $X^{(k)}$ vanishes on the solution set of the originally given system. This condition assures that $u(x)$ is solution of whatever $u^*(x^*)$ is one.

3) Following the procedure as mentioned in section, a system of linear partial differential equations for ξ and η that constitutes a set of determining equations for the infinitesimal generator X admitted by the given system of partial differential equations is obtained.

4) The solutions of the determining equations will lead to the explicit forms of ξ and η.

5) Construct the corresponding characteristics equations and obtain u in terms of n-1 new independent variables.

6) Rewrite the system in these new coordinates to get the reduced from of the system.

Chapter 3

GARDNER EQUATION

3.1 Introduction

The mathematical theory of the nonlinear evolution equations, starting from the Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation is an area of research for the past few decades [28, 31, 32, 33, 34]. The Korteweg-de Vries equation (KdV) is a well known model for description of nonlinear long internal waves in the oceans. Its coefficients are defined by vertical density and currents stratification. In a shelf zone the coefficient of quadratic nonlinearity may tend to zero, and the nonlinear model should be modified. In particular, The Gardner's equation, that is also known as the mixed KdV-mKdV equation. The Gardner equation,which differs from the KdV by presence of an additional term of cubic nonlinearity may be used as the generalized model.

This gardner equation shows up, particularly, in the context of internal gravity waves in a density-stratified ocean. This is commonly described by the KdV equations and its versions with small nonlinearity. However, there are situations when waves with strong nonlinearity is experienced, as in the case of Coastal Ocean Probe Experiment during 1995 in the Oregon Bay, the problem of creating an adequate theoretical model was deemed necessary. This lead to the study of Gardner equation [16].

3.2 Symmetries

Consider the Gardner equation

$$u_t = 6(u+\varepsilon^2 u^2)u_x + u_{xxx}. \tag{3.2.1}$$

Let the group of infinitesimal transformations be defined as

$u^* = u + \varepsilon\eta(x,t,u) + \bigcirc(\varepsilon^2)$

$x^* = x + \varepsilon\xi(x,t,u) + \bigcirc(\varepsilon^2)$

$t^* = t + \varepsilon\tau(x,t,u) + \bigcirc(\varepsilon^2)$

On invoking the invariance criterion the following relation from these coefficient of the first order of ε is deduced,

$$\eta^t - 6u\eta^x - 6u_x\eta - 6\alpha^2u^2\eta^x - 12\alpha^2uu_x\eta - \eta^{xxx} = 0 \tag{3.2.2}$$

where η^t, η^x, η^{xxx} are infinitesimal generator corresponding to u_t, u_x, u_{xxx}.

Substitute the values of η^t, η^x, η^{xxx} in above equation we get

$$\frac{\partial\eta}{\partial t} + 6(u+\varepsilon^2u^2)u_x\frac{\partial\eta}{\partial u} - u_{xxx}\frac{\partial\eta}{\partial u} - 6(u+\varepsilon^2u^2)u_x\frac{\partial\tau}{\partial t} + u_{xxx}\frac{\partial\tau}{\partial t} - \frac{\partial\xi}{\partial t}u_x - \frac{\partial\tau}{\partial u}(36(u+\varepsilon^2u^2)^2u_x^2 + u_{xxx}^2 - 12(u+\varepsilon^2u^2)u_xu_{xxx}) - 6u_x^2\frac{\partial\xi}{\partial u}(u+\varepsilon^2u^2) + \frac{\partial\xi}{\partial u}u_xu_{xxx} - 6\eta u_x - 6\alpha^2u^2\frac{\partial\eta}{\partial x} - 6\alpha^2u^2u_x\frac{\partial\eta}{\partial u} + 6\alpha^2u^2u_x\frac{\partial\xi}{\partial x} + 6\alpha^2u^2\frac{\partial\tau}{\partial x}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + 6\alpha^2u^2\frac{\partial\xi}{\partial u}u_x^2 + 6\alpha^2u^2u_x\frac{\partial\tau}{\partial u}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) - 12\alpha^2u\eta u_x - 6u\frac{\partial\eta}{\partial x} - 6uu_x\frac{\partial\eta}{\partial u} + 6uu_x\frac{\partial\xi}{\partial x} + 6u\frac{\partial\tau}{\partial x}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + 6uu_x^2\frac{\partial\xi}{\partial u} + 6uu_x\frac{\partial\tau}{\partial u}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) - \frac{\partial^3\eta}{\partial x^3} - 3\frac{\partial^3\eta}{\partial x^3}u_x + \frac{\partial^3\xi}{\partial x^3}u_x + \frac{\partial^3\tau}{\partial x^3}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + 3\frac{\partial^3\tau}{\partial x^2\partial u}u_x(6uu_x + 6\varepsilon^2u^2u_x - uxxx) - 3\frac{\partial^3\eta}{\partial u^2\partial x}u_x^2 + 3\frac{\partial^3\xi}{\partial x^2\partial u}u_x^2 + 3u_x^2\frac{\partial^3\tau}{\partial x\partial u^2}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) - \frac{\partial^3\eta}{\partial\partial u^3}u_x^3 + 3\frac{\partial^3\xi}{\partial u^2\partial x}u_x^3 + u_x^3\frac{\partial^3\tau}{\partial u^3}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + \frac{\partial^3\xi}{\partial u^3}u_x^4 + 3\frac{\partial^2\tau}{\partial x^2}u_{xt} - 3\frac{\partial^2\eta}{\partial x\partial u}u_{xx} + 3\frac{\partial^2\xi}{\partial x^2}u_xx - 3\frac{\partial^2\eta}{\partial u^2}u_{xx}u_x + 9\frac{\partial^2\xi}{\partial x\partial u}u_{xx}u_x + 3\frac{\partial^2\tau}{\partial x\partial u}u_{xx}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + 6\frac{\partial^2\tau}{\partial x\partial u}u_{xt}u_x + 6\frac{\partial^2\xi}{\partial u^2}u_{xx}u_x^2 + 3\frac{\partial^2\tau}{\partial u^2}u_{xt}u_x^2 + 3\frac{\partial^2\tau}{\partial u^2}u_{xx}u_x(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + 3\frac{\partial\xi}{\partial u}u_{xx}^2 + 3\frac{\partial\tau}{\partial u}u_{xx}u_{xt} + 3\frac{\partial\tau}{\partial x}u_{xxt} - \frac{\partial\eta}{\partial u}u_{xx} + 3\frac{\partial\xi}{\partial x}u_{xx} + 4\frac{\partial\xi}{\partial u}u_{xxx}u_x - \frac{\partial\tau}{\partial u}u_{xxx}(6uu_x + 6\varepsilon^2u^2u_x - u_{xxx}) + 3\frac{\partial\tau}{\partial u}u_{xxt}u_x = 0$$

Now equate the coefficients of u_x, u_{xx}, u_{xt}, u_{xxt}, constants, etc equal to zero and we obtain

$$\frac{\partial\xi}{\partial u} = \frac{\partial\tau}{\partial x} = \frac{\partial\tau}{\partial u} = 0 \tag{3.2.3}$$

$$\frac{\partial\eta}{\partial t} - 6u\frac{\partial\eta}{\partial x} - 6\alpha^2u^2\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x^3} = 0 \tag{3.2.4}$$

$$\begin{aligned} &6(u+\alpha^2u^2)\left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right) - \frac{\partial\xi}{\partial t} - 6u\left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right) - 6\eta - 6\alpha^2u^2\left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right) - 12\alpha^2u\eta \\ &-3\frac{\partial^3\eta}{\partial x^2\partial u} + \frac{\partial^3\xi}{\partial x^3} = 0 \end{aligned} \tag{3.2.5}$$

$$-3\frac{\partial^3\eta}{\partial u^2\partial x} = 0 \tag{3.2.6}$$

$$-\frac{\partial\tau}{\partial t} + 3\frac{\partial\xi}{\partial x} = 0 \tag{3.2.7}$$

$$\frac{\partial^3\eta}{\partial u^3} = 0 \tag{3.2.8}$$

$$-3\left(\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\right) = 0 \tag{3.2.9}$$

$$-3\frac{\partial^2\eta}{\partial u^2}=0 \tag{3.2.10}$$

From these equations we get

$\xi = h(x,t);$

$\tau = j(t);$

$\eta = f(x,t)u + g(x,t);$

Put these values of ξ,τ,η in above equations from ((3.2.3)-(3.2.10))

$$uf_t(x,t)+g_t(x,t)-6u^2f_x(x,t)-6ug_x(x,t)-6\alpha^2\alpha u^3 f_x(x,t)-6\alpha^2u^2g_x(x,t)-uf_{xxx}(x,t) \\ -g_{xxx}(x,t)=0 \tag{3.2.11}$$

$$6(u+\alpha^2u^2)(f(x,t)-j_t(t)-h_t(x,t)-6u(f(x,t)-h_x(x,t))-6uf(x,t)-6g(x,t)-6\alpha^2u^2 \\ (f(x,t)-h_x(x,t))-12\alpha^2u^2f(x,t)-12\alpha^2ug(x,t)-3f_{xx}(x,t)+h_{xxx}(x,t)=0 \tag{3.2.12}$$

$$-j_t(t)+3h_x(x,t)=0 \tag{3.2.13}$$

$$-3(f_x(x,t)-h_{xx}(x,t))=0 \tag{3.2.14}$$

By equating the coefficient of u in equation (3.2.11) and (3.2.12) we get

$$f_t(x,t)-6g_x(x,t)-f_{xxx}(x,t)=0 \tag{3.2.15}$$

$$6(f(x,t)-j_t(t)-f(x,t)+h_x(x,t)-6f(x,t))-12\alpha^2g(x,t)=0 \tag{3.2.16}$$

By equating the coefficient of u^2 in equation (3.2.11) we get

$$-6f_x(x,t)-6\alpha^2g_x(x,t)=0 \tag{3.2.17}$$

$$6\alpha^2(f(x,t)-j_t(t)-f(x,t)+h_x(x,t))-12\alpha^2f(x,t)=0 \tag{3.2.18}$$

By equating the coefficient of u^3 in (3.2.11) we get

$$f_x(x,t)=0 \tag{3.2.19}$$

By equating the coefficient of constants in equation ((3.2.11)-(3.2.14))

$$g_t(x,t)-g_{xxx}(x,t)=0 \tag{3.2.20}$$

$$-h_t(x,t)-6g(x,t)-3f_{xx}(x,t)+h_{xxx}(x,t)=0 \tag{3.2.21}$$

$$-j_t(t)+3h_x(x,t)=0 \tag{3.2.22}$$

$$-3(f_x(x,t)-h_{xx}(x,t)=0 \tag{3.2.23}$$

Now we get from equation (3.2.19) and then from (3.2.17)

$$f_x(x,t)=0 \tag{3.2.24}$$

$$g_x(x,t) = 0 \tag{3.2.25}$$

We get g=constant

g=a (say)

By equations (3.2.18) and (3.2.17)

$$-j_t(t) + h_x(x,t) = 2f(x,t) \tag{3.2.26}$$

$$-j_t(t) + h_x(x,t) - f(x,t) = 2\alpha^2 g(x,t) \tag{3.2.27}$$

Then

$$f = 2\alpha^2 a. \tag{3.2.28}$$

From equation (3.2.21)

$$h_t(x,t) = -6a \tag{3.2.29}$$

Then

$$h = -6at + k(x) \tag{3.2.30}$$

Put value of equation (3.2.28) in equations (3.2.26) and (3.2.27), we get

$$-j_t(t) + h_x(x,t) = 4\alpha^2 a \tag{3.2.31}$$

$$-j_t(t) + 3h_x(x,t) = 0 \tag{3.2.32}$$

We get from equation (3.2.31) and (3.2.32)

$$h = -2\alpha^2 ax + n(t) \tag{3.2.33}$$

$$h = -2\alpha^2 ax - 6at + b \tag{3.2.34}$$

where a and b are arbitrary constants.

$$j(t) = -6\alpha^2 a \tag{3.2.35}$$

$$j = -6\alpha^2 at + c \tag{3.2.36}$$

where a and c are arbitrary constants.
Therefore the value of f, g, h and j is

$$\begin{aligned} &f = 2\alpha^2 a, \\ &g = a, \\ &j = -6\alpha^2 at + b, \\ &h = -2\alpha^2 ax - 6at + b. \end{aligned} \tag{3.2.37}$$

Then the solution of the determining equations is given by

$$\xi(x,t) = -2\alpha^2 ax - 6at + b$$
$$\tau(x,t) = -6\alpha^2 at + c$$
$$\eta(x,t) = 2\alpha^2 au + a.$$

Where a, b, c are arbitrary parameters. Hence the symmetry generators admitted by gardner equation are

$$V_1 = -2\alpha^2 x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial x} - 6\alpha^2 t\frac{\partial}{\partial t} + (2\alpha^2 u + 1)\frac{\partial}{\partial u}$$
$$V_2 = \frac{\partial}{\partial x}$$
$$V_3 = \frac{\partial}{\partial t}$$

Optimal System

The commutators table and adjoint table for Lie algebra can easily be constructed as follows:
The commutator (Lie bracket) of V_α and V_β is first order operator defined by
$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha$.

Table 3.1 Commutator table

comm	V_1	V_2	V_3
V_1	0	$2\alpha^2 V_2$	$6(V_2 + \alpha^2 V_3)$
V_2	$-2\alpha^2 V_2$	0	0
V_3	$-6(V_2 + \alpha^2 v_3)$	0	0

Formula of adjoint table is
$Adj[exp(\varepsilon V_i)]V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i[V_i, V_j]] + ...$

Table 3.2 Adjoint table

Adj	V_1	V_2	V_3
V_1	V_1	$V_2(1 - 2\alpha^2 \varepsilon e^{\frac{-\varepsilon}{2}})$	$V_3 e^{-6\alpha^2\varepsilon} - 6\varepsilon V_2 e^{-8\alpha^2\varepsilon}$
V_2	$V_1 + 2\varepsilon\alpha^2 V_2$	V_2	V_3
V_3	$V_1 - 6\varepsilon V_2 - 6\alpha^2 \varepsilon V_2$	V_2	V_3

We deduce an optimal system of sub algebra with their corresponding generators as follows:-

$(1)V_1$

$(2)V_2+\mu V_3$

$(3)V_3$

Where μ is arbitrary constant.

3.3 Reduction and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebra, the reductions of PDEs(3.1)into ODEs in terms of similarity variable and the new dependent variables-F, are obtained using the auxiliary equations.some exact solutions of each reduced system are then attempted.

Generator (1)

The generator (1) is V_1

The corresponding characteristic equations are given by:-

$$\frac{dx}{-2\alpha^2 x-6t}=\frac{dt}{-6\alpha^2 t}=\frac{du}{2\alpha^2 u+1}$$

Thus the generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$$\xi=t^{\frac{-1}{3}}x-\frac{3}{2\alpha^2}t^{\frac{2}{3}}$$

$$u(x,t)=\frac{1}{2\alpha^2}(t^{\frac{-1}{3}}F(\xi)-1)$$

Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (PDE) (3.2.1) reduces to the followings system of Ordinary differential equation.

$$6\alpha^2F''+2\xi F'\alpha^2+2\alpha^2F+9F^2F'=0$$

In this case we are able to find only reductions.

Generator (2)

The generator (2) is $V_2+\mu V_3$.

The corresponding characteristic equations are given by:-

$$\frac{dx}{1}=\frac{dt}{\mu}=\frac{du}{0}$$

Thus the generator (2) in the optimal system defines the similarity variable and similarity solution as follows:-

$$\xi = \mu x - t$$

$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (PDE) (3.2.1) reduces to the followings system of Ordinary differential equation.

$$6\mu FF' + 6\varepsilon^2\mu F^2F' + \mu^3F''' + F' = 0$$

By using Maple software, we get some exact solutions of

$$F(\xi) = c_3$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3m)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}\xi}{\mu^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2\mu\varepsilon^2}\left(\sqrt{\mu(2c_3^2\varepsilon^2\mu^3 - 2\varepsilon^2 + 3\mu)}\,jacobiCN\left(c_2 + c_3\xi\frac{\sqrt{\mu(2c_3^2\varepsilon^2\mu^3 - 2\varepsilon^2 + 3\mu)}}{2\mu^2\varepsilon c_3}\right)\right)$$

The solution of the system is given by

$$u(x,t) = c_3$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right) \tag{3.3.1}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right) \tag{3.3.2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{\mu\varepsilon^2}\left(\sqrt{\mu(-2\varepsilon^2+3\mu)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{-\mu(-2\varepsilon^2+3\mu)}(\mu x - t)}{\mu^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2\mu\varepsilon^2}\left(\sqrt{\mu(2c_3^2\varepsilon^2\mu^3 - 2\varepsilon^2 + 3\mu)}\,jacobiCN\left(c_2 + c_3(\mu x - t)\frac{\sqrt{\mu(2c_3^2\varepsilon^2\mu^3 - 2\varepsilon^2 + 3\mu)}}{2\mu^2\varepsilon c_3}\right)\right) \tag{3.3.3}$$

where ε, c_1 and μ are constants.

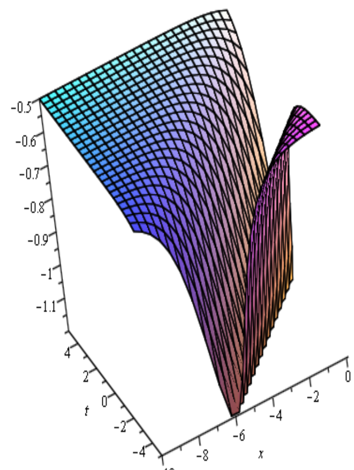

Figure 3.1: Plot of solution (3.3.1) by $\varepsilon = 1, \mu = 1, c_1 = 1, x = -10.....10, t = -5......5$

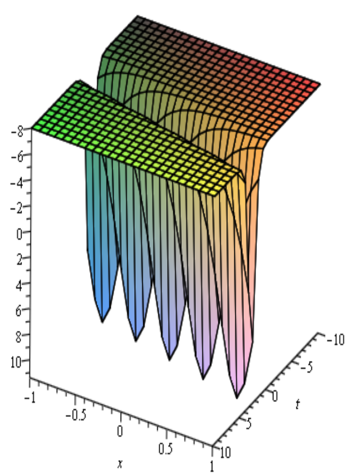

Figure 3.2: Plot of solution (3.3.2) by $\varepsilon = .25, \mu = 2, c_1 = 2, x = -1.....1, t = -10......10$

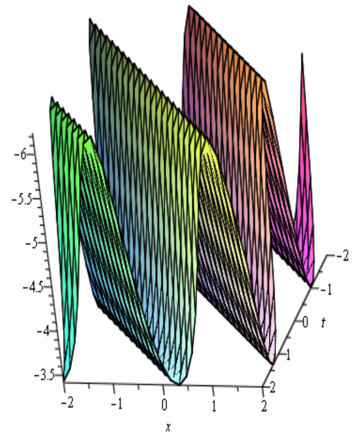

Figure 3.3: Plot of solution (3.3.3) by $\varepsilon = .50, \mu = 1, c_2 = 1, c_3 = 2, x = -2.....2, t = -2......2$

Generator (3)

The generator (3) is V_3.

The corresponding characteristic equations are given by:-

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Thus the generator (3) in the optimal system defines the similarity variable and similarity solution as follows

$$\xi = xu(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (3.2.1) reduces to the followings system of Ordinary differential equation.

$$6FF' + 6\varepsilon^2 F^2 F' + F''' = 0$$

By using Maple software, we get some exact solutions of

$$F(\xi) = c_3$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{4c_3^2\varepsilon^2 + 6}JacobiSN\left(c_2 + c_3\xi\frac{1}{2}\frac{\sqrt{-4c_3^2\varepsilon^2-6}}{\varepsilon c_3}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{-4c_3^2\varepsilon^2 + 6}JacobiND\left(c_2 + c_3\xi\frac{1}{2}\frac{\sqrt{8c_3^2\varepsilon^2-6}}{\varepsilon c_3}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}sech\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{6}sech\left(-c_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{6}sech\left(c_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{6}sech\left(c_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1+\frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1+\frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{3}tanh\left(c_1+\frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{3}tanh\left(c_1+\frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

Thus, the following solution of the system is obtained

$$u(x,t)=c_3$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{4c_3^2\varepsilon^2+6}JacobiSN\left(c_2+c_3x\frac{1}{2}\frac{\sqrt{-4c_3^2\varepsilon^2-6}}{\varepsilon c_3}\right)}{\varepsilon^2} \tag{3.3.4}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{-4c_3^2\varepsilon^2+6}JacobiND\left(c_2+c_3x\frac{1}{2}\frac{\sqrt{8c_3^2\varepsilon^2-6}}{\varepsilon c_3}\right)}{\varepsilon^2} \tag{3.3.5}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{6}sech\left(-c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2} \tag{3.3.6}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{6}sech\left(-c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{6}sech\left(c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}+\frac{1}{2}\frac{\sqrt{6}sech\left(c_1+\frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2} \tag{3.3.7}$$

$$u(x,t)=-\frac{1}{2\varepsilon^2}-\frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1+\frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1 + \frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$
$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$
$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

where ε and c_1 are constants.

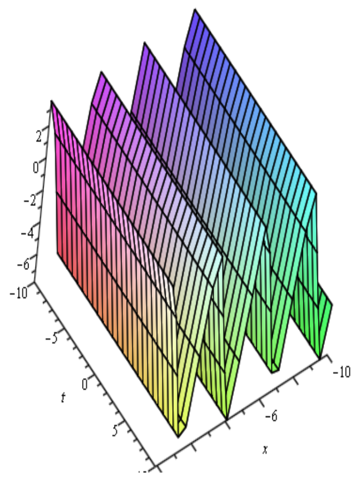

Figure 3.4: Plot of solution (3.3.4) by $\varepsilon = .5, c_2 = 1, c_3 = 1.5, x = -10.....10$

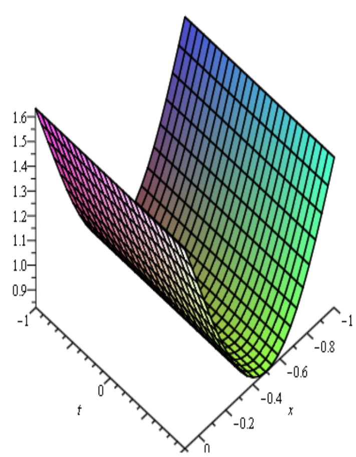

Figure 3.5: Plot of solution (3.3.5) by $\varepsilon = .5, c_3 = 2, c_2 = 1, x = -1.....1$

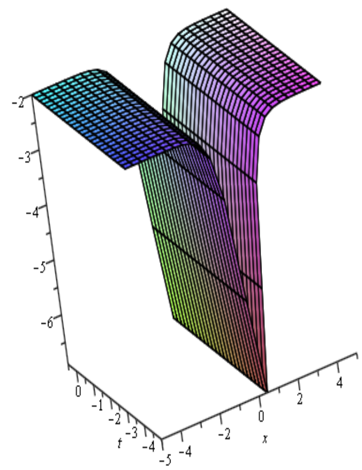

Figure 3.6: Plot of solution (3.3.6) by $\varepsilon = .5, c_1 = 1, x = -5.....5$

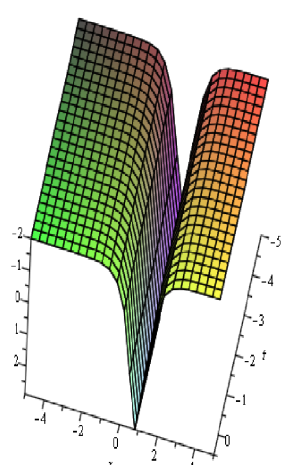

Figure 3.7: Plot of solution (3.3.7) by $\varepsilon = .5, c_1 = 2, x = -5.....5$

Chapter 4

BUCKMASTER EQUATION

4.1 Introduction

Partial differential equations from the basis of many mathematical models of physical, chemical and biological phenomenon, and more recently their use has spread into economics, financial forecasting, image processing and other fields . Their are many methods to solve partial differential equations, such as the collocation method, method of lines etc. In this chapter, the Lie classical method was used for solving Buckmaster equation. Earlier this Buckmaster equtaion was solved by Finite Volume Method [37].

4.2 Vector fields and Optimal Systems

Consider the Buckmaster equation.

$$u_t = (u^4)_{xx} + (u^3)_x.$$

It becomes:-

$$u_t = 12u^2u_x^2 + 4u^3u_{xx} + 3u^2u_x. \tag{4.2.1}$$

Let the group of infinitesimal transformations be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + \bigcirc(\varepsilon^2)$$
$$x^* = x + \varepsilon\xi(x,t,u) + \bigcirc(\varepsilon^2)$$
$$t^* = t + \varepsilon\tau(x,t,u) + \bigcirc(\varepsilon^2)$$

The coefficients of first order of ε are:

$$24uu_x^2\eta + 24u^2u_x\eta^x + 12u^2\eta u_{xx} + 4u^3\eta^{xx} + 3u^2\eta^x + 6u\eta u_x - \eta^t = 0 \tag{4.2.2}$$

Where η^x, η^t, η^{xx} are infinitesimal generator corresponding to u_x, u_t, u_{xx}.
Substitute the values of η^x, η^t, η^{xx} in above equation, also put the value of

$$u_t = 12u^2u_x^2 + 4u^3u_{xx} + 3u^2u_x.$$

And

$$u_{xt} = 24u^2u_xu_{xx} + 24uu_x^3 + 12u^2u_xu_{xx} + 4u^3u_{xxx} + 6uu_x^2 + 3u^2u_{xx}$$

We get:-

$$24u\eta u_x^2 + 24u^2\frac{\partial\eta}{\partial x}u_x + 24u^2\frac{\partial\eta}{\partial u}u_x^2 - 24u^2\frac{\partial\xi}{\partial x}u_x^2 - 24*12u^4\frac{\partial\tau}{\partial x}u_x^3 - 96u^5\frac{\partial\tau}{\partial x}u_xu_{xx} - 72u^4\frac{\partial\tau}{\partial x}u_x^2$$
$$-24u^2\frac{\partial\xi}{\partial u}u_x^3 - 288u^4\frac{\partial\tau}{\partial u}u_x^4 - 96^u\frac{\partial\tau}{\partial u}u_x^2u_{xx} - 72u^4\frac{\partial\tau}{\partial u}u_x^3 + 12u^2\eta u_{xx} + 4u^3\frac{\partial^2\eta}{\partial x^2} + 8u^3\frac{\partial^2\eta}{\partial x\partial u}u_x$$
$$-4u^3\frac{\partial^2\xi}{\partial x^2}u_x - 48u^5\frac{\partial^2\tau}{\partial x^2}u_x^2 - 16u^6\frac{\partial^2\tau}{\partial x^2}u_{xx} - 12u^5\frac{\partial^2\tau}{\partial x^2}u_x + 4u^3\left(\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi}{\partial x\partial u}\right)u_x^2 - 96u^5$$
$$\frac{\partial^2\tau}{\partial x\partial u}u_x^3 - 32u^6\frac{\partial^2\tau}{\partial x\partial u}u_xu_{xx} - 24u^5\frac{\partial^2\tau}{\partial x\partial u}u_x^2 - 4u^3\frac{\partial^2\xi}{\partial u^2}u_x^3 - 48u^5\frac{\partial^2\tau}{\partial u^2}u_x^4 - 16u^6\frac{\partial^2\tau}{\partial u^2}u_x^2u_{xx}$$
$$-12u^5\frac{\partial^2\tau}{\partial u^2}u_x^3 + 4u^3\left(\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi}{\partial x}\right)u_{xx} - 8u^3\frac{\partial\tau}{\partial x}(24u^2u_xu_{xx} + 24uu_x^3 + 12u^2u_xu_{xx} + 4u^3u_{xxx}$$
$$+6uu_x^2 + 3u^2u_{xx}) - 12u^3\frac{\partial\xi}{\partial u}u_xu_{xx} - 48u\partial^5\frac{\partial\tau}{\partial u}u_x^2u_{xx} - 16u^6\frac{\partial\tau}{\partial u}u_{xx}^2 - 12u^5\frac{\partial\tau}{\partial u}u_x^2u_{xx} - 96u^5$$
$$\frac{\partial\tau}{\partial u}u_x^2(24u^2u_xu_{xx} + 24uu_x^3 + 12u^2u_xu_{xx} + 4u^3u_{xxx} + 6uu_x^2 + 3u^2u_{xx}) - 32u^6\frac{\partial\tau}{\partial u}u_{xx}(24u^2u_xu_{xx}$$
$$+24uu_x^3 + 12u^2u_xu_{xx} + 4u^3u_{xxx} + 6uu_x^2 + 3u^2u_{xx}) - 24u^5\frac{\partial\tau}{\partial u}u_x(24u^2u_xu_{xx} + 24uu_x^33 + 12u^2$$
$$u_xu_{xx} + 4u^3u_{xxx} + 6uu_x^2 + 3u^2u_{xx}) + 3u^2\frac{\partial\eta}{\partial x} + 3u^2\left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right)u_x - 36u^4\frac{\partial\tau}{\partial x}u_x^2 - 12u^5\frac{\partial\tau}{\partial x}u_{xx}$$
$$-9u^4\frac{\partial\tau}{\partial x}u_x - 3u^2\frac{\partial\xi}{\partial u}u_x^2 - 36u^4\frac{\partial\tau}{\partial u}u_x^3 - 12u^5\frac{\partial\tau}{\partial u}u_xu_{xx} - 9u^4\frac{\partial\tau}{\partial u}u_x^2 + 6uu_x\eta - \frac{\partial\eta}{\partial t} - 12u^2u_x^2$$
$$\left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right) - 4u^3u_{xx}\left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right) - 3u^2u_x\left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right) - \frac{\partial\xi}{\partial t}u_x - \frac{\partial\tau}{\partial u}(144u^4u_x^4 + 16u^6u_{xx}^2$$
$$+9u^4u_x^2 + 96u^5u_x^2u_{xx} + 24u^5u_xu_{xx} + 72u^4u_x^3) - \frac{\partial\xi}{\partial u}12u^2u_x^3 - 4u^3\frac{\partial\xi}{\partial u}u_xu_{xx} - 3u^2\frac{\partial\xi}{\partial u}u_x^2 = 0.$$

Now equating the coefficient of u_x, u_x^2, u_xu_{xx}, $constants$, u_{xxx}, u_x^5, then we get:-

$$\frac{\partial\tau}{\partial u} = \frac{\partial\tau}{\partial x} = \frac{\partial\xi}{\partial u} = 0 \tag{4.2.3}$$

$$3\eta + 2u\frac{\partial\xi}{\partial x} - u\frac{\partial\tau}{\partial t} = 0 \tag{4.2.4}$$

$$6u\frac{\partial\xi}{\partial x} - 3u\frac{\partial\eta}{\partial u} - 6\eta - u^2\frac{\partial^2\eta}{\partial u^2} - 3u\frac{\partial\tau}{\partial t} = 0 \tag{4.2.5}$$

$$-24u^2\frac{\partial\eta}{\partial x} + 4u^3\frac{\partial^2\xi}{\partial x^2} - 8u^3\frac{\partial^2\eta}{\partial x\partial u} + 3u^2\frac{\partial\xi}{\partial x} - 3u^2\frac{\partial\tau}{\partial t} - 6u\eta - \frac{\partial\xi}{\partial t} = 0 \tag{4.2.6}$$

$$4u^3\frac{\partial^2\eta}{\partial x^2} + 3u^2\frac{\partial\eta}{\partial x} - \frac{\partial\eta}{\partial t} = 0. \tag{4.2.7}$$

From equation (4.2.4)

We get

$$\eta = \frac{1}{3}(u\frac{\partial \tau}{\partial t} - 2u\frac{\partial \xi}{\partial x})$$

And differentiate w.r.t. u, we get

$$\frac{\partial \eta}{\partial u} = \frac{1}{3}(\frac{\partial \tau}{\partial t} - 2\frac{\partial \xi}{\partial x})$$

Again differentiate w.r.t. u, we get,

$$\frac{\partial^2 \eta}{\partial u^2} = 0$$

Put value of $\frac{\partial \eta}{\partial u}$ and $\frac{\partial^2 \eta}{\partial u^2}$ in equation (4.2.5)

From here we get

$$\frac{\partial \tau}{\partial t} = 2\frac{\partial \xi}{\partial x}$$

And differentiate w.r.t. x, we get

$$\frac{\partial^2 \tau}{\partial t \partial x} = \frac{\partial^2 \xi}{\partial x^2} = 0.$$

So value of $\eta = 0$

Put value of η in equation (4.2.5)

Then we get

$$-3u^2\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial t} \qquad (4.2.8)$$

And

$$\frac{\partial \xi}{\partial x} = 0$$

And

$$\frac{\partial \xi}{\partial t} = 0$$

We get

$$\xi = k_1.$$

Also from equation (4.2.8)

$$2\frac{\partial \tau}{\partial t} = 0$$

So

$$\tau = k_2$$

Then the solution of determining equations is given by:-

$$\xi(x,t) = k1$$

$$\tau(x,t) = k2$$

$$\eta(x,t) = 0$$

Hence the point symmetry generators admitted by the Buckmaster equation are:-

$$V_1 = \frac{\partial}{\partial x}$$
$$V_2 = \frac{\partial}{\partial t}$$

Optimal System

The commutator table and adjoint table for Lie algebra can easily be constructed as follows:-
The commutator of $V_\alpha and V_\beta$ is first order operator defined by:-
$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha$

Table 4.1 Commutator table

comm	V_1	V_2
V_1	0	0
V_2	0	0

Formula for adjoint table is:-
$Adj[exp(\varepsilon V_i)]V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i[V_i, V_j]] + ...$

Table 4.2 Adjoint table

adj	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebra with their corresponding generators as follows:-
$(1)V_1 + mV_2$
$(2)V_2$
Where m is arbitrary constant.

4.3 Reduced ODE's and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebra, the reductions of PDEs(3.1)into ODEs in terms of similarity variable and the new dependent variables-F, are obtained using the auxiliary equations.some exact solutions of each reduced system are then attempted.

Generator(1)

The generator(1) is $V_1 + mV_2$.

The corresponding characteristic equations are given by:-
$\frac{dx}{1} = \frac{dt}{m} = \frac{du}{0}$
Thus the generator (1) in the optimal system defines the similarity variable and similarity solution as follows
$\xi = mx - t$
$u(x,t) = F(\xi)$
Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (PDE) (4.2.1) reduces to the followings system of Ordinary differential equation(ODE)
$12m^2F^2F'^2 + 4F^3m^2F'' + 3mF^2F' + F' = 0.$
Using maple we obtain the exact solutions

$$F(\xi) = \frac{1}{c_3}$$

The solution of the system is given by

$$u(x,t) = \frac{1}{c_3}$$

where c_3 is constant.

Generator(2)

The generator (2) is V_2.

The corresponding characteristic equations are given by,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}$$

Thus the generator (2) in the optimal system defines the similarity variable and similarity solution as follows
$\xi = x.$
$u(x,t) = F(\xi).$
Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (PDE) (3.1) reduces to the followings system of Ordinary differential equation(ODE)
$12F^2F'^2 + 4F^3F'' + 3F^2F' = 0.$
Using maple we obtain the exact solutions

$$F(\xi) = \frac{1}{c_3}$$

$$F(\xi) = c_3 - \frac{1}{2}\frac{c_1 + c_2\xi}{c_2}$$

The solution of the system is given by

$$u(x,t) = \frac{1}{c_3}$$

$$u(x,t) = c_3 - \frac{1}{2}\frac{c_1 + c_2 x}{c_2}$$

Where c_1, c_2 and c_3 are constants.

Chapter 5

DEGASPERIS PROCESI EQUATION

5.1 Introduction

Degasperis-Procesi Equation is a real nonlinear partial differential equation which models propagation of nonlinear dispersive waves and is solvable by the methods of soliton theory. The Degasperis-Procesi equation was first introduced by an asymptotic integrability within a family of third order dispersive equations. Then Degasperis [11] proved the exact integrability by constructing a Lax pair. Exact solutions play a vital role in the study of nonlinear phenomena as Degasperis-Procesi equation provide much information n various aspects of the physical phenomena. The classical Lie method is utilized to obtain optimal system of the subalgebras for this equation [17].

5.2 Lie Symmetries

Consider the Degasperis Processi equation

$$u_t - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx} \tag{5.2.1}$$

Let the group of infinitesimal transformations be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + \bigcirc(\varepsilon^2)$$
$$x^* = x + \varepsilon\xi(x,t,u) + \bigcirc(\varepsilon^2)$$
$$t^* = t + \varepsilon\tau(x,t,u) + \bigcirc(\varepsilon^2)$$

On invoking the invariance criterion the following relation from these coefficient of the first order of ε is deduced,

$$\eta^t - \eta^{xxt} + 4\eta u_x + 4u\eta^x - 3u_x x\eta^x - 3u_x\eta^{xx} - u_{xxx}\eta - u\eta^{xxx} = 0 \tag{5.2.2}$$

Put the value of η^x, η^t, η^{xx}, η^{xxx}, η^{xxt} in (5.2.2), We get

$\eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t + 4\eta u_x + 4u\eta_x + 4u(\eta_u - \xi_x)u_x - 4u\tau_x u_t - 4u\xi_u u_x^2 - 4u\tau_u u_x u_t -$

$3u_{xx}\eta_x - 3u_{xx}(\eta_u - \xi_x)u_x + 3u_{xx}\tau_x u_t + 3u_{xx}\xi_u u_x^2 + 3u_{xx}\tau_u u_x u_t - 3u_x\eta_{xx} - 3(2\eta_{xu} - \xi_{xx})u_x^2 + 3u_x u_t\tau_{xx} - 3(\eta_{uu} - 2\xi_x)u_x^3 + 6\tau_{xu}u_x^2 u_t + 3\xi_{uu}u_x^4 - 3(\eta_u - 2\xi_x)u_x u_{xxx} + 6u_x\tau_x u_{xt} + 9\xi_u u_{xx}u_x^2 + 3\tau_u u_{xx}u_x u_t + 6\tau_u u_{xxt}u_x u_t + 3\tau_{uu}u_x^3 u_t - u_{xxx}\eta - u\eta_{xxx} - (3\eta_{xxu} - \xi_{xxx})u_x u + \tau_{xxx}u_t u + 3u\tau_{xxu}u_x u_t - 3(\eta_{uux} - \xi_{xxu})uu_x^2 + 3\tau_{xuu}uu_x^2 u_t - (\eta_{uuu} - 3\xi_{xuu})uu_x^3 + \tau_{uuu}uu_x^3 u_t + \xi_{uuu}uu_x^4 + 3\tau_{xx}u_{xt}u - 3(\eta_{xu} - \xi_{xx})uu_{xx} - 3(\eta_{uu} - 3\xi_{xu})uu_x u_{xx} + \tau_{xu}uu_{xx}u_t + 6\tau_{xu}uu_{xt}u_x + 6\xi_{uu}u_{xx}uu_x^2 + 3\tau_{uu}u_{xt}uu_x^2 + 3\tau_{uu}u_{xx}uu_x u_t + 3\xi_u u_{xx}^2 u + 3\tau_u u_{xx}u_{xt}u + 3\tau_x u_{xxt}u - (\eta_u - 3\xi)uu_{xxx} + 4\xi_u u_{xxx}uux + \tau_u u_{xxx}uu_t + 3\tau_u u_{xxt}u_x u - \eta_{xxt} - (\eta_{xxu} - \tau_{xxt})u_t - (2\eta_{xtu} - \xi_{xxt})u_x - (\eta_{tuu} - 2\xi_{xtu})u_x^2 - (2\eta_{xuu} - \xi_{xxu} - 2\tau_{xtu})u_x u_t + \tau_{xxu}u_t^2 + 2\tau_{xuu}u_x u_t^2 + \xi_{tuu}u_x^3 - (\eta_{uuu} - 2\xi_{xuu} - \tau_{uuu})u_x^2 u_t + \xi_{uuu}u_x^3 u_t + \tau_{uuu}u_x^2 u_t^2 - (\eta_{tu} - 2\xi_{xt})u_{xx} - (2\eta_{xu} - \xi_{xx} - 2\tau_{xt})u_{xt} + \tau_{xx}u_{tt} + 4\tau_{xu}u_{xt}u_t + 2\tau_{xu}u_{tt}u_x + 3\xi_{tu}u_{xx}u_x - 2(\eta_{uu} - 2\xi_{xu} - \tau_{tu})u_{xt}u_x - (\eta_{uu} - 2\xi_{xu} - \tau_{tu})u_{xx}u_t + 3\xi_{uu}u_{xt}u_x^2 + 4\tau_{uu}u_{xt}u_x u_t + \tau_{uu}u_{tt}u_x^2 + 3\xi_{uu}u_{xx}u_x u_t + \tau_{uu}u_{xx}u_t^2 + 3\xi_u u_{xt}u_{xx} + 2\tau_u u_{xt}^2 + \tau_u u_{xx}u_{tt} - (\eta_u - 2\xi_x - \tau_t)u_{xxt} + 2\tau_x u_{xtt} + \xi_t u_{xxx} + 3\xi_u u_{xxt}u_x + \xi_u u_{xxx}u_t + 2\tau_u u_{xxt}u_t + 2\tau_u u_{xtt}u_x = 0$

Put value of $u_{xxt} = u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx}$ in above equation, We get

$\eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t + 4\eta u_x + 4u\eta_x + 4u(\eta_u - \xi_x)u_x - 4u\tau_x u_t - 4u\xi_u u_x^2 - 4u\tau_u u_x u_t - 3u_{xx}\eta_x - 3u_{xx}(\eta_u - \xi_x)u_x + 3u_{xx}\tau_x u_t + 3u_{xx}\xi_u u_x^2 + 3u_{xx}\tau_u u_x u_t - 3u_x\eta_{xx} - 3(2\eta_{xu} - \xi_{xx})u_x^2 + 3u_x u_t\tau_{xx} - 3(\eta_{uu} - 2\xi_x)u_x^3 + 6\tau_{xu}u_x^2 u_t + 3\xi_{uu}u_x^4 - 3(\eta_u - 2\xi_x)u_x u_{xxx} + 6u_x\tau_x u_{xt} + 9\xi_u u_{xx}u_x^2 + 3\tau_u u_{xx}u_x u_t + 6\tau_u(u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx})u_x u_t + 3\tau_{uu}u_x^3 u_t - u_{xxx}\eta - u\eta_{xxx} - (3\eta_{xxu} - \xi_{xxx})u_x u + \tau_{xxx}u_t u + 3u\tau_{xxu}u_x u_t - 3(\eta_{uux} - \xi_{xxu})uu_x^2 + 3\tau_{xuu}uu_x^2 u_t - (\eta_{uuu} - 3\xi_{xuu})uu_x^3 + \tau_{uuu}uu_x^3 u_t + \xi_{uuu}uu_x^4 + 3\tau_{xx}u_{xt}u - 3(\eta_{xu} - \xi_{xx})uu_{xx} - 3(\eta_{uu} - 3\xi_{xu})uu_x u_{xx} + \tau_{xu}uu_{xx}u_t + 6\tau_{xu}uu_{xt}u_x + 6\xi_{uu}u_{xx}uu_x^2 + 3\tau_{uu}u_{xt}uu_x^2 + 3\tau_{uu}u_{xx}uu_x u_t + 3\xi_u u_{xx}^2 u + 3\tau_u u_{xx}u_{xt}u + 3\tau_x(u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx})u - (\eta_u - 3\xi)uu_{xxx} + 4\xi_u u_{xxx}uux + \tau_u u_{xxx}uu_t + 3\tau_u(u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx})u_x u - \eta_{xxt} - (\eta_{xxu} - \tau_{xxt})u_t - (2\eta_{xtu} - \xi_{xxt})u_x - (\eta_{tuu} - 2\xi_{xtu})u_x^2 - (2\eta_{xuu} - \xi_{xxu} - 2\tau_{xtu})u_x u_t + \tau_{xxu}u_t^2 + 2\tau_{xuu}u_x u_t^2 + \xi_{tuu}u_x^3 - (\eta_{uuu} - 2\xi_{xuu} - \tau_{uuu})u_x^2 u_t + \xi_{uuu}u_x^3 u_t + \tau_{uuu}u_x^2 u_t^2 - (\eta_{tu} - 2\xi_{xt})u_{xx} - (2\eta_{xu} - \xi_{xx} - 2\tau_{xt})u_{xt} + \tau_{xx}u_{tt} + 4\tau_{xu}u_{xt}u_t + 2\tau_{xu}u_{tt}u_x + 3\xi_{tu}u_{xx}u_x - 2(\eta_{uu} - 2\xi_{xu} - \tau_{tu})u_{xt}u_x - (\eta_{uu} - 2\xi_{xu} - \tau_{tu})u_{xx}u_t + 3\xi_{uu}u_{xt}u_x^2 + 4\tau_{uu}u_{xt}u_x u_t + \tau_{uu}u_{tt}u_x^2 + 3\xi_{uu}u_{xx}u_x u_t + \tau_{uu}u_{xx}u_t^2 + 3\xi_u u_{xt}u_{xx} + 2\tau_u u_{xt}^2 + \tau_u u_{xx}u_{tt} - (\eta_u - 2\xi_x - \tau_t)(u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx}) + 2\tau_x u_{xtt} + \xi_t u_{xxx} + 3\xi_u(u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx})u_x + \xi_u u_{xxx}u_t + 2\tau_u u_{xxt}u_t + 2\tau_u(u_t + 4uu_x - 3u_x u_{xx} - uu_{xxx})u_x = 0$

On subsituting the coefficients of different differentials equal to zero lead to number of partial differential equations in τ, ξ, and η that need to be satisfied. The set of determining equations for the group of infinitesimals τ, ξ, and η , which is obtained from above equation, after equating the coefficients of various derivative terms to zero, is as follows:

$$\xi_u = 0 \tag{5.2.3}$$

$$\tau_x = \tau_u = 0 \tag{5.2.4}$$

$$\eta_{uu} = 0 \tag{5.2.5}$$

$$2\eta_{xu} - \xi_{xx} = 0 \tag{5.2.6}$$

$$\eta_{xxu} - 2\xi_x = 0 \tag{5.2.7}$$

$$u\tau_t - \xi_t + \eta - u\xi_x = 0 \tag{5.2.8}$$

$$-\tau_t + \xi_x - \eta_u = 0 \tag{5.2.9}$$

$$\eta_{tu} - 2\xi_{tx} + 3\eta_x - 3u\eta_{xu} = 0 \tag{5.2.10}$$

$$\eta_t - \eta_{txx} + 4u\eta_x - u\eta_{xxx} = 0 \tag{5.2.11}$$

$$u\eta_u - 3u\eta_{xxu} + \xi_t - 4\eta + 3\eta_{xx} = 0 \tag{5.2.12}$$

The set of equations (5.2.2-5.2.11) helps us to obtain the infinitesimals ξ, τ and η, as follows:

$$\xi = a_1$$

$$\tau = -a_3 t + a_2$$

$$\eta = a_3 u.$$

Where a_1, a_2 and a_3 are arbitray constants. The Lie algebra associated with equation (5.2.1) consists of the following three vector fields.

$$V_1 = \frac{\partial}{\partial x},$$
$$V_2 = \frac{\partial}{\partial t},$$
$$V_3 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}.$$

5.3 Optimal Systems

The commutators table and adjoint table for Lie algebra can easily be constructed as follows:
The commutator (Lie bracket) of V_α and V_β is first order operator defined by
$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha$.

Table 5.1 Commutator table

comm	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$-V_2$
V_3	0	V_2	0

Formula of adjoint table is

$Adj[exp(\varepsilon V_i)]V_j = V_j - \varepsilon[V_i,V_j] + \frac{\varepsilon^2}{2}[V_i[V_i,V_j]] + ...$

Table 5.2 Adjoint table

Adj	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 + \varepsilon V_2$
V_3	V_1	$V_2 e^{-\varepsilon}$	V_3

We deduce an optimal system of sub algebra with their corresponding generators as follows:-

(1) $V_1 + \mu V_3$

(2) V_3

Where μ is arbitrary constant.

5.4 Reduction and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebra, the reductions of PDEs (5.2.1) into ODEs in terms of similarity variable and the new dependent variables-F, are obtained using the auxiliary equations.some exact solutions of each reduced system are then attempted.

Generator (1)

The generator (1) is $V_1 + \mu V_3$

The corresponding characteristic equations are given by:-

$$\frac{dx}{1} = \frac{dt}{\mu t} = \frac{du}{\mu u}$$

Thus the generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$\xi = te^{\mu x}$

$u(x,t) = \frac{1}{t}F(\xi)$

Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (PDE) (5.2.1) reduces to the followings system of Ordinary differential equation(ODE)

$$-\xi F' + F + \mu^2\xi^3 F''' + 2\mu^2\xi^2 F'' - 4\mu\xi F F' + 3\mu^3\xi^3 F'F'' + 3\mu^3\xi^2 (F')^2 + \mu^3\xi^3 F F''' + 3\mu^3\xi^2 F F'' + \mu^3\xi F F' = 0$$

In this case we are able to find only reductions.

Generator (2)

The generator (1) is V_3

The corresponding characteristic equations are given by:-

$$\frac{dx}{0} = \frac{dt}{-t} = \frac{du}{u}$$

Thus the generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$\xi = x$

$u(x,t) = \frac{1}{t} F(\xi)$

Using the similarity variable, the forms of the similarity solution, the system of Partial differential equation (PDE) (5.2.1) reduces to the followings system of Ordinary differential equation(ODE)

$$FF''' + 3F'F'' - 4FF' - F'' + F = 0 \tag{5.4.1}$$

We have considered the series solution here. Let

$$F(\xi) = \frac{a_2}{\xi^2} + \frac{a_1}{\xi} + a_0 + b_1\xi + b_2\xi^2 \tag{5.4.2}$$

Diff. w.r.t ξ, we get

$$F'(\xi) = -2\frac{a_2}{\xi^3} - \frac{a_1}{\xi^2} + b_1 + 2b_2\xi \tag{5.4.3}$$

Again diff. w.r.t ξ

$$F''(\xi) = 6\frac{a_2}{\xi^4} + 2\frac{a_1}{\xi^3} + 2b_2 \tag{5.4.4}$$

Again diff. w.r.t ξ

$$F'''(\xi) = -24\frac{a_2}{\xi^5} - 6\frac{a_1}{\xi^4} \tag{5.4.5}$$

Put value of $F(\xi)$, $F'(\xi)$, $F''(\xi)$, $F'''(\xi)$ in equation (5.4.1) , we get

Then comparing the coefficients of ξ^{-3}, ξ^{-2}, ξ^{-1}, ξ^0, ξ^1, ξ^2, ξ^3, ξ^4, ξ^5, ξ^6, ξ^7, we get

$$a_1 = 0$$

$$a_2 = 0$$

$$b_2 = 0$$

$$b_1 = \frac{1}{4}$$

$$a_0 = 0$$

$$F(\xi) = \frac{1}{4}\xi.$$

The solution of the system is given by

$$u(x,t) = \frac{x}{4t} \tag{5.4.6}$$

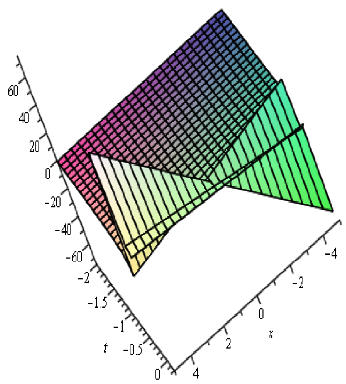

Figure 5.1: Plot of solution (5.4.6) by $x = -5.....5, t = -2......2$

CONCLUSION

In this thesis we carried out the studies of nonlinear partial differential equation due to their immense use in the study of various fields i.e. in pure mathematics, in applied mathematics and in various physical phenomena. To derive the exact solution of nonlinear partial differential equation, Lie Classical Method is used in this thesis via Gardner's equation, Buckmaster equation, Degasperis Processi. We refer to Classical Lie approach to determine the admissible symmetries. After obtaining the point symmetries admitted by a system under investigation, a formal approach of identifying an optimal system of Lie sub algebras has been opted with the help of the adjoint action of the Lie algebra. The basic generator contained in the optimal system have been exploited to achieve the desired reductions of partial differential equations to ordinary differential equations. Some exact solution of each reduced systems are then attempted. The resulting ordinary differential equations hav ebeen examined subsequently for various types of exact solutions. Various exact solutions are obtained in terms of hyperbolic functions, elliptic functions.. Some figures to have an idea about the nature of the solutions are plotted in the thesis.

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