Symmetries and Exact Solutions of Some Nonlinear Partial Differential Equations by Symmetry Reduction Method

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Submitted by

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under

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DEDICATED

TO

GOD, MY PARENTS

AND

SUPERVISOR

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "Symmetries and Exact Solutions of Some Nonlinear Partial Differential Equations by Symmetry Reduction Method" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Rajesh Kumar Gupta.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

The thesis entitled “Symmetries and Exact Solutions of Some Nonlinear Partial Differential Equations by Symmetry Reduction Method” is attempted to find some exact solutions of nonlinear partial differential equations governing some important physical phenomenons by using Symmetry reduction method which is based on Frechet derivatives of the differential operators and we derive Lie algebra which then helps us to obtain the optimal system of generators. Then, we find reduced ordinary differential equation from given nonlinear partial differential equations and some exact solutions of them.

The thesis has been divided into five chapters. The brief outlines of the research work presented chapter wise in the thesis is as follows:

In the first chapter, we have described the introduction of nonlinear partial differential equations and exact solutions. In this chapter, we discussed Lie group of transformations in preliminary material and relevant literature.

In the second chapter, we have described the detailed study of Symmetry reduction method based on Frechet derivatives of differential operators.

In the third chapter, by using Symmetry Reduction method, we deduce the Lie symmetries and find reduced ordinary differential equations and the exact solutions of Gardner equation.

In fourth chapter, ordinary differential equations and solutions of Buckmaster equation with symmetry method of Frechet derivatives techniques are obtained.

In fifth chapter, we apply Symmetry reduction method in order to obtain symmetries of Degasperi Processi equation.

It is worth to mention that all the solutions reported in the thesis are checked by Maple software.

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Chapter 1

INTRODUCTION

Nonlinear problems are of interest to engineers, physicists and mathematicians and many other scientists because most systems are inherently nonlinear in nature. As nonlinear equations are difficult to solve, nonlinear systems are commonly approximated by linear equations. A system of differential equations is said to be nonlinear if it is not a linear system. Problems involving nonlinear differential equations are extremely diverse and methods of solution or analysis are problem dependent. Examples of nonlinear differential equations are the Navier-Stokes equations in fluid dynamics and nonlinear optics. One of the greatest difficulties of nonlinear problems is that it is not generally possible to combine known solutions into new solutions.

A few nonlinear differential equations have known exact solutions, but many which are important in applications do not. Sometimes these equations may be linearized by an expansion process in which nonlinear terms are discarded. In mathematics and physics, nonlinear partial differential equations are partial differential equations with nonlinear terms. They describe many different physical systems, ranging from gravitation to fluid dynamics and have been used in mathematics to solve problems.

In mathematics, a nonlinear system of equations is a set of simultaneous equations in which the unknowns appear as variables of a polynomial of degree higher than one or in the argument of a function which is not a polynomial of degree one. In other words, in a nonlinear system of equations, the equations to be solved cannot be written as linear combination of the unknown variables or functions that appear in it.

The work carried out in this thesis is devoted to the applications of some group theoretic techniques. The prime objective and motivation in carrying out the proposed study is to demonstrate the importance of group theoretic methods over various other methods available in the literature. In the thesis, three nonlinear partial differential equations considered for exact solutions are: Gardner equation, Buckmaster equation, Degasperi Processi equation.

We have used Software like Maple during research and to test the authenticity of solutions.

1.1 Literature Review

The mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The group theoretic techniques based on Lie group theory can be categorized in this class and usually these techniques produce a variety of exact solutions. The analysis and classification of differential equation using group theory goes back to the Norwegian mathematician devoted his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically based sciences.

Nevertheless, anyone who is already familiar with one of these modern manifestations of Lie group theory is perhaps surprised to learn that its original inspirational source was the field of differential equations. The entire subject lay dormant for nearly half a century until G. Birkhoff (1950) [4] called attention to the unexploited application of the lie group to differential equations. Ovsiannikov [5] and his coworkers began a systematic program of successfully applying these methods to wide range of physically important problems.

We have concerned the work of Bluman and Cole [1, 2]. Since then, the theory has witnessed a veritable explosion of research both in the application to physical systems and its development (Olver [12, 13]).

Lie's continuous group theoretic ideas are divided into two categories: direct methods and group theoretic methods. The direct method consists of separation of variables devised by Kline [9] and Miller [11], and dimensional analysis due to Sedov [14]. Group theoretic methods are divided into two categories namely inspectional methods and deductive methods. Inspectional methods are two fold in the sense that the first one is due to Birkhoff [4], and the other is due to Hellums and Churchill [8]. In the class of deductive procedures, there are the following techniques proposed by different authors. Some Nonclassical methods (Bluman and Cole [1]), Classical Lie method (Olver [12]), Symmetry reduction method (Steinberg [17]) etc.

1.2 Preliminary Material

In this section, we are presenting some of basic definitions essential for the techniques to derive exact solutions of the nonlinear systems from Bluman and Anco [3].

1.2.1 Lie Group of Transformations

Definition

A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

(1) **Closure property**: For any elements a and b of G, $\phi(a,b)$ is an element of G.

(2) **Associative property**: For any elements a, b, c of G:

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c)$$

(3) **Identity element**: There exists a unique identity element e of G such that for any element a of G:

$$\phi(a,e) = \phi(e,a) = a$$

(4) **Inverse element**: For any element a of G, there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$$

Definition

Let $x = (x_1, x_2, x_3, ..., x_n)$ lie in a region $D \subset R^n$. The set of transformations

$$x^* = X(x; \varepsilon),$$

defined for each x in D and parameter ε in set $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters ε and δ in S, forms a one-parameter group of transformations on D if the following hold:

(1) For each ε in S the transformations are one-to-one onto D.

(2) S with the law of composition ϕ forms a group G.

(3) For each x in D, $x^* = x$ when $\varepsilon = \varepsilon_0$ corresponds to the identity e, i.e

$$X(x; \varepsilon_0) = x.$$

(4) If $x^* = X(x; \varepsilon)$, $x^{**} = X(x^*; \delta)$, then

$$x^{**} = X(x; \phi(\varepsilon, \delta)).$$

Definition

A one-parameter group of transformations defines a one-parameter Lie group of transformations if, in addition to satisfying the property (1) to (4) of above definition, the following hold:

(5) ε is a continuous parameter, i.e., S is an interval in R. Without loss of generality, $\varepsilon = 0$ corresponds to the identity element e.

(6) X is infinitely differentiable with respect to x in D and an analytic function of ε in S.

(7) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, $\varepsilon \in S$, $\delta \in S$.

Example

Consider

$$x^* = \alpha x,$$

$$y^* = \alpha^2 y, 0 < \alpha < \infty.$$

and $\phi(\alpha, \beta) = \alpha\beta$, this forms a one-parameter lie group of transformations.

Definition

Consider a one-parameter (ε) Lie group of transformations

$$x^* = X(x;\varepsilon) \tag{1.1}$$

with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding (1.1) about $\varepsilon = 0$, in some neighborhood of $\varepsilon = 0$, we get

$$x^* = x + \varepsilon(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}) + \frac{1}{2}\varepsilon^2(\frac{\partial^2 X(x;\varepsilon)}{\partial \varepsilon^2}|_{\varepsilon=0}) + \dots\dots\dots = x + \varepsilon(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}) + O(\varepsilon^2)$$

Let

$$\xi(x) = \frac{\partial X(x;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}$$

The transformation $x + \varepsilon\xi(x)$ is called the infinitesimal transformation of the Lie group of transformations (1.1). The components of $\xi(x)$ are called the infinitesimals of (1.1).

Definition

The infinitesimal generator of the one-parameter Lie group of transformations (1.1) is the operator

$$X = X(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}$$

where ∇ is the gradient operator

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\dots, \frac{\partial}{\partial x_n})$$

For any differentiable function $F(X) = F(x_1, x_2, \dots., x_n)$, one has

$$XF(x) = \xi(x).\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial x_i}$$

Definition

An infinitely differentiable function $F(x)$ is an invariant function of lie group of transformation (1.1) if and only if, for any group of transformation (1.1),

$$F(x^*) = F(x)$$

If $F(x)$ is invariant function of (1.1), then $F(x)$ is called an invariant of (1.1) and $F(x)$ is said to be invariant under (1.1).

Definition

$F(x)$ is invariant under a lie group of transformations (1.1) if and only if $XF(x) = 0$.

1.2.2 Lie Bracket

For an r-parameter Lie group of transformations with infinitesimal generators X_α, $\alpha = 1, 2, ..., r$, the commutator (Lie Bracket) of X_α and X_β is a first order operator defined by

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = \sum_{i,j=1}^{n} [(\xi_{\alpha i}(x)\frac{\partial}{\partial x_i})(\xi_{\beta j}(x)\frac{\partial}{\partial x_j}) - (\xi_{\beta j}(x)\frac{\partial}{\partial x_i})(\xi_{\alpha j}(x)\frac{\partial}{\partial x_j})]$$

$$= \sum_{j=1}^{n} \eta_j(x)\frac{\partial}{\partial x_j}$$

where

$$\eta_j(x) = \sum_{j=1}^{n} \left(\xi_{\alpha i}(x)\frac{\partial \xi_{\beta i}(x)}{\partial x_i} - \xi_{\beta j}(x)\frac{\partial \xi_{\alpha j}(x)}{\partial x_i} \right)$$

It immediately follows that

$$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]$$

1.2.3 Adjoint Vector

Let G be a Lie group with Lie algebra L. For each vector $v \in L$,

the adjoint vector Ad v at $w \in L$ is

$Ad\ v\ |_w = [w,v] = -[v,w]$.

The adjoint representation Ad G of the underlying Lie group can be reconstructed either by integrating the system of linear ordinary differential equations

$\frac{\partial w}{\partial \varepsilon} = Ad\ v\ |_w, w(0) = w_0$

with solution

$$w(\varepsilon) = Ad(exp(\varepsilon v))w_0$$

$$Ad(exp(\varepsilon v))w_0 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(Adv)^n(w_0)$$

$$= w_0 - \varepsilon[v,w_0] + \frac{\varepsilon^2}{2}[v,[v,w_0]] - .....$$

1.2.4 Optimal System

An optimal system of one-parameter subgroups is a list of conjugacy inequivalent one-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub algebras. For one dimensional sub algebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation.

Chapter 2

METHODOLOGY

In this thesis, we deal with the methods of group invariant solution, based on the theory of continuous group of transformations, better known as 'Lie group', acting on the space of independent and dependent variables of the system. The method is due originally to Sophus Lie [10], who unified and extended the bewildering special methods of integration of differential equations. Another application of symmetry methods is to reduce systems of differential equations, finding equivalent systems of differential equations of simpler form.

Sophus Lie introduced the notion of a Lie group in order to study the solutions of ordinary differential equations. He showed the property: the order of an ordinary differential equation can be reduced by one if it is invariant under one-parameter Lie group of point transformations. Lie devoted the remainder of his mathematical career to developing these continuous groups that have now an impact on many areas of mathematically-based sciences. The applications of Lie groups to differential systems were mainly established by Lie.

The work put up in this thesis has primarily been based on certain concepts of group symmetry through the applications of symmetry of Reduction methods.

A Lie point symmetry of a system is a local group of transformations that maps every solution of the system to another solution of the same system. In other words, it maps the solution set of the system to itself. Elementary examples of Lie groups are translations, rotations and scalings. Lie groups and their infinitesimals can be naturally "extended" to act on the space of independent variables, dependent variables and dependent variables up to any finite order.

As mentioned earlier the work comprising this thesis is based primarily on the applications of symmetry reduction methods. The problems are dealt-with in two phases-in the first, the symmetries of the system under investigation are derived using symmetry reduction method and then in second phase, after successfull deduction of the reduced systems of ordinary differential equations, find some exact solutions.

2.1 Symmetry Reduction Method

Symmetry method is often used for reduction of partial differential equations to the equations with fewer number of independent variables and thus for the construction of exact solutions for different mathematical physics problems. Though the technique relies heavily on the theory of sophisticated use of non-linear operators yet it has been cast in the form that it is easy to utilize by specialist and non-specialist alike. The algorithmic representation of the method makes the concepts clear and straight forward. Further, it bears a close relationship to the method of separation of variables in case of linear equations. This algorithm overview has been considered from Steinberg [17].

This method is utilized in chapters 3-5 to investigate symmetries and reductions of Gardener equation, Buckmaster equation and Degasperis Processi equation. The advantage of this approach is that it not only furnishes the group infinitesimals comparatively easily but, also often renders symmetries more generalized than the classical Lie methods.

The analytical execution of the technique can be thought of as following of the three steps:

(a) Find the symmetries of the differential equations.

(b) Determine the canonical coordinate for symmetry or assume a separable form for the differential equation.

(c) Find the reduced problem in terms of the canonical coordinates.

For determining the symmetry operator of a system of a differential equations, we need to proceed as follows:

Let us consider a system of k non-linear partial differential equations in k dependent variables $\bar{u} = (u_1, u_2, ..., u_k)$ and (n+1) independent variables $(t, \bar{x}) = (t, x_1, x_2, ..., x_n)$. Let us assume that our system can be written in terms of non-linear differential operator $\bar{N} = (N_1, N_2, ..., N_K)$ as follows:

$$\bar{N}(\bar{u}) = \frac{\partial^p \bar{u}}{\partial t^p} - \bar{H}(\bar{u}) = \bar{0} \tag{2.1}$$

where $\bar{u} = \bar{u}(t, \bar{x})$. $\bar{H}$ may be defined on the space of t, $\bar{x}$, $\bar{u}$ and any derivative of $\bar{u}$ as long as the derivative of $\bar{u}$ don't contain more than (p-1) derivatives of t. $\bar{H}$ can be nonlinear.

Next, we define symmetry operator $\bar{S} = (S_1, S_2, ..., S_K)$ for the system (2.1) called infinitesimal symmetries. These symmetries are quasi-linear partial differential operators of first order and consequently must have the form.

$$\bar{S} \equiv A(t, \bar{x}, \bar{u}) \frac{\partial \bar{u}}{\partial t} + \sum_{i=1}^{n} B_i(t, \bar{x}, \bar{u}) \frac{\partial \bar{u}}{\partial x_i} + \bar{C}(t, \bar{x}, \bar{u}), \tag{2.2}$$

where $\bar{C} = (C_1, C_2, ..., C_K)$. Frechet derivative $\bar{F} = (F_1, F_2, ..., F_K)$ of $\bar{N}$ at $\bar{u} = (u_1, u_2, ..., u_k)$ in the direction of $\bar{v} = (v_1, v_2, ..., v_k)$ is given by

$$\bar{F}(\bar{N}, \bar{u}, \bar{v}) = \frac{d}{d\varepsilon}[\bar{N}(\bar{u} + \varepsilon \bar{v})]|_{\varepsilon=0} \tag{2.3}$$

(a) The method mainly consists of determining the coefficients A, B_i, i=1,2,...,n and C_j, j=1,2,...,k in the symmetry operator $\bar{S}$, we need to proceed as follows:

We first find Frechet derivative $\bar{F} = (F_1, F_2, ..., F_K)$ of $\bar{N}(\bar{u}) = (N_1, N_2, ..., N_k)$ by the equations (2.3), then $\bar{v} = (v_1, v_2, ..., v_k)$ is substituted by $\bar{S} = (S_1, S_2, ..., S_K)$ in order to evaluate them in the direction of the symmetry operator.

$$\bar{F}(\bar{N}, \bar{S}) = \frac{d}{d\varepsilon}[\bar{N}(\bar{u} + \varepsilon \bar{S})]|_{\varepsilon=0}. \tag{2.4}$$

For invariance of the system (2.1), we require the Frechet derivative (2.4) must vanish on the solution set of (2.1) in the direction of the symmetry operator $\bar{S}$.
That is we must have

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{S})|_{\bar{N}=\bar{0}} = \bar{0}. \tag{2.5}$$

For this we substitute $\bar{H}(\bar{u})$ for $\frac{\partial^p \bar{u}}{\partial t^p}$ in equation (2.5). The equations (2.5) when expanded, result in to polynomial expression in various partial derivatives of $\bar{u}$. Equating the various coefficients of these derivative terms, we will get a set of linear partial differential equations called "determining equations" for the group infinitesimals A, B_i, i=1,2,...,n and C_j, j=1,2,...,k. Solve the resulting "determining equations" for the symmetry of the system (2.1).

(b) Once this resulting set of partial differential equations is solved for coefficients of $\bar{S}$. The associated Lie algebra of infinitesimal symmetries of (2.1) is then the set of vector fields of the form

$$V \equiv A(t,\bar{x},\bar{u})\frac{\partial}{\partial t} + \sum_{i=1}^{n} B_i(t,\bar{x},\bar{u})\frac{\partial}{\partial x_i} - \sum_{j=1}^{k} C_j(t,\bar{x},\bar{u})\frac{\partial}{\partial u_j}. \tag{2.6}$$

Or, equivalently the one-parameter group of point transformations of (2.1) is as follows:

$$t^* = t + \varepsilon A(t,\bar{x},\bar{u}) + O(\varepsilon^2)$$

$$\bar{x}^* = \bar{x} + \varepsilon \bar{B}(t,\bar{x},\bar{u}) + O(\varepsilon^2)$$

$$\bar{u}^* = \bar{u} - \varepsilon \bar{C}(t,\bar{x},\bar{u}) + O(\varepsilon^2)$$

where $\bar{u}^* = (u_1^*, u_2^*, ...u_k^*,)$. Using the infinitesimal generators (2.6), one can obtain a reduction of system (2.1) to a system with number of independent variables one less than the original one. For this, first we solve the "characteristic equations"

$$\frac{dt}{A} = \frac{dx_1}{B_1} = \frac{dx_2}{B_2} = ... = \frac{dx_n}{B_n} = \frac{du_1}{-C_1} = \frac{du_1}{-C_2} = ... = \frac{du_k}{-C_k}$$

From these equations, we obtain the canonical coordinates (similarity form of the solution in terms of new independent variables).

(c) Change the system (2.1) in these new coordinate to get the reduced form of the problem.

Chapter 3

GARDNER EQUATION

The mathematical theory of nonlinear evolution equations, starting from the Korteweg-de-Vries (KDV) equation and modified Korteweg-de-Vries (mKDV) equation, is an area of research for the past few decades [7, 15, 16, 19]. In particular, the Gardner equation [18] that is also known as the mixed KDV-mKDV equation.

This Gardner equation shows up, particularly, in the context of internal gravity waves in a density-stratified ocean. This is commonly described by the KDV equations and its versions with small nonlinearity. This lead to study of Gardner equation.

$$u_t = 6(u + \varepsilon^2 u^2)u_x + u_{xxx}.$$

3.1 Determining Equations for Lie Symmetries

Consider the Gardner equation

$$u_t = 6(u + \varepsilon^2 u^2)u_x + u_{xxx}. \tag{3.1}$$

Let the system (3.1) is defined in terms of non linear operators N_1 as follows:

$$N_1 \equiv u_t - 6(u + \varepsilon^2 u^2)u_x - u_{xxx} = 0. \tag{3.2}$$

The symmetry operator $\bar{S} = (S_1)$ for the equation (3.1) is:

$$S_1(u) \equiv A(\bar{X}, \bar{\eta})\frac{\partial u}{\partial t} + B(\bar{X}, \bar{\eta})\frac{\partial u}{\partial x} + C(\bar{X}, \bar{\eta}), \tag{3.3}$$

with $\bar{X} = (t,x)$, $\bar{\eta} = (u)$.

The Frechet derivative of nonlinear operator N_1 obtained in the direction of the symmetry operator $\bar{S} = (S_1)$ are given, respectively, by the following:

$$F_1(N_1,\bar{\eta},\bar{S}) = \frac{d}{d\varepsilon}[N_1(\bar{\eta}+\varepsilon\bar{S})]|_{\varepsilon=0} = [S_1]_t - 6(u[S_1]_x + [S_1]u_x) - 6\varepsilon^2(u^2[S_1]_x + 2u[S_1]u_x) - [S_1]_{xxx} = 0 \tag{3.4}$$

In equation (3.4), on replacing S_1 with the help of equation (3.3):

$$\begin{aligned}&(Au_t + Bu_x + C)_t - 6(u(Au_t + Bu_x + C)_x + (Au_t + Bu_x + C)u_x)\\&- 6\varepsilon^2(u^2(Au_t + Bu_x + C)_x + 2uu_x(Au_t + Bu_x + C)) - (Au_t + Bu_x + C)_{xxx} = 0\end{aligned} \tag{3.5}$$

on simplify

$$\begin{aligned}&[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t - 6(u([A]_x u_t + Au_{tx} + [B]_x u_x + Bu_{xx} + [C]_x) + u_x(Au_t + Bu_x + C))\\&- 6\varepsilon^2(u^2([A]_x u_t + Au_{tx} + [B]_x u_x + Bu_{xx} + [C]_x) + 2uu_x(Au_t + Bu_x + C)) - (3[A]_{xx}u_{tx} + 3[A]_x u_{txx}\\&+ Au_{txxx} + [A]_{xxx}u_t + 3[B]_{xx}u_{xx} + 3[B]_x u_{xxx} + Bu_{xxx} + [B]_{xxx}u_x + [C]_{xxx} = 0\end{aligned} \tag{3.6}$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc. in equation (3.6) are represent the total differentiation with respect to suffices and are given by the following expressions:

$[A]_t = A_t + A_u u_t$

$[A]_x = A_x + A_u u_x$

$[A]_{xx} = A_{xx} + 2A_{ux}u_x + A_u u_{xx} + A_{uu}u_x^2$

$[A]_{xxx} = A_{xxx} + 3A_{uxx}u_x + 3A_{ux}u_{xx} + A_u u_{xxx} + 3A_{uux}u_x^2 + 3A_{uu}u_{xx}u_x + A_{uuu}u_x^3.$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$. The invariance equation applied on equation (3.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C which are obtained after equating the coefficients of various derivative terms to zero:

$$A_x = A_u = 0$$

$$B_u = 0$$

$$C_{uu} = 0$$

$$A_t + 2Cu + 3B_x = 0 \tag{3.7}$$

$$C_{ux} + Bxx = 0 \tag{3.8}$$

$$6uA_t + 6\varepsilon^2 u^2 A_t + B_t - 6uB_x - 6C - 6\varepsilon^2 u^2 B_x - 12u\varepsilon^2 C + B_{xxx} - 3C_{uxx} = 0 \tag{3.9}$$

$$C_t - 6uCx - 6\varepsilon^2 u^2 C_x + C_{xxx} = 0 \tag{3.10}$$

From (3.7), Differentiate w.r.t. x: $2C_{ux} + 3B_{xx} = 0$.

By solving (3.7) and (3.8) equations, we get $B_{xx} = 0, C_{ux} = 0$.

If $B_{xx} = 0$ then $B = B_1(t)x + B_2(t)$.

Above system becomes:

$$A_x = A_u = 0 \Rightarrow A = A(t)$$

$$B_u = 0, B_{xx} = 0 \Rightarrow B = B_1(t)x + B_2(t)$$

$$C_{uu} = 0, C_{ux} = 0 \Rightarrow C = C_1(t)u + C_2(t)$$

$$A_t + 2C_u + 3B_x = 0$$

$$6uA_t + 6\varepsilon^2 u^2 A_t + B_t - 6uB_x - 6C - 6\varepsilon^2 u^2 B_x - 12u\varepsilon^2 C = 0$$

$$C_t - 6uC_x - 6\varepsilon^2 u^2 C_x + C_{xxx} = 0.$$

Put values of B and C then system becomes:

$$A_t + 2C_1 + 3B_1 = 0 \tag{3.11}$$

$$6uA_t + 6\varepsilon^2 u^2 A_t + B_{1t}x + B_{2t} - 6uB_1 - 6(C_1 u + C_2) - 6\varepsilon^2 u^2 B_1 - 12\varepsilon^2 u(C_1 u + C_2) = 0 \tag{3.12}$$

$$C_{1t}u + C_{2t} = 0 \tag{3.13}$$

From (3.13), coefficient of u:

$$C_{1t} = 0$$

constant:

$$C_{2t} = 0$$

Then

$$C_1 = k_1$$

and

$$C_2 = k_2$$

So,

$$C = k_1 u + k_2$$

Put value of C in (3.11) and (3.12), then we have:

$$A_t + 2k_1 + 3B_1 = 0$$

$$6uA_t + 6\varepsilon^2 u^2 A_t + B_{1t}x + B_{2t} - 6uB_1 - 6(k_1 u + k_2) - 6\varepsilon^2 u^2 B_1 - 12\varepsilon^2 u(k_1 u + k_2) = 0 \tag{3.14}$$

From (3.14), coeff. of u^2:

$$A_t - 2k_1 - B_1 = 0$$

coeff. of u:

$$A_t - k_1 - B_1 - 2\varepsilon^2 k_2 = 0$$

constant:

$$B_{2t} - 6k_2 = 0$$

So,

$$B_2 = 6k_2 t + k_3$$

coeff. of x:

$$B_{1t} = 0$$

So,

$$B_1 = k_4$$

From (3.14), Differentiate w.r.t. t:

$$A_{tt} = 0$$

So,

$$A = k_4 t + k_5$$

where k_1, k_2, k_3, k_4 and k_5 are constants.

System (3.14) becomes:

$$A = k_4 t + k_5$$

$$B = k_4 x + 6k_2 t + k_3$$

$$C = k_1 u + k_2$$

$$k_1 = k_4$$

$$k_1 = 0$$

$$k_2 = 0$$

From these values,we get infinitesimals A, B and C:

$$\begin{aligned} A &= k_5 \\ B &= k_3 \\ C &= 0. \end{aligned} \tag{3.15}$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \tag{3.16}$$

Thus it easily seen that the application of symmetry method to equations (3.1) leads to a two-parameter Lie -group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} \\ V_2 &= \frac{\partial}{\partial t} \end{aligned} \tag{3.17}$$

Now we calculate Optimal system:

The commutator table-3.1 and adjoint table- 3.2 for Lie algebra (3.17) can easily constructed as follows:

The commutator (Lie bracket) of X_a and X_b is a first order operator defined by

$[X_a, X_b] = X_a X_b - X_b X_a$.

Table 3.1: Commutator Table

comm	V_1	V_2
V_1	0	0
V_2	0	0

$$Ad(exp\varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] + ... \quad (3.18)$$

Table 3.2: Adjoint Table

Ad	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) $V_1 + mV_2$

(ii) V_2,

where m is arbitrary constant.

3.2 Reduced ODEs and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of partial differential equations (3.1) into ordinary differential equations in terms of similarity variable ξ and the new dependent variables- F is obtained using the auxiliary equations (3.16). Some exact solutions of each reduce system are then attempted.

Vector field (i)

The vector field (i) in the optimal system defines the similarity variable and similarity solution as follows:

$V_1 + mV_2$.

We use values of system (3.17) and use characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{m} = \frac{du}{0}. \tag{3.19}$$

Then by solving this characteristic equation, we have:

$\xi = mx - t$

$u(x,t) = F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of partial differential equations (3.1) reduces to following system of ordinary differential equations:

$$6mFF' + 6\varepsilon^2 mF^2F' + m^3F''' + F' = 0 \tag{3.20}$$

By using Maple software, we get some exact solutions of (3.20)

$$F(\xi) = c_3$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{-m(-2\varepsilon^2+3m)}\xi}{m^2\varepsilon}\right)\right)$$

The solution of the system (3.1) is given by

$$u(x,t) = c_3$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right) \tag{3.21}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right) \tag{3.22}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 - \frac{1}{2}\frac{\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{1}{m\varepsilon^2}\left(\sqrt{m(-2\varepsilon^2+3m)}tanh\left(c_1 + \frac{1}{2}\frac{\sqrt{m(-2\varepsilon^2+3m)}(mx-t)}{m^2\varepsilon}\right)\right)$$

where ε, c_1 and m are constants.

Vector field (ii)

For this vector field the associated the similarity variable and similarity solutions as follows: V_2.

We use values of system (3.17), and use characteristic equation we have:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}. \tag{3.23}$$

Then by solving this characteristic equation, we have:

$\xi = x$

$u(x,t) = F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of partial differential equations (3.1) reduces to following system of ordinary differential equations:

$$6FF' + 6\varepsilon^2 F^2 F' + F''' = 0 \tag{3.24}$$

By using Maple software, we get some exact solutions of (4.20):

$$F(\xi) = c_3$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}sech\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{6}sech\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{6}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{6}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1 + \frac{1}{2}\frac{\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1 + \frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

$$F(\xi) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}\xi}{\varepsilon}\right)}{\varepsilon^2}$$

Thus, the following solution of the system (3.1) is obtained

$$u(x,t) = c_3$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}sech\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{6}sech\left(-c_1 + \frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{6}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2} \tag{3.25}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{6}sech\left(c_1 + \frac{1}{2}\frac{\sqrt{6}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1 + \frac{1}{2}\frac{\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2} \tag{3.26}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}tanh\left(-c_1 + \frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} - \frac{1}{2}\frac{\sqrt{3}tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

$$u(x,t) = -\frac{1}{2\varepsilon^2} + \frac{1}{2}\frac{\sqrt{3}tanh\left(c_1 + \frac{1}{2}\frac{i\sqrt{3}x}{\varepsilon}\right)}{\varepsilon^2}$$

where ε and c_1 are constants.

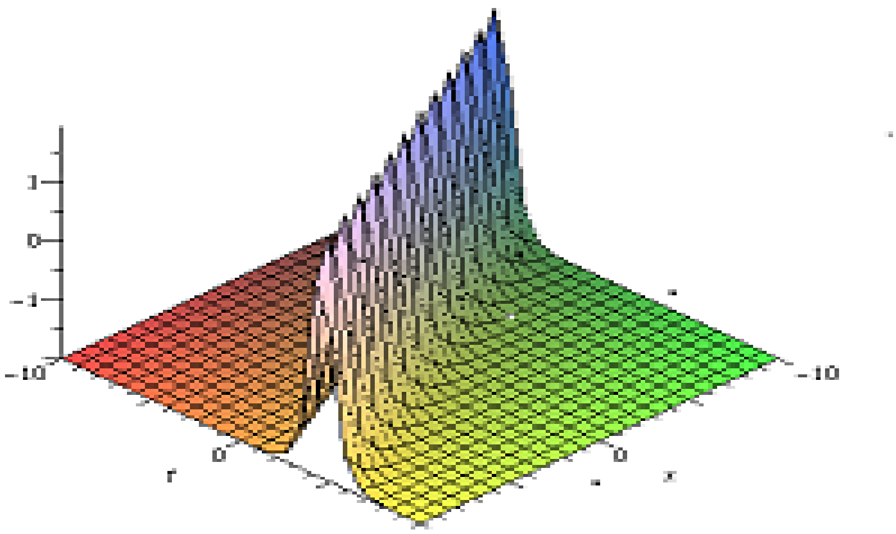

Figure 3.1: Graph for the solution no. (3.21) when ε=0.5, m=0.5, $c_1 = 1$

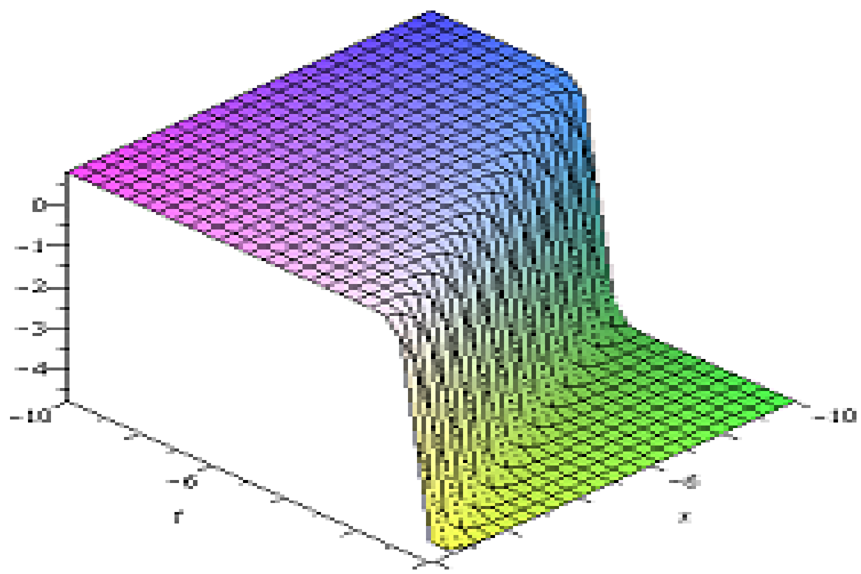

Figure 3.2: Graph for the solution no. (3.22) when ε=0.5, m=0.5, $c_1 = 1$

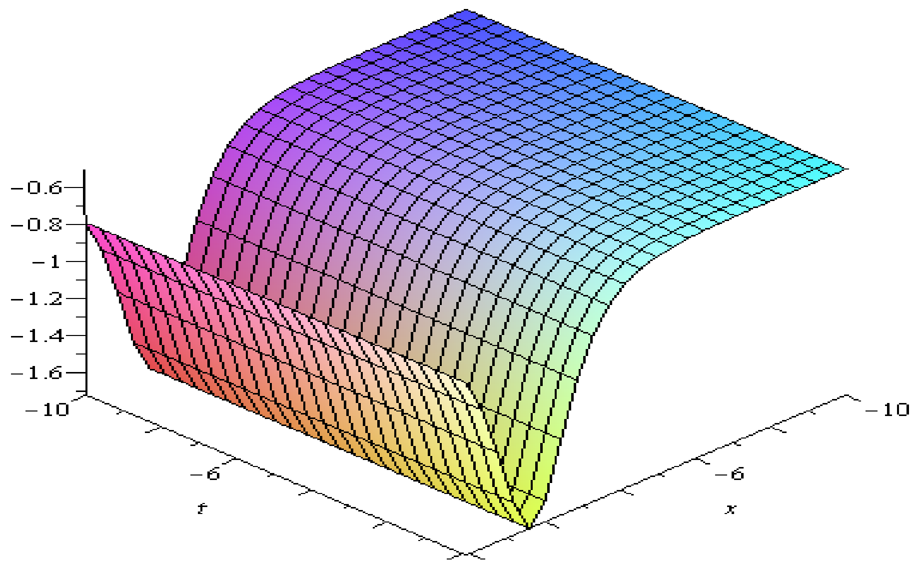

Figure 3.3: Graph for the solution no. (3.25) when ε=1, $c_1 = 2$

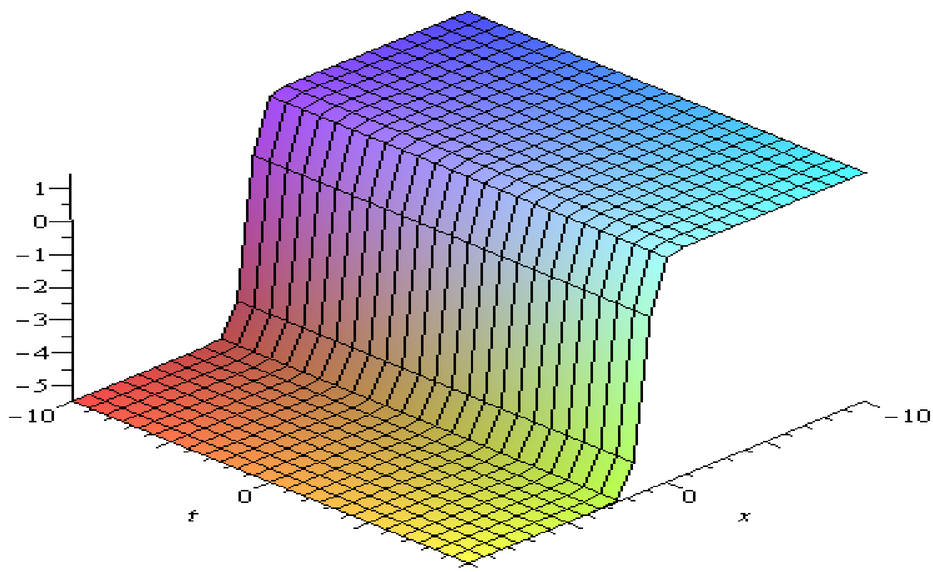

Figure 3.4: Graph for the solution no. (3.26) when ε=0.5, $c_1 = 2$

Chapter 4

BUCKMASTER EQUATION

Partial differential equations are used to mathematically formulate and thus aid the solution of physical and other problems involving functions of severable variables, such as the propagation of heat or sound, elasticity, electrodynamics and electrostatics. In this chapter, Symmetry reduction method was used for solving Buckmaster equation. Earlier this equation was solved by Finite Volume Method [20].

4.1 Symmetry Analysis

Consider Buckmaster equation u$_t = (u^4)_{xx} + (u^3)_x$

Let the system (4.1) is defined in terms of non linear N_1 as follows:

$$N_1 \equiv u_t - 12u^2u_x^2 - 4u^3u_{xx} - 3u^2u_x - u_t = 0 \tag{4.1}$$

The symmetry operator $\bar{S} = (S_1)$ for the equation (4.1) is:

$$S_1(u) \equiv A(\bar{X},\bar{\eta})\frac{\partial u}{\partial t} + B(\bar{X},\bar{\eta})\frac{\partial u}{\partial x} + C(\bar{X},\bar{\eta}) \tag{4.2}$$

with $\bar{X} = (t,x)$, $\bar{\eta} = (u)$. The Frechet derivative of non-linear operator N_1 obtained in the direction of the symmetry operator $\bar{S_1} = (S_1)$ are given, respectively, by the following:

$$\begin{aligned} F_1(N_1,\bar{\eta},\bar{S}) = \frac{d}{d\varepsilon}[N_1(\bar{\eta}+\varepsilon\bar{S})]|_{\varepsilon=0} = [S_1]_t - 24u^2u_x[S_1]_x - 24uu_x^2[S_1] \\ - 12u^2u_{xx}[S_1] - 4u^3[S_1]_{xx} - 6u[S_1]u_x - 3u^2[S_1]_x = 0 \end{aligned} \tag{4.3}$$

In equation (4.3), on replacing S_1 with the help of equation (4.2):

$$
\begin{aligned}
&[Au_t+Bu_x+C]_t-24u^2u_x[Au_t+Bu_x+C]_x-24uu_x^2[Au_t+Bu_x+C]-12u^2u_{xx}[Au_t+Bu_x+C]\\
&-4u^3[Au_t+Bu_x+C]_{xx}-6u[Au_t+Bu_x+C]u_x-3u^2[Au_t+Bu_x+C]_x=0
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
&24[A]_xu^2u_xu_t+24Au^2u_xu_{tx}+24B_xu^2u_x^2+24Bu^2u_xu_{xx}+24[C]_xu^2u_x+24Auu_x^2u_t+24Buu_x^3\\
&+24Cuu_x^2+12Au^2u_{xx}u_t+12Bu_xu^2u_{xx}+12Cu^2u_{xx}+4u^3[A]_{xx}u_t+4u^3[A]_xu_{tx}+4u^3[A]_xu_{tx}\\
&+4u^3Au_{txx}+4u^3[B]_{xx}u_x+8u^3[B]_xu_{xx}+4u^3Bu_{xxx}+4u^3[C]_{xx}+6Auu_xu_t+6Buu_x^2+6Cuu_x\\
&+3[A]_xu^2u_t+3Au^2u_{tx}+3[B]_xu^2u_x+3Bu^2u_{xx}+3[C]_xu^2-[A]_tu_t-Au_{tt}-[B]_tu_x-Bu_{xt}\\
&-[C]_t=0
\end{aligned}
\tag{4.5}
$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}$ etc. in (4.5) are represent the total differentiation with respect to suffices and are given by the following expressions:

$[A]_t = A_t + A_u u_t$

$[A]_x = A_x + A_u u_x$

$[A]_{xx} = A_{xx} + 2A_{ux}u_x + A_u u_{xx} + A_{uu}u_x^2$

$[A]_{xxx} = A_{xxx} + 3A_{uxx}u_x + 3A_{ux}u_{xx} + A_u u_{xxx} + 3A_{uux}u_x^2 + 3A_{uu}u_{xx}u_x + A_{uuu}u_x^3.$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$. The invariance equation applied on equation (4.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C which are obtained after equating the coefficients of various derivative terms to zero:

$$A_u = 0, A_x = 0$$

$$B_u = 0$$

$$3C + 2uB_x - uA_t = 0 \tag{4.6}$$

$$6B_xu + 3C_uu + 6C + C_{uu}u^2 - 3A_tu = 0 \tag{4.7}$$

$$24u^2C_x + 4B_{xx}u^3 + 8C_{ux}u^3 + 3B_xu^2 + 6Cu - 3A_tu^2 - B_t = 0 \tag{4.8}$$

$$4u^3C_{xx} + 3u^2C_x - C_t = 0 \tag{4.9}$$

From equation (4.6), we get

$$C = (\frac{A_t - 2B_x}{3})u \tag{4.10}$$

Differentiate (4.10) w.r.t u, we get

$$C_u = \frac{A_t - 2B_x}{3}$$

So

$$C_{uu} = 0$$

Again diff. w.r.t t., we get

$$C_t = (\frac{A_{tt} - 2B_{xt}}{u})3$$

Now diff. w.r.t. x,

$$C_x = (\frac{-2B_{xx}}{3})u$$

Again diff. w.r.t. x,

$$C_{xx} = (\frac{-2B_{xxx}}{3})u$$

Put these values in above equations,we get

$$(\frac{-52}{3})u^3 B_{xx} - B_x u^2 - A_t u^2 - B_t = 0 \tag{4.11}$$

$$2B_{xt} - A_{tt} = 0 \tag{4.12}$$

From (4.11), coeff. of u^3:

$$B_{xx} = 0$$

Coeff. of u^2:

$$A_t + B_x = 0 \tag{4.13}$$

Const.;

$$B_t = 0$$

Diff. (4.13) w.r.to t, we get

$$A_{tt} = 0$$

$$A = k_1 t + k_2$$

$$B = k_3 x + k_4$$

$$C = (\frac{k_1 - 2k_3}{3})u$$

$$k_1 = 0$$

$$k_3 = 0$$

From these values, we get infinitesimals A, B and C:

$$A = k_2$$

$$B = k_4$$

$$C = 0$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \tag{4.14}$$

Thus it easily seen that the application of symmetry method to equations (4.1) leads to a two-parameter Lie group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} \\ V_2 &= \frac{\partial}{\partial t} \end{aligned} \tag{4.15}$$

Now we calculate Optimal system:

The commutator table-4.1 and adjoint table- 4.2 for Lie algebra (4.15) can easily constructed as follows:

The commutator (Lie bracket) of X_a and X_b is a first order operator defined by

$[X_a, X_b] = X_a X_b - X_b X_a.$

Table 4.1: Commutator Table

comm	V_1	V_2
V_1	0	0
V_2	0	0

$$Ad(exp\varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] + \ldots \tag{4.16}$$

Table 4.2: Adjoint Table

Ad	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) $V_1 + mV_2$

(ii) V_2,

where m is arbitrary constant.

4.2 Reduced ODEs and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of partial differential equations (4.1) into ordinary differential equations in terms of similarity variable ξ and the new dependent variables- F is obtained using the auxiliary equations (4.14). Some exact solutions of each reduce system are then attempted.

Vector field (i)

The vector field (i) in the optimal system defines the similarity variable and similarity solution as follows:

$V_1 + mV_2$.

We use values of system (4.15) and use characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{m} = \frac{du}{0}. \tag{4.17}$$

Then by solving this characteristic equation, we have:

$\xi = mx - t$

$u(x,t) = F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of partial differential equations (4.1) reduces to following system of ordinary differential equations:

$$12mF^2F'^2 + 4m^2F^3F'' + 3mF^2F' + F' = 0 \tag{4.18}$$

By using Maple software, we get exact solutions of (4.18)

$$F(\xi) = \frac{1}{c_3}$$

The solution of the system (4.1) is given by

$$u(x,t) = \frac{1}{c_3}$$

where c_3 is constant.

Vector field (ii)

For this vector field the associated the similarity variable and similarity solutions as follows: V_2.

We use values of system (4.15), and use characteristic equation we have:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}. \tag{4.19}$$

Then by solving this characteristic equation, we have:

$\xi = x$

$u(x,t) = F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of partial differential equations (4.1) reduces to following system of ordinary differential equations:

$$12F'^2 + 4FF'' + 3F' = 0 \tag{4.20}$$

By using Maple software, we get some exact solutions of (4.20):

$$F(\xi) = \frac{1}{c_3}$$

$$F(\xi) = c_3 - \frac{1}{4}\frac{c_1 + c_2\xi}{c_2}$$

Thus, the following solution of system (4.1) is obtained

$$u(x,t) = \frac{1}{c_3}$$

$$u(x,t) = c_3 - \frac{1}{4}\frac{c_1 + c_2 x}{c_2}$$

where c_1, c_2 and c_3 are constants.

Chapter 5

DEGASPERIS PROCESSI EQUATION

The Degasperis Processi equation is nonlinear partial differential equation which models propagation of nonlinear dispersive waves. On the mathematical side the Degasperis Processi equation is very special because it belongs to the class of integrable equations, or solitonic equations (that is, partial differential equations with infinitely many conservation laws), for the investigation of which an analytical tool is available which generalizes to nonlinear partial differential equations the standard Fourier analysis of linear equations.

5.1 Symmetry Analysis

Consider Degasperis processi equation

$$u_t = u_{xxt} - 4uu_x + 3u_x u_{xx} + uu_{xxx} \tag{5.1}$$

Let the system (5.1) is defined in terms of non linear operators N_1 as follows:

$$N_1 \equiv u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{5.2}$$

The symmetry operator $\bar{S} = (S_1)$ for the equation (5.1) is:

$$S_1(u) \equiv A(\bar{X}, \bar{\eta})\frac{\partial u}{\partial t} + B(\bar{X}, \bar{\eta})\frac{\partial u}{\partial x} + C(\bar{X}, \bar{\eta}) \tag{5.3}$$

with $\bar{X} = (t, x)$, $\bar{\eta} = (u)$.

The Frechet derivative of non-linear operator N_1 obtained in the direction of the symmetry operator

$\bar{S_1} = (S_1)$ are given, respectively, by the following:

$$F_1(N_1, \bar{\eta}, \bar{S}) = \frac{d}{d\varepsilon}[N_1(\bar{\eta} + \varepsilon\bar{S})]|_{\varepsilon=0} = [S_1]_t - [S_1]_{xxt} + 4u[S_1]_x + \\ 4[S_1]u_x - 3u_x[S_1]_{xx} - 3[S_1]_x u_{xx} - u[S_1]_{xxx} - [S_1]u_{xxx} \tag{5.4}$$

In equation (5.4), on replacing S_1 with the help of equation (5.3):

$$(Au_t + Bu_x + C)_t - (Au_t + Bu_x + C)_{xxt} + 4u(Au_t + Bu_x + C)_x + 4(Au_t + Bu_x + C)u_x - \\ 3u_x(Au_t + Bu_x + C)_{xx} - 3(Au_t + Bu_x + C)_x u_{xx} - u(Au_t + Bu_x + C)_{xxx} - (Au_t + Bu_x + C)u_{xxx} = 0 \tag{5.5}$$

on simplify

$$[A]_t u_t + Au_{tt} + [B]_t u_x + Bu_{xt} + [C]_t - [A]_{xxt} u_t - [A]_{xx} u_{tt} - 2[A]_{xt} u_{tx} - 2[A]_x u_{txt} - [A]_t u_{txx} - Au_{txxt} \\ - [B]_{xxt} u_x - [B]_{xx} u_{xt} - 2[B]_{xt} u_{xx} - 2[B]_x u_{xxt} - [B]_t u_{xxx} - Bu_{xxxt} - [C]_{xxt} + 4Au_x u_t + 4Bu_x^2 + 4Cu_x \\ + 4[A]_x uu_t + 4Auu_{tx} + 4[B]_x uu_x + 4Buu_{xx} + 4u[C]_x - 3[A]_{xx} u_x u_t - 6[A]_x u_{tx} u_x - 3Au_x u_{txx} - 3[B]_{xx} u_x^2 \\ - 6[B]_x u_{xx} u_x - 3Bu_{xxx} u_x - 3[C]_{xx} u_x - 3[A]_x u_t u_{xx} - 3Au_{tx} u_{xx} - 3[B]_x u_x u_{xx} - 3Bu_{xx}^2 - 3[C]_x u_{xx} \\ - [A]_{xxx} uu_t - 3[A]_{xx} uu_{tx} - 3[A]_x uu_{txx} - Auu_{txxx} - [B]_{xxx} uu_x - 3[B]_{xx} uu_{xx} - 3[B]_x uu_{xxx} - Buu_{xxxx} \\ - u[C]_{xxx} - Au_t u_{xxx} - Bu_x u_{xxx} - Cu_{xxx} = 0 \tag{5.6}$$

where $[A]_t, [A]_x, [A]_{xx}, [A]_{xxx}, A_{xt}$ etc. in equation (5.6) are represent the total differentiation with respect to suffices and are given by the following expressions:

$[A]_t = A_t + A_u u_t$

$[A]_x = A_x + A_u u_x$

$[A]_{xt} = A_{xt} + A_{xu} u_t + A_{ut} u_x + A_{uu} u_t u_x + A_u u_{xt}$

$[A]_{xx} = A_{xx} + 2A_{ux} u_x + A_u u_{xx} + A_{uu} u_x^2$

$[A]_{xxx} = A_{xxx} + 3A_{uxx} u_x + 3A_{ux} u_{xx} + A_u u_{xxx} + 3A_{uux} u_x^2 + 3A_{uu} u_{xx} u_x + A_{uuu} u_x^3.$

Similar expressions have been used for $[B]_t, [B]_x, [B]_{xx}, [B]_{xxx}$ and for $[C]_t, [C]_x, [C]_{xx}, [C]_{xxx}$. The invariance equation applied on equation (5.1) leads to the following simplified set of determining equations for the group infinitesimals A, B and C which are obtained after equating the coefficients of various derivative terms to zero:

$$A_u = 0, A_x = 0$$

$$B_u = 0$$

$$C_{uu} = 0$$

$$A_t u - B_t - B_x u - C = 0 \tag{5.7}$$

$$-B_{xx} - 2C_{ux} = 0 \tag{5.8}$$

$$-C_{xxu} - 2B_x = 0 \tag{5.9}$$

$$C_t - C_{xxt} + 4uC_x - uC_{xxx} = 0 \tag{5.10}$$

$$A_t - B_x - C_u = 0 \tag{5.11}$$

$$-2B_{xt} - C_{ut} - 3C_x - 3C_{ux}u - 3B_{xx}u = 0 \tag{5.12}$$

$$B_t - B_{xxt} - 4B_x u + 4C - 4A_t u - B_{xxx}u - 2C_{uxt} - 3C_{xx} - 3uC_{xxu} = 0 \tag{5.13}$$

By solving these equations, we get

$$A = -k_3 t + k_2$$

$$B = k_1$$

$$C = k_3 u$$

Having determined the infinitesimals, the symmetry variables are found by solving the auxiliary equations

$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{-C}. \tag{5.14}$$

Thus it easily seen that the application of symmetry method to equations (5.1) leads to a two-parameter Lie -group. Associated with this Lie group we have a Lie algebra which can represented by the following generators:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} \\ V_2 &= \frac{\partial}{\partial t} \\ V_3 &= u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} \end{aligned} \tag{5.15}$$

Now we calculate Optimal system:

The commutator table-5.1 and adjoint table- 5.2 for Lie algebra (5.15) can easily constructed as follows:

The commutator (lie bracket) of X_a and X_b is a first order operator defined by

$[X_a, X_b] = X_aX_b - X_bX_a$.

Table 5.1: Commutator Table

$Comm$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$-V_2$
V_3	0	V_2	0

$$Ad(exp\varepsilon V_i)V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] + \dots \tag{5.16}$$

Table 5.2: Adjoint Table

Ad	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 + \varepsilon V_2$
V_3	V_1	$V_2e^{-\varepsilon}$	V_3

We deduce an optimal system of sub algebra with their corresponding generators as follows:

(i) $V_1 + mV_3$

(ii) V_3,

where m is arbitrary constant.

5.2 Reduced ODEs and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebras, the reduction of partial differential equations (5.1) into ordinary differential equations in terms of similarity variable ξ and the new dependent variables- F is obtained using the auxiliary equations (5.14). Some exact solutions of each reduce system are then attempted.

Vector field (i)

The vector field (i) in the optimal system defines the similarity variable and similarity solution as follows:

$V_1 + mV_3$.

We use values of system (5.15) and use characteristic equation we have:

$$\frac{dx}{1} = \frac{dt}{-tm} = \frac{du}{um}. \tag{5.17}$$

Then by solving this characteristic equation, we have:

$\xi = te^{mx}$

$u(x,t) = \frac{1}{t}F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of partial differential equations (5.1) reduces to following system of ordinary differential equations:

$$\begin{aligned} &-\xi F' + F + m^2\xi^3 F''' + 2m^2\xi^2 F'' - 4m\xi F F' + 3m^3\xi^3 F' F'' + \\ &3m^3\xi^2 F'^2 + m^3\xi^3 F F''' + 3m^3\xi^2 F F'' + m^3\xi F F' = 0 \end{aligned} \tag{5.18}$$

In this case, we are able to find only reductions.

Vector field (ii)

For this vector field the associated the similarity variable and similarity solutions as follows: V_3.

We use values of system (5.15), and use characteristic equation we have:

$$\frac{dx}{0} = \frac{dt}{-t} = \frac{du}{u}. \tag{5.19}$$

$\xi = x$

$u(x,t) = \frac{1}{t}F(\xi)$.

Using the similarity variable, the forms of similarity solutions, the system of partial differential equations (5.1) reduces to following system of ordinary differential equations:

$$FF''' + 3F''F' - 4FF' - F'' + F = 0 \tag{5.20}$$

Let

$$F(\xi) = \frac{a_0}{\xi^2} + \frac{a_1}{\xi} + a_0 + b_1\xi + b_2\xi^2 \tag{5.21}$$

Diff. w.r.t ξ, we get

$$F'(\xi) = -2\frac{a_2}{\xi^3} - \frac{a_1}{\xi^2} + b_1 + 2b_2\xi \tag{5.22}$$

Again diff. w.r.t ξ

$$F''(\xi) = 6\frac{a_2}{\xi^4} + 2\frac{a_1}{\xi^3} + 2b_2 \tag{5.23}$$

Again diff. w.r.t ξ

$$F'''(\xi) = -24\frac{a_2}{\xi^5} - 6\frac{a_1}{\xi^4} \tag{5.24}$$

Substitute these values in (5.20) and equating the coefficients of ξ, ξ^{-1}, ξ^2, ξ^3 and so on.

We get $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $b_1 = \frac{1}{4}$, $b_2 = 0$.

Put these values in (5.21), we get

$$F(\xi) = \frac{1}{4}\xi \tag{5.25}$$

The solution of the system (5.1) is given by

$$u(x,t) = \frac{x}{4t} \tag{5.26}$$

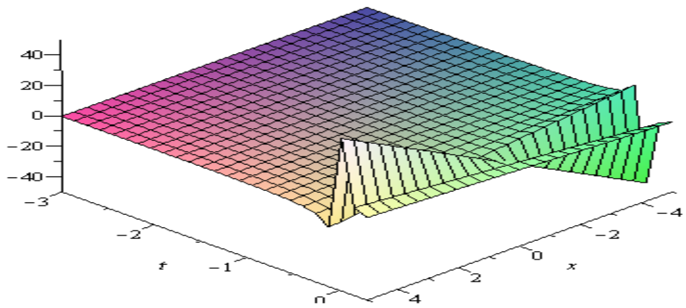

Figure 5.1: Graph for the solution no. (5.26)

CONCLUSION

In summary, the Symmetry reduction method based on Frechet derivatives of differential operators is utilized to investigate the symmetries and invariant solutions of Gardner equation, Buckmaster equation and Degasperis Processi equation. The reduced ordinary differential equations have been examined for various types of exact solutions via techniques which are essentially based on special functions such as Kummer U, Kummer M and hyperbolic functions.

In all three cases. the solutions obtained are of very specific nature and a proper search for the reduction of partial differential equations to ordinary differential equations and then reduction of order of ordinary differential equations based on Lie groups led to the symmetries. As the objective in present work was aimed to the applications of Symmetry reduction method with the view of deducing the symmetries and then attempting some exact analytic solutions.

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