Application of Fuzzy Sets in Time Series Forecasting

A thesis submitted in partial fulfillment of the requirements for award of the degree of

Master of Technology

in

Computer Science and Applications

Submitted By

Anuj Samariya

Roll No. 601303003

Under the Supervision of:

Dr. Pritpal Singh
(Lecturer)

Ms. Vineeta Bassi
(Assistant Professor)

Computer Science and Engineering Department

Thapar University

Patiala-147004

JUNE 2015

Certificate

I, **Anuj Samariya**, hereby certify that the work which is being presented in the thesis entitled, "*Applications of Fuzzy Sets in Time Series Forecasting*", in partial fulfillment of the requirements for the award of degree of Master of Technology (M.Tech) in Computer Science and Applications submitted in Computer Science and Engineering Department of Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of Dr. Pritpal Singh and Ms. Vineeta Bassi and refers other researcher's work which are duly listed in the reference section.

The matter presented in the thesis has not been submitted for award of any other degree of this or any other University.

Signature
(Anuj Samariya)

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

(Dr. Pritpal Singh)
Lecturer
CSE Department

(Ms. Vineeta Bassi)
Assistant Professor
CSE Department

Countersigned by:

(Dr. Deepak Garg)
Head
Computer Science and Engineering Department
Thapar University
Patiala

(Dr. S.S. Bhatia)
Dean (Academic Affairs)
Thapar University
Patiala

Acknowledgement

First and foremost, I would like to thank my supervisors, Lecturer **Pritpal Singh** and Assistant Professor **Vineeta Bassi** of Thapar University for giving me an opportunity to work under him on this challenging and burning topic and providing me ample guidance and support through the course of this research. I also thankful to them for inspiring my interests in bigdata time series forecasting. I am grateful for their guidance, encouragement and support throughout my M.Tech research work at Thapar University and also in my life. This thesis comes from their helpful discussions, supervision and meticulous attention to details.

I would like to thank **Dr. S.S. Bhatia**, Professor and Dean of Academic Affairs, Thapar University, Patiala, for giving provisions of the entire required infrastructure such as computer labs, library facilities, immensely useful for learners to equip themselves with the latest in the field.

I take this opportunity to express my appreciations towards the **Dr. Deepak Garg** for his kind help and cooperation, and other faculty members for their constructive suggestions through out the research. I am also thankful to all my respected teachers in the Department and all my friends, for their direct or indirect help, inspiration and motivation. I am grateful to all the technical and non-technical members of the Department for their support.

I want to express my greatest gratitude to my dear mother and father, my loving sisters for their endless love, constant support, encouragement and patients throughout my research without which I would not have reached this position.

Last but not least, I would like to thank almighty for everything.

Date: 3 July 2015 — Anuj Samariya

Thapar University, Patiala — (Anuj Samariya)

Abstract

In the modern competitive world, government and business organizations have to make the right decision in time depending on the information at hand. As large amounts of historical data are readily available, the need of performing accurate forecasting of future behavior becomes crucial to arrive at good decisions. Therefore, demand for a definition of robust and efficient forecasting techniques is increasing day by day. A successful time series forecasting depends on an appropriate model fitting.

Time series data are highly non-stationary and uncertain in nature. Therefore, forecasting of time series using statistical or mathematical techniques is extremely difficult. The scientific community has been attracted by soft computing (SC) techniques in recent years to overcome these limitations. The SC is an amalgamation of different methodologies, such as fuzzy sets, neural computing, rough sets, evolutionary computing and probabilistic computing, to solve real world problems. The present work is a comprehensive examination of designing models for time series forecasting based on SC techniques, especially fuzzy time series (FTS).

In this thesis, we provide in-depth study of various issues and problems associated with the FTS modeling approaches in time series forecasting. Apart from exhaustive literature survey on applications of FTS in time series forecasting, we provide improved methods for forecasting based on the FTS. Our main contributions are given below:

- ⋆ A new FTS based forecasting model is proposed that can deal with one-factor time series data set very effectively. This model deals with two major issues, *viz.*, determination of the effective length of intervals, and defuzzification operation.
- ⋆ Various theorems are developed based on Type-1 and Type-2 fuzzy sets concept.
- ⋆ Another model is proposed based on Type-2 fuzzy sets concept in this thesis. Comparative studies show that forecasting accuracy of the Type-2 fuzzy set is far better than Type-1 fuzzy set.

Experimental results on real time data sets establish the validity of the proposed models.
Keywords: *Soft computing (SC), Fuzzy set, Fuzzy time series (FTS), Interval, Fuzzy logical relationship (FLR),Discretization, Defuzzification.*

Contents

List of Figures

List of Tables

List of Algorithms

Glossary of Abbreviations/Symbols

List of abbreviations used throughout the thesis:

- A
  - ⋆ ANN: Artificial Neural Network
- F
  - ⋆ FTS: Fuzzy Time Series
  - ⋆ FLR: Fuzzy Logical Relationship
  - ⋆ FLRG: Fuzzy Logical Relationship Group
  - ⋆ FRNA: Finding Root Node Algorithm
  - ⋆ FL: Fuzzy Logic
- N
  - ⋆ NBTSDCT: Node Based Time Series Data Clustering Technique
  - ⋆ NIA: Node Insertion Algorithm
  - ⋆ NCA: Node Clustering Algorithm
  - ⋆ NYSE: New York Stock Exchange
- R
  - ⋆ RBILFA: Rank Based Interval Length Finding Algorithm
- T
  - ⋆ TCA: Tree Construction Algorithm
- W
  - ⋆ WMS: Weight Mean Square

List of symbols for the statistical parameters used throughout the thesis:

- ⋆ U: Theil's Statistic
- ⋆ SD: Standard Deviation
- ⋆ AFER: Average Forecasting Error Rate
- ⋆ RMSE: Root Mean Square Error

1

Introduction

Overview of the chapter: An introduction to fuzzy set is given in Section 1.1. Various fuzzy set operations are discussed in Section 1.2. Properties of fuzzy set are presented in Section 1.3

KEYWORDS: *Fuzzy Sets, Crisp Sets.*

1.1 Introduction

The idea of fuzzy set was first proposed by Dr. Lotfi Zadeh of the University of California at Berkeley in the 1960s. He said that several sets in the universe that close around us are characterize by a non-distinct boundary [1]. In fuzzy sets, there are not only two alternatives, but a full continuity of truth values are available for sets. Consider a set A has the truth value 0.7 and its complement A' has the truth value 0.3 *i.e.*, (1-0.7). Fuzzy set includes 0 and 1 are the acute cases of truth, but also have the other states of truth in between, for example, Milk not sweet, little sweet, sweet and too sweeter can be written as: (0/not sweet, 0.2/little sweet, 0.4/sweet, 1/too sweeter).

In real world, we use fuzzy knowledge, knowledge that is ambiguous, probabilistic, imprecise, uncertain or inexact in nature. There is fuzzy information involved in human thinking and reasoning like young, tall, good, high, cold:

- ⋆ The term young is not defined by single quantitative term.
- ⋆ Some people consider 17 age is young, but, some consider 32 is young.

However, many questions whose information are incomplete, our systems are not able to answer these types of questions. The reason behind this is, mostly systems design are based upon classical set theory and two valued logic (0, 1), which unable to cope with unreliable and incomplete information and also unable to give expert opinions. We want our systems also be able to cope with unreliable and incomplete information and give expert opinions. Fuzzy sets have been able, to provide solutions to many real world problems.

1.1.1 Crisp Sets

A crisp set is specified in such a way that all the human being in a given world can be divided into two sets: one who belong to the set, and the other who do not belong to the set. Mathematically, we give the following definition.

If U is the universe, then the set of elements in U having property P (the property is such that each element of the universe either has the property or does not have the property) is denoted by B, and can be written as:

B = x : x $\in$ U and x has property P. If A and B are two sets of the universe U, then A is said to be a subset of (or contained in) B, denoted by A $\subset$ B or B $\supset$ A, if and only if x $\in$ A => x $\in$ B.

Explanation: All crisp sets can be a fuzzy sets. Let universe of discourse U can be defined as: U = 1, 2, 3, 4, 5. Let crisp set, A = 1, 3, 4. Let fuzzy set, B = 1/1, 0/2, 1/3, 1/4, 0/5. But all fuzzy sets can't be Crisp Set. Let a fuzzy set B = 0.2/1, 0.5/2, 0.7/3, 1/4.
Note: The degree used in fuzzy set B is different and we can't use it in crisp set.

1.1.2 Fuzzy Sets

Fuzzy set theory is an extended version of classical set theory where elements contain degree of memberships. It is a set with a continuity of degree of membership. The members of this set have the degree of membership ranging between 0 and 1.

In mathematics a set, by definition, is a collection of things that belong to some definition. Any item either belongs to that set or does not belong to that set. Let us look at simple example, the set of tall men. We shall say that people taller than or equal to 6 feet are tall. This set can be represented graphically in Fig. 1.1. The Fig. 1.1 describes

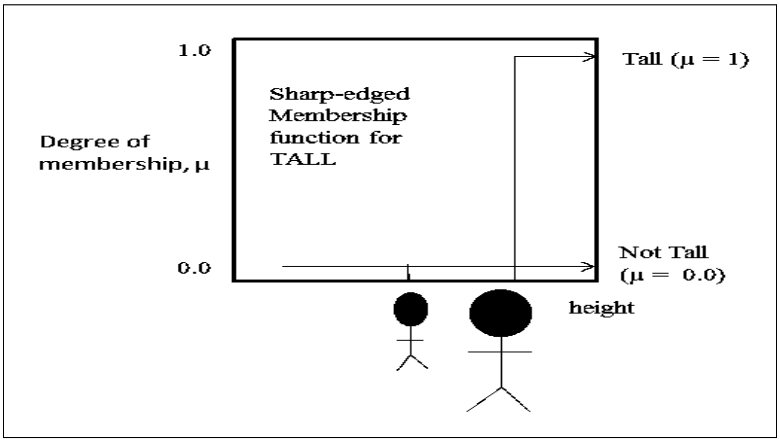

Figure 1.1: Classical Set theory representation.

the membership of the "tall" set, you are either within it or not. The sharp edge describing binary operations and mathematics nicely, but it doesn't formulate the real world problems nicely. There is no difference between one who is 6'3" and other who is 7'3", they are both simply tall. Clearly there is a much difference between the two heights. The other side of this, we can easily seen that there is only one inch difference between the height of 5'9" and 6'0" man, however the crisp set membership function says that one is tall and the other is not tall. Fuzzy set approach provides a much better representation of the tallness of a person. The set is defined by a continuously inclining function which is shown in Fig 1.2. The membership function defines the fuzzy set for the possible values underneath of it on

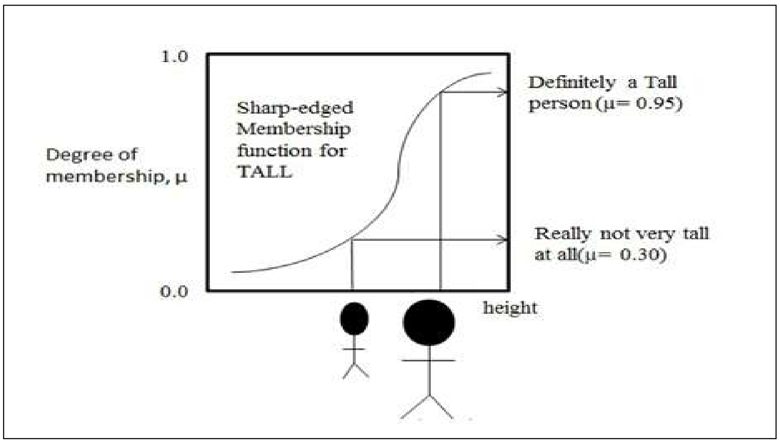

Figure 1.2: Fuzzy Set Theory Representation.

the horizontal axis. The vertical axis, on a scale of 0 to 1, provides the membership value of the height in the fuzzy set. So for the two people shown above the first person has a membership of 0.3 and so is not very tall. The second person has a membership of 0.95 and so he is definitely tall. He does not, however, belong to the set of tall men in the way that bivalent sets work, he has a high degree of membership in the fuzzy set of tall men.

1.2 Fuzzy Set Operations

The important operations of fuzzy set are:

i) **Complement:** Creates the complement of the set. The operation shown in Fig. 1.3 $A' = \mu_{A'}(x) = 1 - \mu_A(x), \forall x \in X$

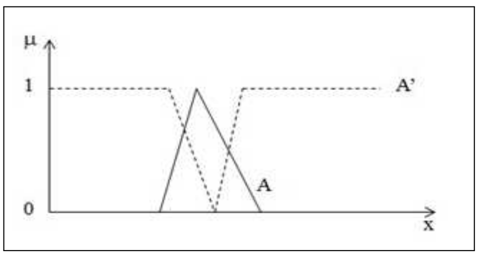

Figure 1.3: A and A'.

Example:
AGE1(x) = (0.75/James, 0.4/Kate, 0.6/Harry, 1/Peter)
AGE1(x)' = (0.25/James, 0.6/Kate, 0.4/Harry, 0/Peter)

ii) **Union:** Creates the union of the (two) sets operated. The operation shown in Fig. 1.4
$\mu_{A\cup B} = max\left[\mu_A\left(x\right), \mu_B\left(x\right)\right], \forall x \in X$

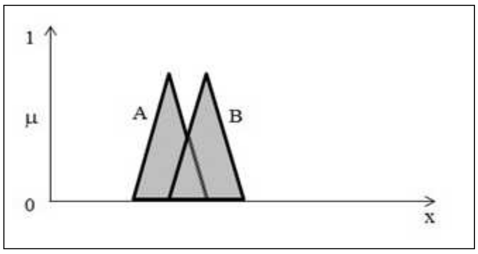

Figure 1.4: A ∪ B
.

Example:
AGE1(x) = (0.75/James, 0.4/Kate, 0.6/Harry, 1/Peter)
AGE2(x) = (0.85/James, 0.6/Kate, 0/Harry, 0.8/Peter)
(AGE1 ∪ AGE2)(x) = max [AGE1(x), AGE2(x)]
(AGE1 ∪ AGE2)(x) = (0.85/James, 0.6/Kate, 0.6/Harry, 1/Peter)

iii) **Intersection:** Creates the intersection of the (two) sets operated. The operation shown in Fig. 1.5
$\mu_{A\cap B} = max\left[\mu_A\left(x\right), \mu_B\left(x\right)\right], \forall x \in X$

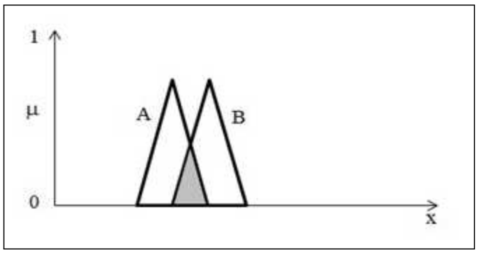

Figure 1.5: A ∩ B.

Example:
AGE1(x) = (0.75/James, 0.4/Kate, 0.6/Harry, 1/Peter)
AGE2(x) = (0.85/James, 0.6/Kate, 0/Harry, 0.8/Peter)
(AGE1 ∩ AGE2)(x) = min [AGE1(x), AGE2(x)]
(AGE1 ∩ AGE2)(x) = (0.75/James, 0.4/Kate, 0/Harry, 0.8/Peter)

1.3 Properties of Crisp and Fuzzy Sets

Property 1.3.1. A ∪ A' ≠ X

Proof: Suppose we take an example of river and measure its quality of water in different seasons. For example, in the summer when the flows are lowest, the quality of water can be the highest. River quality of water in different seasons are shown in the below example. **Example:** Let X is a set of Universe in which X = (winter, spring, summer, fall). Now, assign the membership values to set A as: A = (0.15/winter + 0.33/spring + 0.52/summer + 0.25/fall). The complement of the set A can be defined as: A' = 1 - μ_A(x), where μ_A(x) = degree of membership of x in Set A. Hence, the complement of set A is: A' = (0.85/winter + 0.67/spring + 0.48/summer + 0.75/fall). This operation is shown in Fig. 1.6. Take the union of A and A', which can be defined as:

$$\mu_{A\cup B}(x) = \mu_A(x) \cup \mu_B(x) = max(\mu_A(x), \mu_B(x)).$$

Hence, A ∪ A' = (0.85/winter + 0.67/spring + 0.52/summer + 0.75/fall). But this is not equals to X, because in X all the members have degree 1. But in set A ∪ A', the degree of membership of each element is greater than or equal to 0.5, which is shown in Fig. 1.6. Hence we can say that fuzzy set A and union of its complement can't cover the whole universe set X, *i.e.*, A ∪ A' $\neq$ X.

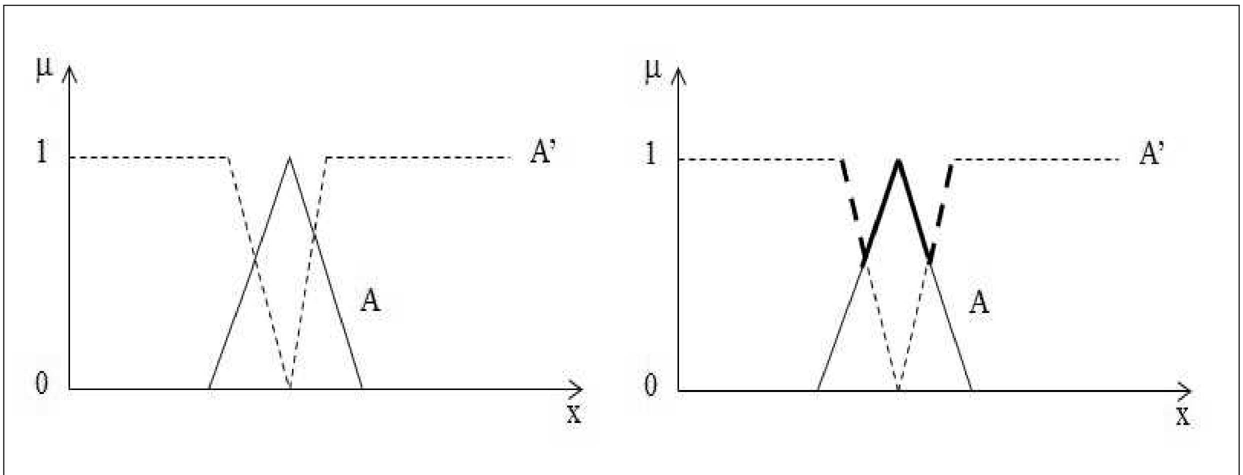

Figure 1.6: A and A' with A ∪ A'.

Property 1.3.2. A ∩ A' $\neq \phi$

Proof: Suppose we take an example of river and measure its quality of water in different seasons. For example, in the summer when the flows are lowest, the quality of water can be the highest. River quality of water in different seasons are shown in the below example. **Example:** Let X is a set of Universe in which X = winter, spring, summer, fall. Now, assign the membership values to set A as: A = 0.15/winter + 0.33/spring + 0.52/summer + 0.25/fall. The complement of the set A can be defined as: A' = 1 - μ_A(x), where μ_A(x) =

degree of membership of x in Set A. Hence, the complement of set A is: A' = (0.85/winter + 0.67/spring + 0.48/summer + 0.75/fall). This operation is shown in Fig. 1.7. Take the intersection of A and A', which can be defined as:

$$\mu_{A \cap B}(x) = \mu_A(x) \cap \mu_B(x) = min(\mu_A(x), \mu_B(x)).$$

Hence, A ∩ A' = (0.15/winter + 0.33/spring + 0.48/summer + 0.25/fall). But this is not equals to Φ. In set A ∩ A', the degree of membership of each element is less than or equal to 0.5, which is shown in Fig. 1.7. Hence we can say that fuzzy set A and intersection of its complement can't be null or ϕ, *i.e.*, A ∩ A' $\neq \phi$.

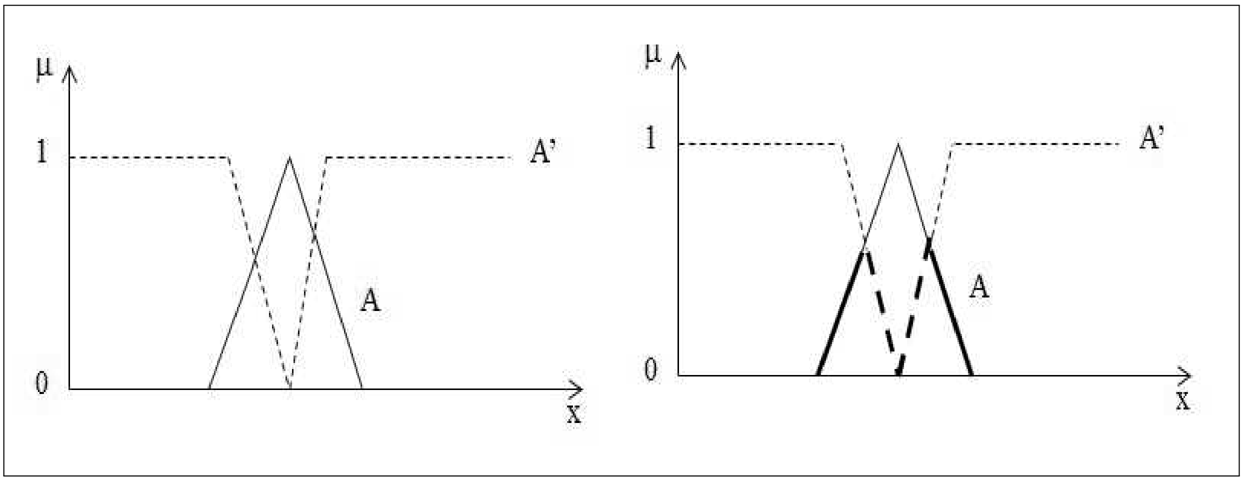

Figure 1.7: A and A' with $A \cap A'$

2

Graph Based Clustering and Improved Interval Lengths Techniques

Overview of the chapter: An introduction to Fuzzy Forecasting is given in Section 2.1 followed by some essential definitions associated with fuzzy sets and its extended application in time series forecasting in Section 2.2. In Section 2.3, a new time series data clustering technique is introduced. The architecture of the proposed FTS model followed by it's detail explanation is discussed in Section 2.4 and Section 2.5, respectively. An another algorithm for finding the best interval lengths is presented in Section 2.6. Experimental set up and results are discussed in Section 2.7. Time complexity of the proposed model is discussed in Section 2.8. In Section 2.9, we provide discussion. In section 2.10, we provide appendix where the formula's are described.
KEYWORDS: *FTS, FLR, FLRs, FLRG.*

2.1 Fuzzy Forecasting: An Introduction

The prediction of future events of time series has always influence people from the ancient times. However, it may be impossible to predict these events with full accuracy, but, one can improved their accuracy and forecast processing speed. To solve this problem, Song and Chissom [2] proposed a model in 1993 based on ambiguity and estimated knowledge encorporated in the data of time series. Firstly, they gave a model by the help of fuzzy sets which represent or take care of all these ambiguity, and called this concept as "Fuzzy Time Series (FTS)".

In 1996, Chen [3] developed FTS model based on first-order fuzzy logical relationships (FLRs), and obtained the forecasted results with simplified arithmetic operations rather than complicated max-min composition operations. Chen's [3] forecasting results were far better than the models proposed by Song and Chissom [2, 4, 5]. Recently, many studies provided some improvements in Chen's [3] model in terms of finding effective lengths of intervals [6–10], fuzzification of time series data set [11], establishment of FLRs [12], and defuzzification [13].

To improve the forecasting accuracy, just now various FTS models are proposed by many researchers. For example, Chen et al. [14] introduced a new FTS model for stock price forecasting by using the theory of the fibonacci sequence. This model is based on the framework of the conventional FTS models, whose forecasting accuracy is outperformed than these models [4, 15]. Recently, researchers in these articles [16–20] introduced computational methods for forecasting by which high order FLR's are blown away the fault of first-order FTS models [3, 21]. To minimize the time of complicated computations of fuzzy relational matrices or to find the steady state of fuzzy relational matrix, Singh [22] proposed a new method in FTS modeling approach. Li and Cheng [23] introduced a new fuzzy deterministic model to solve three major issues, *viz.*, to restraint the ambiguity in forecasting, to divide the intervals adequately, and to achieve the forecasted accuracy with various interval lengths. For these purpose, they designed an optimized FTS forecasting procedure by which the forecasted data is treated as a trapezoidal fuzzy number rather than a single-point data. The proposed model gives better forecasting results than the conventional FTS models [3, 11, 24]. In [25], Singh and Borah proposed two-factors high-order FTS model to resolve the problem associated with determination of lengths of intervals and data defuzzification. The proposed model is based on the hybridization of artificial neural network (ANN) with FTS. The comparative analyzes signify that this model exhibits higher accuracy than those of existing two-factors models [26–31]. In [32], Singh and Borah presented a new model based on hybridization of FTS theory with ANN. In this model, an ANN based architecture is incorporated to defuzzify the fuzzified time series values and to obtain the results. Chen [33] presented a novel high-order FTS model to solve nonlinear problem of time series data set. In this model, entropy-based clustering technique and ANN based architecture are employed to overcome the problems of data partition and representation of FLRs, respectively.

In this thesis, we present a new method to forecast the time series data set based on the FTS modeling approach. In this approach, initially, we cluster the historical data into different length of intervals by applying a graph based clustering algorithm. Then, we define various fuzzy sets based on these evolved intervals. Then, the historical data are fuzzified into fuzzy sets. Based on these fuzzified values, we derive the FLRs. Then, we obtain fuzzy logical relationship groups (FLRGs) from the FLRs. Later, all these FLRGs are used to obtain the forecasting results based on the centroid defuzzification method. From the proposed clustering algorithm, numerous intervals are generated with different lengths of intervals. Huarng [34] shows that how different lengths of intervals affect the forecasting results. Motivated by this critical issue, we propose a new algorithm for finding the best interval lengths in this work. In this model, the main motive of this algorithm is to improve the accuracy of the forecasting by managing the the corresponding degree of membership and simultaneously adjusting the length of every interval in the universe of discourse.

2.2 Fuzzy Sets and Fuzzy Time Series

In 1965, Zadeh [1] introduced fuzzy sets theory involving continuous set membership for processing data in presence of uncertainty. He also presented fuzzy arithmetic theory and its application [35].

Definition 2.2.1. (Fuzzy Set) [1]. A fuzzy set is a class with varying degrees of membership in the set. Let U be the universe of discourse, which is discrete and finite, then fuzzy set A can be defined as follows :

$$A = \{\mu_{A(x_1)}/x_1 + \mu_{A(x_2)}/x_2 + \ldots\} = \Sigma_i \mu_A(x_i)/x_i \tag{2.2.1}$$

where μ_A is the membership function of A, μ_A: $U \to [0,1]$, and $\mu_{A(x_i)}$ is the degree of membership of the element x_i in the fuzzy set A. Here, the symbol "+" indicates the operation of union and the symbol "/" indicates the separator rather than the commonly used summation and division in algebra, respectively.

When U is continuous and infinite, then the fuzzy set A of U can be defined as:

$$A = \{\int \mu_{A(x_i)}/x_i\}, \forall x_i \in U \tag{2.2.2}$$

where the integral sign stands for the union of the fuzzy singletons, $\mu_{A(x_i)}/x_i$.

Definition 2.2.2. (Fuzzy Time Series) [2, 4, 5]. Let $Y(t)(t = 0, 1, 2, \ldots)$ be a subset of Z and the universe of discourse U on which fuzzy sets $\mu_i(t)(i = 1, 2, \ldots)$ are defined and let $F(t)$ be a collection of $\mu_i(t)(i = 1, 2, \ldots)$. Then, $F(t)$ is called a fuzzy time series on $Y(t)(t = 0, 1, 2, \ldots)$.

Definition 2.2.3. (Fuzzy logical relationship) [2, 3, 5]. Assume that $F(t-1) = A_i$ and $F(t) = A_j$. The relationship between $F(t)$ and $F(t-1)$ is referred as a fuzzy logical relationship (FLR), which can be represented as:

$$A_i \to A_j, \tag{2.2.3}$$

where A_i and A_j refer to the left-hand side and right-hand side of the FLR, respectively.

Definition 2.2.4. (Fuzzy logical relationship group) [2, 3, 5]. Assume the following

FLRs:

$$\begin{aligned} &A_i \to A_{k1}, \\ &A_i \to A_{k2}, \\ &\quad \dots \\ &A_i \to A_{km} \end{aligned}$$

Chen [3] suggested that FLRs having same fuzzy sets on left-hand side can be grouped into a fuzzy logical relationship group (FLRG). So, based on Chen's model [3], these FLRs can be grouped into the FLRG as:

$$A_i \to A_{k1}, A_{k2}, \dots, A_{km}.$$

Definition 2.2.5. (High-order FLR) [36]. Assume that $F(t)$ is caused by $F(t-1), F(t-2), \dots,$ and $F(t-n)$ $(n > 0)$, then high-order FLR can be expressed as:

$$F(t-n), \dots, F(t-2), F(t-1) \to F(t) \tag{2.2.4}$$

2.3 NBTSDCT: A New Time Series Data Clustering Technique

Cluster analysis plays an important role in a wide variety of research domains, such as social sciences, bio-informatics, psychology, pattern recognition, image processing, machine learning, and data mining. Cluster analysis partitions data into various groups (user-specified and non-specified) which discover meaningful and useful information [37]. We people have the skilled to classify or clustering the objects into groups based on their similarities and dissimilarities. But in case of vast amounts of data, human's analytical skill is failure to determine classes or clusters. In such case, we use the machine learning techniques that automatically finding classes.

Clustering of data can be regarded as a form of groups having certain class (cluster) labels [38]. This determination of class labels may be *supervised* or *unsupervised*. If the classification is done in the absence of any knowledge class labels, then it is referred as *unsupervised* clustering. When the class labels are taken into account, then such type of classification is termed as *supervised* clustering. In literature, various *supervised* and *unsupervised* clustering algorithms are presented [39].

A complete sets of clusters are commonly termed as a **Clustering**. We can distinguish various types of clusterings into five categories: partitioning method, hierarchical method, density based method, grid-based method, and model-based method. A brief description of each category of method is given below:

a) Partitioning method: A partitional clustering divides the set of data objects into distinct non-overlapping clusters. For example, K-means [40], K-medoids [41].

b) Hierarchical method: In this method, we allow each cluster have subclusters, which can be organized in the form of tree. For example, CURE [42], BIRCH [43].

c) Density based method: This method allows the clusters to grow as long as the density (number of objects or data points) in the "neighborhood" exceeds some threshold. For example, DBSCAN [44], OPTICS [45].

d) Grid-based method: This method allows the object space to form finite number of cells (a grid structure) where all the operations associated with clustering are performed. For example, STING [46].

e) Model-based method: This method assumes a model for each of the clusters and attempt to best fit the data to the assumed model. For example, ART [47], SOFM [48].

Graph-based method is also used to represent the data objects into clusters. In this method, data objects are represented as a graph, where each node is an object and objects are associated with links. Here, a cluster is made when a group of objects is linked to one another, but have no connectivity to objects outside the group. An example of such type of approach for clustering is CLIQUE [49].

The Node Based Time Series Data Clustering Technique (NBTSDCT) is also a class of graph-based method to represent the time series data set into clusters. The technique automatically generates the clusters without any supervision. The NBTSDCT represents the data set in the form of a tree by which the representation and visualization of data set is very easy. The NBTSDCT divides into four parts, *viz.*, (a) Finding Root Node Algorithm (FRNA) inputs the data set and find the root node; (b) Tree Construction Algorithm (TCA) inputs the data set, root node and output the tree; (c) Node Insertion Algorithm (NIA) inputs one element of data set and root node, and set the elements in the tree at their proper position; and (d) Node Clustering Algorithm (NCA) inputs the tree which is generated by the TCA, and makes logical clustering of the nodes.

2.3.1 NBTSDCT: An Overview

In this subsection, we present a new time series data clustering technique entitled as "NBTSDCT", which comprises of four algorithms, *viz.*, Finding Root Node Algorithm (FRNA), Tree Construction Algorithm (TCA), Node Insertion Algorithm (NIA), and Node Clustering Algorithm (NCA). To demonstrate the applicability of the technique, the university enrollments data set of Alabama [4], shown in Table 2.1, is employed. A few significant steps of the NBTSDCT are explained in Example 1.

Example 1:

Algorithm 1: Finding Root Node Algorithm (FRNA)

1: Input: $S(X_1, X_2, \cdots, X_N)$

2: Calculate range (R_g) as:

$$R_g = MAX_{value} - MIN_{value} \tag{2.3.1}$$

3: Calculate standard deviation (SD) of the S.

4: Calculate width (W) as:

$$W = \frac{R_g}{SD \times N} \tag{2.3.2}$$

5: Compute universe of discourse (U) of the S as:

$$U = [L_{bound}, U_{bound}], \tag{2.3.3}$$

where $L_{bound} = MIN_{value} - W$ and $U_{bound} = MAX_{value} + W$.

6: Calculate mid-point (X_{mid}) as:

$$X_{mid} = \frac{L_{bound} + U_{bound}}{2} \tag{2.3.4}$$

7: Assign the X_{mid} as root node (T_{root}):

$$T_{root} = X_{mid} \tag{2.3.5}$$

Algorithm 2: Node Insertion Algorithm (NIA)

```
Input: T_root, X_i
  if (X_i < T_root) then
    if T_root.LEFT ≠ NULL then
      Call: NIA(T_root.LEFT, X_i)
    else
      T_root.LEFT = NULL
    end if
  else if (X_i > T_root) then
    if T_root.RIGHT ≠ NULL then
      Call: NIA(T_root.RIGHT, X_i)
    else
      T_root.RIGHT = NULL
    end if
  end if
```

Algorithm 3: Tree Construction Algorithm (TCA)

```
Input: T_root, S
for i = 1 to N do
  | NIA(T_root, X_i)
end
```

Algorithm 4: Node Clustering Algorithm (NCA)

Input: T_{root}
if ($T_{root} = NULL$) **then**
  "No Tree Found"
  return
else if $T_{root}.RIGHT \neq NULL$ && $T_{root}.LEFT \neq NULL$ **then**
  if T_{root} is not presented in Cluster **then**
    $minDiffnode = makeDiff(T_{root}, T_{root}.RIGHT, T_{root}.LEFT)$
    $makeCluster(T_{root}, minDiffnode)$
    if $minDiffnode = T_{root}.RIGHT$ **then**
      if $((T_{root}.RIGHT).LEFT \neq NULL)$ **then**
        add $(T_{root}.RIGHT).LEFT$ child in the Cluster
      end if
      if $((T_{root}.RIGHT).RIGHT \neq NULL)$ **then**
        Call: NCA($(T_{root}.RIGHT).RIGHT$)
      end if
      Call: NCA($T_{root}.LEFT$)
    else
      if $((T_{root}.LEFT).LEFT \neq NULL)$ **then**
        Call: NCA($(T_{root}.LEFT).LEFT$)
      end if
      if $((T_{root}.LEFT).RIGHT \neq NULL)$ **then**
        add $((T_{root}.LEFT).RIGHT)$ child in the Cluster
      end if
      Call: NCA($T_{root}.RIGHT$)
    end if
  end if
else if $T_{root}.RIGHT \neq NULL$ && $T_{root}.LEFT == NULL$ **then**
  if T_{root} is not presented in Cluster **then**
    $makeCluster(T_{root}, T_{root}.RIGHT)$
    if $((T_{root}.RIGHT).LEFT \neq NULL)$ **then**
      add $(T_{root}.RIGHT).LEFT$ child in the Cluster
    end if
    if $((T_{root}.RIGHT).RIGHT \neq NULL)$ **then**
      Call: NCA($(T_{root}.RIGHT).RIGHT$)
    end if
  end if
else if $T_{root}.RIGHT == NULL$ && $T_{root}.LEFT \neq NULL$ **then**
  if T_{root} is not presented in Cluster **then**
    $makeCluster(T_{root}, T_{root}.LEFT)$
    if $((T_{root}.LEFT).LEFT \neq NULL)$ **then**
      Call: NCA($(T_{root}.LEFT).LEFT$)
    end if
    if $((T_{root}.LEFT).RIGHT \neq NULL)$ **then**
      add $((T_{root}.LEFT).RIGHT)$ child in the Cluster
    end if
  end if
else
  if T_{root} is not presented in the Cluster **then**
    $makeCluster(T_{root})$
  end if
  return
end if

Table 2.1: Data set of the enrollments of the University of Alabama.

Year	Actual Enrollments	Year	Actual Enrollments
1971	13055	1982	15433
1972	13563	1983	15497
1973	13867	1984	15145
1974	14696	1985	15163
1975	15460	1986	15984
1976	15311	1987	16859
1977	15603	1988	18150
1978	15861	1989	18970
1979	16807	1990	19328
1980	16919	1991	19337
1981	16388	1992	18876

Step 1. Input the university enrollments data set of Alabama as:
$S(13055, 13563, 13867, 14696, 15460, \ldots, 19328, 19337, 18876)$.

Step 2. From Eq. 2.3.1, value of R_g is:

$R_g = 19337 - 13055 = 6282.$

Step 3. The value of SD is:
$SD = 1774.72.$

Step 4. From Eq. 2.3.2, the value of W is:
$W = \frac{6282}{1774.72 \times 22} = 0.16.$

Step 5. From Eq. 2.3.3, the U is:
$U = [13054.84, 19337.16].$

Step 6. From Eq. 2.3.4, the X_{mid} is:
$X_{mid} = \frac{13054.84+19337.16}{2} = 16196.$

Step 7. Thus, the T_{root} in Eq. 2.3.5 is 16196.

Step 8. A tree is designed based on the input data set S and T_{root}. For this purpose, we have employed two algorithms, *viz.*, Algorithm 2 and Algorithm 3. The output of these two algorithms are shown in Fig. 2.1.

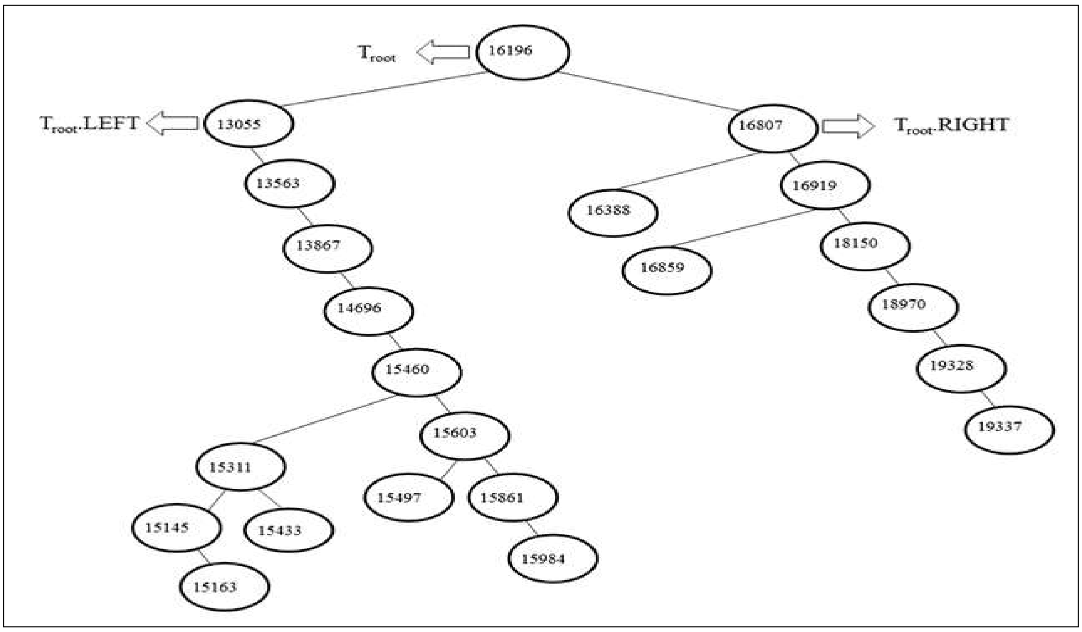

Figure 2.1: A tree of the input data set. In this tree, $T_{root} = 16196$.

Step 9. Make the clusters from the tree shown in Fig. 2.1. For this purpose, Algorithm 4 is used. Output of this algorithm is represented by Fig. 2.3. A brief explanation of this algorithm is given below:

Step 9.1 A symbolic representation of a tree formed by Algorithm 2 and Algorithm 3 is shown in Fig. 2.2. In this tree, initially we check that T_{root} exists or not. If

T_{root} exists, then check that T_{root} has left ($T_{root}.LEFT$) and right ($T_{root}.RIGHT$) children both. If both children exist, then compute the difference between values of the T_{root} and $T_{root}.RIGHT$, and T_{root} and $T_{root}.LEFT$. Make a cluster with corresponding child (either $T_{root}.LEFT$ or $T_{root}.RIGHT$) and T_{root}, which have the minimum difference. In Fig. 2.3, $T_{root} = 16196$, it has both $T_{root}.LEFT = 13055$ and $T_{root}.RIGHT = 16807$. Now, the $minDiffnode = T_{root}.RIGHT - T_{root}$ has the minimum difference, which is 611. Hence, make the cluster with $T_{root}.RIGHT$ and T_{root}, and entitle this cluster as C_1 (see Fig. 2.3).

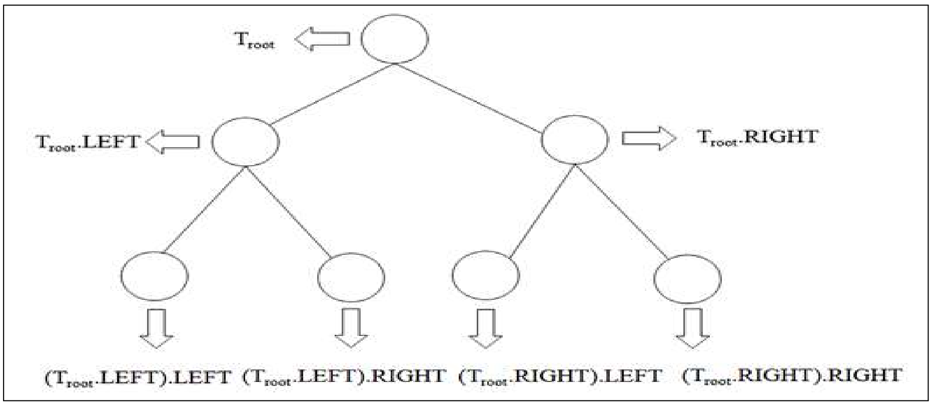

Figure 2.2: A symbolic representation of a tree formed by Algorithm 2 and Algorithm 3.

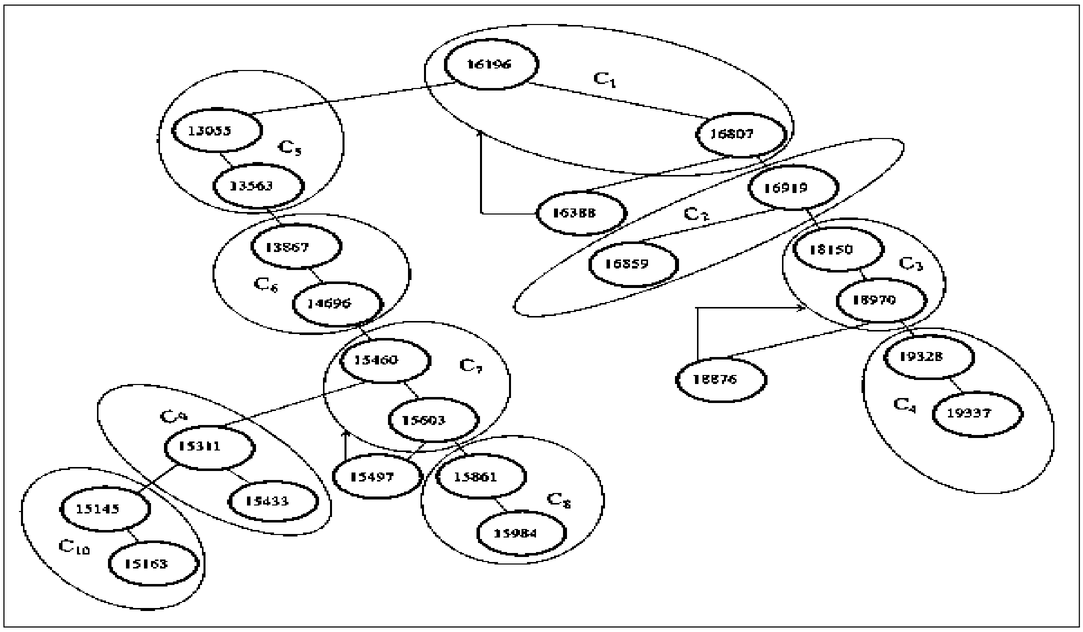

Figure 2.3: Clusters of the input data set.

Step 9.2 Now, according to Algorithm 4, $T_{root}.RIGHT$ has $(T_{root}.RIGHT).LEFT$ child with value 16388, so we add this value to the cluster C_1. Note that if the $minDiffnode = T_{root} - T_{root}.LEFT$ has the minimum difference, then we will make the cluster with $T_{root}.LEFT$ and T_{root}. Again, according to Algorithm 4, if $T_{root}.LEFT$ has $(T_{root}.LEFT).RIGHT$ with certain value, then we add this value to this cluster.

Step 9.3 If only one child (either $T_{root}.LEFT$ or $T_{root}.RIGHT$) exist for each T_{root}, then make the cluster with either T_{root} and $T_{root}.LEFT$ or T_{root} and $T_{root}.RIGHT$.

Step 9.4 Repeat Steps 9.1 - 9.3, until all the nodes of the tree are processed.

All the clusters which are evolved from the above technique is shown in Table 2.2. In this table, lower and upper bounds of the intervals are formed from the minimum and maximum values of the corresponding clusters, respectively.

Table 2.2: Clusters, their corresponding elements, intervals and mid-points.

Cluster	Corresponding Element	Initial Interval	Mid-Point
C_1	$(16196, 16807, 16388)$	$a_1(0) = [16196, 16807]$	16292
C_2	$(16919, 16859)$	$a_2(0) = [16859, 16919]$	16889
C_3	$(18150, 18970, 18876)$	$a_3(0) = [18150, 18970]$	18560
C_4	$(19328, 19337)$	$a_4(0) = [19328, 19337]$	19332.5
C_5	$(13055, 13563)$	$a_5(0) = [13055, 13563]$	13309
C_6	$(13867, 14696)$	$a_6(0) = [13867, 14696]$	14281.5
C_7	$(15460, 15603, 15497)$	$a_7(0) = [15460, 15603]$	15531.5
C_8	$(15861, 15984)$	$a_8(0) = [15861, 15984]$	15922.5
C_9	$(15311, 15433)$	$a_9(0) = [15311, 15433]$	15372
C_{10}	$(15145, 15163)$	$a_{10}(0) = [15145, 15163]$	15154

2.4 Architecture of the FTS Model

Most of the existing FTS models as discussed earlier use the following six common steps to deal with the forecasting problems [3]:

Step 1. Partition the universe of discourse into intervals.

Step 2. Define linguistic terms for each of the interval.

Step 3. Fuzzify the time series data set.

Step 4. Establish the FLRs based on Definition 2.2.3.

Step 5. Construct the FLRGs based on Definition 2.2.4.

Step 6. Defuzzify and compute the forecasted values.

In this chapter, an improved FTS forecasting model is proposed, which is designed for handling the research issues discussed in Section 2.1. Therefore, above steps are modified, which is represented in Algorithm 5.

Algorithm 5: Proposed FTS Model

Step 1: Select the time series data set.

Step 2: Apply the "NBTSDCT" to cluster the time series data set, and determine the corresponding interval.

Step 3: Define linguistic terms for each of the interval.

Step 4: Fuzzify the time series data set.

Step 5: Establish the FLRs based on Definition 2.2.3.

Step 6: Construct the FLRGs based on Definition 2.2.4.

Step 7: Defuzzify the fuzzified time series data set.

2.5 Detail of the Proposed FTS Model

In this section, we propose a new FTS model to forecast time series data set. For explaining each phase of the model, the university enrollments data set of Alabama (Table 2.1) is used. The functionality of each phase of the model is explained in step-wise as follows:

Phase 2.5.1. Apply the "NBTSDCT" to cluster the time series data set, and determine the corresponding interval.

[Explanation] The "NBTSDCT" is used to cluster the time series data of enrollment. From each of the cluster, corresponding interval is generated. Experimental results are depicted in Table 2.2.

Phase 2.5.2. Define linguistic terms for each of the interval. Assume that the historical time series data set is distributed among n initial intervals (*i.e.*, $a_1(0), a_2(0), \ldots,$ and $a_n(0)$). Therefore, define n linguistic variables $A_1, A_2, \ldots, A_n$, which can be represented by fuzzy sets, as shown below:

$$\begin{aligned}
A_1 &= 1/a_1(0) + 0.5/a_2(0) + 0/a_3(0) + \ldots + 0/a_{n-2}(0) + 0/a_{n-1}(0) + 0/a_n(0),\\
A_2 &= 0.5/a_1(0) + 1/a_2(0) + 0.5/a_3(0) + \ldots + 0/a_{n-2}(0) + 0/a_{n-1}(0) + 0/a_n(0),\\
A_3 &= 0/a_1(0) + 0.5/a_2(0) + 1/a_3(0) + \ldots + 0/a_{n-2}(0) + 0/a_{n-1}(0) + 0/a_n(0),\\
&\vdots\\
A_n &= 0/a_n(0) + 0/a_2(0) + 0/a_3(0) + \ldots + 0/a_{n-2}(0) + 0.5/a_{n-1}(0) + 1/a_n(0). \quad (2.5.1)
\end{aligned}$$

Here, maximum degree of membership of fuzzy set A_i occurs at interval $a_i(0)$, and $1 \leq i \leq n$.

[Explanation] We define 10 linguistic variables $A_1, A_2, \ldots, A_{10}$ for the university enrollments data set, because total 10 intervals are generated. All these defined linguistic variables are shown as follow:

$$
\begin{aligned}
A_1 &= 1/a_1(0) + 0.5/a_2(0) + 0/a_3(0) + \ldots + 0/a_{n-2}(0) + 0/a_{n-1}(0) + 0/a_{10}(0),\\
A_2 &= 0.5/a_1(0) + 1/a_2(0) + 0.5/a_3(0) + \ldots + 0/a_{n-2}(0) + 0/a_{n-1}(0) + 0/a_{10}(0),\\
A_3 &= 0/a_1(0) + 0.5/a_2(0) + 1/a_3(0) + \ldots + 0/a_{n-2}(0) + 0/a_{n-1}(0) + 0/a_{10}(0),\\
&\vdots\\
A_{10} &= 0/a_1(0) + 0/a_2(0) + 0/a_3(0) + \ldots + 0/a_{n-2}(0) + 0.5/a_{n-1}(0) + 1/a_{10}(0). \quad (2.5.2)
\end{aligned}
$$

For ease of computation, the degree of membership values of fuzzy set $A_j (j = 1, 2, \ldots, 10)$ are considered as either 0, 0.5 or 1, and $1 \leq j \leq 10$. In Eq. 2.5.2, for example, A_1 represents a linguistic value, which denotes a fuzzy set $= \{a_1(0), a_2(0), \ldots, a_{10}(0)\}$. This fuzzy set consists of ten members with different degree of membership values $= \{1, 0.5, 0, \ldots, 0\}$. Similarly, the linguistic value A_2 denotes the fuzzy set $= \{a_1(0), a_2(0), \ldots, a_{10}(0)\}$, which also consists of ten members with different degree of membership values $= \{0.5, 1, 0.5, \ldots, 0\}$. The descriptions of remaining linguistic variables, *viz.*, $A_3, A_4, \ldots, A_{10}$, can be provided in a similar manner.

Table 2.3: Fuzzified enrollments values.

Year	Actual Enrollments	Fuzzified Enrollments	Year	Actual Enrollments	Fuzzified Enrollments
1971	13055	A_5	1982	15433	A_9
1972	13563	A_5	1983	15497	A_7
1973	13867	A_6	1984	15145	A_{10}
1974	14696	A_6	1985	15163	A_{10}
1975	15460	A_7	1986	15984	A_8
1976	15311	A_9	1987	16859	A_2
1977	15603	A_7	1988	18150	A_3
1978	15861	A_8	1989	18970	A_3
1979	16807	A_1	1990	19328	A_4
1980	16919	A_2	1991	19337	A_4
1981	16388	A_1	1992	18876	A_3

Phase 2.5.3. Fuzzify the time series data set. If one day's datum belongs to the interval a_i, then datum is fuzzified into A_i, where $1 \leq i \leq n$.

[Explanation] In Step 2.5.2, each fuzzy set contains 10 intervals, and each interval corresponds to all fuzzy sets with different degree of membership values. For example, interval $a_1(0)$ corresponds to linguistic variables A_1 and A_2 with degree of membership values 1 and 0.5, respectively, and remaining fuzzy sets with degree of membership value 0. Similarly, interval $a_2(0)$ corresponds to linguistic variables A_1, A_2 and A_3 with degree of membership values 0.5, 1, and 0.5, respectively, and remaining fuzzy sets with degree of membership value 0. The descriptions of remaining intervals, *viz.*, $a_3(0), a_4(0), \ldots, a_{10}(0)$, can be provided in a similar manner.

In order to fuzzify the historical time series data, it is essential to obtain the degree of membership value of each observation belonging to each A_j $(j = 1, 2, \ldots, n)$ for each year. If the maximum membership value of one year's observation occurs at interval $a_i(0)$ and $1 \leq i \leq n$, then the fuzzified value for that particular year is considered as A_i. For example, the university enrollments value for the year 1971 is 13055, which belongs to the interval $a_5(0)$. Hence, the university enrollments value is fuzzified into fuzzy set A_5. All these fuzzified enrollments values are shown in Table 2.3.

Phase 2.5.4. Establish the FLR between the fuzzified time series values.

[Explanation] Based on Definition 2.2.3, we can establish FLRs between two consecutive fuzzified enrollments values. For example, in Table 2.4, fuzzified enrollments values for Years 1973 and 1974 are A_6 and A_6, respectively. So, we can establish a FLR between A_6 and A_6 as: $A_6 \rightarrow A_6$. In this way, we have obtained FLRs for the fuzzified enrollments values, which are presented in Table 2.4.

Table 2.4: FLRs of the university enrollments data set.

FLR	FLR
$A_5 \rightarrow A_5$	$A_1 \rightarrow A_9$
$A_5 \rightarrow A_6$	$A_7 \rightarrow A_{10}$
$A_6 \rightarrow A_6$	$A_{10} \rightarrow A_{10}$
$A_6 \rightarrow A_7$	$A_{10} \rightarrow A_8$
$A_7 \rightarrow A_9$	$A_8 \rightarrow A_2$
$A_9 \rightarrow A_7$	$A_2 \rightarrow A_3$
$A_7 \rightarrow A_8$	$A_3 \rightarrow A_3$
$A_8 \rightarrow A_1$	$A_3 \rightarrow A_4$
$A_1 \rightarrow A_2$	$A_4 \rightarrow A_4$
$A_2 \rightarrow A_1$	$A_4 \rightarrow A_3$

Phase 2.5.5. Construct the FLRG.

[Explanation] Based on the same previous state of the FLRs, the FLRs can be grouped into a FLRG. For example, the FLRG "$A_i \rightarrow A_m, A_n$" (*i.e.*, Group i) indicates that there are following FLRs:

$$A_i \rightarrow A_m,$$
$$A_i \rightarrow A_n.$$

In Table 2.4, there are 2 FLRs with the same previous state, $A_5 \rightarrow A_5$ and $A_5 \rightarrow A_6$. These FLRs are used to form the FLRG, $A_5 \rightarrow A_5, A_6$ (*i.e.*, Group 1). All these FLRGs are shown in Table 2.5. In this study, we have discarded the repeated FLRs in the FLRGs.

Table 2.5: FLRGs of the university enrollments data set.

Group	FLRG
Group 1:	$A_5 \rightarrow A_5, A_6$
Group 2:	$A_6 \rightarrow A_6, A_7$
Group 3:	$A_7 \rightarrow A_9, A_8, A_{10}$
Group 4:	$A_9 \rightarrow A_7$
Group 5:	$A_8 \rightarrow A_1, A_2$
Group 6:	$A_1 \rightarrow A_2, A_9$
Group 7:	$A_2 \rightarrow A_1, A_3$
Group 8:	$A_{10} \rightarrow A_{10}, A_8$
Group 9:	$A_3 \rightarrow A_3, A_4$
Group 10:	$A_4 \rightarrow A_4, A_3$

Phase 2.5.6. Defuzzify the fuzzified time series data set.

To defuzzify the fuzzified time series data set, following principles are used. Based on the application of technique, it is categorized as: **Principle 1** and **Principle 2**. The steps involve in **Principle 1** are explained next.

⋆ **Principle 1**: For forecasting year $Y(t)$, the fuzzified value for year $Y(t-1)$ is required, where "t" is the current year which we want to forecast. The **Principle 1** is applicable only if there are more than one fuzzified values available in the current state. The steps under **Principle 1** are explained next.

Step 1 Obtain the fuzzified value for year $Y(t-1)$ as $A_i(i = 1, 2, 3 \ldots, n)$.

Step 2 Obtain the FLRG whose previous state is $A_i(i = 1, 2, 3 \ldots, n)$, and the current state is $A_k, A_s, \ldots, A_n$, *i.e.*, the FLRG is in the form of $A_i \rightarrow A_k, A_s, \ldots, A_n$.

Step 3 Find the intervals where the maximum membership values of the fuzzy sets $A_k, A_s, \ldots, A_n$ occur. Let these intervals be $a_k, a_s, \ldots, a_n$. All these intervals have the corresponding mid-points $C_k, C_s, \ldots, C_n$.

Step 4 Apply following formula to calculate the forecasted value for year, $Y(t)$:

$$Forecast\,(t) \quad = \quad \frac{C_k + C_s + C_n}{n}, \tag{2.5.3}$$

where n is the total number of mid-points to be used.

⋆ **Principle 2**: This rule is applicable if there is only one fuzzified value in the current state. The steps involve in **Principle 2** are explained next.

Step 1 Obtain the fuzzified value for year $Y(t-1)$ as $A_i (i = 0, 1, \ldots, n)$.

Step 2 Find the FLRG whose previous state is A_i, and the current state is A_j, *i.e.*, the FLRG is in the form of $A_i \rightarrow A_j$.

Step 3 Find the interval where the maximum value of the fuzzy set A_j occurs. Let this interval be a_j. This interval a_j have the corresponding mid-point C_j. For forecasting year, $Y(t)$, this mid-point C_j is the forecasted value.

Table 2.6: Actual Enrollments Vs. Forecasted Enrollments.

Year	Actual Enrollments	Forecasted Enrollments
1971	13055	
1972	13563	13795.250
1973	13867	13795.250
1974	14696	14900.750
1975	15460	14900.750
1976	15311	15482.833
1977	15603	15520.000
1978	15861	15482.833
1979	16807	16676.332
1980	16919	16130.500
1981	16388	17564.500
1982	15433	16130.500
1983	15497	15520.000
1984	15145	15482.833
1985	15163	15538.250
1986	15984	15538.250
1987	16859	16676.332
1988	18150	17564.500
1989	18970	18998.918
1990	19328	18998.918
1991	19337	18998.918
1992	18876	18998.918
AFER		2.124%

In this way, we obtain the forecasted values for the university enrollments data set based on the proposed model. To measure the performance of the model, average forecasting error rate (AFER) is used as an evaluation criterion. The AFER can be defined as follows:

$$AFER = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{F_i - A_i}{A_i} \right| \times 100\% \tag{2.5.4}$$

Here, F_i represents the forecasted value at time i, A_i represents the actual value at time i and n represents the total number of days to be forecasted. The AFER value of the forecasted enrollments is presented in Table 2.6.

2.6 Algorithm for Finding the Best Interval Lengths

In this section, we present an algorithm for finding the best interval lengths from the initial intervals. The algorithm is entitled as "Rank Based Interval Lengths Finding Algorithm (RBILFA)", and presented next.

Step 1. Take clusters, $C_1(d_{11}, d_{12}, \ldots, d_{1m})$, $C_2(d_{21}, d_{22}, \ldots, d_{2m})$, ..., $C_n(d_{n1}, d_{n2}, \ldots, d_{nm})$ as input data sequence, where $\{d_{11}, d_{12}, \ldots, d_{1m}\}$, $\{d_{21}, d_{22}, \ldots, d_{2m}\}$, ..., $\{d_{n1}, d_{n2}, \ldots, d_{nm}\}$ are the elements of the clusters $C_1, C_2, \ldots, C_n$, respectively. Here, m represents the total number of elements in each cluster, and n represents the total number of clusters.

Step 2. Calculate the weight mean square (WMS) for the corresponding set of clusters as:

$$WMS = \sqrt{\frac{\sum_{i=1}^{n}(C_i - \rho)}{n}}, \tag{2.6.1}$$

where $\rho = \frac{p}{n}$, and $p = \left[\frac{\{d_{11}+d_{12}+\ldots+d_{1m}\}}{m} + \frac{\{d_{21}+d_{22}+\ldots+d_{2m}\}}{m} + \ldots + \frac{\{d_{n1}+d_{n2}+\ldots+d_{nm}\}}{m}\right]$. Here, $C_i = \left[\frac{\{d_{i1}+d_{i2}+\ldots+d_{im}\}}{m}\right]$.

Step 3. Select the initial intervals as:

$$\begin{aligned} a_1(0) &= [min(C_1), max(C_1)], \\ a_2(0) &= [min(C_2), max(C_2)] \\ &\cdots \\ a_n(0) &= [min(C_n), max(C_n)]. \end{aligned} \tag{2.6.2}$$

Here, $Lower_{bound} = min(C_j)$ and $Upper_{bound} = max(C_j)$, where $j = 1, 2, \ldots, n$.

Step 4. Generate the next set of intervals as:

$$\begin{aligned} a_1(1) &= [min(C_1) - WMS, max(C_1) + WMS], \\ a_2(1) &= [min(C_2) - WMS, max(C_2) + WMS] \\ &\cdots \\ a_n(1) &= [min(C_n) - WMS, max(C_n) + WMS]. \end{aligned}$$

Step 5. Compute the fitness for all set of intervals, and select the best one from Step 3 - Step4.

Step 6. Compute the ranks "$R_{k(upper)}$" and "$R_{k(lower)}$" only for the set of intervals which have the best fitness (obtained from Step 5) as:

$$R_{k(upper)} = \left\lceil \frac{\left|1 - \frac{Upper_{bound_k}}{n(n+1)}\right|}{n} \right\rceil, and$$

$$R_{k(lower)} = \left\lceil \frac{\left|1 - \frac{Lower_{bound_k}}{n(n-1)}\right|}{n} \right\rceil. \tag{2.6.3}$$

where, $k = 1, 2, \ldots, n.$

Step 7. Generate final set of intervals as:

$$a_1(1) = [min(C_1) - (R_{1(lower)} \times 1), max(C_1) + (R_{1(upper)} \times 1)],$$
$$a_2(1) = [min(C_2) - (R_{2(lower)} \times 1), max(C_2) + (R_{2(upper)} \times 1)],$$
$$\ldots$$
$$a_n(1) = [min(C_n) - (R_{n(lower)} \times 1), max(C_n) + (R_{n(upper)} \times 1)].$$

$$a_1(2) = [min(C_1) - (R_{1(lower)} \times 2), max(C_1) + (R_{1(upper)} \times 2)],$$
$$a_2(2) = [min(C_2) - (R_{2(lower)} \times 2), max(C_2) + (R_{2(upper)} \times 2)],$$
$$\ldots$$
$$a_n(2) = [min(C_n) - (R_{n(lower)} \times 2), max(C_n) + (R_{n(upper)} \times 2)].$$

$$\ldots$$

$$a_1(n) = [min(C_1) - (R_{1(lower)} \times n), max(C_1) + (R_{1(upper)} \times n)],$$
$$a_2(n) = [min(C_2) - (R_{2(lower)} \times n), max(C_2) + (R_{2(upper)} \times n)],$$
$$\ldots$$
$$a_n(n) = [min(C_n) - (R_{n(lower)} \times n), max(C_n) + (R_{n(upper)} \times n)]. \tag{2.6.4}$$

Step 8. Compute the fitness for all set of intervals from Step 7, and select the best one.

Step 9. Stop the whole process.

2.6.1 Implementation of the RBILFA

The RBILFA algorithm can be applied to search the best set of intervals. Array representation of selected initial intervals is shown in Fig. 2.4, where interval is positioned as $a_1(0), a_2(0), \ldots, a_n(0)$. Next set of intervals is generated with the help of "WMS", whose array representation is shown in Fig. 2.5. Finally, with the help of ranks "$R_{k(upper)}$" and "$R_{k(lower)}$", final set of intervals are defined, which are depicted in Fig. 2.6.

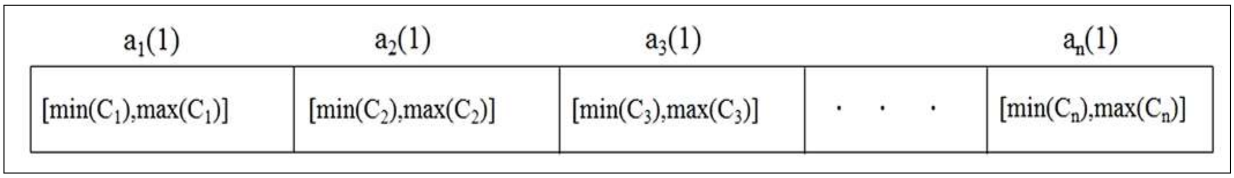

Figure 2.4: Array representation of initial intervals.

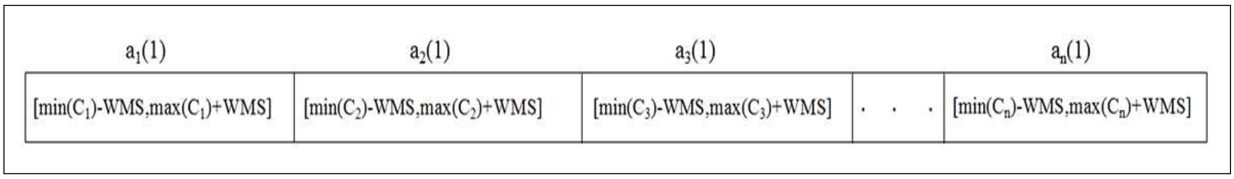

Figure 2.5: Array representation of next set of intervals.

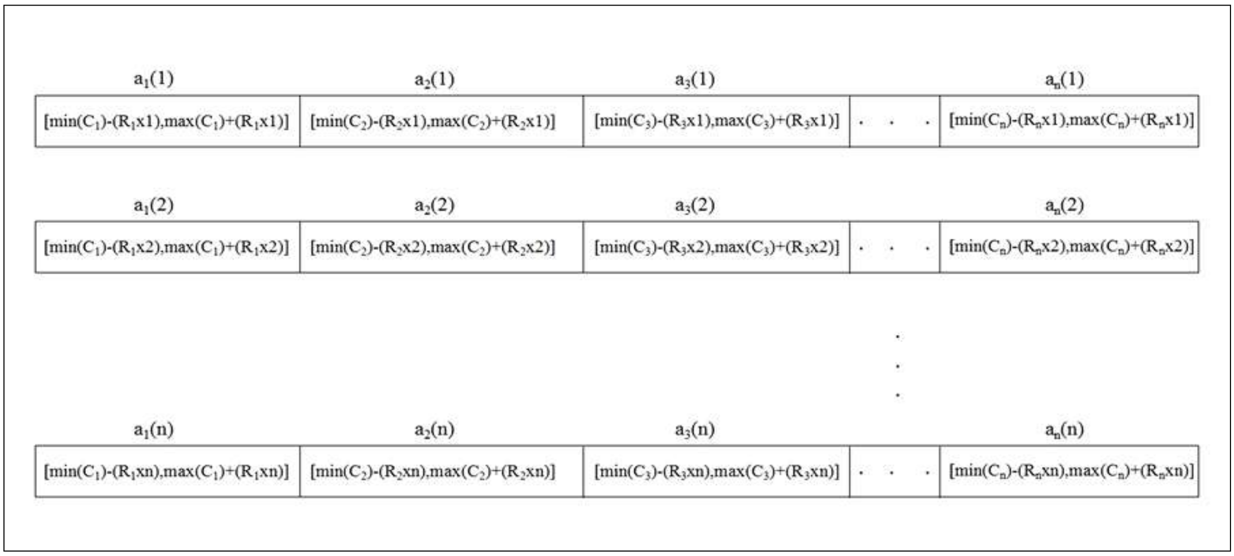

Figure 2.6: Array representation of final set of intervals.

The main features of the searching procedure of the RBILFA can be summarized as follows:

- ⋆ The RBILFA can be used to solve the continuous as well as discontinues set of problems.

⋆ RBILFA selects the initial set of intervals based on Eq. 2.6.2 and generate the next set of intervals based on Eq. 2.6.3.

⋆ After that, final set of intervals is generated based on Eq. 2.6.4.

⋆ It can be implemented easily using linear array.

⋆ It can find the best set of intervals in a very least number of iterations.

2.6.2 Improved Hybridized Forecasting Model

The main downside of fuzzy based forecasting model is that increase in the number of interval increases accuracy rate of forecasting, and decreases the fuzziness of time series data sets [50]. Singh and Borah [51] show that appropriate selection of intervals also increases the forecasting accuracy of the model. Therefore, to improve the forecasting accuracy of the proposed FTS model, we have hybridized the RBILFA with Algorithm 5. The main function of the RBILFA in Algorithm 5 is to adjust the length of initial intervals and membership values simultaneously, without increasing the number of intervals in the model. We have entitled this model as "FTS-RBILFA". The detailed description of the algorithm is presented in Algorithm 6.

Algorithm 6: FTS-RBILFA

Step 1 Select the initial set of intervals.

Step 2 Define linguistic terms for the above set of intervals.

Step 3 Fuzzify the historical time series according to the linguistic terms defined above.

Step 4 Establish FLRs based on Definition 2.2.3.

Step 5 Establish FLRGs based on Definition 2.2.4.

Step 6 Defuzzify the fuzzified time series data set.

Step 7 Compute the AFER value for initial set of intervals based on Eq. 2.5.4.

Step 8 Generate the next set of intervals based on Eq. 2.6.3.

Step 9 Compute the fitness for all set of intervals, and select the best one from Step 1 and Step 8.

Step 10 Compute the ranks "$R_{k(upper)}$" and "$R_{k(lower)}$" only for the set of intervals which have the best fitness in terms of Eq. 2.5.4 (obtained from Step 9).

Step 11 Generate final set of intervals based on Eq. 2.6.4.

Step 12 Compute the fitness (in terms of Eq. 2.5.4) for all set of intervals from Step 11, and select the best one.

2.7 Experiment Set Up and Results Based on the FTS-RBILFA Model

The FTS-RBILFA model employs the RBILFA to obtain the best interval lengths as well as FLRs for the historical enrollments data set. In subsection 2.3.1, we have the universe of discourse $U = [13054.84, 19337.16]$ for the enrollments data set, where lower bound $L_{bound} = 13054.84$ and upper bound $U_{bound} = 19337.16$. On the universe of discourse, total 10 intervals are defined based on the "NBTSDCT". Therefore, for the representation of *initial intervals* in the RBILFA, we use these 10 intervals. For finding the best interval lengths, *i.e.*, *final set of intervals*, we need to iterate the RBILFA only once.

The *final set of intervals* obtained from the RBILFA is depicted in Table 2.7. In this table, interval $a_2(9)$ does not cover any historical datum, so this interval doesn't employ for defining the fuzzy set. Hence, only 9 intervals (*i.e., final set of intervals*) are further employed to define the fuzzy sets, and to obtain the FLRs. All these FLRGs (obtained from FLRs) based on these *final set of intervals* are shown in Table 2.8.

Table 2.7: Final set of intervals and their corresponding elements.

Cluster	Corresponding Element	Final Set of Intervals
$\acute{C}_1$	$(16807, 16388, 16919, 16859)$	$a_1(9) = [16034.94, 16943.611]$
$\acute{C}_2$	(Nil)	$a_2(9) = [16691.31, 17056.527]$
$\acute{C}_3$	$(18150, 18970, 18876)$	$a_3(9) = [17969.4, 19124.309]$
$\acute{C}_4$	$(19328, 19337)$	$a_4(9) = [19135.62, 19494.312]$
$\acute{C}_5$	$(13055, 13563)$	$a_5(9) = [12925.35, 13673.07]$
$\acute{C}_6$	$(13867, 14696)$	$a_6(9) = [13729.23, 14815.34]$
$\acute{C}_7$	$(15460, 15311, 15603, 15433, 15497)$	$a_7(9) = [15306.3, 15729.761]$
$\acute{C}_8$	$(15861, 15984)$	$a_8(9) = [15703.29, 16113.878]$
$\acute{C}_9$	(15163)	$a_9(9) = [15158.79, 15558.37]$
$\acute{C}_{10}$	(15145)	$a_{10}(9) = [14994.45, 15286.161]$

Based on the FLRGs as shown in Table 2.8, forecasted results for the university enrollments data are obtained. These results are shown in the last column of Table 2.9. To judge the efficiency of the proposed model, we can compare the forecasted result of the proposed model with the previous existed FTS models [2, 3, 24, 52–54]. In Table 2.9, we give the forecasted results for compare with the existing FTS mdoels $AFER$. From Table 2.9, it is obvious that the proposed FTS-RBILFA model gets lower $AFER$ in comparison to the competing models. It is also found that integration of the RBILFA with the FTS model reduces the AFER for the historical university data set from 2.124% (refer to Table 2.6) to 2.09% (refer to Table 2.9).

In the next set of experiment, we apply the FTS-RBILFA model to forecast the New York Stock Exchange (NYSE) data set from the period $11/01/1999 - 12/31/1999$

Table 2.8: FLRGs of the university enrollments data set obtained from the *final set of intervals*.

Group	FLRG
Group 1:	$A_5 \rightarrow A_5, A_6$
Group 2:	$A_6 \rightarrow A_6, A_7$
Group 3:	$A_7 \rightarrow A_9, A_8, A_{10}$
Group 4:	$A_8 \rightarrow A_1$
Group 5:	$A_1 \rightarrow A_1, A_7, A_3$
Group 6:	$A_{10} \rightarrow A_9$
Group 7:	$A_9 \rightarrow A_8$
Group 8:	$A_3 \rightarrow A_3, A_4$
Group 9:	$A_4 \rightarrow A_4, A_3$

Table 2.9: A comparison of the existing models in Enrollment Data Set with the proposed FTS-RBILFA Model.

Year	Actual Enrollments	Model [2]	Model [3]	Model [24]	Model [52] (MEPA)	Model [52] (TFA)	Model [53]	Model [54]	FTS-RBILFA Model
1971	13055	–	–	–	–	–	–	–	–
1972	13563	14000	14000	14025	15430	14230	14195	14242.0	13785.748
1973	13867	14000	14000	14568	15430	14230	14424	14242.0	13785.748
1974	14696	14000	14000	14568	15430	14230	14593	14242.0	14891.652
1975	15460	15500	15500	15654	15430	15541	15589	15474.3	14891.652
1976	15311	16000	16000	15654	15430	15541	15645	15474.3	15519.970
1977	15603	16000	16000	15654	15430	15541	15634	15474.3	15519.970
1978	15861	16000	16000	15654	15430	16196	16100	15474.3	15519.970
1979	16807	16000	16000	16197	16889	16196	16188	16146.5	16455.518
1980	16919	16813	16833	17283	16871	16196	17077	16988.3	16874.370
1981	16388	16813	16833	17283	16871	17507	17105	16988.3	16874.370
1982	15433	16789	16833	16197	15447	16196	16369	16146.5	16874.370
1983	15497	16000	16000	15654	15430	15541	15643	15474.3	15519.970
1984	15145	16000	16000	15654	15430	15541	15648	15474.3	15519.970
1985	15163	16000	16000	15654	15430	15541	15622	15474.3	15358.580
1986	15984	16000	16000	15654	15430	15541	15623	15474.3	15908.584
1987	16859	16000	16000	16197	16889	16196	16231	16146.5	16455.518
1988	18150	16813	16833	17283	16871	17507	17090	16988.3	16874.370
1989	18970	19000	19000	18369	19333	18872	18325	19144.0	18985.766
1990	19328	19000	19000	19454	19333	18872	19000	19144.0	18985.766
1991	19337	19000	19000	19454	19333	18872	19000	19144.0	18985.766
1992	18876		19000		19333	18872	19000	19144.0	18985.766
AFER	–	3.22%	3.11%	2.67%	2.75%	2.66%	2.66%	2.40%	2.09%

(format:mm/dd/yyyy). Table 2.10 shows a comparison of the AFERs for the different existing FTS models. From Table 2.10, we can see that the proposed model gets higher forecasting accuracy rates in comparison to Chen [3] model, Yu [15] model, and Teoh et al. [55] model. From Table 2.10, it is also obvious that forecasting accuracy of the FTS-RBILFA model is better than the proposed FTS model. In Fig. 2.7, curves clearly indicate that the forecasted NYSE values obtained from the proposed models (both FTS model FTS-RBILFA model) are very close to that of actual NYSE values.

To verify the results obtained from the proposed FTS-RBILFA model is statically significant or not in case of the NYSE data set, we use various statistical parameters, *viz.*, means and standard deviations (SD) of the observed and predicted values, root mean square error

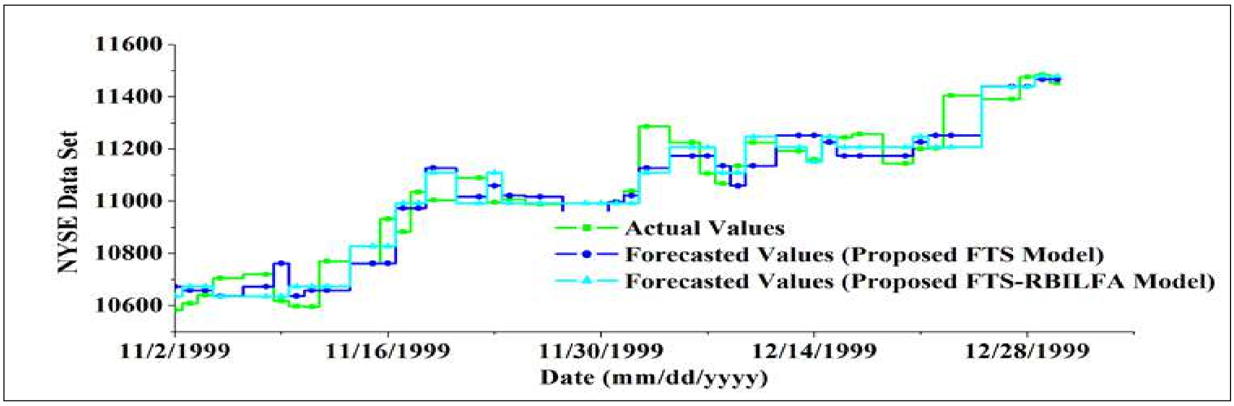

Figure 2.7: Comparison curves of actual NYSE values and forecasted NYSE values.

Table 2.10: A comparison of the existing models in NYSE Data Set with the proposed FTS-RBILFA Model.

Date	Stock index	Chen [3] Model	Yu [15] Model	Teoh et al. [55] Model	Proposed FTS Model	Proposed FTS-RBILFA Model
11/1/1999	10648.51	–	–	–	–	–
11/2/1999	10581.84	10581	10640.00	10648.11	10672.81	10635.46
11/3/1999	10609.06	10581	10640.00	10581.84	10658.04	10673.29
11/4/1999	10639.64	10581	10613.33	10608.90	10658.04	10673.29
11/5/1999	10704.48	10581	10600.00	10639.29	10636.24	10635.46
11/8/1999	10718.85	10581	10640.00	10705.26	10672.81	10635.46
11/9/1999	10617.32	10581	10693.33	10719.55	10761.94	10635.46
11/10/1999	10597.74	10581	10666.67	10617.11	10636.24	10673.29
11/11/1999	10595.3	10581	10640.00	10597.65	10658.04	10673.29
11/12/1999	10769.32	10581	10624.00	10595.22	10658.04	10673.29
11/15/1999	10760.75	10910	10600.00	10769.71	10761.94	10827.80
11/16/1999	10932.33	10910	10706.67	10761.19	10761.94	10827.80
11/17/1999	10883.09	10910	10760.00	10931.75	10972.55	10992.19
11/18/1999	11035.7	10910	10920.00	10882.8	10972.55	10992.19
11/19/1999	11003.89	10910	10880.00	11037.73	11126.90	11109.22
11/22/1999	11089.52	10910	11040.00	11006.11	11017.06	10992.19
11/23/1999	10995.63	11074	11000.00	11091.23	11058.87	11109.22
11/24/1999	11008.17	10910	11066.67	10994.67	11021.71	10992.19
11/26/1999	10988.91	10910	11033.33	11010.36	11017.06	10992.19
11/29/1999	10947.92	10910	11020.00	10987.99	10923.11	10992.19
11/30/1999	10877.81	10910	11000.00	10947.24	10923.11	10992.19
12/1/1999	10998.39	10910	10946.67	10877.55	10997.01	10992.19
12/2/1999	11039.06	10910	10960.00	11000.64	11021.71	10992.19
12/3/1999	11286.18	10910	10960.00	11041.07	11126.90	11109.22
12/6/1999	11225.01	11074	11040.00	11286.71	11173.57	11206.81
12/7/1999	11106.65	11074	11280.00	11225.9	11173.57	11206.81
12/8/1999	11068.12	11074	11240.00	11108.25	11135.53	11109.22
12/9/1999	11134.79	10910	11080.00	11069.95	11058.87	11109.22
12/10/1999	11224.7	11074	11133.33	11136.22	11135.53	11247.33
12/13/1999	11192.59	11074	11173.33	11225.60	11252.00	11206.81
12/14/1999	11160.17	11074	11240.00	11193.68	11252.00	11151.12
12/15/1999	11225.32	11074	11200.00	11161.45	11226.56	11247.33
12/16/1999	11244.89	11074	11166.67	11226.21	11173.57	11206.81
12/17/1999	11257.43	11074	11196.00	11245.66	11173.57	11206.81
12/20/1999	11144.27	11074	11212.00	11258.13	11173.57	11206.81
12/21/1999	11200.54	11074	11226.67	11145.65	11226.56	11247.33
12/22/1999	11203.6	11074	11186.67	11201.58	11252.00	11206.81
12/23/1999	11405.76	11074	11193.33	11204.62	11252.00	11206.81
12/27/1999	11391.08	11406	11405.76	11405.57	11439.55	11438.42
12/28/1999	11476.71	11074	11333.33	11390.98	11439.55	11438.42
12/29/1999	11484.66	11477	11476.71	11476.09	11466.77	11479.55
12/30/1999	11452.86	11485	11484.66	11484.00	11466.77	11479.55
12/31/1999	11497.12	11453	11452.86	11452.39	11497.12	11479.55
AFER		0.97%	0.80%	0.59%	0.54%	0.52%

(RMSE) and Theil's U Statistic. All these parameters are defined in Appendix. Statistical analyzes results are depicted in Table 2.11. From Table 2.11, it is clear that the mean of observed values is close to mean of forecasted values. The comparison of SD values

Table 2.11: Statistical analyzes of forecasting results obtained from the FTS-RBILFA model for the NYSE data set.

Statistics	Value
$\bar{A}$ Observed (in dollar)	11042.31
$\bar{A}$ Forecasted (in dollar)	11045.30
SD Observed (in dollar)	269.12
SD Forecasted (in dollar)	257.63
Forecasted RMSE (in dollar)	72.57
U	0.0033

between observed and forecasted values show that predictive skill of our proposed model is satisfactory. Forecasted results in terms of $RMSE$ indicate the very small error rate. In Table 2.11, U value is closer to 0, which indicates the effectiveness of the proposed model.

2.8 Time complexity analysis of the proposed models

This study consists of two models, *viz.*, FTS model and FTS-RBILFA model. Later, we have ensembled the RBILFA with the FTS model, which is entitled as "FTS-RBILFA model". Therefore, we will initially discuss the time complexity of the FTS model followed by the FTS-RBILFA model. Hence, the time complexity of the FTS model is explained below:

1. The FTS model mainly consists of two parts, *viz.*, NBTSDCT and basic steps of Chen [3] model. Now, the time complexity of the NBTSDCT is: This technique consists of four main parts. The first is the FRNA, the second part is the TCA, the third part is the NIA, and the fourth part is the NCA. The time complexity of each different part is given below:

   ⋆ The time complexity of the FRNA (O_{FRNA}) is as follows:

   $$O_{FRNA} = O(N)log(N). \tag{2.8.1}$$

   Here, N is the sample size.

   ⋆ The time complexity of the TCA (O_{TCA}) is as follows:

   $$O_{TCA} = O(N)log(N). \tag{2.8.2}$$

   Here, N is the sample size.

   ⋆ The time complexity of the NIA (O_{NIA}) is as follows:

   $$O_{NIA} = O(log(N)). \tag{2.8.3}$$

   Here, N is the sample size.

⋆ The time complexity of the NCA (O_{NCA}) is as follows:

$$O_{NCA} = O(N). \tag{2.8.4}$$

Here, N is the sample size.

2. Hence, the overall time complexity of the NBTSDCT $(O_{NBTSDCT})$ is as follows:

$$\begin{aligned} O_{NBTSDCT} &= O_{FRNA} + O_{TCA} + O_{NIA} + O_{NCA} \\ &= O(N)log(N) + O(N)log(N) + O(log(N)) + O(N) \\ &= O(NlogN). \end{aligned} \tag{2.8.5}$$

3. The time complexity of Chen [3] model (O_{chen}) is as follows:

$$O_{chen} = O(C). \tag{2.8.6}$$

Here, C is the total number of clusters.

4. Overall time complexity of the FTS model (O_{FTS}) is the integration of the time complexities of the NBTSDCT and Chen [3] model, which is as follows:

$$\begin{aligned} O_{FTS} &= (O_{NBTSDCT}) + (O_{chen}) \\ &= O(NlogN) + O(C) \\ &= O(C + NlogN). \end{aligned} \tag{2.8.7}$$

5. The time complexity of RBILFA (O_{RBILFA}) is as follows:

$$O_{RBILFA} = O(C). \tag{2.8.8}$$

Here, C is the total number of clusters.

6. Now, the overall time complexity of the FTS-RBILFA model $(O_{FTS-RBILFA})$ is the integration of the time complexities of the FTS model and RBILFA, which is as follows:

$$\begin{aligned} O_{FTS-RBILFA} &= O_{FTS} + O_{RBILFA} \\ &= O(C + NlogN) + O(C) \\ &= O(C + NlogN). \end{aligned} \tag{2.8.9}$$

2.9 Discussion

Researchers in the domain of FTS modeling indicate that the lengths of intervals affect the forecasting results [50, 51]. In this thesis, we have presented a new fuzzy time series forecasting model which adjusts the lengths of intervals without increasing their numbers by which forecasting accuracy can be improved. The proposed model is designed on the framework of Chen [3] model.

In this proposed model, first we apply the NBSCDT to cluster the data set for generating the different lengths of intervals. Based on these intervals, we have defined fuzzy sets. These fuzzy sets are further employed to establish the FLRs. From these FLRs, FLRGs are defined, which are further used for forecasting the historical data. This model is entitled as "FTS Model". Moreover, to improve it's forecasting accuracy, another technique entitled as "RBILFA" is proposed. This technique is ensembled with the "FTS model". Hence, this new hybrid model is referred to as "FTS-RBILFA model". Both "FTS Model" and "FTS-RBILFA model" are verified with the university enrollments data set of Alabama and the NYSE data sets.

From Table 2.9, we can see that the FTS-RBILFA model has higher average forecasting accuracy rate than the existing FTS models [2, 3, 24, 52–54] in case of forecasting the university enrollments data set of Alabama. From Table 2.10, we can see that the proposed model also gets a higher average forecasting accuracy rate than existing competing models, such as Chen [3] model, Yu [15] model, and Teoh et al. [55] model.

Forecasting accuracy is not the only way to compare the models; however, time complexity analysis also plays a significant role. In this thesis, the time complexity of the proposed models is evaluated, and it is found that running time of both FTS model and FTS-RBILFA model grows close to the linear. The most time consuming part of the proposed model is the NBTSDCT (which is used for clustering data) and RBILFA (which is used to find the best interval lengths).

2.10 Appendix

1. The mean can be defined as:

$$\bar{A} = \frac{\sum_{i=1}^{n} Actual_i}{d} \tag{2.10.1}$$

2. The SD can be defined as:

$$SD = \sqrt{\frac{1}{d}\sum_{i=1}^{d}(Actual_i - \bar{A})^2} \tag{2.10.2}$$

3. The $RMSE$ can be defined as:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{d}(Forecasted_i - Actual_i)^2}{d}} \tag{2.10.3}$$

4. The formula used to calculate Theil's U statistic is:

$$U = \frac{\sqrt{\sum_{i=1}^{d}(Actual_i - Forecasted_i)^2}}{\sqrt{\sum_{i=1}^{d} Actual_i^2} + \sqrt{\sum_{i=1}^{d} Forecasted_i^2}} \tag{2.10.4}$$

Here, each $Forecasted_i$ and $Actual_i$ is the forecasted and final price of day i respectively, d is the total number of days to be forecasted. In Eqs. 2.10.1 and 2.10.2, $\{A_1, A_2, \ldots, A_n\}$ are the observed values of the final price and $\bar{A}$ is the mean value of these observations. Similarly, mean and SD for predicted time series data set are computed. For a good forecasting, the observed means and SDs should be close to predicted means and SDs. In Eq. 2.10.3, a small RMSE value indicates good forecasting. In Eq. 2.10.4, U is bound between 0 and 1, with values closer to 0 indicating good forecasting accuracy.

3

A Study on Effects of Length of Intervals on Fuzzy Time Series Forecasting

Overview of the chapter: In Section 3.1 and 3.2, we review the theory of fuzzy set with an overview of FTS, and it's applications in FTS modeling. In Section 3.3, effects of length of intervals in Type-1 and Type-2 fuzzy sets are discussed. Various theorems associated with length of intervals are presented in Section 3.4. Data fuzzification and defuzzification methods are presented in Section 3.5. Experimental results are discussed in Section 3.6. In Section 3.7, we provide discussion.
KEYWORDS: *FTS, ANN, RS, EC.*

3.1 Introduction

Based on recent development in the applications of fuzzy sets concept, it can be categorized into Type-1 and Type-2 fuzzy sets. Both these concepts were introduced by Zadeh in 1965 [1] and 1975 [56], respectively. The difference between Type-1 and Type-2 fuzzy sets are that in Type-1 fuzzy set, the degree of membership is characterized by a crisp value; whereas in Type-2 fuzzy set, the degree of membership is regarded as a fuzzy set [57]. That is, when there are more uncertainty in the events, and we have a difficulty to determine its exact degree of membership in the Type-1 fuzzy sets, then we can simply use Type-2 fuzzy sets. Hence, Type-2 fuzzy sets can be regarded as fuzzy sets within a fuzzy set. However, Type-2 fuzzy sets are not easy to understand and use as compare to the Type-1 fuzzy sets, which is the reason the use and application of Type-2 fuzzy sets are not quite popular among the researchers. In this study, we try to demonstrate the understanding and use of both Type-1 and Type-2 fuzzy sets in terms of time series forecasting with the faith that they will be used extensively.

Both Type-1 [13, 25, 58] and Type-2 [59–62] fuzzy sets are highly used in time series forecasting. In Type-2 modeling approach, observations that are handled by Type-1 model can be termed as "main-factor / Type-1 observations", whereas observations that are handled by Type-2 model can be termed as "secondary-factors / Type-2 observations". Later, both these observations are combined together to take the final decision. But, due to involvement of Type-2 observations with Type-1 observations, massive fuzzy relations are generated in Type-2 model. For this reason, Type-2 model suffers from the burden of extra computation. Determination of degree of membership of time series data in Type-2 model is very difficult, because there is no well-known method or function available in literature. Moreover, degree of membership representation in Type-1 fuzzy sets are two-dimensional, whereas degree of membership representation of type-2 fuzzy sets are three-dimensional [63]. In this study, we show the use of triangular membership function to determine the degree of membership of data in case of both Type-1 and Type-2 fuzzy sets in a very lucid manner.

Although all these difficulties, Type-2 fuzzy sets are still used by the researchers in time series forecasting [51, 64–68]. Other domains where Type-2 fuzzy sets are used as: for finding the solution of fuzzy equations [69], for solving the problem of community transport scheduling [70], for analyzing the radiographic tibia images [71]. Detail descriptions of Type-2 fuzzy sets and their various operations can be found in this chapter [72].

3.2 Preliminaries

In 1965, Zadeh [1] introduced fuzzy sets theory involving continuous set membership for processing data in presence of uncertainty. He also presented fuzzy arithmetic theory and its application [35].

Definition 3.2.1. (Fuzzy Set) [1]. A fuzzy set is a class with varying degrees of membership in the set. Let U be the universe of discourse, which is discrete and finite, then fuzzy set A can be defined as follows:

$$A = \{\mu_{A(x_1)}/x_1 + \mu_{A(x_2)}/x_2 + \ldots\} = \Sigma_i \mu_A(x_i)/x_i \tag{3.2.1}$$

where μ_A is the membership function of A, μ_A: $U \rightarrow [0, 1]$, and $\mu_{A(x_i)}$ is the degree of membership of the element x_i in the fuzzy set A. Here, the symbol "+" indicates the operation of union and the symbol "/" indicates the separator rather than the commonly used summation and division in algebra, respectively.

When U is continuous and infinite, then the fuzzy set A of U can be defined as:

$$A = \{\int \mu_{A(x_i)}/x_i\}, \forall x_i \in U \tag{3.2.2}$$

where the integral sign stands for the union of the fuzzy singletons, $\mu_{A(x_i)}/x_i$.

Definition 3.2.2. (Fuzzy Time Series) [2, 4, 5]. Let $Y(t)(t = 0, 1, 2, \ldots)$ be a subset of Z and the universe of discourse U on which fuzzy sets $\mu_i(t)(i = 1, 2, \ldots)$ are defined and let $F(t)$ be a collection of $\mu_i(t)(i = 1, 2, \ldots)$. Then, $F(t)$ is called a fuzzy time series (FTS) on $Y(t)(t = 0, 1, 2, \ldots)$.

Definition 3.2.3. (Fuzzy logical relationship) [2, 3, 5]. Assume that $F(t-1) = A_i$ and $F(t) = A_j$. The relationship between $F(t)$ and $F(t-1)$ is referred as a fuzzy logical relationship (FLR), which can be represented as:

$$A_i \rightarrow A_j, \tag{3.2.3}$$

where A_i and A_j refer to the left-hand side and right-hand side of the FLR, respectively.

Definition 3.2.4. (Fuzzy logical relationship group) [2, 3, 5]. Assume the following FLRs:

$$\begin{aligned} &A_i \rightarrow A_{k1}, \\ &A_i \rightarrow A_{k2}, \\ &\quad \ldots \\ &A_i \rightarrow A_{km} \end{aligned} \tag{3.2.4}$$

Chen [3] suggested that FLRs having same fuzzy sets on left-hand side can be grouped into a fuzzy logical relationship group (FLRG). So, based on Chen's model [3], these FLRs can be grouped into the FLRG as:

$$A_i \rightarrow A_{k1}, A_{k2}, \ldots, A_{km} \tag{3.2.5}$$

3.3 Effects of Length of Intervals in Type-1 and Type-2 Fuzzy Sets

Most of the researchers still use to prefer Type-1 modeling approach for forecasting. But, as far as accuracy of forecasting is concerned, Type-2 based model produces better result than Type-1 based models [50]. However, this accuracy highly dependent on effective lengths of intervals. Here, intervals that are associated with Type-1 and Type-2 fuzzy sets are termed as "Primary Intervals (PIs)" and "Secondary Intervals (SIs)", respectively. Now, how these PIs and SIs affect in the forecasting results, are explained with the examples below.

Let us consider the example of university grading system. At the end of final examination, faculties make the intervals of marks based on the performance of the students. Suppose, in the paper of physics, Prof. Mathew makes the following intervals of marks as: $A_1 = [1, 30]$, $A_2 = [31, 50]$, $A_3 = [51, 70]$, and $A_4 = [71, 100]$. Here, all these intervals are termed as the PIs. Now, Prof. Mathew defines the fuzzy sets for these intervals as:

1. If any student resides in the interval A_1, then he/she will get D grade (*i.e.*, Poor).
2. If any student resides in the interval A_2, then he/she will get C grade (*i.e.*, Average).
3. If any student resides in the interval A_3, then he/she will get B grade (*i.e.*, Good).
4. If any student resides in the interval A_4, then he/she will get A grade (*i.e.*, Best).

Here, linguistic variables "Poor", "Average", "Good", and "Best" can be considered as Type-1 fuzzy sets. The intervals A_1, A_2, A_3, and A_4 are associated with the fuzzy sets "Poor", "Average", "Good", and "Best", respectively.
Effects of PIs in the forecasting results: Suppose, Herry got 78 marks in physics, which resides in PI: A_4. According to this interval, Herry will receive A grade (*i.e.*, Best). The degree of membership for the fuzzy set "Best" is shown in Fig. 3.1. Here, we have a crisp degree of membership value 0.4 for the marks of Herry. Now, in case of decision making, if we use centroid method, then final result will be: $c = \frac{a+b}{2} = \frac{71+100}{2} = 85.5$. Here, "a" and "b" represent the lower and upper bounds of the PI, and "c" represents its centroid. Hence, in case of centroid defuzzification method, centroid of the PI gives the defuzzified value.

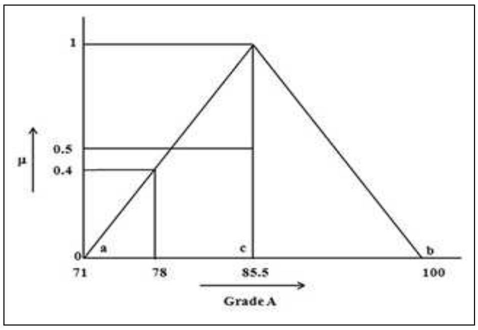

Figure 3.1: A Type-1 fuzzy set.

Effects of SIs in the forecasting results: Let us consider another situation here. In the same PI, there may be a few more students, whose marks are better than Herry. That means, grade A can have more than one fuzzy sets, which can be represented as: average best, very best, very very best, super best, super super best, super super super best, *etc*. In this case, linguistic variable "Best", which is a member of Type-1 fuzzy set, can be termed as "primary fuzzy set", and the linguistic variables which are generated from the primary fuzzy set (*e.g.*, average best, very best, very very best, super best, super super best, super super super best, *etc.*) can be termed as "secondary fuzzy sets". Hence, we can say that the secondary fuzzy sets are the subset of primary fuzzy set.

Above explanation indicates that the fuzzy set "Best" for the Grade A can have more than one degree of memberships as: (0.4, 0.5 and 0.6) (shown in Fig. 3.2). In other words, there can be multiple degrees of membership for the same "Best", as shown in Fig. 3.2.

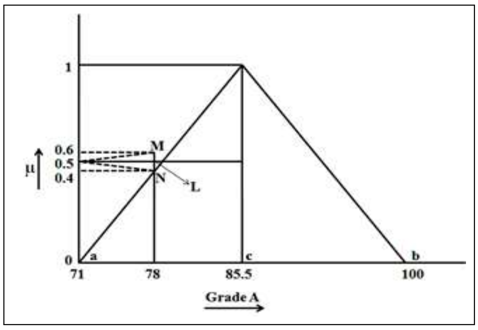

Figure 3.2: A Type-2 fuzzy set.

In Fig. 3.2, the highest degree of membership (0.6) indicates the positive view about the occurrence of event, whereas the lowest degree of membership (0.4) indicates the negative view about the occurrence of event. To represent these degree of memberships together, we can employ Type-2 fuzzy sets. Hence, for the representation of Type-1 fuzzy set "Best" into Type-2 fuzzy set, we need to discretize the PI: $A_4 = [71, 99]$ further as: $I_1 = [71, 75]$, $I_2 = [76, 80]$, $I_3 = [81, 85]$, $I_4 = [86, 90]$, $I_5 = [91, 95]$, and $I_6 = [96, 100]$. Here, all these intervals are termed as SIs. Now, we can define the Type-2 fuzzy sets for these intervals as:

1. If any student resides in the interval I_1, then he/she can be represented with fuzzy set "average best".

2. If any student resides in the interval I_2, then he/she can be represented with fuzzy set "very best".

3. If any student resides in the interval I_3, then he/she can be represented with fuzzy set "very very best".

4. If any student resides in the interval I_4, then he/she can be represented with fuzzy set "super best".

5. If any student resides in the interval I_5, then he/she can be represented with fuzzy set "super super best".

6. If any student resides in the interval I_6, then he/she can be represented with fuzzy set "super super super best".

Now, the marks 78 of Herry resides in the SI: $I_2 = [76, 80]$. For the final decision making, if we use centroid method, then result will be: $L = \frac{N+M}{2} = \frac{76+80}{2} = 78$. Here, both "N" and "M" represent the lower and upper bounds of the SI, and "L" represents its centroid. Hence, in case of centroid defuzzification method, centroid of the SI also gives the defuzzified value. Here, the value of "L" is equal to the marks obtained by Herry. It indicates that the SIs take better decision than the PIs. However, to find out the best interval lengths for both the Type-1 and Type-2 fuzzy sets is a heuristic search approach.

3.4 Theorems and Corollaries

Theorem 3.4.1. Type-2 fuzzy set is the subset of Type-1 fuzzy set.

Proof: The proof is obvious.

Theorem 3.4.2. Elements of a type-2 fuzzy set can have more than one degree of memberships.

Proof. The proof is obvious.

Theorem 3.4.3. If PI length is changed, then degree of membership of an element corresponding to SI would also be changed.

Proof: Consider the similar example of grading system, where Herry obtains the marks 78, which belongs to the PI, $A_4 = [71, 100]$. Here, the degree of membership of Herry's grade (which is A) is 0.4 which is shown in Fig. 3.1. To obtain SIs, we further discretize this A_4 interval into six equal lengths of intervals as: $I_1 = [71, 75]$, $I_2 = [76, 80]$, $I_3 = [81, 85]$, $I_4 = [86, 90]$, $I_5 = [91, 95]$, and $I_6 = [96, 100]$. Now, in case of SIs, Herry's marks lie in the interval I_2. The positive and negative views for this event are 0.6 and 0.4, respectively, which is shown in Fig. 3.1.

Let us change the length of considered PI, $A_4 = [71, 100]$ to $\bar{A}_4 = [76, 100]$. According to this new interval length, the degree of membership of Herry would be changed from 0.4 to 0.2 which is shown in Fig. 3.3.

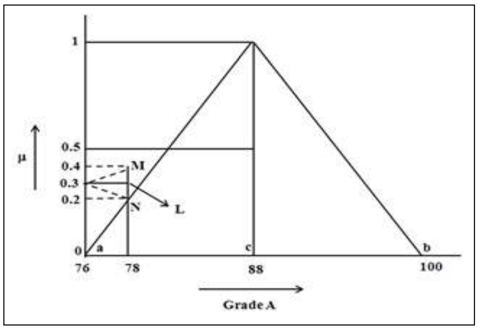

Figure 3.3: Degree of memberships of Herry's marks in terms of Type-1 and Type-2 fuzzy sets.

Now, again discretize this A_4 interval into five equal lengths of intervals as: $I_1 = [76, 80]$, $I_2 = [81, 85]$, $I_3 = [86, 90]$, $I_4 = [91, 95]$, and $I_5 = [96, 100]$. In case of SIs, Herry's marks lie in the interval I_1. The positive view of this event is changed from 0.6 to 0.4, and negative view for this event is changed from 0.4 to 0.2 which is shown in Fig. 3.1. Hence, it can be concluded that degree of membership of Type-2 fuzzy set would be changed if the corresponding length of PI would be changed.

Corollary 1. Degree of membership of an element in a Type-2 fuzzy set always depends on PIs.

Proof. The proof is obvious.

Theorem 3.4.4. An element of a Type-2 fuzzy set can't have upper bound in it's corresponding interval, if it's corresponding Type-1 fuzzy set have the highest degree of membership is closed to 1 in the PIs.

Proof: Consider the similar example of grading system, where Herry obtains the marks 83. Since, marks 83 belongs to the interval, $A_4 = [71, 100]$. Degree of membership for this fuzzy set is 0.9, which is closed to 1 which is shown in Fig. 3.4.

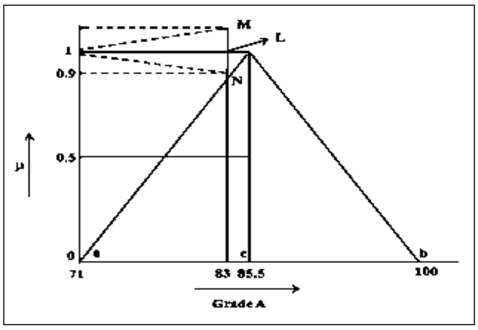

Figure 3.4: A Type-2 fuzzy set with upper bound above 1.

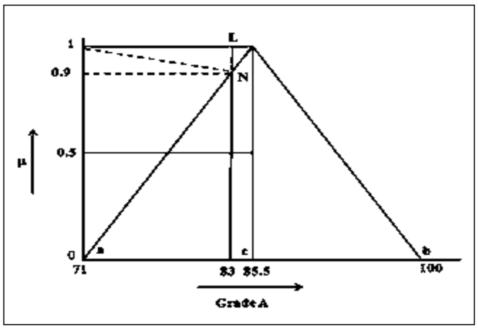

Figure 3.5: A Type-2 fuzzy set without upper bound.

To obtain SIs, we further discretize this A_4 interval into six equal lengths of intervals as: $I_1 = [71, 75]$, $I_2 = [76, 80]$, $I_3 = [81, 85]$, $I_4 = [86, 90]$, $I_5 = [91, 95]$, and $I_6 = [96, 100]$. Now, in case of SI, Herry's marks lie in the interval I_3. The positive and negative views for this event are above 1 and 0.9, respectively which is shown in Fig. 3.4. Since, the positive view of any event is represented by the upper bound of any SI. In this case, positive view crosses the maximum boundary of degree of membership of fuzzy set, which is 1. This representation violates the limitation of assigning the degree of membership to any event, which have positive view. Hence, we can say that Type-2 fuzzy set can't have the upper bound; if its corresponding PI has the degree of membership tends to 1.

In this scenario, we will take the final decision based on the two points, *viz.*, N (negative view) and L (midpoint of N and M (positive view)) which is shown in Fig. 3.5. The defuzzification formula is: (N + L)/2 = (81 + 83)/2 = 82.

Theorem 3.4.5. An element of a Type-2 fuzzy set can't have upper bound in SIs, if its corresponding Type-1 fuzzy set have the degree of membership greater than 0.8.

Proof. From Theorem 3.4.4, we get the proof.

Theorem 3.4.6. The defuzzified value of a corresponding Type-2 fuzzy set would be changed, if we change the lower and upper bound of a PI.

Proof: Consider the similar example of grading system where, Herry obtain the marks 80. Since, marks 80 belongs to the interval, $A_4 = [71, 100]$. Now, in case of decision making, if we use centroid method, then final result will be c = (a +b)/2 = (71 + 100)/2 = 85.5.

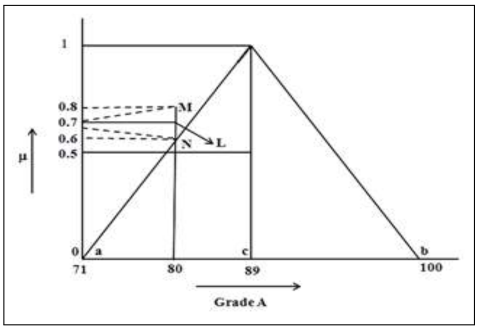

Figure 3.6: Defuzzification of Type-2 fuzzy set without changing of lower and upper bounds.

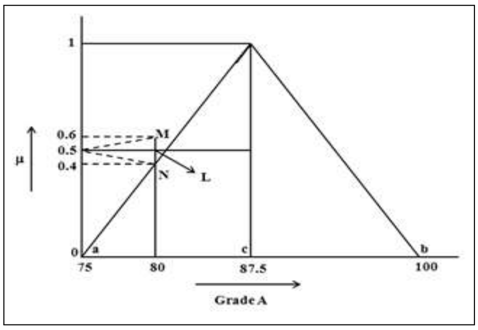

Figure 3.7: Defuzzification of Type-2 fuzzy set with changing of lower and upper bounds.

To obtain SIs, we further discretize this A_4 interval into six equal lengths of intervals as: $I_1 = [71, 75]$, $I_2 = [76, 80]$, $I_3 = [81, 85]$, $I_4 = [86, 90]$, $I_5 = [91, 95]$, and $I_6 = [96, 100]$. Now, in case of SIs, Herry's marks lie in the interval I_2. The positive and negative views for this event are at 0.8 and 0.6, respectively (refer to Fig. 3.6), and the defuzzified value for SI is (76 + 80)/2 = 78. Let us change the length of considered PI, $A_4 = [71, 100]$ to $\bar{A}_4 = [75, 100]$. Now, again discretize this $\bar{A}_4$ interval into five equal lengths of intervals as: $I_1 = [75, 80]$, $I_2 = [81, 85]$, $I_3 = [86, 90]$, $I_4 = [91, 95]$, and $I_5 = [96, 100]$. Now, in case of SIs, Herry's marks lie in the interval I_1. The positive view of this event is changed from 0.8 to 0.6, and negative view for this event is changed from 0.6 to 0.4 as shown in Fig 3.7. According to the centroid method its defuzzified value is (75 + 80)/2 = 77.5. Hence, it can be concluded that if there is a change in the upper and lower bound of a PI, then the defuzzified value of a corresponding Type-2 fuzzy set would also be changed.

3.5 Data Fuzzification and Defuzzification

In the following, we demonstrate how a data set can be fuzzified based on PIs and SIs, and defuzzified based on both Type-1 and Type-2 fuzzy sets. To explain this approach, an algorithm is proposed which is presented in Algorithm 7. To explain this algorithm, the university enrollments data set of Alabama [4], shown in Table 3.1, is employed. Each step of the algorithm is elucidated next.

Algorithm 7: Proposed time series forecasting model

Step 1: Determine PIs and SIs for time series data set.

Step 2: Define linguistic terms for each of the PIs and SIs.

Step 3: Fuzzify the time series data set.

Step 4: Establish the FLRs between the fuzzified time series values.

Step 5: Construct the FLRGs from the FLRs.

Step 6: Defuzzify the fuzzified time series data set.

Table 3.1: The data set of the enrollments of the University of Alabama.

Year	Actual Enrollments	Year	Actual Enrollments
1971	13055	1982	15433
1972	13563	1983	15497
1973	13867	1984	15145
1974	14696	1985	15163
1975	15460	1986	15984
1976	15311	1987	16859
1977	15603	1988	18150
1978	15861	1989	18970
1979	16807	1990	19328
1980	16919	1991	19337
1981	16388	1992	18876

Phase 3.5.1. Determine PIs and SIs for time series data set.

[Explanation for PIs]: Let E_{min} and E_{max} be the minimum and maximum enrollment values of the historical data set. Based on E_{min} and E_{max}, we can define the universe of discourse U as: $[E_{min} - P_1, E_{max} + P_2]$, where P_1 and P_2 are two positive numbers. From Table 3.1, we can see that $E_{min} = 13055$ and $E_{max} = 19337$. Thus, initially, we can let $P_1 = 55$ and $P_2 = 663$. Hence, the universe of discourse $U = [13000, 20000]$. Now, we partition $U = [13000, 20000]$ into seven equal intervals as: A_1, A_2, ..., and A_7, where $A_1 = [13000, 14000]$, $A_2 = [14000, 15000]$, $A_3 = [15000, 16000]$, $A_4 = [16000, 17000]$, $A_5 = [17000, 18000]$, $A_6 = [18000, 19000]$, and $A_7 = [19000, 20000]$. All these intervals can be regarded as PIs.

[Explanation for SIs]: In case of Type-2 fuzzy set, each PI can be considered as the universe of discourse. For example, interval $I_1 = [13000, 14000]$ can be considered as the universe of discourse U, and can be further partitioned into four equal lengths of intervals to generate SIs. Thus, SIs for the PI, $A_1 = [13000, 14000]$ are: $I_1 = [13000, 13250]$, $I_2 = [13250, 13500]$, $I_3 = [13500, 13750]$, and $I_4 = [13750, 14000]$. Here, each $I_i (i = 1, 2, \ldots, n)$ represents the SI. In this manner, we have partitioned all the PIs into SIs.

Phase 3.5.2. Define linguistic terms for each of the PIs and SIs.

[Explanation for PIs]: In this step, we assume that the historical time series data set is distributed among n PIs (*i.e.*, $A_1, A_2, \ldots,$ and A_n). Then, define n linguistic variables for the PIs as: $E_1, E_2, E_3, \ldots, E_n$, which can be represented by fuzzy sets, as shown below:

$$\begin{aligned}
E_1 &= 1/A_1 + 0.5/A_2 + 0/A_3 + \ldots + 0/A_{n-2} + 0/A_{n-1} + 0/A_n, \\
E_2 &= 0.5/A_1 + 1/A_2 + 0.5/A_3 + \ldots + 0/A_{n-2} + 0/A_{n-1} + 0/A_n, \\
E_3 &= 0/A_1 + 0.5/A_2 + 1/A_3 + \ldots + 0/A_{n-2} + 0/A_{n-1} + 0/A_n, \\
&\vdots \\
E_n &= 0/A_1 + 0/A_2 + 0/A_3 + \ldots + 0/A_{n-2} + 0.5/A_{n-1} + 1/A_n
\end{aligned} \tag{3.5.1}$$

[Explanation for SIs]: Similarly, define n linguistic variables for the SIs as: $F_1, F_2, F_3, \ldots, F_n$, which can be represented by fuzzy sets, as shown below:

$$\begin{aligned}
F_1 &= 1/I_1 + 0.5/I_2 + 0/I_3 + \ldots + 0/I_{n-2} + 0/I_{n-1} + 0/I_n, \\
F_2 &= 0.5/I_1 + 1/I_2 + 0.5/I_3 + \ldots + 0/I_{n-2} + 0/I_{n-1} + 0/I_n, \\
F_3 &= 0/I_1 + 0.5/I_2 + 1/I_3 + \ldots + 0/I_{n-2} + 0/I_{n-1} + 0/I_n, \\
&\vdots \\
F_n &= 0/I_1 + 0/I_2 + 0/I_3 + \ldots + 0/I_{n-2} + 0.5/I_{n-1} + 1/I_n
\end{aligned} \tag{3.5.2}$$

Obtain the degree of membership of each year's time series value belonging to each fuzzy set A_i. Here, maximum degree of membership of fuzzy set E_i occurs at interval A_i, and $1 \leq i \leq n$. Similarly, maximum degree of membership of fuzzy set F_i occurs at interval I_i, and $1 \leq i \leq n$.

Phase 3.5.3. Fuzzify the historical time series data set.

[Explanation for PIs]: In order to fuzzify the historical time series data set *w.r.t.* PIs, it is essential to obtain the degree of membership value of each observation belonging to each A_i $(i = 1, 2, 3, \ldots, n)$, for each year. If the maximum membership value of one year's observation occurs at "A_i", and $1 \leq i \leq n$, then the fuzzified value for that particular year is considered as E_i. For example, the enrollment value for the year 1971 belongs to "A_1" with the highest degree of membership value 1 (based on Eq. 3.5.1), so it is fuzzified to E_1. In this way, we have fuzzified each observation of the historical time series data set based on PIs, and are presented in Table 3.2.

Table 3.2: Fuzzified data set based on PIs.

Year	Actual Enrollments	Fuzzified Enrollments	Year	Actual Enrollments	Fuzzified Enrollments
1971	13055	E_1	1982	15433	E_3
1972	13563	E_1	1983	15497	E_3
1973	13867	E_1	1984	15145	E_3
1974	14696	E_2	1985	15163	E_3
1975	15460	E_3	1986	15984	E_3
1976	15311	E_3	1987	16859	E_4
1977	15603	E_3	1988	18150	E_6
1978	15861	E_3	1989	18970	E_6
1979	16807	E_4	1990	19328	E_7
1980	16919	E_4	1991	19337	E_7
1981	16388	E_4	1992	18876	E_6

[Explanation for SIs]: In order to fuzzify the historical time series data set *w.r.t.* SIs, it is essential to obtain the degree of membership value of each observation belonging to each I_i $(i = 1, 2, 3, \ldots, n)$, for each year. If the maximum membership value of one year's observation occurs at "I_i", and $1 \leq i \leq n$, then the fuzzified value for that particular year is considered as F_i. For example, the enrollment value for the year 1971 belongs to "I_1" with the highest degree of membership value 1 (based on Eq. 3.5.2), so it is fuzzified to F_1. The fuzzified historical time series data set in terms of SIs are presented in Table 3.3.

Phase 3.5.4. Establish the FLRs between the fuzzified time series values.

[Explanation for PIs]: Based on Definition 3.2.3, we can establish the FLRs between the fuzzified time series values. For establishment of these relations, we will consider that current event at time "t" is caused by previous events at time "$t-1$". For example, in Table 3.3, the fuzzified enrollment value "E_1" for the year 1972 is caused by the previous year 1971 fuzzified enrollment value "E_1" (refer to Table 3.2). Hence, the FLRs for the PI is represented in the following form:

$$E_1 \rightarrow E_1 \tag{3.5.3}$$

[Explanation for SIs]: Similarly, in case of SIs, we have considered that the fuzzified enrollment value "F_3" for the year 1972 is caused by the previous year 1971 fuzzified enrollment

Table 3.3: Fuzzified data set based on SIs.

Year	Actual Enrollments	Fuzzified Enrollments	Year	Actual Enrollments	Fuzzified Enrollments
1971	13055	F_1	1982	15433	F_{10}
1972	13563	F_3	1983	15497	F_{10}
1973	13867	F_4	1984	15145	F_9
1974	14696	F_7	1985	15163	F_9
1975	15460	F_{10}	1986	15984	F_{12}
1976	15311	F_{10}	1987	16859	F_{16}
1977	15603	F_{11}	1988	18150	F_{21}
1978	15861	F_{12}	1989	18970	F_{24}
1979	16807	F_{16}	1990	19328	F_{26}
1980	16919	F_{16}	1991	19337	F_{26}
1981	16388	F_{14}	1992	18876	F_{24}

value "F_1" (refer to Table 3.3). Hence, the FLRs for the SI is represented in the following form:

$$F_1 \rightarrow F_3 \tag{3.5.4}$$

Here, LHS of the FLR is called the previous state, whereas RHS of the FLR is called the current state. FLRs for PIs and SIs obtained for the fuzzified enrollment values are listed in Tables 3.4 and 3.5, respectively.

Table 3.4: FLRs based on PIs.

FLRs	FLRs
$E_1 \rightarrow E_1$	$E_4 \rightarrow E_3$
$E_1 \rightarrow E_2$	$E_4 \rightarrow E_6$
$E_2 \rightarrow E_3$	$E_6 \rightarrow E_6$
$E_3 \rightarrow E_3$	$E_6 \rightarrow E_7$
$E_3 \rightarrow E_4$	$E_7 \rightarrow E_7$
$E_4 \rightarrow E_4$	$E_7 \rightarrow E_6$

Phase 3.5.5. Construct the FLRGs from the FLRs.

[Explanation for PIs]: Based on the same previous state of the FLRs for the PIs, the FLRs can be grouped into a FLRG. For example, the FLRG "$E_i \rightarrow E_m, E_n$" (*i.e.*, Group i) indicates that there are following FLRs:

$$E_i \rightarrow E_m,$$
$$E_i \rightarrow E_n$$

In Table 3.4, there are two FLRs with the same previous state, $E_1 \rightarrow E_1$ and $E_1 \rightarrow E_2$.

Table 3.5: FLRs based on SIs.

FLRs	FLRs
$F_1 \rightarrow F_3$	$F_{14} \rightarrow F_{14}$
$F_3 \rightarrow F_4$	$F_{10} \rightarrow F_{10}$
$F_4 \rightarrow F_7$	$F_9 \rightarrow F_9$
$F_7 \rightarrow F_{10}$	$F_9 \rightarrow F_9$
$F_{10} \rightarrow F_{10}$	$F_{16} \rightarrow F_{12}$
$F_{10} \rightarrow F_{11}$	$F_{21} \rightarrow F_{21}$
$F_{11} \rightarrow F_{11}$	$F_{24} \rightarrow F_{24}$
$F_{12} \rightarrow F_{12}$	$F_{26} \rightarrow F_{26}$
$F_{16} \rightarrow F_{16}$	$F_{26} \rightarrow F_{26}$
$F_{16} \rightarrow F_{16}$	-

Table 3.6: FLRGs based on PIs.

Groups	FLRGs
Group 1:	$E_1 \rightarrow E_1, E_2$
Group 2:	$E_2 \rightarrow E_3$
Group 3:	$E_3 \rightarrow E_3, E_4$
Group 4:	$E_4 \rightarrow E_4, E_3, E_6$
Group 5:	$E_6 \rightarrow E_6, E_7$
Group 6:	$E_7 \rightarrow E_7, E_6$

These FLRs are used to form the FLRG as: $E_1 \rightarrow E_1, E_2$ (*i.e.*, Group 1). All these FLRGs are depicted in Table 3.6. In this study, we have discarded the repeated FLRs in the FLRGs.
[Explanation for SIs]: Based on the same previous state of the FLRs for the SIs, the FLRs can be grouped into a FLRG. For example, the FLRG "$F_i \rightarrow F_m, F_n$" (*i.e.*, Group i) indicates that there are following FLRs:

$$F_i \rightarrow F_m, \tag{3.5.5}$$

$$F_i \rightarrow F_n \tag{3.5.6}$$

In Table 3.5, there are two FLRs with the same previous state, $F_{10} \rightarrow F_{10}$ and $F_{10} \rightarrow F_{11}$. These FLRs are used to form the FLRG as: $F_{10} \rightarrow F_{10}, F_{11}$ (*i.e.*, Group 5). All these FLRGs are depicted in Table 3.7. In this study, we have discarded the repeated FLRs in the FLRGs.

Phase 3.5.6. Defuzzify the fuzzified time series data set.

To defuzzify the fuzzified time series data set and to obtain the forecasted values, defuzzification technique is proposed here. Based on the application of technique, it is categorized as: **Principle 1** and **Principle 2**. The **Principle 1** is applicable to defuzzify the fuzzified time series values based on Type-1 fuzzy sets, whereas **Principle 2** is applicable to defuzzify the fuzzified time series values based on Type-2 fuzzy sets.

Table 3.7: FLRGs based on SIs.

Groups	FLRGs	Groups	FLRG
Group 1:	$F_1 \rightarrow F_3$	Group 9:	$F_{14} \rightarrow F_{10}$
Group 2:	$F_3 \rightarrow F_3$	Group 10:	$F_{10} \rightarrow F_9$
Group 3:	$F_4 \rightarrow F_7$	Group 11:	$F_9 \rightarrow F_9, F_{12}$
Group 4:	$F_7 \rightarrow F_{10}$	Group 12:	$F_{16} \rightarrow F_{21}$
Group 5:	$F_{10} \rightarrow F_{10}, F_{11}$	Group 13:	$F_{21} \rightarrow F_{24}$
Group 6:	$F_{11} \rightarrow F_{12}$	Group 14:	$F_{24} \rightarrow F_{26}$
Group 7:	$F_{12} \rightarrow F_{16}$	Group 15:	$F_{24} \rightarrow F_{26}$
Group 8:	$F_{16} \rightarrow F_{16}, F_{14}$		–

⋆ The **Principle 1** is given as follows:

Step 1. *Obtain the fuzzified forecasting data for forecasting day $D(t)$, whose previous state is $E_{i1}, E_{i2}, \ldots, E_{ip}$ $(i = 1, 2, 3, \ldots, n)$, and current state is E_{jp} $(j = 1, 2, 3, \ldots, n)$, i.e., the FLR is in the form of $E_{i1}, E_{i2}, \ldots, E_{ip} \rightarrow E_{jp}$.*

Step 2. *Obtain defuzzified forecasting value for the previous state as:*

$$Defuzz_{prev} = C_{i1} + C_{i2} + \ldots + C_{ip} \tag{3.5.7}$$

where C_{i1}, C_{i2}, ..., C_{ip} denote mid-values of the intervals A_{i1}, A_{i2}, ..., A_{ip} $(i = 1, 2, 3, \ldots, n)$, respectively, and the maximum membership values of E_{i1}, E_{i2}, ..., E_{ip} occur at intervals A_{i1}, A_{i2}, ..., A_{ip}, respectively.

Step 3. *Obtain defuzzified forecasting value for the current state as:*

$$Defuzz_{curr} = C_{jp} \tag{3.5.8}$$

where C_{jp} denotes mid-value the interval A_{jp} $(j = 1, 2, 3, \ldots, n)$, and the maximum membership value of E_{jp} occurs at interval A_{jp}.

Step 4. *Compute the forecasted value for the proposed model as:*

$$Forecast_{D(t)} = \frac{Defuzz_{prev} + Defuzz_{curr}}{K} \tag{3.5.9}$$

Here, K is the total number of mid-values corresponding to $Defuzz_{prev}$ and $Defuzz_{curr}$.

⋆ The **Principle 2** is given as follows:

Step 1. *Obtain the fuzzified forecasting data for forecasting day $D(t)$, whose previous state is $F_{i1}, F_{i2}, \ldots, F_{ip}$ $(i = 1, 2, 3, \ldots, n)$, and current state is F_{jp} $(j = 1, 2, 3, \ldots, n)$, i.e., the FLR is in the form of $F_{i1}, F_{i2}, \ldots, F_{ip} \rightarrow F_{jp}$.*

Step 2. *Obtain defuzzified forecasting value for the previous state as:*

$$Defuzz_{prev} = M_{i1} + M_{i2} + \ldots + M_{ip} \tag{3.5.10}$$

where M_{i1}, M_{i2}, …, M_{ip} denote mid-values the intervals I_{i1}, I_{i2}, …, I_{ip} $(i = 1, 2, 3, \ldots, n)$, respectively, and the maximum membership values of F_{i1}, F_{i2}, …, F_{ip} occur at intervals I_{i1}, I_{i2}, …, I_{ip}, respectively.

Step 3. *Obtain defuzzified forecasting value for the current state as:*

$$Defuzz_{curr} = M_{jp} \tag{3.5.11}$$

where M_{jp} denotes mid-value the interval I_{jp} $(j = 1, 2, 3, \ldots, n)$, and the maximum membership value of F_{jp} occurs at interval jp.

Step 4. *Compute the forecasted value for the proposed model as:*

$$Forecast_{D(t)} = \frac{Defuzz_{prev} + Defuzz_{curr}}{K} \tag{3.5.12}$$

Here, K is the total number of mid-values corresponding to $Defuzz_{prev}$ and $Defuzz_{curr}$.

⋆ In case of applications of **Principle 1** and **Principle 2**, if there is a “Null” value in the current state of the FLR, then forecasted value for day $D(t)$ can be computed as:

$$Forecast_{D(t)} = \frac{Defuzz_{prev}}{K} \tag{3.5.13}$$

Here, K is the total number of mid-values corresponding to $Defuzz_{prev}$.

3.6 Experimental Results

Based on the Algorithm 7, we obtain the forecasted values for the PIs and SIs individually. To measure the performance of the algorithm, mean absolute percentage error (MAPE) is used as an evaluation criterion. The MAPE can be defined as follows:

$$MAPE = \frac{1}{z}\sum_{i=1}^{z}\left|\frac{Forecasted_i - Actual_i}{Actual_i}\right| \times 100\% \tag{3.6.1}$$

Here, $Forecasted_i$ represents the forecasted value at time i, $Actual_i$ represents the actual value at time i, and z represents the total number of years to be forecasted. The MAPE value of the forecasted enrollment is presented in Table 3.8. Table 3.8 indicates that forecasted results obtained from SIs are better than the PIs. The results obtained by the PIs and SIs are

Table 3.8: A comparison of the existing FTS models with the proposed method.

Year	Actual Enrollments	Model [2]	Model [3]	Model [24]	Model [52] (MEPA)	Model [52] (TFA)	Model [53]	Model [54]	Results from PIs	Results from SIs
1971	13055	–	–	–	–	–	–	–	–	–
1972	13563	14000	14000	14025	15430	14230	14195	14242.0	14000	13375
1973	13867	14000	14000	14568	15430	14230	14424	14242.0	14000	13750
1974	14696	14000	14000	14568	15430	14230	14593	14242.0	14000	14250
1975	15460	15500	15500	15654	15430	15541	15589	15474.3	15500	15000
1976	15311	16000	16000	15654	15430	15541	15645	15474.3	16000	15458
1977	15603	16000	16000	15654	15430	15541	15634	15474.3	16000	15458
1978	15861	16000	16000	15654	15430	16196	16100	15474.3	16000	15750
1979	16807	16000	16000	16197	16889	16196	16188	16146.5	16000	16375
1980	16919	16813	16833	17283	16871	16196	17077	16988.3	16813	16708
1981	16388	16813	16833	17283	16871	17507	17105	16988.3	16813	16708
1982	15433	16789	16833	16197	15447	16196	16369	16146.5	16789	15875
1983	15497	16000	16000	15654	15430	15541	15643	15474.3	16000	15458
1984	15145	16000	16000	15654	15430	15541	15648	15474.3	16000	15458
1985	15163	16000	16000	15654	15430	15541	15622	15474.3	16000	15375
1986	15984	16000	16000	15654	15430	15541	15623	15474.3	16000	15375
1987	16859	16000	16000	16197	16889	16196	16231	16146.5	16000	16375
1988	18150	16813	16833	17283	16871	17507	17090	16988.3	16833	16708
1989	18970	19000	19000	18369	19333	18872	18325	19144.0	19000	18500
1990	19328	19000	19000	19454	19333	18872	19000	19144.0	19000	19125
1991	19337	19000	19000	19454	19333	18872	19000	19144.0	19000	19208
1992	18876	–	19000	–	19333	18872	19000	19144.0	19000	19208
MAPE	–	3.22%	3.11%	2.67%	2.75%	2.66%	2.66%	2.40%	3.22%	2.08%

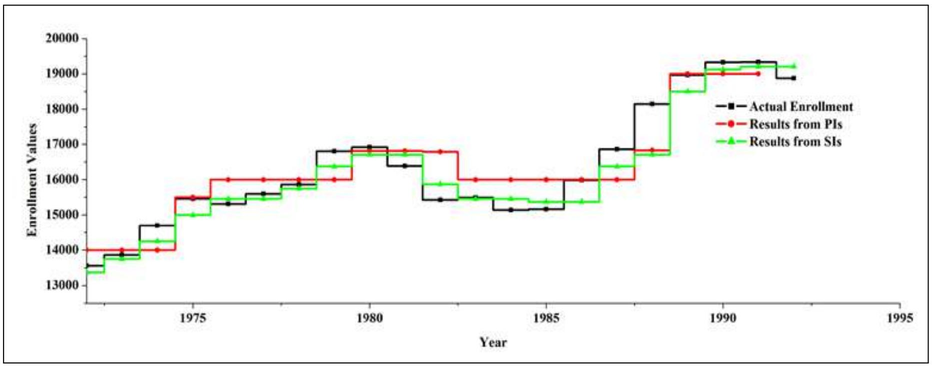

Figure 3.8: Comparison curves of actual enrollment values and forecasted enrollment values based on the PIs and SIs.

further compared by the existing FTS models [2, 3, 24, 52–54]. These comparative results are also presented in Table 3.8. We can see that the proposed algorithm produces more precise results than the existing competing models.

Graphical representation of the actual enrollment values and the forecasted enrollment values based on the PIs and SIs are presented in Fig. 3.8. Curves in Fig. 3.8 signify the effectiveness of the proposed model for forecasting enrollments based on the SIs.

Table 3.9: Statistical analysis of the forecasted enrollments based on the existing model and the proposed model.

Evaluation Criterion	Model [20]	Proposed model
$RMSE$	583.55	212.22
δ_r	0.14	0.07
R	0.95	0.99
R^2	0.89	0.98
PP	0.64	0.87
M_{ad}	393.15	110.55
TS	-0.5867	0.0629

3.7 Discussion

The proposed method efficiently fuzzified and defuzzified the historical time series data set. This method is a modification of the method presented by Song and Chissom [2]. Song and Chissom [2] only employ the primary fuzzy sets for the data fuzzification. In this study, we show that data can be fuzzified based on secondary fuzzy sets also, where each secondary fuzzy set is the subset of primary fuzzy set. The main contributions of this study are presented as follows:

1. In this study, authors develop various theorems for the Type-1 and Type-2 fuzzy sets based on intervals, *viz.*, PIs and SIs.

2. In this study, authors explain how time series data can be fuzzified based on intervals, *viz.*, PIs and SIs.

3. In this study, authors explain how fuzzified time series values can be defuzzified based on both Type-1 and Type-2 fuzzy sets.

4. Authors show that forecasting results obtained from the SIs are better than the results obtained from the PIs.

5. Authors demonstrate that forecasting accuracy of the proposed method is better than existing FTS models [2, 3, 24, 52–54].

4

Conclusion

The final chapter of the thesis concludes (a) the contributions in the domain (refer to Section 4.1), and (b) those future research works that are associated with the domain, which require further investigations by the scientific community (refer to Section 4.2).

4.1 Research Contributions and Conclusion

The main motivation for the research work is the growing need of time series forecasting in nearly all fields of natural and social sciences and engineering, especially weather and financial forecasting. However, the main problem in the time series is to choose the best methodology for fulfilling the desired goals and objectives under reasonable time. To deal these issues, an extensive literature reviews were carried out and concluded that out of the various methodologies, SC is the most appropriate and efficient technique for resolving it. The SC is the amalgamated domain of different methodologies such as fuzzy sets, ANN, EC and probabilistic computing. Therefore, among these techniques, to find which technique is most suitable for our problem, consumed lots of time. As most of data under prediction are uncertain in nature or it may represent the past behavior of the system, but unable to predict the future behavior. So to handle these uncertainties in the data, fuzzy sets theory is the most appropriate one. Song and Chissom [2] used this theory in the forecasting of time series and popularly named as “FTS forecasting model”. Motivated from this, we have extended their idea in the present thesis on resolving various domain specific problems based on time series forecasting. Hence, in this thesis, we contribute the following:

1. Two new FTS forecasting models based on the framework of Song and Chissom’s model [2],

2. Different techniques to resolve problems associated with the FTS modeling approach, such as data discretization, fuzzification, and defuzzification, and

3. Various theorems based on Type-1 and Type-2 fuzzy sets.

4.2 Future Work

In this thesis, we introduced two models based on the SC techniques (especially FTS modeling approach). However, this study deserve further studies, therefore the final Section is dedicated to confer a few significant future works closely related to our study.

- In observation of certain event, recorded time series values not only depend on previous values but also on current values. Therefore, representation of FLR in terms of high-order is a worthy idea in FTS modeling approach [52]. However, defining FLR in high-order is more complicated and computationally more expensive than first-order [73]. Therefore, many researchers employ ANN based method to define FLRs in high-order[1]. But, still there is no method suggested to find out the optimal order of the high-order FLRs. Therefore, there is a need to put more stress on development of new methods that can automatically determine the optimal order of the high-order FLRs to deal with the forecasting problems.

- The multivariate FTS models are based on the prior assumption that one-factor always dependent on another factors. Therefore, in order to fuzzify all these factors together, it is very much essential to extract the hidden information from the data, and then try to explore the membership values of each datum. To tackle this problem, many researchers use FCM technique[2]. While some researchers[3] introduce unsupervised clustering techniques that determine the membership values efficiently. In spite of all these development, there is the need for future research on developing more robust data clustering algorithm for multivariate FTS model.

- The FTS models developed so far can only predict the future values, but they don't consider the change in trend associated with the time series values in terms of upward, downward or unchanged. Thirteen years later, some researchers[4] considered these trends, and proposed trend-based FTS models. However, these trend-based models are very few in numbers, so still some attention is needed in this approach. In future more robust trend-based models can be expected from the researchers.

[1] References are: [74–77]
[2] References are: [54, 54, 78–82]
[3] References are: [7, 58, 83–87]
[4] References are: [85, 88–91]

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] Q. Song and B. S. Chissom, "Forecasting enrollments with fuzzy time series – Part I," *Fuzzy Sets and Systems*, vol. 54, no. 1, pp. 1–9, 1993.

[3] S. M. Chen, "Forecasting enrollments based on fuzzy time series," *Fuzzy Sets and Systems*, vol. 81, pp. 311–319, 1996.

[4] Q. Song and B. S. Chissom, "Fuzzy time series and its models," *Fuzzy Sets and Systems*, vol. 54, no. 1, pp. 1–9, 1993.

[5] ——, "Forecasting enrollments with fuzzy time series – Part II," *Fuzzy Sets and Systems*, vol. 62, no. 1, pp. 1–8, 1994.

[6] H.-T. Liu and M.-L. Wei, "An improved fuzzy forecasting method for seasonal time series," *Expert Systems with Applications*, vol. 37, no. 9, pp. 6310–6318, 2010.

[7] S.-M. Chen and K. Tanuwijaya, "Multivariate fuzzy forecasting based on fuzzy time series and automatic clustering techniques," *Expert Systems with Applications*, vol. 38, no. 8, pp. 10 594–10 605, 2011.

[8] Y.-L. Huang, S.-J. Horng, T.-W. Kao, R.-S. Run, J.-L. Lai, R.-J. Chen, I.-H. Kuo, and M. K. Khan, "An improved forecasting model based on the weighted fuzzy relationship matrix combined with a PSO adaptation for enrollments," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 7 (A), pp. 4027–4045, 2011.

[9] L. Wang, X. Liu, and W. Pedrycz, "Effective intervals determined by information granules to improve forecasting in fuzzy time series," *Expert Systems with Applications*, vol. 40, no. 14, pp. 5673–5679, 2013.

[10] W. Lu, X. Chen, W. Pedrycz, X. Liu, and J. Yang, "Using interval information granules to improve forecasting in fuzzy time series," *International Journal of Approximate Reasoning*, vol. 57, pp. 1–18, 2015.

[11] J. R. Hwang, S. M. Chen, and C. H. Lee, "Handling forecasting problems using fuzzy time series," *Fuzzy Sets and Systems*, vol. 100, pp. 217–228, 1998.

[12] K.-H. Huarng and T. H.-K. Yu, "Modeling fuzzy time series with multiple observations," *International Journal of Innovative Computing, Information and Control*, vol. 8, no. 10(B), pp. 7415–7426, 2012.

[13] P. Singh and B. Borah, "An efficient time series forecasting model based on fuzzy time series," *Engineering Applications of Artificial Intelligence*, vol. 26, pp. 2443–2457, 2013.

[14] T.-L. Chen, C.-H. Cheng, and H. J. Teoh, "Fuzzy time-series based on fibonacci sequence for stock price forecasting," *Physica A: Statistical Mechanics and its Applications*, vol. 380, pp. 377–390, 2007.

[15] H.-K. Yu, "Weighted fuzzy time series models for TAIEX forecasting," *Physica A: Statistical Mechanics and its Applications*, vol. 349, no. 3–4, pp. 609–624, 2005.

[16] S.-M. Chen, "Forecasting enrollments based on high-order fuzzy time series," *Cybernetics and Systems: An International Journal*, vol. 33, no. 1, pp. 1–16, 2002.

[17] T.-L. Chen, C.-H. Cheng, and H.-J. Teoh, "High-order fuzzy time-series based on multi-period adaptation model for forecasting stock markets," *Physica A: Statistical Mechanics and its Applications*, vol. 387, no. 4, pp. 876–888, 2008.

[18] S. R. Singh, "A computational method of forecasting based on high-order fuzzy time series," *Expert Systems with Applications*, vol. 36, no. 7, pp. 10 551–10 559, 2009.

[19] S.-M. Chen and K. Tanuwijaya, "Fuzzy forecasting based on high-order fuzzy logical relationships and automatic clustering techniques," *Expert Systems with Applications*, vol. 38, no. 12, pp. 15 425–15 437, 2011.

[20] S. S. Gangwar and S. Kumar, "Partitions based computational method for high-order fuzzy time series forecasting," *Expert Systems with Applications*, vol. 39, no. 15, pp. 12 158–12 164, 2012.

[21] K. Huarng, "Heuristic models of fuzzy time series for forecasting," *Fuzzy Sets and Systems*, vol. 123, pp. 369–386, 2001.

[22] S. R. Singh, "A robust method of forecasting based on fuzzy time series," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 472–484, 2007.

[23] S.-T. Li and Y.-C. Cheng, "Deterministic fuzzy time series model for forecasting enrollments," *Computers and Mathematics with Applications*, vol. 53, no. 12, pp. 1904–1920, 2007.

[24] H. S. Lee and M. T. Chou, "Fuzzy forecasting based on fuzzy time series," *International Journal of Computer Mathematics*, vol. 81, no. 7, pp. 781–789, 2004.

[25] P. Singh and B. Borah, "An effective neural network and fuzzy time series-based hybridized model to handle forecasting problems of two factors," *Knowledge and Information Systems*, vol. 38, no. 3, pp. 669–690, 2012.

[26] S. M. Chen and J. R. Hwang, "Temperature prediction using fuzzy time series," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 30, pp. 263–275, 2000.

[27] Z.-Y. Lee, "Method of bilaterally bounded to solution blasius equation using particle swarm optimization," *Applied Mathematics and Computation*, vol. 179, no. 2, pp. 779–786, 2006.

[28] L.-W. Lee, L.-H. Wang, and S.-M. Chen, "Temperature prediction and TAIFEX forecasting based on fuzzy logical relationships and genetic algorithms," *Expert Systems with Applications*, vol. 33, no. 3, pp. 539–550, 2007.

[29] ——, "Temperature prediction and TAIFEX forecasting based on high-order fuzzy logical relationships and genetic simulated annealing techniques," *Expert Systems with Applications*, vol. 34, no. 1, pp. 328–336, 2008.

[30] Y. C. Chang and S. M. Chen, "Temperature prediction based on fuzzy clustering and fuzzy rules interpolation techniques," in *Proceedings of the 2009 IEEE International Conference on Systems, Man, and Cybernetics*, San Antonio, TX, USA, October 2009, pp. 3444–3449.

[31] N.-Y. Wang and S.-M. Chen, "Temperature prediction and TAIFEX forecasting based on automatic clustering techniques and two-factors high-order fuzzy time series," *Expert Systems with Applications*, vol. 36, no. 2, Part 1, pp. 2143–2154, 2009.

[32] P. Singh and B. Borah, "Indian summer monsoon rainfall prediction using artificial neural network," *Stochastic Environmental Research and Risk Assessment*, vol. 27, no. 7, pp. 1585–1599, 2013.

[33] M.-Y. Chen, "A high-order fuzzy time series forecasting model for internet stock trading," *Future Generation Computer Systems*, vol. 37, pp. 461–467, 2014.

[34] K. Huarng, "Effective lengths of intervals to improve forecasting in fuzzy time series," *Fuzzy Sets and Systems*, vol. 123, pp. 387–394, 2001.

[35] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Sciences*, vol. 3, pp. 177–200, 1971.

[36] S. M. Chen and C. D. Chen, "Handling forecasting problems based on high-order fuzzy logical relationships," *Expert Systems with Applications*, vol. 38, no. 4, pp. 3857–3864, 2011.

[37] P.-N. Tan, M. Steinbach, and V. Kumar, *Introduction to Data Mining*. San Francisco: Pearson Education, 2006.

[38] R. Xu and D. W. II, "Survey of clustering algorithms," *IEEE Transactions on Neural Networks*, vol. 16, no. 3, pp. 645–678, 2005.

[39] L. Kurgan and K. Cios, "CAIM discretization algorithm," *IEEE Transactions on Knowledge and Data Engineering*, vol. 16, no. 2, pp. 145–153, 2004.

[40] A. K. Jain, "Data clustering: 50 years beyond k-means," *Pattern Recognition Letters*, vol. 31, no. 8, pp. 651–666, 2010.

[41] X. Wu *et al.*, "Top 10 algorithms in data mining," *Knowledge and Information Systems*, vol. 14, pp. 1–37, 2008.

[42] S. Guha, R. Rastogi, and K. Shim, "CURE: An efficient clustering algorithm for large databases," in *Proceedings of the 1998 ACM SIGMOD International Conference on Management of Data*, New York, NY, USA, 1998, pp. 73–84.

[43] T. Zhang, R. Ramakrishnan, and M. Livny, "BIRCH: An efficient data clustering method for very large databases," in *Proceedings of the 1996 ACM SIGMOD International Conference on Management of Data*, New York, NY, USA, 1996, pp. 103–114.

[44] M. Ester, H.-P. Kriegel, J. Sander, and X. Xu, "A density-based algorithm for discovering clusters in large spatial databases with noise," in *Second International Conference on Knowledge Discovery and Data Mining*, Portland, Oregon, 1996, pp. 226–231.

[45] M. Ankerst, M. M. Breunig, H.-P. Kriegel, and J. Sander, "OPTICS: Ordering points to identify the clustering structure," in *Proceedings of the 1999 ACM SIGMOD International Conference on Management of Data*, 1999, pp. 49–60.

[46] W. Wang, J. Yang, and R. R. Muntz, "STING: A statistical information grid approach to spatial data mining," in *Proceedings of the 23rd International Conference on Very Large Data Bases*, San Francisco, CA, USA, 1997, pp. 186–195.

[47] G. A. Carpenter and S. Grossberg, "Adaptive resonance theory," CAS/CNS Technical Report Series, Tech. Rep., 2009.

[48] T. Kohonen, "The self organizing maps," in *Proc. IEEE*, vol. 78, no. 9, 1990, pp. 1464–1480.

[49] R. Agrawal, J. Gehrke, D. Gunopulos, and P. Raghavan, "Automatic subspace clustering of high dimensional data," *Data Mining and Knowledge Discovery*, vol. 11, no. 1, pp. 5–33, 2005.

[50] P. Singh, "A brief review of modeling approaches based on fuzzy time series," International Journal of Machine Learning and Cybernetics, 2015.

[51] P. Singh and B. Borah, "Forecasting stock index price based on M-factors fuzzy time series and particle swarm optimization," *International Journal of Approximate Reasoning*, vol. 55, pp. 812–833, 2014.

[52] C. Cheng, J. Chang, and C. Yeh, "Entropy-based and trapezoid fuzzification-based fuzzy time series approaches for forecasting IT project cost," *Technological Forecasting and Social Change*, vol. 73, pp. 524–542, 2006.

[53] W. Qiu, X. Liu, and H. Li, "A generalized method for forecasting based on fuzzy time series," *Expert Systems with Applications*, vol. 38, no. 8, pp. 10 446–10 453, 2011.

[54] C. H. Cheng, G. W. Cheng, and J. W. Wang, "Multi-attribute fuzzy time series method based on fuzzy clustering," *Expert Systems with Applications*, vol. 34, pp. 1235–1242, 2008.

[55] H. J. Teoh *et al.*, "Fuzzy time series model based on probabilistic approach and rough set rule induction for empirical research in stock markets," *Data and Knowledge Engineering*, vol. 67, no. 1, pp. 103–117, 2008.

[56] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning – I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975.

[57] T.-Y. Chen, "A signed-distance-based approach to importance assessment and multi-criteria group decision analysis based on interval type-2 fuzzy set," *Knowledge and Information Systems*, pp. 1–39, 2012.

[58] P. Singh and B. Borah, "High-order fuzzy-neuro expert system for daily temperature forecasting," *Knowledge-Based Systems*, vol. 46, pp. 12–21, 2013.

[59] N. N. Karnik, J. M. Mendel, and Q. Liang, "Type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 6, pp. 643–658, 1999.

[60] A. Mencattini, M. Salmeri, S. Bertazzoni, R. Lojacono, E. Pasero, and W. Moniaci, "Local meteorological forecasting by type-2 fuzzy systems time series prediction," in *IEEE International Conference on Computational Intelligence for Measurement Systems and Applications*, Giardini Naxos, Italy, July 2005, pp. 20–22.

[61] A. Mencattini *et al.*, "Short term local meteorological forecasting using type-2 fuzzy systems," in *Neural Nets Lecture Notes in Computer Science*, vol. 3931, 2006, pp. 95–104.

[62] M. Almaraashi and R. John, "Tuning of type - 2 fuzzy systems by simulated annealing to predict time series," in *Proceedings of the International Conference of Computational Intelligence and Intelligent Systems*, London, July 2011.

[63] J. M. Mendel and R. I. B. John, "Type-2 fuzzy sets made simple," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 117–127, 2002.

[64] K. Huarng and H.-K. Yu, "A Type 2 fuzzy time series model for stock index forecasting," *Physica A: Statistical Mechanics and its Applications*, vol. 353, pp. 445–462, 2005.

[65] N. S. Bajestani and A. Zare, "Forecasting TAIEX using improved type 2 fuzzy time series," *Expert Systems with Applications*, vol. 38, no. 5, pp. 5816–5821, 2011.

[66] Y. Lertworaprachaya, Y. Yang, and R. John, "High-order Type-2 fuzzy time series," in *2010 International Conference of Soft Computing and Pattern Recognition*, 2010, pp. 363–368.

[67] J. M. Mendel, "Uncertainty, fuzzy logic, and signal processing," *Signal Processing*, vol. 80, no. 6, pp. 913–933, 2000.

[68] Q. Liang and J. Mendel, "Interval type-2 fuzzy logic systems: theory and design," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 5, pp. 535–550, 2000.

[69] M. Wagenknecht and K. Hartmann, "Application of fuzzy sets of type 2 to the solution of fuzzy equations systems," *Fuzzy Sets and Systems*, vol. 25, no. 2, pp. 183–190, 1988.

[70] R. I. John, "Type-2 inferencing and community transport scheduling," in *4th Euro. Congress Intelligent Techniques Soft Computing*, Aachen, Germany, Sept. 1996, pp. 1369–1372.

[71] R. John, P. R. Innocent, and M. R. Barnes, "Type 2 fuzzy sets and neuro-fuzzy clustering of radiographic tibia images," in *Proceedings of the Sixth IEEE International Conference on Fuzzy Systems*, vol. 3, Jul 1997, pp. 1375–1380.

[72] J. M. Mendel, "Advances in type-2 fuzzy sets and systems," *Information Sciences*, vol. 177, no. 1, pp. 84–110, 2007.

[73] C. H. Aladag, U. Yolcu, and E. Egrioglu, "A high order fuzzy time series forecasting model based on adaptive expectation and artificial neural networks," *Mathematics and Computers in Simulation*, vol. 81, no. 4, pp. 875–882, 2010.

[74] C. H. Aladag *et al.*, "Forecasting in high order fuzzy times series by using neural networks to define fuzzy relations," *Expert Systems with Applications*, vol. 36, no. 3, pp. 4228–4231, 2009.

[75] E. Egrioglu *et al.*, "A new hybrid approach based on SARIMA and partial high order bivariate fuzzy time series forecasting model," *Expert Systems with Applications*, vol. 36, no. 4, pp. 7424–7434, 2009.

[76] ——, "A new approach based on artificial neural networks for high order multivariate fuzzy time series," *Expert Systems with Applications*, vol. 36, no. 7, pp. 10 589–10 594, 2009.

[77] E. Egrioglu, V. Uslu, U. Yolcu, M. A. Basaran, and A. C. Hakan, "A new approach based on artificial neural networks for high order bivariate fuzzy time series," in *Applications of Soft Computing*, ser. Advances in Intelligent and Soft Computing, 2009, vol. 58, pp. 265–273.

[78] S.-T. Li, Y.-C. Cheng, and S.-Y. Lin, "A FCM-based deterministic forecasting model for fuzzy time series," *Computers and Mathematics with Applications*, vol. 56, no. 12, pp. 3052–3063, 2008.

[79] S.-M. Chen and Y.-C. Chang, "Multi-variable fuzzy forecasting based on fuzzy clustering and fuzzy rule interpolation techniques," *Information Sciences*, vol. 180, no. 24, pp. 4772–4783, 2010.

[80] S.-T. Li *et al.*, "Deterministic vector long-term forecasting for fuzzy time series," *Fuzzy Sets and Systems*, vol. 161, no. 13, pp. 1852–1870, 2010.

[81] S. T. Li *et al.*, "A vector forecasting model for fuzzy time series," *Applied Soft Computing*, vol. 11, no. 3, pp. 3125–3134, 2011.

[82] C. H. Aladag *et al.*, "A new time invariant fuzzy time series forecasting method based on particle swarm optimization," *Applied Soft Computing*, vol. 12, no. 10, pp. 3291–3299, 2012.

[83] K. Huarng and T. H.-K. Yu, "Ratio-based lengths of intervals to improve fuzzy time series forecasting," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 36, no. 2, pp. 328–340, 2006.

[84] Y.-K. Bang and C.-H. Lee, "Fuzzy time series prediction using hierarchical clustering algorithms," *Expert Systems with Applications*, vol. 38, no. 4, pp. 4312–4325, 2011.

[85] S.-M. Chen and N.-Y. Wang, "Fuzzy forecasting based on fuzzy-trend logical relationship groups," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 40, no. 5, pp. 1343–1358, 2010.

[86] E. Egrioglu *et al.*, "Fuzzy time series forecasting method based on Gustafson-Kessel fuzzy clustering," *Expert Systems with Applications*, vol. 38, no. 8, pp. 10 355–10 357, 2011.

[87] M. Bahrepour *et al.*, "An adaptive ordered fuzzy time series with application to FOREX," *Expert Systems with Applications*, vol. 38, no. 1, pp. 475–485, 2011.

[88] C. H. Cheng, T. L. Chen, and C. H. Chiang, "Trend-weighted fuzzy time-series model for TAIEX forecasting," in *Neural Information Processing*, I. King, J. Wang, L.-W. Chan, and Others, Eds., vol. 4234. Springer-Verlag, 2006, pp. 469–477.

[89] C.-H. Cheng, Y.-S. Chen, and Y.-L. Wu, "Forecasting innovation diffusion of products using trend-weighted fuzzy time-series model," *Expert Systems with Applications*, vol. 36, no. 2, Part 1, pp. 1826–1832, 2009.

[90] M. Shah, "Fuzzy based trend mapping and forecasting for time series data," *Expert Systems with Applications*, vol. 39, no. 7, pp. 6351–6358, 2012.

[91] S. M. Chen *et al.*, "Fuzzy forecasting based on two-factors second-order fuzzy-trend logical relationship groups and particle swarm optimization techniques," *IEEE Transactions on Cybernetics*, vol. 43, no. 3, pp. 1102–1117, 2013.

Communicated Articles

Detail of communicated articles is given below:

- ⋆ P. Singh and A. Samariya. A New Fuzzy Time Series Model Based on Graph Based Clustering and Improved Interval Lengths Techniques. *IEEE Transactions on Fuzzy Systems*, **Under Review**.

- ⋆ P. Singh and A. Samariya. A study on effects of length of intervals on fuzzy time series forecasting. *In 5th International Conference on Fuzzy and Neural Computing*, Hyderabad, India, IDRBT, Springer (2015), **Under Review**.

Video Presentation

Below is the link of my research video:

https://youtu.be/RTfSluQVU_s