EFFECT OF MAGNETIC FIELD AND TIDAL FORCE ON THE MOTION OF A PARTICLE ORBITING A PRESTON-POISSON BLACK HOLE

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Submitted by

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CERTIFICATE

I hereby certify that the work which has been presented in the thesis entitled **"EFFECT OF MAGNETIC FIELD AND TIDAL FORCE ON THE MOTION OF A PARTICLE ORBITING A PRESTON-POISSON BLACK HOLE"** in partial fulfillment of the requirements for the award of the degree of Master of Science in "Mathematics and Computing" to the School of Mathematics, Thapar University, Patiala is an authentic record of my own work studied under the supervision of Dr. Arvind Kumar Lal. The matter presented in this report has not been submitted in part or full to any other University or Institution for the award of any degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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(Sahil Kumar Bhardwaj)

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ABSTRACT

Black holes are the most unusual and enthralling objects of the universe. The acute conditions found in them drives our understanding of the space and time to its very limits. These astrophysical compact objects can be used as natural testing grounds for exploring the nature of matter in very powerful gravitational fields. Thus, in the universe, a very significant role is played by the black holes. By investigating how a particle orbits around a black hole, one can get a lot of information about how the black hole interacts with its surroundings. The present work focuses on examining the motion of a massless and massive charged particle orbiting a Preston-Poisson black hole which is a non-rotating black hole and has an external mechanical structure (a giant solenoid) surrounding it which contributes to the uniform magnetic field. The Preston-Poisson metric has tidal force ϵ as a parameter which characterizes the surrounding mechanical structure. This metric describes the space-time here. The presence of both the magnetic field B and the tidal force ϵ affects the motion of the particle. For a massless particle, one circular orbit is found to exist at B = $\epsilon = 0$, but as soon as either the magnetic field B or the tidal force ϵ is increased, the second orbit shows up from $r \longrightarrow \infty$ and advances towards the first circular orbit and finally, the two orbits meld into one another, which represents the last stable orbit. On further increasing either the magnetic field B or the tidal force ϵ to some values, no circular orbits are found. For a massive charged particle, we restricted our study to investigate how the effective potential U_{eff} evolves under the influence of the magnetic field B and the tidal force ϵ.

The thesis is divided into three chapters and chapter wise summary of each chapter is presented below :

Chapter 1 This chapter is introductory in nature. In this chapter, the literature available on the subject and a summary of the present work has been presented and the basic concepts

related to classical mechanics have been discussed. The derivation of the Hamilton-Jacobi equations in classical mechanics has been discussed and it is further extended to derive the Hamilton-Jacobi equations in curved space-time for both massive and massive charged particles.

Chapter 2 In this chapter, a method for discussing the geodesic characteristics of a particle around a black hole has been described.

- Through Hamilton-Jacobi equations

With the help of this method, the expressions for the effective potential U_{eff} of a massive, massless and massive charged particle have been found and the conditions for stable and unstable orbits have also been presented. The expressions for the propagation and trajectory equations for massive and massless particles have finally been found.

Chapter 3 In this chapter an introduction to Preston-Poisson black holes and the Preston-Poisson metric has been presented. For **a massless particle** orbiting around a Preston-Poisson black hole, the radii of the circular orbits and hence the number of circular orbits have been calculated. (With the help of the equation $\frac{dU_{eff}}{dr} = 0$ and putting different values of the magnetic field B and the tidal force ϵ in it, different values of r have been obtained). A graph has then been plotted between the magnetic field B and the tidal force ϵ showing the number of circular orbits at different values of B and ϵ. After that curves have also been plotted between the effective potential U_{eff} and the tidal force ϵ at B = 0 (where $\epsilon=$ 0, 0.003, 0.007, 0.009, 0.012, 0.015, 0.018, 0.02) and curves have also been plotted between the effective potential U_{eff} and the magnetic force B at $\epsilon = 0$ (where B = 0, 0.04, 0.044, 0.088, 0.132, 0.176, 0.22, 0.264). Then for **a massive charged particle** orbiting around a Preston-Poisson black hole, curves have been plotted which show how the effective potential U_{eff} changes with change in the values of the magnetic field B and the tidal force ϵ for a positively charged particle.

Contents

Chapter 1

Introduction

This chapter is introductory in nature and is organized into 4 sections. In section 1.1, the literature available on this subject is presented in brief. In section 1.2, the summary of the present work is presented. In section 1.3, basic definitions and concepts of classical mechanics have been discussed. The concluding observations have been discussed in section 1.4

1.1 Literature Review

Zhang et al.(2005) [1] and Han (2006) [2] discussed that black holes in the centres of galaxies are immersed in a strong magnetic field due to the charged matter surrounding them. The strong magnetic field in the centre of galaxies is guaranteed by toroidal currents around galactic black holes . So, an exact solution of Einstein-Maxwell describing a black hole immersed in a uniform magnetic field, known as the Ernst solution discussed by Ernst (1976) [3] was of interest. The light and the particle motion around Ernst-Schwarzschild black hole was examined by Esteban and Ramos (1988) [4] where the motion of neutral particles was considered for a more general situation of electromagnetized Kerr background. There it was shown that the release of binding energy is considerably increased because of the presence of electromagnetic field, and the binding energy for circular orbits was calculated. Yet, in a more realistic situation, the strong magnetic field in the central region near black hole is created by some surrounding matter, such as accretion disk or an active galactic nuclei. This surrounding structure exerts strong gravitational tidal force on particles moving near black holes, so that magnetic influence of the structure might be even much smaller than its gravitational influence. Therefore a more physical situation should include into consideration, the corrections to the black hole metric due-to that structure. Preston and Poisson (2006) [5] found such a corrected metric. This is the solution to the perturbative Einstein-Maxwell equations depending on three parameters: the black hole mass M, magnetic field B, and a new parameter ϵ, which characterize the surrounding structure. The solution is very accurate for $r^2B^2 \ll, \frac{M}{a} \ll 1$, and $r^2\epsilon \ll 1$, where r is the distance from the black hole, a is the length scale of the mechanical structure. Indeed, comparison with the exact Ernst solution shows that next order corrections are of order B^4, and are very small. Kanoplya (2006) [6], analyzed the motion of massless and massive particle around black holes immersed in a uniform magnetic field and surrounded by some mechanical structure, which provides the magnetic field. The space-time was described by the Preston-Poisson metric which is the generalization of the Ernst metric with a new parameter, tidal force, characterizing the surrounding structure.(The Preston-Poisson metric was reduced to the Ernst-like form, by going over to a new coordinate system). Also, the Hamilton-Jacobi equations in the equatorial plane were considered and used for the analysis of the massless particles and motion of the massive particles was described, where the binding energy of particles in circular was calculated.

1.2 Summary of the Present Work

In the present work, firstly for **a massless particle** orbiting around a Preston-Poisson black hole, the radii of the circular orbits and hence the number of circular orbits have been calculated. (With the help of the equation $\frac{dU_{eff}}{dr} = 0$ and putting different values of the magnetic field B and the tidal force ϵ in it, different values of r have been obtained). A graph has then been plotted between the magnetic field B and the tidal force ϵ showing the number of circular orbits at different values of B and ϵ. After that curves have also been plotted between the effective potential U_{eff} and the tidal force ϵ at B = 0 (where ϵ= 0, 0.003, 0.007, 0.009, 0.012, 0.015, 0.018, 0.02) and curves have also been plotted between the effective potential U_{eff} and the magnetic force B at $\epsilon = 0$ (where B = 0, 0.04, 0.044, 0.088, 0.132, 0.176, 0.22, 0.264). Secondly, for **a massive charged particle** orbiting around a Preston-Poisson black hole, curves have been plotted which show how the effective potential U_{eff} changes with change in the values of the magnetic field B and the tidal force ϵ for a positively charged particle.

1.3 Basic Preliminaries

The basics of classical mechanics following Goldstein (2003) [8] have been discussed in the following sections.

1.3.1 Generalized coordinates

The minimum number of coordinates required to define the position of a particle in a space are called generalized coordinates (also called degrees of freedom). The number of degrees of freedom is an important characteristic of any mechanical system. For N free objects, then there are 3N degrees of freedom but if there are some constraints on the objects, then for each constraint one degree of freedom is removed.For a system with N objects and n constraints, there are 3N-n degrees of freedom. For example, consider a rigid body with 3 free particles. So, the number of degrees of freedom of the system will be 9. But, if we constrain the separation between the objects to be fixed, then we have to subtract 3 degrees of freedom and therefore, our system has only $9 - 3 = 6$ degrees of freedom(for example, the 3 coordinates of the centre of mass and the three Euler angles).

1.3.2 Generalized velocities

Consider a dynamical system containing N particles with masses $m_1, m_2, \ldots, m_N$ and let the number of generalized coordinates be 'n'. Let $r_1, r_2, \ldots, r_N$ be the position vectors of these N particles and the generalized coordinates are represented by $q_1, q_2, \ldots, q_n$.Then the generalized position vectors r_i are given by:

$$
\begin{aligned}
r_1 &= r_1(q_1, q_2, \ldots, q_n) \\
r_2 &= r_1(q_1, q_2, \ldots, q_n) \\
&\vdots \\
r_N &= r_1(q_1, q_2, \ldots, q_n)
\end{aligned}
$$

In general,

$$r_i = r_i(q_1, q_2, \ldots, q_n) \qquad\qquad 1 \le i \le N$$

The velocity of the i^{th} particle is given by:

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\delta q_1}.\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}.\frac{dq_2}{dt} + \ldots + \frac{\partial r_i}{\partial q_n}.\frac{dq_n}{dt}.$$

There are N such velocity vectors characterizing the N-particle system.

$$\dot{r}_i = \frac{\partial r_i}{\partial q_1}.\dot{q}_1 + \frac{\partial r_i}{\delta q_2}.\dot{q}_2 + \ldots + \frac{\partial r_i}{\partial q_n}.\dot{q}_n$$

or,

$$v_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}.\dot{q}_j \qquad\qquad 1 \le i \le N \tag{1.3.1}$$

Invariance of dot :

$$
\begin{aligned}
\frac{\partial \dot{r}_i}{\partial \dot{q}_1} &= \frac{\partial r_i}{\partial q_1} \\
\frac{\partial \dot{r}_i}{\partial \dot{q}_2} &= \frac{\partial r_i}{\partial q_2} \\
&\vdots \\
\frac{\partial \dot{r}_i}{\partial \dot{q}_j} &= \frac{\partial r_i}{\partial q_j}
\end{aligned} \tag{1.3.2}
$$

1.3.3 Generalized Force Components

Consider that under the effect of some external force, the dynamical system is displaced to some new location. Let $\delta r_1, \delta r_2, \ldots, \delta r_N$ represent the generalized displacement of the particles and $F_1, F_2, \ldots, F_N$ represent the forces experienced by these N particles. Let δW be the virtual work done on the system.Then,

$$\begin{aligned}
\delta W &= \delta W_1 + \delta W_2 + \ldots + \delta W_N \\
&= F_1.\delta r_1 + F_2.\delta r_2 + \ldots + F_N.\delta r_N
\end{aligned}$$

$$\begin{aligned}
W = F_1\left(\frac{\partial r_1}{\partial q_1}\delta q_1 + \frac{\partial r_1}{\partial q_2}\delta q_2 + \ldots + \frac{\partial r_1}{\partial q_n}\delta q_n\right) + F_2\left(\frac{\partial r_2}{\partial q_1}\delta q_1 + \frac{\partial r_2}{\partial q_2}\delta q_2 + \ldots + \frac{\partial r_2}{\partial q_n}\delta q_n\right) + \\
\ldots + F_N\left(\frac{\partial r_N}{\partial q_1}\delta q_1 + \frac{\partial r_N}{\partial q_2}\delta q_2 + \ldots + \frac{\partial r_N}{\partial q_n}\delta q_n\right) \\
= \left(F_1\frac{\partial r_1}{\partial q_1} + F_2\frac{\partial r_2}{\partial q_1} + \ldots + F_N\frac{\partial r_N}{\partial q_1}\right)\delta q_1 + \left(F_1\frac{\partial r_1}{\partial q_2} + F_2\frac{\partial r_2}{\partial q_2} + \ldots + F_N\frac{\partial r_N}{\partial q_2}\right)\delta q_2 + \\
\ldots + \left(F_1\frac{\partial r_1}{\partial q_N} + F_2\frac{\partial r_2}{\partial q_N} + \ldots + F_N\frac{\partial r_N}{\partial q_N}\right)\delta q_N
\end{aligned}$$

therefore,

$$\delta W = Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_j\delta q_j + + \ldots + Q_n\delta q_n \tag{1.3.3}$$

where,

$$Q_j = \sum_{i=1}^{N} F_i\frac{\partial r_i}{\partial q_j}; \qquad 1 \leq j \leq n \tag{1.3.4}$$

In the expression for the virtual work done, the coefficients of generalized displacements i.e. $Q_1, Q_2, \ldots, Q_n$ are called the generalized force components.

1.3.4 Lagrange Equation

Joseph-Louis Lagrange (1736-1813) began his life by studying law but when he read a book by Halley (of comet fame), his interest shifted towards mathematics. He was an autodidact and joined as a professor in his native town of Turin at the age of 19. He moved to Italy and lived an isolated life there for the next 11 years although he kept in touch with Euler. In the year 1766, he left for Berlin as he accepted Euler's recently evacuated position. There he did his most famous work on mechanics and the calculus of variations. In the year 1788, his work was published in a book titled "Mechanique Analytique".

$$F_i = m_i\ddot{r}_i; \qquad 1 \le i \le N \tag{1.3.5}$$

$$Q_j = \sum_{i=1}^{N} F_i \frac{\partial r_i}{\partial q_j}; \qquad 1 \le j \le n \tag{1.3.6}$$

From the above equations, one gets,

$$F_i \frac{\partial r_i}{\partial q_j} = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \tag{1.3.7}$$

Now consider,

$$\begin{aligned}
\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) &= \ddot{r}_i \frac{\partial r_i}{\partial q_j} + \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \\
&= \ddot{r}_i \frac{\partial r_i}{\partial q_j} + \dot{r}_i \frac{\partial}{\partial q_j}\left(\frac{dr_i}{dt}\right) \\
\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) &= \ddot{r}_i \frac{\partial r_i}{\partial q_j} + \dot{r}_i \frac{\partial}{\partial q_j}(\dot{r}_i) \\
\ddot{r}_i \frac{\partial r_i}{\partial q_j} &= \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i \frac{\partial}{\partial q_j}(\dot{r}_i)
\end{aligned} \tag{1.3.8}$$

Substituting equation (1.3.8) in equation (1.3.7),we get,

$$\begin{aligned}
F_i \frac{\delta r_i}{\partial q_j} &= m_i\left[\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_j}\right] \\
&= m_i\left[\frac{d}{dt}\left(\dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_j}\right] && \left(\because \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}\right) \\
&= m_i\left[\frac{d}{dt}\left(\frac{1}{2}.\frac{\partial}{\partial \dot{q}_j}(\dot{r}_i.\dot{r}_i)\right) - \frac{1}{2}\frac{\partial}{\partial q_j}(\dot{r}_i.\dot{r}_i)\right] && \left(\because \vec{a}.\frac{d\vec{a}}{dx} = \frac{1}{2}\frac{d}{dx}(\vec{a}.\vec{a})\right)
\end{aligned} \tag{1.3.9}$$

Substituting equation (1.3.9) in equation (1.3.6),we get,

$$Q_j = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\right)\left(\frac{m_i\dot{r}_i^2}{2}\right) - \frac{\partial}{\partial q_j}\left(\frac{m_i\dot{r}_i^2}{2}\right)\right] \tag{1.3.10}$$

or,

$$Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}; \qquad 1 \le j \le n \tag{1.3.11}$$

1.3.5 Lagrange Equation for Conservative Forces

Conservative force - A force $\vec{F}$ is said to be conservative if the work done by the force is independent of the the path.For example, Gravitational force. If the work done does not depend on the path followed, then the force is called a Non-conservative force. For example, frictional force.

Some points about conservative forces:

1. If $\vec{F}$ is a conservative force, then $\nabla X\vec{F} = 0$ and so $\vec{F}$ can be expressed as a gradient of some scalar function ϕ i.e.,$\vec{F} = \nabla\phi$ or $\vec{F} = -(\nabla V)$, where V is called the potential energy function.

2. The relationship between workdone and potential energy is given by : dW = - dV

   It can be proved as,

   $dW = \vec{F}.\vec{ds} = -\nabla\vec{V}.\vec{dr}$

   $= -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right).\left(\hat{i}.dx + \hat{j}.dy + \hat{k}.dz\right)$

   $= -\left(\frac{\partial V}{\partial x}.dx + \frac{\partial V}{\partial y}.dy + \frac{\partial V}{\partial z}.dz\right)$

   $= -dV$

Lagrange equation for conservative forces:

We have,

$$Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}; \qquad 1 \le j \le n \tag{1.3.12}$$

$$\delta W = Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_j\delta q_j + \ldots + Q_n\delta q_n \tag{1.3.13}$$

$$F = -\nabla V \tag{1.3.14}$$

$$\delta W = -\delta V \tag{1.3.15}$$

where V is the potential energy of the system. As potential energy is always a function of displacemts. So,

$$\delta W = -\frac{\partial V}{\partial q_1}\delta q_1 - \frac{\partial V}{\partial q_2}\delta q_2 - \ldots - \frac{\partial V}{\partial q_n}\delta q_n \tag{1.3.16}$$

Using equations (1.3.13) and (1.3.16),we get,

$$Q_1 = -\frac{\partial V}{\partial q_1}, Q_2 = -\frac{\partial V}{\partial q_2}, \ldots, Q_j = -\frac{\partial V}{\partial q_j}, \ldots, Q_n = -\frac{\partial V}{\partial q_n} \tag{1.3.17}$$

Now substituting the value of Q_j from equation (1.3.17) in equation (1.3.12), we get,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} &= -\frac{\partial V}{\partial Q_j} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) &= 0 \end{aligned} \tag{1.3.18}$$

As potential is never a function of velocities, $\therefore$ $\frac{\partial V}{\partial \dot{q}_j} = 0$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T-V)\right) - \frac{\partial}{\partial q_j}(T-V) = 0 \tag{1.3.19}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad 1 \le j \le n \tag{1.3.20}$$

where, L = T- V

1.3.6 Hamiltonian or Hamilton's Equations

From Lagrange's equation we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad 1 \le i \le n$$

where, L = T- V So,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \tag{1.3.21}$$

Also, as $L(q_i, \dot{q}_i)$ is a function of q_i and $\dot{q}_i$. $\therefore$

$$\frac{dL}{dt} = \sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \tag{1.3.22}$$

Now, by substituting equation (1.3.21) in equation (1.3.23), we get,

$$\begin{aligned}
\frac{dL}{dt} &= \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{dq_i}{dt} + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \\
&= \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \\
&= \sum_{i=1}^{n} \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) \\
\frac{dL}{dt} &= \frac{d}{dt}\sum_{i=1}^{n}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) \\
0 &= \frac{d}{dt}\left(-L + \sum_{i=1}^{n}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) \\
H &= -L + \sum_{i=1}^{n}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} \\
H &= \sum_{i=1}^{n} p_i\dot{q}_i - L \qquad (1.3.23) \\
&= 2T - L \\
&= 2T - (T - V) \\
H &= T + V \qquad (1.3.24)
\end{aligned}$$

To order to show that Hamiltonian H is a function of q_i and p_i only, we have :

From equation (1.3.23), we have,

$$\begin{aligned}
H &= \sum_{i=1}^{n} p_i\dot{q}_i - L \\
dH &= \sum_{i=1}^{n} d(p_i\dot{q}_i) - dL \\
&= \sum_{i=1}^{n}(p_i d\dot{q}_i + \dot{q}_i dp_i) - dL \\
&= \sum_{i=1}^{n}(p_i d\dot{q}_i + \dot{q}_i dp_i) - \left(\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i\right) \\
&= \sum_{i=1}^{n} p_i d\dot{q}_i + \sum_{i=1}^{n}\dot{q}_i dp_i - \sum_{i=1}^{n}\frac{\partial L}{\partial q_i}dq_i - \sum_{i=1}^{n} p_i d\dot{q}_i \\
dH &= \sum_{i=1}^{n}(\dot{q}_i dp_i - \dot{p}_i dq_i) \qquad (1.3.25)
\end{aligned}$$

$\Longrightarrow$ H = H(p$_i, q_i$), i.e.,H is a function of p_i and q_i only.

Now,

$$\begin{aligned} dH &= \dot{q}_i dp_i - \dot{p}_i dq_i \\ dH &= \frac{\partial H}{\partial p_i}.dp_i + \frac{\partial H}{\partial q_i}.dq_i \end{aligned}$$

,

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i \end{aligned} \tag{1.3.26}$$

1.3.7 Hamilton's Principle

According to this principle, of all the possible paths between two points along which a dynamical system may move from one point to another within a time interval from t_1 to t_2 , the actual path followed by the system is the one which minimizes the line integral of Langrangian within the given time interval. In other words, of all arbitrary trajectories that a mechanical system can move along between two fixed points t_1 and t_2, the actual path will take place where the denite integral

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0 \tag{1.3.27}$$

becomes stationary.

1.3.8 Cyclic Coordinates

The generalized coordinates of a certain physical system that do not occur explicitly in the expression of the characteristic of that system are called cyclic coordinates of the system. In other words, if the Lagrangian of a dynamical system does not contain the coordinate q_i, then q_i is called a cyclic coordinate or, ignorable coordinate. However, Langrangian L may contain $\dot{q}_I$. For example, if the Langrangian function $L(q_i, \dot{q}_i, t)$ where q_i are the generalized coordinates, $\dot{q}_i$ are the generalized velocities and t is the time, does not contain q_i explicitly, then q_i is a cyclic coordinate. i.e.,

$$\frac{\partial L}{\partial q_i} = 0 \tag{1.3.28}$$

So, the Lagrange Equation in mechanics

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1.3.29}$$

becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{1.3.30}$$

or,

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_i} &= constant \\ p_i &= constant \end{aligned}$$

which means that the generalized momentum associated with a cyclic coordinate is constant. Note that : If a variable q_i is cyclic in $L(q_i, \dot{q}_i, t)$, then its Hamiltonian $H(p_i, q_i, t)$, also does not contain that variable.

1.3.9 Canonical Transformations

Definition (1) : A canonical transformation is a change of the canonical coordinate $(q_i, p_i, t) \rightarrow (Q_i, P_i, t)$ that preserves the form of Hamilton's equations, i.e., one can get the new Hamilton's equations arising from the changed Hamiltonian by replacing the old coordinates by the new coordinates.

Definition (2) : A transformation of the type :

$$Q_k = Q_k(p_k, q_k) \tag{1.3.31}$$

and

$$P_k = P_k(p_k, q_k) \tag{1.3.32}$$

connecting the old phase space (p_k, q_k, t) with the new phase space is said to be a canonical transformation if it satisfies the following properties :

Old	New
q_k, p_k	Q_k, P_k
L, H	L', H'
$\delta \int L dt = 0$	$\delta \int L' dt = 0$
$\frac{\partial H}{\partial p_k} = \dot{q}_k$, $\frac{\delta H}{\delta q_k} = -\dot{p}_k$	$\frac{\partial H'}{\partial P_k} = \dot{Q}_k$, $\frac{\delta H'}{\delta Q_k} = -\dot{P}_k$

There exists an operator L' in the new coordinate system which satisfies the Hamilton's principle i.e.,actual path followed in the new coordinate system is obtained by Hamilton's principle

$$\delta \int L' dt = 0 \tag{1.3.33}$$

There exists an operator H' in the new coordinate system which satisfies the Hamilton's equations i.e.,

$$\frac{\partial H'}{\partial P_k} = \dot{Q}_k \tag{1.3.34}$$

$$\frac{\delta H'}{\delta Q_k} = -\dot{P}_k \tag{1.3.35}$$

Modified Hamilton's Principle : If L is the Lagrangian of a dynamical system i.e.,

$$\delta \int_{t_1}^{t_2} L dt = 0, \tag{1.3.36}$$

then a function of the form

$$L' = L + \frac{dF}{dt} \tag{1.3.37}$$

also satisfies the above equation. The above statement can be proved as:
We start with

$$\begin{aligned} \delta \int L' dt & \\ &= \delta \int_{t_1}^{t_2} \left(L + \frac{dF}{dt} \right) \\ &= \delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \frac{dF}{dt} dt \\ &= \delta \int_{t_1}^{t_2} L dt + [\delta F]_{t_1}^{t_2} \end{aligned}$$

$$\delta \int L' dt = \delta \int_{t_1}^{t_2} L dt + \left(\frac{\partial F}{\partial q_k} \delta q_k + \frac{\partial F}{\partial p_k} \delta p_k + \frac{\partial F}{\partial t} \delta t \right)_{t_1}^{t_2} \tag{1.3.38}$$

The delta variations at the end points are zero i.e. in delta variations there is no change in q_k,p_k, and t at the end points.Therefore,

$$\delta \int_{t_1}^{t_2} L' dt = \delta \int_{t_1}^{t_2} L dt = 0 \tag{1.3.39}$$

from the above result we can conclude that the "two Lagrangians of the same system differ by $\frac{dF}{dt}$ So, substituting

$$L = \sum_{k=1}^{n} p_k \dot{q}_k - H \tag{1.3.40}$$

and

$$L' = \sum_{k=1}^{n} P_k \dot{Q}_k - H' \tag{1.3.41}$$

into

$$L - L' = \frac{dF}{dt}$$

we have

$$\left(\sum_{k=1}^{n} p_k \dot{q}_k - H \right) - \left(\sum_{k=1}^{n} P_k \dot{Q}_k - H' \right) = \frac{dF}{dt} \tag{1.3.42}$$

The LHS in the above equation has $4n + 1$ variables ($2n$ for old coordinates,$2n$ for new coordinates and one for time t but out of these only $2n+1$ variables are independent variables). Depending upon the combination of these old and new variables, four different transformations are obtained which are given below.

$$F_1(q_k, Q_k, t) \text{ type transformation } : \frac{\partial F_1}{\partial q_k} = p_k \quad , \quad \frac{\partial F_1}{\partial Q_k} = -P_k \tag{1.3.43}$$

$$F_2(q_k, P_k, t) \text{ type transformation } : \frac{\partial F_2}{\partial q_k} = p_k \quad , \quad \frac{\partial F_2}{\partial P_k} = Q_k \tag{1.3.44}$$

$$F_3(p_k, Q_k, t) \text{ type transformation } : \frac{\partial F_3}{\partial p_k} = -q_k \quad , \quad \frac{\partial F_3}{\partial Q_k} = -P_k \tag{1.3.45}$$

$$F_4(p_k, P_k, t) \text{ type transformation } : \frac{\partial F_4}{\partial p_k} = -q_k \quad , \quad \frac{\partial F_4}{\partial Q_k} = Q_k \tag{1.3.46}$$

Since F generates these four transformations, so F is called the generating function of these transformations. Equation (1.3.42) can be written as :

$$\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H = \frac{dF}{dt} \tag{1.3.47}$$

1. $F_1(q_k, Q_k, t)$ type transformation: From equation (1.3.47), we have,

$$\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H = \frac{dF_1}{dt}$$
$$= \sum_{k=1}^{n} \frac{\partial F_1}{\partial q_k}\dot{q}_k - \sum_{k=1}^{n} \frac{\partial F_1}{\partial Q_k}\dot{Q}_k + H' - H \quad (1.3.48)$$

On comparing both sides of the above written equations, we get,

$$\frac{\partial F_1}{\partial q_k} = p_k \quad , \quad \frac{\partial F_1}{\partial Q_k} = -P_k \quad (1.3.49)$$

which is F_1 type transformation.

To find the equations for the remaining three types of transformations, we have to study another transformation called Legendre Transformation which converts a function with one set of variables to another function with a conjugate set of variables. For example, consider

$$dF = \frac{\partial F}{\partial x}.dx + \frac{\partial F}{\partial y}.dy$$

where,

$$u = \frac{\partial F}{\partial x} \qquad v = \frac{\partial F}{\partial y}$$

If

$$F' = F - ux$$

then,

$$dF' = dF - udx + xdu$$
$$= (udx + vdy) - udx + xdu$$
$$dF' = -xdu + vdy$$

The transformation $F' = F - ux$, changes the base from (x, y) to (u, v). Therefore, F' is a function of u and v.

2. $F_2(q_k, P_k, t)$ type transformation : $F_1(q_k, Q_k, t) \rightarrow F_2(q_k, P_k, t)$

Putting $y = q_k$ and $x = Q_k$

we have,

$$u = \frac{\partial F_1}{\partial x} = \frac{\partial F_1}{\partial Q_k} = -P_k \quad (1.3.50)$$

Therefore,

$$\begin{aligned} F_2 &= F_1(q_k, Q_k) - \sum_{k=1}^{n} (-P_k) Q_k \\ F_2 &= F_1(q_k, Q_k) + \sum_{k=1}^{n} P_k Q_k \end{aligned}$$

or,

$$F_1 = F_2(q_k, P_k, t) - \sum_{k=1}^{n} P_k Q_k \tag{1.3.51}$$

Substituting equation (1.3.51) in equation (1.3.47), we get,

$$\begin{aligned} \sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{d}{dt}\left(F_2(q_k, P_k, t) - \sum_{k=1}^{n} P_k Q_k \right) \\ \sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{d}{dt}(F_2(q_k, P_k, t) - \sum_{k=1}^{n} P_k \dot{Q}_k - \sum_{k=1}^{n} \dot{P}_k Q_k \end{aligned}$$

So,

$$\begin{aligned} \frac{d}{dt}(F_2(q_k, P_k, t) &= \sum_{k=1}^{n} p_k \dot{q}_k + \sum_{k_1}^{n} Q_k \dot{P}_k + H - H' \\ \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k} . \dot{q}_k + \sum_{k=1}^{n} \frac{\partial F_2}{\partial P_k} . \dot{P}_k + \frac{\partial F_2}{\partial t} &= \sum_{k=1}^{n} p_k \dot{q}_k + \sum_{k_1}^{n} Q_k \dot{P}_k + H - H' \end{aligned}$$

On comparing the LHS with the RHS of the above equation, we get,

$$\frac{\partial F_2}{\partial q_k} = p_k \qquad , \qquad \frac{\partial F_2}{\partial P_k} = Q_k \tag{1.3.52}$$

which is F_2 type transformation.

3. $F_3(p_k, q_k, t)$ type transformation : $F_1(q_k, Q_k, t) \to F_2(p_k, q_k, t)$

Putting $y = Q_k$ and $x = q_k$

we have,

$$u = \frac{\partial F_1}{\partial x} = \frac{\partial F_1}{\partial q_k} = p_k \tag{1.3.53}$$

Therefore,

$$F_3 = F_1(q_k, Q_k) - \sum_{k=1}^{n} p_k q_k$$

or,

$$F_1 = F_3(p_k, Q_k, t) + \sum_{k=1}^{n} p_k q_k \tag{1.3.53}$$

$$\begin{aligned}
\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{d}{dt}\left(F_3(p_k, Q_k, t) + \sum_{k=1}^{n} p_k q_k\right) \\
\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{d}{dt}(F_3(p_k, Q_k, t) + \sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} \dot{p}_k q_k
\end{aligned}$$

So,

$$\begin{aligned}
\frac{d}{dt}(F_3(p_k, Q_k, t) &= -\sum_{k=1}^{n} q_k \dot{p}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H - H' \\
\sum_{k=1}^{n} \frac{\partial F_3}{\partial p_k} . \dot{p}_k + \sum_{k=1}^{n} \frac{\partial F_3}{\partial Q_k} . \dot{Q}_k + \frac{\partial F_3}{\partial t} &= \sum_{k=1}^{n} (-q_k) \dot{p}_k + \sum_{k=1}^{n} (-P_k) \dot{Q}_k + H - H'
\end{aligned}$$

On comparing the LHS with the RHS of the above equation, we get,

$$\frac{\partial F_3}{\partial p_k} = -q_k \qquad , \qquad \frac{\partial F_3}{\partial Q_k} = -P_k$$

which is F_3 type transformation.

4. $F_4(p_k, P_k, t)$ type transformation : $F_3(q_k, Q_k, t) \to F_4(p_k, P_k, t)$

Putting $y = p_k$ and $x = Q_k$

we have,

$$u = \frac{\partial F_3}{\partial x} = \frac{\partial F_3}{\partial Q_k} = -P_k$$

Therefore,

$$\begin{aligned}
F_4 &= F_3(q_k, Q_k) - \sum_{k=1}^{n} (-P_k) Q_k \\
F_4 &= F_3(q_k, Q_k) + \sum_{k=1}^{n} P_k Q_k \\
F_4 &= F_1 - \sum_{k=1}^{n} p_k q_k + \sum_{k=1}^{n} P_k Q_k
\end{aligned}$$

or,

$$F_1 = F_4 + \sum_{k=1}^{n} p_k q_k - \sum_{k=1}^{n} P_k Q_k$$

$$\begin{aligned}
\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{d}{dt}\left(F_4(p_k, P_k, t) + \sum_{k=1}^{n} p_k q_k - \sum_{k=1}^{n} P_k Q_k\right) \\
\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{d}{dt}(F_4(p_k, P_k, t) + \frac{d}{dt}\left(\sum_{k=1}^{n} p_k q_k - \sum_{k=1}^{n} P_k Q_k\right) \\
\sum_{k=1}^{n} p_k \dot{q}_k - \sum_{k=1}^{n} P_k \dot{Q}_k + H' - H &= \frac{dF_4}{dt} + \sum_{k=1}^{n} p_k \dot{q}_k + \sum_{k=1}^{n} \dot{p}_k q_k - \sum_{k=1}^{n} p_k \dot{Q}_k - \sum_{k=1}^{n} \dot{P}_k Q_k
\end{aligned}$$

So,

$$\begin{aligned}
\frac{d}{dt} F_4(p_k, P_k, t) &= -\sum_{k=1}^{n} \dot{p}_k q_k + \sum_{k=1}^{n} \dot{P}_k Q_k \\
\sum_{k=1}^{n} \frac{dF_4}{dp_k} + \sum_{k=1}^{n} \frac{\partial F_4}{\partial P_k} + \frac{\partial F_4}{\partial t} &= -\sum_{k=1}^{n} \dot{p}_k q_k + \sum_{k=1}^{n} \dot{P}_k Q_k + H - H'
\end{aligned}$$

On comparing the LHS with the RHS of the above equation, we get,

$$\frac{\partial F_4}{\partial p_k} = -q_k \quad , \quad \frac{\partial F_4}{\partial P_k} = Q_k$$

which is F_4 type transformation.

Hamilton-Jacobi Method : This method is based on the construction of $F_2(q_k, P_k, t)$ type of transformation in such a way that $H' = 0$. We have the Hmilton's Equations as :

$$\begin{aligned}
\frac{\partial H}{\partial p_k} = \dot{q}_k \quad &, \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k \\
\frac{\partial H'}{\partial P_k} = \dot{Q}_k = 0 \quad &, \quad \frac{\partial H'}{\partial Q_k} = -\dot{P}_k = 0
\end{aligned}$$

So Q_k = constant = β_k and P_k = constant = α_k

Now, as

$$H' - H = \frac{\partial F}{\partial t}$$

or,

$$\begin{aligned}
H' = H + \frac{\partial F}{\partial t} &= 0 \\
H(q_k, p_k, t) + \frac{\partial F_2}{\partial t} &= 0
\end{aligned}$$

Replacing $F_2(q_k, P_k, t)$ by $S(q_k, P_k, t)$ where S is an F_2 type of transformation, we get,

$$H(q_k, p_k, t) + \frac{\partial}{\partial t} S(q_k, P_k, t) = 0$$

Now, as S is an F_2 type of transformation, $\therefore$

$$\frac{\partial S}{\partial q_k} = p_k \qquad , \qquad \frac{\partial S}{\partial P_k} = Q_k$$

Finally we have,

$$H(q_k, \frac{\partial S}{\partial q_k}, t) + \frac{\partial}{\partial t} S(q_k, P_k, t) = 0$$

TheEquation represents Hamilton-Jacobi Equation.

1.3.10 Tensors

As discussed by Spiegel and Seymour (1959), physical laws should not depend on any particular coordinate systems used in describing them mathematically, if they are to be valid. A study of the consequences of this requirement leads to tensor analysis which are of great use in the general theory of relativity.

Spaces of N Dimensions : In 3 dimensional space a point is a set of three numbers, called coordinates, determined by specifying a particular coordinate system or frame of reference. For example (x, y, z), (ρ, ϕ, z), (r, θ, ϕ) are coordinates of a point in rectangular, cylindrical and spherical coordinate systems respectively. A point in N dimensional space is a set of N numbers denoted by $(x^1, x^2, x^3, \ldots, x^N)$ where 1,2,3,..., N are superscripts and not to be taken as exponents.

Coordinate Transformations : Let $(x^1, x^2, x^3, \ldots, x^N)$ and $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots, \bar{x}^N)$ be coordinates of a point in two different frames of reference. Suppose there exists N independent relations between the coordinates of the two systems having the form

$$\begin{aligned}
\bar{x}^1 &= \bar{x}^1(x^1, x^2, x^3, \ldots, x^N) \\
\bar{x}^2 &= \bar{x}^2(x^1, x^2, x^3, \ldots, x^N) \\
&\vdots \\
\bar{x}^N &= \bar{x}^N(x^1, x^2, x^3, \ldots, x^N)
\end{aligned}$$

or generally,

$$\bar{x}^k = \bar{x}^k(x^1, x^2, x^3, \ldots, x^N) \qquad k = 1, 2, \ldots, N \tag{1.3.24}$$

where it is supposed that the functions involved are single-valued, continuous, and have continuous derivatives. Then conversely to each set of coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots, \bar{x}^N)$ there will correspond a unique set $(x^1, x^2, x^3, \ldots, x^N)$ given by

$$x^k = x^k(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots, \bar{x}^N) \qquad k = 1, 2, \ldots, N \tag{1.3.25}$$

The equations or define transformation of coordinates from one frame of reference to another. **The Summation Convention** : In writing an expression like $a_1x^1 + a_2x^2 + \ldots + a_Nx^N$, we can write $\sum_{i=1}^{N} a_ix^i$. A shorter notation is to write a_ix^i , where we adopt the convention that whenever an index (subscript or superscript) is repeated in a given term we are to sum over that index from 1 to N unless otherwise specified. This is called the summation convention. Any index which is repeated in a given term, so that the summation convention applies, is called a dummy index. An index occurring only once in a given term is called a free index and can stand for any of the numbers 1, 2, ..., N such as k in equation (1.3.24) or (1.3.25), each of which represents N equations.

Contravariant and Covariant tensors : If N quantities $A^1, A^2, \ldots, A^N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}^1, \bar{A}^2, \ldots, \bar{A}^N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{x^q} A^q \qquad p = 1, 2, \ldots, N$$

which by the summation convention can simply be written as

$$\bar{A}^p = \frac{\partial \bar{x}^p}{x^q} A^q$$

they are called components of a contravariant vector or contravariant tensor of the first rank or first order.

If N quantities $A_1, A_2, \ldots, A_N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}_p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{x^q} A_q \qquad p = 1, 2, \ldots, N$$

which by the summation convention can simply be written as

$$\bar{A}_p = \frac{\partial \bar{x}^p}{x^q} A_q$$

they are called components of a covariant vector or covariant tensor of the first rank or first order. Note that a superscript is used to indicate contravariant components whereas a subscript is used to indicate covariant components; an exception occurs in the notation for coordinates. Instead of speaking of a tensor whose components are A^p or A_p we shall often refer simply to the tensor A^p or A_p.

Contravariant, Covariant and Mixed tensors : If N^2 quantities A^{qs} in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{x^q} \frac{\bar{x}^r}{x^s} A^{qs} \qquad p, r = 1, 2, \ldots, N.$$

which by the summation convention can simply be written as

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{x^q} \frac{\bar{x}^r}{x^s} A^{qs}$$

they are called contravariant components of a tensor of the second rank or rank two.

The N^2 quantities A_{qs} are called covariant components of a tensor of the second rank if

$$\bar{A}_{pr} = \frac{\partial \bar{x}^p}{x^q} \frac{\bar{x}^r}{x^s} A_{qs}$$

Similarly the N^2 quantities A^q_s are called components of a mixed tensor of the second rank if

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{x^q} \frac{x^s}{\bar{x}^r} A^q_s \qquad (1.3.19)$$

1.3.11 Hamilton-Jacobi Equations (Classical Mechanics)

1.3.11.1 Lagrangian

The action of a mechanical system is described by

$$S = \int_{t_1}^{t_2} L dt$$

Here, $S(q_1, q_2, ..., q^\mu; t)$ also called the Hamilton's Principal Function, is the functional of an actual path in n-dimensional space , q^μ is the generalized coordinate and L is the Lagrangian of the mechanical system.

$$L = (q_1, q_2, ..., q_\mu; t) = T - U$$

where, T is the kinetic energy and U is the potential energy of the mechanical system.The Langrangian determines the entire dynamics of the system.

Hamilton's Principle - According to the Hamilton principle, of all arbitrary trajectories that a mechanical system can move along between two fixed points t_1 and t_2, the actual path will take place where the denite integral

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0$$

becomes stationary.

Lagrange's Equation from Hamilton principle - To find the necessary and sufficient conditions for the action integral to be stationary :

We start with

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L dt \\
&= \int_{t_1}^{t_2} \frac{\delta L}{\delta q^\mu} \delta q^\mu dt + \int_{t_1}^{t_2} \frac{\delta L}{\delta \dot{q}^\mu} \delta \dot{q}^\mu dt \\
&= \int_{t_1}^{t_2} \frac{\delta L}{\delta q^\mu} \delta q^\mu dt + \int_{t_1}^{t_2} \frac{\delta L}{\delta \dot{q}^\mu} \frac{d}{dt}(\delta q^\mu) dt \qquad (\delta \dot{q}^\mu = \frac{d}{dt}(\delta q^\mu)) \\
&= \int_{t_1}^{t_2} \frac{\delta L}{\delta q^\mu} \delta q^\mu dt + \int_{t_1}^{t_2} \frac{\delta L}{\delta \dot{q}^\mu} d(\delta q^\mu) \\
&= \int_{t_1}^{t_2} \frac{\delta L}{\delta q^\mu} \delta q^\mu dt + \left[\frac{\delta L}{\delta \dot{q}^\mu} d(\delta q^\mu) \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{q}^\mu} \right) \delta \dot{q}^\mu dt \\
&= \left[\frac{\delta L}{\delta \dot{q}^\mu} d(\delta q^\mu) \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{\delta L}{\delta q^\mu} \delta q^\mu dt - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{q}^\mu} \right) \delta \dot{q}^\mu dt \\
&= \left[\frac{\partial L}{\partial \dot{q}^\mu} d(\delta q^\mu) \right]_{t_1}^{t_2} + \left(\int_{t_1}^{t_2} \frac{\partial L}{\partial q^\mu} \delta q^\mu - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\mu} \right) \delta \dot{q}^\mu \right) dt \qquad (1.3.14)
\end{aligned}$$

As both the ends are fixed, the first term in the above equation vanishes.So, the necessary and sufficient conditions for the action integral to be stationary are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^{\mu}}\right) - \frac{\partial L}{\partial q^{\mu}} = 0 \qquad (1.3.15)$$

where $\mu = 0, 1, 2, ..., n-1$ in an n-dimensional configuration space.

1.3.11.2 Hamiltonian

In some cases, Hamiltonian is more preferable than the Lagrangian as instead of having 'n' 2^{nd} order differential equations of motion, we have '$2n$' 1^{st} order differentials of Hamiltonian canonical equations.

We define generalized momentum as : $p_{\mu} = \frac{\partial L}{\partial \dot{q}^{\mu}}$.

For a single point in three dimensions, momentum can be expressed as components of $m\vec{v}$.

$$\dot{p}_{\mu} = \frac{d}{dt} p_{\mu} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{\mu}} = \frac{\partial L}{\partial q^{\mu}}$$

The differential of Lagrangian $L(q^{\mu}, \dot{q}_{\mu}, t)$ is :

$$\begin{aligned} dL &= \frac{\partial L}{\partial q^{\mu}} dq^{\mu} + \frac{\partial L}{\partial \dot{q}^{\mu}} d\dot{q}^{\mu} \\ &= \dot{p}_{\mu} dq^{\mu} + p_{\mu} d\dot{q}^{\mu} \\ &= \dot{p}_{\mu} dq^{\mu} + p_{\mu} d(\dot{q}^{\mu}) + \dot{q}^{\mu} d(p_{\mu}) - \dot{q}^{\mu} d(p_{\mu}) \\ dL &= \dot{p}_{\mu} dq^{\mu} + d(p_{\mu}\dot{q}^{\mu}) - \dot{q}^{\mu} d(p_{\mu}) \end{aligned}$$

or,

$$d(p_{\mu}\dot{q}^{\mu} - L) = \dot{q}^{\mu} d(p_{\mu}) - \dot{p}_{\mu} dq^{\mu}$$

From the above equation, we define the Hamiltonian $H(p_{\mu}, q^{\mu})$ as:

$$H(p_{\mu}, q^{\mu}) \equiv (p_{\mu}\dot{q}^{\mu} - L) \qquad (1.3.11)$$

Finally, we get,

$$dH = \dot{q}^{\mu} d(p_{\mu}) - \dot{p}_{\mu} d(q^{\mu})$$
$$dH = \frac{\partial H}{\partial p_{\mu}} dp_{\mu} - \frac{\partial H}{\partial q^{\mu}} dq^{\mu}$$

Thus,

$$\frac{\partial H}{\partial p_\mu} = \dot{q}^\mu \qquad , \qquad \frac{\partial H}{\partial q^\mu} = -\dot{p}_\mu \tag{1.3.10}$$

1.3.11.3 Hamilton-Jacobi Equations

Instead of expressing the dynamics of a mechanical system through either the Lagrangian or the Hamiltonian, an alternative is to convey the dynamics directly though its own action $S(q^\mu; t)$ of a system. The corresponding equations are called the Hamilton-Jacobi equations. In order to get the Hamilton-Jacobi equations, we make an assumption that, instead of having "both ends" fixed, we only have the first end fixed. Starting the derivation from equation (1.3.14), since the criterium for the action integral to be stationary is equation (1.3.15), we get,

$$\begin{aligned}
\partial S &= \left[\frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu\right]_{t_1}^{t} + \int_{t_1}^{t}\left(\frac{\partial L}{\partial q^\mu}\delta q^\mu - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu\right) dt \\
&= \left[\frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu\right]_{t_1}^{t_2} \\
&= \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu(t) - \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu(t_1)
\end{aligned}$$

As t_1 is fixed, so $\delta q^\mu(t_1)$ vanishes.Therefore, the above equation reduces to :

$$\partial S = \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu(t) = p_\mu \partial q^\mu$$

Thus, we get,

$$\frac{\partial S}{\partial q^\mu} = p_\mu$$

The action $S(q^\mu; t)$ has differential as :

$$dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial q^\mu}dq^\mu \tag{1.3.6}$$

Here, the parameter t is regarded as the time coordinate q^0, and q^μ is regarded as the spatial coordinate with $\mu = 1, 2, 3, \ldots, n-1$.

With

$$\begin{aligned}
S &= \int_t^{t_1} L dt \\
\frac{dS}{dt} &= L
\end{aligned} \tag{1.3.6}$$

On combining equations (1.3.6), (1.3.6) and (1.3.11), we have,

$$L = \frac{\partial S}{\partial r} + p_\mu \dot{q}^\mu \qquad - H = \frac{\partial S}{\partial r}$$

So, we obtain the Hamilton-Jacobi equations as :

$$\frac{\partial S}{\partial r} + H\left(q^\mu, \frac{\partial S}{\partial q^\mu}, t\right) = 0 \tag{1.3.6}$$

1.3.12 Hamilton-Jacobi Equations in Curved Space Time

1.3.12.1 Massive particles

Velocity and momentum :

Take line element of the space time as $ds = (g_{\mu\nu} dq^\mu dq^\nu)^{\frac{1}{2}}$.

The four velocity $\dot{q}^\mu$ is defined as : $\dot{q}^\mu = \frac{dq^\mu}{ds}$

For the case of (-, +, +, +) metric,

$$\begin{aligned} -ds^2 &= g_{\mu\nu} dq^\mu dq^\nu \\ &= g_{\mu\nu} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds} ds^2 \\ &= g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu ds^2 \end{aligned}$$

Therefore,

$$g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = -1 \tag{1.3.4}$$

The four momentum p^μ is defined through the four velocity $\dot{q}^\mu$ as: $\dot{q}^\mu = m\dot{q}^\mu$

On the other hand, canonical momentum p_μ, which conjugates to $\dot{q}^\mu$ is defined as :$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$

Hamilton-Jacobi Equations :

The variation of the action of a free particle can be expressed by the line element ds as :L

$$\delta S = \delta \int_{s_1}^{s_2} C ds = 0$$

where $ds = (g_{\mu\nu} dq^\mu dq^\nu)^{\frac{1}{2}}$ and C is a constant.

The action of the system is then,

$$\begin{aligned} S[q(S)] &= \int_{s_1}^{s_2} C ds \\ &= \int_{s_1}^{s_2} C\left(g_{\mu\nu} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds}\right) ds \\ &= \int_{s_1}^{s_2} C(g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu) ds \end{aligned}$$

Therefore, the Lagrangian of the system is :

$$L = C(g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}) \qquad \left(\because S[q(S)] = \int Lds\right)$$

Setting $C = \frac{1}{2}$, the Lagrangian becomes :

$$L = \frac{1}{2}mg_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}$$

Also, the canonical momentum becomes :

$$p_{\mu} = \frac{\partial L}{\partial \dot{q}^{\mu}} = \frac{1}{2}mg_{\mu\nu}\dot{q}^{\nu}$$

As we know that,

$$H = \frac{\partial L}{\partial \dot{q}^{\mu}}\dot{q}^{\mu} - L$$

First term in the above equation :

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}^{\mu}}\dot{q}^{\mu} &= \frac{\partial}{\partial \dot{q}^{\mu}}\left[\frac{1}{2}m\left(g_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}\right)\right].\dot{q}^{\mu} \\ &= mg_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} \end{aligned}$$

Therefore, H becomes,

$$\begin{aligned} H &= mg_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} - \frac{1}{2}mg_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} \\ &= \frac{1}{2}mg_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} \qquad (1.3.\text{-}4) \\ H &= -\frac{1}{2}m \qquad (1.3.\text{-}3) \end{aligned}$$

From equation (1.3.-4), we have,

$$\begin{aligned} H &= \frac{1}{2}mg_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} \\ &= \frac{1}{2}mg_{\mu\nu}\frac{^{\mu}}{m}\frac{p^{\nu}}{m} \qquad (\because p^{\mu} = m\dot{q}\mu \qquad p^{\nu} = m\dot{q}\nu) \\ &= \frac{1}{2m}\left(g_{\mu\nu}p^{\mu}p^{\nu}\right) \qquad (1.3.\text{-}4) \end{aligned}$$

From equations (1.3.-3) and (1.3.-4), we get,

$$\begin{aligned} \frac{1}{2m}\left(g_{\mu\nu}p^{\mu}p^{\nu}\right) &= -\frac{1}{2}m \\ g_{\mu\nu}p^{\mu}p^{\nu} &= -m^{2} \qquad (1.3.\text{-}4) \end{aligned}$$

Now, as $\frac{\partial S}{\partial q^\mu} = p^\mu$, therefore the equation (1.3.-4) cab be written in the Hamilton-Jacobi equations as :

$$g^{\mu\nu}\frac{\partial S}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} = -m^2 \tag{1.3.-3}$$

1.3.12.2 Massless particles

Putting $m = 0$ in equation (1.3.-3), we get, Hamilton-Jacobi equations of the masless particle as :

$$g^{\mu\nu}\frac{\partial S}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} = 0 \tag{1.3.-2}$$

1.3.12.3 Massive charged particles

The action of a charged particle in an electromagnetic field is given by :

$$S = \int_{s_1}^{s_2} C ds + \int_{s_1}^{s_2} q ds$$

Therefore, action of the system is given by :

$$S[q(s)] = \int_{s_1}^{s_2} \frac{1}{2}m\left(g_{\mu\nu}\frac{dq^\mu}{ds}\frac{dq^\nu}{ds}\right)ds - \int_{s_1}^{s_2} e g_{\mu\nu}A^\nu \dot{q}^\mu ds$$

where $C = \frac{1}{2}m$ and $q = -e$

$$S[q(s)] = \int_{s_1}^{s_2}\left(\frac{1}{2}m g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu - e g_{\mu\nu}A^\nu\dot{q}^\mu\right)ds$$

In the above equation, the first part : $\frac{1}{2}m g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu ds$ represents the action of the free particle and the second part : $\int_{s_1}^{s_2} -e g_{\mu\nu}A^\nu\dot{q}^\mu ds$ represents the interaction of the field with the charged particle. A^ν is the four potential of the electromagnetic field.

The Lagrangian can be written as :

$$L = \frac{1}{2}m g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu - e g_{\mu\nu}A_\mu\dot{q}^\mu \equiv L_1 + L_2$$

where,

$$L_1 = \frac{1}{2}m g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu \qquad L_2 = -e g_{\mu\nu}A^\nu\dot{q}^\mu$$

Therefore, the canonical momentum which conjugates to p_μ is defined as :

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = \frac{1}{2}m g_{\mu\nu}\dot{q}^\nu - e g_{\mu\nu}A^\nu \tag{1.3.-6}$$

The Hamiltonian is given by :

$$\begin{aligned}
H &= \frac{\partial L}{\partial \dot{q}^\mu}\dot{q}^\mu - L \\
&= \left(\frac{\partial L_1}{\partial \dot{q}^\mu}\dot{q}^\mu - L_1\right) + \left(\frac{\partial L_2}{\partial \dot{q}^\mu}\dot{q}^\mu - L_2\right) \\
&= \left[\frac{\partial\left(\frac{1}{2}mg_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right)}{\partial \dot{q}^\mu}\dot{q}^\mu - \frac{1}{2}mg_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right] + \frac{\partial\left(-eg_{\mu\nu}A^\nu\dot{q}^\mu\right)\dot{q}^\mu}{\partial \dot{q}^\mu} - \left(-eg_{\mu\nu}A^\nu\dot{q}^\mu\right) \\
&= \left(mg_{\mu\nu}\dot{q}^\mu\dot{q}^\nu - \frac{1}{2}mmg_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right) - eg_{\mu\nu}A^\nu\dot{q}^\mu + eg_{\mu\nu}A^\nu\dot{q}^\mu \\
&= \frac{1}{2}m\left(g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right) \\
&= -\frac{1}{2}m \qquad (1.3.\text{-}10)
\end{aligned}$$

Now, as,

$$\begin{aligned}
p_\mu &= \frac{\partial L}{\partial \dot{q}^\mu} \\
&= \frac{\partial\left(L_1 + L_2\right)}{\partial \dot{q}^\mu} \\
&= \frac{1}{2}mg_{\mu\nu}\dot{q}^\nu - eg_{\mu\nu}A^\nu
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dot{q}^\nu &= \frac{2}{mg_{\mu\nu}}\left(p_\mu + eg_{\mu\nu}A^\nu\right) \\
\dot{q}^\nu &= \frac{2}{m}\left(g^{\mu\nu}p_\mu + eA^\nu\right) \qquad (1.3.\text{-}13)
\end{aligned}$$

$$\dot{q}^\mu = \frac{2}{m}\left(g^{\mu\nu}p_\nu + eA^\mu\right) \qquad (1.3.\text{-}12)$$

As,

$$H = \frac{1}{2}m\left(g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\right)$$

Using equations (1.3.-13) and (1.3.-12), we get,

$$\begin{aligned}
-\frac{1}{2}m &= \frac{1}{2}mg_{\mu\nu}\left(\frac{4}{m^2}\right)\left(g^{\mu\nu}p_\nu + eA^\mu\right)\left(g^{\mu\nu}p_\mu + eA^\nu\right) \\
-m^2 &= g_{\mu\nu}\left(g^{\mu\nu}p_\nu + eA^\mu\right)\left(g^{\mu\nu}p_\mu + eA^\nu\right) \qquad (1.3.\text{-}13)
\end{aligned}$$

Using $\frac{\partial S}{\partial q^\mu} = p^\mu$, in the above equation, we get the Hamilton-Jacobi equations for the massive charged particle as :

$$g_{\mu\nu}\left(g^{\mu\nu}\frac{\partial S}{\partial q^\nu} + eA^\mu\right)\left(g^{\mu\nu}\frac{\partial S}{\partial q^\mu} + eA^\nu\right) = -m^2 \qquad (1.3.\text{-}12)$$

1.4 Concluding observations

The basic preliminaries discussed in the chapter will be helpful in the derivation of certain other concepts discussed in the following chapters. The Hamilton-Jacobi equations discussed here will be used in deriving the expressions for the effective potential of a massive, massless and massive charged particle.

Chapter 2

Particle motion around a black hole

In general relativity, a geodesic is a generalization of the notion of a straight line to curved spaces. Geodesics play an important role in describing the motion of particles around a black hole. They have many interesting properties such as they preserve the direction of a particle on a surface and define path of an object whose mass is relatively small to that of the black hole. In order to discuss the various features of the geodesic motion of a particle which is orbiting a black hole (including the innermost stable circular orbit, or gravitational lensing properties of a massless particle such as the distance of closest approach r_{min}) and the bending angle or time delay, we must know the geodesic characteristic of the particle moving in the black hole space time.

Keeping this in mind, this chapter has been organized into 3 sections in which we will be discussing here an approach to find the geodesic characteristics through Hamilton-Jacobi equations which will give us the expressions for the effective potential of a massive, massless and massive charged particle in section 2.1. Then we discuss the various features of the geodesic motion such as the criteria for circular orbits, propagation and trajectory equations of the particle and the bending angle and time delay in section 2.2. The concluding observations have been discussed in section 2.3.

2.1 Approach : Derivation through Hamilton-Jacobi Equations

2.1.1 Massive Particles

Following Kanoplya (2006) [6], the metric that is discussed here has a diagonal form and the signature $(-,+,+,+)$ will be used for the metric.

The momentum in a curved space-time is given by :

$$\frac{\partial S}{\partial q^{\mu}} = p_{\mu}$$

Also,

$$p_{\nu} = \frac{\partial L}{\partial \dot{q}^{\nu}} = m g_{\mu\nu} \dot{q}^{\mu}$$

or,

$$p_{\mu} = \frac{\partial L}{\partial \dot{q}^{\mu}} = m g_{\mu\nu} \dot{q}^{\nu}$$

Therefore,

$$\frac{\partial S}{\partial q^{\mu}} = m g_{\mu\nu} \dot{q}^{\mu} = p_{\mu} \tag{2.1.-2}$$

The Hamilton-Jacobi equations for the massive particles in the curved spacetime can be expressed as :

$$g^{\mu\nu} \frac{\partial S}{\partial q^{\mu}} \frac{\partial S}{\partial q^{\nu}} = -\frac{\partial S}{\partial s} = -m^2 \tag{2.1.-1}$$

The action can be written as :

$$S = S_s(s) + S_t(t) + S_r(r) + S_{\theta}(\theta) + S_{\varphi}(\varphi)$$

In the equitorial plane $\theta = \frac{\pi}{2}$, so $S_{\theta}(\theta)$ vanishes.

Therefore, the action now becomes :

$$\begin{aligned} S &= S_s(s) + S_t(t) + S_r(r) + S_{\varphi}(\varphi) \\ &= ms - Et + S_r(r) + L\varphi \end{aligned}$$

where, E and L are the integrals of motion and remain constant during motion.

1. E, the energy of the system, is a constant quantity and results from the homogeneity of time, i.e., Lagrangian of a closed system does not depend explicitly on time.

2. L, the angular momentum of the system, is a constant quantity that results from the isotropy of space, i.e., the mechanical properties of a closed system do not change with φ.

3. The plus-minus sign of E and L depends on the signature of the metric.

Now, from equation (2.1.-2), the components of momentum can be expressed as :

$$p_t = \frac{\partial S}{\partial t} = \frac{\partial S_t(t)}{\partial t} = -E = m g_{tt} \frac{dt}{ds} \tag{2.1.-3}$$

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial S_r(r)}{\partial t} = m g_{rr} \frac{dr}{ds} \tag{2.1.-2}$$

$$p_\varphi = \frac{\partial S}{\partial \varphi} = \frac{\partial S_\varphi(\varphi)}{\partial \varphi} = L = m g_{\varphi\varphi} \frac{d\varphi}{ds} \tag{2.1.-1}$$

Using equations (2.1.-3), (2.1.-2) and (2.1.-1) in equation (2.1.-1), we get,

$$g^{tt}\left(\frac{\partial S}{\partial t}\right)^2 + g^{rr}\left(\frac{\partial S}{\partial r}\right)^2 + g^{\varphi\varphi}\left(\frac{\partial S}{\partial \varphi}\right)^2 = \frac{\partial S}{\partial s} = -m^2$$

$$g^{tt}\left(\frac{\partial S_t(t)}{\partial t}\right)^2 + g^{rr}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + g^{\varphi\varphi}\left(\frac{\partial S_\varphi(\varphi)}{\partial \varphi}\right)^2 = \frac{\partial S_s(s)}{\partial s} = -m^2$$

Therefore,

$$p_r^2 = \frac{(-m^2 - g^{tt}E^2 - g^{\varphi\varphi}L^2)}{g^{rr}}$$

$$= -\frac{g^{tt}}{g^{rr}}\left(\frac{m^2}{g^{tt}} + E^2 + \frac{g^{\varphi\varphi}}{g^{tt}}L^2\right)$$

$$p_r^2 = -\frac{g^{rr}}{g^{tt}}\left(E^2 + \frac{g^{tt}}{g^{\varphi\varphi}}L^2 + m^2 g^{tt}\right) \tag{2.1.-4}$$

From equation (2.1.-11), we have,

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial S_r(r)}{\partial t} = m g_{rr} \frac{dr}{ds}$$

Therefore,

$$\frac{dr}{ds} = \frac{p_r}{m g_{rr}}$$

$$\left(\frac{dr}{ds}\right)^2 = \frac{p_r^2}{m^2 g_{rr}^2} \tag{2.1.-5}$$

Using equation (2.1.-4) in equation (2.1.-5), we get,

$$\begin{aligned}\left(\frac{dr}{ds}\right)^2 &= \frac{1}{m^2 g_{rr}^2}\left(-\frac{g^{rr}}{g^{tt}}\right)\left(E^2 + \frac{g^{tt}}{g^{\varphi\varphi}}L^2 + m^2 g^{tt}\right) \\ &= -\frac{1}{m^2 g_{rr} g_{tt}}\left(E^2 + \frac{g^{tt}}{g^{\varphi\varphi}}L^2 + m^2 g^{tt}\right) \\ &= -\frac{1}{m^2 g_{rr} g_{tt}}\left[E^2 - \left(-m^2 g_{tt} - \frac{g^{tt}}{g^{\varphi\varphi}}L^2\right)\right] \\ &= -\frac{1}{m^2 g_{rr} g_{tt}}\left[E^2 - U_{eff}^2\right]\end{aligned}$$

where,

$$U_{eff} = \sqrt{\left(-m^2 g_{tt} - \frac{g^{tt}}{g^{\varphi\varphi}}L^2\right)}$$

or,

$$U_{eff} = \sqrt{-m^2 g_{tt}\left(1 + \frac{L^2}{m^2 g_{\varphi\varphi}}\right)}$$

Here, U_{eff} is the effective potential and the metric signature used is (-, +, +, +).

Note : If the metric has signature (+, -, -, -), then the effective potential will become

$$U_{eff} = \sqrt{-m^2 g_{tt}\left(1 + \frac{L^2}{m^2 g_{\varphi\varphi}}\right)} \qquad (2.1.\text{-}10)$$

2.1.2 Massless Particles

For massless particles, $m = 0$, so equation (2.1.-10) becomes

$$\begin{aligned}U_{eff} &= \sqrt{-m^2 g_{tt}\left(1 + \frac{L^2}{m^2 g_{\varphi\varphi}}\right)} \\ &= \sqrt{-\frac{g_{tt}}{g_{\varphi\varphi}}L^2}\end{aligned} \qquad (2.1.\text{-}10)$$

2.1.3 Massive charged Particles

The action of the massive charged particle can be expressed in the form :

$$S = S_s(s) + S_t(t) + S_r(r) + S_\theta(\theta) + S_\varphi(\varphi)$$

In the equatorial plane $\theta = \frac{\pi}{2}$, so $S_\theta(\theta)$ vanishes.

Therefore, the action now becomes :

$$\begin{aligned} S &= S_s(s) + S_t(t) + S_r(r) + S_\varphi(\varphi) \\ &= -\frac{1}{2}m^2 s + Et + S_r(r) + L\varphi \end{aligned}$$

The momentum is defined according to equation (2.1.-18).The background magnetic field is axially symmetric, $A_\mu = (0,0,0,A_\varphi)$ (an object is axially symmetric if its appearance remains unchanged when rotated about an axis).

The components of the momentum are :

$$p_t = \frac{\partial S}{\partial t} = \frac{\partial S_t(t)}{\partial t} = -E = mg_{tt}\frac{dt}{ds} \tag{2.1.-12}$$

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial S_r(r)}{\partial t} = mg_{rr}\frac{dr}{ds} \tag{2.1.-11}$$

$$p_\varphi = \frac{\partial S}{\partial \varphi} = \frac{\partial S_\varphi(\varphi)}{\partial \varphi} = L + eA_\varphi = mg_{\varphi\varphi}\frac{d\varphi}{ds} \tag{2.1.-10}$$

The Hamilton-Jacobi equations for a massive charged particle are given by :

$$\begin{aligned} g_{\mu\nu}\left(g^{\mu\nu}\frac{\partial S}{\partial q^\nu} + eA^\mu\right)\left(g^{\mu\nu}\frac{\partial S}{\partial q^\mu} + eA^\nu\right) &= -m^2 \\ g_{\mu\nu}\left(\frac{\partial S}{\partial q^\mu} + eA^\mu\right)\left(\frac{\partial S}{\partial q^\nu} + eA^\nu\right) &= -m^2 \end{aligned}$$

$$g_{\mu\nu}\left[\frac{\partial S}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} + \frac{\partial S}{\partial q^\mu}eA^\nu + \frac{\partial S}{\partial q^\nu}eA^\mu + e^2A^\mu A^\nu\right] = -m^2$$

$$g_{tt}\frac{\partial S}{\partial t}^2 + \frac{\partial S}{\partial t}eA^t + \frac{\partial S}{\partial t}eA^t + e^2A^tA^t + g_{rr}\frac{\partial S}{\partial r}^2 + \frac{\partial S}{\partial r}eA^r +$$
$$\frac{\partial S}{\partial r}eA^r + e^2A^rA^r + g_{\varphi\varphi}\frac{\partial S}{\partial \varphi}^2 + \frac{\partial S}{\partial \varphi}eA^\varphi + \frac{\partial S}{\partial t}eA^\varphi + e^2A^\varphi A^\varphi = -m^2$$

Now as,

$$\begin{aligned} p_t &= \frac{\partial S}{\partial t} = \frac{\partial S_t(t)}{\partial t} = -E \\ p_r &= \frac{\partial S}{\partial r} = \frac{\partial S_r(r)}{\partial t} \\ p_\varphi &= \frac{\partial S}{\partial \varphi} = \frac{\partial S_\varphi(\varphi)}{\partial \varphi} = L + eA_\varphi \end{aligned}$$

Using the above equations in equation (2.1.3), we get,

$$g_{tt}E^2 - 2eA^tE + e^2(A^t)^2 + g_{rr}p_r^2 + 2eA^rp_r^2 + e^2(A^r)^2 + g_{\varphi\varphi}(L + eA_\varphi)^2$$
$$+2eA_\varphi(L + eA_\varphi) + e^2(A^\varphi)^2 = -m^2$$

which reduces to :

$$g_{tt}E^2 + g_{rr}p_r^2 + g_{\varphi\varphi}(L + eA_\varphi)^2 = -m^2$$

Therefore,

$$\begin{aligned} p_r^2 &= \frac{(-m^2 - g_{tt}E^2 - g_{\varphi\varphi}(L + eA_\varphi)^2)}{g_{rr}} \\ &= -\frac{g_{tt}}{g_{rr}}\left(\frac{m^2}{g_{tt}} + E^2 + \frac{g_{\varphi\varphi}}{g_{tt}}(L + eA_\varphi)^2\right) \\ &= -\frac{g^{rr}}{g^{tt}}\left(E^2 + m^2g^{tt} + \frac{g^{tt}}{g^{\varphi\varphi}}(L + eA_\varphi)^2\right) \end{aligned} \quad (2.1.\text{-}18)$$

From equation (2.1.-11), we have,

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial S_r(r)}{\partial t} = mg^{rr}\frac{dr}{ds}$$

Therefore,

$$\begin{aligned} \frac{dr}{ds} &= \frac{p_r}{mg_{rr}} \\ \left(\frac{dr}{ds}\right)^2 &= \frac{p_r^2}{m^2(g^{rr})^2} \end{aligned} \quad (2.1.\text{-}19)$$

Using equation (2.1.-18) in equation (2.1.-19), we get,

$$\begin{aligned} \left(\frac{dr}{ds}\right)^2 &= \frac{1}{m^2(g^{rr})^2}\left(-\frac{g^{rr}}{g^{tt}}\right)\left[E^2 + m^2g^{tt} + \frac{g^{tt}}{g\varphi\varphi}(L + eA_\varphi)^2\right] \\ &= -\frac{1}{m^2g^{rr}g^{tt}}\left[E^2 + \frac{g^{tt}}{g\varphi\varphi}(L + eA_\varphi)^2 + m^2g^{tt}\right] \\ &= -\frac{1}{m^2g^{rr}g^{tt}}\left(E^2 - U_eff^2\right) \end{aligned}$$

where,

$$\begin{aligned} U_{eff} &= \sqrt{-m^2g^{tt} - \frac{g^{tt}}{g^{\varphi\varphi}}(L + eA_\varphi)^2} \\ &= \sqrt{-m^2g^{tt}\left(1 + \frac{(L + eA_\varphi)^2}{m^2g\varphi\varphi}\right)} \end{aligned}$$

Here, U_{eff} representsthe effective potential and the metric signature used is (-, +, +, +).

2.2 Geodesic Motion

We have discussed the geodesic characteristics, now we discuss the features of the geodesic motion such as circular motion, propagation and geodesic trajectory equations, and calculate the distance of the minimal approach.

2.2.1 Circular Orbits

Two criteria for circular orbits :
A circular orbit of both massless and massive particles must satisfy the two conditions :

$$\frac{dr}{ds} = 0 \qquad and \qquad \frac{d^2r}{dr^2} = 0$$

First condition : $\frac{dr}{ds} = 0$
For massive particles, we have,

$$\left(\frac{dr}{ds}\right)^2 = -\frac{1}{m^2 g_{rr} g_{tt}}\left(E^2 - U_{eff}^2\right)$$

Using $\frac{dr}{ds} = 0$, we get,

$$-\frac{1}{m^2 g_{rr} g_{tt}}\left(E^2 - U_{eff}^2\right) = 0$$

$\implies$

$$E^2 = U_{eff}^2$$

and since $E \geq 0$ and $L \geq 0$

$$E = U_{eff}$$

1. When $E < U_{eff}$: the particle will be scattered by the effective potential.

2. When $E > U_{eff}$: the particle will eventually be captured by the black hole.

Dividing $E^2 = U_{eff}^2$ by L^2, we get,

$$\frac{E^2}{L^2} = \frac{U_{eff}^2}{L^2}$$

Now putting $b = \frac{L}{E}$ in the above equation we get,

$$\frac{1}{b^2} = \frac{U_{eff}^2}{L^2}$$

Thus,

$$b = \sqrt{\frac{L^2}{U_{eff}^2}} = \frac{L}{E}$$

Here, b represents the impact parameter.

1. When $b \;\; > \;\; \frac{L}{E}$: the particle will be scattered by the effective potential.

2. When $b \;\; < \;\; \frac{L}{E}$: the particle will eventually be captured by the black hole.

Second condition : $\frac{d^2r}{ds^2} = 0$

Again for the massive particles, we have,

$$\begin{aligned} \left(\frac{dr}{ds}\right)^2 &= -\frac{1}{m^2 g_{rr} g_{tt}} \left(E^2 - U_{eff}^2\right) \\ \left(\frac{dr}{ds}\right) &= \left(-\frac{1}{m^2 g_{rr} g_{tt}} \left(E^2 - U_{eff}^2\right)\right)^{\frac{1}{2}} \end{aligned} \tag{2.2.-3}$$

Now,

$$\begin{aligned} \frac{d^2r}{ds^2} &= \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d}{ds}\left(-\frac{1}{m^2 g_{rr} g_{tt}} \left(E^2 - U_{eff}^2\right)\right)^{\frac{1}{2}} \\ &= \frac{U_{eff}\left(\frac{dU_{eff}}{ds}\right)}{m^2 g_{rr} g_{tt}\left(\frac{dr}{ds}\right)} \\ &= \frac{U_{eff}}{m^2 g_{rr} g_{tt}} \frac{dU_{eff}}{dr} \end{aligned}$$

Now as $\frac{d^2r}{ds^2} = 0$, therefore, from the above equation we have,

$$\begin{aligned} \frac{U_{eff}}{m^2 g_{rr} g_{tt}} \frac{dU_{eff}}{dr} &= 0 \\ \frac{dU_{eff}}{dr} &= 0 \end{aligned}$$

$\Longrightarrow$

$$U_{eff} = constant \tag{2.2.-7}$$

The stability of the circular orbits also depends on the value of $\frac{d^2U_{eff}}{dr^2}$ as :

1. $\frac{d^2U_{eff}}{dr^2} \;\; > \;\; 0$: corresponds to a stable circular orbit.

2. $\frac{d^2U_{eff}}{dr^2} \quad = \quad 0$: corresponds to the innermost stable circular orbit.

3. $\frac{d^2U_{eff}}{dr^2} \quad < \quad 0$: corresponds to an unstable circular orbit.

Calculation of momentum and energy :

The circular orbit takes place at $E = U_{eff}$ and $\frac{dU_{eff}}{dr} = 0$. From these two relations, we can derive the energy E and the angular momentum L of the particle performing circular orbit. (First L is calculated from $\frac{dU_{eff}}{dr} = 0$ and then substituting the value of the obtained L in $E = U_{eff}$, we obtain the energy of the particle).

2.2.2 Propagation and geodesic trajectory equations

Following Kanoplya (2006) [6], the propagation and geodesic trajectory equations are :

Massive Particles :

For massive particles, we have,

$$p_t = -E = mg_{tt}\frac{dt}{ds} \implies \frac{dt}{ds} = -\frac{E}{mg_{tt}} \tag{2.2.-6}$$

$$p_\varphi = L = mg_{\varphi\varphi}\frac{d\varphi}{ds} \implies \frac{d\varphi}{ds} = \frac{L}{mg_{\varphi\varphi}} \tag{2.2.-5}$$

$$\left(\frac{dr}{ds}\right)^2 = \frac{-1}{m^2 g_{tt} g_{rr}}\left(E^2 + m^2 g_{tt} + L^2\frac{g_{tt}}{g_{\varphi\varphi}}\right) = \frac{-1}{m^2 g_{tt} g_{rr}}\left(E^2 - U_{eff}^2\right) \tag{2.2.-4}$$

Combining equations (2.2.-6) and (2.2.-4), we get,

$$\begin{aligned}
\left(\frac{dr}{dt}\right)^2 &= \frac{\left(\frac{dr}{ds}\right)^2}{\left(\frac{dt}{ds}\right)^2} \\
&= \frac{\frac{-1}{m^2 g_{tt} g_{rr}}\left(E^2 + m^2 g_{tt} + L^2\frac{g_{tt}}{g_{\varphi\varphi}}\right)}{\left(\frac{-E}{mg_{tt}}\right)^2} \\
&= -\frac{m^2 g_{tt}^2}{m^2 g_{rr} g_{tt}}\left(\frac{E^2 + m^2 g_{tt} + L^2\frac{g_{tt}}{g_{\varphi\varphi}}}{E^2}\right)
\end{aligned}$$

$$\left(\frac{dr}{dt}\right)^2 = -\frac{g_{tt}}{g_{rr}}\left(1 + \frac{g_{tt}}{g_{\varphi\varphi}}\left(\frac{L}{E}\right)^2 + \frac{m^2}{E^2}g_{tt}\right) \tag{2.2.-6}$$

Equation (2.2.-6) represents the propagation equation.

Combining equations (2.2.-5) and (2.2.-4), we get,

$$
\begin{aligned}
\left(\frac{dr}{d\varphi}\right)^2 &= \frac{\left(\frac{dr}{ds}\right)^2}{\left(\frac{d\varphi}{ds}\right)^2} \\
&= \frac{\frac{-1}{m^2 g_{tt} g_{rr}}\left(E^2 + m^2 g_{tt} + L^2 \frac{g_{tt}}{g_{\varphi\varphi}}\right)}{\left(\frac{L}{m g_{\varphi\varphi}}\right)^2} \\
&= -\frac{m^2 g_{\varphi\varphi}^2}{m^2 g_{rr} g_{tt}}\left(\frac{E^2}{L^2} + \frac{m^2}{L^2} g_{tt} + \frac{g_{tt}}{g_{\varphi\varphi}}\right) \\
&= -\frac{g_{\varphi\varphi}}{g_{rr}}\frac{g_{\varphi\varphi}}{g_{tt}}\left(\frac{E^2}{L^2} + \frac{m^2}{L^2} g_{tt} + \frac{g_{tt}}{g_{\varphi\varphi}}\right) \\
&= -\frac{g_{\varphi\varphi}}{g_{rr}}\left(\frac{E^2}{L^2}\frac{g_{\varphi\varphi}}{g_{tt}} + \frac{m^2}{L^2} g_{\varphi\varphi} + 1\right) \\
\left(\frac{dr}{d\varphi}\right)^2 &= -\frac{g_{\varphi\varphi}}{g_{rr}}\left(1 + \frac{g_{\varphi\varphi}}{g_{tt}}\frac{E^2}{L^2} + g_{\varphi\varphi}\frac{m^2}{L^2}\right) \qquad (2.2.\text{-}10)
\end{aligned}
$$

Equation (2.2.-10) represents the geodesic trajectory equation.

Masless Particles :

For the case of massless particles, $m = 0$ so the equations (2.2.-6) and (2.2.-10) become :

Propagation equation :

$$\left(\frac{dr}{dt}\right)^2 = -\frac{g_{tt}}{g_{\varphi\varphi}}\left(1 + \frac{g_{tt}}{g_{\varphi\varphi}}\left(\frac{L}{E}\right)^2\right)$$

Geodesic trajectory equation :

$$\left(\frac{dr}{dt}\right)^2 = -\frac{g_{\varphi\varphi}}{g_{rr}}\left(1 + \frac{g_{\varphi\varphi}}{g_{tt}}\left(\frac{E}{L}\right)^2\right)$$

2.2.3 Bending angle and time delay

Distance of closest approach r_{min} :

To find the bending angle and the time delay, the following two lensing equations need to be performed (for which the distance of closest approach should be known accurately). The expressions for calculating the bending angle and time delay as given by Kanoplya (2006) [6] are :

Bending angle :

$$\alpha = -\int_{r_s}^{r_{min}} \frac{d\varphi}{dr} dr + \int_{r_{min}}^{r_o} \frac{d\varphi}{dr} dr - \pi$$

Time delay :

$$\triangle t = -\int_{r_s}^{r_{min}} \frac{dt}{dr} dr + \int_{r_{min}}^{r_o} \frac{dt}{dr} dr - \frac{d_{s-o}}{cos\beta}$$

Here, r_o is the radial distance between the observer and black hole and r_s is the radial distance between the black hole and the source

When there is no gravitational lens - the particle travels in a flat spacetime with propagation angle π and $\frac{d_{s-o}}{cos\beta}$ represents the time interval for the propagating massless particle.

r_{min} is the distance of closest approach of the particle which is the closest radial distance between the source and the ray of light during its travel.The bending angle α is shown in the figure. The turning points are obtained from $\frac{dr}{dt} = 0$. ($\frac{dr}{dt} \quad < \quad 0$: means that the particle is moving towards the black hole and $\frac{dr}{dt} \quad > \quad 0$: means that the particle is moving away from the black hole).

Furthest distance r_{max} : The furthest distance, r_{max}, is the distance between the source and the furthest turning point that a particle can travel. After reaching r_{max} the particle will turn its propagation direction and start heading towards the lens again. Therefore, r_s and r_o must be well defined within the furthest turning point, otherwise the particle can change its direction before reaching the observer.

2.3 Concluding observations

The expressions for the effective potential U_{eff} derived in this chapter and the features of the geodesic motion of the particle discussed here, will be helpful in calculating the radii of the circular orbits of a massless and massive charged particle orbiting a Preston-Poisson black hole.

Chapter 3

Particle motion around a Preston - Poisson black hole under the influence of magnetic field B and a tidal force ϵ

The Preston-Poisson black hole as discussed by Preston and Poisson (2006) is a neutral, non-rotating black hole which is surrounded by an external mechanical structure (like a giant solenoid). This structure produces the magnetic field and causes the tidal gravity which deforms the configuration of the black hole.

The Preston-Poisson solution describes a black hole surrounded by a solenoid, as shown in the figure. In this model, a giant solenoid of mass M' and radius 'a' which produces a uniform magnetic field B is considered around the black hole of mass M. We will now focus only on the events inside the region which the solenoid surrounds and hence, there is a restriction on the particle black hole distance, i.e., $r < a$. As the solenoid has a mass M' and is situated at a distance 'a' from the black hole, so the tidal field it produces near the black hole is : $\epsilon \sim \frac{M'}{a^3}$. The homogeneous magnetic field creates a perturbation which is considered to be small enough such that $r^2B^2 \ll 1$, so that one can calculate it by the Black Hole Perturbation Theory.So, we have $r < a$ and $r^2B^2 \ll 1$. An assumption that the solenoid structure is positioned at the black hole's weak field region is also made so that $r^2B^2 \ll 1$.

Although the two criteria $r^2B^2 \ll 1$ and $\frac{M}{a} \ll 1$ are considered to be small but their relative sizes are not constrained.

This chapter is organized into 4 sections. The Preston-Poisson metric is discussed in section 3.1. In section 3.2, with the help of the effective potential U_{eff} and features of the geodesic motion, the radii of the circular orbits for a massless particle orbiting a Preston-Poisson black hole under the influence of magnetic field B and tidal force ϵ has been calculated. Also, the curves showing how the effective potential of a massless particle around a Preston-Poisson black hole changes with varying ϵ (at B = 0) and varying B (at $\epsilon = 0$) have been plotted. In section 3.3, the curves showing how the effective potential of a massive charged particle around a Preston-Poisson black hole varies with change in the values of the magnetic field b and the tidal force ϵ have been plotted. Finally, the concluding observations have been discussed in section 3.4.

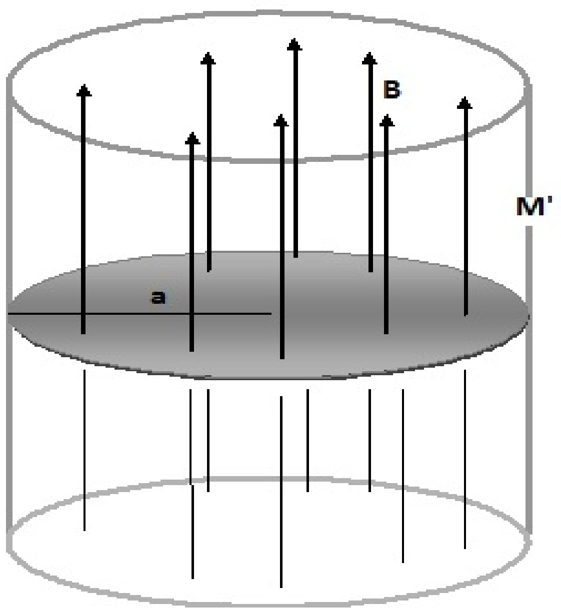

Figure 4.1 : The Preston-Poisson Black Hole. (The cylinder is considered as a giant solenoid around the black hole. The solenoid has mass M' and radius 'a' and produces a homogeneous magnetic field B)

3.1 The Preston-Poisson Metric

Following Preston and Poisson (2006) and Kanoplya (2006), the metric describing the space-time of the Preston-Poisson model, written in the light-cone gauge, is :

$$ds^2 = -g_{vv}dv^2 + 2dvdr + g_{v\theta}dvd\theta + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi} \tag{3.1.1}$$

where

$$g_{vv} = -\left(1 - \left(\frac{2M}{r}\right)\right) - \frac{1}{9}B^2r(3r - 8M) - \left(\frac{1}{9}B^2(3r^2 - 14Mr + 18M^2) + \epsilon(r - 2M)^2\right)(3cos^2\theta - 1)$$

$$g_{v\theta} = \left(\frac{2}{3}B^2(r - 3M) - 2\epsilon\right) r^2 sin\theta cos\theta \tag{3.1.2}$$

$$g_{\theta\theta} = r^2 + \left(-\frac{1}{3}B^2r^2 + B^2M^2 + \epsilon(r^2 - 2M^2)\right) r^2 sin^2\theta \tag{3.1.3}$$

$$g_{\varphi\varphi} = r^2 sin^2\theta + B62r^2\left(\left(-\frac{1}{3}r^2 - M^2\right) - \epsilon(r^2 - 2M^2)r^2\right) sin^4\theta \tag{3.1.4}$$

The metric in equation (3.1.1) is accurate through order (B^2, ϵ), whenever we have $r^2B^2 \ll 1$ and $r^2\epsilon \ll 1$. The parameter ϵ which is the tidal gravity when $r \gg M$ is a characteristic of the mechanical structure (solenoid) which contains the black hole. Now considering the motion in the equitoria plane and putting $\theta = \frac{\pi}{2}$ in equatins (3.1.1) to (3.1.4) and making the coordinate transformations :

$$v = t + \bar{r} + 2M \ln\left|\frac{\bar{r}}{2M} - 1\right|, \tag{3.1.5}$$

$$r = \bar{r}\left[1 + \frac{1}{6}B^2\bar{r}^2 sin^2\theta + O(B^4)\right] \tag{3.1.6}$$

After performing these transformations, the metric in (3.1.1) is transformed into the diagonal form :

$$ds^2 = g_{tt}dt^2 + g_{\bar{r}\bar{r}d\bar{r}^2} + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 \tag{3.1.7}$$

where the components of the metric are given as (In the metric components the $O(B^4, \epsilon^2)$ are neglected):

$$g_{\bar{r}\bar{r}} = \left(1 - \frac{2M}{\bar{r}}\right)^{-1} + \frac{(4M - 3\bar{r})\bar{r}^2}{6M - 3\bar{r}}\epsilon - \frac{2M\bar{r}^2}{6M - 3\bar{r}}B^2 \quad (3.1.8)$$

$$g_{tt} = \left(1 - \frac{2M}{\bar{r}}\right) + \frac{1}{3}(-8M^2 + 10M\bar{r} - 3\bar{r}^2)\epsilon + \frac{2}{3}M(2M - \bar{r})B^2 \quad (3.1.9)$$

$$g_{\theta\theta} = \bar{r}^2 + \frac{1}{3}\bar{r}^2(-6M^2 - 4M\bar{r} + 5\bar{r}^2)\epsilon + \frac{1}{3}\bar{r}^2(3M - \bar{r})(M + \bar{r})B^2 \quad (3.1.10)$$

$$g_{\varphi\varphi} = \bar{r}^2 - \frac{1}{3}\bar{r}^2(-6M^2 + 4M\bar{r} + 5\bar{r}^2)\epsilon - \frac{1}{3}\bar{r}^2(3M^2 - 2M\bar{r} + \bar{r}^2)B^2 \quad (3.1.11)$$

In the equitorial plane, this metric contains only diagonal elements and is therefore, much simpler for consideration.(The non-diagonals components are of the order $O(B^4, \epsilon^2)$ and are neglected).

3.2 For Massless particles

For a massless particle, orbiting around a Preston-Poisson black hole the effective potential can be written as :

$$U_{eff} = \sqrt{\left(L^2 \frac{6(B^2 - 2\epsilon)M^2 r + M(6 - B^2 r^2 + 12\epsilon r^2 - 3(r + \epsilon r^3)}{r^3(-3 - 6\epsilon M^2 + 3\epsilon r^2 + B^2(3M^2 + r^2))}\right)} \quad (3.2.1)$$

And by putting, $\frac{dU_{eff}}{dr} = 0$, we can obtain the radii of the circular orbits.

$$\frac{dU_{eff}}{dr} = \left(\frac{-2B^2 r + 6(B^2 - 2\epsilon) + 24r\epsilon - 3(1 + 3r^2\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)} - \frac{(2B^2 r + 6r\epsilon)(6 - B^2 r^2 + 6r(B^2 - 2\epsilon)) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)^2} - \frac{3(6 - B^2 r^2 + 6r(B^2 - 2\epsilon)) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^4(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)}\right) \Bigg/ \left(2\sqrt{\frac{3(6 - B^2 r^2 + 6r(B^2 - 2\epsilon)) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)}}\right)$$

On solving the above equation for r and putting different values of the magnetic field B and the tidal force ϵ, values of the radii of circular orbits and hence the number of circular orbits were obtained. These values are given in tables 3.1 and 3.2

On the basis of tables 3.1 and 3.2, a graph was plotted between B (on y-axis) and ϵ (on x-axis) showing the values of B and ϵ where the circular orbits were obtained. Refer figure 3.1.

Table 3.1: Different values of the radii of the circular orbits of a massless particle orbiting a Preston-Poisson black hole at different values of the magnetic field B and the tidal force ϵ.

B	ϵ	radius of the particle orbiting the black hole (r)	r (positive real value)
0	0	3	3
0.05	0	-1594.38, -24.7497, 3.01906, 24.1141	3.01906, 24.1141
0.10	0	-394.409, -12.4510, 3.08047, 11.7799	3.08047, 11.7799
0.15	0	-172.231,-8.32152, 3.20265, 7.57185	3.20265, 7.57185
0.20	0	-94.5155, -6.23603, 3.4683, 5.28318	3.4683, 5.28318
0.25	0	-58.5980, -4.96877, 3.78340 - 0.723229 i, 3.78340 + 0.723229 i	-
0	0.005	-9.37641, 1.81003 - 21.8991 i, 1.81003 + 21.8991 i, 3.08847, 8.66789	3.08847, 8.66789
0.05	0.005	-8.76922, 1.67204 - 21.6347 i, 1.67204 + 21.6347 i, 3.11304, 8.06208	3.11304, 8.06208
0.10	0.005	-7.48364, 1.27318 - 21.0939 i, 1.27318 + 21.0939 i, 3.19567, 6.74161	3.19567, 6.74161
0.15	0.005	-6.21189, 0.623903 - 20.5567 i, 0.623903 + 20.5567 i, 3.38575, 5.32834	3.38575, 5.32834
0.20	0.005	-5.17869, -0.279911 - 20.0547 i, -0.279911 + 20.0547 i, 3.86926 - 0.517136 i, 3.86926 + 0.517136 i	-
0.25	0.005	-4.37663, -1.44269 - 19.5215 i, -1.44269 + 19.5215 i, 3.50601 - 0.911275 i, 3.50601 + 0.911275 i	-
0	0.01	-6.70608, 1.80939 - 15.4321 i, 1.80939 + 15.4321 i, 3.22296, 5.86434	3.22296, 5.86434
0.05	0.01	-6.47407, 1.73777 - 15.3228 i, 1.73777 + 15.3228 i, 3.26067, 5.61286	3.26067, 5.61286
0.1	0.01	-5.90104, 1.52819 - 15.0538 i, 1.52819 + 15.0538 i, 3.40537, 4.93929	3.40537, 4.93929
0.15	0.01	-5.21082, 1.18761 - 14.7238 i, 1.18761 + 14.7238 i, 3.85529 - 0.419753 i, 3.85529 + 0.419753 i	-
0.2	0.01	-4.55023, 0.716998 - 14.3816 i, 0.716998 + 14.3816 i, 3.55812 - 0.833527 i, 3.55812 + 0.833527 i	-
0.25	0.01	-3.973, 0.11404 - 14.0274 i, 0.11404 + 14.0274 i, 3.30996 - 0.979977 i, 3.30996 + 0.979977 i	-

Table 3.2: Different values of the radii of the circular orbits of a massless particle orbiting a Preston-Poisson black hole at different values of the magnetic field B and the tidal force ϵ.

B	ϵ	radius of the particle orbiting the black hole (r)	r (positive real value)
0	0.015	-5.52418, 1.80874 - 12.5569 i, 1.80874 + 12.5569 i, 3.52609, 4.3806	3.52609, 4.38060
0.05	0.015	-5.39249, 1.75945 - 12.4906 i, 1.75945 + 12.4906 i, 3.68266, 4.1076	3.68266, 4.10760
0.1	0.015	-5.04726, 1.61417 - 12.3159 i, 1.61417 + 12.3159 i, 3.7428 - 0.529628 i, 3.7428 + 0.529628 i	-
0.15	0.015	-4.59302, 1.37725 - 12.0802 i, 1.37725 + 12.0802 i, 3.54426 - 0.792525 i, 3.54426 + 0.792525 i	-
0.2	0.015	-4.11874, 1.05044 - 11.8174 i, 1.05044 + 11.8174 i, 3.34227 - 0.939994 i, 3.34227 + 0.939994 i	-
0.25	0.015	-3.67319, 0.632842 - 11.5383 i, 0.632842 + 11.5383 i, 3.16209 - 1.00532 i, 3.16209 + 1.00532 i	-
0	0.02	-4.82076, 1.80809 - 10.8369 i, 1.80809 + 10.8369 i, 3.60229 - 0.654178 i, 3.60229 + 0.654178 i	-
0.05	0.02	-4.73248, 1.76999 - 10.7898 i, 1.76999 + 10.7898 i, 3.565 - 0.702681 i, 3.565 + 0.702681 i	-
0.1	0.02	-4.49361, 1.65719 - 10.6611 i, 1.65719 + 10.6611 i, 3.46461 - 0.809269 i, 3.46461 + 0.809269 i	-
0.15	0.02	-4.16300, 1.47258 - 10.4780 i, 1.47258 + 10.4780 i, 3.32767 - 0.913461 i, 3.32767 + 0.913461 i	-
0.2	0.02	-3.79844, 1.21774 - 10.2641 i, 1.21774 + 10.2641 i, 3.18148 - 0.983519 i, 3.18148 + 0.983519 i	-
0.25	0.02	-3.43863, 0.892474 - 10.0308 i, 0.892474 + 10.0308 i, 3.04559 - 1.01066 i, 3.04559 + 1.01066 i	-

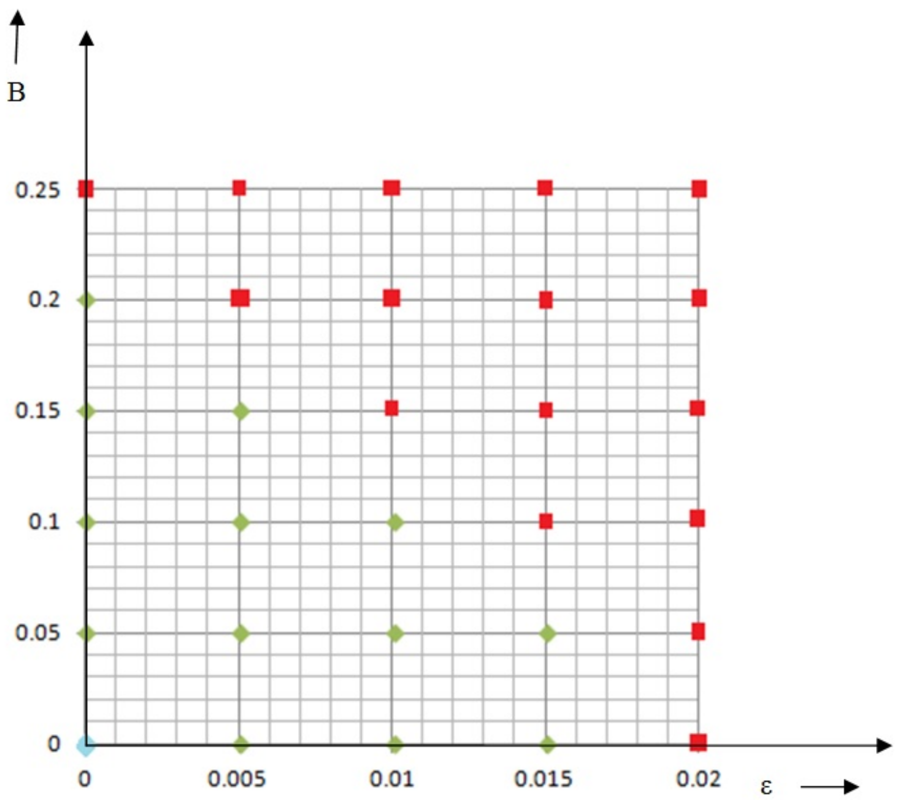

Figure 3.1: The roots of the equation $\frac{dU_{eff}}{dr} = 0$ for a massless particle. Blue dot represents that one positive root, i.e. one circular orbit is obtained, green dots indicate that 2 positive roots, i.e. two circular orbits are obtained and the red dots indicate that no root is obtained, i.e. no circular orbits.

From figure 3.1, we conclude that :

1. When B = ϵ = 0, one positive value of r is obtained which shows that one circular orbit is obtained. Here, the metric reduces to the Schwarzschild and so the result is an unstable circular orbit.

2. On either increasing the value of B or ϵ, the curve tail starts to twist upwards which represents a stable circular orbit.

3. On further increasing the value of either B or ϵ, the curve keeps bending more upwards.The last stable orbit occurs at the boundary of area of the green and red dots regions.

4. If we continue to increase either B or ϵ, the curve continues to twist upwards and no roots (no circular orbits) are obtained then.

Figure 5.2 shows how the effective potential of the particle around a Preston-Poisson black hole changes with varying tidal force ϵ keeping the magnetic field B = 0 and figure 5.3 shows how the effective potential of the particle around a Preston-Poisson black hole changes with varying magnetic field B keeping the tidal force $\epsilon = 0$.

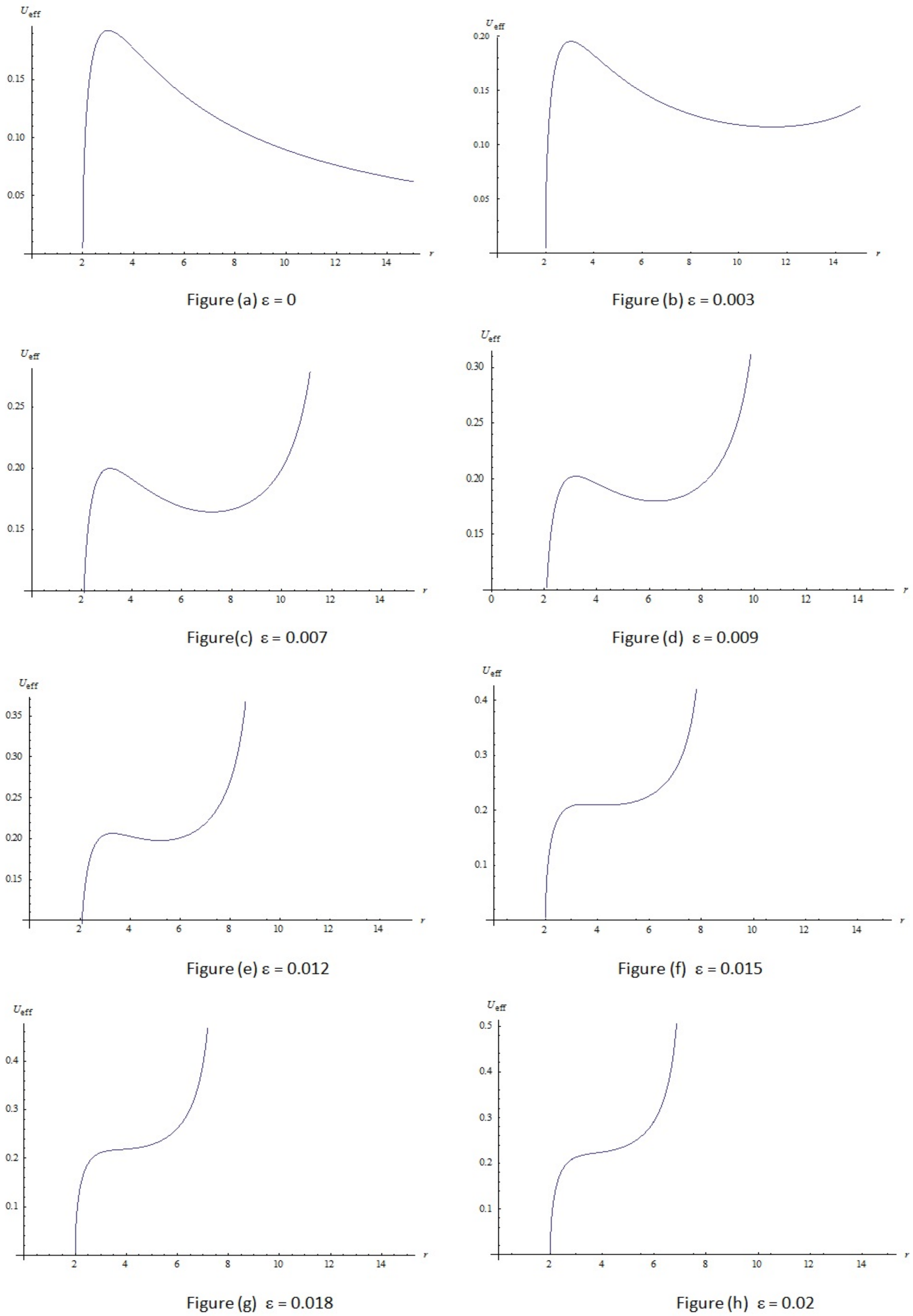

Figure 5.2 : The effective potential U_{eff} with varying ϵ at B $= 0$. Starting from figure (a) to figure (h), $\epsilon = 0$, 0.003, 0.007, 0.009, 0.012, 0.015, 0.018 and 0.02.

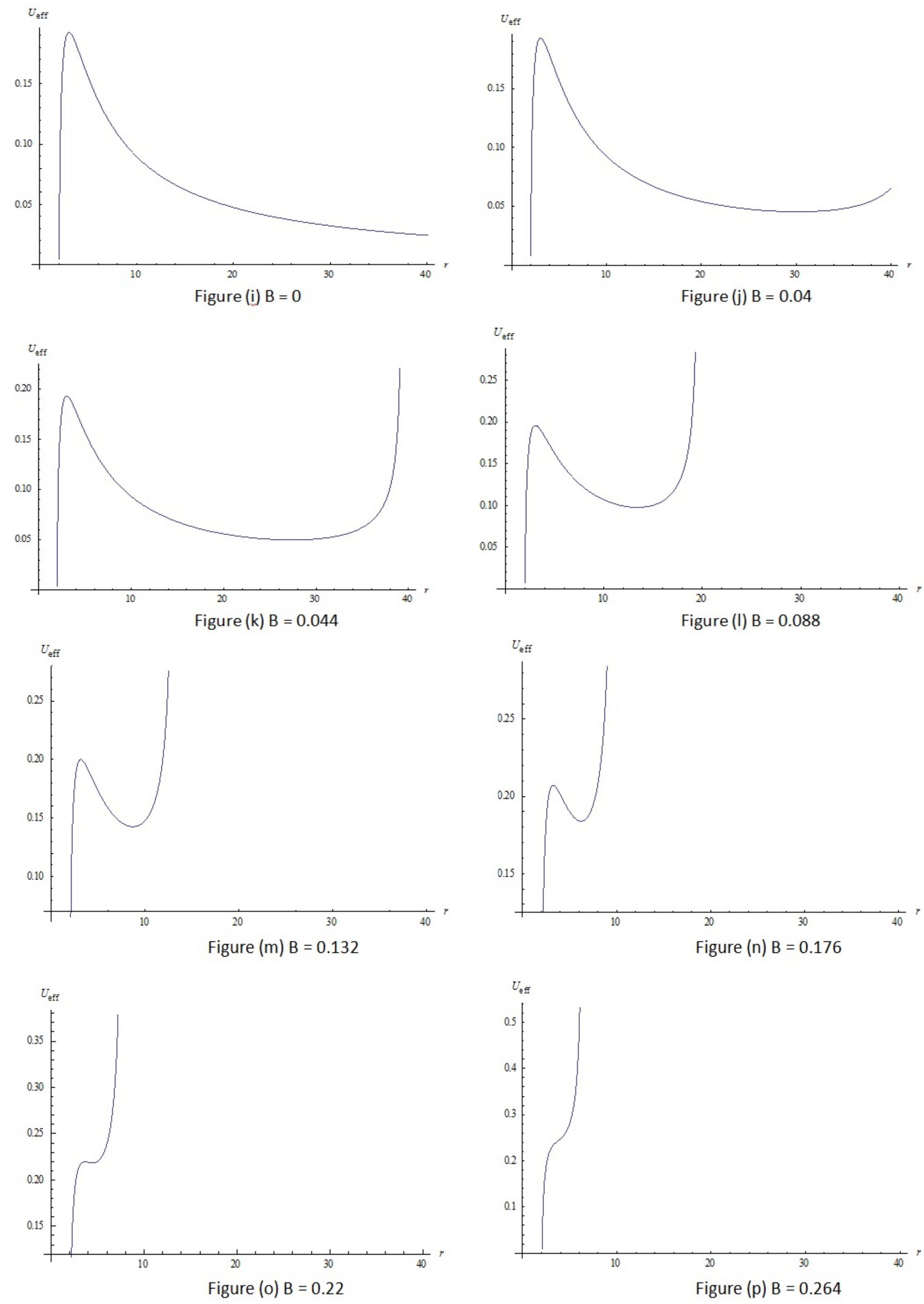

Figure 5.3 : The effective potential U_{eff} with varying B at $\epsilon = 0$. Starting from figure (i) to figure (p), B = 0, 0.04, 0.044, 0.088, 0.132, 0.176, 0.22 and 0.264.

From figures 5.2 and 5.3, it can be seen that as either the tidal force ϵ (at B = 0) or the magnetic field B (at $\epsilon = 0$) is increased, the effective potential of the particle increases.

3.3 For Massive Charged particle

For a massive charged particle, orbiting around a Preston-Poisson black hole the effective potentialcan be written as :

$$U_{eff} = -mc\sqrt{-1 + \frac{2B^2M(3M-r)}{3} + \frac{2M}{r} - \epsilon(-2M+r)^2} \sqrt{-1 + \frac{3(2Bqr^2 + cL(4+B^2r^2))^2}{c^4M^2r^2(4+B^2r^2)^2(-3-6\epsilon M^2+3\epsilon r^2+B^2(3M^2+r^2))}} \tag{3.3.1}$$

And by putting, $\frac{dU_{eff}}{dr} = 0$, we can obtain the radii of the circular orbits.

Figures 5.4 and 5.5 show how the effective potential U_{eff} changes with the change in the values of magnetic field B and tidal force ϵ for a positively charged particle.

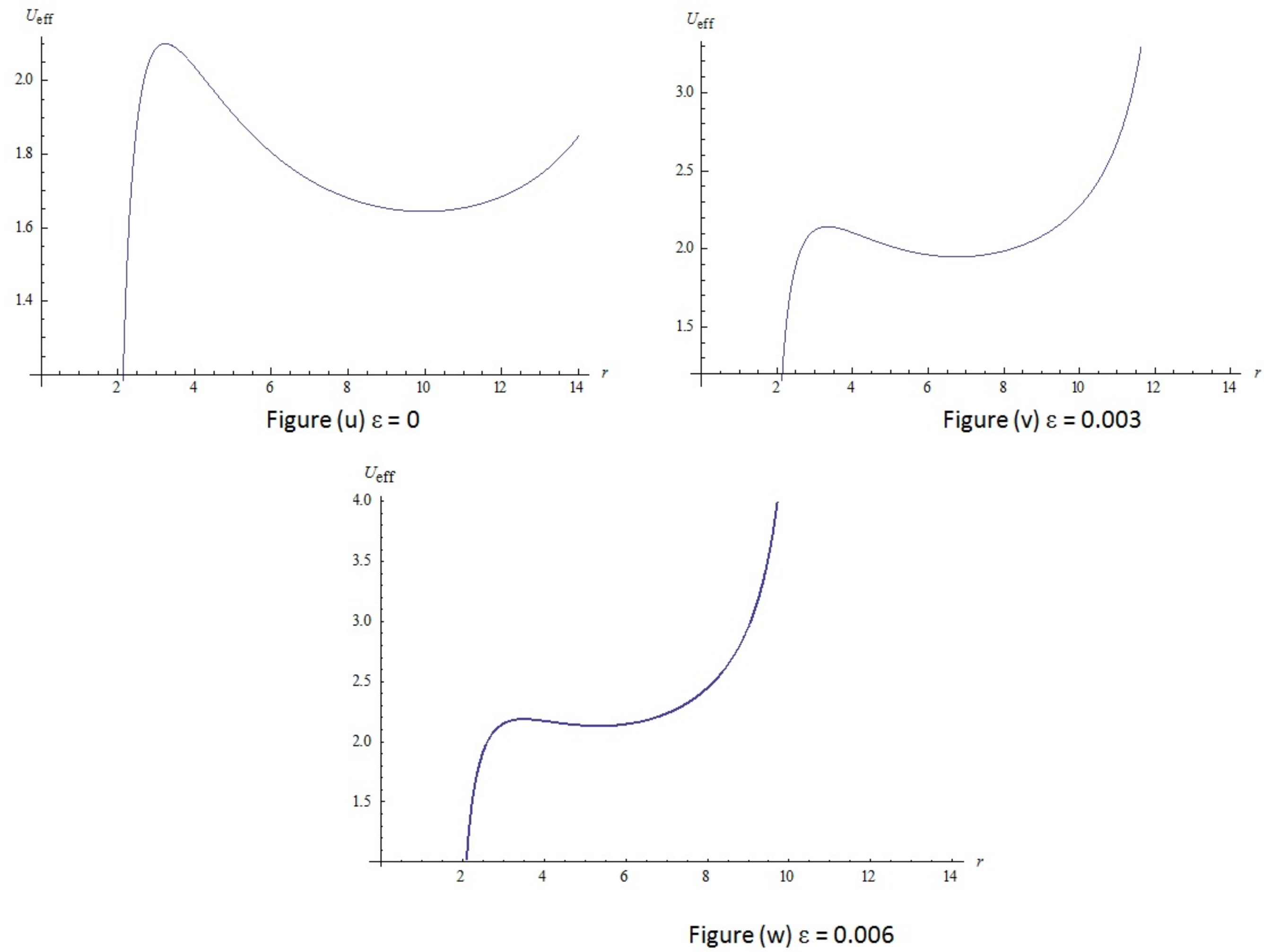

Figure 5.4 : The effective potential U_{eff} curves for q = 0.5 and changing ϵ. (ϵ= 0, 0.003, 0.006) and B = 0.1 $and L = 10, where$ m = M = c = 1 is considered .

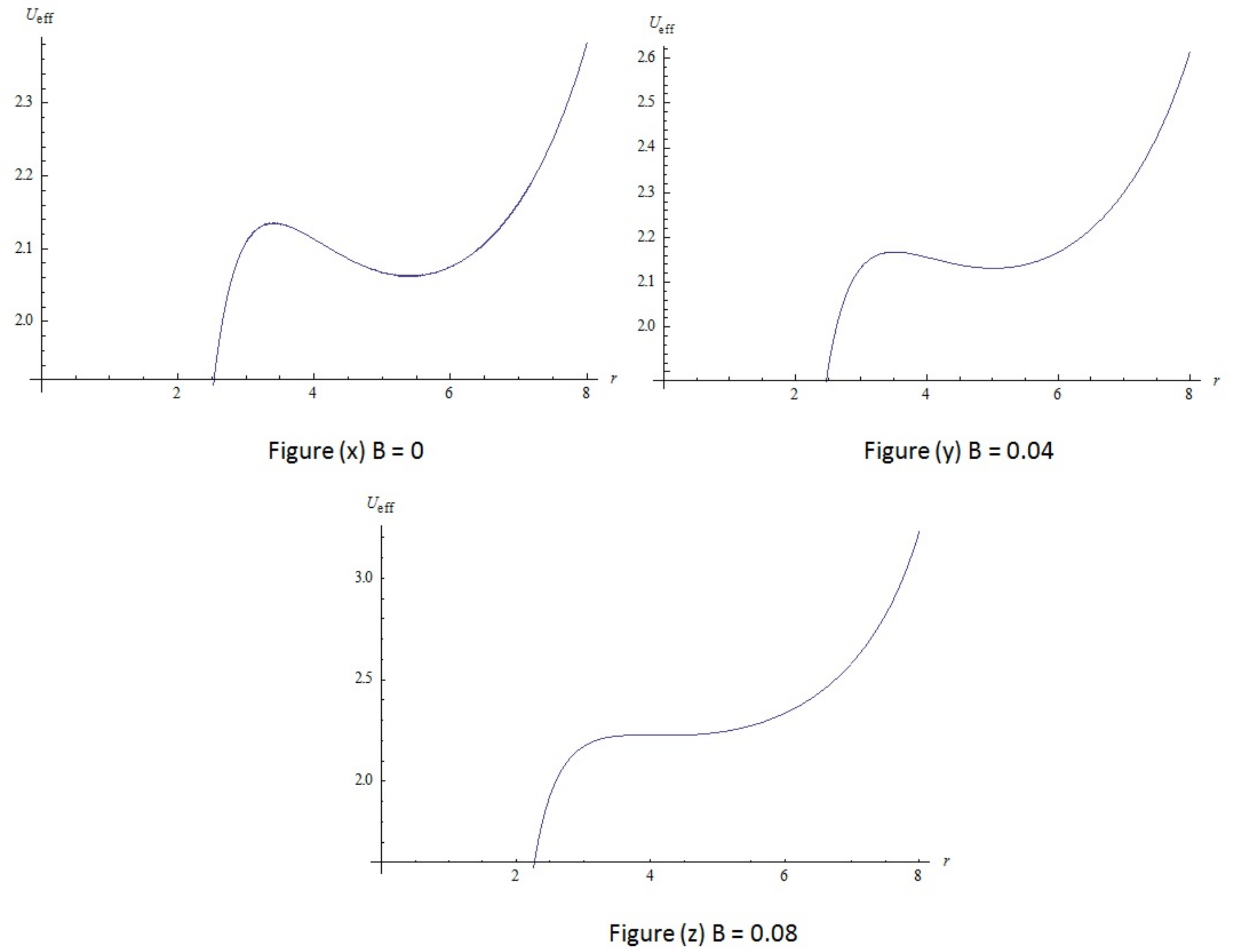

Figure 5.5 : The effective potential U_{eff} curves for q = 0.5 and changing B. (B = 0, 0.04, 0.08) and ϵ = 0.01 and L = 10, where m = M = c = 1 is considered.

From figures 5.2 and 5.3, it can be seen that on increasing either the tidal force ϵ with $B \neq 0$ or the magnetic field B with $\epsilon \neq 0$, the effective potential of the particle increases.

3.4 Concluding observations

The motion of massless and massive charged particle was considered near the Preston-Poisson Black hole. For massless particles, the influence of the magnetic field B and the tidal force ϵ on the radii of circular orbits was studied by examining how the effective potential U_{eff} evolves when either the magnetic field B or the tidal force ϵ is changed.

1. It was found that at B = $\epsilon = 0$, there exists a circular orbit.

2. When either the magnetic field B or the tidal force ϵ is increased, the second orbit emerges from $r \longrightarrow \infty$ which approaches the first circular orbit and finally they merge together into one.

3. Further increasing the values of the magnetic field B or the tidal force ϵ to some values, no circular orbits were found.

For massive charged particles, it was studied how the effective potential U_{eff} emerges under the effect of the magnetic field B and the tidal force ϵ. It was found that, on increasing either the tidal force ϵ with $B \neq 0$ or the magnetic field B with $\epsilon \neq 0$, the effective potential U_{eff} increases which means that because of the presence of ϵ (the BH configuration distortion) or the interaction of q with B, the particle does not fall into the black hole.

Conclusion

The motion of massless and massive charged particle was considered near the Preston-Poisson Black hole.

Massless Particles :

The influence of the magnetic field B and the tidal force ϵ on the radii of circular orbits was studied by examining how the effective potential U_{eff} evolves when either the magnetic field B or the tidal force ϵ is changed.

1. It was found that at B $= \epsilon = 0$, there exists a circular orbit.

2. When either the magnetic field B or the tidal force ϵ is increased, the second orbit emerges from $r \longrightarrow \infty$ which advents the first circular orbit and finally they meld together into one.

3. Further increasing the values of the magnetic field B or the tidal force ϵ to some values, no circular orbits were found.

Massive charged particles :

It was studied how the effective potential U_{eff} emerges under the effect of the magnetic field B and the tidal force ϵ. It was found that, on increasing either the tidal force ϵ with $B \neq 0$ or the magnetic field B with $\epsilon \neq 0$, the effective potential U_{eff} increases which means that because of the presence of ϵ (the BH configuration distortion) or the interaction of q with B, the particle does not fall into the black hole.

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Appendix A

A.1 Massless particle

A.1.1 Calculations for Figure 5.1 (the radii of the circular orbits)

For a massless particle, orbiting around a Preston-Poisson black hole the effective potential can be written as :

$$U_{eff} = \sqrt{\left(L^2 \frac{6(B^2 - 2\epsilon)M^2 r + M(6 - B^2 r^2 + 12\epsilon r^2 - 3(r + \epsilon r^3)}{r^3(-3 - 6\epsilon M^2 + 3\epsilon r^2 + B^2(3M^2 + r^2))}\right)}$$

And by putting, $\frac{dU_{eff}}{dr} = 0$, we can obtain the radii of the circular orbits.

$$\begin{aligned}\frac{dU_{eff}}{dr} = &\left(\frac{-2B^2 r + 6(B^2 - 2\epsilon) + 24r\epsilon - 3(1 + 3r^2\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)}\right.\\ &- \frac{(2B^2 r + 6r\epsilon)(6 - B^2 r^2 + 6r(B^2 - 2\epsilon)) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)^2}\\ &\left.- \frac{3(6 - B^2 r^2 + 6r(B^2 - 2\epsilon)) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^4(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)}\right)\\ &/\left(2\sqrt{\frac{3(6 - B^2 r^2 + 6r(B^2 - 2\epsilon)) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)}}\right)\end{aligned}$$

On solving the above equation for r and putting different values of the magnetic field B and the tidal force ϵ, values of the radii of circular orbits and hence the number of circular orbits were obtained. From table 3.1 and table 3.2, we can see that on putting different values of B and ϵ in equation (3.2), different values of r are obtained.

A.1.2 Calculations for Figure 5.2 and Figure 5.3

From equation (3.2.1), we have

$$U_{eff} = \sqrt{\left(L^2 \frac{6(B^2 - 2\epsilon)M^2 r + M(6 - B^2 r^2 + 12\epsilon r^2 - 3(r + \epsilon r^3)}{r^3(-3 - 6\epsilon M^2 + 3\epsilon r^2 + B^2(3M^2 + r^2))}\right)}$$

Putting L = m = 1, in the above equation, we get

$$U_{eff} = \sqrt{\left(\frac{6 - 2B^2 r^2 + 6r(B^2 - 2\epsilon) + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^3(-3 + B^2(3 + r^2) - 6\epsilon + 3r^2\epsilon)}\right)} \tag{A.1.-2}$$

For Figure 5.2

Putting B = 0 in the equation (A.1.-2), we get

$$U_{eff} = \sqrt{\left(\frac{6 - 12r\epsilon + 12r^2\epsilon - 3(r + r^3\epsilon)}{r^3(-3 - 6\epsilon + 3r^2\epsilon)}\right)} \tag{A.1.-1}$$

Putting $\epsilon = 0$, 0.003, 0.007, 0.009, 0.012, 0.015, 0.018 and 0.02 one by one in equation (A.1.-1), curves showing the variation of the effective potential U_{eff} with changing values of ϵ in Figure 5.2 are obtained.

For Figure 5.3

Putting $\epsilon = 0$ in the equation (A.1.-2), we get

$$U_{eff} = \sqrt{\left(\frac{6 - 3r + 6B^2 r - 2B^2 r^2}{r^3(-3 + B^2(3 + r^2))}\right)} \tag{A.1.0}$$

Putting B = 0, 0.04, 0.044, 0.088, 0.132, 0.176, 0.22 and 0.264 one by one in equation (A.1.0), curves showing the variation of the effective potential U_{eff} with changing values of B in Figure 5.3 are obtained.

A.2 Massive charged particle

A.2.1 Calculations for Figure 5.4 and Figure 5.5

For a massive charged particle, orbiting around a Preston-Poisson black hole the effective potential can be written as :

$$U_{eff} = -mc\sqrt{-1+\frac{2B^2M(3M-r)}{3}+\frac{2M}{r}-\epsilon(-2M+r)^2}\sqrt{-1+\frac{3(2Bqr^2+cL(4+B^2r^2))^2}{c^4M^2r^2(4+B^2r^2)^2(-3-6\epsilon M^2+3\epsilon r^2+B^2(3M^2+r^2))}}$$

Putting m = M = c = 1 in the above equation, we get

$$U_{eff} = -\sqrt{-1+\frac{2B^2(3-r)}{3}+\frac{2}{r}-\epsilon(-2+r)^2}\sqrt{-1+\frac{3(2Bqr^2+L(4+B^2r^2))^2}{r^2(4+B^2r^2)^2(-3-6\epsilon+3\epsilon r^2+B^2(3+r^2))}} \tag{A.2.0}$$

For Figure 5.4

Putting B = 0.1, L = 10 and q = 0.5 in equation (A.2.0), we get

$$U_{eff} = -\sqrt{-1+0.00666667(3-r)+\frac{2}{r}-\epsilon(-2+r)^2}\sqrt{-1+\frac{3(0.1r^2+10(4+0.01r^2))^2}{r^2(4+0.01r^2)^2(-3-6\epsilon+3\epsilon r^2+0.01(3+r^2))}} \tag{A.2.1}$$

Putting ϵ = 0, 0.003, 0.006 one by one in equation (A.2.1), curves showing the variation of the effective potential U_{eff} with changing values of ϵ in Figure 5.4 are obtained.

For Figure 5.5

Putting ϵ = 0.01, L = 10 and q = 0.5 in equation (A.2.1), we get

$$U_{eff} = -\sqrt{-1+\frac{2B^2(3-r)}{3}+\frac{2}{r}-0.01(-2+r)^2}\sqrt{-1+\frac{3(Br^2+10(4+B^2r^2))^2}{r^2(4+B^2r^2)^2(-3.06+0.03r^2+B^2(3+r^2))}} \tag{A.2.2}$$

Putting B = 0, 0.04, 0.08 one by one in equation (A.2.2), curves showing the variation of the effective potential U_{eff} with changing values of B in Figure 5.5 are obtained.