SUBGROUPS GENERATED BY SMALL CLASSES IN FINITE GROUPS

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Submitted by
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Dedicated to

God,

Parents and Teachers.

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"Subgroups Generated by Small Classes in finite Groups"** in partial fulfillment of the requirements for the award of **Master of Science,** School of Mathematatics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Deepak Gumber.**

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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Chapter 1

INTRODUCTION

Let G be a finite group. We denote by G', $Z(G)$, $M(G)$, $F(G)$, respectively the commutator subgroup, the center, the subgroup of G generated by all elements that lie in conjugacy classes of the two smallest sizes, the unique maximal normal nilpotent subgroup of G. N. It$\hat{o}$ [2] in 1953 that finite groups having just two conjugacy class sizes must be nilpotent.

Up to my knowledge there was very little progress in understanding these groups until K. Ishikawa showed [1] that they have nilpotency class at most 3. Using a very much simpler proof, A. Mann [3] dramatically improved Ishikawa's result by showing that if G is an arbitrary finite nilpotent group, then $M(G)$ has nilpotent class atmost 3.

Of course, if G has just two class sizes, then $M(G) = G,$ and since G is nilpotent by It$\hat{o}$'s theorem, Mann's result applies, and it shows that G has nilpotence class at most 3.

In this thesis we present a slight simplification of Mann's argument, and this allows us to prove results for finite groups that are not necessarily nilpotent. We also prove later in the main theorem that the Fitting subgroup of $M(G)$ has class at most 4.

Chapter 2

PRELIMINARIES AND NOTATIONS

Definition 2.1 (***Centralizer of a subgroup***)

Let $H \leq G$, if $x \in G$ s.t $xh = hx \quad \forall \ h \in H$, then we say that x centralizes H. The *centralizer* of H in G is $C_G(H) = \{x \in G \mid xh = hx \quad \forall \ h \in H\}$.

For example, let S_3 be a symmetric group and $A_3 < S_3$ then $C_{S_3}(A_3) = A_3$.

Note: Let $H \leq G$,where G is any abelion group then $C_G(H) = G$.

Proposition 2.2 $C_G(H)$ is a subgroup of G.

Proof: Let $x \in C_G(H)$. Then $xh = hx \quad \forall \ h \in H$

$\Rightarrow hx^{-1} = x^{-1}h \quad \forall \ h \in H$

$\Rightarrow x^{-1} \in C_G(H)$

Let $x, y \in C_G(H)$. Then $xh = hx$ and $yh = hy \quad \forall \ h \in H$

Now, $xyh = xhy = hxy \quad \forall \ h \in H$

So $xy \in C_G(H)$

Therefore $C_G(H) \leq G$. □

Definition 2.3 (***Normalizer of a subgroup***)

Let $H \leq G$, if $x \in G$ s.t $xH = Hx$, then we say that x normalizes H. The *normalizer* of H in G is $N_G(H) = \{x \in G \mid xH = Hx\}$.

For example, $N_{S_3}(A_3) = S_3$.

Note: Let $H \leq G$,where G is any abelion group then $N_G(H) = G$.

Proposition 2.4 $N_G(H)$ is a subgroup of G.

Proof: Let $x \in N_G(H)$.

Then $xH = Hx \Rightarrow Hx^{-1} = x^{-1}H$

so $x^{-1} \in N_G(H)$.

Let $x, y \in N_G(H)$, then $xH = Hx$ and $yH = Hy$

Now $xyH = xHy = Hxy$. So $xy \in N_G(H)$

Therefore $N_G(H) \leq G$. □

Proposition 2.5 If $H \leq G$, then H is normal in $N_G(H)$.

Proof: $\forall\ a \in H$ we have $aH = Ha$.

Since $H \subseteq N_G(H)$ i.e H is a subgroup of G contained in $N_G(H)$.

Also $\forall\ a \in N_G(H)$, we have $aH = Ha$.

So H is normal in $N_G(H)$. □

Proposition 2.6 If H and K are subgroups of G and H is a normal subgroup of K, then $K \subseteq N_G(H)$, i.e $N_G(H)$ is the largest subgroup of G in which H is normal.

Proof: Let $H \subseteq K \subseteq G$ s.t H is a normal subgroup of K and K is a subgroup of G.

Now we will show that $K \subseteq N_G(H)$.

Let $k \in K$ be any element, then $Hk = kH$, since H is a normal subgroup of K

So by the definition of $N_G(H), k \in N_G(H)$

Therefore $k \in N_G(H)\ \ \forall\ \ k \in K$

Hence $K \subseteq N_G(H)$. □

Proposition 2.7 H is a normal subgroup of G iff $N_G(H) = G$.

Proof: Firstly, let H is a normal subgroup of G.

Then $\forall\ g \in G$, we have $Hg = gH$. So $g \in N_G(H)$.

Therefore $G \subseteq N_G(H)$.

But $N_G(H) \subseteq G$ always, hence $G = N_G(H)$.

Conversely, let $N_G(H) = G$.

Therefore $N_G(H) = \{x \in G \mid xH = Hx\} = G$.

So $xH = Hx \ \ \forall\, x \in G$.

Hence H is a normal subgroup of G. □

Proposition 2.8 $C_G(H)$ is normal in $N_G(H)$.

Proof: Let $a \in C_G(H)$, then $ah = ha \ \ \forall\, h \in H$.

Let $g \in N_G(H)$, then $gH = Hg$. Therefore $g^{-1}H = Hg^{-1}$.

So $g^{-1}h = h_1 g^{-1}$ for some $h, h_1 \in H$

Now $(gag^{-1})h = gah_1g^{-1} = gh_1ag^{-1} = h(gag^{-1})$.

So $C_G(H)$ is normal in $N_G(H)$. □

Definition 2.9 (**Center of a Group**)

The *center* $Z(G)$, of a group G is the subset of elements in G that commute with every element of G. In symbols, $Z(G) = \{a \in G \mid ax = xa \ \ \forall\, x \in G\}$.

For examles, $Z(S_3) = \{\, I \,\}$.

Let Q_8 be a quarternion group, then $Z(Q_8) = \{\, I, -I \,\}$.

Remark: G is abelian iff $Z(G) = G$.

Proposition 2.10 $Z(G)$ is a normal subgroup of G.

Proof: Since e $\in Z(G)$, so $Z(G)$ is non-empty.

Now we will show that $Z(G)$ is a subgroup of G.

Let $a \in Z(G)$, so $ax = xa \ \forall x \in G$

$\Rightarrow a^{-1}x = xa^{-1} \ \forall x \in G$

$\Rightarrow a^{-1} \in Z(G)$.

Let $a, b \in Z(G)$, then $ax = xa$ and $bx = xb \ \forall x \in G$

$\Rightarrow abx = axb = xab \ \forall x \in G$

$\Rightarrow ab \in Z(G)$

Therefore $Z(G) \leq G$.

Now we will show that $Z(G)$ is normal in G.

Let $a \in Z(G), g \in G$ be any element.

Now $gag^{-1} = gg^{-1}a = a \in Z(G)$. So $Z(G)$ is normal in G. □

Definition 2.11 (***Conjugate elements***)

If a, b are any two elements of a group G, then b is said to be *conjugate* to a if ∃ an element $x \in G$ s.t $b = xax^{-1}$, and we write it as $b \sim a$.

Lemma 2.12 The relation of conjugacy in a group G is an equivalence relation.

Proof: Define a relation $\sim$ on G as follows

$a \sim b$ iff $a = gbg^{-1}$ for some $g \in G$.

Let a, b, c be any arbitrary elements of G.

Since $a = eae^{-1}$. Thus $a \sim a$.

Therefore $\sim$ is reflexive.

Let $a \sim b$. So there exists $g \in G$ such that $a = gbg^{-1}$.

Thus $g^{-1}ag = g^{-1}a(g^{-1})^{-1} = b$. So $b \sim a$.

Therefore $\sim$ is symmetric.

Let $a \sim b$ and $b \sim c$. So there exists $g, h \in G$ such that $a = gbg^{-1}$ and $b = hch^{-1}$.

Now $a = gbg^{-1} = g(hch^{-1})g^{-1} = (gh)c(gh)^{-1}$. So $a \sim c$.

Therefore $\sim$ is transitive.

Hence the relation of conjugacy in a group G is an equivalence relation. ☐

Lemma 2.13 Let G be a group. Then the set of conjugacy classes of G is a partition of G.

Proof: Define a relation $\sim$ on G as follows

$a \sim b$ iff $a = gbg^{-1}$ for some $g \in G$.

By lemma 2.12, $\sim$ is an equivalence relation on G.

The equivalence class of a in G is the set $cl(a) = \{gag^{-1} \mid g \in G\}$, which is also the conjugacy class of a .

Thus the set of conjugacy classes of G is a partition of G.

Lemma 2.14 (***The Number of Conjugates of a***)

Let G be a finite group and let a be an element of G. Then $|cl(a)| = [G: C_G(a)]$.

Proof: Consider a function $T: cl(a) \to G/C_G(a)$ s.t $T(xax^{-1}) = xC_G(a)$.

Let $xax^{-1}, yay^{-1} \in cl(a)$, where $x, y \in G$

Such that $T(xax^{-1}) = T(yay^{-1})$

$\Rightarrow\ xC(a) = yC(a) \Rightarrow\ x^{-1}y \in C(a) \Rightarrow\ x^{-1}ya = ax^{-1}y$

$\Rightarrow\ xx^{-1}yay^{-1} = xax^{-1}yy^{-1} \Rightarrow\ yay^{-1} = xax^{-1}$

Therefore T is one-one.

Clearly for $xC_G(a) \in G/C_G(a)$, where $x \in G$, we have $T(xax^{-1}) = xC_G(a)$

Therefore T is onto. Hence $|cl(a)| = [G: C_G(a)]$. ☐

Theorem 2.15 (***The Class Equation***)

For any finite group G, $|G| = \sum [G: C_G(a)]$, where the sum runs over one element a from each conjugacy class of G. ☐

Definition 2.16 (***Maximal Subgroup***)

Let G be a group. A subgroup N of G is called a *maximal subgroup* if

1. $N \neq G$.
2. If $N \leq H \leq G$, then $H = N$ or $H = G$.

Definition 2.17 (***p-group***)

A group of order p^n, where p is a prime, is called a p*-group*.

Theorem 2.18 Let G be a p-group. Then

1. $Z(G)$ is non-trivial
2. $Z(G) \cap N$ is non-trivial for any non-trivial normal subgroup N of G.
3. If H is a proper subgroup of G, then H is properly contained in $N_G(H)$.
4. Every maximal subgroup of G is normal.

Proof: First observe that $cl(a) = \{a\}$ if and only if $a \in Z(G)$. Thus by culling out these elements, we may write the class equation in the form

$|G| = |Z(G)| + \sum_{a\in X}[G: C_G(a)]$, where X is a subset of G contains exactly one element from each conjugacy class with more than one element.

1. By *Lagrange's Theorem*, $|C_G(a)| \mid |G|$

Now $a \in Z(G) \Leftrightarrow C_G(a) = G$. If $a \notin \mathrm{Z}(G)$ then $C_G(a) < G$

$\Rightarrow |C_G(a)| \leq p^{n-1}$

$$\Rightarrow \frac{|G|}{|C_G(a)|} \geq p$$

So $[G: C_G(a)] \geq p$

$\Rightarrow p \mid [\mathrm{G}: C_G(a)] \Rightarrow p \mid \sum_{a\in X}[G: C_G(a)]$

$\Rightarrow p \mid (|G| - \sum_{a\in X}[G: C_G(a)])$ i.e $p \mid |Z(G)|$

So $Z(G)$ is non-trivial.

2. We have $G = Z(G) \cup (\bigcup_{a\in X} cl(a))$

$N = N \cap G = N \cap (Z(G) \cup (\bigcup_{a\in X} cl(a)))$

$= \left(N \cap Z(G)\right) \cup (N \cap (\bigcup_{a\in X} cl(a))$

Hence $|N| = |Z(G) \cap N| + \sum_{a\in X} |cl(a) \cap N|$ ... (1)

If $a \in N$, then $cl(a) \subset N$. In this case $cl(a) \cap N = cl(a)$.

Suppose $a \notin N$ and $cl(a) \cap N \neq \emptyset$. Let $y \in cl(a) \cap N$.

Then $y \in cl(a)$ and $y \in N$. Let $y = gag^{-1}$, where $g \in G$

Then $gag^{-1} \in N \Rightarrow g^{-1}(gag^{-1})g = a \in N$, a contradiction, so $cl(a) \cap N = \emptyset$.

Thus $cl(a) \cap N = cl(a)$ or $\emptyset \quad \forall\, a \in X$

$\Rightarrow |cl(a)) \cap N|$ is 0 or $|cl(a)|$ i.e 0 or $[G\colon C_G(a)]$

But $p \mid [G\colon C_G(a)] \quad \forall\, a \in X$. So $p \mid |cl(a) \cap N| \quad \forall\, a \in X$

$\Rightarrow \; p \mid \sum_{a\in X} |cl(a) \cap N|$

Also $p \mid |N|$

Therefore by (1), $p \mid |Z(G) \cap N|$. Hence $Z(G) \cap N$ is non-trivial.

3. Let K be a maximal normal subgroup of G contained in H.

The quotient group G/K is of order $p^r (r > 0)$

Now by part-1, $Z(G/K)$ is non-trivial.

Let $L/K = Z(G/K)$.

Now $L/K \lhd G/K$, so $L \lhd G$.

Clearly L cannot be in H, because otherwise maximality of K is lost.

Let $h \in H, l \in L$, then $hK \in G/K$ and $lK \in L/K$

Because $L/K = Z(G/K)$, so

$(lK)(hK) = (hK)(lK)$

$\Rightarrow lhK = hlK \Rightarrow l^{-1}hl \in hK \subset H$.

Therefore $L \subset N_G(H)$. This implies that $H \neq N_G(H)$.
Because $H \subset N_G(H)$, it follows that H is properly contained in $N_G(H)$.

4. If H is a maximal subgroup of G, then by part-3, $H < N_G(H)$ implies that $N_G(H) = G$; therefore by proposition 2.7, $H \lhd G$. ☐

Definition 2.19 (***Group Homomorphism***)
A *homomorphism* ϕ from a group G to a group $\bar{G}$ is a mapping from G into $\bar{G}$ that preserves the group operation; that is $\phi(ab) = \phi(a)\phi(b)\ \forall\, a, b \in G$.

Definition 2.20 (***Kernel of a Homomorphism***)
The *kernel* of a homomorphism ϕ from a group G to a group $\bar{G}$ with identity e is the set $\{x \in G \mid \phi(x) = e\}$. The kernel of ϕ is denoted by Ker ϕ.

Lemma 2.21 Let f be a group homomorphism from G to a group $\bar{G}$. Then Ker f is a normal subgroup of G.

Proof: For $e \in G$, we have $f(e) = e$. So Ker f is nonempty.
Let $x_1, x_2 \in$ Ker f. So $f(x_1) = e$, $f(x_2) = e$.
Now $f(x_1{x_2}^{-1}) = f(x_1)f({x_2}^{-1}) = f(x_1)(f(x_2))^{-1} = ee^{-1} = e$.
Thus $x_1{x_2}^{-1} \in$ Ker f. So Ker f is a subgroup of G.

Now we will show that Ker f is normal in G.
Let $a \in$ Ker f and $g \in G$. So $f(a) = e$.
Now $f(gag^{-1}) = f(g)f(a)f(g^{-1}) = f(g)f(a)\big(f(g)\big)^{-1} = f(g)\big(f(g)\big)^{-1} = e$.
Thus $gag^{-1} \in$ Ker f. So Ker f is normal in G. ☐

Lemma 2.22 Let f be a homomorphism from a group G to a group H and K be a subgroup of G. Then

1. $f(K) = \{f(k) |\, k \in K\}$ is a subgroup of H.
2. If K is normal in G then $f(K)$ is normal in $f(G)$.

Proof: 1. Since $e = f(e) \in f(K)$, so $f(K)$ is non-empty.

If $f(k_1), f(k_2) \in f(K)$,

Then, $f(k_1)\big(f(k_2)\big)^{-1} = f(k_1)f\big({k_2}^{-1}\big)$

$$= f\big(k_1{k_2}^{-1}\big) \in f(K), \text{ since } k_1{k_2}^{-1} \in K.$$

2. Let $f(k) \in f(K)$ and $f(g) \in f(G)$.

Then $f(g)f(k)f(g)^{-1} = f(gkg^{-1}) \in f(K)$, since K is normal in G. ☐

Theorem 2.23 (***First Isomorphism Theorem***)

Let $\phi: G \to G'$ be a homomorphism of groups. Then $G/\text{Ker } \phi \cong \text{Im } \phi$

Hence, in particular, if ϕ is surjective, then $G/\text{Ker } \phi \cong G'$.

Proof: Consider the mapping $\psi: G/K \to \text{Im } \phi$ given by $xK \mapsto \phi(x)$, where $K = \text{Ker } \phi$ for any $x, y \in G$,

Now $xK = yK \Leftrightarrow y^{-1}x \in K \Leftrightarrow \phi(y^{-1}x) = e' \Leftrightarrow \phi(x) = \phi(y)$

Hence ψ is well defined and injective.

Let $xK, yK \in G/K$.

Then $\psi(xKyK) = \psi(xyK) = \phi(xy) = \phi(x)\phi(y) = \psi(xK)\psi(yK)$

Hence ψ is a homomorphism.

ψ is clearly surjective, we conclude that ψ is an isomorphism of groups. ☐

Lemma 2.24 Let N is a normal subgroup of a group G, and H is any subgroup of G, then NH is a subgroup of G.

Proof: $e = ee \in NH$, Thus NH is non-empty.

Let $a, b \in NH$. Then $a = n_1h_1,\ b = n_2h_2$ for some $n_1, n_2 \in N$ and $h_1, h_2 \in H$

Now $ab^{-1} = n_1h_1{h_2}^{-1}{n_2}^{-1} = n_1h_3{n_2}^{-1}$, where $h_1{h_2}^{-1} \in H$

$= n_1h_3{n_2}^{-1}{h_3}^{-1}h_3$

$= n_1n_3h_3$ where $h_3{n_2}^{-1}{h_3}^{-1} = n_3$, since N is a normal subgroup.

$= n_4h_3 \in NH$, where $n_1n_3 = n_4$

Hence $ab^{-1} \in NH$. This proves that NH is a subgroup.

Lemma 2.25 Let $A \lhd B$ and $A \lhd C$, then $A \lhd BC$

Proof: Let $x \in A, y \in BC$

$yay^{-1} = bcac^{-1}b^{-1} = ba_1b^{-1}$, where $cac^{-1} = a_1$, since $A \lhd C$

$= a_2 \in A,$ since $A \lhd B$

Hence $yay^{-1} \in A$. This proves that $A \lhd BC$.

Lemma 2.26 Let H and K be subgroups of a group G. Then HK is a subgroup of G iff $HK = KH$.

Proof: Let $HK = KH$. Since $e = ee \in HK, HK$ is not empty.

Let $a, b \in HK$. Then $a = h_1k_1, b = h_2k_2$for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$

Now $ab^{-1} = h_1k_1{k_2}^{-1}{h_2}^{-1} = h_1k_3{h_2}^{-1}$, where $k_3 = k_1{k_2}^{-1} \in K$

Now $k_3{h_2}^{-1} \in KH = HK$. Hence $k_3{h_2}^{-1} = h_3k_4$ for some $h_3 \in H, k_4 \in K$

Therefore $ab^{-1} = h_1h_3k_4 = h_4k_4$, where $h_4 = h_1h_3 \in H$

Hence $ab^{-1} \in HK$. This proves that HK is a subgroup.

Conversely suppose that HK is a subgroup.

Let $a \in KH$. So $a = kh$ for some $h \in H,\ k \in K$.

Then $a^{-1} = h^{-1}k^{-1} \in HK$. Hence $a \in HK$.

Therefore $KH \subset HK$. Next let $b \in HK$. Then $b^{-1} \in HK$.

Hence $b^{-1} = h'k'$ for some $h' \in H, k' \in K$.

Therefore $b = k'^{-1}h'^{-1} \in KH$. Hence $HK \subset KH$. This proves that $HK = KH$. □

Definition 2.27 (***Commutator***)

If $a, b \in G$, the *commutator* of a, b is denoted by $[a, b]$ and defined by $[a, b] = a^{-1}b^{-1}ab$. The commutator subgroup (or derived subgroup) of G, denoted by G', is the subgroup of G generated by all the commutators. Thus $G' = \langle [a, b] \mid a, b \in G \rangle$.

Remark: G is abelian iff $G' = \{e\}$

Definition 2.28 (***Subgroup generated by subset of a Group***)

Let S be a subset of a group G. A subgroup H of G is said to be *subgroup generated by* S if it satisfies the following conditions.

1. $S \subseteq H$
2. If K is any subgroup of G such that $S \subseteq K$ then $H \subseteq K$.

We denote the subgroup generated by S by $\langle S \rangle$.

For example, in S_3,$\langle\ (12), (123)\ \rangle$ generates S_3.

Lemma 2.29 If S is any non-void subset of a group G, then the subgroup $\langle S \rangle$ of G generated by S is the set of all finite products of the form $a_1a_2a_3.\dots.a_n$, where for each i, either $a_i \in S$ or ${a_i}^{-1} \in S$ and n is any positive integer.

Proof: Let H be the set of all finite products of the form $a_1a_2a_3.\dots.a_n$, where for each i, either $a_i \in S$ or ${a_i}^{-1} \in S$ and n be any positive integer.

Consider $x = a_1a_2a_3.\dots.a_n$, $y = b_1b_2b_3.\dots.b_m$ in H.

Then $xy = a_1a_2a_3.\dots.a_nb_1b_2b_3.\dots.b_m$ is a product of finite number of elements a_ib_j such that either the factor or its inverse is in S, consequently $xy \in H$.

Further $x^{-1} = {a_n}^{-1}{a_{n-1}}^{-1}.\dots.{a_2}^{-1}{a_1}^{-1}$.

Consider any ${a_i}^{-1}$. Since a_i or ${a_i}^{-1}$ is in S, and $a_i = ({a_i}^{-1})^{-1}$, we see that either ${a_i}^{-1}$ or $({a_i}^{-1})^{-1}$ is in S, hence $x^{-1} \in H$. This proves that H is a subgroup of G.

Clearly, $S \subseteq H$. Consider any subgroup K of G containing S. Then for each $a \in S$, we have $a \in K$ and hence $a^{-1} \in K$. Thus if $x = a_1 a_2 a_3 \ldots\ldots a_n$; $a_i \in S$ or ${a_i}^{-1} \in S$, is any element of H, then $x \in K$, since $a_i \in K \;\forall\; i$.

Hence $H \subseteq K$. This proves that H is the subgroup of G generated by S. ☐

Theorem 2.30 Let G be a group and G' be its commutator subgroup, then

1. G' is a normal subgroup of G.
2. For any normal subgroup H of $G, G/H$ is an abelian group if and only if H contains G'.

Proof: 1. Let $a, b \in G$, since $(a^{-1}b^{-1}ab)^{-1} = b^{-1}a^{-1}ba$ is again a commutator, it follows from the above lemma that each element of G' is a product of finite number of commutators.

Consider $x \in G'$, then $x = g_1 g_2 g_3 \ldots\ldots g_t$ where for each $i = 1, 2, \ldots\ldots, t,\; g_i$ is a commutator, so that $g_i = {a_i}^{-1}{b_i}^{-1}a_i b_i$ for some $a_i, b_i \in G$.

Now for any $a \in G, a^{-1}xa = (a^{-1}g_1a)(a^{-1}g_2a)\ldots\ldots(a^{-1}g_ta)$.

Further

$a^{-1}g_ia = a^{-1}{a_i}^{-1}{b_i}^{-1}a_ib_ia = (a^{-1}a_ia)^{-1}(a^{-1}b_ia)^{-1}(a^{-1}a_ia)(a^{-1}b_ia) = c^{-1}d^{-1}cd;$ where $c = a^{-1}a_ia,\; d = a^{-1}b_ia$. Thus $a^{-1}g_ia$ is again a commutator.

Hence $a^{-1}xa$ is a product of commutators; by definition $a^{-1}xa \in G'$.

2. Consider $a, b \in G$. Let G/H be abelian. So $abH = baH \;\forall\; aH, bH \in G/H$.

Thus $a^{-1}b^{-1}ab \in H$. Thus H contains every commutator $a^{-1}b^{-1}ab$.

Consequently as G' is generated by all the commutators, $G' \subseteq H$.

Conversely let $G' \subseteq H$. Then $a^{-1}b^{-1}ab \in G'$ gives $a^{-1}b^{-1}ab \in H$ i.e $abH = baH$.

Thus $(aH)(bH) = (bH)(aH) \;\forall\; aH, bH \in G/H$. So G/H is abelian. ☐

Definition 2.31 (**Normal Series**)

A sequence $(G_0, G_1, G_2 \ldots\ldots G_{r-1}, G_r)$ of subgroups of a group G is called a *normal series* of G if

$\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \ldots\ldots\ldots\ldots\ldots\ldots \lhd G_{r-1} \lhd G_r = G$.

The factors of a normal series are the quotient groups G_i/G_{i-1} , $1 \leq i \leq r$.

Definition 2.32 (Composition Series)

A *composition series* of a group G is a normal series $(G_{0,\ldots\ldots\ldots,}G_r)$ without repetition whose factors G_i/G_{i-1} are simple groups. The factors G_i/G_{i-1} are called composition factors of G.

Remark: For any group $G, \{e\} = G_0 \lhd G_1 = G$ is trivially a normal series of $G.$ If G is a simple group, then $\{e\} \lhd G$ is the only composition series of G.

Some Examples of Composition series:

1. $\{e\} \lhd \{e,(123),(132)\} \lhd S_3$ is a composition series for S_3.
2. $\{0\} \lhd \{0,9\} \lhd \{0,3,6,9,12,15\} \lhd \{0,1,2,\ldots\ldots\ldots..17\} = Z_{18}$ is a composition for Z_{18}.

Definition 2.33 (Derived Series)

$$G^{(0)} = G$$
$$G^{(1)} = \left[G^{(0)}, G^{(0)}\right] = [G,G] = G'$$
$$G^{(2)} = \left[G^{(1)}, G^{(1)}\right] = [G',G'] = G''$$

Where $G^{(1)}, G^{(2)}$ are first and second derived subgroup repectively.

In general, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ where $n \in \mathbb{N}$

$G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \ldots\ldots\ldots\ldots$, decending *derived series.*

Definition 2.34 (Solvable Group)

The subgroup G' generated by the set of all commutators $a^{-1}b^{-1}ab$ in a group G is called the derived group of $G.$ For any positive integer n, we define the nth derived group of G, written $G^{(n)}$, as follows $G^{(1)} = G'$, $G^{(n)} = \left(G^{(n-1)}\right)'$ $(n > 1)$

A group G is *solvable* if it has the derived series,

$G^{(0)} = G \rhd G^{(1)} \rhd G^{(2)} \ldots\ldots\ldots$, where every subgroup is the commutator of the previous one eventually reaches the trivial subgroups $\{e\}$ of G.

The least n s.t $G^{(n)} = \{e\}$ is called the derived length of the solvable group G.

Some Examples of Solvable Groups:

(1) All abelian groups are solvable, In this case $G' = \{e\}$.

(2) S_3 is solvable because its derived series ends with a trivial subgroup

$$S_3 \rhd {S_3}^{(1)} = A_3 \rhd {S_3}^{(2)} = \{e\}$$

Definition 2.35 (**Lower Central Series**)

$\gamma_1(G) = G$;

$\gamma_2(G) = [\gamma_1(G),\ G]$

In general, $\gamma_{i+1}(G) = [\gamma_i(G),\ G]$.

The lower central series (or descending central series) of G is the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots .$$

Definition2.37(NilpotentGroup)

A group G is said to be nilpotent if there is an positive integer c such that $\gamma_{c+1}(G) = \{e\}$; the least c is called the class of the nilpotent group G.

If G is abelian group, then $\gamma_2(G) = \{e\}$. Thus, trivially, every abelian group is nilpotent.

For examples,

A_3 is a nilpotent group of class 1.

Q_8 is a nilpotent group of class 2.

Theorem 2.38 Every nilpotent group G is solvable.

Proof: Firstly we will prove by induction that $G^{(i)} \leq \gamma_{i+1}(G)$ for all i .

For $i = 0, G^{(0)} \leq \gamma_{1}(G)$.

Let us assume result is true for $i = n - 1$. Therefore $G^{(n-1)} \leq \gamma_{n}(G)$.

Now $G^{(n)} = \left[G^{(n-1)}, G^{(n-1)}\right] \leq \left[\gamma_{n}(G), G\right] = \gamma_{n+1}(G)$.

Hence by induction $G^{(i)} \leq \gamma_{i+1}(G)$ for all i .

Let nilpotency class of G is c, $\gamma_{c+1}(G) = 1$ then $G^{(c)} = 1$.

Hence G is solvable (with derived length $\leq c$) □

Definition 2.40 (**M(G)**)

The subgroup of G generated by all elements that lie in conjugacy class of two smallest sizes,

Which is denoted by $M(G)$.

For example, $M(S_3) = A_3$.

Definition 2.41 (***Fitting subgroup***)

Every finite group G has a unique maximal normal nilpotent subgroup $F(G)$, which is called the *Fitting subgroup* of G.

For example, $F(S_3) = A_3$.

Chapter-3

MAIN RESULT

Lemma 3.1 Let $K \lhd G$, where G is arbitrary finite group and K is abelian. Let x be the non central element of G, and let $y = [t, x]$ for some element $t \in K$. Then $|C_G(y)| > |C_G(x)|$, and so the G-class of y is smaller than that of x.

Proof: Since x is a non central in G, the result is trivial if $y \in Z(G)$, and so, we assume that $y \notin Z(G)$.

Now by lemma 2.24, $\mathrm{H} = \mathrm{K}C_G(x)$ is a subgroup of G.

Observe that $y = [t, x] = txt^{-1}x^{-1} \in K \subseteq H$.

Now we will prove that $C_H(x) = C_G(x)$

Let $g \in C_H(x)$, therefore $gx = xg$ where $g \in H$

Now $H \subseteq G$

$\Rightarrow g \in G$ and so $gx = xg$

$\Rightarrow g \in C_G(x)$

$\Rightarrow C_H(x) \subseteq C_G(x)$ ... (1)

Let $g \in C_G(x)$ be any element.

Now $C_G(x) \subseteq H$

$\Rightarrow g \in H$ and so $gx = xg$

$\Rightarrow g \in C_H(x)$

$\Rightarrow C_G(x) \subseteq C_H(x)$ ... (2)

By (1) and (2), $C_H(x) = C_G(x)$.

It suffices to show that $|C_H(y)| > |C_H(x)|$, or equivalently, that $|H : C_H(y)| < |H : C_H(x)|$.

If $u, v \in K$, then we will prove that $[uv, x] = [u, x]^v [v, x] = [u, x][v, x]$.

Now $[uv, x] = (uv)^{-1} x^{-1} (uv) x$

$= v^{-1} u^{-1} x^{-1} uvx \qquad \ldots (3)$

Also $[u, x]^v [v, x] = v^{-1} [u, x] v [v, x]$

$= v^{-1} u^{-1} x^{-1} uxvv^{-1} x^{-1} vx$

$= v^{-1} u^{-1} x^{-1} uvx \qquad \ldots (4)$

From (3) & (4) $[uv, x] = [u, x]^v [v, x]$.

Now $[u, x]^v [v, x] = v^{-1} [u, x] v [v, x]$

$= [u, x][v, x]$, since $[u, x] \in K$ and K is abelian

Hence $[uv, x] = [u, x]^v [v, x] = [u, x][v, x]$.

Define the map $\theta : K \to K$ s.t $\theta(k) = [k, x]$.

Now we have to show that θ is homomorphism.

Now $\theta(k_1 k_2) = [k_1 k_2, x] = [k_1, x][k_2, x] = \theta(k_1)\theta(k_2)$.

Hence θ is a homomorphism, so by using lemma 2.22 (part-1), $\theta(K)$ is a subgroup.

Now by using theorem 2.23 (*First Isomorphism Theorem*), $|\theta(K)| = |K : \text{Ker}\ \theta|$.

But Ker $\theta = \{ y \in K \mid \theta(y) = 1\}$

$= \{ y \in K \mid [y, x] = 1\}$

$= \{ y \in K \mid y^{-1} x^{-1} yx = 1$

$= \{ y \in K \mid yx = xy \ \}$

$= C_K(x)$

Thus $|\theta(K)| = |K : \text{Ker}\ \theta| = |K : C_K(x)|$.

Now we have to prove that $|K : C_K(x)| = |H : C_H(x)|$ or equivalently $|x^K| = |x^H|$.

Let $y \in x^H$ be any arbitrary element.

Then $y = h^{-1} x\, h$ where $h = kx_1 \in H$, since $\mathrm{H} = \mathrm{K}C_G(x)$

$= (x_1 k)^{-1} x (x_1 k)$

$= k^{-1} {x_1}^{-1} x x_1 k$

$= k^{-1}x\,k$

Therefore $y \in x^K$ and hence $x^H \subseteq x^K$.

Let $y \in x^K$ be arbitrary element. Then $y = k^{-1}x\,k$ where $k \in K$.

Now $K \subseteq H$, so $k \in H$.

Therefore $y = k^{-1}x\,k \in x^H$ where $k \in H$

$\Rightarrow x^K \subseteq x^H$

$\Rightarrow x^K = x^H$

$\Rightarrow |x^K| = |x^H|$.

Now we have to prove that $\theta(K)$ is normalized by $C_G(x)$.

Let $x_1 \in C_G(x)$

This gives $xx_1 = xx_1$.

Let $[k,x] \in \theta(K)$ for some $k \in K$.

Now $[k,x]x_1 = (k^{-1}x^{-1}kx)x_1$

$= (k^{-1}k_2)x_1$, since $x^{-1}kx \in K$

$= x_1({x_1}^{-1}k^{-1}k_2x_1)$

$= x_1({x_1}^{-1}k_3x_1)$

Now we have to prove that $({x_1}^{-1}k_3x_1) \in \theta(K)$.

Now ${x_1}^{-1}k_3x_1 = {x_1}^{-1}k^{-1}k_2x_1 = (kx_1)^{-1}(x^{-1}kx)x_1$

$= (kx_1)^{-1}x^{-1}(kx)x_1 = (kx_1)^{-1}x^{-1}(kx_1)x$

$= [kx_1, x] = [x_1{x_1}^{-1}kx_1, x]$

$= [x_1, x][x_1, x, {x_1}^{-1}kx_1][{x_1}^{-1}kx_1, x]$

$= [{x_1}^{-1}kx_1, x]$, since $x_1 \in C_G(x)$

$= [{x_1}^{-1}kx_1, x] \in \theta(K)$, since ${x_1}^{-1}kx_1 \in K$

So $\theta(K)$ is normalized by $C_G(x)$. ….(5)

Now, K normalizes $\theta(K)$ because K is abelion ….(6)

From(5)&(6) $\theta(K)$ is normal in $KC_G(x) = H$

We know that H-Class of y is $y^H = \{h^{-1}y\,h \mid h \in H\}$.

Let $a \in y^H$ be anyarbitrary element.

Then $a = h^{-1}y\,h$ for some $h \in H$.

$\Rightarrow a = h^{-1}y\,h \in \theta(K)$

$\Rightarrow y^H \subseteq \theta(K)$

But y is a non-identity element of $\theta(K)$,and thus the entire H class of y consists of non-identity element of $\theta(K)$.

$\Rightarrow |y^H| = |H : C_H(y)| < |\theta(K)| = |H : C_H(x)|$ ☐

Corollary 3.2 Let $K \lhd G$, where G is arbitrary finite group and K is abelian. Then $[K, M(G)] \subseteq Z(G)$.

Proof: If x lies in a class of G of size m, where m is the smallest class size exceeding 1.

Now $y = [k, x] \in [K, M(G)]$

$\Rightarrow |C_G(y)| > |C_G(x)|$

$\Rightarrow |y^G| < |x^G|$

$\Rightarrow y \in Z(G)$

$\Rightarrow [K, M(G)] \subseteq Z(G)$. ☐

Lemma 3.3 (***Hall-Witt's Identity***)

$$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1.$$

Proof: Let $u = xzx^{-1}yx, v = yxy^{-1}zy,\ w = zyz^{-1}xz$.

Then $[x, y^{-1}, z]^y = [x^{-1}yxy^{-1}, z]^y$

$$= y^{-1}(yx^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}z)y$$

$$= (x^{-1}y^{-1}xz^{-1}x^{-1})(yxy^{-1}zy)$$

$$= u^{-1}v$$

Similarly $[y, z^{-1}, x]^z = v^{-1}w, [z, x^{-1}, y]^x = w^{-1}u$.

Therefore $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = u^{-1}vv^{-1}ww^{-1}u = 1.$ ☐

Lemma 3.4 (***Three Subgroup Lemma***)

Let X, Y and Z be three subgroups of a group G, and let N be a normal subgroup of G. If $[X, Y, Z]$ and $[Y, Z, X]$ are both contained within N, then so is $[Z, X, Y]$.

Proof: Let $x \in X, y \in Y$ and $z \in Z$. Since $[X, Y, Z]$ and $[Y, Z, X]$ are both contained in N, then$[x, y^{-1}, z]^{y}$and $[y, z^{-1}, x]^{z}$ and elements of N(Since N is normal), and so by Witt's Identity, $([x, y^{-1}, z]^{y}[y, z^{-1}, x]^{z})^{-1} = [z, x^{-1}, y]^{x} \in N$.

Since N is normal, so we can conjugate by x^{-1} to get $[z, x^{-1}, y] \in N$. But by writing $x' = x^{-1}$, we have $[z, x', y] \in N$ for all $z \in Z, x' \in X$ and $y \in Y$. Since $[Z, X, Y]$ is generated by such elements, $[Z, X, Y] \leq N$.

Theorem 3.5 Let G be a finite group that contains a normal abelian subgroup A such that $C_G(A) = A$. Then $M(G)$ is nilpotent, and it has nilpotency class at most 3.

Proof: We are given an abelian subgroup $A \lhd G$ with $C_G(A) = A$, and we write $M = M(G)$.

By Corollary 3.2, we have $[A, M] \subseteq Z(G)$, and thus $[M, A, M] = [A, M, M] = 1$. By three subgroup lemma, $[M, M, A] = 1$. Therefore $M' \subseteq C_G(A) = A$.

Now, $M^4 = [M', M, A] \subseteq [A, M, M] = 1$, and thus M is nilpotent with class at most 3.

Theorem 3.6 Let G be a finite group. Then the Fitting subgroup $F(M(G))$ has nilpotence class at most 4.

Proof: Let $M = M(G)$ and $F = F(M)$, and let n be the nilpotence class of F, so that $F^n > 1$ and $F^{n+1} = 1$. Now we will prove that $n \leq 4$, and so we can certainly assume that $n \geq 3$, and thus the subgroup F^{n-2} is defined. If F^{n-2} normal subgroup of G is abelian, then by Corollary 3.2, we have $[F^{n-2}, M] \subseteq Z(G)$, since $F \subseteq M$, thus $[F^{n-2}, F] \subseteq Z(G)$

$F^n = [F^{n-1}, F] = [F^{n-2}, F, F] \subseteq [Z(G), F] = 1$, which is a contradiction.

Thus F^{n-2} is nonabelian, and so $1 < [F^{n-2}, F^{n-2}] \subseteq F^{2n-4}$, this implies $F^{2n-4} > 1$.

Now we will prove that $2n - 4 < n + 1$.

If $2n - 4 \geq n + 1$, then $F^{2n-4} \leq F^{n+1} = 1$, this implies $F^{2n-4} = 1$, which is not so.

Therefore $2n - 4 < n + 1$. Thus $n < 5$, as required.

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