EFFECTS OF ECCENTRIC ORBITS AND ASYNCHRONOUS ON THE EQUILLIBRIUM STRUCTURES OF ROTATIONALLY AND TIDALLY DISTORTED POLYTROPIC MODELS OF STARS

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This is to certify that the thesis "Effects of eccentric orbits and asynchronous on the equilibrium structures of rotationally and tidally distorted polytropic models of stars" submitted by Nikita Madaan of M.Sc (Mathematics And Computing), Thapar University, Patiala, was carried out by me under supervision of Dr. A. K. Lal. She has not submitted this material for credit towards any other degree at Thapar University, Patiala or any other University.

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Dedicated to my
Parents and God

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Abstract

In astrophysics it has great importance to study the problems of equilibrium structures of stars distorted by rotational and tidal effects. So, The aim of the present thesis is to study the effects of eccentric orbits and asynchronous on the equilibrium structure of distorted stars.This types of problem helps in better understanding to inner structure and stability of binary system.

It is quite a complex problem to determine the equilibrium structure of distorted stellar models analytically. Therefore, researchers tried to solve such problems by approximations. In one of those attempts, Mohan, Saxena and Aggarwal used Kippenhahn and Thomas averaging technique together with the results of Kopal on Roche Equipotential, to determine the effects of rotational and tidal distortion on the equilibrium structure of binary stars. However, the problems of determining the equilibrium structures of eccentric orbits and asynchronous on the rotationally and tidally distorted polytropic models of stars have not been satisfactory tackled so far.In the present thesis an attempt has been made to study the effects of eccentric orbits and asynchronous on the equilibrium structures of rotationally and tidally distorted polytropic models of stars.

This thesis consists of three chapters. Chapter one is introductory in nature where we discussed the astrophysical importance of studying the equilibrium structure of rotationally and tidally distorted stellar models. Chapter two deals with the concept of Roche equipotential of distorted stars which accounts for the effects of eccentric orbits and asynchronous rotation of the binary system. Along with the results of Roche equipotential, the explicit expressions for distortional parameters u, v, w, f_P and f_T are obtained by using Kippenhahn and Thomas averaging technique. In this chapter we also discussed about how to find the equilibrium structure of rotationally and tidally distorted stellar models in presence of eccentric orbits and asynchronous rotation.

The methodology discussed in chapter-II is next used in chapter-III to determine the equilibrium structure of rotationally and tidally distorted polytropic models of stars. Computation is carried out to determine the structure as well as various parameters like volume and surface area of the rotationally and tidally distorted polytropic models with

various values of degree of non synchronous parameter F. Conclusion is based on study has finally been discussed.

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Chapter 1

Introduction

This chapter is fundamentally introductory in nature. In section 1.1 we explain in brief the astrophysical significance of the problem for determining the effects of rotational and tidal distortions on the equilibrium structures of gaseous spheres. A brief survey of the literature available on the subject is presented in Section 1.2. Section 1.3 deals with some basic equations required to study the equilibrium structure of gaseous sphere and section 1.4 shows how Kippenhahn and Thomas (24) used an averaging technique be used in determining the equilibrium structures of rotationally and tidally distorted stellar models.

1.1 Astrophysical significance of determining the effects of rotation and tidal distortions on the equilibrium structures of gaseous spheres

Theoretical studies of the problems of the equilibrium structure of gaseous spheres have been offenly carried out to understand the nature of the internal structures, responsible for various observed phenomena of the stars. The theoretical model of a star is essentially a self gravitating gaseous sphere in hydrostatic and thermal equilibrium. It is observed that some of the stars are either found as single stars or in groups of two or more stars. Investigations also confirm that some of the stars are rotating about their axes of rotation. This rotation can be either a solid body rotation or a differential rotation. A large number of stars observed in the sky are rotating stars or binary stars. In a binary system of stars the two stars normally rotate about their own axis as well as revolve about their common center of mass. In a majority of binary stars, one star called primary, is generally more massive compared to its companion star. Contact binaries also been observed in which outermost surfaces of the two stars just touch each other. Observations shows most of the stars in binary and multiple systems are also known as rotating about their axes as well as revolving around each other. Thus, if we consider the equilibrium structure of a single non rotating star as a gaseous sphere, then the structure of such rotating star will be rotationally distorted gaseous sphere. Similarly, the equilibrium structure of a star appearing in a binary or a multiple system will be a tidally distorted gaseous sphere if it is not rotating and a rotationally and tidally distorted gaseous sphere if the star is rotating as well. The brightness of certain observed stars varies with time. These stars are called variable stars. In some of these variable stars, the variations in luminosity are periodic. It is reasonable to assume in the case of such regular variable stars that these are pulsating gaseous spheres in which the variation in luminosity is being caused by the periodic contraction and expansion of the gaseous mass. The regular variable stars gained importance in astrophysics when it was discovered that there exists a definite relation between the periods of pulsation and the luminosities of such stars. This relationship has been utilized to determine the distance of stars. Such an important use of the regular variable stars motivated astrophysicists to investigate the problems of small oscillations of the equilibrium structure of the variable stars so as to have a clear idea of the mechanism which could possibly be sustaining pulsations in these stars. Such investigations are also expected to help us in better understanding the nature of the internal structure of the stars. Analytic studies of the problems of rotating stars and stars in binary systems have engaged the attention of astrophysicist since long with a view to analyze and understand the observational behavior of such stars. Keeping this in view an attempt has been made in the present thesis to investigate certain aspects of the problems of equilibrium structures of rotationally and tidally distorted gaseous spheres which needs further investigations.

1.2 Brief survey of the literature

Most of the theoretical studies about the equilibrium structures and oscillations of the stars have been carried out in literature by assuming the star to be an undistorted spherical gaseous sphere. Extensive literature is now available on this subject (see for instance Chandrasekhar (8), Schwarzschild (52), Eddington et al. (14), Cox and Giuli (12), Kippenhahn and Weigert (25), Clement (9), Kopal (28), Cox (11), Bohm-Vitense (7), Horedt (21).

Sepinsky, Willems et al. (23) under the hypothesis of quasi-static equilibrium investigated the existence and properties of equipotential surfaces and Lagrangian points in non-synchronous, eccentric binary star and planetary systems. They calculated the volume-equivalent radius of the Roche lobe as a function of the four parameters, that is, the Lagrangian points as functions of the mass ratio, the degree of asynchronism, the orbital eccentricity, and the position of the stars or planets in their relative orbit. They provided generalized analytic fitting formulae for the volume-equivalent Roche lobe radius appropriate for non synchronous, eccentric binary star and planetary systems. They adopted a binary potential that accounts for asynchronous rotation and eccentric orbits and they computed the orbital eccentricity.

Wilson (58) discusses the computation of binary star light and radial velocity curves, including the effects of eccentric orbits and non synchronous rotation. Avni (3) discussed a generalization of Roche potential for eccentric binary orbits in the approximation of quasi-static equilibrium. Kruszewski (29) investigated the influence of deviations from synchronism between rotational and orbital motions on the shape of components in close binary systems. When the star is smaller than its Roche limit, the flow of gas from the surface of the star can occur. Kopal and Ali (26) studied the integrability of the Roche coordinates.

A number of binary systems present evidence of enhanced activity around periastron passage, suggesting a connection between tidal interactions and these periastron effects. The region around a star in a binary system within which orbiting material is gravitationally bound to that star is Roche Lobe. It is an approximately tear-drop-shaped region bounded by a critical gravitational equipotential.

Kippenhahn and Thomas (24) suggested a practical way of analyzing the effects of rotation and tidal distortions on the equilibrium structures of stars by approximating the actual equipotential surfaces of the star by Roche equipotential.

Kopal (27) introduced a system of coordinates, which he called Roche coordinates, to study the problems of rotating stars and stars in binary system. Mohan and Singh (45) considered the use of Roche coordinates in solving the problems of small adiabatic oscillations of rotationally and tidally distorted stellar models. Mohan and Saxena (41) used the Kippenhahn and Thomas (24) averaging technique in conjunction with Kopal's results on Roche equipotential to determine the combined effects of rotation and tidal distortions on the equilibrium structures and oscillations of the polytropic models of the stars. This approach is presented in detail by Saxena (51). Later this approach was also used by Mohan and Agarwal (40) to study the effects of rotation and tidal distortions on the structure and periods of small adiabatic oscillations of composite models of stars. The technique was subsequently formalized by Mohan et al. (43) and used to study the problems of equilibrium structures and oscillations of rotationally and tidally distorted main sequence stars. Singh (45) also studied the oscillations of differentially rotating stars in binary system. Later on Lal (31) studied in detail the equilibrium structures and periods of oscillations of differentially rotating stellar models.

In literature the basic equations of the system of equilibrium structure of gaseous sphere in hydrostatic and thermal equilibrium are well established. On stellar models these equations govern the problem of equilibrium structure.

1.3 Basic equations to determine the equilibrium structure of gaseous sphere

1.3.1 Mass conservation

Let P and ρ are the pressure and density of a point respectively, and r be the distant from the center of the sphere. Let M(r) be the mass enclosed within radius r. So, the mass enclosed with the shell from r to r+dr is given by

$$M(r+dr) - M(r) = \rho(r)\ dV \tag{1.3.1}$$

where $dV = 4\ \pi\ r^2\ dr$ be the volume of the shell

Now, equation (1.3.1) can be rewritten as,

$$dM(r)dr\ dr = \rho(r)\ 4\ \pi\ r^2\ dr \tag{1.3.2}$$

So we get,

$$\frac{dM(r)}{dr} = \rho(r)\ 4\ \pi\ r^2 \tag{1.3.3}$$

1.3.2 Hydrostatic equilibrium

In hydrostatic equilibrium, for a star the gravity is balanced by pressure. Area for a small element in a shell from r to r+dr is dA. Where the radial component increases outwards in a coordinate system establishes. Then, the pressure force acting on the inner side of radius r is positive, while on the outer side of radius r+dr is negative. Then the total pressure force is

$$P(r)\ dA - P(r+dr)\ dA = [P(r) - P(r+dr)]\ dA = -\frac{dP}{dr}\ dr\ dA \tag{1.3.4}$$

Due to spherical symmetry, the gravity force points toward the center and has a negative sign i.e.

$$-\frac{G\ M(r)\ dm}{r^2} \tag{1.3.5}$$

where dm be the mass of the element. Then,

$$dm = \rho(r)dV = \rho(r)\ dA\ dr \tag{1.3.6}$$

So, the gravity is given as

$$-\frac{G\ M(r)\ \rho(r)\ dA\ dr}{r^2} \tag{1.3.7}$$

By Newton's Second law,

$$\rho(r)\ dA\ dr\ \frac{d^2\ r}{dt^2} = -\frac{dP}{dr}\ dr\ dA - \frac{G\ M(r)\ \rho(r)\ dA\ dr}{r^2} \tag{1.3.8}$$

$$\rho(r)\ \frac{d^2\ r}{dt^2} = -\frac{dP}{dr} - \frac{G\ M(r)\rho(r)}{r^2} \tag{1.3.9}$$

Sum of all forces must vanish, if the star is in hydrostatic equilibrium, i.e.,

$$-\frac{dP}{dr} - \frac{GM(r)\rho(r)}{r^2} = 0 \tag{1.3.10}$$

which gives,

$$\frac{dP}{dr} = -\frac{G\ M(r)\ \rho(r)}{r^2} \tag{1.3.11}$$

1.3.3 Energy Conversation

Via radiation stars loose its energy. The radiation loss must be balanced by energy generated by nuclear reactions. In mathematical equation, the energy conservation express in term as:

Let $L(r)$ be the energy flow over the sphere with radius r, in units of W, then the total energy loss in the shell from r to r+dr is:

$$L(r + dr) - L(r) = \frac{dL(r)}{dr} dr \tag{1.3.12}$$

The energy generated in the shell is

$$dE = \epsilon\rho(r) 4\pi r^2 dr \tag{1.3.13}$$

if ϵ is the energy generation per kg.
For the gas to be in thermal equilibrium, radiation loss must be equals to the energy gain for the nuclear burning. So, we have

$$\frac{dL(r)}{dr} dr = \epsilon\rho(r)\ 4\ \pi r^2\ dr \tag{1.3.14}$$

$$\frac{dL(r)}{dr} = \epsilon\rho(r) 4\pi r^2 \tag{1.3.15}$$

In problems, where the thermal properties of the model are either not to be investigated or are not important, the equilibrium structures of the gaseous sphere may be resolved by dealing with equations (1.3.3) and (1.3.11) using some suitable equations of state together with boundary conditions
At center r=0, M(r)=0 and M(r)=M, P=0, or P_s , ρ=0, or ρ_s at surface
A number of the theoretical as well as numerical studies regarding the equilibrium structure of gaseous spheres, particularly those which have particular reference to the problems of the gaseous spheres, particularly those which have particular reference to the problems of the equilibrium structures of the stars are available in literature (Chandershekhar (8), Kippenhahn and Wiegert (25), Eddington (12), Cox and Giuli (12), Mentzel et al. (38))

1.4 Average technique of Kippenhahn and Thomas

In order to study the effects of rotation and tidal distortions on the equilibrium structure of gaseous spheres, Kippenhahn and Thomas (24) developed the concept of topologically equivalent spherical surfaces corresponding to actual equipotential surfaces of a rotationally and tidally distorted model. They define on these equivalent spherical surfaces, quantities such as $\bar{f}$, $\bar{g}$ etc. which denote certain averages of the quantities f, g, respectively on the actual equipotential surfaces. If ψ denotes the total potential (gravitation, rotation and tidal forces) of a rotationally and tidally distorted model at an arbitrary point P(x,y,z) then $\psi(x, y, z)$ = constant is an equipotential surface. Let V_ψ be the volume enclosed by the equipotential surface. ψ = constant and S_ψ the surface area of this equipotential surface. For any function f(x,y,z) they define $\bar{f}$ as its mean value over the equipotential surfaces ψ= constant by the relation

$$\bar{f} = \frac{1}{S_\psi} \int_{\psi=constant} f d\sigma \tag{1.4.1}$$

where $d\sigma$ denotes the surface element of the equipotential surface ψ = constant. Clearly $\bar{f}$ is a function of equipotential surface ψ only and can be obtained as equation (1.4.1) for each equipotential surface ψ = constant. Kippenhahn and Thomas (24) also define a variable r_ψ in analogy with the radius of sphere by the relation

$$V_\psi = \frac{4}{3}\ \pi\ r_\psi^3 \tag{1.4.2}$$

Also by definition

$$S_\psi = \int_{\psi=constant} d\sigma \tag{1.4.3}$$

Obviously, in general, S_ψ is not equal to 4 π r_ψ^2. Kippenhahn and Thomas (24) define a function g(x,y,z) by the relation as

$$g = \frac{d\psi}{dn} \tag{1.4.4}$$

This g corresponds to the force of gravity of a sphere. dn be the distance between two neighboring surfaces ψ = constant and $\psi + d\psi$ = constant is, in general, not constant (i.e. not same at all points of the surface). They used equation (1.4.4) to compute the mean values $\bar{g}$ and $\bar{g^{-1}}$ by using relations

$$\bar{g} = \frac{1}{S_\psi} \int_{\psi=constant} \frac{d\psi}{dn}\ d\sigma \qquad \bar{g^{-1}} = \frac{1}{S_\psi} \int_{\psi=constant} (\frac{d\psi}{dn})^{-1}\ d\sigma \tag{1.4.5}$$

Both $\bar{g}$ and $\bar{g^{-1}}$ are functions of ψ alone and represent the value of g and g^{-1} respectively over the topologically equivalent spherical surface. The volume dV_ψ between the surface ψ = constant and $\psi +\ d\psi$ = constant is given by

$$dV_\psi = \int_{\psi=constant} dn d\sigma = \int_{\psi=constant} (\frac{d\psi}{dn})^{-1} dn = S_\psi \bar{g^{-1}} d\psi \tag{1.4.6}$$

Kippenhahn and Thomas (24) also defined dimensionless parameters u, v and w as

$$u = \frac{S_\psi}{4\ \pi\ r_\psi^2}, \quad v = \frac{\bar{g}\ r_\psi^2}{G\ M_\psi}, \quad w = \frac{\bar{g}^{-1} G\ M_\psi}{r_\psi^2} \tag{1.4.7}$$

where M_ψ is the mass enclosed by equipotential surface ψ = constant. We may thus regard the equipotential surface ψ = constant to be topologically equivalent to a sphere of radius r_ψ for which various functions are defined by the above relations.It may be noticed that if ψ is the gravitational potential of a sphere then the surface ψ = constant is spherical surface with r_ψ = r for which u = 1 and $g = \frac{G\ M_\psi}{r_\psi^2}$ is constant on these spheres and therefore v and w are constants and equal to 1.
Above equations (1.4.1) and (1.4.7) are purely mathematical definitions, which have been applied by Kippenhahn and Thomas (24) to gravitational fields of gaseous spheres distorted by rotational and tidal forces. In hydrostatic equilibrium the equipotential surfaces are also surface of equipressure and equidensity. Therefore, on an equipotential surface the pressure P_ψ and the density ρ_ψ are also constant. Using these concepts, Kippenhahn and Thomas (24) obtained the equations governing the equilibrium structure of a rotationally and tidally distorted stellar model in the following manner.

From equation (1.4.2) the mass dM_ψ between the equipotential surface ψ= constant and $\psi + d\psi$ = constant is given by

$$dM_\psi = dV_\psi\ \rho_\psi = 4\ \pi\ r_\psi^2\ \rho_\psi\ dr_\psi \tag{1.4.8}$$

Thus, we get

$$\frac{dM_\psi}{dr_\psi} = 4\ \pi r_\psi^2\ \rho_\psi \tag{1.4.9}$$

From equation (1.4.6) and (1.4.8) we have

$$d\psi = \frac{d\psi}{dV_\psi}\ dV_\psi = \left(\frac{dV_\psi}{d\psi}\right)^{-1} \frac{dM_\psi}{\rho_\psi} = \frac{dM_\psi}{S_\psi \bar{g}^{-1} \rho_\psi} \tag{1.4.10}$$

Using relations (1.4.7), we get

$$d\psi = \frac{GM_\psi dM_\psi}{4\pi r_\psi^4 \rho_\psi uw} \tag{1.4.11}$$

The conditions for hydrostatic equilibrium, $\frac{dP_\psi}{d\psi} = -\rho_\psi$,can now be written with equation (1.4.7) in the form

$$\frac{dP_\psi}{dM_\psi} = -\frac{GM_\psi}{4\pi r_\psi^4} f_p \tag{1.4.12}$$

where

$$f_p = \frac{1}{uw} = \frac{4\pi r_\psi^4}{GM_\psi} \frac{1}{S_\psi \bar{g^{-1}}} \tag{1.4.13}$$

The factor f_p is a function of ψ only. If ψ is known the equipotential surface can be determined, and then consequently values of S_ψ, r_ψ, $\bar{g}$ and $\bar{g^{-1}}$ for each equipotential surface can be obtained simply from the geometry of the equipotential. The mass M_ψ which depends on the density distribution ρ_ψ can be determined by integrating the equation (1.4.9). Similarly the other structure equations derived by Kippenhahn and Thomas (24), which includes the effects of rotation and tidal distortions on the equilibrium structure of gaseous spheres are as follows.
For chemically homogenous spheres, the nuclear energy generation rate depends only upon density ρ_ψ and the temperature T_ψ and are, therefore, constant on equipotential surface. Thus, if L_ψ is the energy which passes per second through the equipotential surface ψ = constant, then

$$\frac{dL_\psi}{dM_\psi} = \epsilon \tag{1.4.14}$$

Using equation (1.4.8), it can be written as

$$\frac{dL_\psi}{dM_\psi} = 4\pi r_\psi^2 \rho_\psi \epsilon \tag{1.4.15}$$

If the energy is transported by radiation, then the energy transport equation is

$$F_\psi = -\frac{4acT_\psi^3}{3\kappa} \frac{d\psi}{dn} \frac{dT_\psi}{dM_\psi} \frac{4\pi r_\psi^4 uw}{GM_\psi} \tag{1.4.16}$$

where F_ψ is the radiative flux on the equipotential surface ψ = constant. By integrating F_ψ over the equipotential surface ψ = constant, we get

$$L_\psi \;=\; \int_{\psi=constant} F_\psi d\sigma$$

$$= -\frac{4acT_\psi^3}{3\kappa}\frac{d\psi}{dn}\frac{dT_\psi}{dM_\psi}uw\frac{4\pi r_\psi^4}{GM_\psi}\int_{\psi=constant}\frac{d\psi}{dn}d\sigma$$

$$= \frac{64\pi^2 acT_\psi^3 r_\psi^4}{3\kappa}u^2vw\frac{dT_\psi}{dM_\psi} \quad (1.4.17)$$

so that

$$\frac{dT_\psi}{dM_\psi} = -\frac{3\kappa L_\psi}{64\pi^2 acT_\psi^3 r_\psi^4}f_T \quad (1.4.18)$$

Using equation (1.4.8), this equation can be expressed as

$$\frac{dT_\psi}{dr_\psi} = -\frac{3\kappa\rho_\psi L_\psi}{16\pi acT_\psi^3 r_\psi^2}f_T \quad (1.4.19)$$

where

$$f_T = \frac{1}{u^2vw} \quad (1.4.20)$$

Equations (1.4.8), (1.4.12), (1.4.13) and (1.4.18) which are the four basic equations governing the equilibrium structure of a gaseous sphere distorted by rotational and tidal forces may now be collected together and written as.

$$\frac{dM_\psi}{dr_\psi} = 4\pi r_\psi^2\rho_\psi \quad (1.4.21)$$

$$\frac{dP_\psi}{dM_\psi} = -\frac{GM_\psi}{4\pi r_\psi^4}{}^{f_p} \quad (1.4.22)$$

$$\frac{dL_\psi}{dM_\psi} = \epsilon \quad (1.4.23)$$

and

$$\frac{dT_\psi}{dM_\psi} = -\frac{3\kappa L_\psi}{64\pi^2 acT_\psi^3 r_\psi^4}f_T \quad (1.4.24)$$

where $F_P = \frac{1}{uw}$ and $F_T = \frac{1}{u^2w}$

These reduced to the normal equations used for determining the equilibrium structures of spherical models of stars when distortion parameters u,v,w are set one each. The boundary conditions which the above equations has to satisfy are

$$M_\psi = 0, L_\psi = 0, \quad at \;\; the \;\; center \quad r_\psi = 0 \tag{1.4.25}$$

$$M_\psi = M_0, L_\psi = L_{\psi_s}, P_\psi = 0, T_\psi = 0 \tag{1.4.26}$$

or

$$P_\psi = P_{\psi s}, \qquad T_\psi = T_{\psi s} \tag{1.4.27}$$

at the free surface $r_\psi = R_\psi$
where M_0 is the total mass of the model and $L_{\psi s}, \;\; P_{\psi s}, \;\; T_{\psi s}$ are the values of $L_\psi, \;\; P_\psi, \;\; T_\psi$ respectively, on the outermost equipotential surface.

1.5 Present Work

In astrophysics it has great importance to study the problems of equilibrium structures of stars distorted by rotational and tidal effects. This types of problem helps in better understanding to inner structure and stability of binary system. So, the aim of the present thesis is to study the effects of eccentric orbits and asynchronous on the equilibrium structure of distorted stars.

It is quite a complex problem to determine the equilibrium structure of distorted stellar models analytically. Therefore, researchers tried to solve such problems by approximations. In one of those attempts, Mohan, Saxena and Aggarwal used Kippenhahn and Thomas averaging technique together with the results of Kopal on Roche Equipotential, to determine the effects of rotational and tidal distortion on the equilibrium structure of binary stars. However, the problems of determining the equilibrium structures of eccentric orbits and asynchronous on the rotationally and tidally distorted polytropic models of stars have not been satisfactory tackled so far. In the present thesis an attempt has been made to study the effects of eccentric orbits and asynchronous on the equilibrium structures of rotationally and tidally distorted polytropic models of stars.

This thesis consists of three chapters. Chapter one is introductory in nature where we discussed the astrophysical importance of studying the equilibrium structure of rotationally and tidally distorted stellar models. Chapter two deals with the concept of Roche equipotential of distorted stars which accounts for the effects of eccentric orbits and asynchronous rotation of the binary system. Along with the results of Roche equipotential, the explicit expressions for distortional parameters u, v, w, f_P and f_T are obtained by using Kippenhahn and Thomas averaging technique. In this chapter we also discussed about how to find the equilibrium structure of rotationally and tidally distorted stellar models in presence of eccentric orbits and asynchronous rotation.

The methodology discussed in chapter-II is next used in chapter-III to determine the equilibrium structure of rotationally and tidally distorted polytropic models of stars. Computation is carried out to determine the structure as well as various parameters like volume and surface area of the rotationally and tidally distorted polytropic models with various values of degree of non synchronous parameter F. Conclusion based on present study has finally been discussed.

Chapter 2

Equilibrium Structure Of Rotationally And Tidally Distorted Stellar Models

2.1 Modified Roche Equipotential Of Distorted Stars

Roche equipotential have often been used to analyze the problems of rotationally and tidally distorted models of stars. In order to introduce the concept of Roche equipotential, we assume two components of a close binary system known as primary and secondary star. The primary star is supposed to be much more massive than the secondary which is assumed as a point mass causing tidal effects on the more massive primary component. Both the components of binary system are assumed to be rotating about their axes as well as revolving about their common center of mass. Following Kopal (28), Mohan and Singh (45), Mohan et al. (43) certain results on Roche equipotential which are of practical interest to the present study, are summarized below:
Suppose that M_1 and M_2 are the masses of the two components of a close binary system separated by distance D. Further suppose that the primary component of this system of mass M_1 is much larger than its companion star of mass M_2 ($M_1 \geq M_2$) which can be regarded as a point mass. Next suppose that the position of the two components is referred to a rectangular system of Cartesian coordinates with origin at the center of gravity of mass M_1, the x-axis along the line joining the mass centers of two components and z-axis perpendicular to the plane of the orbit of the two components. Then the total potential ψ due to the gravitational and disturbing force acting at an arbitrary point

P(x,y,z), which is not inside any of these two gaseous spheres is given by:

$$\psi = \frac{GM_1}{r} + \frac{GM_2}{r_2} + \frac{\Omega^2}{2}[(x - \frac{M_2 D}{M_1 + M_2})^2 + y^2] \tag{2.1.1}$$

where $r^2 = x^2 + y^2 + z^2$ and $r_2^2 = (D - x)^2 + y^2 + z^2$ represent the squares of the distances of P from the center of gravity of the two components, Ω denotes the angular velocity of rotation of the system about an axis perpendicular to the xy-plane and passing through the center of gravity of the system and G the constant of gravitation.
The first, second and third term on the right hand side of equation (2.1.1) respectively represent the potential which arises due to the mass M_1 of the primary component, the disturbing potential of its companion of mass M_2 and the potential arising from the centrifugal force respectively. Equation (2.1.1) strictly holds at points which are outside the components of binary system. In case we assume Roche model for the primary (In Roche model it is assumed that the total mass of a star is concentrated at its center and this point mass is surrounded by an evanescent envelope in which density varies inversely as the square of the distance from its center) and a point mass for the secondary component, equation (1.4.21-1.4.24) holds everywhere. In order to study some kind of effects of eccentric orbit and non-synchronous rotations, Wilson (1971) developed the generalised potential at point P which is given by

$$\psi = \frac{1}{r} + q\big[(D^2 + r^2 - 2r\lambda D)^{\frac{-1}{2}} - \frac{r\lambda}{D^2}\big] + \frac{1}{2}F^2(1+q)r^2(1-\nu^2) \tag{2.1.2}$$

In roche model the potential formulation should reduces when e=0 and F=1. Where F is the ratio of angular rotation rate to the synchronous rate. Here, F is a constant, not phase dependent. Palvec (47) gives the solution to the non-synchronous problem. Plavec-Limber solution accounts for non-synchronous by including a factor F^2 in the centrifugal term of the Roche binary potential equation, the synchronous form of which can be found in various references, such as Kopal (27) et.al.The binary dimensionless potential generalized to account for asynchronous and non circular orbits is

$$\psi^* = \frac{1}{r^*} + q\left[(1 + r^{*2} - 2r^*\lambda)^{\frac{-1}{2}} - r^*\lambda\right] + \frac{1}{2}F^2(1+q)r^{*2}(1-\nu^2) \tag{2.1.3}$$

where $q = \frac{M_2}{M_1}$ is a non-dimensional parameter representing the ratio of mass of the secondary over primary and 2n represents the square of the normalized angular velocity Ω .

where

$$\frac{D\psi}{GM_1} - \frac{M_1^2}{2M_1(M_1+M_2)}$$

is the non-dimensional form of total potential ψ and $r^* = r/D$ is non-dimensional form of. Also $\lambda = \sin\theta\cos\phi$, $\mu = \sin\theta\sin\phi$ and $\nu = \cos\theta$, (r,θ,ϕ) being the polar spherical coordinate of the point P.

Also if we assume that the angular velocity Ω is identical with Keplerian angular velocity, that is,

$$\Omega^2 = \Omega_k^2 = G\frac{M_1+M_2}{D^3} \tag{2.1.4}$$

then we get a relation of the type

$$n = \frac{q+1}{2} \tag{2.1.5}$$

The equation (2.1.1) reduces to the potential of a purely rotating spherical model if q=0. For n=0, it reduces to the potential of a non-rotating spherical model distorted by the tidal effects of the companion alone. Equation (2.1.3) reduces to the potential earlier obtained by Kopal (27) and Mohan and Saxena (41) for F=1.

The surfaces generated by setting ψ= constant on the left hand side of equation (2.1.1) are referred to as Roche equipotential. Roche equipotential in dimensionless form may be represented by ψ^* = constant where ψ^* is same as defined in equation (2.1.3). The form of Roche-equipotential depends entirely upon the values of ψ. If ψ is large the corresponding equipotential consist of two separate ovals, closed around each of the two mass points. For specified values of M_1, M_2, Ω and D the right hand side of equation (2.1.1) can be large only if r and r_2 becomes small. Therefore, large values of ψ correspond to equipotential which differ little from spheres surrounding each of the two mass centers. With decreasing values of ψ, these spherical equipotential surfaces become oval shaped and get elongated in the direction of the center of gravity of the system until for a certain critical value of ψ, which is characteristic of each mass ratio, both oval shaped surfaces unite at a single point on the x-axis to form a dumbbell like configuration. These limiting values of ψ are called Roche limits. For certain mass ratios Kopal (27) computed the numerical values of Roche limits in the case of synchronous binary stars for values of q ranging from zero to one. Defining a non-dimensional variable r_0 by the relation

$$r_0 = \frac{1}{\psi^* - q} \tag{2.1.6}$$

Kopal (63) has also shown that on the surface of Roche equipotential, (r, θ, ϕ) are connected through the relation

$$r^* = r_0[1 + c_1 r_0^3 + c_2 r_0^4 + c_3 r_0^5 + c_4 r_0^6 + c_5 r_0^7 + c_6 r_0^8 + c_7 r_0^9 + c_8 r_0^{10} + ...] \tag{2.1.7}$$

where

$$\begin{aligned}
c_1 &= qP_2 + nF^2(1 - \nu^2), \quad c_2 = qP_3, \quad c_3 = qP_4 \\
c_4 &= qP_5 + 3C_1^2, c_5 = qP_6 + 7qP_3C_1 \\
c_6 &= qP_7 + 8qP_4c_1 + 4q^2P_3^2 \\
c_7 &= qP_8 + 9qP_5C_1 + 9q^2P_3P_4 \\
c_8 &= qP_6 + 3qP_6C_1 + 10q^2P_3P_5 + 5q^2P_4^2
\end{aligned}$$

Here, $P_j = P_j(\lambda)$ are the Legendre polynomials and terms up to second order of smallness in n and q have been retained in equation (2.1.7). This relation helps to obtain the shape of a Roche equipotential $\psi^* =$ constant. The volumde enclosed by the equipotential surface $\psi^* =$ constant is given by

$$V_\psi = \frac{2}{3}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^3}{\mu} d\lambda d\nu \tag{2.1.8}$$

Kopal has shown that the explicit expression of V_ψ in terms of r_0 defined by equation (2.1.6), can be represented as

$$V_\psi = \frac{4}{3}\Pi D^3 r_0^3[1 + 2nF^2r_0^3 + (\frac{12}{5}q^2 + \frac{8}{5}F^2nq + \frac{32}{5}F^4n^2)r_0^6 + \frac{15}{7}q^2r_0^8 + 2q^2r_0^{10} + ...] \tag{2.1.9}$$

where terms up to second order of smallness in n and q are retained. Following the approach of Kopal (27), Mohan and Singh (45) obtained the explicit expressions for the surface area S_ψ and the values of averages or parameters r_ψ, g and $\bar{g}^{-1}$ on the Roche equipotential ψ^* constant. These are

$$\begin{aligned}
S_\psi &= 2\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^2}{\mu} d\lambda d\nu \qquad (2.1.10) \\
&= 4\Pi D^2 r_0^2[1 + \frac{4n}{3}F^2r_0^3 + (\frac{7}{5}q^2 + \frac{14}{15}F^2nq + \frac{56}{15}F^4n^2)r_0^6 + \frac{9}{7}q^2r_0^8 + \frac{11}{9}q^2r_0^{10} + ...]
\end{aligned}$$

$$r_\psi = (\frac{3V_\psi}{4\Pi})^{\frac{1}{3}} \tag{2.1.11}$$
$$= Dr_0[1 + \frac{2n}{3}F^2r_0^3 + (\frac{4}{5}q^2 + \frac{8}{15}F^2nq + \frac{76}{45}F^4n^2)r_0^6 + \frac{5}{7}q^2r_0^8 + \frac{2}{3}q^2r_0^{10} + ...]$$

$$\bar{g} = \frac{2}{S_\psi}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}}(\frac{d\psi}{dn})\frac{r^2}{\mu}d\lambda d\nu \tag{2.1.12}$$
$$= \frac{GM_\psi}{D^2r_0^2}[1 - \frac{8n}{3}F^2r_0^3 - (3q^2 + 2F^2nq + \frac{40}{9}F^4n^2)r_0^6 - \frac{51}{14}q^2r_0^8 - \frac{13}{3}q^2r_0^{10} + ...]$$

and

$$\bar{g}^{-1} = \frac{2}{S_\psi}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}}(\frac{d\psi}{dn})^{-1}\frac{r^2}{\mu}d\lambda d\nu \tag{2.1.13}$$
$$= \frac{D^2r_0^2}{GM_\psi}[1 + \frac{8n}{3}F^2r_0^3 + (\frac{26}{5}q^2 + \frac{52}{15}F^2 + \frac{524}{45}F^4n^2)r_0^6 + \frac{40}{7}q^2r_0^8 + \frac{11}{9}q^2r_0^{10} + ...]$$

$$r_0 = r_\psi^*[1 - \frac{2n}{3}F^2r_0^3 - (\frac{4}{5}q^2 + \frac{8}{15}F^2nq - \frac{4}{45}F^4n^2)r_0^6 - \frac{5}{7}q^2r_0^8 - \frac{2}{3}q^2r_0^{10} + ...] \tag{2.1.14}$$

where $r_\psi^* = \frac{r_\psi}{D}$, r_ψ^* being the non dimensional form r_ψ. In all the above expressions terms up to second order of smallness in n and q have been retained.

2.2 Mohan, Saxena and Agarwal's approach for computing the effects of rotationally and tidally distortions on the equilibrium structures of gaseous spheres

Mohan, Saxena and Agarwal (43) used the concept of Roche equipotential proposed by Kopal (27) in conjunction with Kippenhahn and Thomas's (24) averaging approach to explicitly obtain equations governing the equilibrium structures and periods of radial and non radial oscillations of rotationally and tidally distorted stars and applied these to analyze the problems of rotating stars and stars in binary systems.

In order to determine the inner structure of a rotationally and tidally distorted gaseous sphere, the system of equations (1.4.21-1.4.24) has to be integrated numerically subject to the boundary conditions equation (1.4.27) specified there in. Therefore, the evaluation of the actual equipotential surface of a rotationally and tidally distorted gaseous sphere is complicated. Kippenhahn and Thomas (24) proposed that for evaluation of the distortion parameters u, v, w, f_P, f_T etc., the actual equipotential surface may be replaced by Roche equipotential surface (It may be noted that this approximation is

reasonably valid for most of the models of the actual stars. In fact as far back as 1933, Chandrasekhar (8) had shown that for stars whose central density bears to the mean density a ratio of 100 or more, the Roche model of a rotating configuration will represent the actual equipotential surfaces of the star within an error of less than one percent).
Once the equipotential surfaces of a rotationally and tidally distorted star are approximated by the Roche equipotential, the results obtained by Kopal (27) and Mohan and Singh (45) may be used to evaluate explicitly the values of the distortion parameters u, v, w, f_P and f_T appearing in stellar structure equations (1.4.12) and (1.4.19). Following Mohan et al. (40) using equations (1.4.7), (1.4.12), (1.4.19) and (2.1.9- 2.2.14) the explicit expressions of the distortions parameters u, v, w, f_P and f_T incorporating the effects of eccentricity orbit and asynchronous rotation on the equipotential surface are as follows:

$$\begin{aligned} u &= [1-(\frac{1}{5}q^2+\frac{2}{15}F^2nq+\frac{4}{45}F^4n^2)r_0^6 \\ &- \frac{1}{7}q^2r_0^8-\frac{1}{9}q^2r_0^{10}+...] \end{aligned} \tag{2.2.1}$$

$$\begin{aligned} v &= [1-\frac{4n}{3}F^2r_0^3-(\frac{7}{5}q^2+\frac{14}{15}F^2nq+\frac{188}{45}F^4n^2)r_0^6 \\ &- \frac{31}{14}q^2r_0^8-3q^2r_0^{10}+...] \end{aligned} \tag{2.2.2}$$

$$\begin{aligned} w &= [1+\frac{4n}{3}F^2r_0^3+(\frac{18}{5}q^2+\frac{12}{5}F^2nq+\frac{272}{45}F^4n^2)r_0^6 \\ &+ \frac{30}{7}q^2r_0^8-\frac{17}{9}q^2r_0^{10}+...] \end{aligned} \tag{2.2.3}$$

$$\begin{aligned} f_P &= [1-\frac{4n}{3}F^2r_0^3-(\frac{17}{5}q^2+\frac{34}{15}F^2nq+\frac{188}{45}F^4n^2)r_0^6 \\ &- \frac{29}{7}q^2r_0^8-2q^2r_0^{10}+...] \end{aligned} \tag{2.2.4}$$

$$\begin{aligned} \textit{and} \quad f_T &= [1-(\frac{14}{15}q^2+\frac{28}{15}F^2nq+\frac{133}{45}F^4n^2)r_0^6 \\ &- \frac{46}{14}q^2r_0^8-\frac{34}{9}q^2r_0^{10}+...] \end{aligned} \tag{2.2.5}$$

where $r_\psi^* = \frac{r_\psi}{D}$, r_ψ^* being the dimensionless form r_ψ. In all the above expressions terms up to second order of smallness in n and q have been retained.
The values of M_ψ, P_ψ, $L\psi$, etc. on the various equipotential surfaces of a rotationally and tidally distorted gaseous spheres may now be obtained by solving the system of differential equations (1.4.21-1.4.24) with boundary conditions (1.4.27) and using the values of distortion parameters f_P and f_T as given in (2.2.1-2.2.5).

It may be noted that while approximating the equipotential surfaces of a rotationally and tidally distorted model by Roche equipotential, the structure of the star is not approximated by the structure of a Roche model. In the case of no distortion (n=q=0), equations (2.2.1-2.2.5) gives u=v=w=f_P=f_T=1 and the system of differential equations (1.4.21-1.4.24) reduce to the equations governing the equilibrium structure of the original undistorted star and not of the Roche model.
Usual methods for stellar structure equations can be now used to integrate the system of differential equations (1.4.21-1.4.24) governing the equilibrium structure of a rotationally and tidally distorted gaseous sphere. At every step, the values of the distortion parameters u, v, w, f_P and f_T have to be computed using (2.2.1-2.2.5).
In case the thermal properties are not considered important and only hydrostatic equilibrium of a rotationally and tidally distorted gaseous spheres is to be investigated then we need only to integrate equations (1.4.9) and (1.4.12) subject to the boundary conditions: At the center $r_\psi = 0, M_\psi = 0$
and at the free surface $r_\psi = R_\psi$,

$$\begin{aligned} M_\psi &= M_0, P_\psi = 0 \\ \rho_\psi &= 0 \;\; or \;\; P_\psi = P_{\psi S}, \; \rho_\psi = \rho_{\psi S} \end{aligned} \tag{2.2.6}$$

In case the star is being distorted by rotational forces alone (or tidal forces alone) we may set q=0, n=0 and still use the above approach to determine the equilibrium structure of corresponding purely rotationally distorted or purely tidally distorted model. For obtaining the structure of the primary component of a synchronous binary system we may set $n = \frac{q+1}{2}$. Mohan and Saxena (41) found it more convenient to work with r_0 in place of M_ψ or r_ψ as independent variable by using (1.3.4) which is connected with variable through relation (2.1.14). Saxena (51) expressed the system of differential equations governing the equilibrium structure of a rotationally and tidally distorted stellar model as

$$\frac{dM_\psi}{dr_0} = 4\pi D^3 \rho_\psi r_0^2 f_1 \tag{2.2.7}$$

$$\frac{dP_\psi}{dr_0} = -\frac{GM_\psi}{Dr_0^2}\rho_\psi f_2 \tag{2.2.8}$$

$$\frac{dL_\psi}{dr_0} = 4\pi\epsilon D^3 \rho_\psi r_0^2 f_1 \tag{2.2.9}$$

and

$$\frac{dT_\psi}{dr_0} = -\frac{3\kappa L_\psi}{16\pi DacT_\psi^3}\frac{\rho_\psi}{r_0^2}f_3 \tag{2.2.10}$$

Here f_1, f_2 and f_3 are functions of n, q and r_0 incorporating the effects of rotation and tidal distortions on the equilibrium structure equations of a stellar model. The explicit expressions for these distortion parameters as given by Saxena (51) are

$$\begin{aligned} f_1 &= [1+4nF^2r_0^3+(\frac{36}{5}q^2+\frac{24}{5}F^2nq+\frac{864}{45}F^4n^2)r_0^6 \\ &+ \frac{55}{7}q^2r_0^8+\frac{26}{3}q^2r_0^{10}+...] \end{aligned} \tag{2.2.11}$$

$$\begin{aligned} f_2 &= [1+(\frac{3}{5}q^2+\frac{8}{15}F^2nq+\frac{112}{45}F^4n^2)r_0^6-\frac{6}{7}q^2r_0^8 \\ &- 4q^2r_0^{10}+...] \end{aligned} \tag{2.2.12}$$

$$\begin{aligned} f_3 &= [1+\frac{4n}{3}F^2r_0^3+(\frac{6}{5}q^2+\frac{4}{5}F^2nq+\frac{199}{15}F^4n^2)r_0^6 \\ &+ \frac{24}{15}q^2r_0^8+\frac{20}{9}q^2r_0^{10}+...] \end{aligned} \tag{2.2.13}$$

In these above expressions terms up to second order of smallness in n, q, F, and r_0 up to r_0^{10} in r_0 are retained. The boundary conditions governing the system of differential equations are: At the center

$$r_0 = 0, M_\psi = 0, L_\psi = 0 \tag{2.2.14}$$

and at the free surface $r_0 = r_{0s}$

$$\begin{aligned} M_\psi &= M_0, L_\psi = L_{\psi s} \\ P_\psi &= 0, \rho_\psi = 0, T_\psi = 0 \;\; or \;\; P_\psi = P_{\psi s}, \;\; \rho_\psi = \rho_{\psi s}, \;\; T_\psi = T_{\psi s} \end{aligned} \tag{2.2.15}$$

where r_{0s} being the value of r_0 at the free surfaces. In fact

$$r_{0s} = \frac{1}{\psi_s^* - q}$$

where ψ_s^* is the non dimensional form of the total potential ψ on the outermost equipotential surface of a rotationally and tidally distorted stellar model. In the case of no distortion $f_p = f_T = 1$ and the above equations reduce to the usual equations governing the equilibrium structure of an undistorted gaseous sphere.

Chapter 3

Effects Of Eccentric Orbits And Asynchronous On The Equilibrium Structures Of Rotationally And Tidally Distorted Polytropic Models Of Stars

Polytropic models have been mathematically exploited in literature to perceive the inner structures of realistic stars. Chandershekar(8) developed the theory of distorted polytropes in order to study the effects of eccentric orbits and asynchronous on the equilibrium structures of rotationally and tidally distorted polytropic models of stars.

3.1 Introduction To Polytropic Models Of Stars

In a Polytropic Model the pressure P and the density ρ are linked with

$$P = P_c \theta^{N+1} \qquad and \qquad \rho = \rho_c \theta^N \tag{3.1.1}$$

where ρ_c and P_c are the density and pressure of the model at the center and θ is a parameter depending upon the distance of the arbitrary point from the center. The term N represents polytropic index which lies between 0 and 1. Index N is a measure of central concentration of the model. A polytropic model with N=0 has a homogeneous structure in which density is uniform whereas if N=5 means highly centrally condensed model whose radius extends to infinity.
The second order non linear differential equation determining the equilibrium structure of a polytropic model of index N is given by the solution of non-linear differential

equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^N \tag{3.1.2}$$

subject to boundary conditions

$$At \quad the \quad center, \quad \theta = 1, \frac{d\theta}{d\xi} = 0 \tag{3.1.3}$$

$$At \quad the \quad surface, \quad \theta = 0, \xi = \xi_s$$

Equation (3.1.2) is represents Lane-Emden Equation. For N=0, 1 and 5 only the analytical solutions are possible. To calculate the parameters of stars having inner structure as a polytrope of index N, the solution of Lane-Emden equation satisfies the boundary conditions (3.1.3) and thus helps to compute parameters of the star having inner structure as a polytrope of index N.

3.2 Polytropic Models For Determining The Equilibrium Structure Of Eccentric Orbits And Asynchronous Rotationally And Tidally Distorted Star

Assume ρ_ψ and P_ψ be the density and pressure on the equipotential surface $\psi =$ constant of the distorted model. Then the values of the pressure and the density on the equipotential surface of the corresponding spherical model will be ρ_ψ and P_ψ respectively. It is assumed that the distorted model behave as polytropic model so that ρ_ψ and P_ψ are linked through the polytropic type of relations

$$\rho_\psi = \rho_{c\psi}\theta_\psi^N \qquad and \qquad P_\psi = P_{c\psi}\theta_\psi^{N+1} \tag{3.2.1}$$

Where $\rho_{c\psi}$ and $P_{c\psi}$ are values of ρ_ψ and P_ψ and $\theta\psi$ is some average of θ on the equipotential surface $\psi = constant$. The index N used in equation (3.2.1) is called the polytropic index of the model which lies between 0 to 5 for the practical interest to the problem of stellar structure. The fact, N represents the central condensation of the model. Equations which govern the hydrostatic equilibrium rotationally and tidally distorted gaseous sphere can be combined to yield

$$\frac{1}{r_0^2}\frac{d}{dr_0}[\frac{r_0^2}{\rho_\psi f_2}\frac{dP_\psi}{dr_0}] = -4\Pi G D^2 P_\psi f_1 \tag{3.2.2}$$

On substituting the values of P_ψ and ρ_ψ from equation (3.2.1), this relation can be expressed in the dimensionless form as

$$\frac{1}{r_0^2}\frac{d}{dr_0}[\frac{r_0^2}{\rho_\psi f_2}\frac{d\theta_\psi}{dr_0}] = -\frac{D^2}{\alpha^2}\theta_\psi^N f_1 \tag{3.2.3}$$

where

$$\alpha^2 = \frac{(N+1)P_{c\psi}}{4\pi G\rho_{c\psi}^2}$$

and f_1 and f_2 are the functions of distortion parameters n, q, F and r_0. The boundary conditions are specified as follows: P_ψ and ρ_ψ must be maximum at the center and zero at free surface, we should have θ_ψ maximum at the center and the zero at the free surface. Thus the boundary conditions satisfied by (3.2.2) are

At the center :$r_0 = 0, \quad \theta_\psi = 1, \quad \frac{d\theta_\psi}{dr_0} = 0$
At the surface: $r_0 = r_{os}$, $\theta_\psi = 0$ where r_{0s} is the value of r_0 at surface.

In the absence of any distortion ($f_1 = f_2 = 1$), equation (3.2.3) becomes Lane-Emden equation for an undistorted polytropic model of index N.

The quantity α defined as defined in equation (3.2.3) is of dimension of length. The term ξ will become non dimensional defined for equivalent spherical model if $r_\psi = \alpha\xi$. It coincides with Emden variable ξ of the Lane-Emden equation for an undistorted spherical polytropic model.The dimensionless k is the ratio of the undistorted radius R_ψ of primary and D the distance between the centers of the two components.

$$\frac{D}{\alpha} = \frac{D\xi_u}{\alpha\xi_u} = \frac{D}{R_\psi\xi_u} = \frac{1}{k}\xi_u \tag{3.2.4}$$

where ξ_u is the value of ξ at the outermost surface of the undistorted polytropic model. For k=1, only rotational forces will distort polytropic model. In case the polytropic model being the primary component then K must be such that the outermost surface of the primary component lies within the Roche lobe.

From equation (3.2.3) and equation (3.2.3-3.2.4), we finally obtained

$$\frac{d}{dr_0}\left(A(r_0,n,q)\frac{d\theta_\psi}{dr_0}\right) = -\frac{\xi_u^2}{k^2}r_0^2\theta_\psi^N B(r_0,n,q) \tag{3.2.5}$$

where

$$\begin{aligned} A(r_0,n,q) &= \frac{r_0^2}{f_2} = r_0^2\Bigg[1-\left(\frac{3}{5}q^2+\frac{18}{15}qnF^2+\frac{112}{45}n^2F^4\right)r_0^6 \\ &- \frac{6}{7}q^2r_0^8 - 4q^2r_0^10...\Bigg] \\ B(r_0,n,q) &= f_1 = \Bigg[1+4nF^2r_0^3+(\frac{36}{5}q^2+\frac{24}{5}qnF^2+\frac{96}{5}n^2F^4)r_0^6 \\ &+ \frac{55}{7}q^2r_0^8+\frac{26}{3}q^2r_0^{10}...\Bigg] \end{aligned}$$

Equation (3.2.5) subject to boundary conditions determines the equilibrium structure of asynchronously rotationally and tidally distorted polytropic model. For q=0, this can only be used to determine the effect of rotational forces. For n=0, then it allows to study the equilibrium structure of polytropic model distorted by tidal forces alone. If F=1, the above expressions reduces to one earlier obtained by Mohan and Saxena (44). In this section we represent explicit executions for measures the volume and surface area of a rotationally and tidally distorted polytropic model. The approach of Mohan and Saxena (44) the total volume enclosed by rotationally and tidally distorted polytropic model is given by

$$\begin{aligned} V_\psi &= \frac{4\pi}{3}\left(\frac{\alpha\epsilon_u}{K}\right)^3 r_{0S}^3[1+2nF^2r_{0S}^3+(\frac{12}{5}q^2+\frac{8}{5}F^2nq+\frac{32}{5}F^4n^2)r_{0S}^6 \\ &+ \frac{15}{7}q^2r_{0S}^8+...] \\ S_\psi &= 4\Pi r_{0S}^2\left(\frac{\alpha\epsilon_u}{K}\right)^2[1+\frac{4n}{3}F^2r_{0S}^3+(\frac{7}{5}q^2+\frac{14}{15}F^2nq+\frac{56}{15}F^4n^2)r_{0S}^6 \\ &+ \frac{9}{7}q^2r_{0S}^8+...] \end{aligned}$$

3.3 Numerical Computation

In order to determine the numerical solution of the equation (3.2.5), we start integrate from the center r_0, using $\theta_\psi = 1$ and $\frac{d\theta_\psi}{dr_0} = 0$ as the initial conditions. N, ξ_u ,n ,q and K respectively denotes the polytropic index, the radius of undistorted polytropic model, angular velocity of rotation, the ratio of the mass of companion and the ratio of the undistorted radius of the primary to the distance D between primary and secondary component. Therefore, for obtaining the numerical solution, equation (3.2.5) has been integrated numerically using fourth order Runge-Kutta method. The series solution is

$$\theta_\psi = 1 - \frac{k^2}{6} r_0^2 + \frac{Nk^4}{120} r_0^4 - \frac{2nk^2}{15} r_0^5 - \frac{N(8N-5)k^6}{15120} r_0^6 + \frac{Nnk^4}{70} r_0^7 ... \tag{3.3.1}$$

This series solution coincides with the one obtained for undistorted polytropic model obtained by Chandrasekhar. The value of r_0 when θ_ψ becomes zero determines the outermost free surface of model. Further pressure P_ψ and density ρ_ψ on various equipotential surface of the distorted model can be obtained and the other physical parameters of distorted polytropic stars are also computed.

TABLE 3.1(a) UNDISTORTED MODEL

Polytropic index N=1.5, ξ_u=3.653750 n=0.0, q=0.0, k=1.0			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	1.000001	0.24318	1.677600
0.2	1.000001	0.24318	1.677600
0.4	1.000001	0.24318	1.677600
0.6	1.000001	0.24318	1.677600
0.8	1.000001	0.24318	1.677600
1.0	1.000001	0.24318	1.677600

TABLE 3.1(b)TIDALLY DISTORTED MODEL

Polytropic index N=1.5, ξ_u=3.653750 n=0.0, q=0.0, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499974	0.24383	1.677893
0.2	0.499974	0.24383	1.677893
0.4	0.499974	0.24383	1.677893
0.6	0.499974	0.24383	1.677893
0.8	0.499974	0.24383	1.677893
1.0	0.499974	0.24383	1.677893

TABLE 3.1(c)ROTATIONALLY DISTORTED MODEL

Polytropic index N=1.5, ξ_u=3.653750 n=0.1, q=0.0, k=1.0			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	1.00001	0.204323	1.677630
0.2	0.998733	0.205184	1.682338
0.4	0.994681	0.207726	1.696181
0.6	0.987189	0.211690	1.717707
0.8	0.975492	0.216479	1.743861
1.0	0.959126	0.221067	1.769665

TABLE 3.1(d) ROTATIONALLY AND TIDALLY DISTORTED MODEL (NON-SYNCHRONOUSLY)

Polytropic index N=1.5, ξ_u=3.653750 n=0.1, q=0.1, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.49974	0.204383	1.677893
0.2	0.499894	0.204491	1.678484
0.4	0.499651	0.204814	1.680246
0.6	0.499239	0.205348	1.683160
0.8	0.498649	0.206091	1.687203
1.0	0.497868	0.207033	1.692326

TABLE 3.1(e) ROTATIONALLY AND TIDALLY DISTORTED MODEL (SYNCHRONOUSLY)

Polytropic index N=1.5, ξ_u=3.653750 n=0.55, q=0.1, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499974	0.204383	1.677893
0.2	0.499528	0.204974	1.681121
0.4	0.498131	0.206720	1.690627
0.6	0.495622	0.210228	1.709661
0.8	0.491785	0.213109	1.725302
1.0	0.486426	0.217061	1.746959

TABLE 3.2(a) UNDISTORTED MODEL

Polytropic index N=3.0, ξ_u=6.896850 n=0.0, q=0.0, k=1.0			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	1.000001	1.374172	5.7977388
0.2	1.000001	1.374172	5.7977388
0.4	1.000001	1.374172	5.7977388
0.6	1.000001	1.374172	5.797738
0.8	1.000001	1.374172	5.7977388
1.0	1.000001	1.374172	5.7977388

TABLE 3.2(b) TIDALLY DISTORTED MODEL

Polytropic index N=3.0, ξ_u=6.896850 n=0.0, q=0.0, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499991	1.374680	5.978636
0.2	0.499991	1.374680	5.978636
0.4	0.499991	1.374680	5.978636
0.6	0.499991	1.374680	5.978636
0.8	0.499991	1.374680	5.978636
1.0	0.499991	1.374680	5.978636

TABLE 3.2(c) ROTATIONALLY DISTORTED MODEL

Polytropic index N=3.0, ξ_u=6.896850 n=0.1, q=0.0, k=1.0			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	1.000000	1.374172	5.977388
0.2	0.999296	1.382358	6.001097
0.4	0.996861	1.406612	6.071020
0.6	0.991790	1.445351	6.182046
0.8	0.988731	1.494017	6.321402
1.0	0.968885	1.543376	6.464971

TABLE 3.2(d) ROTATIONALLY AND TIDALLY DISTORTED MODEL (NON-SYNCHRONOUSLY)

Polytropic index N=3.0, ξ_u=6.896850 n=0.1, q=0.1, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499982	1.374680	5.978636
0.2	0.499937	1.375698	5.981582
0.4	0.499799	1.378743	5.990393
0.6	0.499560	1.383801	6.005006
0.8	0.499209	1.390858	6.025364
1.0	0.498727	1.399855	6.051264

TABLE 3.2(e) ROTATIONALLY AND TIDALLY DISTORTED MODEL (SYNCHRONOUSLY)

Polytropic index N=3.0, ξ_u=6.896850 n=0.55, q=0.1, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499982	1.374680	5.978636
0.2	0.499728	1.380259	5.994774
0.4	0.498891	1.312820	5.797824
0.6	0.497252	1.423845	6.120091
0.8	0.494488	1.459466	6.221963
1.0	0.490242	1.500091	6.338468

TABLE 3.3(a) UNDISTORTED MODEL

Polytropic index N=4.0, ξ_u=14.97155 n=0.0, q=0.0, k=1.0			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	1.000001	14.056920	28.167240
0.2	1.000001	14.056920	28.167240
0.4	1.000001	14.056920	28.167240
0.6	1.000001	14.056920	28.167240
0.8	1.000001	14.056920	28.167240
1.0	1.000001	14.056920	28.167240

TABLE 3.3(b) TIDALLY DISTORTED MODEL

Polytropic index N=4.0, ξ_u=14.97155 n=0.0, q=0.0, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499991	14.062840	28.174080
0.2	0.499991	14.062840	28.174080
0.4	0.499991	14.062840	28.174080
0.6	0.499991	14.062840	28.174080
0.8	0.499991	14.062840	28.174080
1.0	0.499991	14.062840	28.174080

TABLE 3.3(c) ROTATIONALLY DISTORTED MODEL

Polytropic index N=4.0, ξ_u=14.97155 n=0.1, q=0.0, k=1.0			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	1.000001	14.056920	28.167240
0.2	0.999705	14.158140	28.302250
0.4	0.998468	14.460790	28.703850
0.6	0.995299	14.955200	29.354760
0.8	0.988731	15.600760	30.200510
1.0	0.977249	16.299240	31.122250

TABLE 3.3(d) ROTATIONALLY AND TIDALLY DISTORTED MODEL (NON-SYNCHRONOUSLY)

Polytropic index N=4.0, ξ_u=14.97155 n=0.1, q=0.1, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499991	14.062840	28.174080
0.2	0.499971	14.075370	28.190790
0.4	0.499909	14.113000	28.240950
0.6	0.499797	14.175740	28.324460
0.8	0.499621	14.263450	28.440970
1.0	0.499363	14.378930	28.594000

TABLE 3.3(e) ROTATIONALLY AND TIDALLY DISTORTED MODEL (SYNCHRONOUSLY)

Polytropic index N=4.0, ξ_u=14.97155 n=0.55, q=0.1, k=0.5			
F	r_{0S}	$V_\psi * 10^{-3}$	$S_\psi * 10^{-2}$
0.0	0.499991	14.062840	28.174080
0.2	0.499877	14.131860	28.266070
0.4	0.499453	14.338520	28.540490
0.6	0.498482	14.679390	28.990420
0.8	0.496594	15.140580	29.595600
1.0	0.493351	15.687170	30.311760

3.4 Conclusion

In order to study the effects of eccentricity and asynchronous rotation on the equilibrium structure of polytropic models, certain physical parameters have been computed and presented in tables. Results presented in tables (3.1-3.3) are the values of F, r_{0s}, volume V_ψ and surface area s_ψ at surface for all the rotationally and tidally distorted polytropic models of indices 1.5, 3.0, 4.0. On comparing the volumes and surface areas of distorted models are larger than the corresponding values for the undistorted models. And on comparing to their tidal effects, the effects produced by rotational distortion, the volume and surface area of the distorted models are larger. It has also been seen that compared to their both rotationally and tidally effects,the effects produced by synchronously, the volume and surface area of the distorted models are larger. The F which is the ratio of angular rotations rate to the synchronous rate is an important parameter. If F=1, then it represents a synchronous rotations. So, we have chosen F as 0.0, 0.2, 0.4, 0.6, 0.8 and 1.0 to analyze its impact on the structure of distorted polytropic stars. Our results are verifies with all the results earlier obtained by Mohan and Saxena(38) for F=1. As F decreases from 1 to 0, the respective values decrease with respect to the corresponding values earlier obtained by Mohan and Saxena(38).

Bibliography

[1] Alexender, M. E., Astrophysics and Space Science, 23, 259, 1973.

[2] Agarwal, S. R., The effect of rotation and tidal distortion on the structure and periods of small adibatics oscillations of composite models of stars, Ph.D. Thesis, Roorkee University, Roorkee, india. 1987.

[3] Avni, Y., Astrophysical journal, 209, 574, 1976

[4] Batten, Binary and Multiple Systems of stars, A. H. 1973.

[5] Bhatnagar, P. L. and Kothari D.S., Monthly Notices of Royal Astronomical Society, London, 104, 292, 1949.

[6] Bhatnagar, P. L. and Kushwaha, R. S., Anharmonic pulsations of Roche's model, Bull. Calcutta Math. Society., 43, 95, 1951.

[7] Bohm-Vitense, E. ,Introduction to stellar Astrophysics Volume1 - Basic stellar observations and data, Cambridge University Press Publication,1989.

[8] Chandrashekhar, S., Steller Structure, Dover publication., 1939.

[9] Clement, M. J., Normal modes of oscillations for rotating stars. II Variational solutions, Astrophysical Journal, 276, 724-736, 1984.

[10] Cowling, T. G., The nonradial oscillations of polytopic stars, Monthly Notices of Royal Astronomical Society, London, 101, 367-375, 1941.

[11] Cox J. P., Theory of stellar pulsation, Princeton University. Press, 1980.

[12] Cox, J. P., Giuli, H. T, Principles of steller structure, Vol I and II, Gordon and Breach, New York, Lodon, Paris, 1968.

[13] Eddigton, A. S., Monthly Notices of Royal Astronomical Society, London , 79, 177, 1918.

[14] Eddigton, A. S., Monthly Notices of Royal Astronomical Society, London, 101, 182, 1941.

[15] Eddigton, A. S., Internal constitution of the stars, Dover publication, 1959.

[16] Fadeyev Yu. A., Monthly Notices of Royal Astronomical Society, London, 262, 119-127, 1993.

[17] Goodricke, j., Phil. Trans., 76, 48, 1786.

[18] Gurm, H. S., Monthly Notices of Royal Astronomical Society, London, 126, 425, 1963.

[19] Hansen, C. J., Cox, J. P. and Carroll, B. W., The quasi-adibatic analysis of non-radial models of stellar oscillation in the presence os slow rotation, Astrophysical Journal, 226, 210-221, 1978.

[20] Hilditch, R. W., An Introduction to Close Binary stars, Cambridge Astrophysics, 2001.

[21] Horedt, G. P., Polytropes: Applications in Astrophysics and Related field, Kluwer Academic Publishers, Dordrecht, 2004.

[22] Hurley, M., and Roberts, P. H. 1964, ApJ, 180, 583

[23] J. F. Spinsky, B.Willems, and V. Kalogera, Equipotential surfaces and lagrangian points in non synchronous, evventric binary and planetary systems, Astrophysical Journal,660, 1624-1635, 2007.

[24] Kippenhahn, R. and Thomas, H. C., A simple method for the solution of the stellar structure equation including rotation and tidal forces, Stellar Rotation, Proceedings of IAU Colloq 4, Ohio State University, Columbus, ed. Slettebak, A., D., Gordon and Breach Science Publishers, 1970.

[25] Kippenhahn, R. and Weigert, A., Stellar Structure and Evolution, Springer Verlag Publ., 1990.

[26] Kopal, Z. and Ali, A. K. M. Sekender., On the integrability of the Rochr coordinates, Astrophysics and space science, 11, 423-429, 1971.

[27] Kopal, Z., Nonradial oscillations of the standard model, Astrophysical Journal, 109, 509 - 527, 1949.

[28] Kopal, Z., Roche Model and its Application to Close Binary Systems, Advances in Astronomy and Astrophysics, 9, 1-65, 1972.

[29] Kruszewski, A., Exchange of matter in close binary systems. Equilibrium configurations in the case of deviations from synchronism, Acta Astronomica, 13, 106-117, 1963.

[30] Lal, A. K., Mohan, C. and Singh, V. P., Equilbrium structures of differentially rotating and tidally distorted white dwarfs models of stars, Astrophysics and Space Science, 301, 51-60, 2006.

[31] Lal, A. K., Structure oscillations and stability of differentially rotating and tidally distorted stellar models, PhD Thesis, Roorkee University, Roorkee,India, 1993.

[32] Lal, A. K., Pathania, A., Bhalla, A. and Mohan, C., Effects of differential rotations on the eigenfrequencies of small adiabatic barotropic modes of oscillations of polytropic models of stars, Journal of Physics A Mathematical and Theoretical (IOP), 42, 485212, 2009.

[33] Lal, A. K., Pathania, A. and Mohan, C., Effect of Coriolis force on the equilibrium structures of rotating stars and stars in binary systems, Astrophysics and Space Science, 315, 157 165, 2008.

[34] Lal, A. K., Kumar, T., and Pathania, A., Eigenfrequencies of rotationally and tidally distorted white dwarf models of stars, Astrophysics and Space Science, 345, 155-167, 2013.

[35] Linnell, A. P. 1977, Ap and SS, 48, 165

[36] Linnell, A. P. 1981, Ap and SS, 80, 501

[37] Malhotra, C. L, Ph.D Thesis, Roorkee University, 1968.

[38] Mentzel, D.H., Bhatnagar, P.L. and Sen, H.K., Stellar Interiors, Chapman and Hall Ltd., London,1963.

[39] Mohan, C. and Saxena, R. M., Effect of rotation and tidal distortion on the adiabatic barotropic modes of oscillations of polytropic models of stars, Astrophys. Space Sci., 113, 155, 1985.

[40] Mohan, C. and Agarwal, S. R., Effects of rotation and tidal distortions on the periods of adiabatic oscillations of composite models of stars, Astrophysics and Space Science, 129, 73 - 94, 1987.

[41] Mohan, C. and Saxena, R. M., Effect of rotation and tidal distortion on the structure of polytropic stars, Astrophysics and Space Science, 95, 369-381, 1983.

[42] Mohan, C. and Saxena, R. M., Effect of rotation and tidal distortion on the periods of small adiabatics oscillations of polytropics models of stars , Astrophysics and Space Science, 113, 155-169, 1985.

[43] Mohan, C., Saxena, R. M. and Agarwal, S. R., Equilibrium structures of rotationally and tidally distorted stellar models, Astrophysics and Space Science, 163, 23 - 39, 1990.

[44] Mohan, C., Saxena, R. M., and Aggarwal, S. R., Eigenfrequencies of small adiabatic barotropic modes of oscillation of roationally and tidally distorted stars, Astrophysics and Space Science., 178, 89, 1991.

[45] Mohan, C. and Singh, V. P., Use of Roche coordinates in the problems of small oscillations of rotationally distorted stellar models, Astrophysics and Space Science, 54, 293 - 304, 1978.

[46] Pathania, A., Lal, A. K and Mohan, C, On the use of Roche equipotentials in analyzing the problems of binary and rotating stars, Journal of Astrophysics and Astronomy (JOAA), 34, 69-73, 2013.

[47] Plavec,M.,Tables for the Roche model of closed binaries,Bulletin of the Astronomical Institute of Czechoslovakia,15,165-170,1864.

[48] Plavec,M.1958,Mem.Soc.Roy.Sci.Liege,20,411.

[49] Roberts,P.H.,On highly rotating polytropes,I.,Astrophysical Journal,137,1129-1141,1963.

[50] Saio, H., Rotational and tidal perturbation of nonradial oscillations in a polytropic star, Astrophysical Journal, 244, 299-315, 1981.

[51] Saxena, R. M., The effects of rotation and tidal distortions on the structure and periods of small oscillations of gaseous spheres, PhD Thesis, Roorkee University, Roorkee, India, 1984.

[52] Schwarzchild, M., structure and evolution of the stars, Princeton University Press, 1958.

[53] Van der Borght, R. and Murphy, J. O., Anharmonic Pulsation of an early main-sequence Star, Astrophysics and Space Science, 131, 225-235, 1966. bibitem Wilson, R. E., and Biermann, P.1976, Astr. Ap., 48, 349.

[54] Wilson, R. E., and Caldwell, C.N., Ap. J., 221, 917, 1978.

[55] Wilson, R. E., and Devinney, E. J., Ap. J.,166, 605, 1971.

[56] Wilson, R. E., Eccentric Orbit Generalization And Simultaeous Solution Of Binary Star Light And Velocity Curves, Astrophysical Journal, 234, 1054-1066, 1979.

[57] Wilson, R. E., and Sofia, S., Ap. J., 203, 182, 1976.

[58] Wilson, R. E., and Wilson, A. T., Ap. J., 204, 551, 1976.