On Automorphism Groups of Finitely Generated Groups

Thesis

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SANDEEP SINGH
(Regn. No. 901111003)

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Declaration of Authorship

I hereby declare that the work which is being presented in this thesis entitled *"On Automorphism Groups of Finitely Generated Groups"*, submitted by me, for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar University, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Deepak Gumber, Associate Professor, School of Mathematics, Thapar University, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or Abroad and that the ideas and references cited herein have been duly acknowledged.

(Sandeep Singh)

Regd. No. 901111003

CERTIFICATE

This is to certify that the thesis *"On Automorphism Groups of Finitely Generated Groups"* which is submitted by Mr. Sandeep Singh, in fulfillment of the requirement for the award of the degree of *Doctor of Philosophy* in the School of Mathematics, Thapar University, Patiala, is a record of the candidate's own independent and original research work carried out by him under my supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any University or Institute for the award of any degree.

Attestation by supervisor

(Dr. Deepak Gumber)
Associate Professor
School of Mathematics
Thapar University
Patiala 147 004
INDIA.

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September, 2015 *(Sandeep Singh)*

Dedicated
To
my
Spiritual Guru
Saint Dr. Gurmeet Ram Rahim Singh Ji Insan

Dera Sacha Sauda is a Social Welfare and Spiritual Organization that preaches and practices humanitarianism and selfless services to others. **Beparawah Mastana Ji Maharaj** founded Dera Sacha Sauda on April 29th, 1948 to encourage spiritual awakening among the masses, to uplift humanity, and to create a better world. With devotion and hard work, Maharaj Ji transmuted the barren land of Sirsa into a spiritual garden and imparted the glorious method of meditation to HIS followers. The slogan **"Dhan Dhan Satguru Tera Hi Asra,"** ('Dhan Dhan' (only God is worth praising) — 'Satguru' (the Supreme Being) which is synonymous with the words Om, Allah, Waheguru, Ram, Parmatma, God, etc., as used in many of the world's religions — 'Tera Hi Asra' (we are solely dependent on HIM and only HE can help us everywhere in all eventualities)), which Dera's devotees chant at the Ashram, elucidates the kindness of God Almighty as our saviour. His Excellency Guruji preached about following the path of truth, humanity, and hard work. Shortly thereafter, His Holiness appointed **Param Pita Shah Satnam Singh Ji Maharaj** as his successor on 28th February 1960. After a few decades, on 23rd September 1990, Param Pita Shah Satnam Ji Maharaj announced **Saint Dr. Gurmeet Ram Rahim Singh Ji Insan**, the present Spiritual Master, as his successor.

Presently, Dera has been undertaking around 106 social welfare activities starting with helping in road accidents and extending to the protection of daughters from heinous fetal murder and solemnizing the marriages of harlots by inspiring them to quit this abhorrent profession. All these social works are very surprising things for a humankind to do. Here, i would like mention some of works for example, **HEART DONATION, KIDNEY DONATION, BODY DONATION, TRUE BLOOD PUMP(Blood donation), Sukh Dua Samaj(Hijra/ Kinnars), HOMELY SHELTER, FOOD BANK, CLOTH BANK, CLEANSE CAMPAIGN** etc. For more detail of their social works and about them visit the official website "http://www.derasachasauda.org/" of Dera Saucha Sauda Sirsa.

Abstract

Let G be a group and let $\mathrm{Aut}(G)$ denote the group of all automorphisms of G. An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\varphi(g) \in Z(G)$, the center of G, for all $g \in G$. The group of all central automorphisms of G is denoted as $\mathrm{Aut}^{z}(G)$. An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1}xg_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group of all inner automorphisms of G is denoted by $\mathrm{Inn}(G)$ and by $\mathrm{Aut}_c(G)$, we denote the group of all class-preserving automorphisms of G.

The main objective of the thesis is to study isomorphism between two automorphism groups of a finitely generated group. Chapter 1, contains the introductory part and some basic definitions. Following Bachmuth [10], we call an automorphism φ of G an IA-automorphism if it induces the identity automorphism on the abelianized group G/G'. In other words, if $g^{-1}\varphi(g) \in G'$, the commutator subgroup of G, for all $g \in G$. In chapter 2, we classify all finitely generated nilpotent groups G of class 2 for which $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$. In particular, we classify all finite nilpotent groups of class 2 for which each IA-automorphism is inner.

Following Hegarty, we analogously call an automorphism α an absolute central automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$, where $L(G)$ is the absolute center of G. Let $\mathrm{Aut}^{l}(G)$ denote the group of all absolute central automorphisms of G and let $\mathrm{Aut}_z^{l}(G)$ denote the group of all those absolute central automorphisms of G which fix $Z(G)$ element-wise. In chapter 3, we obtain very short and easy proofs of main results of Moghaddam and Safa [49]. We also study finitely generated groups G with $G' \leq L(G)$ such that $\mathrm{Aut}^{l}(G) \simeq \mathrm{Inn}(G)$ and, as a consequence, obtain the necessary and sufficient conditions on a finite p-group G such that $\mathrm{Aut}^{l}(G) = \mathrm{Inn}(G)$. Our results generalize the results obtained by Nasrabadi and Farimani [52, Main Theorem].

In chapter 4, we prove a technical lemma, Lemma 4.2.1, and as a consequence give a short and easy proof of main theorem of Azhdari and Malayeri [8, Theorem 0.1]. We also obtain short and alternate proofs of Corollary 2.1 of Azhdari and Malayeri [9], Propostion 1.11 and Theorem 2.2(i) of Azhdari [7]. Some other related results for finitely generated and finite p-groups are also obtained.

An automorphism α of a group G is called a commuting automorphism if each element x in G commutes with its image $\alpha(x)$ under α. Let $A(G)$ denote the set of all commuting automorphisms of G. A group G is said be of co-class 2 if it is nilpotent group of class $n-2$. Rai [Proc. Japan Acad., Ser. A **91** (2015), no. 5, 57-60] has given some sufficient conditions on a finite p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ and, as a consequence, has proved that in a finite p-group G of co-class 2, where p is an odd prime, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. In chapter 5, we give very elementary and short proofs of main results of Rai and obtain some other related results.

List of Research Papers

(1) S. Singh, D. Gumber, H. Kalra, *IA-automorphisms of finitely generated nilpotent groups*, J. Algebra Appl., **13** (2014), 1450027 (5 pages).

(2) S. Singh, D. Gumber, *Autocentral automorphisms of finitely generated groups* (Communicated).

(3) S. Singh, D. Gumber, *Finite p-groups whose absolute central automorphisms are inner*, Math. Commun. **20** (2015), 125-130.

(4) S. Singh, D. Gumber, *Isomorphism between automorphism groups of finitely generated groups* (Communicated).

(5) S. Singh, D. Gumber, *A note on commuting automorphisms of some finite p-groups* (Communicated).

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CHAPTER 1

Introduction and Basics

1.1 — Introduction

Let G be an arbitrary group and let G' and $Z(G)$ respectively denote the commutator subgroup and the center of G. Let $\mathrm{Aut}(G)$ denote the full automorphism group of G. An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1} x g_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\mathrm{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut}_c(G)$ of all class-preserving automorphisms of G. An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\varphi(g) \in Z(G)$, the center of G, for all $g \in G$. The group of all central automorphisms of G is denoted as $\mathrm{Aut}^{z}(G)$.

Our main objective in the thesis is to study different automorphism groups of finitely generated groups and to study the structure of finitely generated groups whose different automorphism groups are isomorphic. Interest in the equality of two different automorphism groups dates back to 1908, when Hilton [34, p. 233] asked the following question: Whether a non-abelian group can have an abelian group of isomorphisms (automorphisms), that is, when $\mathrm{Aut}(G) = \mathrm{Aut}^{z}(G)$? An affirmative answer to this question was given by Miller [48] in 1913. He constructed a non-abelian group G of order 64 for which $\mathrm{Aut}(G)$ was abelian of order 128. More examples of such finite 2-groups were constructed by Struik [69] in 1982, Curran [18] in 1987 and Jamali [39] in 2002. In 1911 Burnside [14, Note B] also posed the following question: Does there exist a finite group G such that G has a non-inner class preserving automorphism? In 1913, Burnside [15] himself gave a positive answer to his question. He constructed a group G of order p^6 isomorphic to the group W consisting of all 3×3 matrices of the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}$$

with x, y, z in the field $\mathbb{F}_{p^2}$ of p^2 elements, where p is an odd prime. For this group G, $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$. In 1947, Wall [73] constructed some smaller and simpler groups G for which $\mathrm{Aut}_c(G) \neq \mathrm{Inn}(G)$. The smallest group constructed by Wall is of order 32. In 1968, Sah [63] explored many basic properties of $\mathrm{Aut}_c(G)$ for a finite group G. It is a well known result of Gaschutz [26] that every finite p-group has a non-inner automorphism. Therefore, it becomes interesting topic to study when two different automorphism groups of a group are equal or isomorphic. For more details about this problem, one can see the survey article by Yadav [77] and also papers of some other authors [40, 53, 66].

IA-automorphisms

Following Bachmuth [10], we call an automorphism of G an IA-automorphism if it induces the identity automorphism on the abelianized group G/G'. Let $\mathrm{IA}(G)$ denote the group of all IA-automorphisms of G and let $\mathrm{IA}(G)^*$ denote its subgroup consisting of those IA-automorphisms which fix $Z(G)$ element-wise. In 1917, Nielsen [55] proved that if G is a free group of rank 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$. For free metabelian groups of rank 2, Bachmuth [10] in 1965, obtained that $\mathrm{IA}(G) = \mathrm{Inn}(G)$. A group G is called semicomplete if $\mathrm{IA}(G) = \mathrm{Inn}(G)$. In 1969, Andreadakis [3] proved that a free product $A * B$ of two non-trivial groups is semicomplete if and only if both A and B are abelian. In 1981, Gupta [28] proved that if G is a 2-generator metabelian group, then $\mathrm{IA}(G)$ is always a metabelian group. In 2002, Panagopoulos [58], studied the semicompleteness of the direct product $G = A \times B$ of two groups A and B in relation to the semicompleteness of its direct factors. Recently in 2011, Attar [5, Theorem 2.1] has proved that if G is a finite p-group of class 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic and $\mathrm{IA}(G) = \mathrm{IA}(G)^*$. In **Chapter 2**, we classify finitely generated nilpotent groups of class 2 for which $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ and $\mathrm{IA}^*(G) \simeq \mathrm{Inn}(G)$. In particular, we classify all finite nilpotent groups G of class 2 for which (i) $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and (ii) $\mathrm{IA}(G)^* = \mathrm{Inn}(G)$.

The results of this Chapter have appeared in, " Sandeep Singh, Deepak Gumber and Hemant Kalra, *IA- automorphisms of finitely generated nilpotent groups*, J. Algebra Appl., 13 (2014), 1450027 (5 pages)."

Absolute central automorphisms

Let G be any group. For $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. Inductively, define $[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n]$, where $\alpha_i \in \mathrm{Aut}(G)$. Observe that

$$Z(G) = \{g \in G \mid \alpha(g) = g, \text{ for all } \alpha \in \mathrm{Inn}(G)\},$$

and

$$G' = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in \mathrm{Inn}(G)\rangle.$$

In an analogous manner, Hegarty [29] defined the absolute center $L(G)$ of G as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\}.$$

The autocommutator subgroup G^* of G is defined as

$$G^* = \langle [g, \alpha] \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle.$$

It is easy to see that $L_n(G) \leq Z_n(G)$, the nth term of the upper central series of G, for all $n \geq 1$ and $G' \leq G^*$. An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\varphi(g) \in Z(G)$ for all $g \in G$. Let $\mathrm{Aut}^z(G)$ denote the group of all central automorphisms of G and let $\mathrm{Aut}^z_z(G)$ denote the group of all those central automorphisms which fix the center of G element-wise. Analogous to central automorphisms group, Hegarty [29] in 1994, defined absolute central automorphism group. He called an automorphism

α of G to be absolute central automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. The set of all absolute central automorphisms forms a normal subgroup of $\mathrm{Aut}(G)$ and is denoted by $\mathrm{Aut}^l(G)$. In 2010, Moghaddam and Safa [49] have obtained some results about the nature of absolute central automorphisms. In **Chapter 3**, we give very short and easy proofs of main results of Moghaddam and Safa [49]. Recently in 2015, Nasrabadi and Farimani [52, Main Theorem] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. In this Chapter, we also give necessary and sufficient conditions for a finitely generated group G with $G' \leq L(G)$ such that $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$ and, as a consequence, obtain the necessary and sufficient conditions on a finite p-group G such that $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$, thereby generalizing the result of Nasrabadi and Farimani.

Some results of this Chapter have appeared in, " Sandeep Singh and Deepak Gumber, *Finite p-groups whose absolute central automorphisms are inner*, Math. Commun., 20 (2015), 125-130."

Central automorphisms

An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G; or equivalently $g^{-1}\varphi(g) \in Z(G)$, the center of G, for all $g \in G$. The group of all central automorphisms of G is denoted as $\mathrm{Aut}^{\mathrm{z}}(G)$. The central automorphism group can be as large as possible when all automorphisms are central i.e. when $\mathrm{Aut}^{\mathrm{z}}(G) = \mathrm{Aut}(G)$, and can be as small as possible when $\mathrm{Aut}^{\mathrm{z}}(G) = \mathrm{Z}(\mathrm{Inn}(G))$. If G is abelian, then $\mathrm{Inn}(G)$ is trivial and hence $\mathrm{Aut}^{\mathrm{z}}(G) = \mathrm{Aut}(G)$. If G is non-abelian and if $\mathrm{Aut}^{\mathrm{z}}(G) = \mathrm{Aut}(G)$, then since $\mathrm{Inn}(G)$

is abelian, G is a nilpotent group of class 2. Non-abelian p-groups G for which all automorphisms are central, that is when $\mathrm{Aut}^{z}(G) = \mathrm{Aut}(G)$, have been well studied. If $\mathrm{Aut}(G)$ is abelian, then necessarily $\mathrm{Aut}^{z}(G) = \mathrm{Aut}(G)$, and various authors have considered this situation (for example see [11, 23, 30, 31, 32, 35, 37, 50, 51]). If $\mathrm{Aut}(G)$ is non-abelian, even then all automorphisms may be central and this case has been explored, for example, in [17, 27, 38, 45]. In 2001, Curran and McCaughan [19] gave necessary and sufficient conditions for a finite p-group G such that $\mathrm{Aut}^{z}(G) = \mathrm{Inn}(G)$. They proved that for any finite p-group G, $\mathrm{Aut}^{z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. In 2007, Attar [4] established that for any finite p-group G, $\mathrm{Aut}^{z}_{z}(G) = \mathrm{Inn}(G)$ if and only if G is abelian or nilpotency class of G is 2 and $Z(G)$ is cyclic. In 2009, Yadav [76] gave necessary and sufficient conditions on a finite p-group G of nilpotency class 2 such that $\mathrm{Aut}^{z}(G) = \mathrm{Aut}^{z}_{z}(G)$. Attar [6] in 2012 and Jafari [36] in 2011 have generalized this result. They have given necessary and sufficient conditions on a finite p-group G of arbitrary nilpotence class such that each central automorphism of G fixes the center element-wise. In 2011, Azhdari and Malayeri [8, Theorem 0.1] (see also [9, Theorem 2.3] for correct version) generalized the result of Attar [4] and proved that if G is a finitely generated nilpotent group of class 2, then $\mathrm{Aut}^{z}_{z}(G) \simeq \mathrm{Inn}(G)$ if and only if $Z(G)$ is infinite cyclic or $Z(G) \simeq C_m \times H \times \mathbb{Z}^r$, where $C_m \simeq \prod_{p\in\pi(G/Z(G))} Z(G)_p$, $H \simeq \prod_{p\notin\pi(G/Z(G))} Z(G)_p$, $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ is of finite exponent dividing m. In **Chapter 4**, We prove a more general technical lemma, Lemma 4.2.1 and as a consequence give a short and easy proof of this main theorem of Azhdari and Malayeri. We also obtain short and alternate proofs of Corollary 2.1 of Azhdari and

Malayeri [9], Propostion 1.11 and Theorem 2.2(i) of Azhdari [7]. Some other related results for finitely generated and finite p-groups are also obtained.

Commuting automorphisms

An automorphism α of a group G is called a commuting automorphism if each element x in G commutes with its image $\alpha(x)$ under α. Let $A(G)$ denotes the set of all commuting automorphisms of G. A group is said to be an $A(G)$ group if the set of all commuting automorphisms of G forms a subgroup of $\mathrm{Aut}(G)$. The commuting automorphisms were first studied for various classes of rings [22, 12, 44]. In the year 1984, Herstein [33] proposed the following problem to American Mathematical Monthly: If G is a simple non-abelian group, then prove that $A(G) = 1$. In 1986, Laffey [42] observed that $A(G) = 1$ provided G has no nontrivial abelian normal subgroups, while Pettet [60] proved that $A(G) = 1$ if $Z(G) = 1$ and $\gamma_2(G) = G$. Deaconescu, Silberberg and Walls [21] in 2002, raised the following questions about $\mathrm{A}(G)$:

1. Is it true that the set $\mathrm{A}(G)$ is always a subgroup of $\mathrm{Aut}(G)$?

2. What conditions on G imply the equality $\mathrm{A}(G) = \mathrm{Aut}^{\mathrm{z}}(G)$?

3. Is it true that $\mathrm{A}(G) = 1$ if and only if $\mathrm{Aut}^{\mathrm{z}}(G) = 1$?

Regarding to question 1, Deaconescu *et al.* gave an example of a group of order 2^5 in which $\mathrm{A}(G)$ doesn't form a subgroup. In 2007 , Mihai and Codruta [47] have also proved some related results. In 2013, Vosooghpour and Malayeri [72] showed that minimum order of a non-$A(G)$ p-group G is p^5. in 2013, Fouladi and Orfi [24] have proved that if G is either a finite AC-group or a p-group of maximal class or a metacyclic p-group, then G is an $A(G)$-group. Recently in 2015, Rai [Proc. Japan

Acad., Ser. A **91** (2015), no. 5, 57-60] has given some sufficient conditions on a finite p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ and, as a consequence, has proved that in a finite p-group G of co-class 2, where p is an odd prime, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. In **Chapter 5**, we give very elementary and short proofs of main results of Rai and some other related results are also obtained. We also give two Gap algorithms for finding the set of all commuting automorphisms of a finite group G, and to check whether an automorphism of a finite group is commuting or not.

1.2 — Basics

This section has been taken from thesis of Kalra [41]. In this section, we give a quick review of some of the basic facts of group theory that are assumed in the foregoing chapters. The definitions and proofs of results presented here can be found in any standard book on group theory. We of course suppose a familiarity of more basic group theoretic terms and concepts like abelian, cyclic, coset, normal subgroup, factor or quotient group, homomorphism, isomorphism, direct product et cetera.

Let G be an arbitrary group and X be a subset of G. The intersection of the family of subgroups of G which contain X is a subgroup of G and is denoted by $\boldsymbol{\langle X \rangle}$. In other words, $\langle X \rangle$ is the smallest subgroup of G which contains X. The subgroup $\langle X \rangle$ is called the subgroup generated by X. If X is non-empty, then $\langle X \rangle$ contains every finite product of the type

$$x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}, \;\; r \geq 1, \; x_i \in X, \; m_i \in \mathbb{Z},$$

and conversely all such products form a subgroup of G containing X. It follows that $\langle X \rangle$ consists of all such products. A cyclic group is thus generated by a single element. We shall denote a cyclic group of order m by $\boldsymbol{C_m}$. The ***rank*** of a group G is the smallest cardinality of a generating set of G and is denoted by $\boldsymbol{d(G)}$. The least common multiple of the orders of the elements of a finite group G is called the ***exponent*** of G and is denoted by $\boldsymbol{exp(G)}$.

The ***commutator*** of two elements $a, b \in G$ is the element $\boldsymbol{[a,b] = a^{-1}b^{-1}ab}$ of G and the ***commutator subgroup*** or the ***derived subgroup*** $\boldsymbol{G'}$ of G is the subgroup of G generated by all commutators of G. It is easy to see that G' is a normal subgroup of G. If X and Y are two subsets of G, then we define $\boldsymbol{[X,Y] = \langle [x,y] | x \in X, y \in Y \rangle}$. Thus $[X,Y]$ is always a subgroup of G. For $x \in G$, $\boldsymbol{[x,G]}$ denotes the set of all commutators $[x,g]$, where $g \in G$. By $\boldsymbol{K(G)}$ we denote the set of all commutators of G. The followings are well known commutator identities

$$[x, yz] = [x,z][x,y][x,y,z]; \quad [xy, z] = [x,z][x,z,y][y,z],$$

where $x, y, z \in G$ and will be frequently used in the thesis without any reference.

A series

$$1 = G_0 \leq G_1 \leq \ldots \leq G_l = G$$

of subgroups of G is called a ***normal series*** if each G_i is a normal subgroup of G. The normal series above is called a ***central series*** if for each i, $G_{i+1}/G_i \leq Z(G/G_i)$. Let $Z_0 = 1$ and let $Z_{i+1}/Z_i = Z(G/Z_i)$ for $i \geq 0$. Observe that Z_1 is the center of G and Z_{i+1}/Z_i, being the center of G/Z_i, is normal in G/Z_i and hence Z_{i+1} is normal in G for all $i \geq 0$. It follows that the series

$$1 = Z_0 \leq Z_1 \leq Z_2 \leq \ldots$$

is a central series of G. The subgroup Z_i is called the ***i-th center*** and this series is called the ***upper central series*** of G.

We define subgroups $\boldsymbol{\gamma_i(G)}$, $i \geq 1$, of G by setting

$$\gamma_1(G) = G, \ \ \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Observe that $\gamma_2(G) = G'$, each $\gamma_i(G)$ is normal in G and $\gamma_{i+1}(G) \leq \gamma_i(G)$. The series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \gamma_n(G) \geq \dots$$

is called the ***lower central series*** of G. If the lower central series of a group G terminates in a finite number of steps at 1, and if c is the least natural number such that $\gamma_{c+1}(G) = 1$, then G is called a ***nilpotent group*** of class c. The class of a nilpotent group is denoted by $\boldsymbol{cl(G)}$. Observe that if $cl(G) = 2$, then $G' \leq Z(G)$.

A ***maximal subgroup*** of G is a proper subgroup M such that there is no subgroup H of G with $M < H < G$. The intersection of all the maximal subgroups of G is the ***Frattini subgroup*** $\boldsymbol{\Phi(G)}$ of G. If G has no maximal subgroup, then we set $\Phi(G) = G$. An element a of G is called a ***non-generator*** of G if whenever $G = \langle a, X\rangle$, then $G = \langle X\rangle$. An interesting property of $\Phi(G)$ is that it is exactly the set of all non-generators of G.

Two elements a and b of G are called ***conjugate*** if there exists an element g of G such that $b = g^{-1}ag$. It is easily seen that "conjugacy" is an equivalence relation on G and therefore it partitions G into equivalence classes. The equivalence class that contains the element a of G is called the ***conjugacy class*** of a and is denoted as $\boldsymbol{a^G}$. Let A be a non-empty subset of G. The set of elements of G which commute with every element of A is called the ***centralizer*** of A in G, and is denoted as

$\boldsymbol{C_G(A)}$. If $A = \{a\}$ is singleton, then $C_G(\{a\})$ is simply denoted as $C_G(a)$. It is easy to see that $C_G(A)$ is a subgroup of G. For any $a \in G$, $|a^G| = |G|/|C_G(a)|$.

A finite group G is called a ***purely non-abelian*** group if it has no non-trivial abelian direct factor. If $Z(G)$ is cyclic or if $Z(G) \leq \Phi(G)$, then G is purely non-abelian.

An isomorphism α of G to itself is called an ***automorphism*** of G. The set of all automorphisms of G is a group under the usual operation of compositions of mappings. We call this group the full automorphism group of G and denote it by $\mathbf{Aut}(\boldsymbol{G})$.

Let A be an abelian group and let $\mathbf{Hom}(\boldsymbol{G}, \boldsymbol{A})$ denote the set of all homomorphisms of G into A. For $f, g \in \text{Hom}(G, A)$, define $fg(x) = f(x)g(x)$. Then $\text{Hom}(G, A)$ becomes an abelian group under this operation. If A, B, C are all finite abelian groups, then $\text{Hom}(A, B \times C) \simeq \text{Hom}(A, B) \times \text{Hom}(A, C)$ and $\text{Hom}(A, B) \simeq \text{Hom}(B, A)$. Also, $\text{Hom}(C_m, C_n) \simeq C_d$, where $d = \gcd(m, n)$.

For a fix prime number p, a group G is called a $\boldsymbol{p}$**-group** if order of every element of G is a power of p. If G is finite, then G is a p-group if and only if $|G|$, the order of G, is a power of p. The followings are well known facts about p-groups: (i) $G' \leq \Phi(G)$ and (ii) $G/\Phi(G)$ is elementary abelian of rank $d(G)$.

CHAPTER 2

IA-automorphisms of finitely generated nilpotent groups

2.1 — Introduction.

Let G be any group and let G' and $Z(G)$ respectively denote the derived group and the center of G. By $\mathrm{Inn}(G)$ we denote the inner automorphism group of G. Following Bachmuth [10], we call an automorphism of G an IA-automorphism if it induces the identity automorphism on the abelianized group G/G'. Let $\mathrm{IA}(G)$ denote the group of all IA-automorphisms of G and let $\mathrm{IA}(G)^*$ denote its subgroup consisting of those IA-automorphisms which fix $Z(G)$ element-wise. It is easy to see that $\mathrm{Inn}(G) \leq \mathrm{IA}(G)$. A group G is called complete if it has trivial centre and $\mathrm{Aut}(G) = \mathrm{Inn}(G)$; and semicomplete if $\mathrm{IA}(G) = \mathrm{Inn}(G)$. IA-automorphisms have been well studied in past. If G is a free group of rank 2, then it is a classical result of Nielsen [55] that $\mathrm{IA}(G) = \mathrm{Inn}(G)$. Bachmuth [10] obtained the same result for free metabelian groups of rank 2. Gupta [28] proved that if G is a 2-generator metabelian group then $\mathrm{IA}(G)$ is always a metabelian group. Andereadkis [3] proved that a free product $A * B$ of two non-trivial groups is semicomplete if and only if both A and B

are abelian. Panagopoulos [58] studied the semicompleteness of the direct product $G = A \times B$ of two groups A and B in relation to the semicompleteness of its direct factors. Many other authors (see for example [78, 64, 13, 70, 46, 71, 56, 57, 59]) have also studied the structure of $\mathrm{IA}(G)$. Recently Attar [5, Theorem 2.1] has proved that if G is a finite p-group of class 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic and $\mathrm{IA}(G) = \mathrm{IA}(G)^*$. In section 2, we classify all finitely generated nilpotent groups G of class 2 for which (i) $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ and (ii) $\mathrm{IA}(G)^* \simeq \mathrm{Inn}(G)$. In particular, we classify all finite nilpotent groups G of class 2 for which (i) $\mathrm{IA}(G) = \mathrm{Inn}(G)$ and (ii) $\mathrm{IA}(G)^* = \mathrm{Inn}(G)$. As a consequence we prove that if G is a finite 2-generated nilpotent group of class 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$.

It is a well known theorem of Schur [65] that for any group G if $G/Z(G)$ is finite, then G' is finite. Neumann [54, Corollary 5.41] proved the converse of it for finitely generated groups. We call these results as Schur theorem and Neumann theorem respectively.

By X^n we denote the direct product of n copies of a group X. The torsion rank and torsion-free rank of a group G are respectively denoted as $d(G)$ and $\rho(G)$. For a finite group G, G_p and $\pi(G)$ respectively denote the Sylow p-subgroup and the set of primes dividing the order of G.

The following results will be used quite frequently.

Lemma 2.1.1 *If U and V are finite abelian groups and $\exp(U) = \exp(V)$, then $\mathrm{Hom}(U, V) \simeq U$ if and only if V is a cyclic group.*

Lemma 2.1.2 *Let G be a finitely generated nilpotent group of class 2. Then $\exp(T(G')) = \exp(T(G/Z(G)))$, where $T(H)$ denotes the torsion part of a group H.*

2.2 — IA-automorphisms.

Let G be a finitely generated nilpotent group of class 2. Let

$$G/Z(G) \simeq A \times \mathbb{Z}^a,$$

$$G/G' \simeq B \times \mathbb{Z}^b$$

and

$$G' \simeq C \times \mathbb{Z}^c,$$

where A, B, C are respective torsion parts and a, b, c are respective torsion-free ranks of $G/Z(G), G/G'$ and G'. Let

$$\pi(A) = \{p_1, p_2, \ldots, p_d, p'_1, p'_2, \ldots, p'_{d'}\}$$

and

$$\pi(B) = \{p_1, p_2, \ldots, p_e\},$$

where $d \leq e$ because $G/Z(G)$ is a quotient group of G/G'. Suppose that

$$A_{p_i} \simeq \prod_{j=1}^{m_i} C_{p_i^{\alpha_{ij}}},$$

$$A_{p'_i} \simeq \prod_{j=1}^{m'_i} C_{{p'_i}^{\alpha'_{ij}}}$$

and

$$B_{p_i} \simeq \prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}.$$

Then

$$A \simeq \prod_{i=1}^{d} \prod_{j=1}^{m_i} C_{p_i^{\alpha_{ij}}} \times \prod_{i=1}^{d'} \prod_{j=1}^{m'_i} C_{{p'_i}^{\alpha'_{ij}}}$$

and

$$B \simeq \prod_{i=1}^{e}\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}},$$

where $\alpha_{11} \geq \alpha_{12} \geq \ldots \geq \alpha_{1m_1}, \alpha_{21} \geq \alpha_{22} \geq \ldots \geq \alpha_{2m_2}, \ldots, \alpha_{d1} \geq \alpha_{d2} \geq \ldots \geq \alpha_{dm_d}$, $\alpha'_{11} \geq \alpha'_{12} \geq \ldots \geq \alpha'_{1m'_1}, \alpha'_{21} \geq \alpha'_{22} \geq \ldots \geq \alpha'_{2m'_2}, \ldots, \alpha'_{d'1} \geq \alpha'_{d'2} \geq \ldots \geq \alpha'_{d'm'_{d'}}$ and $\beta_{11} \geq \beta_{12} \geq \ldots \geq \beta_{1n_1}, \beta_{21} \geq \beta_{22} \geq \ldots \beta_{2n_2}, \ldots, \beta_{e1} \geq \beta_{e2} \geq \ldots \geq \beta_{en_e}$ are positive integers. Observe that if G is finite, then $m_i \leq n_i$ and $\alpha_{ij} \leq \beta_{ij}$ for all $i, 1 \leq i \leq d$ and for all $j, 1 \leq j \leq m_i$. We fix these notations, unless or otherwise stated, for the rest of this chapter.

We start with the following important lemma.

Lemma 2.2.1 *Let G be a nilpotent group of class 2. Then*

(i) $\mathrm{IA}(G) \simeq \mathrm{Hom}(G/G', G')$ *and*

(ii) $\mathrm{IA}(G)^* \simeq \mathrm{Hom}(G/Z(G), G')$.

Proof. We prove only (i) as the proof of (ii) is similar. Let

$$f : \mathrm{IA}(G) \longrightarrow \mathrm{Hom}(G/G', G')$$

be defined by $f(\alpha) = f_\alpha$, where $f_\alpha : G/G' \longrightarrow G'$ is defined by $f_\alpha(gG') = g^{-1}\alpha(g)$. It is easy to see that the mapping f is well defined. Let $\alpha_1 \neq \alpha_2$ be any two elements of $\mathrm{IA}(G)$. If possible, let $f_{\alpha_1} = f_{\alpha_2}$. Then $g^{-1}\alpha_1(g) = g^{-1}\alpha_2(g)$ for all $g \in G$ and therefore $\alpha_1 = \alpha_2$, a contradiction. Hence f is one-one. Also, for any $g \in G$, $f_{\alpha_1.\alpha_2}(gG') = g^{-1}(\alpha_1.\alpha_2)(g) = g^{-1}\alpha_1(\alpha_2(g)) = g^{-1}\alpha_1(g.g^{-1}\alpha_2(g)) = g^{-1}\alpha_1(g).g^{-1}\alpha_2(g) = f_{\alpha_1}.f_{\alpha_2}(gG')$ implies that f is a homomorphism. We next show that f is an onto mapping. For any $\gamma \in \mathrm{Hom}(G/G', G')$, the map $\beta : G \longrightarrow G$ defined by $\beta(g) = g\gamma(gG')$ is clearly an IA(G) automorphism and $f_\beta = \gamma$ shows that f is onto. This proves the result. □

Theorem 2.2.2 *Let G be a finitely generated nilpotent group of class 2. Then*

(*i*) *if G is torsion-free, then $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ if and only if G' is cyclic and $\rho(G/Z(G)) = \rho(G/G')$.*

(*ii*) *if G is not torsion-free, then $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ if and only if one of the following five conditions holds:*

(*a*) *G is finite, $m_i = n_i$ for $1 \leq i \leq d$, G' is cyclic, and for each i, $1 \leq i \leq d$, either $(G/Z(G))_{p_i}$ is homocyclic or $\alpha_{i1} = \alpha_{it_i}$ for $i1 \leq it_i \leq i(r_i - 1)$ and $\beta_{it_i} = \alpha_{it_i}$ for $ir_i \leq it_i \leq in_i$, where ir_i is the smallest positive integer between $i1$ and in_i such that $\beta_{ir_i} < \alpha_{i1}$.*

(*b*) *$G' \simeq \mathbb{Z}$ and $\rho(G/Z(G)) = \rho(G/G')$.*

(*c*) *G' is torsion, G/G' is torsion-free and $A \simeq C^b$.*

(*d*) *G' is finite and $G/Z(G) \simeq \mathrm{Hom}(B, C) \times C^b$.*

(*e*) *$G/Z(G)$ and $G' \simeq C \times \mathbb{Z}$ are mixed groups, G/G' is torsion-free, $A \simeq C^b$ and $\rho(G/Z(G)) = \rho(G/G')$.*

(*iii*) *$\mathrm{IA}(G)^* \simeq \mathrm{Inn}(G)$ if and only if G' is cyclic.*

Proof. (i) First Suppose that $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$. Then $\mathrm{Hom}(G/G', G') \simeq G/Z(G)$ by Lemma 2.2.1 (i). If G/G' is finite, then $G/Z(G)$ is finite and hence by Schur theorem G' is finite, which is not so. Thus $G/G' \simeq B \times \mathbb{Z}^b$, where $b > 0$. If $G/Z(G)$ has a torsion element, then so does G', which is not so. Thus $G/Z(G)$ is torsion-free. It then follows that $\mathbb{Z}^{bc} \simeq \mathbb{Z}^a$. Since $a \leq b$, $c = 1$ and $a = b$. Hence G' is cyclic and $\rho(G/Z(G)) = \rho(G/G')$. The converse follows easily.

(ii) If $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$, then

$$\mathrm{Hom}(B \times \mathbb{Z}^b, C \times \mathbb{Z}^c) \simeq A \times \mathbb{Z}^a \tag{2.1}$$

by Lemma 2.2.1 (i). First suppose that G is finite. Then $\mathrm{Hom}(B, C) \simeq A$. Since $d(A) \leq d(B)$ and $\exp(A) = \exp(C)$, G' is cyclic and thus $G' \simeq C_{{p_1}^{\alpha_{11}}} \times C_{{p_2}^{\alpha_{21}}} \times \ldots \times C_{{p_d}^{\alpha_{d1}}}$. Observe that

$$\mathrm{Hom}(B, C) \simeq \mathrm{Hom}(\prod_{i=1}^{e}\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\alpha_{i1}}}) \simeq \prod_{i=1}^{d} \mathrm{Hom}(\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\alpha_{i1}}}),$$

and $A \simeq \prod_{i=1}^{d}\prod_{j=1}^{m_i} C_{p_i^{\alpha_{ij}}}$. It thus follows that $\mathrm{Hom}(\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\alpha_{i1}}}) \simeq \prod_{j=1}^{m_i} C_{p_i^{\alpha_{ij}}}$ for each $i, 1 \leq i \leq d$. This immediately implies that $m_i = n_i$. Now either $\beta_{ij} \geq \alpha_{i1}$ for each $j, 1 \leq j \leq n_i$, or there exist smallest positive integer ir_i between $i1$ and in_i such that $\beta_{ir_i} < \alpha_{i1}$. In the first case $(G/Z(G))_{p_i}$ is homocyclic, and in the second case $\alpha_{i1} = \alpha_{it_i}$ for $i1 \leq it_i \leq i(r_i - 1)$ and $\beta_{it_i} = \alpha_{it_i}$ for $ir_i \leq it_i \leq in_i$.

Now suppose that G is infinite. The derived group G' can be torsion, mixed, or torsion-free. First assume that G' is torsion-free. Then $G/Z(G)$ is torsion-free and hence G/G' is not torsion. Equation (2.1) then implies that $\mathbb{Z}^{bc} \simeq \mathbb{Z}^a$. It follows that $a = b$ and $c = 1$. Next assume that G' is torsion. Then $G/Z(G)$ is torsion by Neumann theorem. If G/G' is torsion, then G is finite. If G/G' is torsion free, then (2.1) implies that $A \simeq C^b$; and if G/G' is a mixed group, then (2.1) implies that $A \simeq \mathrm{Hom}(B, C) \times C^b$. Finally assume that G' is a mixed group. Then G/G' is either mixed or torsion-free by Schur theorem. If G/G' is torsion-free, then (2.1) implies that $A \simeq C^b$, $a = b$, $c = 1$; and if G/G' is mixed, then (2.1) implies that $A \simeq \mathrm{Hom}(B, C) \times C^b$ and $a = bc$. It follows that $a = b = 0$. Therefore both $G/Z(G)$ and G/G' are finite. By Schur theorem G' is finite and hence G is finite, a contradiction. Conversely it is not very hard to see that if G satisfies any of the five conditions, then $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$.

(iii) If G' is cyclic, finite or infinite, then $\mathrm{IA}(G)^* \simeq \mathrm{Inn}(G)$ by Lemma 2.2.1 (ii). Conversely suppose that $\mathrm{IA}(G)^* \simeq \mathrm{Inn}(G)$. Then $\mathrm{Hom}(G/Z(G), G') \simeq G/Z(G)$ by Lemma 2.2.1 (ii). If G is torsion-free or torsion, then it is easy to see that G' is cyclic. Suppose that G is a mixed group. If G' is torsion-free or torsion, then again it is easily seen that G' is cyclic. So suppose that G' is a mixed group. Then $G/Z(G)$ is either a mixed or a torsion-free group by Schur theorem. If $G/Z(G)$ is mixed, then $\mathrm{Hom}(A, C) \times C^a \simeq A$, which is not possible because $\exp(A) = \exp(C)$. Finally if $G/Z(G)$ is torsion-free, then $\mathrm{Hom}(\mathbb{Z}^a, C \times \mathbb{Z}^c) \simeq \mathbb{Z}^a$ implies that $G' \simeq \mathbb{Z}$. □

Corollary 2.2.3 *Let G be a 2-generated finite nilpotent group of class 2. Then any* IA*-automorphism of G is an inner automorphism.*

Proof. Observe that G' is cyclic and $d(G/Z(G)) = d(G/G') = 2$. Also $G/Z(G)$ is homocyclic by [51, Lemma 0.4]. The result now follows from Theorem 2.2.2. □

Corollary 2.2.4 *Let G be a 2-generated torsion-free nilpotent group of class 2. Then* $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$.

Proof. Observe that $G' \simeq \mathbb{Z}$ and $\rho(G/Z(G)) = \rho(G/G') = 2$. Thus $\mathrm{IA}(G) \simeq \mathrm{Inn}(G)$ by Theorem 2.2.2. □

Let $\mathrm{Aut}_c(G)$ denote the group of all conjugacy class preserving automorphisms of a group G. As a consequence of Theorem 2.2.2 (iii), which also generalizes Corollary 2.3 of Attar [5], we have the following small but significant generalization of Corollary 3.6 of Yadav [75].

Corollary 2.2.5 *Let G be a finitely generated nilpotent group of class 2 such that G' is cyclic and $G/Z(G)$ is finite. Then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$.

Proof. The proof follows from the fact that $\mathrm{Inn}(G) \leq \mathrm{Aut}_c(G) \leq \mathrm{IA}(G)^*$. □

Let G be a finite p-group of nilpotency class 2 and let $G/Z(G) \simeq \prod_{i=1}^{t} C_{p^{\alpha_i}}$ and $G/G' \simeq \prod_{j=1}^{s} C_{p^{\delta_j}}$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_t$ and $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_s$ are positive integers. Since $G/Z(G)$ is a quotient group of G/G', $t \leq s$ and $\alpha_i \leq \delta_i$ for $1 \leq i \leq t$. Attar [5, Theorem 2.1] has proved that if G is a finite p-group of class 2, then $\mathrm{IA}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic and $\mathrm{IA}(G) = \mathrm{IA}(G)^*$. Our next corollary, which is a particular case of Theorem 2.2.2, gives an explicit interpretation of this result.

Corollary 2.2.6 *Let G be a finite p-group of class 2. Then $\mathrm{IA}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic, $d(G/Z(G)) = d(G/G')$, and either $G/Z(G)$ is homocyclic or $\alpha_i = \alpha_1$ for $1 \leq i \leq r-1$ and $\delta_i = \alpha_i$ for $r \leq i \leq s$, where r is the smallest positive integer such that $\delta_r < \alpha_1$.*

CHAPTER 3

Finite p-groups whose absolute central automorphisms are inner

3.1 — Introduction

Let G be a finitely generated group and let G' and $Z(G)$ respectively denote the commutator subgroup and the center of G. Let $\mathrm{Aut}(G)$ denote the full automorphism group and let $\mathrm{Inn}(G)$ denote the inner automorphism group of G. For $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. Inductively, define

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n],$$

where $\alpha_i \in \mathrm{Aut}(G)$. Hegarty [29] defined the absolute center $L(G)$ of G as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\}.$$

The autocommutator subgroup G^* of G is defined as

$$G^* = \langle [g, \alpha] \mid g \in G, \alpha \in \mathrm{Aut}(G) \rangle.$$

It is easy to see that $L_n(G) \leq Z_n(G)$, the nth term of the upper central series of G, for all $n \geq 1$ and $G' \leq G^*$. An automorphism α of G is called an absolute central automorphism if it induces the identity automorphism on $G/L(G)$; or equivalently, $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. Let $\mathrm{Aut}^l(G)$ denote the group of all absolute central automorphisms of G and let $\mathrm{Aut}^l_z(G)$ denote the group of all those absolute central automorphisms of G which fix $Z(G)$ element-wise. A group G is called autonilpotent of class at most n if $L_n(G) = G$ for some natural number n. Observe that if G is autonilpotent of class 2, then $G^* \leq L(G)$. Let $\mathrm{Aut}^{\mathrm{z}}(G)$ denote the group of all central automorphisms of G and let $\mathrm{Aut}^{\mathrm{z}}_{\mathrm{z}}(G)$ denote the group of all those central automorphisms of G which fix $Z(G)$ element-wise. It is well known (see section 2) that $\mathrm{Aut}^{\mathrm{z}}_{\mathrm{z}}(G) \simeq \mathrm{Hom}(G/G'Z(G), Z(G))$ and if G is a purely non-abelian finite group, then there is a one-to-one correspondence between $\mathrm{Aut}^{\mathrm{z}}(G)$ and $\mathrm{Hom}(G/G', Z(G))$.

Let $E(G) = \langle g^{-1}\alpha(g) | g \in G, \alpha \in C_{\mathrm{Aut}(G)}(\mathrm{Aut}^l(G))\rangle$. It is easy to see that $G' \leq E(G)$ and $\mathrm{Aut}^l(G)$ fixes $E(G)$ element-wise. Moghaddam and Safa [49, Prop. 1, Prop. 2 and Theorem B] proved that

$$\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/L(G), L(G)),$$

$$\mathrm{Aut}^l_z(G) \simeq \mathrm{Hom}(G/E(G)Z(G), L(G))$$

and if $L(G) \leq E(G)$, then

$$\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/E(G), L(G)).$$

They also proved [49, Theorem C] that if G is a purely non-abelian finite group, then

$$\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G, L(G)).$$

In section 2, we prove two lemmas and, as an immediate consequence, obtain these results of Moghaddam and Safa.

Also recently, Nasrabadi and Farimani [52, Main Theorem] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. In Propositions 3.2.10 and 3.2.11, we give necessary and sufficient conditions for a finitely generated group G with $G' \leq L(G)$ such that $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$ and, as a consequence, obtain the necessary and sufficient conditions on a finite p-group G such that $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$.

The following well known results will be used very frequently without further reference.

Lemma 3.1.1 *Let U, V and W be abelian groups. Then*

(i) $\mathrm{Hom}(U \times V, W) \simeq \mathrm{Hom}(U, W) \times \mathrm{Hom}(V, W)$*,*

(ii) $\mathrm{Hom}(U, V \times W) \simeq \mathrm{Hom}(U, V) \times \mathrm{Hom}(U, W)$*,*

(iii) $\mathrm{Hom}(C_m, C_n) \simeq C_d$*, where d is the g.c.d. of m and n,*

(iv) if U is torsion-free of rank m, then $\mathrm{Hom}(U, V) \simeq V^m$*, and*

(v) if U is torsion and V is torsion-free, then $\mathrm{Hom}(U, V) = 1$.

3.2 — Main Results

We start with a crucial lemma which is a small but very useful modification of a lemma of Alperin [2, Lemma 3].

Lemma 3.2.1 *Let G be any group and M be a central subgroup of G contained in a normal subgroup N of G. Then the group of all automorphisms of G that induce the identity on both N and G/M is isomorphic to* $\mathrm{Hom}(G/N, M)$.

Proof. Let $\mathrm{Aut}_N^M(G)$ denote the group of all automorphisms of G that induce identity on both N and G/M. For any $\mu \in \mathrm{Aut}_N^M(G)$, the map

$$\psi_\mu : G/N \longrightarrow M$$

defined as $\psi_\mu(bN) = b^{-1}\mu(b)$ is a homomorphism. The map

$$\psi : \mathrm{Aut}_N^M(G) \longrightarrow \mathrm{Hom}(G/N, M)$$

defined as $\psi(\mu) = \psi_\mu$ is a monomorphism. For any $\tau \in \mathrm{Hom}(G/N, M)$, the map $\mu : G \to G$ defined as $\mu(g) = g\tau(gN)$ is an automorphism of G inducing identity on both N and G/M and $\psi(\mu) = \tau$. Thus ψ is onto as well. □

The following results of Curran [20, Theorem 2.3] and Moghaddam and Safa [49, Prop. 1, Prop. 2 and Theorem B] are immediate consequences of this lemma.

Corollary 3.2.2 $\mathrm{Aut}_z^z(G) \simeq \mathrm{Hom}(G/G'Z(G), Z(G))$.

Corollary 3.2.3 $\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/L(G), L(G))$.

Corollary 3.2.4 $\mathrm{Aut}_z^l(G) \simeq \mathrm{Hom}(G/E(G)Z(G), L(G)) \simeq \mathrm{Hom}(G/Z(G), L(G))$.

Corollary 3.2.5 $\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/L(G)E(G), L(G))$ *and if* $L(G)$ *is contained in* $E(G)$, *then* $\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/E(G), L(G))$.

The next lemma generalizes a well known result of Adney and Yen [1, Theorem 1] and Theorem C of Moghaddam and safa [49].

Lemma 3.2.6 *Let G be a purely non-abelian group satisfying maximal and minimal conditions on normal subgroups. Let M be a central subgroup of G and let $Aut^M(G)$ denote the group of all those automorphisms of G which induce the identity on G/M. Then* (*i*) *there is a one-one correspondence between* $\mathrm{Aut}^M(G)$ *and* $\mathrm{Hom}(G, M)$.

(*ii*) *if* M *is contained in* $L(G)$, *then* $Aut^M(G) \simeq \mathrm{Hom}(G, M)$.

Proof. Define $f : Aut^M(G) \to \mathrm{Hom}(G, M)$ by $f(\alpha) = f_\alpha$, where $f_\alpha : G \to M$ is defined as $f_\alpha(g) = g^{-1}\alpha(g)$. It is easy to see that $f_\alpha \in \mathrm{Hom}(G, M)$ and f is one-one. For $\gamma \in \mathrm{Hom}(G, M)$, the map $\beta : G \to G$ defined as $\beta(g) = g\gamma(g)$ is a normal endomorphism of G. It follows by Fitting's Lemma [62, 3.3.4] that $G = \mathrm{Ker}\,\beta^m \times \mathrm{Im}\,\beta^m$ for some $m \geq 1$. Since G is purely non-abelian, $\mathrm{Ker}\,\beta^m = 1$ and $\mathrm{Im}\,\beta^m = G$. Thus β is an automorphism of G inducing the identity on G/M. It is easy to see that $f(\beta) = \gamma$ and hence f is onto.

(ii) If M is contained in $L(G)$, then f is easily seen to be a homomorphism and hence the result. □

Corollary 3.2.7 *[1, Theorem 1] For a purely non-abelian group* G, *the correspondence* $\sigma \longrightarrow f_\sigma$ *is a one-to-one map of* $\mathrm{Aut}^z(G)$ *onto* $\mathrm{Hom}(G, Z(G))$.

Corollary 3.2.8 *[49, Theorem C] Let G be a purely non-abelian finite group, Then*

$$\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G, L(G)).$$

Lemma 3.2.9 *Let* G *be an autonilpotent group of class* 2. *Then*

(*i*) *for any* $g \in G$ *and* $\alpha \in \mathrm{Aut}(G)$, $[g, \alpha]^m = [g^m, \alpha]$, *for all* $m \geq 1$.

(ii) if $G/L(G) = \langle y_1L(G), y_2L(G), ..., y_nL(G)\rangle$*, then* $G^* = \langle [y_i, \alpha] \mid 1 \le i \le n, \alpha \in \mathrm{Aut}(G)\rangle$.

(iii) $\exp(T(G/L(G))) = \exp(T(G^*))$*, where* $T(G)$ *denotes the torsion part of group* G.

Proof. (i) By definition of G^*, $[g, \alpha] \in G^*$, where $g \in G$ and α is an automorphism of G. Since G is an autonilpotent group of class 2, $G^* \le L(G)$. Thus $[g, \alpha] \in L(G)$. Let $[g, \alpha] = l$ for some $l \in L(G)$. Then $g^{-1}\alpha(g) = l$ and hence $\alpha(g) = gl$. Observe that $[g, \alpha]^m = (g^{-1}\alpha(g))^m = l^m$ and $[g^m, \alpha] = g^{-m}\alpha(g^m) = g^{-m}(gl)^m = g^{-m}g^ml^m = l^m$. Thus $[g^m, \alpha] = [g, \alpha]^m$.

(ii) Let $gL(G) \in G/L(G) = \langle y_1L(G), y_2L(G), \ldots, y_nL(G)\rangle$. Then

$$gL(G) = (y_1L(G))^{m_1}(y_2L(G))^{m_2} \ldots (y_nL(G))^{m_n},$$

where $m_1, m_2, \ldots, m_n$ are positive integers. Therefore $g = y_1^{m_1}y_2^{m_2} \ldots y_n^{m_n}l$ for some $l \in L(G)$. Thus

$$\begin{aligned}
G^* &= \langle [g, \alpha] \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle \\
&= \langle [{y_1}^{m_1}{y_2}^{m_2} \ldots {y_n}^{m_n}l, \alpha] \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle \\
&= \langle ({y_1}^{m_1}{y_2}^{m_2} \ldots {y_n}^{m_n}l)^{-1}\alpha({y_1}^{m_1}{y_2}^{m_2}...{y_n}^{m_n}l) \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle \\
&= \langle l^{-1}y_n^{-1} \ldots {y_2}^{-1}[y_1, \alpha]\alpha(y_2) \ldots \alpha(y_n)l \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle \\
&= \langle [y_1, \alpha][y_2, \alpha] \ldots [y_n, \alpha] \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle.
\end{aligned}$$

(iii) Let $\exp(T(G/L(G))) = d$ and $\exp(T(G^*)) = k$. Then $y_i^d \in L(G)$ for all $i = 1, \ldots, n$. By (i), $[y_i, \alpha]^d = [y_i^d, \alpha] = 1$. Thus $k \le d$. Again as $T(G^*) \le T(L(G))$, so by (i), $[y_i, \alpha]^d = [y_i^d, \alpha] = 1 \Rightarrow y_i^k \in L(G)$. Thus $d \le k$ and hence $k = d$. $\square$

Let G be a finitely generated non-abelian group such that $G' \leq L(G)$. Suppose that $\pi(G/Z(G)) = \{p_1, p_2, \ldots, p_d, p'_1, p'_2, \ldots, p'_{d'}\}$, $\pi(G/L(G)) = \{p_1, p_2, \ldots, p_e\}$ and $\pi(L(G)) = \{q_1, q_2, \ldots, q_f\}$. Let X, Y, Z be respective torsion parts and a, b, c be respective torsion-free ranks of $G/Z(G), G/L(G)$ and $L(G)$. Let $X_{p_i} \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, $X_{p'_i} \simeq \prod_{j=1}^{l'_i} C_{{p'_i}^{\alpha'_{ij}}}$, $Y_{p_i} \simeq \prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}$ and $Z_{q_i} \simeq \prod_{j=1}^{n_i} C_{q_i^{\gamma_{ij}}}$, where for each i, l_i, l'_i, m_i, n_i, $\alpha_{ij} \geq \alpha_{i(j+1)}$, $\alpha'_{ij} \geq \alpha'_{i(j+1)}$, $\beta_{ij} \geq \beta_{i(j+1)}$ and $\gamma_{ij} \geq \gamma_{i(j+1)}$ are positive integers, respectively denote the Sylow subgroups of X, Y and Z. Then

$$G/Z(G) \simeq X \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} X_{p_i} \times \prod_{i=1}^{d'} X_{p'_i} \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}} \times \prod_{i=1}^{d'} \prod_{j=1}^{l'_i} C_{{p'_i}^{\alpha'_{ij}}} \times \mathbb{Z}^a,$$

$$G/L(G) \simeq Y \times \mathbb{Z}^b \simeq \prod_{i=1}^{e} Y_{p_i} \times \mathbb{Z}^b \simeq \prod_{i=1}^{e} \prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}} \times \mathbb{Z}^b$$

and

$$L(G) \simeq Z \times \mathbb{Z}^c \simeq \prod_{i=1}^{f} Z_{q_i} \times \mathbb{Z}^c \simeq \prod_{i=1}^{f} \prod_{j=1}^{n_i} C_{q_i^{\gamma_{ij}}} \times \mathbb{Z}^c.$$

Since $G/Z(G)$ is a quotient group of $G/L(G)$, it follows from [16, Section 25] that $d(X) + a \leq d(Y) + b$, $a \leq b$, $d \leq e$ and if G is finite, then $l_i \leq m_i$ and $\alpha_{ij} \leq \beta_{ij}$ for all $i, 1 \leq i \leq d$ and for all $j, 1 \leq j \leq l_i$.

Proposition 3.2.10 *Let G be a finitely generated non-abelian torsion or torsion-free group such that $G' \leq L(G)$. Then $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$ if and only if one of the following conditions holds:*

(i) G is torsion-free, $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$.

(ii) G is finite, $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i}$, $q_i \neq p_i$ for $d+1 \leq i \leq f$, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$.

Proof. Since $L(G)$ is a central subgroup and is fixed by all automorphisms, therefore $\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/L(G), L(G))$ by Lemma 3.2.1. It is not very hard to see that if any of the two conditions is satisfied, then $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$. Conversely, suppose that $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$. Then $\mathrm{Hom}(G/L(G), L(G)) \simeq G/Z(G)$ and thus

$$\mathrm{Hom}(Y \times \mathbb{Z}^b, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a. \tag{3.1}$$

First assume that G is torsion-free. Then $L(G)$ is torsion-free and also $G/Z(G)$ is torsion-free by Lemma 3.1.1 (i) & (v) and equation (3.1). As $\mathrm{Hom}(Y \times \mathbb{Z}^b, \mathbb{Z}^c) \simeq \mathbb{Z}^a$ by (3.1), we have $bc = a$. Since $a \leq b$, $c = 1$ and $a = b$. It follows that $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$.

Next assume that G is torsion. Then G is finite, because both $L(G)$ and $G/L(G)$, being torsion and abelian, are finite. Thus $\mathrm{Hom}(Y, Z) \simeq X$ by (3.1). Since $d(X_{p_i}) \leq d(Y_{p_i})$, $q_i = p_i$ and $n_i = 1$ for $1 \leq i \leq d$ and $q_i \neq p_i$ for $i > d$. Thus $L(G) \simeq \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i}$ and therefore

$$\begin{aligned} \mathrm{Hom}(Y, Z) &\simeq \mathrm{Hom}(\prod_{i=1}^{e}\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f}\prod_{j=1}^{n_i} C_{q_i^{\gamma_{ij}}}) \\ &\simeq \mathrm{Hom}(\prod_{i=1}^{d}\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}}) \\ &\simeq \prod_{i=1}^{d} \mathrm{Hom}(\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}). \end{aligned}$$

Since $X \simeq \prod_{i=1}^{d}\prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, it thus follows that for $1 \leq i \leq d$,

$$\mathrm{Hom}(\prod_{j=1}^{m_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$$

and hence $l_i = m_i$. We thus have

$$\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}} \tag{3.2}$$

for $1 \leq i \leq d$. Two cases now arise here. Either $\exp(Y_{p_i}) \leq \exp(Z_{p_i})$ or $\exp(Y_{p_i}) > \exp(Z_{p_i})$. If $\exp(Y_{p_i}) \leq \exp(Z_{p_i})$, then $\beta_{ij} \leq \gamma_{i1}$ for each j and thus

$$\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}.$$

It therefore follows from (3.2) that $\alpha_{ij} = \beta_{ij}$ for each j, and thus $(G/Z(G))_{p_i} = (G/L(G))_{p_i}$. And, if $\exp(Y_{p_i}) > \exp(Z_{p_i})$, then there exists a largest integer r_i between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ and $\beta_{ij} \leq \gamma_{i1}$ for each $j, r_i + 1 \leq j \leq l_i$. Then $\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{r_i} C_{p_i^{\gamma_{i1}}} \times \prod_{j=r_i+1}^{l_i} C_{p_i^{\beta_{ij}}}$ and hence, by (3.2), $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$. □

Proposition 3.2.11 *Let G be a finitely generated non-abelian infinite mixed group such that $G' \leq L(G)$. Then $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$ if and only if one of the following conditions holds:*

(*i*) *$L(G)$ is infinite cyclic, $G/Z(G)$ is torsion-free and $\rho(G/Z(G)) = \rho(G/L(G))$.*

(*ii*) *$L(G)$ is finite and (a) $G/L(G)$ is mixed and $G/Z(G) \simeq \mathrm{Hom}(Y, Z) \times Z^b$ or (b) $L(G)$ is cyclic, $G/L(G)$ is torsion-free and $G/Z(G) \simeq Z^b$.*

(*iii*) *$L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i} \times \mathbb{Z}^c$ (c is either 1 or arbitrary), $q_i \neq p_i$ for $d+1 \leq i \leq f$, both $G/Z(G)$ and $G/L(G)$ are finite, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$.*

Proof. It is easy to see that if any of the three conditions hold, then $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$. For the converse part, we proceed according to the structure of $L(G)$. First suppose that $L(G)$ is torsion-free. Then $G/Z(G)$ is torsion-free and $G/L(G)$ is either mixed or torsion-free by Lemma 3.1.1 (v) and equation (3.1). In both cases, it is easy to see from (3.1) that $c = 1$ and $a = b$. Thus $L(G)$ is infinite cyclic and $\rho(G/Z(G)) = \rho(G/L(G))$.

Next suppose that $L(G)$ is torsion. If $G/L(G)$ is torsion, then G is finite, which is not so. If $G/L(G)$ is mixed, then $\mathrm{Hom}(Y \times \mathbb{Z}^b, Z) \simeq X \times \mathbb{Z}^a$ by (3.1) and hence $a = 0$. Thus $L(G)$ is finite and $G/Z(G) \simeq \mathrm{Hom}(Y, Z) \times Z^b$. Finallly, if $G/L(G)$ is torsion-free, then, by equation (3.1), $\mathrm{Hom}(\mathbb{Z}^b, Z) \simeq X \times \mathbb{Z}^a$ and thus $a = 0$ and $X \simeq Z^b$. Since $d(X) + a \leq d(Y) + b$, $d(X) \leq b$ and hence $Z = L(G)$ is cyclic.

In the end, we suppose that $L(G)$ is a mixed group. First assume that $G/L(G)$ is torsion-free. Then $\mathrm{Hom}(\mathbb{Z}^b, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$ by (3.1) and hence $b\,c = a$, implying that $a = b$ and $c = 1$. As $d(X) + a \leq b$, $d(X) = 0$ and hence $d(Z) = 0$. Thus $L(G)$ is infinite cyclic, $G/Z(G)$ is torsion-free and $\rho(G/Z(G)) = \rho(G/L(G))$. We next assume that $G/L(G)$ is a mixed group. Then $\mathrm{Hom}(Y, Z) \times Z^b \simeq X$ and $\mathbb{Z}^{bc} \simeq \mathbb{Z}^a$, again implying that $a = b$ and $c = 1$. Thus $d(X) \leq d(Y)$ and hence $a = b = 0$. Therefore (3.1) reduces to $\mathrm{Hom}(Y, Z) \simeq X$. Now proceeding as in Proposition 1(ii), we can prove that $L(G) \simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i} \times \mathbb{Z}$, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$.

Finally assume that $G/L(G)$ is torsion. Then $\mathrm{Hom}(Y, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$ implying $a = 0$, c arbitrary and $\mathrm{Hom}(Y, Z) \simeq X$. Again, proceeding as in Proposition 3.2.10 (ii), we can prove that $L(G) \simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{f} Z_{q_i} \times \mathbb{Z}^c$, and for $1 \leq i \leq d$, either $(G/L(G))_{p_i} = (G/Z(G))_{p_i}$ or $d((G/L(G))_{p_i}) = d((G/Z(G))_{p_i})$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$. $\square$

Let G be a finite p-group such that $G' \leq L(G)$. Let $G/Z(G) \simeq \prod_{i=1}^{r} C_{p^{\alpha_i}}$, $G/L(G) \simeq \prod_{i=1}^{s} C_{p^{\beta_j}}$ and $L(G) \simeq \prod_{i=1}^{t} C_{p^{\gamma_i}}$, where $\alpha_i \geq \alpha_{i+1}$, $\beta_i \geq \beta_{i+1}$ and $\gamma_i \geq \gamma_{i+1}$ are positive integers. Since $G/Z(G)$ is a quotient group of $G/L(G)$, $r \leq s$ and $\alpha_i \leq \beta_i$ for $1 \leq i \leq r$.

Corollary 3.2.12 *Let G be a finite non-abelian p-group. Then* $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$ *if and only if $G' \leq L(G)$, $L(G)$ is cyclic and either $L(G) = Z(G)$ or $d(G/L(G)) = d(G/Z(G))$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest integer such that $\beta_k > \gamma_1$.*

Proof. Observe that if $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$, then for any commutator $[a, b] \in G'$, $[a, b] = a^{-1}I_b(a) \in L(G)$, where I_b is the inner automorphism of G induced by b, and thus $G' \leq L(G)$. The result now follows from Proposition 3.2.10 (ii). $\square$

Corollary 3.2.13 ([52, Theorem 3.2]) *Let G be a non-abelian autonilpotent finite p-group of class* 2. *Then* $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$ *if and only if $L(G) = Z(G)$ and $L(G)$ is cyclic.*

Proof. Since G is autonilpotent group of class 2, then $\exp(G/L(G)) | \exp(L(G))$, by Lemma 3.2.9 (iii). The result now follows from Corollary 3.2.12. $\square$

CHAPTER 4

Isomorphism between automorphism groups of finitely generated groups

4.1 — Introduction

Let G be a finitely generated group and let $\mathrm{Inn}(G)$ denote the inner automorphism group of G. For normal subgroups X and Y of G, let $\mathrm{Aut}^X(G)$ and $\mathrm{Aut}_Y(G)$ denote the subgroups of $\mathrm{Aut}(G)$ centralizing G/X and Y respectively. We denote the intersection $\mathrm{Aut}^X(G) \cap \mathrm{Aut}_Y(G)$ by $\mathrm{Aut}_Y^X(G)$. Let C^*, in particular, denote the group $\mathrm{Aut}_{Z(G)}^{Z(G)}(G)$, where $Z(G)$ is the center of G. For a finite group G, let G_p and $\pi(G)$ respectively denote the Sylow p-subgroup and the set of prime divisors of G. For a finite p-group G, Attar [4, Main Theorem] proved that $C^* = \mathrm{Inn}(G)$ if and only if either G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic. Azhdari and Malayeri [8, Theorem 0.1] (see also [9, Theorem 2.3] for correct version) generalized this result of Attar and proved that if G is a finitely generated nilpotent group of class 2, then $C^* \simeq \mathrm{Inn}(G)$ if and only if $Z(G)$ is infinite cyclic or $Z(G) \simeq C_m \times H \times \mathbb{Z}^r$, where $C_m \simeq \prod_{p \in \pi(G/Z(G))} Z(G)_p$, $H \simeq \prod_{p \notin \pi(G/Z(G))} Z(G)_p$, $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ is of finite exponent dividing m. We prove a technical lemma, Lemma 4.2.1, and as a consequence give a short and easy proof of this main theorem of Azhdari and Malayeri. We also obtain short and alternate proofs of Corollary 2.1 of [9], Propostion 1.11 and Theorem 2.2(i) of [7]. Some other related results for finitely generated and finite p-groups are also obtained.

The following well known results will be used very frequently without further referring.

Lemma 4.1.1 *Let U, V and W be abelian groups. Then*

(i) if U is torsion-free of rank m, then $\mathrm{Hom}(U, V) \simeq V^m$, *and*

(ii) if U is torsion and V is torsion-free, then $\mathrm{Hom}(U, V) = 1$.

4.2 — Main results.

Let G be a finitely generated group and M be an abelian subgroup of G with $\pi(M) = \{q_1, q_2, \ldots, q_e\}$. Let L and N be normal subgroups of G such that $\pi(G/L) = \pi(G/N) = \{p_1, p_2, \ldots, p_d\}$ and $G' \leq N \leq L$. Let $G/L \simeq X \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} X_{p_i} \times \mathbb{Z}^a$, $G/N \simeq Y \times \mathbb{Z}^b \simeq \prod_{i=1}^{d} Y_{p_i} \times \mathbb{Z}^b$ and $M \simeq Z \times \mathbb{Z}^c \simeq \prod_{i=1}^{e} Z_{q_i} \times \mathbb{Z}^c$, where X, Y, Z are respective torsion parts and a, b, c are respective torsion-free ranks of $G/L, G/N$ and M. Suppose that $X_{p_i} \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, $Y_{p_i} \simeq \prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}$ and $Z_{q_i} \simeq \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$, where for each i, $\alpha_{ij} \geq \alpha_{i(j+1)}$, $\beta_{ij} \geq \beta_{i(j+1)}$ and $\gamma_{ij} \geq \gamma_{i(j+1)}$ are positive integers. Then $X \simeq \prod_{i=1}^{d}\prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, $Y \simeq \prod_{i=1}^{d}\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}$ and $Z \simeq \prod_{i=1}^{e}\prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$. If G is finite, as G/L is a quotient group of G/N, it follows that $l_i \leq n_i$ and $\alpha_{ij} \leq \beta_{ij}$ for all $i, 1 \leq i \leq d$ and for all $j, 1 \leq j \leq l_i$. We start with the following technical lemma.

Lemma 4.2.1 *Let G, L, M and N be as above. Then* $\mathrm{Hom}(G/N, M) \simeq G/L$ *if and only if one of the following conditions hold:*

(i) *G is torsion-free, M is infinite cyclic and both G/L and G/N are torsion-free of same rank.*

(ii) *G is torsion, $M \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$, $l_i = n_i$ and either $\alpha_{ij} = \beta_{ij} \leq \gamma_{i1}$ for each j or $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

(iii) *G is a mixed group, $M \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}} \times \mathbb{Z}^c$, both G/L and G/N are finite, $l_i = n_i$ and either $\alpha_{ij} = \beta_{ij} \leq \gamma_{i1}$ for each j or $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

Proof. It is easy to see that if any of the three conditions hold, then $\mathrm{Hom}(G/N, M) \simeq G/L$. Conversely suppose that $\mathrm{Hom}(G/N, M) \simeq G/L$. Then

$$\mathrm{Hom}(Y \times \mathbb{Z}^b, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a. \tag{4.1}$$

First assume that G is torsion-free. Then M is also torsion-free and therefore by (4.1) $\mathrm{Hom}(Y \times \mathbb{Z}^b, \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$. Thus $X = 1$ and since $a \leq b$, $c = 1$ and $a = b$. It follows that M is infinite cyclic and both G/N and G/L are torsion-free of same rank. Next assume that G is torsion. Then $\mathrm{Hom}(Y, Z) \simeq X$ by (4.1). Since $\pi(X) = \pi(Y)$ and $d(X) \leq d(Y)$, therefore $q_i = p_i$ and $m_i = 1$ for all $i, 1 \leq i \leq d$.

Thus $M \simeq \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$. Also, observe that

$$\begin{aligned}\mathrm{Hom}(Y,Z) &\simeq \mathrm{Hom}(\prod_{i=1}^{d}\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e}\prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}) \\ &\simeq \mathrm{Hom}(\prod_{i=1}^{d}\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}}) \\ &\simeq \prod_{i=1}^{d} \mathrm{Hom}(\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}})\end{aligned}$$

and $X \simeq \prod_{i=1}^{d}\prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$. It thus follows that $\mathrm{Hom}(\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$ for each $i, 1 \leq i \leq d$, and hence $l_i = n_i$. It thus follows that for each fixed $i, 1 \leq i \leq d$,

$$\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}. \tag{4.2}$$

Now, if $\exp(Y_{p_i}) \leq \exp(Z_{p_i})$, then $\beta_{ij} \leq \gamma_{i1}$ for each j and $\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}$. It therefore follows from (4.2) that $\alpha_{ij} = \beta_{ij}$ for each j. And, if $\exp(Y_{p_i}) > \exp(Z_{p_i})$, then there exists largest positive integer r_i between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ and $\beta_{ij} \leq \gamma_{i1}$ for each $j, r_i + 1 \leq j \leq l_i$. Therefore $\mathrm{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{r_i} C_{p_i^{\gamma_{i1}}} \times \prod_{j=r_i+1}^{l_i} C_{p_i^{\beta_{ij}}}$. It then follows by (4.2) that $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$.

Finally assume that G is a mixed group. First suppose that M is torsion. Then (4.1) gives that $\mathrm{Hom}(Y \times \mathbb{Z}^b, Z) \simeq X \times \mathbb{Z}^a$. It follows that $\mathrm{Hom}(Y, Z) \times Z^b \simeq X$. Since $\pi(X) = \pi(Y)$, $b = 0$ and hence it reduces to case (ii). Next suppose that M is torsion-free. Then (4.1) implies that $\mathrm{Hom}(Y \times \mathbb{Z}^b, \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$. It thus follows that $X = 1$ and hence $Y = 1$, because $\pi(X) = \pi(Y)$. Also $bc = a$ gives that $a = b$ and $c = 1$. Therefore it reduces to case (i). Lastly assume that M is mixed. Then by (4.1), $\mathrm{Hom}(Y,Z)\times Z^b \simeq X$ and $\mathbb{Z}^{bc}\simeq \mathbb{Z}^a$. It therefore follows that $b=0$ and hence $a=0$. Thus $\mathrm{Hom}(Y,Z\times\mathbb{Z}^c)\simeq \mathrm{Hom}(Y,Z)$. As $\mathrm{Hom}(Y,Z)\simeq X$, so $M\simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}}\times \prod_{i=d+1}^{e}\prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}\times\mathbb{Z}^c$ □

Remark 4.2.2 The Lemma 4.2.1, in its general form, is very technical in its nature but a particular form of it is very nice. Observe that if $N=L$ and $\exp(T(G/N))\mid \exp(T(M))$, then $\exp(Y_{p_i})\le \exp(Z_{p_i})$ for all i and hence $\mathrm{Hom}(G/L,M)\simeq G/L$ if and only if either M is infinite cyclic or $M\simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}}\times \prod_{i=d+1}^{e}\prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}\times\mathbb{Z}^c$, where $c\ge 0$ is the torsion-free rank of M.

We can now obtain the following corollaries.

Corollary 4.2.3 ([8, Theorem 0.1] (see [9, Theorem 2.3] for correct version)) *Let G be a finitely generated nilpotent group of class 2. Then $C^*\simeq \mathrm{Inn}(G)$ if and only if either $Z(G)$ is infinite cyclic or $Z(G)\simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}}\times \prod_{i=d+1}^{e}\prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}\times\mathbb{Z}^c$, where c is the torsion-free rank of $Z(G)$.*

Proof. Observe that $C^*\simeq \mathrm{Hom}(G/Z(G),Z(G))$ by Lemma 3.2.1 and since G is nilpotent of class 2, $\exp(T(G'))=\exp(T(G/Z(G)))$. The result now follows from Lemma 4.2.1 by taking $L=M=N=Z(G)$. □

Corollary 4.2.4 ([9, Corollary 2.1]) *Let G be a finitely generated non-abelian group and let M and N be normal subgroups of G such that $M\le Z(G)\le N$ and $G/Z(G)$ is finite. Then $\mathrm{Aut}_N^M(G)=\mathrm{Inn}(G)$ if and only if G is a nilpotent group of class 2, $N=Z(G)$, $G'\le M$ and $M\simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}}\times \prod_{i=d+1}^{e}\prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}\times\mathbb{Z}^c$, where $c\ge 0$ is the torsion-free rank of M.*

Proof. First suppose that $\mathrm{Aut}_N^M(G) = \mathrm{Inn}(G)$. Observe that $\mathrm{Aut}_N^M(G)$ is isomorphic to $\mathrm{Hom}(G/N, M)$ by Lemma 3.2.1. It follows that $\mathrm{Inn}(G)$ is abelian and therefore nilpotence class of G is 2. For any $[a, b] \in G'$, $[a, b] = a^{-1}I_b(a) \in M$ and thus $G' \leq M$. Also, for any $n \in N$, $I_x(n) = n$ for all $x \in G$ and therefore $N = Z(G)$. Now since $\exp(T(G/Z(G))) = \exp(T(G'))$ divides $\exp(T(M))$, the result follows from Lemma 4.2.1 by taking $L = Z(G)$. The converse follows easily. □

In 1911, Burnside [14, Note B, p.463] gave the notion of pointwise inner automorphism of a group G. An automorphism α of G is called pointwise inner automorphism of G if x and $\alpha(x)$ are conjugate for each $x \in G$. Let H be a characteristic subgroup of G. As defined in [7], an automorphism α of G is called $H-$pointwise inner if for each element $x \in G$, there exists $h \in H$ such that $\alpha(x) = x^h = x[x, h]$. For convenience we denote $\gamma_k(G)-$pointwise inner automorphism of G by $\mathrm{Aut}_{k-pwi}(G)$. As another application of Lemma 4.2.1, we get the following two results of Azhdari [7]. The second one generalizes Theorem 2.2(i) of [7].

Corollary 4.2.5 ([7, Prop. 1.11]) *Let G be a finitely generated nilpotent group of class $k+1 \geq 2$. Then $\mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq G/\zeta_k(G)$ if and only if $\gamma_{k+1}(G)$ is cyclic. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$ is cyclic, then $\mathrm{Aut}_{k-pwi}(G)$ is isomorphic to a quotient group of $\mathrm{Inn}(G)$.*

Proof. It follows from [74, Cor. 2.6, Cor. 3.16, Cor. 3.17] that $\exp(T(G/\zeta_k(G)) = \exp(T(\gamma_{k+1}(G))$ and $G/\zeta_k(G)$ is finite if and only if $\gamma_{k+1}(G)$ finite. The result now follows from Lemma 4.2.1 (see Remark 4.2.2) by taking $L = N = \zeta_k(G)$ and $M = \gamma_{k+1}(G)$. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$ is cyclic, then using the arguments as in [75, Prop. 3.1], we can prove that $\mathrm{Aut}_{k-pwi}(G) \simeq \mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G))$. □

Corollary 4.2.6 (*cf.* [7, Theorem 2.2(i)]) *Let G be a finitely generated nilpotent group of class $k+1 \geq 2$. Then $\mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq \mathrm{Inn}(G)$ if and only if G is nilpotent of class 2 and G' is cyclic. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$, then $\mathrm{Aut}_{k-pwi}(G) \simeq \mathrm{Inn}(G)$ if and only if G is nilpotent of class 2 and G' is cyclic.*

Proof. Observe that if $\mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq \mathrm{Inn}(G)$, then $G/Z(G)$ is abelian, and therefore nilpotence class of G is 2. It follows that $\zeta_k(G) = Z(G)$ and $\gamma_{k+1}(G) = G'$. The result now follows from above corollary by taking $k = 1$. □

For $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. Inductively, define

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n],$$

where $\alpha_i \in \mathrm{Aut}(G)$. The absolute center $L(G)$ of G is defined as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in \mathrm{Aut}(G)\}$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\}.$$

The autocommutator subgroup G^* of G is defined as

$$G^* = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle.$$

It is easy to see that $L_n(G) \leq Z_n(G)$ for all $n \geq 1$ and $G' \leq G^*$. An automorphism α of G is called an absolute central automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. The group of all absolute central automorphisms of G is denoted by $\mathrm{Aut}^l(G)$. By $\mathrm{Aut}^l_z(G)$ we denote the set of those absolute central automorphism of G, which fixes the center of G element-wise. A group G is called autonilpotent of class at most n if $L_n(G) = G$ for some natural number n. Observe that if G is autonilpotent of class 2, then $G^* \leq L(G)$. Nasrabadi and Farimani [52] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. As a final consequence of Lemma 4.2.1, we prove the following general result and, as a by-product, generalize this main result of Nasrabadi and Farimani.

Corollary 4.2.7 *Let G be a finitely generated non-abelian group such that $G' \leq L(G)$ and $\pi(G/L(G)) = \pi(G/Z(G))$. Then $\mathrm{Aut}^l(G) \simeq \mathrm{Inn}(G)$ if and only if one of the following holds*

(i) *G is torsion-free, $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$;*

(ii) *G is torsion, $L(G) \simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$, either $L(G) = Z(G)$ or $l_i = n_i$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

(iii) *G is a mixed group, both $G/L(G)$ and $G/Z(G)$ are finite, $L(G) \simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}} \times \mathbb{Z}^c$, either $L(G) = Z(G)$ or $l_i = n_i$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

Proof. Observe that $\mathrm{Aut}^l(G) \simeq \mathrm{Hom}(G/L(G), L(G))$ by lemma 3.2.1. The proof now follows from Lemma 4.2.1 by taking $M = N = L(G)$ and $L = Z(G)$. □

Corollary 4.2.8 *Let G be finitely generated group such that $G' \leq L(G)$. Then*

$$\mathrm{Aut}_z^l(G) \simeq \mathrm{Inn}(G)$$

if and only if either $L(G)$ is infinite cyclic or $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}} \times \mathbb{Z}^c$, where c is the torsion-free rank of $L(G)$.

Proof. The result follows from Remark 4.2.2. □

Let G be a finite p-group such that $G' \leq M \leq N$, where N is central subgroup of G. Let $G/Z(G) \simeq \prod_{i=1}^{r} C_{p^{\alpha_i}}$, $G/N \simeq \prod_{i=1}^{s} C_{p^{\beta_j}}$ and $M \simeq \prod_{i=1}^{t} C_{p^{\gamma_i}}$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_s$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_t$ are positive integers. Since $G/Z(G)$ is a quotient group of G/N, $r \leq s$ and $\alpha_i \leq \beta_i$ for $1 \leq i \leq r$.

Proposition 4.2.9 *Let G be a finite non-abelian p-group such that $M \leq N$, where N is central subgroup of G. Then $\mathrm{Aut}_{\mathrm{N}}^{\mathrm{M}}(G) = \mathrm{Inn}(G)$ if and only if $G' \leq M$, M is cyclic and either $N = Z(G)$ or $d(G/N)) = d(G/Z(G))$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest positive integer such that $\beta_k > \gamma_1$.*

Proof. Observe that if $\mathrm{Aut}_{\mathrm{N}}^{\mathrm{M}}(G) = \mathrm{Inn}(G)$, then for any $[a,b] \in G'$, $[a,b] = a^{-1}I_b(a) \in M$ and thus $G' \leq M$. As $\mathrm{Aut}_{\mathrm{N}}^{\mathrm{M}}(G) \simeq \mathrm{Hom}(G/N, M)$ by Lemma 3.2.1, the result follows from Lemma 4.2.1. □

Remark 4.2.10 *Let G be an abelian p-group. Then $\mathrm{Aut}_{\mathrm{N}}^{\mathrm{M}}(G) = \mathrm{Inn}(G)$ if and only if either $M = 1$ or $N = G$.*

Proof. Since G is abelian p-group, $\mathrm{Inn}(G) = 1$. Therefore $\mathrm{Hom}(G/N, M) = 1$. Thus either $G = N$ or $M = 1$. □

Corollary 4.2.11 ([4, Main Theorem]) *If G is a finite p-group, then $C^* = \mathrm{Inn}(G)$ if and only if either G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic.*

Corollary 4.2.12 *Let G be finite p-group and $L(G)$ is a non-trivial subgroup. Then $\mathrm{Aut}^l(G) = \mathrm{Inn}(G)$ and if and only if either $|G| \leq 2$ or $G' \leq L(G)$, $L(G)$ is cyclic, $d(G/L(G)) = d(G/Z(G))$ and either $L(G) = Z(G)$ or $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest positive integer such that $\beta_k > \gamma_1$.*

CHAPTER 5

On commuting automorphisms of some p-groups

5.1 — Introduction

An automorphism α of G is called commuting automorphism if each element in G commutes with it's image under α or $[x, \alpha(x)] = 1$ for all $x \in G$. The set of all commuting automorphisms is denoted by $\mathrm{A}(G)$. It is easy to see that $\mathrm{Aut}^{\mathrm{z}}(G)$ is always contained in $\mathrm{A}(G)$. In the year 1984, Herstein [33] proposed the following problem to American Mathematical Monthly: If G is a simple non-abelian group, then prove that $A(G) = 1$. An answer to this problem was given by Laffey [42] in 1986. He in fact, proved a little more than what was asked. He proved that $A(G) = 1$ if G has no non-trivial abelian normal subgroup. Pettet (see [60]), however, observed that it is sufficient to assume that $Z(G) = 1$ and $G' = G$. In 2002, Deaconescu, Silberberg and Walls [21] proved some interesting properties of commuting automorphisms and raised the natural question: Is it true that the set $A(G)$ is always a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G? An example of an extra-special group of order 32 in [21] shows that $A(G)$ is not always a subgroup of $\mathrm{Aut}(G)$. Let us call a group G an $A(G)$-group if $A(G)$ is a subgroup of $\mathrm{Aut}(G)$.

Vosooghpour and Akhavan-Malayeri [72] showed that for a given prime p, minimum order of a non-$A(G)$ p-group is p^5. They also proved that there exists a non-$A(G)$ p-group of order p^n for all $n \geq 5$. Fouladi and Orfi [24] have given that if G is either a finite AC-group or a p-group of maximal class or a metacyclic p-group, then G is an $A(G)$-group.

Recently, Rai [Proc. Japan Acad., Ser. A **91** (2015), no. 5, 57-60] has given some sufficient conditions on a finite p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ and, as a consequence, has proved that in a finite p-group G of co-class 2, where p is an odd prime, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. In this chapter we give very elementary and short proofs of main results of Rai and some other related results are also obtained. We also give two Gap algorithms for finding the set of all commuting automorphisms of a finite group G, and to check whether an automorphism of a finite group is commuting or not. A group G is said be of co-class k if it is nilpotent group of class $n-k$. Also let $|G|$ denote the order of group G and $\mathrm{Aut}(G)$ is full automorphism group of G.

The following results will be used very frequently without further reference.

Lemma 5.1.1 (Laffey) *If φ is a commuting automorphism of a group G, then $[\varphi(x), y] = [x, \varphi(y)]$ for all $x, y \in G$.*

Proof. Observe that $[\varphi(x), y] = \varphi(x)^{-1}y^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}\varphi(y)x\varphi(x)^{-1}y^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}\varphi(y)\varphi(x)^{-1}xy^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}xy^{-1}\varphi(y)\varphi(x)^{-1}\varphi(x)y = x^{-1}\varphi(y)^{-1}xy^{-1}\varphi(y)y = [x, \varphi(y)]$. □

Lemma 5.1.2 (Pettet) *If φ is a commuting automorphism of a group G, then $x^{-1}\varphi(x) \in C_G(G')$ for all $x \in G$.*

Proof. Observe that $[\varphi(xy), z] = [xy, \varphi(z)]$ for all $x, y, z \in G$ by Lemma 1. It follows that $[\varphi(x), z]^{\varphi(y)}[\varphi(y), z] = [x, \varphi(z)]^y[y, \varphi(z)]$ and hence $[\varphi(x), z]^{\varphi(y)} = [x, \varphi(z)]^y$. Since x, z are arbitrary, $y^{-1}\varphi(y) \in C_G(G')$ for all $y \in G$. □

5.2 — Proof of the main theorems

Let G be a finite non-abelian p-group of order p^n, and let $\gamma_k(G)$ and $Z_k(G)$ respectively denote the kth terms of the lower and upper central series of G. For convenience, $\gamma_2(G)$ and $Z_1(G)$ are respectively denoted as G' and $Z(G)$. Let $\alpha, \beta \in A(G)$. Suppose that G is a maximal class group. If $|G| = p^3$, then $|G/Z(G)| = p^2$. Let $x \in G - Z(G)$. Then $Z(G) < Z(C_G(x)) \leq C_G(x) < G$ implies that $C_G(x)$ is abelian. It follows that $1 = [\alpha(x), \beta(x)] = [x, \alpha\beta(x)]$ and hence $\alpha\beta \in A(G)$. Assume that $n \geq 4$. Observe that $|\gamma_i(G)| = p^{n-i}$ for all $i \geq 2$. If $|C_G(G')| = p^{n-1}$, then $G' \leq C_G(G')$ and hence G' is a central subgroup of $C_G(G')$. Since $C_G(G')/G'$ is cyclic, $C_G(G')$ is abelian. It follows that $[x, \alpha\beta(x)] = [\alpha(x), \beta(x)] = [x^{-1}\alpha(x), x^{-1}\beta(x)] = 1$ and hence $\alpha\beta \in A(G)$. We therefore suppose that $p^3 \leq |C_G(G')| \leq p^{n-2}$. Let $|G/C_G(G')| = p^k$, where $2 \leq k \leq n-3$. Then $\gamma_k(G/C_G(G')) = 1$ and hence $\gamma_k(G) \leq C_G(G')$. But $|\gamma_k(G)| = |C_G(G')| = p^{n-k}$, and therefore $\gamma_k(G) = C_G(G')$. Now $\gamma_k(G) \leq G'$ and it commutes with G', therefore $\gamma_k(G)$ and hence $C_G(G')$ is abelian. We thus have the following theorem (cf. [24, Theorem 3.4]).

Theorem 5.2.1 *If G is a finite p-group of maximal class, then G is an $A(G)$-group.*

Now suppose that p is an odd prime. Then $G^2 = G$ and hence $x^{-1}\alpha(x), x^{-1}\beta(x) \in Z_2(G)$ by [21, Theorem 1.4]. If $Z_2(G)$ is abelian, then as in the case of $C_G(G')$, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. Suppose that $|Z_2(G)/Z(G)| = p^2$ and $Z(G) = \gamma_k(G)$ for some $k \geq 2$. Then G is of nilpotence class k and hence $\gamma_{k-1}(G) \leq Z_2(G)$. If $k = 2$, then $|G/Z(G)| = p^2$ and hence, as above, G is an $A(G)$-group. Assume that $k \geq 3$. Since $\gamma_{k-1}(G)$ commutes with $Z_2(G)$, $\gamma_{k-1}(G)$ is a central subgroup of $Z_2(G)$. It follows that $Z_2(G)$ is abelian, because $Z_2(G)/\gamma_{k-1}(G)$ is cyclic. We have thus proved the following theorem of Rai.

Theorem 5.2.2 ([61, Theorem 3.3]) *Let p be an odd prime and G be a finite p-group such that $|Z_2(G)/Z(G)| = p^2$ and $Z(G) = \gamma_k(G)$ for some $k \geq 2$. Then G is an $A(G)$-group.*

Observe that if G is of co-class 2, then $|Z_2(G)/Z(G)| = p$ or p^2. It follows that $Z_2(G)$ is abelian as explained above. We thus have the following theorem of Rai.

Theorem 5.2.3 ([61, Theorem A]) *Let G be a finite p-group of co-class 2 for an odd prime p. Then G is an $A(G)$-group.*

5.3 — Algorithms

In this section, we give two Gap algorithms. Algorithm 1 checks whether an automorphism α of a group G is commuting or not. Algorithm 2 can be used to find the set of all commuting automorphisms of G.

Algorithm 1 To check wether α is commuting or not.

```
IsComAut := function(G, α)
local x, ok ;
for x in Elements(G) do
    if Comm(ImageElm(α, x), x) = One(G) then
        ok := true;
    else
        ok := false;
    end if
end for
if ok = true then
    Print("true");
else
    Print("false");
end if
```

Algorithm 2 To find the set of commuting automorphism of a group.

```
CommAuts := function(G)
local A, gens, ok, x, α;
A := AutomorphismGroup(G);
gens := [One(A)];
for α in Elements(A) do
    for x in Elements(G) do
        if Comm(ImageElm(α, x), x) = One(G) then
            ok := true;
        else
            ok := false;
        end if
    end for
    if ok = true and not (α in gens); then
        Add(gens, α);
    end if
end for
return gens;
```

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