ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS

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By

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CERTIFICATE

Certified that the thesis entitled, **"ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES WITH SPECIAL COEFFICIENTS"**, which is being submitted by **Ms. Jatinderdeep Kaur** (Registration No. 9051401), in the fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** (Mathematics), to the School of Mathematics and Computer Applications, Thapar University, Patiala, comprises of candidate's own research work carried out under our supervision and guidance during the period from February 2006 to September 2009.The work presented in this thesis has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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ABSTRACT

The existence of sine and cosine series as a Fourier series, their integrability and L^1-convergence appears to be one of the most difficult question in theory of convergence of trigonometric series in L^1-metric norm.

One of the most prominent question is to find the class of coefficient sequences which satisfy the sufficient conditions for the sine series to be Fourier series.

The present thesis entitled, **"On L^1-Convergence of Certain Trigonometric Series with Special Coefficients"** comprises certain investigations carried out by me at the School of Mathematics and Computer Applications, Thapar University, Patiala, under the supervision of Dr. S.S. Bhatia, Professor and Head, School of Mathematics and Computer Applications, Thapar University, Patiala and Dr. Babu Ram, Professor(Retd.), Department of Mathematics, Maharshi Dayanand University, Rohtak.

The aim of this work is, to investigate classes of sequences formed by the coefficients of real / complex trigonometric series, thereby improving the existing classes of sequences, to obtain sharper results on integrability and L^1-convergence of trigonometric series with special coefficients.

The thesis embodies nine chapters. The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in the sequel, we have presented some known results interrelated to our results alongwith a brief plan of our results presented in the subsequent chapters. The purpose of chapter II is to generalize the results of Hooda and Ram, and that of Teljakovskǐi for a more general classes S^{**} and S^{**}_r of coefficient sequences. In chapter III, we have extended the result of Ram and have studied the L^1-convergence of the cosine series belonging to an extended class S_r, $r = 0, 1, 2, 3.....$ of coefficient sequences. We have also obtained the necessary and sufficient conditions for the L^1-convergence of the r^{th} derivative of cosine trigonometric series under the same class S_r. In chapter IV, we have introduced new modified cosine and sine sums and have studied their integrability and L^1-convergence under a newly defined class SJ of coefficient sequences. Also, the L^1-convergence of cosine and sine series have been deduced as corollary. We have also obtained L^1-convergence of r^{th} derivative of new modified cosine and sine sums under an extended class SJ_r of coefficient

sequences. The object of chapter V is to study the L^1-convergence of complex form of the newly modified trigonometric sums introduced in chapter IV under a new class J^* of coefficient sequences. We have also studied L^1-convergence of r^{th} derivative of new modified complex form under an extended class J_r^*.

In the literature so far available, most of the authors have studied the integrability and L^1-convergence of cosine trigonometric series using different classes of coefficient sequences. However, very few of them have studied about the L^1-convergence of sine trigonometric series. In this direction, we have, in chapter VI studied the L^1-convergence of sine trigonometric series by using a newly introduced modified cosine sum under a new class J of coefficient sequences. In chapter VII, L^1-convergence of the complex form of the, modified trigonometric cosine and sine sums introduced by the author and that of the Kaur, Bhatia and Ram under the class J^* of coefficient sequences as introduced in chapter V have been studied.

In chapter VIII, we have extended the results discussed in chapter IV from one dimensional cosine and sine series to two dimensional ones. In this chapter, we have obtained necessary and sufficient conditions for the integrability and L^1-convergence of double cosine, sine and mixed series by using modified double cosine, sine and mixed trigonometric sums with coefficients satisfying certain conditions.

Chapter IX is devoted to the study of L^1-convergence of double cosine series under an extended class S_d^2 of two dimensional coefficient sequences. Towards the end, references of various publications cited in the present thesis have been reported.

All the results reported in the thesis have been published/communicated in various international/ national journals.

List of Research Papers published/ communicated to various international/ national journals

1. On L^1-Convergence of Certain Trigonometric Sums, **Global Journal of Pure and Applied Mathematics, 2(2), (2006), 111-116.**

2. Integrability and L^1- Convergence of Certain Cosine Sums, **Kyungpook Mathematical Journal, 47 (2007), 323-328.**

3. Convergence of Rees-Stanojević Sums in the Metric space L, **Journal of Punjab Academy of Sciences, 4(1&2), (2007), 1-3.**

4. Convergence of New Modified Trigonometric Sums in the Metric Space L, **The Journal of Nonlinear Sciences and Applications, 1(3), (2008), 179-188.**

5. The extension of the theorem of J.W. Garrett, C.S. Rees and C.V. Stanojević from one dimension to two Dimension, **International Journal of Mathematical Analysis, 3(26), (2009), 1251-1257.**

6. On L^1-Convergence of Modified Cosine Sums, **accepted** for publication in **Southeast Asian Bulletin of Mathematics.**

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3. On L^1-convergence of modified cosine sum, **National conference on History of Mathematics and Recent Developments**, M.D. University, Rohtak, Dec 20-22, 2006.

4. Integrability and Convergence of double sine series, **Eighteenth International Conference of Forum for Interdisciplinary Mathematics on Interdisciplinary Mathematical and Statistical Techniques**, J.U.I.T, Waknaghat, Solan, August 2-4, 2009.

CHAPTER I

INTRODUCTION

1.1 The present thesis comprises certain investigations carried out by the author **"On L^1-Convergence of Certain Trigonometric Series with Special Coefficients"**. It is well known that if a trigonometric series converges in L^1-metric to a function $f \in L^1(T)$, then it is the Fourier series of the function f. Riesz {[2], Vol.II, Ch. VIII §22} gave a counter example to show that in L^1 metric, the converse of the above said result does not hold good. This motivated various authors to study the L^1-convergence of trigonometric series with special coefficients.

Integrability and L^1-convergence of trigonometric series with special coefficients have been studied by number of authors. The work on this topic was initiated by Young W.H. [69] and Kolmogorov A.N. [30] by taking classes of convex sequences ($\triangle^2 a_n \geq 0$) and quasi-convex sequences ($\sum_{n=1}^{\infty} n|\triangle^2 a_n| < \infty$) respectively. Teljakovskiĭ S.A. [68] yet considered another class S introduced by Sidon [49] for L^1-convergence of trigonometric series. The results obtained by these authors were further generalized and extended by Hardy G.H. and Littlewood J.E. [21], Kano T. [23], Garrett J.W. and Stanojević Č.V. ([18], [19]), Ram B. ([39], [41]), Singh N. and Sharma K.M. ([50], [51], [52]), Bojanic R. and Stanojević Č.V. [8], Chen C.P. [13], Bala R. and Ram B. [1], Móricz F. [34], Bhatia S.S. and Ram B. [3], Tomovski Ž. ([59], [60], [61], [64]), Hooda N. and Ram B. [22], Kaur K., Bhatia S.S. and Ram B. [24] and others by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric series.

During their investigations, some authors introduced modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1-metric to the sum of trigonometric series whereas the classical series itself may not. In this concern, various authors like Rees, C.S. and Stanojević Č.V. [45], Bor. H. ([9], [10], [11], [12]), Chen C.P. [14], Kumari S. and Ram B. [31], Ram B. and Kumari S. [42], Hooda N., Ram B. and Bhatia S.S. [22], Kaur K., Bhatia S.S. and Ram B. [27] have introduced various new modified trigonometric sums and have studied their L^1-convergence under various classes of coefficient sequences. Bhatia S.S. and Ram B. [3] generalized the results of Kumari S. and Ram B. [31] and have obtained their results as corollary.

L^1-convergence of complex trigonometric series was considered by Stanojević Č.V. and Stanojević V.B. [54], Sheng Shu Yun [48], Móricz F. [38], Chen C.P. [16], Tomovski Ž. ([62], [65], [66]) for various classes of complex sequences by considering some additional conditions to control the sine and mixed part of the complex trigonometric series. Bhatia S.S. and Ram B. [4], Tomovski Ž. [63] studied the L^1-convergence of complex form of the modified sums introduced by Kumari S. and Ram B. [31] for new classes of coefficient sequences.

Further, Bhatia S.S. and Ram B. [5] extended the results of Móricz F. [38] and studied the integrability of the r-times differentiated complex trigonometric series. They obtained a new necessary and sufficient condition for the L^1-convergence of the Fourier series.

Móricz F. [33] studied the convergence and integrability of double trigonometric series with coefficients of bounded variation ($\sum\sum|\triangle_{11}a_{jk}| < \infty$). In 1991, Móricz [37] extended the classical theorems of Kolmogorov [30] and Young [69] from one dimensional cosine and sine series to two dimensional cosine and sine series by considering the special cases, where the double sequence of coefficients is monotone decreasing, convex or quasi-convex.

Móricz F. [36], Ram B. and Bhatia S.S. [44] have further extended the results of Hardy G.H. and Littlewood J.E. [21] on L^p-integrability of the sum f of the trigonometric series from one dimensional series to two dimensional series.

Chen C.P. and Hsieh P.H. [15] studied the point-wise convergence of rectangular partial sums of certain type of double trigonometric series under certain conditions on the finite order differences of its coefficients. Kaur K., Bhatia S.S. and Ram B. [28] have studied the double trigonometric series with coefficients of bounded variation of higher order.

Ram B. and Bhatia S.S. [43] have further studied the L^1-convergence of complex double trigonometric series under a new class S_2 of coefficient sequences which is analogous to class S of Sidon [49] for single trigonometric series. Further, Kaur K., Bhatia S.S. and Ram B. [26] studied the L^1-convergence of complex double trigonometric series under different conditions on coefficient sequences.

In the present thesis, number of results have been proved by the author, most of which are directly associated with the works of above mentioned authors.

To provide sufficient background for later chapters, a summary of basic concepts, techniques and a brief chapter wise résumé of the results contained in the thesis has been given in this introductory chapter. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS

Let $\{a_n\}$ be a sequence. Then we write

$$\triangle a_n = a_n - a_{n+1}$$

$$\triangle^2 a_n = \triangle(\triangle a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation ([2], Vol.I, p.1). If $a_0, a_1, a_2, ..., v_0, v_1, ..., v_n, ...$ are any real numbers, let us assume that

$$V_n = v_0 + v_1 + ... + v_n.$$

Then for any values of m and n we find that

$$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \triangle a_k V_k + a_n V_n - a_m V_{m-1}$$

(under the condition that if $m = 0, V_{-1} = 0$).

Convex sequence. A sequence $\{a_n\}$ is said to be convex if $\triangle^2 a_n \geq 0$.

Quasi-convex sequence ([2], Vol.II, p.202). A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty} n|\triangle^2 a_n| < \infty.$$

Semi-convex sequence [23]. A null sequence $\{a_n\}$ is said to be semi-convex if

$$\sum_{n=1}^{\infty} n|\triangle^2 a_{n-1} + \triangle^2 a_n| < \infty, \quad (a_0 = 0)$$

It may be remarked here that every quasi-convex null sequence is semi-convex.

Generalized semi-convex sequence [25]. A null sequence $\{a_n\}$ is said to be generalized semi-convex, if

$$\sum_{n=1}^{\infty} n^{\alpha}|\triangle^{\alpha+1} a_{n-1} + \triangle^{\alpha+1} a_n| < \infty, \quad for \quad \alpha > 0, \quad (a_0 = 0)$$

For $\alpha = 1$, this class reduces to the semi-convex sequence as in [23].

Sequence of bounded variation ([2], Vol.I, p.3). A sequence $\{a_n\}$ is said to be of bounded variation if

$$\sum_{n=1}^{\infty} |\triangle a_n| < \infty.$$

We denote the class of all null-sequences of bounded variation by **BV**.

Sequence of bounded variation of order m≥ 1 [20]. A null sequence $\{a_n\}$ is said to be bounded variation of order $m \geq 1$, if

$$\sum_{n=1}^{\infty} |\triangle^m a_n| < \infty,$$

where

$$\triangle^m a_n = \triangle(\triangle^{m-1} a_n) = \triangle^{m-1} a_n - \triangle^{m-1} a_{n+1}.$$

We denote the class of null sequences of bounded variation of order m by $(\mathbf{BV})^m$. Clearly for $m = 1$, the class $(\mathbf{BV})^m$ reduces to the class $\mathbf{BV}$. Also,

$$\{a_n\} \in (\mathbf{BV})^m \Rightarrow \{a_n\} \in (\mathbf{BV})^{m+1},$$

but the converse is not true [20].

Sequence of bounded variation order (m,p) [55]. A null sequence $\{a_n\}$ is said to be of bounded variation of order (m,p) if

$$\sum_{n=1}^{\infty} |\triangle^m a_n|^p < \infty, \text{ for some } m \geq 1 \text{ and } 1 \leq p \leq 2.$$

We denote such a class of all null sequences of bounded variation of order (m,p) by $\mathbf{BV}$(m,p).

Quasi-monotone sequence ([47], [56]). A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone if $a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n}\right)$ for some $\alpha > 0$ and all $n > n_0(\alpha)$. An equivalent definition is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

Weakly even sequence [54]. A complex null sequence $\{c_n\}$ satisfying

$$\sum_{n=1}^{\infty} |\triangle(c_n - c_{-n})| \log n < \infty,$$

is called weakly even sequence and is denoted by $\{c_n\} \in \mathbf{W}$. Clearly if $\{c_n\}$ is an even sequence $(c_n = c_{-n})$, then it is weakly even.

If $\{c_n\}$ satisfies

$$\sum_{n=1}^{\infty} |\triangle(c_n - c_{-n})| n^r \log n < \infty, \qquad r \in 0, 1, 2, 3, ...,$$

then it is denoted by $\{c_n\} \in \mathbf{W}_r$, obviously $\mathbf{W}_0 = \mathbf{W}$.

Fourier series. A trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

the coefficients a_n and b_n of which are determined by the Fourier formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \ \cos nx \ dx,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \ \sin nx \ dx$$

derived from the function $f(x)$, is called the Fourier series of the function $f(x)$. We then write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

If we set

$$c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2},$$

then the trigonometric series takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

known as the complex form of the trigonometric series. The partial sums

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

then takes the form

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx},$$

and the coefficients c_n are determined by the formulae

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int} dt \qquad (n = 0, \pm 1, ...).$$

Dirichlet kernel ([2], Vol.I, p.85). Let

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

then

$$\begin{aligned} 2\sin\frac{x}{2} D_n(x) &= \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + ... + 2\sin\frac{x}{2}\cos nx \\ &= \sin\left(n + \frac{1}{2}\right) x, \end{aligned}$$

hence

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

This expression is known as Dirichlet's kernel. Moreover,

$$\begin{aligned}\tilde{D}_n(x) &= \sin x + \sin 2x + ... + \sin nx \\ &= \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\end{aligned}$$

is called the kernel conjugate to the Dirichlet kernel.

If $x \neq 0(\mod 2\pi)$, then

$$|D_n(x)| \leq \frac{\pi}{2x}, \quad for \quad 0 < |x| \leq \pi$$

and

$$|\tilde{D}_n(x)| \leq \frac{\pi}{x}, \quad for \quad 0 < |x| \leq \pi$$

Also, we shall use the uniform estimate

$$|D_n(x)| \leq n + \frac{1}{2}, \quad for \quad any \quad x$$

and the estimate

$$L_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(x)|dx \approx \frac{4}{\pi^2}\log n$$

for Lebesgue constant.

We have similarly

$$\tilde{L}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\tilde{D}_n(x)|dx \approx \log n.$$

Fejér kernel ([2], [71]). The Fejér kernel $K_n(x)$ is defined as

$$\begin{aligned}K_n(x) &= \frac{1}{n+1}\sum_{j=0}^{n} D_j(x) \\ &= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left(j+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.\end{aligned}$$

Using $|D_n(x)| \leq n+1$, it follows that $K_n(x) \leq n+1$.

It has the properties

$$(i) \quad K_n(x) \geq 0,$$

$$(ii) \quad \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1.$$

The conjugate Fejér kernel is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{D}_j(x)$$

We have

$$\tilde{K}_n(x) > 0 \quad for \quad 0 < x < \pi, \quad n = 1, 2, 3, ...$$

and

$$|\tilde{K}_n(x)| < \frac{1}{2}n.$$

The class S ([49], [68]). A sequence $\{a_n\}$ belongs to class S, if $a_n \to 0 \ \ as \ \ n \to \infty$ and there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, and $|\Delta a_n| \leq A_n$, for all n.

The class S is usually called as Sidon-Teljakovskĭi class. Since, firstly, Teljakovskĭi expressed Sidon's conditions [49] in a succinct equivalent form and secondly, he showed that the class S is also a class of L^1-convergence.

Obviously, S$\subset$ **BV**. Further, letting $A_n = \sum_{k=n}^{\infty} |\triangle^2 a_k|$, we observe that every quasi-convex null sequence satisfies the class S.

Singh and Sharma [51] gave a definition of one more class namely S′ in the following manner:

The class S′ [51]. A sequence $\{a_n\}$ of numbers is said to belongs to class S′, if $a_n \to 0 \ \ as \ \ n \to \infty$ and there exists a sequence $\{A_n\}$ such that $\{A_n\}$ is quasi-monotone, $\sum_{n=0}^{\infty} A_n < \infty$ and $|\triangle a_n| \leq A_n$, for all n.

However, Leindler [32] proved that class S and class S′ are equivalent.

The class S_p [52]. A sequence $\{a_n\}$ of numbers is said to belongs to class S_p, if $a_n \to 0 \;\; as \;\; n \to \infty$ and there exists a sequence $\{A_n\}$ such that $\{A_n\}$ is monotone, $\sum_{n=0}^{\infty} A_n < \infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\triangle a_k|^p}{A_k^p} = O(1), \quad 1 < p \leq 2, \quad n \to \infty.$$

Clearly, this class is also a generalization of the class S.

The class $\mathrm{S}_{p\alpha}$ [48]. A sequence $\{a_n\}$ of numbers is said to belong to class $\mathrm{S}_{p\alpha}$, if $a_n \to 0 \;\; as \;\; n \to \infty$ and there exists a sequence $\{A_n\}$ such that

$$\begin{aligned}
&(i) \quad A_n \downarrow 0, \quad n \to \infty,\\
&(ii) \quad \sum_{n=0}^{\infty} n^{\alpha} A_n < \infty, \quad for \quad some \quad \alpha \geq 0,\\
&(iii) \quad \frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k{}^p} = O(1), \quad 1 < p \leq 2, \quad n \to \infty.
\end{aligned}$$

For $\alpha = 0$, this class reduces to the class S_p.

In [20], Garrett, Rees and Stanojević gave an equivalent definition of class S in the following manner:

The class S^2 [20]. A null sequence $\{a_n\}$ belong to the class S^2 if there exists a null-sequence $\{A_n\}$ of non-negative numbers such that $\sum_{n=1}^{\infty} n|\triangle A_n| < \infty$ and $|\triangle a_n| \leq A_n$ for all n.

The class S^{**} [70]. A null sequence $\{a_n\}$ of numbers belongs to class S^{**} if

$$n\triangle a_n = o(1), \quad n \to \infty.$$

The class S_r^{**}. A null sequence $\{a_n\}$ is said to belong to class S_r^{**}, $\; r = 0, 1, 2, 3, ...$ if

$$n^{r+1}\triangle a_n = o(1), \quad n \to \infty.$$

For $r = 0$, this class reduces to the class S^{**} [70]. Clearly $S_{r+1}^{**} \subset S_r^{**} \;\; \forall \;\; r = 0, 1, 2, 3, ...,$ but the converse of this inclusion is not true. (cf. Chapter II)

The class $\mathcal{C}$ of Garrett and Stanojević [19]. A null sequence $\{a_n\}$ belongs to class $\mathcal{C}$ if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, independent of n, and such that

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k(x) \right| dx < \epsilon, \quad for \;\; all \;\; n \geq 0.$$

The Class R [23]. A null sequence $\{a_n\}$ belongs to the class R, if

$$\sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{a_k}{k} \right) \right| < \infty.$$

The Class $\mathbf{R}_r$ [3]. A null sequence $\{a_n\}$ belongs to the class R_r, $\;r = 0, 1, 2, 3, ...$, if

$$\sum_{k=1}^{\infty} k^{r+2} \left| \triangle^2 \left(\frac{a_k}{k} \right) \right| < \infty.$$

Clearly, for $r = 0$ this class reduces to the class R.

The class K [27]. If $a_n = o(1), \;\; n \to \infty$ and

$$\sum_{n=1}^{\infty} n|\triangle^2 a_{n-1} - \triangle^2 a_{n+1}| < \infty \quad (a_0 = 0),$$

then we say that $\{a_n\}$ belongs to the class K.

The class $\mathbf{K}^\alpha$ [29]. If $a_n = o(1), \;\; n \to \infty$ and

$$\sum_{n=1}^{\infty} n^\alpha |\triangle^{\alpha+1} a_{n-1} - \triangle^{\alpha+1} a_{n+1}| < \infty \quad for \;\; \alpha > 0, \;\; (a_0 = 0),$$

then we say that $\{a_n\}$ belongs to the class K^α.

For $\alpha = 1$, the class K^α is same as the class K.

The class $\mathbf{S}_r$ [59]. A null sequence $\{a_n\}$ is said to belong to class S_r, $\;r = 0, 1, 2, 3, ...$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum\limits_{k=1}^{\infty} k^r A_k < \infty$, and $|\Delta a_k| \leq A_k$, for all k.

When $r = 0$, we denote $S_r = S$. Clearly $S_{r+1} \subset S_r \;\; \forall \;\; r = 0, 1, 2, 3, ...$, but the converse of this inclusion is not true. (cf. Chapter III)

The class SJ. A null sequence $\{a_n\}$ of positive numbers belongs to class SJ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $\left|\triangle\left(\frac{a_k}{k}\right)\right| \leq \frac{A_k}{k}$, for all k.

The class SJ$_r$. A null sequence $\{a_n\}$ of positive numbers belongs to class SJ$_r$, $r = 0, 1, 2, 3, ...$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{k=1}^{\infty} k^r A_k < \infty$ and $\left|\triangle\left(\frac{a_k}{k}\right)\right| \leq \frac{A_k}{k}$, for all k.

Clearly, for $r = 0$, SJ$_r$ = SJ. It is obvious that SJ$_{r+1} \subset$ SJ$_r$, for all $r = 0, 1, 2, 3, ...$, but the converse is not true. (cf. Chapter IV)

The class S$_p^*$ [54]. A weakly even null-sequence $\{c_n\}$ of complex numbers belongs to the class S$_p^*$, if for some $1 < p \leq 2$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, the condition

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\triangle c_k|^p}{A_k^p} = O(1), \quad n \to \infty$$

holds. Clearly, this class is further generalization of the class S for complex series.

The class S$_{p\alpha r}^*$ [48]. A null sequence of complex numbers $\{c_n\} \in W_r$ belongs to class S$_{p\alpha r}^*$, if for some $1 < p \leq 2$ there exists a monotone sequence $\{A_n\}$ such that

$$(i) \ \sum_{n=1}^{\infty} n^{\alpha} A_n < \infty, \quad for \quad some \quad \alpha \geq 0,$$

$$(ii) \ \frac{1}{n^{p(\alpha-r)+1}}\sum_{k=1}^{n}\frac{|\triangle c_k|^p}{A_k^p} = O(1), \quad r \in \{0, 1, 2, ..., [\alpha]\}, \quad n \to \infty$$

The case $\alpha = r = 0$ of this class reduces to the class S$_p^*$.

The Class R* [4]. A null sequence $\{c_n\}$ of complex numbers belongs to the class R*, if

$$(i) \ \sum_{k=1}^{\infty}\left|\triangle\left(\frac{c_{-k} - c_k}{k}\right)\right| k \log k < \infty,$$

$$(ii) \ \sum_{k=1}^{\infty} k^2 \left|\triangle^2\left(\frac{c_k}{k}\right)\right| < \infty.$$

The class S* ([7], [63]). A null sequence $\{c_n\}$ of complex numbers belongs to the class S* if

$$(i)\ \sum_{k=1}^{\infty}\left|\triangle\left(\frac{c_{-k}-c_k}{k}\right)\right|k^{r+1}\log k<\infty,$$

$$(ii)\ \sum_{k=1}^{\infty}k^{r+2}\left|\triangle^2\left(\frac{c_k}{k}\right)\right|<\infty \quad r\in\{0,1,2,...\}.$$

Clearly, for $r=0$, this class reduces to the class R*. In [63], this class has been named as R*(r).

The class K* [6]. A null sequence $\{c_n\}$ of complex numbers belongs to the class K*, if for some $1<p\le 2$,

$$(i)\ \frac{1}{[\lambda n]}\sum_{k=1}^{[\lambda n]}\left(\frac{c_k-c_{-k}}{k}\right)k\log k=o(1), \qquad (n\to\infty)$$

$$(ii)\ \lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}\triangle\left(\frac{c_k-c_{-k}}{k}\right)k\log k=0,$$

$$(iii)\ \lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]}k^{p-1}\left|\triangle\left(\frac{c_k}{k}\right)\right|^p=0.$$

The class J*. A null sequence $\{c_n\}$ of complex numbers belongs to class J^* if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty}kA_k<\infty$ and $\left|\triangle\left(\frac{c_k-c_{-k}}{k}\right)\right|\le\frac{A_k}{k}$, for all k.

The class $\mathbf{J}^*_r$. A null sequence $\{c_n\}$ of complex numbers belongs to class J^*_r, $r=0,1,2,3,...$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty}k^{r+1}A_k<\infty$ and $\left|\triangle\left(\frac{c_k-c_{-k}}{k}\right)\right|\le\frac{A_k}{k}$, for all k.

Clearly, for $r=0$, this class reduces to the class J^*. It is obvious that $J^*_{r+1}\subset J^*_r$, but converse of this inclusion is false. (cf. Chapter V)

Double Fourier Series ([57], [46]). Let $f(x,y)$ be an absolutely integrable function defined on the square $K(-\pi\le x\le\pi,-\pi\le y\le\pi)$. Then the Fourier series of $f(x,y)$ is given by

$$f(x,y)\sim\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}c_{mn}e^{i(mx+ny)},$$

where

$$c_{mn} = \frac{1}{(2\pi)^2} \int_K \int f(x,y) e^{-i(mx+ny)} dx\ dy$$

This can further be written as

$$f(x,y) \sim \sum_{m=0}^{\infty} \sum_{0}^{\infty} \lambda_{mn} (a_{mn}\ \cos mx\ \cos ny\ +\ \alpha_{mn}\ \cos mx\ \sin ny$$
$$+\ b_{mn}\ \sin mx\ \cos ny\ +\ \beta_{mn}\ \sin mx\ \sin ny)$$

where

$$\begin{aligned} a_{mn} &= \frac{1}{\pi^2} \int_K \int f(x,y)\ \cos mx\ \cos ny\ dx\ dy, \\ \alpha_{mn} &= \frac{1}{\pi^2} \int_K \int f(x,y)\ \cos mx\ \sin ny\ dx\ dy, \\ b_{mn} &= \frac{1}{\pi^2} \int_K \int f(x,y)\ \sin mx\ \cos ny\ dx\ dy, \\ \beta_{mn} &= \frac{1}{\pi^2} \int_K \int f(x,y)\ \sin mx\ \sin ny\ dx\ dy, \end{aligned}$$

$$\lambda_{mn} = \left\{ \begin{array}{ll} \frac{1}{4} & for\ \ m = n = 0 \\ \frac{1}{2} & for\ \ m > 0,\ \ n = 0\ \ or\ \ m = 0,\ \ n > 0 \\ 1 & for\ \ m > 0,\ \ n > 0. \end{array} \right\}$$

Let $\{a_{jk}\}$ be a double sequence of real numbers, then we set

$$\triangle_{10} a_{jk} = a_{jk} - a_{j+1,k},$$

$$\triangle_{01} a_{jk} = a_{jk} - a_{j,k+1},$$

$$\triangle_{11} a_{jk} = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}.$$

The class BV_2 [35]. A sequence $\{a_{jk}\}$ is said to belong to the class BV_2 if

(i) $\quad a_{jk} \to 0\ \ as\ \ j + k \to \infty,$

(ii) $\quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{11} a_{jk}| < \infty.$

Condition (i) expresses that $\{a_{jk}\}$ is a null sequence while (ii) expresses that $\{a_{jk}\}$ is a sequence of bounded variation.

The class $\mathcal{C}_2$ [35]. A null sequence $\{a_{jk}\}$ belongs to class $\mathcal{C}_2$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $0 \leq m \leq M$ and $0 \leq n \leq N$

we have

$$C(m, M; n, N; \delta) := \int\int_{D_\delta} \left| \sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x) D_k(y) \triangle_{11} a_{jk} \right| dx \quad dy \leq \epsilon$$

where

$$D_\delta := Q - (\delta, \pi] \times (\delta, \pi] = \{(x, y) : 0 \leq x, y \leq \pi \ \ \& \ \ \min(x, y) \leq \delta\}$$

or

$$\int\int_{D_\delta} \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} D_j(x) D_k(y) \triangle_{11} a_{jk} \right| dx \quad dy \leq \epsilon \quad \forall \ m, n \geq 0.$$

The class S_d^2. A double null sequence $\{a_{jk}\}$ belongs to S_d^2 if there exists a null sequence $\{A_{jk}\}$ of non-negative numbers such that

$$(i) \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} jk|\triangle_{11} A_{jk}| < \infty,$$

$$(ii) \quad |\triangle_{11} a_{jk}| \leq A_{jk} \ \forall \ j, k.$$

This class is an extension of the class S^2 from one dimension to two dimension.

The class $\mathbf{J}_d$. A double null sequence $\{a_{jk}\}$ of positive numbers is said to belong to class J_d if there exists a double sequence $\{A_{jk}\}$ such that

$$(i) \quad A_{jk} \downarrow 0, \ \ j + k \to \infty$$

$$(ii) \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} jk A_{jk} < \infty,$$

$$(iii) \quad \left| \triangle_{pq} \left(\frac{a_{jk}}{jk} \right) \right| \leq \frac{A_{jk}}{jk}, \quad 1 \leq p + q \leq 2, \quad \forall \ \ p, q \geq 0 \ \ \textit{and} \ \ j, k \in \{1, 2, 3, ...\}$$

1.3 The following results about the behavior of cosine and sine series are known:

Theorem I ([2], [30], [69]). If $\{a_k\}$ is a quasi-convex null sequence, then

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0, \pi]. \tag{1.3.1}$$

Theorem II ([2], [67]) . If $\{a_k\}$ is a quasi-convex null sequence, then

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1.3.2}$$

is a Fourier series if and only if $\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$.

In 1968, Kano [23] generalized Theorem I and Theorem II in the following form:

Theorem III. If $\{a_k\}$ is a null sequence such that

$$\sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{a_k}{k} \right) \right| < \infty, \tag{1.3.3}$$

then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.

Concerning the integrability of trigonometric series belonging to the class S (introduced already in §1.2), Teljakovskiǐ [68] established the following theorems:

Theorem IV. Let the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

belongs to the class S. Then this is a Fourier series and the following relation holds:

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \leq C \sum_{k=0}^{\infty} A_k,$$

where C is an absolute constant.

Theorem V. Let the sine series

$$\sum_{k=1}^{\infty} a_k \sin kx$$

belongs to the class S. Then the following relation holds for $p = 1, 2, 3, ...$

$$\int_{\pi/(p+1)}^{\pi} |\sum_{k=1}^{\infty} a_k \sin kx| dx = \sum_{k=1}^{p} \frac{|a_k|}{k} + O(\sum_{k=1}^{\infty} A_k).$$

We observe that Theorem I and Theorem III provide just only the sufficient conditions for the integrability of cosine series. Rees and Stanojević [45] showed that $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$ is a necessary and sufficient condition for $L^1[0, \pi]$ integrability but for a different type of cosine sums. They proved the following results:

Theorem VI. Let $b_k = \frac{a_k}{k} \downarrow 0$. Then

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{b_k}{2} + \left(\sum_{j=k}^{n} b_j \right) \cos kx \right]$$

exists for $x \in (0, \pi]$ and $g \in L^1[0, \pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Theorem VII. Let $b_k = \frac{a_k}{k} \downarrow 0$. Then

$$\frac{1}{x} \sum_{k=1}^{\infty} \frac{b_k}{2} \sin(k + \frac{1}{2})x = \frac{h(x)}{x},$$

converges for $x \neq 0$ and $\frac{h(x)}{x} \in L^1[0, \pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Theorem VIII. Let $(k + 1)|\triangle^2 a_k| \downarrow 0$. Then

$$h(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{2}(k + 1)|\triangle^2 a_k| + (\sum_{j=k}^{n} (j + 1)|\triangle^2 a_j|) \cos kx \right]$$

exists for $x \in (0, \pi]$ and $h \in L^1[0, \pi]$ if and only if $\{a_k\}$ is quasi-convex.

Ram [40] showed that the condition S is sufficient for the integrability of Rees-Stanojević sums [45]

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_j) \cos kx \qquad (1.3.4)$$

He proved the following theorems:

Theorem IX. Let the sequence $\{a_k\}$ satisfy the condition S. Then $g(x) = \lim_{n\to\infty} g_n(x)$ exists for $x \in (0, \pi]$, and

$$\int_0^\pi |g(x)|dx \leq C \sum_{k=0}^{\infty} A_k.$$

Theorem X. Let $\{a_k\}$ be a sequence satisfying the condition S. Then

$$\frac{1}{x}\sum_{k=1}^{\infty} \triangle a_k \sin\left(k+\frac{1}{2}\right)x = \frac{h(x)}{x}$$

converges for $x \in (0, \pi]$ and $\frac{h(x)}{x} \in L^1[0, \pi]$.

The above theorems were further studied by Ram [41], under a condition where the monotonicity of the sequence in the definition of the class S is replaced by quasi-monotonicity. Consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.3.5}$$

Let the partial sums of (1.3.5) is denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$. Denote the class of sequences of Fourier coefficients $\{a_k\}$ by $\mathbf{F}$. There are subclasses of $\mathbf{F}$ for which $a_n \log n = o(1), \quad n \to \infty$ is a necessary and sufficient condition for $||S_n - f||_{L^1} = o(1), \quad n \to \infty$.

A subclass $\mathbf{G}$ of $\mathbf{F}$ is called a class of L^1-convergence if $||S_n - f||_{L^1} = o(1), \quad n \to \infty$ if and only if $a_n \log n = o(1), \quad n \to \infty$.

There are three classical examples of classes of L^1-convergence. The first one is due to Young [69], where $\mathbf{G}$ is defined to be the class of all convex sequences $\{a_k\}$. The second one is the class of all quasi-convex sequences $\{a_k\}$, introduced by Kolmogorov [30]. The third example is class S due to Teljakovskiǐ [68]. We have already pointed out that S$\subset$ $\mathbf{BV}$. These classical classes have been extended by various authors. We present now a brief summary of the results obtained by various authors in this direction.

Concerning the L^1-convergence of the cosine series, we have the following classical result of Kolmogorov [30].

Theorem XI. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.3.5) in the metric space L, it is necessary and sufficient that $a_k \log k = o(1), \quad k \to \infty$.

The case, in which the sequence $\{a_k\}$ is convex, of this theorem was established by Young [69].

Generalizing the above classical result, Teljakovskiǐ [68] proved the following result:

Theorem XII. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (1.3.5) is $a_k \log k = o(1), \quad k \to \infty$.

Rees and Stanojević [45] introduced modified cosine sums (1.3.4) and obtained an analogue of Theorem XI for these sums. These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1-metric to the sum of the cosine series whereas the classical cosine series itself may not. They proved the following result:

Theorem XIII. Let f be the sum of the cosine series (1.3.5). Then $g_n(x)$ converges to f in L^1-metric if and only if $\{a_k\}$ belongs to the class $\mathcal{C}$.

Ram [39] proved the following result on L^1-convergence of Rees-Stanojević sums (1.3.4).

Theorem XIV. If (1.3.5) belongs to class S. Then

$$||f - g_n||_{L^1} = o(1), \quad n \to \infty.$$

Theorem XII of Teljakovskiǐ [68] follows as corollary of this theorem.

Singh and Sharma [51] proved the above theorem by replacing the monotonicity of sequence $\{A_n\}$ in the definition of class S by quasi-monotonocity of $\{A_n\}$. Their result reads as:

Theorem XV. Let $a_n \in \mathrm{S}'$, then $f_n(x)$ converges to $f(x)$ in L^1-metric.

Further, Ram and Kumari ([31], [42]) introduced new modified cosine and sine sums as

$$(1.3.6) \qquad f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k \cos kx$$

and

$$(1.3.7) \qquad g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k \sin kx$$

and studied their L^1-convergence under the condition that the cosine series and sine series belong to the classes R and S. They also deduced the results about L^1-convergence of cosine and sine series. Their results state as below:

Theorem XVI. Let $\{a_n\}$ belong to the class S. If $\lim_{n\to\infty} |a_{n+1}| \log n = 0$, then $||f - f_n||_{L^1} = o(1) \;\; n \to \infty$.

Theorem XVII. Let $\{a_n\}$ belong to the class R. If $t_n(x)$ represents $f_n(x)$ and $g_n(x)$, then $||f - t_n||_{L^1} = o(1) \;\; n \to \infty$.

Later, Hooda and Ram [22] have proved the following theorem:

Theorem XVIII. Let $\{a_n\}$ belong to the class S$'$. Then $\quad ||f - f_n||_{L^1} = o(1), \quad n \to \infty$.

In chapter II, we have generalized the Theorems XII, XVI and XVIII for the cosine series with more general class S^{**} and S_r^{**} by using the modified cosine sums (1.3.6) of Ram and Kumari [42].

The objective of chapter III is to generalize Theorem XIV for the cosine series for an extended class S_r, $\; r = 0, 1, 2....$ of coefficient sequences and also to study the L^1-convergence of the r^{th} derivative of cosine series.

In chapter IV, we have introduced new modified cosine and sine sums as

$$(1.3.8) \qquad f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx)$$

and

$$(1.3.9) \qquad g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \sin jx)$$

and have studied their integrability and L^1-convergence under a newly defined class SJ of coefficient sequences. Also, the L^1-convergence of cosine and sine series have been deduced as corollary.

Further, in this chapter, we have obtained L^1-convergence of r^{th} derivative of modified cosine and sine sums (1.3.8) and (1.3.9) under an extended class SJ_r of coefficient sequences.

1.4 Let

$$S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt}, \quad t \in T = \mathbb{R}/2\pi Z,$$

denotes the partial sums of the complex trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$. In case the trigonometric series is a Fourier series of some Lebesgue integrable function f, we write $c_n = \hat{f}(n)$ for all n and $S_n(C,t) = S_n(f)$.

Concerning the integrability and L^1-convergence of complex trigonometric series, Stanojević, Č.V and Stanojević, V.B. [54] proved the following result, which in turn generalizes the Teljakovskiĭ [68] result.

Theorem XIX. Let $\{c_n\} \in S_p^*$. Then

$$(i) \qquad for \ \ t \neq 0, \quad \lim_{n\to\infty} S_n(C,t) = f(t) \quad exists,$$

$$(ii) \qquad f(t) \in L^1(T);$$

$$(iii) \qquad ||S_n(f) - f|| = o(1), \ \ n \to \infty, \ is \ \ equivalent \ \ to \ \ \hat{f}(n)\log|n| = o(1), \ \ |n| \to \infty.$$

In proving this theorem the authors have used the complex form

$$g_n(C,t) = S_n(C,t) - (c_n E_n(t) + c_{-n} E_{-n}(t))$$

of Rees-Stanojević sums (1.3.4).

Sheng [48] extended the results of Stanojević and Stanojević [54] by studying the L^1-convergence of the r^{th} differential of complex form of Rees-Stanojević sums (1.3.4) with coefficients belonging to the class $S^*_{p\alpha r}$.

Further, in 1995, Bhatia and Ram [4] introduced the complex form

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}[c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)] \tag{1.4.1}$$

of modified trigonometric sums (1.3.6) and (1.3.7) and studied their L^1-convergence under a new class R* of coefficient sequences and obtained the necessary and sufficient condition for the L^1-convergence of complex trigonometric series.

Later, Bhatia and Ram [5] introduced the complex form of r^{th} derivative of complex trigonometric sums (1.4.1) as

$$g_n{}^r(C,t) = S_n{}^r(C,t) + \frac{i}{n+1}[c_{n+1}E_n^{r+1}(t) - c_{-(n+1)}E_{-n}^{r+1}(t)] \tag{1.4.2}$$

and gave the sufficient condition for the integrability of r times differentiated trigonometric series using the complex trigonometric sums (1.4.2) and obtained new necessary and sufficient condition for L^1-convergence of r^{th} derivative of complex trigonometric series.

In chapter V, we have considered the L^1-convergence of the complex form

$$g_n(C,t) = S_n(C,t) - n(c_{n+1}e^{i(n+1)t} + c_{-(n+1)}e^{-i(n+1)t}) \tag{1.4.3}$$

of the newly defined modified trigonometric sums (1.3.8) and (1.3.9) and have studied the L^1-convergence of complex trigonometric series with coefficients belonging to a new class J^* of coefficient sequences. Further, the L^1-convergence of r^{th} derivative of the complex trigonometric sums (1.4.3) under an extended class J_r^* have also been studied in this chapter.

1.5 In the literature so far available, most of the authors have studied the integrability and L^1-convergence of cosine trigonometric series using different classes of coefficient sequences. However, very few of them have studied the L^1-convergence of sine trigonometric series.

In this direction, we have, in chapter VI studied the L^1-convergence of sine trigonometric series by using a newly introduced modified cosine trigonometric sum

$$(1.5.1) \qquad \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_{j+1}-\triangle a_{j-1})\cos kx, \qquad (a_0=a_1=0)$$

under a new class J of coefficient sequences.

In chapter VII, the results of chapter VI have been extended by introducing the complex form

$$(1.5.2) \qquad S_n(C,t)+\frac{i}{2\sin t}\begin{bmatrix} c_n e^{i(n+1)t}-c_{-n}e^{-i(n+1)t}+c_{n+1}e^{int}-c_{-(n+1)}e^{-int} \\ \\ +(c_n-c_{n+2})E_n(t)+(c_{-(n+2)}-c_{-n})E_{-n}(t) \end{bmatrix}$$

of the modified trigonometric cosine and sine sums introduced by the author and that of Kaur, Bhatia and Ram [27]. In this chapter the integrability and L^1-convergence of complex trigonometric series have been studied by making use of the complex forms (1.5.2) under the class J^* of coefficient sequences.

1.6 Concerning the L^1-convergence of the double cosine series Móricz [35] introduced the following modified double cosine trigonometric sum

$$u_{mn}(x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k(\sum_{i=j}^{m}\sum_{l=k}^{n}\triangle_{11}a_{il})\cos jx\cos ky.$$

and studied the L^1-convergence of double cosine trigonometric series whose coefficients belong to class BV_2, class $\mathcal{C}_2$ and the class of quasi-convex coefficients by making use of L^1-convergence of these modified double cosine trigonometric sums. He proved the following result:

Theorem XX. If $\{a_{jk}\}\in BV_2\cap C_2$, then the sum $f(x,y)$ of series

$$(1.6.1) \qquad \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k \quad a_{jk}\cos jx\cos ky$$

belongs to $L^1(Q)$, where $Q = [0, \pi] \times [0, \pi]$ and (1.6.1) is a Fourier series of $f(x, y)$.

In chapter VIII, we have extended the results discussed in chapter IV from one dimensional cosine and sine series to two dimensional ones. In this chapter, we have obtained necessary and sufficient conditions for the integrability and L^1-convergence of double cosine, sine and mixed series by using modified double cosine, sine and mixed trigonometric sums with coefficients satisfying certain conditions. Also, we have considered the special cases where the double sequence of coefficients is of bounded variation and quasi convex.

In chapter IX, we have extended the class S^2 of Garrett, Rees and Stanojević [20] from one dimensional coefficient sequences to a new class S_d^2 of two dimensional coefficient sequence and the result of Garrett, Rees and Stanojević [20] on the L^1-convergence of the cosine trigonometric series have been extended from one dimensional to two dimensional cosine series.

CHAPTER II

ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS

2.1 Introduction. Consider the cosine trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{2.1.1}$$

and let the partial sums of (2.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Regarding L^1-convergence of (2.1.1), the following theorem of Teljakovskiĭ [68] is well known:

Theorem A. If the coefficient sequence $\{a_n\}$ of the cosine series (2.1.1) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (2.1.1) is $a_n \log n = o(1)$, $n \to \infty$.

Further, Rees and Stanojević [45] introduced the modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\triangle a_j) \cos kx \tag{2.1.2}$$

and obtained a necessary and sufficient condition for the integrability of the limit of (2.1.2).

Ram [41] proved the following theorem in which he showed that class S′ is sufficient for the integrability of limit of (2.1.2).

Theorem B. Let the sequence $\{a_k\}$ satisfy the condition S′. Then

$$g(x) = \lim_{n\to\infty} [\frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\triangle a_j) \cos kx]$$

exists for $x \in (0, \pi]$, and $g(x) \in L(0, \pi)$.

The results obtained in this chapter have been published in **Global Journal of Pure and Applied Mathematics, 2(2), (2006), 111-116.**

Further, Zahid and Hasan [70] defined a new class S^{**} as:

Definition. A sequence $\{a_n\}$ of numbers belongs to class S^{**} if $a_n \to 0$ as $n \to \infty$ and

$$(2.1.3) \qquad n\triangle a_n = o(1), \quad n \to \infty.$$

Class S^{**} is more general than the class S$'$ of Singh and Sharma [51]. Since,

The sequence

$$a_n = \frac{(-1)^{n+1}}{n(\log(n+1))} \quad (n = 1, 2, 3....)$$

does not satisfy the class S$'$ of Singh and Sharma, as $|\triangle a_n| \geq (n\log(n+1))^{-1}$ and so $\sum |\triangle a_n| = \infty$. This contradicts the conditions of class S$'$. However, this sequence satisfies the condition (2.1.3) of S^{**}. Zahid and Hasan [70] showed that the class S^{**} is sufficient for the integrability of limit of (2.1.2) and proved the following theorem:

Theorem C. Let the sequence $\{a_k\}$ satisfy the condition S^{**}. Then

$$g(x) = \lim_{n\to\infty} [\frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_j)\cos kx]$$

exists for $x \in (0, \pi]$, and $g(x) \in L(0, \pi)$.

Later, in 1988, Kumari and Ram [31] introduced the modified cosine sum

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j}{j}\right) k\cos kx$$

and proved the following theorem concerning the L^1-convergence of (2.1.1).

Theorem D. Let $\{a_n\}$ belong to the class S. If $\lim\limits_{n\to\infty} |a_{n+1}|\log n = 0$, then

$$||f - f_n|| = o(1) \quad n \to \infty$$

Further, Hooda and Ram [22] proved the following theorem:

Theorem E. Let $\{a_n\}$ belong to the class S$'$. Then

$$||f - f_n|| = o(1) \quad n \to \infty$$

We, now define a new class S_r^{**}, $r = 0, 1, 2, 3, ...$ of coefficient sequences:

Definition. A null sequence $\{a_n\}$ is said to belong to class S_r^{**}, $r = 0, 1, 2, 3, ...$, if

$$(2.1.4) \qquad n^{r+1}\triangle a_n = o(1), \quad n \to \infty,$$

If $r = 0$, the class S_r^{**} reduces to the class S^{**} of Zahid and Hasan [70]. Clearly, $S_{r+1}^{**} \subset S_r^{**}$, but the converse is not true.

Example. For $n = 1, 2, 3...$, define $a_n = \dfrac{(-1)^{n+1}}{n^{r+2}}$, $r = 0, 1, 2, 3, ...$ Now, $n^{r+1}\triangle a_n = n^{r+1}(-1)^{n+1}\left[\frac{1}{n^{r+2}} + \frac{1}{(n+1)^{r+2}}\right] = o(1)$, $n \to \infty$. But, $n^{r+2}\triangle a_n = n^{r+2}(-1)^{n+1}\left[\frac{1}{n^{r+2}} + \frac{1}{(n+1)^{r+2}}\right]$ does not tends to zero as n tends to infinity.

The objective of this chapter is to generalize the Theorems A, D and E and also to study the L^1-convergence of the modified sums $f_n(x)$ and ${f_n}^r(x)$ under the condition that (2.1.1) belongs to class S^{**} and S_r^{**} respectively, where ${f_n}^r(x)$ represents the r^{th} derivative of $f_n(x)$. Also, a necessary and sufficient condition for the L^1-convergence of the cosine series is deduced as a corollary. These results also generalize the results of Teljakovskiĭ [68] on L^1-convergence of the cosine series.

2.2 Lemma. The following lemma of Sheng [48] will be required for the proof of our results:

Lemma 1 [48]. $||\tilde{D}_n^r(x)||_{L^1} = O(n^r \log n)$, $r = \{0, 1, 2, 3....\}$, where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet kernel $\tilde{D}_n(x) = \sum_{k=1}^{n} \sin kx$.

2.3 Results. We prove the following theorems:

Theorem 1. Let $\{a_n\} \in S^{**}$. Then $||f - f_n|| = o(1)$, $n \to \infty$ if and only if $|a_n| \log n = o(1)$, $n \to \infty$.

Theorem 2. Let $\{a_n\} \in S^{**}$. Then $||f - S_n|| = o(1)$, $n \to \infty$ if and only if $|a_n| \log n = o(1)$, $n \to \infty$.

Theorem 3. Let $\{a_n\} \in S_r^{**}$, $r \in \{0, 1, 2, 3.....\}$. Then $||f^r - {f_n}^r|| = o(1)$, $n \to \infty$ if and only if $n^r|a_n|\log n = o(1), \quad n \to \infty.$

Theorem 4. Let $\{a_n\} \in S_r^{**}, \quad r \in \{0, 1, 2, 3.....\}$. Then $||f^r - {S_n}^r|| = o(1), \quad n \to \infty$ if and only if $n^r|a_n|\log n = o(1), \quad n \to \infty.$

Remark. The case $r = 0$, in Theorem 3 and 4 yields the Theorem 1 and 2.

2.4 Proofs of the Theorems

Proof of Theorem 1. We have

$$\begin{aligned} f_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k}{k} - \frac{a_{k+1}}{k+1} + ... + \frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right) k\cos kx \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n'(x) \end{aligned}$$

Since, $|\tilde{D}_n'(x)| = O(n)$ in $(0, \pi]$ and also $\{a_n\}$ is a null sequence. Therefore, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x)$ for $x \in (0, \pi]$.

Now consider,

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k\cos kx + \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)$$

Applying Abel's transformation, we have

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} \triangle a_k D_k(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)$$

Therefore,

$$\begin{aligned} ||f - f_n|| &= \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} \triangle a_k D_k(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right| dx \\ &\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} \triangle a_k D_k(x)\right| dx + \int_0^{\pi} |a_{n+1}D_n(x)|\ dx + \int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right| dx \end{aligned}$$

Since, $\int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n'(x)\right| dx \approx |a_{n+1}|\log n$, by Zygmund's Theorem ([2], Vol.II, p.458).

By the hypothesis of the theorem that $\{a_n\} \in S^{**}$ and lemma 1, we get

$$\begin{aligned} ||f - f_n|| &= O(\sum_{k=n+1}^{\infty} k\triangle a_k) + o(|a_{n+1}|\log n) + o(|a_{n+1}|\log n) \\ &= o(1) + o(|a_{n+1}|\log n). \end{aligned}$$

Hence, $||f - f_n|| = o(1),\ n \to \infty$ if and only if $|a_n|\log n = o(1),\ n \to \infty$. □

Proof of Theorem 2. Consider,

$$\begin{aligned} ||f - S_n|| &= ||f - f_n + f_n - S_n|| \\ &\leq ||f - f_n|| + ||f_n - S_n|| \\ &= ||f - f_n|| + \int_0^{\pi} \left| \frac{a_{n+1}}{n+1}\ \tilde{D}'_n(x)\ dx \right| \end{aligned}$$

Further, $||f - f_n|| = o(1),\ \ n \to \infty$ (by theorem 1) and also we know that $\int_0^{\pi} \left| \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right|\ dx \approx |a_{n+1}|\log n$, by Zygmund's theorem ([2], Vol.II, p.458). Thus $||f - S_n|| = o(1),\quad n \to \infty$ if and only if $|a_n|\log n = o(1),\ \ n \to \infty$. □

Proof of Theorem 3. Consider,

$$\begin{aligned} f_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j}{j}\right) k \cos kx \\ &= S_n(x) - \frac{a_{n+1}}{n+1}\tilde{D}'_n(x). \end{aligned}$$

We have then,

$$f_n{}^r(x) = S_n{}^r(x) - \frac{a_{n+1}}{n+1}\ \tilde{D}_n^{r+1}(x), \tag{2.4.1}$$

where $\tilde{D}_n^r$ represents the r^{th} derivative of conjugate Dirichlet kernel. Since $\{a_n\} \in S_r^{**},\ \ r \in \{0, 1, 2, ...\}$, therefore, $\lim_{n\to\infty} f_n{}^r(x) = f^r(x)$ exists for $x \in (0, \pi]$.

Now, it follows from (2.4.1) that

$$f^r(x) - f_n{}^r(x) = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)$$

Making use of Abel's transformation, we have

$$f^r(x) - {f_n}^r(x) = \sum_{k=n+1}^{\infty} \triangle a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x)$$

Where ${D_n}^r(x)$ denotes the r^{th} derivative of Dirichlet kernel.

We notice further that

$$\begin{aligned} ||f^r - {f_n}^r|| &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k^r(x) - a_{n+1} D_n^r(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| \, dx \\ &\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k^r(x) \right| \, dx + \int_0^\pi |a_{n+1} D_n^r(x)| \, dx + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| \, dx \end{aligned}$$

Moreover, by use of lemma 1 and given hypothesis, we have

$$\begin{aligned} ||f^r - {f_n}^r|| &= O(\sum_{k=n+1}^{\infty} k^{r+1} \triangle a_k) + o(n^r |a_{n+1}| \log n) + o(n^r |a_{n+1}| \log n) \\ &= o(1) + o(n^r |a_{n+1}| \log n). \end{aligned}$$

This completes the theorem 3. □

Proof of Theorem 4. We notice that,

$$\begin{aligned} ||f^r - {S_n}^r|| &= ||f^r - {f_n}^r + {f_n}^r - {S_n}^r|| \\ &\leq ||f^r - {f_n}^r|| + ||{f_n}^r - {S_n}^r|| \\ &= ||f^r - {f_n}^r|| + \int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| \, dx \end{aligned}$$

Since, $||f^r - {f_n}^r|| = o(1), \quad \to \infty$, by theorem 3 and $\int_0^\pi \left| \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x) \right| \, dx$ behaves like $n^r |a_{n+1}| \log n$ for large values of n by lemma 1, the conclusion of the theorem 4 follows. □

CHAPTER III

INTEGRABILITY AND L^1-CONVERGENCE OF CERTAIN COSINE SUMS

3.1 Introduction. Consider cosine series

(3.1.1) $$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx$$

Let the partial sums of (3.1.1) be denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty}S_n(x)$. Further, let $f^r(x)=\lim_{n\to\infty}S_n^r(x)$ where $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Concerning the L^1-convergence of Rees-Stanojević sums [45]

(3.1.2) $$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\ \Delta a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}\ \Delta a_j\ \cos kx$$

Ram [39] proved the following result:

Theorem A. If (3.1.1) belongs to class S. Then

$$||f-g_n||_{L^1}=o(1),\quad n\to\infty.$$

Recently, Tomovski [60] extended the Sidon-Teljakovskǐi class S to a new class S_r, $r=0,1,2,3...$, as follows:

Definition. A null sequence $\{a_n\}$ is said to belong to class S_r, $r=0,1,2,3,...$, if there exists a sequence $\{A_n\}$ such that

(3.1.3) $$A_n\downarrow 0,\qquad n\to\infty,$$

(3.1.4) $$\sum_{n=0}^{\infty}n^rA_n<\infty,$$

(3.1.5) $$|\Delta a_n|\le A_n,\qquad \forall\quad n.$$

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For $r = 0$, this class reduces to class S. We note that, by $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$, it follows that $k^{r+1} A_k = o(1), \quad k \to \infty$.

It is obvious that $S_{r+1} \subset S_r, \quad \forall \; r = 0, 1, 2, 3....$, but the converse of that inclusion is not true. In support of his assertion, Tomovski [60] presented the following example:

Example. For $n = 1, 2, 3, ...$ define $\triangle a_n = \dfrac{1}{n^{r+2}}, \quad r = 0, 1, 2, 3, ...$

Really, $a_n = \sum_{k=n}^{\infty} \triangle a_k = \sum_{k=n}^{\infty} \dfrac{1}{k^{r+2}} \to 0 \;\; as \;\; n \to \infty$.

Let we define $A_n^* = \max_{i \geq n} |\triangle a_i|$. Then $A_n^* = |\triangle a_n|$ and $A_n^* \downarrow 0 \;\; as \;\; n \to \infty$. Let A_n is an arbitrary positive sequence such that $A_n \downarrow 0$ and $A_n^* \leq A_n$.

Then $\sum_{n=1}^{\infty} n^{r+1} A_n \geq \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, i.e. $\{a_n\} \notin S_{r+1}$.

Now, for all n, let $A_n = \frac{1}{n^{r+2}}, \quad r = 0, 1, 2, ...$

Then $A_n \downarrow 0$, $|\Delta a_n| \leq A_n$ and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, i.e. $\{a_n\} \in S_r$.

We consider another example in support of Tomovski assertion as given below:

Example. For $n = 1, 2, 3, ...$ define $\triangle a_n = \dfrac{\ln n}{n^{r+3/2}}, \quad r = 0, 1, 2, 3, ...$

Really, $a_n = \sum_{k=n}^{\infty} \triangle a_k = \sum_{k=n}^{\infty} \dfrac{\ln k}{k^{r+3/2}} \to 0 \;\; as \;\; n \to \infty$.

Let we define $A_n = \dfrac{\ln n}{n^{r+5/4}}$. Then $A_n \downarrow 0 \;\; as \;\; n \to \infty$ and $|\triangle a_n| \leq A_n$, for all n.

Also $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \dfrac{\ln n}{n^{5/4}}$ is convergent, i.e. $\{a_n\} \in S_r$. But, $\sum_{n=1}^{\infty} n^{r+1} A_n = \sum_{n=1}^{\infty} \dfrac{\ln n}{n^{1/4}}$ is divergent, i.e. $\{a_n\} \notin S_{r+1}$.

The aim of this chapter is to generalize Theorem A for the cosine series with extended class $S_r, \quad r = 0, 1, 2, 3....$ of coefficient sequences and also to study the L^1-convergence of the r^{th} derivative of cosine series.

3.2 Lemmas. The proofs of our results are based on the following lemmas:

Lemma 1 [17]. If $|a_k| \leq 1$, then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k \; D_k(x) \right| \, dx \leq C(n+1),$$

where C is positive absolute constant.

Lemma 2 [60]. Let $\{a_k\}$ be a sequence of real numbers such that $|a_k| \leq 1, \ \forall \ k$. Then there exists a constant $C > 0$ such that for any $n \geq 0$ and $r = 0, 1, 2, 3....$

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k \ D_k^r(x) \right| \ dx \leq C(n+1)^{r+1}.$$

Lemma 3 [48]. $||D_n^r(x)||_{L^1} = O(n^r \log n), \ r = 0, 1, 2, 3....$, where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

3.3 Results. We prove the following two theorems:

Theorem 1. If (3.1.1) belongs to class S_r, then

$$||f - g_n||_{L^1} = o(1), \ n \to \infty.$$

Proof. We have,

$$\begin{aligned} g_n(x) &= \frac{1}{2} \sum_{k=0}^{n} \ \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle a_j \cos kx \\ &= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \end{aligned}$$

Thus, $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} S_n(x) = f(x)$. Since, $D_n(x)$ is bounded in $(0, \pi]$ and $\{a_n\} \in S_r$.

Now, we consider

$$f(x) - g_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1} D_n(x)$$

Making use of Abel's transformation and lemma 1, we have

$$\begin{aligned} \int_0^{\pi} |f(x) - g_n(x)| dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k(x) \right| dx \\ &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\triangle a_k}{A_k} D_k(x) \right| dx \\ &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| dx \\ &\leq \sum_{k=n+1}^{\infty} \frac{k^r}{k^r} \triangle A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| dx \\ &\leq \frac{1}{(n+1)^r} \sum_{k=n+1}^{\infty} k^r \triangle A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| dx \end{aligned}$$

$$\int_0^\pi |f(x) - g_n(x)|dx \leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \triangle A_k$$

(3.1.3) and (3.1.4) now imply the conclusion of the theorem 1. □

Corollary 1. If (3.1.1) belongs to class S_r, $r = 0, 1, 2, 3....$ then

$||f - S_n||_{L^1} = o(1), \quad n \to \infty$ if and only if $|a_{n+1}| \log n = o(1), \quad n \to \infty$.

Proof. Consider,

$$||f - S_n|| = ||f - g_n + g_n - S_n||$$

$$\begin{aligned} ||f - S_n|| &\leq ||f - g_n|| + ||g_n - S_n|| \\ &= ||f - g_n|| + ||a_{n+1} D_n(x)|| \\ &\leq ||f - g_n|| + |a_{n+1}| \int_0^\pi |D_n(x)| \, dx \end{aligned}$$

Further, $||f - g_n||_{L^1} = o(1), \quad n \to \infty$ (by theorem 1) and by lemma 3, the conclusion of the corollary follows. □

Remark. Case $r = 0$ yields the result of Ram [39].

Theorem 2. If (3.1.1) belongs to class S_r, then

$$||f^r - {g_n}^r||_{L^1} = o(1), \quad n \to \infty.$$

Proof. Consider,

$$g_n(x) = S_n(x) - a_{n+1} D_n(x)$$

We have then,

$$g_n{}^r(x) = S_n{}^r(x) - a_{n+1} D_n{}^r(x) \tag{3.3.1}$$

where $g_n{}^r(x)$ represents the r^{th} derivative of $g_n(x)$ and $D_n{}^r(x)$ represents the r^{th} derivative of Dirichlet kernel. Since $\{a_n\}$ is a null sequence and $D_n{}^r(x)$ is bounded in $(0, \pi]$.

Therefore,

$$\lim_{n\to\infty} g_n{}^r(x) = \lim_{n\to\infty} S_n{}^r(x) = f^r(x).$$

For $x \neq 0$, it follows from (3.3.1) that

$$f^r(x) - g_n{}^r(x) = \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1} D_n{}^r(x)$$

Making use of Abel's transformation, we get

$$f^r(x) - g_n{}^r(x) = \sum_{k=n+1}^{\infty} \triangle a_k D_k{}^r(x)$$

We notice further that

$$\int_0^{\pi} |f^r(x) - g_n{}^r(x)|\, dx \;\; = \;\; \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k{}^r(x) \right| dx$$

Again, an application of Abel's transformation and lemma 2 yield

$$\begin{aligned}
\int_0^{\pi} |f^r(x) - g_n{}^r(x)|\, dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\triangle a_k}{A_k} D_k{}^r(x) \right| dx \\
&\leq \sum_{k=n+1}^{\infty} \triangle A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i{}^r(x) \right| dx \\
&\leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \triangle A_k,
\end{aligned}$$

use of (3.1.3) and (3.1.4) imply the conclusion of the theorem 2. □

Corollary 2. If (3.1.1) belongs to class S_r, $\;r = 0, 1, 2, 3....$ then

$||f^r - S_n{}^r||_{L^1} = o(1), \;\; n \to \infty$ if and only if $|a_{n+1}| n^r \log n = o(1), \;\; n \to \infty$.

Proof. Consider,

$$\begin{aligned}
||f^r - S_n{}^r|| &= ||f^r - g_n{}^r + g_n{}^r - S_n{}^r|| \\
&\leq ||f^r - g_n{}^r|| + ||g_n{}^r - S_n{}^r|| \\
&\leq ||f^r - g_n{}^r|| + ||a_{n+1} D_n{}^r(x)|| \\
&\leq ||f^r - g_n{}^r|| + |a_{n+1}| \int_0^{\pi} |D_n{}^r(x)|\, dx
\end{aligned}$$

Since, $||f^r - {g_n}^r||_{L^1} = o(1), \quad n \to \infty$ (by theorem 2) and $\displaystyle\int_0^\pi |a_{n+1} D_n^r(x)| \; dx$ behaves like $n^r |a_{n+1}| \log n$ for large values of n by lemma 3, the conclusion of the corollary follows. $\square$

CHAPTER IV

CONVERGENCE OF NEW MODIFIED TRIGONOMETRIC SUMS IN THE METRIC SPACE L

4.1 Introduction. Consider cosine and sine series

$$(4.1.1) \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

and

$$(4.1.2) \qquad \sum_{k=1}^{\infty} a_k \sin kx$$

or together

$$(4.1.3) \qquad \sum_{k=1}^{\infty} a_k \phi_k(x)$$

where $\phi_k(x)$ is $\cos kx$ or $\sin kx$ respectively. Let the partial sums of (4.1.3) be denoted by $S_n(x)$ and $t(x) = \lim_{n\to\infty} S_n(x)$. Further, let $t^r(x) = \lim_{n\to\infty} S_n^r(x)$ where $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Concerning L^1-convergence of (4.1.1) and (4.1.2), the following theorems are known:

Theorem A ([2], Vol.II, p.204). If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (4.1.1) in the metric space L^1, it is necessary and sufficient that $a_k \log k = o(1), \quad k \to \infty$.

This theorem is due to Kolmogorov [30]. Teljakovskiǐ [68] generalized Theorem A for the cosine series (4.1.1) with coefficients $\{a_k\}$ satisfying the conditions of the class S in the following form:

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Theorem B. If the coefficient sequence $\{a_k\}$ of the cosine series (4.1.1) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (4.1.1) is $a_k \log k = o(1), \quad k \to \infty$.

Theorem C ([2], Vol.II, p.201). If $a_k \downarrow 0$ and $\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right) < \infty$, then (4.1.3) is a Fourier series.

Concerning the L^1-convergence of Rees-Stanojević cosine sums [45]

$$u_n(x) = \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle a_k \cos kx \tag{4.1.4}$$

to a cosine trigonometric series, belonging to the class S, Ram [39] proved the following theorem:

Theorem D. If (4.1.1) belongs to class S. Then

$$||f - u_n||_{L^1} = o(1), \quad n \to \infty.$$

In the present chapter, we have introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx)$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \sin jx)$$

and have studied their integrability and L^1-convergence under a new class SJ of coefficient sequences defined as follows:

Definition. A null sequence $\{a_n\}$ of positive numbers belongs to class SJ if there exists a sequence $\{A_n\}$ such that

$$A_n \downarrow 0, \quad as \quad n \to \infty, \tag{4.1.5}$$

$$\sum_{n=1}^{\infty} A_n < \infty, \tag{4.1.6}$$

$$\left|\triangle\left(\frac{a_n}{n}\right)\right| \leq \frac{A_n}{n}, \quad \forall \quad n. \tag{4.1.7}$$

Now, we define a new class SJ_r of coefficient sequences which is an extension of class SJ.

Definition. A null sequence $\{a_n\}$ of positive numbers belongs to class SJ_r, $\;r = 0, 1, 2, 3, ...,$ if there exists a sequence $\{A_n\}$ such that

$$(4.1.8) \qquad A_n \downarrow 0, \quad as \quad n \to \infty,$$

$$(4.1.9) \qquad \sum_{n=1}^{\infty} n^r A_n < \infty,$$

$$(4.1.10) \qquad \left| \triangle \left(\frac{a_n}{n} \right) \right| \le \frac{A_n}{n} \quad \forall \;\; n.$$

Clearly, for $r = 0$, $\mathrm{SJ}_r = \mathrm{SJ}$. It is obvious that $\mathrm{SJ}_{r+1} \subset \mathrm{SJ}_r$, but converse is not true.

Example. For $n = 1, 2, 3...,$ define $b_n = \frac{1}{n^{r+2}}$, $\quad r = 0, 1, 2, 3, ...$ Firstly, we shall show that $\{b_n\}$ does not belong to SJ_{r+1}.

Really, $b_n = \frac{1}{n^{r+2}} \to 0 \quad as \quad n \to \infty.$

Let there exists $A_n = \frac{1}{n^{r+2}}$, $\quad r = 0, 1, 2, 3, ...$ s.t. $\sum_{n=1}^{\infty} n^{r+1} A_n = \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, i.e. $\{b_n\}$ does not belong to SJ_{r+1}.

But, $A_n \downarrow 0, \;\; as \;\; n \to \infty$, and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} n^r \frac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$

Also $\left| \triangle \left(\frac{b_n}{n} \right) \right| \le \frac{A_n}{n}, \quad \forall \;\; n.$

Therefore, $\{b_n\}$ belongs to SJ_r.

In what follows, $t_n(x)$ will represents $f_n(x)$ or $g_n(x)$, while $t_n^r(x)$ will represents r^{th}-derivative of $t_n(x)$.

4.2 Lemmas. We require the following lemmas in the proof of our results.

Lemma 1 [48]. Let $n \ge 1$ and let r be a nonnegative integer, $x \in [\epsilon, \pi]$. Then $|\tilde{D}_n^r(x)| \le C_\epsilon \frac{n^r}{x}$ where C_ϵ is a positive constant depending on ϵ, $0 < \epsilon < \pi$ and $\tilde{D}_n(x)$ is the conjugate Dirichlet kernel.

Lemma 2 [68]. Let $\{a_n\}$ be a sequence of real numbers such that $|a_n| \le 1$ for all n. Then the relation

$$\int_{\pi/n+1}^{\pi} \left| \sum_{k=0}^{n} a_k \tilde{D}_k(x) \right| \; dx \le M(n+1),$$

holds. Where M is absolute constant. Moreover by Bernstein's inequality, for $r = 0, 1, 2, 3.....$

$$\int_{\pi/n+1}^{\pi} \left| \sum_{k=0}^{n} a_k \tilde{D}_k^r(x) \right| \; dx \le M(n+1)^{r+1}.$$

Lemma 3 [48]. $||\tilde{D}_n^r(x)||_{L^1} = O(n^r \log n),\ r = 0, 1, 2, 3....,$ where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet kernel.

4.3 Results. We establish the following theroems:

Theorem 1. Let the coefficients of the series (4.1.3) belongs to class SJ, then the series (4.1.3) is a Fourier series.

Proof. Making use of Abel's transformation on $\sum_{k=1}^{n} \left(\frac{a_k}{k}\right)$, we get

$$\begin{aligned} \sum_{k=1}^{n} \left(\frac{a_k}{k}\right) &= \sum_{k=1}^{n-1} k \triangle \left(\frac{a_k}{k}\right) + a_n \\ &\le \sum_{k=1}^{n-1} k \left(\frac{A_k}{k}\right) + a_n \end{aligned}$$

But (4.1.3) belongs to class SJ, therefore, the series $\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right)$ converges.
Also sequence $\{a_n\}$ is positive and null, therefore by theorem C the result holds. □

Theorem 2. Let the coefficients of the series (4.3.1) belongs to class SJ, then

(4.3.1) $\quad \lim_{n\to\infty} t_n(x) = t(x), \quad exists\ for\ x \in (0, \pi],$

(4.3.2) $\quad t(x) \in L^1(0, \pi],$

(4.3.3) $\quad ||t(x) - S_n(x)|| = o(1), \quad n \to \infty.$

Proof. We will consider only cosine sums, as the proof for the sine sums follows the same line.

To prove (4.3.1), we notice that

$$\begin{aligned} t_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx) \\ t_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}[a_k \ \cos kx - a_{k+1} \ \cos(k+1)x + a_{k+1} \ \cos(k+1)x \\ & \quad -a_{k+2}\cos(k+1)x + ... + a_n \cos nx - a_{n+1}\cos(n+1)x] \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \sum_{k=1}^{n} a_{n+1}\cos(n+1)x \end{aligned}$$

(4.3.4) $$t_n(x) = S_n(x) - na_{n+1}\cos(n+1)x$$

Since $A_k \downarrow 0, \quad as \quad k \to \infty$ and $\sum_{k=1}^{\infty} A_k < \infty$, therefore, by Oliver's theorem we have, $kA_k \to 0, \ as \ k \to \infty$ and so

(4.3.5) $$na_n = n^2 \sum_{k=n}^{\infty} \triangle\left(\frac{a_k}{k}\right) \leq \sum_{k=n}^{\infty} k^2 \left(\frac{A_k}{k}\right) = o(1)$$

Also $\cos(n+1)x$ is finite in $(0,\pi]$. Hence

$$\lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) = t(x).$$

Moreover,

$$\begin{aligned} t(x) &= \lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\right) \\ &= \frac{a_0}{2} + \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \cos kx\right), \end{aligned}$$

use of Abel's transformation yields

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \cos kx\right) = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \triangle\left(\frac{a_k}{k}\right)\tilde{D}'_k(x) + \frac{a_n}{n}\tilde{D}'_n(x)\right],$$

where $\tilde{D}'_n(x)$ is the derivative of conjugate Dirichlet kernel.

$$\begin{aligned} \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \cos kx\right) &= \sum_{k=1}^{\infty} \triangle\left(\frac{a_k}{k}\right)\tilde{D}'_k(x) \\ &\leq \sum_{k=1}^{\infty}\left(\frac{A_k}{k}\right)\tilde{D}'_k(x) \end{aligned}$$

By the given hypothesis and lemma 1, the series $\sum_{k=1}^{\infty}\left(\frac{A_k}{k}\right)\tilde{D}'_k(x)$ converges. Therefore, the limit $t(x)$ exists for $x \in (0,\pi]$ and thus (4.3.1) follows.

For $x \neq 0$, it follows from (4.3.4) that

$$\begin{aligned} t(x)-t_n(x) &= \sum_{k=n+1}^{\infty} a_k \cos kx + na_{n+1}\cos(n+1)x \\ &= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m}\left(\frac{a_k}{k}\right) k\cos kx\right] + na_{n+1}\cos(n+1)x \end{aligned}$$

Now application of Abel's transformation, lemma 2 and 3 yield

$$\begin{aligned} \int_0^{\pi}|t(x)-t_n(x)|dx &= \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\triangle\left(\frac{a_k}{k}\right)\tilde{D}'_k(x) - \frac{a_{n+1}}{n+1}\tilde{D}'_n(x) + na_{n+1}\cos(n+1)x\right| dx \\ &\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\left(\frac{A_k}{k}\right)\frac{\triangle\left(\frac{a_k}{k}\right)}{\left(\frac{A_k}{k}\right)}\tilde{D}'_k(x)\right| dx + \int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx \\ &\quad + na_{n+1}\int_0^{\pi}|\cos(n+1)x|dx \\ &= \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\triangle\left(\frac{A_k}{k}\right)\sum_{j=1}^{k}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}'_j(x)\right| dx + \int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| dx \\ &\quad + na_{n+1}\int_0^{\pi}|\cos(n+1)x|dx \\ &\leq \sum_{k=n+1}^{\infty}\triangle\left(\frac{A_k}{k}\right)\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}'_j(x)\right| dx + \frac{a_{n+1}}{n+1}\int_0^{\pi}\left|\tilde{D}'_n(x)\right| dx \\ &\quad + na_{n+1}\int_0^{\pi}|\cos(n+1)x|dx \end{aligned}$$

$$\begin{aligned} \int_0^{\pi}|t(x)-t_n(x)|dx &\leq M\sum_{k=n+1}^{\infty}(k+1)^2\triangle\left(\frac{A_k}{k}\right) + \frac{a_{n+1}}{n+1}\int_0^{\pi}\left|\tilde{D}'_n(x)\right| dx \\ &\quad + na_{n+1}\int_0^{\pi}|\cos(n+1)x|dx \\ &= O\left(\sum_{k=n+1}^{\infty}(k+1)^2\triangle\left(\frac{A_k}{k}\right)\right) + o(\log na_{n+1}) + o(na_{n+1}) \end{aligned}$$

But

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\triangle\left(\frac{A_k}{k}\right) + \frac{n(n+1)}{2}\frac{A_n}{n}$$

since $\{a_k\} \in$SJ, we have

$$k(k+1)\frac{A_k}{k} = (k+1)A_k = o(1) \;\; as \;\; k \to \infty,$$

and therefore the series $\sum_{k=1}^{\infty}(k+1)^2\triangle\left(\frac{A_k}{k}\right)$ converges.

Moreover, for all $n \geq 1$, $a_n \log n \leq na_n = o(1)$, $n \to \infty$ by (4.3.5) Hence, it follows that $||t(x) - t_n(x)|| = o(1)$ as $n \to \infty$, and since $t_n(x)$ is a polynomial, therefore $t(x) \in L^1(0, \pi]$. This proves (4.3.2).

We now turn to the proof of (4.3.3), We have

$$\begin{aligned} ||t - S_n|| &= ||t - t_n + t_n - S_n|| \\ &\leq ||t - t_n|| + ||t_n - S_n|| \\ &= ||t - t_n|| + ||na_{n+1}\cos(n+1)x|| \\ &\leq ||t - t_n|| + n|a_{n+1}|\int_0^{\pi} |\cos(n+1)x| \; dx \end{aligned}$$

The assertion (4.3.2) and equation (4.3.5) proves the assertion (4.3.3). This completes the proof of the theorem 2. □

Remark. Of course, the above theorem has been established for a weaker class than the class S, but the results have been obtained for L^1-convergence without using any condition like $a_n \log n \to 0$, as $n \to \infty$.

Theorem 3. Let the coefficients of the series (4.1.3) belongs to class SJ$_r$, then

(4.3.7) $$\lim_{n\to\infty} t_n^r(x) = t^r(x), \;\; exists \; for \; x \in (0, \pi]$$

(4.3.8) $$t^r(x) \in L^1(0, \pi], \qquad (r = 0, 1, 2, ...)$$

(4.3.9) $$||t^r(x) - S_n^r(x)|| = o(1), \;\; n \to \infty.$$

Proof. We will consider only cosine sums as the proof for the sine sums follows the same line. As in the proof of the theorem 2, we have

$$\begin{aligned} t_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx) \\ &= S_n(x) - n a_{n+1}\cos(n+1)x. \end{aligned}$$

We have, then

$$t_n^r(x) = S_n^r(x) - n(n+1)^r a_{n+1} \cos\left((n+1)x + \frac{r\pi}{2}\right)$$

Since $A_k \downarrow 0$, $\ as \ \ k \to \infty$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$, therefore, we have, $k^{r+1}A_k \to 0$, $\ as \ k \to \infty$ and so

$$\text{(4.3.10)} \qquad n^{r+1}a_n = n^{r+2}\sum_{k=n}^{\infty} \triangle\left(\frac{a_k}{k}\right) \leq \sum_{k=n}^{\infty} k^{r+2}\left(\frac{A_k}{k}\right) = o(1),\ n \to \infty.$$

Also $\cos\left((n+1)x + \frac{r\pi}{2}\right)$ is finite in $(0,\pi]$. Hence,

$$\begin{aligned} t^r(x) &= \lim_{n\to\infty} t_n^r(x) \\ &= \lim_{n\to\infty} S_n^r(x) \\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right). \end{aligned}$$

Use of Abel's transformation yields

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right) = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \triangle\left(\frac{a_k}{k}\right)\tilde{D}_k^{r+1}(x) + \frac{a_n}{n}\tilde{D}_n^{r+1}(x)\right],$$

where $\tilde{D}_n^{r+1}(x)$ represents the $(r+1)^{th}$ derivative of conjugate Dirichlet kernel.

$$\begin{aligned} \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right) &= \sum_{k=1}^{\infty} \triangle\left(\frac{a_k}{k}\right)\tilde{D}_k^{r+1}(x) + \lim_{n\to\infty}\left[\frac{a_n}{n}\tilde{D}_n^{r+1}(x)\right] \\ &\leq \sum_{k=1}^{\infty}\left(\frac{A_k}{k}\right)\tilde{D}_k^{r+1}(x) + \lim_{n\to\infty}\left[\frac{a_n}{n}\tilde{D}_n^{r+1}(x)\right] \end{aligned}$$

By making use of the given hypothesis, lemma 1 and (4.3.10), the series $\sum_{k=1}^{\infty}\left(\frac{A_k}{k}\right)\tilde{D}_k^{r+1}(x)$ converges. Therefore, the limit $t^r(x)$ exists for $x \in (0,\pi]$ and thus (4.3.7) follows.

To prove (4.3.8), we consider

$$t^r(x) - t_n^r(x) = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + n(n+1)^r a_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right)$$

Making use of Abel's transformation, we obtain

$$\begin{aligned}
t^r(x) - t_n^r(x) &= \sum_{k=n+1}^{\infty} \triangle\left(\frac{a_k}{k}\right)\tilde{D}_k^{r+1}(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \\
&\quad + n(n+1)^r a_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right) \\
&= \sum_{k=n+1}^{\infty} \left(\frac{A_k}{k}\right)\frac{\triangle\left(\frac{a_k}{k}\right)}{\left(\frac{A_k}{k}\right)}\tilde{D}_k^{r+1}(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) \\
&\quad + n(n+1)^r a_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right) \\
&= \sum_{k=n+1}^{\infty} \triangle\left(\frac{A_k}{k}\right)\sum_{j=1}^{k}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x) - \left(\frac{A_{n+1}}{n+1}\right)\sum_{j=1}^{n}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x) \\
&\quad - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) + n a_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right)
\end{aligned}$$

Thus from lemma 2 and 3, we obtain

$$\begin{aligned}
||t^r(x) - t_n^r(x)|| &\leq \sum_{k=n+1}^{\infty} \triangle\left(\frac{A_k}{k}\right)\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x)\right| dx \\
&\quad + \left(\frac{A_{n+1}}{n+1}\right)\int_0^{\pi}\left|\sum_{j=1}^{n}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x)\right| dx + \int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)\right| dx \\
&\quad + n(n+1)^r|a_{n+1}|\int_0^{\pi}|\cos\left((n+1)x + \frac{r\pi}{2}\right)|\, dx \\
||t^r(x) - t_n^r(x)|| &= O\left(\sum_{k=n+1}^{\infty} k^{r+2}\triangle\left(\frac{A_k}{k}\right)\right) + O\left(n^{r+2}\left(\frac{A_{n+1}}{n+1}\right)\right) + O(n^r a_{n+1}\log n) \\
&\quad + n(n+1)^r|a_{n+1}|\int_0^{\pi}|\cos\left((n+1)x + \frac{r\pi}{2}\right)|\, dx
\end{aligned}$$

Using the same argument as in the proof of theorem 2, it can easily be shown that the series $\sum_{k=n+1}^{\infty} k^{r+2}\triangle\left(\frac{A_k}{k}\right)$ converges. Moreover,

$$\int_0^{\pi}|\cos\left((n+1)x + \frac{r\pi}{2}\right)|\, dx \leq \frac{2}{n+1}$$

and for $n \geq 1$, $n^r a_n \log n \leq n^{r+1}a_n = o(1)$ *by* (4.3.10). Hence it follows that

$$||t^r(x) - t_n^r(x)|| = o(1) \quad as \quad n \to \infty. \tag{4.3.11}$$

and since $t_n^r(x)$ is a polynomial, therefore $t^r(x) \in L^1(0,\pi]$. This proves (4.3.8).

We now turn to the proof of (4.3.9). We have

$$\begin{aligned}
||t^r - S_n^r|| &= ||t^r - t_n^r + t_n^r - S_n^r|| \\
&\leq ||t^r - t_n^r|| + ||t_n^r - S_n^r|| \\
&= ||t^r - {t_n}^r|| + ||n(n+1)^r a_{n+1} \cos\left((n+1)x + \frac{r\pi}{2}\right)|| \\
&\leq ||t^r - t_n^r|| + n(n+1)^r |a_{n+1}| \int_0^\pi |\cos\left((n+1)x + \frac{r\pi}{2}\right)|dx
\end{aligned}$$

Further,
$||t^r - t_n^r|| = o(1),\ n \to \infty$ (by (4.3.11)), $\int_0^\pi |\cos\left((n+1)x + \frac{r\pi}{2}\right)|dx \leq \frac{2}{n+1}$ and $\{a_n\}$ is a null sequence, the conclusion of theorem 3 follows. □

Remark. The case $r = 0$, in theorem 3 yields the theorem 2.

CHAPTER V

$\mathbf{L^1}$-CONVERGENCE OF MODIFIED COMPLEX TRIGONOMETRIC SUMS

5.1 Introduction. Let $\{c_n : n = 0, \pm 1, ...\}$ be a sequence of complex numbers and it is said to be odd sequence if $c_n = -c_{-n}$. Let the partial sums of complex trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ be denoted by $S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt}, \ t \in T = \mathbb{R}/2\pi z$. If the trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$ for all n and $S_n(C,t) = S_n(f,t) = S_n(f)$.

Concerning L^1-convergence of complex Fourier series with asymptotically even coefficients, Stanojević [53] proved the following theorem:

Theorem A. Let $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be the Fourier series of $f \in L^1(T)$ with asymptotically even coefficients, that is,

$$\frac{1}{n}\sum_{j=1}^{n} |\hat{f}(j) - \hat{f}(-j)| \log j = o(1), \ n \to \infty,$$

$$\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda_n]} |\triangle(\hat{f}(j) - \hat{f}(-j))| \log j = 0.$$

If for some $1 < p \leq 2$

$$\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda_n]} j^{p-1} |\triangle \hat{f}(j)|^p = 0,$$

then $||S_n(f) - f|| = o(1), n \to \infty$, if and only if $||\hat{f}(n)E_n + \hat{f}(-n)E_{-n}|| = o(1), \ n \to \infty$, where $E_n(t) = \sum_{k=0}^{n} e^{ikt}$.

Later on, in 1987, Stanojević and Stanojević [54] considered the integrability and L^1-convergence of complex trigonometric series with coefficients satisfying a specific condition. To control the

The work reported in this chapter have been communicated in **Mathematical Communications** for publication.

sine part of the complex trigonometric series, they needed a technical condition along with a class S_p^* of sequences respectively defined as:

A complex null sequence $\{c_n\}$ satisfying

$$\sum_{n=1}^{\infty} |\triangle(c_n - c_{-n})| \log n < \infty,$$

is called weakly even. Obviously if $\{c_n\}$ is an even sequence ($c_n = c_{-n}$), then it is weakly even.

Definition. A weakly even null sequence $\{c_n\}$ of complex numbers belongs to the class S_p^* if, for some $1 < p \leq 2$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$, the condition

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|\triangle c_k|^p}{A_k^p} = O(1), \quad n \to \infty \tag{5.1.1}$$

holds.

Teljakovskiĭ [68] proved the following result:

Theorem B. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (1.3.5) is $a_k \log k = o(1), \quad k \to \infty$.

The main result of Stanojević and Stanojević [54] is the following theorem that generalizes the above result of Teljakovskiĭ [68].

Theorem C. Let $\{c_n\} \in S_p^*$. Then

$$for \quad t \neq 0, \quad \lim_{n\to\infty} S_n(C,t) = f(t) \quad exists, \tag{5.1.2}$$

$$f \in L^1(T); \tag{5.1.3}$$

$$||S_n(f) - f|| = o(1), \ n \to \infty, \ \textit{is equivalent to} \ \hat{f}(n)\log|n| = o(1), \ |n| \to \infty. \tag{5.1.4}$$

In the proof of Theorem C, the authors used the complex form

$$g_n(C,t) = S_n(C,t) - (c_nE_n(t) + c_{-n}E_{-n}(t))$$

of Rees-Stanojević modified trigonometric sums [45]

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$$

Further, in 1995, Bhatia and Ram [4] introduced the complex form

$$(5.1.5) \qquad g_n(C,t) = S_n(C,t) + \frac{i}{n+1}[c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)]$$

of modified trigonometric sums

$$\frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\cos kx$$

and

$$\sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\sin kx$$

introduced by Ram and Kumari ([31], [42]) and studied their L^1-convergence under the class R* of coefficient sequences and obtained the necessary and sufficient condition for the L^1-convergence of complex trigonometric series.

As pointed out in Chapter IV, we have introduced new modified cosine and sine sums as

$$(5.1.6) \qquad \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx)$$

and

$$(5.1.7) \qquad \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \sin jx)$$

and have studied their L^1-convergence. The aim of this chapter is to study the L^1-convergence of complex form of the above modified trigonometric sums with coefficients belonging to a new class J^* of coefficient sequences.

Let

$$
\begin{aligned}
D_n(t) &= \frac{1}{2} + \sum_{m=1}^{n} \cos mt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}, \\
\tilde{D}_n(t) &= \sum_{m=1}^{n} \sin mt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},
\end{aligned}
$$

denote the Dirichlet and the conjugate Dirichlet kernel respectively. Let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$. Then the first differentials $D'_n(t)$ and $\tilde{D}'_n(t)$ of $D_n(t)$ and $\tilde{D}_n(t)$ can be written as

$$
\begin{aligned}
D'_n(t) &= E'_n(t) + E'_{-n}(t), \\
2i\tilde{D}'_n(t) &= E'_n(t) - E'_{-n}(t),
\end{aligned}
$$

where $E'_n(t)$ denotes the first differential of $E_n(t)$. The complex form of modified trigonometric sums (5.1.6) and (5.1.7) is

$$
g_n(C,t) = S_n(C,t) - n(c_{n+1}e^{i(n+1)t} + c_{-(n+1)}e^{-i(n+1)t})
$$

We introduce here a new class J^* of coefficient sequences as:

Definition. A null sequence $\{c_n\}$ of complex numbers belongs to class J^* if there exists a sequence $\{A_n\}$ such that

(5.1.8) $\quad A_n \downarrow 0, \quad as \quad n \to \infty,$

(5.1.9) $\quad \sum_{n=1}^{\infty} nA_n < \infty,$

(5.1.10) $\quad \left| \triangle \left(\frac{c_n - c_{-n}}{n} \right) \right| \le \frac{A_n}{n}, \quad \forall \;\; n = \{1, 2, 3, ...\}.$

Now, we define a new class J^*_r of coefficient sequences which is an extension of class J^* .

Definition. A null sequence $\{c_n\}$ of complex numbers belongs to class J^*_r, $\;r = 0, 1, 2, 3, ...,$ if there exists a sequence $\{A_n\}$ such that

(5.1.11) $\quad A_n \downarrow 0, \quad as \quad n \to \infty,$

(5.1.12) $$\sum_{n=1}^{\infty} n^{r+1} A_n < \infty,$$

(5.1.13) $$\left| \triangle \left(\frac{c_n - c_{-n}}{n} \right) \right| \leq \frac{A_n}{n}, \quad \forall \;\; n = \{1, 2, 3, ...\}.$$

Clearly, for $r = 0$, the class J_r^* reduces to the class J^*. It is obvious that $J_{r+1}^* \subset J_r^*$, but converse is not true.

Example. For $n = 1, 2, 3...$, define $c_n = \dfrac{1+i}{n^{r+5}}, \quad r = 0, 1, 2, 3, ...$ Firstly, we shall show that $\{c_n\}$ does not belong to J_{r+1}^*.

Really, $c_n = \dfrac{1+i}{n^{r+2}} \to 0 \quad as \quad n \to \infty.$

Let there exists $A_n = \dfrac{8}{n^{r+3}}, \quad r = 0, 1, 2, 3, ...$ s.t. $\displaystyle\sum_{n=1}^{\infty} n^{r+2} A_n = \sum_{n=1}^{\infty} n^{r+2} \frac{8}{n^{r+3}} = \sum_{n=1}^{\infty} \frac{8}{n}$ is divergent, i.e. $\{c_n\}$ does not belong to J_{r+1}^*.

But, $A_n \downarrow 0, \quad as \;\; n \to \infty$, and $\displaystyle\sum_{n=1}^{\infty} n^{r+1} A_n = \sum_{n=1}^{\infty} n^{r+1} \frac{8}{n^{r+3}} = \sum_{n=1}^{\infty} \frac{8}{n^2} < \infty,$

Also $\left| \triangle \left(\dfrac{c_n - c_{-n}}{n} \right) \right| \leq \dfrac{A_n}{n}, \quad \forall \;\; n.$

Therefore, $\{c_n\}$ belongs to J_r^*.

5.2 Lemma. The proof of our result is based upon the following lemma.

Lemma 1. [5]. Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $M_{r\epsilon} > 0$ such that for all $\epsilon \leq |t| \leq \pi$ and all $n \geq 1$,

(i) $|E_n^r(t)| \leq M_{r\epsilon} n^r / |t|$,

(ii) $|E_{-n}^r(t)| \leq M_{r\epsilon} n^r / |t|$,

(iii) $|D_n^r(t)| \leq 2M_{r\epsilon} n^r / |t|$,

(iv) $|\tilde{D}_n^r(t)| \leq M_{r\epsilon} n^r / |t|$.

5.3 Result. The main results of this chapter reads as:

Theorem 1. Let an odd sequence $\{c_n\}$ belongs to class J^*, then

(5.3.1) $$for \;\; t \neq 0, \quad \lim_{n\to\infty} g_n(C, t) = f(t) \quad exists,$$

(5.3.2) $$f(t) \in L^1(0,\pi] \ \ and \ \ ||f-g_n||_{L^1} = o(1), \ n \to \infty,$$

(5.3.3) $$||S_n(C,t) - f(t)||_{L^1} = o(1), \quad n \to \infty.$$

Proof. Consider

$$g_n(C,t) = S_n(C,t) - n(c_{n+1}e^{i(n+1)t} + c_{-(n+1)}e^{-i(n+1)t})$$

Since, $\{c_n\}$ is odd, therefore, we get

$$g_n(C,t) = S_n(C,t) - 2inc_{n+1}\sin(n+1)t$$

Further, we know that $\sin nt$ is bounded in $(0,\pi]$ and by the given hypothesis, we have

(5.3.4) $$2nc_{n+1} = n(c_{n+1} - c_{-(n+1)}) \ = \ n(n+1)\sum_{k=n+1}^{\infty} \triangle\left(\frac{c_k - c_{-k}}{k}\right) \leq \sum_{k=n+1}^{\infty} k^2\frac{A_k}{k} = o(1).$$

Therefore, $\lim_{n\to\infty} g_n(C,t) = \lim_{n\to\infty} S_n(C,t) = f(t)$.

Next, we show that $f(t)$ exists in $(0,\pi]$. Making use of Abel's transformation, lemma 1 and (5.1.10), we have

$$\begin{aligned}
S_n(C,t) &= c_0 + \sum_{k=1}^{n}[c_k e^{ikt} + c_{-k}e^{-ikt}] \\
&= c_0 + \sum_{k=1}^{n-1}\triangle\left(\frac{c_k}{k}\right)E'_k(t) - \frac{c_n E'_n}{n} + \sum_{k=1}^{n-1}\triangle\left(\frac{c_{-k}}{k}\right)(-E'_{-k}(t)) - \frac{c_{-n}(-E'_{-n})}{n} \\
&\leq c_0 + \sum_{k=1}^{n-1}k\triangle\left(\frac{c_k - c_{-k}}{k}\right)\frac{M_\epsilon}{t} - \frac{M_\epsilon n}{t}\left(\frac{c_n - c_{-n}}{n}\right) \\
&\leq c_0 + \sum_{k=1}^{n-1}k\frac{A_k}{k}\frac{M_\epsilon}{t} + \frac{M_\epsilon}{t}(c_n - c_{-n}) \\
&= O\left(\sum_{k=1}^{n-1}A_k\right) + o\left(c_n - c_{-n}\right)
\end{aligned}$$

Thus, by the given hypothesis, $\lim_{n\to\infty} S_n(C,t) = f(t)$ exists in $(0,\pi]$ and therefore (5.3.1) follows.

By using Abel's transformation, lemma 1 and (5.1.10), we have

$$\int_0^\pi |f(t) - g_n(C,t)|dt \ = \ \int_0^\pi \left|\sum_{|k|>n}^{\infty} c_k e^{ikt} + n(c_{n+1}e^{i(n+1)t} + c_{-(n+1)}e^{-i(n+1)t})\right| dt$$

$$
\begin{aligned}
\int_0^\pi |f(t) - g_n(C,t)|dt &\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \left\{ \triangle\left(\frac{c_k}{k}\right) E'_k(t) + \triangle\left(\frac{c_{-k}}{k}\right)(-E'_{-k}(t)) \right\} \right| dt \\
&\quad + \int_0^\pi \left| \frac{c_n}{n} E'_n(t) + \frac{c_{-n}}{n}(-E'_{-n}(t)) \right| dt \\
&\quad + 2nc_{n+1} \int_0^\pi |\sin(n+1)t| dt \\
&\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \triangle\left(\frac{c_k - c_{-k}}{k}\right) \frac{kM_\epsilon}{t} \right| dt \\
&\quad + \int_0^\pi \left| \left(\frac{c_{n+1} - c_{-(n+1)}}{n}\right) \frac{nM_\epsilon}{t} \right| dt \\
&\quad + 2nc_{n+1} \int_0^\pi |\sin(n+1)t| dt \\
&\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \frac{A_k}{k} \frac{kM_\epsilon}{t} \right| dt + \int_0^\pi \left| (c_{n+1} - c_{-(n+1)}) \frac{M_\epsilon}{t} \right| dt \\
&\quad + 2nc_{n+1} \int_0^\pi |\sin(n+1)t| dt \\
&= O(\sum_{k=n+1}^{\infty} \log k A_k) + o((c_n - c_{-n}) \log n) + o(2nc_{n+1}) \qquad (5.3.5)
\end{aligned}
$$

Now, (5.1.9) and (5.3.4) imply that all the terms on the right side of (5.3.5) are of $o(1)$ as $n \to \infty$. Since, $g_n(C,t)$ is a polynomial, therefore, it follows that $f \in L^1(0,\pi]$, which proves the assertion (5.3.2).

To prove (5.3.3), we notice further that

$$
\begin{aligned}
\int_0^\pi |f(t) - S_n(C,t)|dt &= \int_0^\pi |f(t) - g_n(C,t) + g_n(C,t) - S_n(C,t)|dt \\
&\leq \int_0^\pi |f(t) - g_n(C,t)|dt + \int_0^\pi |g_n(C,t) - S_n(C,t)|dt \\
&= \int_0^\pi |f(t) - g_n(C,t)|dt + \int_0^\pi |n(c_{n+1}e^{i(n+1)t} + c_{-(n+1)}e^{-i(n+1)t})|dt \\
&= \int_0^\pi |f(t) - g_n(C,t)|dt + \int_0^\pi |2nc_{n+1}\sin(n+1)t|dt
\end{aligned}
$$

and

$$
\int_0^\pi |2nc_{n+1}\sin(n+1)t|dt = o(2nc_{n+1}) = o(1), \; n \to \infty, \quad by \;\; (5.3.4)
$$

But $||f - g_n||_{L^1} = o(1)$, $n \to \infty$, by (5.3.2). Therefore, the assertion (5.3.3) follows. □

Theorem 2. Let an odd sequence $\{c_n\}$ belongs to class J_r^*, $\;r = 0, 1, 2, ...$, then

(5.3.6) $$for \quad t \neq 0, \quad \lim_{n\to\infty} g_n^r(C,t) = f^r(t) \quad exists,$$

(5.3.7) $$f^r(t) \in L^1(0,\pi] \;\; and \;\; ||f^r - g_n^r||_{L^1} = o(1), \; n \to \infty,$$

(5.3.8) $$||S_n^r(C,t) - f^r(t)||_{L^1} = o(1), \quad n \to \infty.$$

Proof. As in the proof of the theorem 1, we have

$$g_n(C,t) = S_n(C,t) - 2inc_{n+1}\sin(n+1)t.$$

We have then

$$g_n^r(C,t) = S_n^r(C,t) - 2in(n+1)^r c_{n+1}\sin((n+1)t + r\pi/2)$$

Since, $\sin nt$ is bounded in $(0,\pi]$ and by the hypothesis of the theorem, we get

(5.3.9) $$\begin{aligned} 2n(n+1)^r c_{n+1} &= n(n+1)^r(c_{n+1} - c_{-(n+1)}) \\ &\leq n(n+1)(n+1)^r \sum_{k=n+1}^{\infty} \triangle\left(\frac{c_k - c_{-k}}{k}\right) \\ &\leq \sum_{k=n+1}^{\infty} k^{r+2}\frac{A_k}{k} = o(1) \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} g_n^r(C,t) = \lim\limits_{n\to\infty} S_n^r(C,t) = f^r(t)$.

Next, we show that $f^r(t)$ exists in $(0,\pi]$. Making use of Abel's transformation, lemma 1 and (5.1.13), we have

$$\begin{aligned} S_n^r(C,t) &= \frac{1}{i}\left[\sum_{k=1}^{n-1}\triangle\left(\frac{c_k}{k}\right)E_k^{r+1}(t) - \frac{c_n E_n^{r+1}}{n} + \sum_{k=1}^{n-1}\triangle\left(\frac{c_{-k}}{k}\right)(-E_{-k}^{r+1}(t)) - \frac{c_{-n}(-E_{-n}^{r+1})}{n}\right] \\ &\leq \frac{1}{i}\left[\sum_{k=1}^{n-1}\triangle\left(\frac{c_k - c_{-k}}{k}\right)\frac{k^{r+1}M_\epsilon}{t} - \frac{n^{r+1}M_\epsilon}{t}\left(\frac{c_n - c_{-n}}{n}\right)\right] \\ &\leq \sum_{k=1}^{n-1}\frac{A_k}{k}\frac{k^{r+1}M_\epsilon}{t} - \frac{n^{r+1}M_\epsilon}{t}\left(\frac{c_n - c_{-n}}{n}\right) \\ &= O\left(\sum_{k=1}^{n-1}k^r A_k\right) + o(n^r(c_n - c_{-n})) = o(1), \; n \to \infty, \quad \text{by (5.1.12) and (5.3.9)} \end{aligned}$$

Therefore, the limit $f^r(t)$ exists in $(0,\pi]$ and this proves the assertion (5.3.5).

For $t \neq 0$, we note that

$$f^r(t) - g_n^r(C,t) = \sum_{k=n+1}^{\infty}[(ik)^r c_k e^{ikt} + (-ik)^r c_{-k}e^{-ikt}] + 2in(n+1)^r c_{n+1}\sin\left((n+1)t + \frac{r\pi}{2}\right).$$

Use of Abel's transformation and lemma 1, yield

$$\begin{aligned} f^r(t) - g_n^r(C,t) &= \frac{1}{i}\left[\sum_{k=n+1}^{\infty}\left\{\triangle\left(\frac{c_k}{k}\right)E_k^{r+1}(t) + \triangle\left(\frac{c_{-k}}{k}\right)(-E_{-k}^{r+1}(t))\right\}\right] \\ &\quad + \frac{1}{i}\left[\frac{c_n}{n}E_n^{r+1}(t) + \frac{c_{-n}}{n}E_{-n}^{r+1}(t)\right] + 2in(n+1)^r c_{n+1}\sin\left((n+1)t + \frac{r\pi}{2}\right) \\ &\leq \frac{1}{i}\left[\sum_{k=n+1}^{\infty}\triangle\left(\frac{c_k - c_{-k}}{k}\right)\frac{k^{r+1}M_\epsilon}{t} + \left(\frac{c_n - c_{-n}}{n}\right)\frac{n^{r+1}M_\epsilon}{t}\right] \\ &\quad + 2in(n+1)^r c_{n+1}\sin\left((n+1)t + \frac{r\pi}{2}\right) \end{aligned}$$

Further, from (5.1.13), we notice that

$$\begin{aligned} f^r(t) - g_n^r(C,t) &\leq \frac{1}{i}\left[\sum_{k=n+1}^{\infty} k^{r+1}\frac{A_k}{k}\frac{M_\epsilon}{t} + \left(\frac{c_n - c_{-n}}{n}\right)\frac{n^{r+1}M_\epsilon}{t}\right] \\ &\quad + 2in(n+1)^r c_{n+1}\sin\left((n+1)t + \frac{r\pi}{2}\right) \end{aligned}$$

Now, consider

$$\begin{aligned} \|f^r - g_n^r\| &\leq \int_0^{\pi}\frac{1}{i}\left[\left|\sum_{k=n+1}^{\infty} k^{r+1}\frac{A_k}{k}\frac{M_\epsilon}{t} + \left(\frac{c_n - c_{-n}}{n}\right)\frac{n^{r+1}M_\epsilon}{t}\right| dt\right] \\ &\quad + \int_0^{\pi}\left|2in(n+1)^r c_{n+1}\sin\left((n+1)t + \frac{r\pi}{2}\right)\right| dt \\ &= O\left(\sum_{k=n+1}^{\infty} k^r A_k\right) + o(n^r \log n(c_n - c_{-n})) + o(2nc_{n+1}(n+1)^r) \\ &= o(1),\ n \to \infty \ \text{ by (5.1.12) and (5.3.9)} \end{aligned}$$

Since, $g_n^r(C,t)$ is a polynomial, therefore, it follows that $f^r \in L^1(0,\pi]$, which proves the assertion (5.3.7).

To prove (5.3.8), we note that

$$\begin{aligned} \|f^r - S_n^r\| &= \|f^r - g_n^r + g_n^r - S_n^r\| \\ &\leq \|f^r - g_n^r\| + \|g_n^r - S_n^r\| \\ &= \|f^r - g_n^r\| + \int_0^{\pi}\left|2in(n+1)^r c_{n+1}\sin\left((n+1)t + \frac{r\pi}{2}\right)\right| dt \end{aligned}$$

Use of (5.3.7) and (5.3.9), proves the assertion (5.3.8). □

Remark. The case $r = 0$, in theorem 2 yields the theorem 1.

CHAPTER VI

ON L^1-CONVERGENCE OF MODIFIED COSINE SUM

6.1 Introduction. Consider the sine series

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{6.1.1}$$

Let the partial sums of (6.1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Concerning the series (6.1.1) following result is known:

Theorem A. ([2], Vol.II, p. 201) For the series (6.1.1) to be Fourier series, it is necessary and sufficient that $\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right) < \infty$.

Kaur, Bhatia and Ram [27] studied the L^1-convergence of cosine trigonometric series by introducing modified sine sums as

$$\frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_{j-1} - \triangle a_{j+1})\sin kx$$

They have studied these results under a new class **K** of coefficient sequences defined as:

Definition. If $a_k = o(1), \quad k \to \infty$ and

$$\sum_{k=1}^{\infty} k|\triangle^2 a_{k-1} - \triangle^2 a_{k+1}| < \infty \quad (a_0 = 0),$$

then we say that $\{a_k\}$ belongs to class **K**.

We, in this chapter, introduce new modified cosine sum as

$$f_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_{j+1} - \triangle a_{j-1})\cos kx, \qquad (a_0 = a_1 = 0)$$

The results obtained in this chapter have been **accepted** for publication in **Southeast Asian Bulletin of Mathematics.**

and study its integrability and L^1-convergence under a new class J of coefficient sequences defined as:

Definition. A null sequence $\{a_n\}$ of positive numbers belongs to class J if there exists a sequence $\{A_n\}$ such that

$$A_n \downarrow 0, \quad as \quad n \to \infty, \tag{6.1.2}$$

$$\sum_{n=1}^{\infty} nA_n < \infty, \tag{6.1.3}$$

$$\left|\triangle\left(\frac{a_n}{n}\right)\right| \leq \frac{A_n}{n} \quad \forall \ n = \{1, 2, 3, ...\}. \tag{6.1.4}$$

In the literature so far available, most of the authors have studied the integrability and L^1-convergence of cosine trigonometric series using different classes of coefficient sequences. However, very few of them have studied about the L^1-convergence of sine trigonometric series. In this chapter we study the L^1-convergence of sine trigonometric series by using a newly introduced modified cosine sum defined as above under a new class J of coefficient sequences.

6.2 Lemmas. The proof of our results are based upon the following lemmas, of which the first two are due to Sheng [48].

Lemma 1. $||D_n^r(x)|| = O(n^r \log n), \quad r = 0, 1, 2, 3, ...$ where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.

Lemma 2. $||\tilde{D}_n^r(x)|| = O(n^r \log n), \quad r = 0, 1, 2, 3, ...$ where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet kernel.

Lemma 3. (i) $\left|\left|\dfrac{D_n(x)}{2\sin x}\right|\right| = o(n \log n), \ n \to \infty$ and

(ii) $\left|\left|\dfrac{\cos nx}{2\sin x}\right|\right| = o(\log n), \ n \to \infty.$

Proof. Since, we know that, $|D_n(x)| < n + \frac{1}{2}$, for all $n = 1, 2, 3, ...$ and for $0 < x \leq \dfrac{\pi}{2}$, $\dfrac{\sin x}{x} \geq \dfrac{2}{\pi}$. Therefore,

$$
\begin{aligned}
\left|\left|\frac{D_n(x)}{2\sin x}\right|\right| &= \int_0^\pi \left|\frac{D_n(x)}{2\sin x}\right| \, dx \\
&< \left(n+\frac{1}{2}\right)\int_0^\pi \left|\frac{1}{2\sin x}\right| \, dx \\
&\le 2\left(n+\frac{1}{2}\right)\int_0^{\frac{\pi}{2}} \frac{1}{2|\sin x|} \, dx \\
&\le \lim_{n\to\infty} 2\left(n+\frac{1}{2}\right)\int_{\pi/n}^{\frac{\pi}{2}} \frac{1}{2|x|} \, dx \\
&= o(n\log n), \; n\to\infty
\end{aligned}
$$

This proves (i) and to prove (ii), consider,

$$
\begin{aligned}
\left|\left|\frac{\cos nx}{2\sin x}\right|\right| &= \int_0^\pi \left|\frac{\cos nx}{2\sin x}\right| \, dx \\
&\le \lim_{n\to\infty} 2\int_{\pi/n}^{\pi/2} \frac{1}{2\,|x|} \, dx \\
&= o(\log n), \; n\to\infty.
\end{aligned}
$$

□

6.3 Results. We prove the following results:

Theorem 1. Let the coefficients of the series (6.1.1) belongs to the class J, then the series (6.1.1) is a Fourier series.

Proof. Making use of Abel's transformation on $\sum_{k=1}^{n}\left(\frac{a_k}{k}\right)$, we get

$$
\begin{aligned}
\sum_{k=1}^{n}\left(\frac{a_k}{k}\right) &= \sum_{k=1}^{n-1} k\triangle\left(\frac{a_k}{k}\right) + a_n \\
&\le \sum_{k=1}^{n-1} k\left(\frac{A_k}{k}\right) + a_n \qquad \textit{by } (6.1.4)
\end{aligned}
$$

Therefore, the series $\sum_{k=1}^{\infty}\left(\frac{a_k}{k}\right)$ is convergent (by (6.1.3)). Hence, the series (6.1.1) is a Fourier series by theorem C. □

Theorem 2. Let the sequence $\{a_n\}$ belongs to the class J. Then $f_n(x)$ converges to $f(x)$ in L^1-norm.

Proof. We have

$$\begin{aligned} f_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_{j+1}-\triangle a_{j-1})\cos kx \\ &= \frac{1}{2\sin x}\left(\sum_{k=1}^{n}(a_{k+1}-a_{k-1})\cos kx+(a_n-a_{n+2})D_n(x)\right) \\ &= \sum_{k=1}^{n}a_k\sin kx+\frac{1}{2\sin x}\left(a_n\cos(n+1)x+a_{n+1}\cos nx+(a_n-a_{n+2})\,D_n(x)\right) \end{aligned}$$

$$(6.3.1)\qquad f_n(x) = S_n(x)+\frac{1}{2\sin x}\left(a_n\cos(n+1)x+a_{n+1}\cos nx+(a_n-a_{n+2})\,D_n(x)\right)$$

Since $\left|\frac{\cos nx}{2\sin x}\right|$ and $\left|\frac{D_n(x)}{2\sin x}\right|$ are bounded in $(0,\pi]$. Also, $\{a_n\}$ is null sequence, so the last three terms of the right hand side of (6.3.1) tends to zero as $n\to\infty$.

Thus

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}S_n(x).$$

Now, we show that the limit f exists in $(0,\pi]$. Consider,

$$S_n(x)=\sum_{k=1}^{n}a_k\sin kx$$

An application of Abel's transformation yields

$$(6.3.2)\qquad S_n(x)=\sum_{k=1}^{n-1}\triangle a_k\tilde{D}_k(x)+a_n\tilde{D}_n(x).$$

Since, $|\tilde{D}_k(x)|=O(k)$ and

$$(6.3.3)\qquad \left|\triangle\left(\frac{a_k}{k}\right)\right|\le\frac{A_k}{k}\Rightarrow|\Delta a_k|\le A_k,\ \forall\ k,\quad \text{by the given hypothesis}$$

Thus (6.3.2) imply that

$$\lim_{n\to\infty}S_n(x)=O(\sum_{k=1}^{\infty}kA_k)+o(na_n)$$

But,

$$(6.3.4)\qquad \log na_n\le na_n=n\sum_{k=n}^{\infty}\triangle a_k\le\sum_{k=n}^{\infty}kA_k=o(1),\ n\to\infty$$

Hence, $f(x)$ exists.

Now, consider

$$||f(x) - f_n(x)|| = \left\| \sum_{k=n+1}^{\infty} a_k \sin kx - \frac{1}{2\sin x} \begin{pmatrix} a_n \cos(n+1)x + a_{n+1}\cos nx \\ + (a_n - a_{n+2})\, D_n(x) \end{pmatrix} \right\|$$

Using Abel's transformation, (6.3.3), lemma 2 and 3, we get

$$\begin{aligned} ||f(x) - f_n(x)|| &\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \triangle a_k \tilde{D}_k(x) \right| dx + \int_0^{\pi} \left| a_{n+1} \tilde{D}_n(x) \right| dx \\ &\quad + \int_0^{\pi} \left| \frac{1}{2\sin x} \left(a_n \cos(n+1)x + a_{n+1}\cos nx + (a_n - a_{n+2})\, D_n(x)\right) \right| dx \\ &\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \tilde{D}_k(x) \right| dx + \int_0^{\pi} \left| a_{n+1} \tilde{D}_n(x) \right| dx \\ &\quad + \int_0^{\pi} \left| \frac{1}{2\sin x} \left(a_n \cos(n+1)x + a_{n+1}\cos nx\right) \right| dx \\ &\quad + \int_0^{\pi} \left| \frac{1}{2\sin x} (a_n - a_{n+2})\, D_n(x) \right| dx \end{aligned}$$

$$\begin{aligned} ||f(x) - f_n(x)|| &= O\left(\sum_{k=n+1}^{\infty} \log k A_k \right) + o(\log n \; a_{n+1}) + o(\log n \; a_{n+1}) + o(\log n \; a_n) \\ &\quad + o(n^2 (a_n - a_{n+2})) \\ &= o(1) + o(1) + o(1) + o(1) + o(n^2(a_n - a_{n+2})) \\ &\qquad \textit{by } (6.1.3) \textit{ and } (6.3.4) \end{aligned}$$

But,

$$\begin{aligned} a_n - a_{n+2} &= a_n - a_{n+1} + a_{n+2} - a_{n+2} \\ &= \triangle a_n + \triangle a_{n+1} \\ &\leq A_n + A_{n+1}, \end{aligned}$$

and if $A_k \downarrow 0,\ k \to \infty$, and $\sum_{k=1}^{\infty} k A_k < \infty$, then $k^2 A_k \to 0,\ k \to \infty$ (by Oliver's theorem). We get

$$||f(x) - f_n(x)|| = o(1),\ n \to \infty.$$

This completes the proof of the theorem 2. □

Corollary. If $\{a_n\}$ belongs to class J , then $||f - S_n|| = o(1), \quad n \to \infty$.

Proof. Consider

$$\begin{aligned} ||f(x) - S_n(x)|| &= \int_0^\pi |f(x) - f_n(x) + f_n(x) - S_n(x)| dx \\ &\leq \int_0^\pi |f(x) - f_n(x)| dx + \int_0^\pi |f_n(x) - S_n(x)| dx \\ &\leq \int_0^\pi |f(x) - f_n(x)| dx + \int_0^\pi \left| \frac{1}{2\sin x} (a_n \cos(n+1)x + a_{n+1} \cos nx) \right| dx \\ &\quad + \int_0^\pi \left| \frac{1}{2\sin x} (a_n - a_{n+2}) D_n(x)) \right| dx \end{aligned}$$

The conclusion of the corollary follows, by using the same argument as in the proof of the theorem 2. □

CHAPTER VII

CONVERGENCE OF NEW MODIFIED COMPLEX TRIGONOMETRIC SUMS IN THE METRIC SPACE L

7.1 Introduction. Let $\{c_k : k = 0, \pm 1, ...\}$ be a sequence of complex numbers. Let the partial sums of complex trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be denoted by $S_n(C,t) = \sum_{k=-n}^{n} c_k e^{ikt}$, $t \in T = \mathbb{R}/2\pi z$. If the trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$ for all n and $S_n(C,t) = S_n(f,t) = S_n(f)$.

Kaur, Bhatia and Ram [27] studied the L^1-convergence of cosine trigonometric series by introducing a new modified sine sum as

$$(7.1.1) \qquad \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_{j-1} - \triangle a_{j+1}) \sin kx$$

They have studied these results under a new class **K** of coefficient sequences.

In chapter VI, we have introduced new modified cosine sum as

$$(7.1.2) \qquad \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_{j+1} - \triangle a_{j-1}) \cos kx \qquad (a_0 = a_1 = 0)$$

and have studied its integrability and L^1-convergence under a new class J of coefficient sequences.

The aim of this chapter is to study the L^1-convergence of complex form of the modified sums (7.1.1) and (7.1.2) with coefficients belonging to the class J^* of coefficient sequences.

The results obtained in this chapter have been **accepted** for publication in **Lobachevskii Journal of Mathematics** for publications.

Definition. A null sequence $\{c_n\}$ of complex numbers belongs to class J^* if there exists a sequence $\{A_n\}$ such that

(7.1.3) $\quad A_n \downarrow 0, \quad as \quad n \to \infty,$

(7.1.4) $\quad \sum_{n=1}^{\infty} nA_n < \infty,$

(7.1.5) $\quad \left| \triangle \left(\frac{c_n - c_{-n}}{n} \right) \right| \leq \frac{A_n}{n}, \quad \forall \; n = \{1, 2, 3, ...\}.$

The complex form of modified sums (7.1.1) and (7.1.2) is

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t} \left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n} e^{-i(n+1)t} + c_{n+1} e^{int} - c_{-(n+1)} e^{-int} \\ + (c_n - c_{n+2}) E_n(t) + (c_{-(n+2)} - c_{-n}) E_{-n}(t) \end{array} \right]$$

7.2 Lemmas. The proof of our result is based upon the following lemmas.

Lemma 1 [5]. Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $M_{r\epsilon} > 0$ such that for all $\epsilon \leq |t| \leq \pi$ and all $n \geq 1$,

(i) $|E_n^r(t)| \leq M_{r\epsilon} n^r / |t|,$

(ii) $|E_{-n}^r(t)| \leq M_{r\epsilon} n^r / |t|,$

(iii) $|D_n^r(t)| \leq 2M_{r\epsilon} n^r / |t|,$

(iv) $|\tilde{D}_n^r(t)| \leq M_{r\epsilon} n^r / |t|.$

Lemma 2. For $n \geq 1$, we have

(i) $\left\| \frac{E_n(t)}{2\sin t} \right\| = o(n), \; n \to \infty$

(ii) $\left\| \frac{E_{-n}(t)}{2\sin t} \right\| = o(n), \; n \to \infty$

(iii) $\left\| \frac{e^{int}}{2\sin t} \right\| = o(\log n), \; n \to \infty.$

Proof. For $t \neq 0$, we note that $\frac{\sin t}{t} \geq \frac{2}{\pi}$ in $(0, \pi/2)$ and using lemma 1, we get

$$\left|\left|\frac{E_n(t)}{2\sin t}\right|\right| = \int_0^\pi \left|\frac{E_n(t)}{2\sin t}\right| dt \leq \int_0^\pi \frac{M_\epsilon}{2|t\sin t|}dt \leq 2\int_0^{\pi/2} \frac{M_\epsilon}{2|t||\sin t|}dt \leq \int_0^{\pi/2} \frac{M_\epsilon}{t^2}dt$$
$$= \lim_{n\to\infty} \left[\frac{-M_\epsilon}{t}\right]_{\pi/n}^{\pi/2} = o(n),\ n \to \infty.$$

Similarly, $\left|\left|\frac{E_{-n}(t)}{2\sin t}\right|\right| = o(n)$, and to prove (iii), we consider,

$$\left|\left|\frac{e^{int}}{2\sin t}\right|\right| = \int_0^\pi \left|\frac{e^{int}}{2\sin t}\right| dt = 2\int_0^{\pi/2} \left|\frac{1}{2\sin t}\right| dt = \int_0^{\pi/2} \left|\frac{1}{t}\right| dt$$
$$= \lim_{n\to\infty} [\log t]_{\pi/n}^{\pi/2} = o(\log n),\ n \to \infty.$$

□

7.3 Result The main result of this chapter is the following theorem:

Theorem. Let $\{c_k\}$ belongs to the class J^* , then

(7.3.1) $\lim_{n\to\infty} g_n(C,t) = f(t)\ exists\ for\ |t| \in (0,\pi],$

(7.3.2) $f(t) \in L^1(0,\pi]\ \ and\ \ ||f - g_n||_{L^1} = o(1),\ n \to \infty,$

(7.3.3) $||S_n(C,t) - f(t)||_{L^1} = o(1),\ \ n \to \infty.$

Proof. Consider,

$$g_n(C,t) = S_n(C,t) + \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (c_n - c_{n+2})E_n(t) + (c_{-(n+2)} - c_{-n})E_{-n}(t) \end{array}\right]$$

Since, $\frac{e^{int}}{\sin t}, \frac{E_n(t)}{\sin t}$ and $\frac{E_{-n}(t)}{\sin t}$ are bounded in $(0,\pi]$. Also, $\{c_n\}$ is null sequence. Therefore, $\lim_{n\to\infty} g_n(C,t) = \lim_{n\to\infty} S_n(C,t) = f(t)$.

Next, we show that $f(t)$ exists in $(0,\pi]$. Consider,

$$S_n(C,t) = \sum_{|k|\leq n} c_k e^{ikt}$$
$$= c_0 + \sum_{k=1}^{n} \left\{\frac{c_k}{k}ke^{ikt} + \frac{c_{-k}}{k}ke^{-ikt}\right\}$$

Making use of Abel's transformation, lemma 1 and (7.1.5), we get

$$\begin{aligned}
S_n(C,t) &= c_0 + \sum_{k=1}^{n-1} \triangle\left(\frac{c_k}{k}\right)(-iE_k'(t)) + \frac{c_n(-iE_n'(t))}{n} + \sum_{k=1}^{n-1} \triangle\left(\frac{c_{-k}}{k}\right)(iE_{-k}'(t)) \\
&\quad + \frac{c_{-n}(iE_{-n}'(t))}{n} \\
&\leq c_0 - \sum_{k=1}^{n-1} k\triangle\left(\frac{c_k - c_{-k}}{k}\right)\frac{M_\epsilon}{t} - \frac{M_\epsilon n}{t}\left(\frac{c_n - c_{-n}}{n}\right) \\
&\leq \sum_{k=1}^{n-1} k\frac{A_k M_\epsilon}{kt} + \frac{M_\epsilon(c_n - c_{-n})}{t} \\
&= O(\sum_{k=1}^{n-1} A_k) + \frac{M_\epsilon}{t}(c_n - c_{-n})
\end{aligned}$$

Hence, by the given hypothesis, $f(t) = \lim_{n\to\infty} S_n(C,t)$ exists and thus (7.3.1) follows.

Further, for $t \neq 0$, we have

$$f(t) - g_n(C,t) = \sum_{|k|>n} c_k e^{ikt} - \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (c_n - c_{n+2})E_n(t) + (c_{-(n+2)} - c_{-n})E_{-n}(t) \end{array}\right]$$

By making use of Abel's transformation, lemma 1 and 2, we get

$$\begin{aligned}
f(t) - g_n(C,t) &= \sum_{k=n+1}^{\infty}\left\{\triangle\left(\frac{c_k}{k}\right)(-iE_k'(t)) + \triangle\left(\frac{c_{-k}}{k}\right)(iE_{-k}'(t))\right\} \\
&\quad - \frac{c_{n+1}(-iE_n'(t))}{n+1} - \frac{c_{-(n+1)}(iE_{-n}'(t))}{n+1} \\
&\quad - \frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t} - c_{-n}e^{-i(n+1)t} + c_{n+1}e^{int} - c_{-(n+1)}e^{-int} \\ + (c_n - c_{n+2})E_n(t) + (c_{-(n+2)} - c_{-n})E_{-n}(t) \end{array}\right] \\
f(t) - g_n(C,t) &\leq \sum_{k=n+1}^{\infty} -i\triangle\left(\frac{c_k - c_{-k}}{k}\right)\frac{M_\epsilon}{|t|} + \frac{iM_\epsilon n}{|t|}\left(\frac{c_{n+1} - c_{-(n+1)}}{n}\right) \\
&\quad + \frac{(c_n - c_{-n}) + (c_{n+1} - c_{-(n+1)})}{|\sin t|} + M_\epsilon\frac{(c_n - c_{-n}) - (c_{n+2} - c_{-(n+2)})}{|t||\sin t|}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=n+1}^{\infty} k\frac{A_k}{k}\frac{M_\epsilon}{|t|} + \frac{iM_\epsilon}{|t|}\left(c_{n+1} - c_{-(n+1)}\right) \\
&\quad + \frac{(c_n - c_{-n}) + (c_{n+1} - c_{-(n+1)})}{|\sin t|} + M_\epsilon\frac{(c_n - c_{-n}) - (c_{n+2} - c_{-(n+2)})}{|t||\sin t|}
\end{aligned}$$

Now, consider

$$\begin{aligned} ||f(t)-g_n(C,t)||_{L^1} &\le \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} k\frac{A_k}{k}\frac{M_\epsilon}{|t|}\right|dt + \int_0^{\pi}\left|\frac{iM_\epsilon}{|t|}\left(c_{n+1}-c_{-(n+1)}\right)\right|dt \\ &+ \int_0^{\pi}\left|\frac{(c_n-c_{-n})+(c_{n+1}-c_{-(n+1)})}{|\sin t|}\right|dt \\ &+ \int_0^{\pi}\left|M_\epsilon\frac{(c_n-c_{-n})-(c_{n+2}-c_{-(n+2)})}{|t||\sin t|}\right|dt \\ &= O(\sum_{k=n+1}^{\infty} A_k\log k)+o(\log n(c_n-c_{-n})) \\ &+o(\log n(c_n-c_{-n}))+o(n(c_n-c_{-n})) \end{aligned}$$

But, for all $n \ge 1$, we note that

$$\begin{aligned} \log n(c_n-c_{-n}) &\le n^2\frac{(c_n-c_{-n})}{n} \le \sum_{k=n}^{\infty} k^2\triangle\left(\frac{c_k-c_{-k}}{k}\right) \le \sum_{k=n}^{\infty} k^2\left(\frac{A_k}{k}\right) \\ &= o(1), \text{ by the hypothesis of the theorem.} \end{aligned}$$

Therefore, $||f(t)-g_n(C,t)||_{L^1} = o(1),\ n\to\infty$ and since $g_n(C,t)$ is a polynomial, it follows that $f \in L^1(0,\pi]$, which proves the assertion (7.3.2).

We notice further that

$$\begin{aligned} ||f-S_n|| &= ||f-g_n+g_n-S_n|| \\ &\le ||f-g_n||+||g_n-S_n|| \end{aligned}$$

$$||f-S_n|| \le ||f-g_n|| + \left\|\frac{i}{2\sin t}\left[\begin{array}{l} c_n e^{i(n+1)t}-c_{-n}e^{-i(n+1)t}+c_{n+1}e^{int}-c_{-(n+1)}e^{-int} \\ +(c_n-c_{n+2})E_n(t)+(c_{-(n+2)}-c_{-n})E_{-n}(t) \end{array}\right]\right\|$$

The assertion (7.3.3) follows using the same argument as in proof of assertion (7.3.2). □

CHAPTER VIII

INTEGRABILITY AND L^1-CONVERGENCE OF DOUBLE TRIGONOMETRIC SERIES

8.1 Introduction. In chapter IV, we have considered the L^1-convergence of the new modified cosine and sine sums

$$(8.1.1) \qquad f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle(a_j \cos jx)$$

and

$$(8.1.2) \qquad g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle(a_j \sin jx)$$

and have obtained the L^1-convergence of the trigonometric series

$$(8.1.3) \qquad \sum_{k=1}^{\infty} a_k \, \phi_k(x)$$

where $\phi_k(x)$ is $\cos kx$ or $\sin kx$, under the condition that $\{a_k\}$ belongs to class SJ.

In the present chapter, we extend the results of one dimensional series (as discussed in chapter IV) to two dimensional trigonometric series in an essentially more general setting. To reveal the essence, we have formulated a new class J_d of coefficient sequences as:

Definition. A double null sequence $\{a_{jk}\}$ of positive numbers is said to belong to class J_d if there exists a double sequence $\{A_{jk}\}$ such that

$$(8.1.4) \qquad A_{jk} \downarrow 0, \;\; j + k \to \infty,$$

Part of the contents of this chapter have been communicated in Journal of **Analysis in Theory and Applications erstwhile known as J. of Approx. Theory and its Applications** for publication.

$$(8.1.5)\qquad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jkA_{jk} < \infty,$$

$$(8.1.6)\qquad \left|\triangle_{pq}\left(\frac{a_{jk}}{jk}\right)\right| \leq \frac{A_{jk}}{jk}, \quad 1 \leq p+q \leq 2 \quad \forall \;\; p,q \geq 0 \;\; and \;\; j,k \in \{1,2,3,....\}$$

In the sequel, we have studied double cosine, sine and mixed series whose coefficients belong to class J_d. Moreover, we have considered the special cases where the double sequence of coefficient is of bounded variation and quasi convex. We are mainly concerned about with the following problems:

(i) the series converges pointwise,

(ii) the sum of the series is integrable,

(iii) the series is the Fourier series of its sum,

(iv) the series converges in L^1-norm.

8.2 Cosine series We consider the double cosine series

$$(8.2.1)\qquad f(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda_j \lambda_k \;\; a_{jk} \cos jx \cos ky$$

on the positive quadrant $Q = [0,\pi] \times [0,\pi]$ of the two dimensional torus, where $\lambda_0 = \frac{1}{2}$ and $\lambda_j = 1$ for $j = 1,2,3,...$ and $\{a_{jk}\}$ is a double sequence of real numbers.

We denote

$$S_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_j \lambda_k \;\; a_{jk} \cos jx \cos ky \qquad (m,n \geq 0)$$

as the rectangular partial sum of the series (8.2.1) and $f(x,y) = \lim_{m,n\to\infty} S_{mn}$.

Definition [35]. We say that $\{a_{jk}\}$ belongs to the class BV_2 if

$$(8.2.2)\qquad a_{jk} \to 0 \;\; as \;\; j{+}k \to \infty,$$

and

$$(8.2.3)\qquad \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\triangle_{11} a_{jk}| < \infty,$$

where

$$\triangle_{11}a_{j,k} = a_{j,k} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}$$

Condition (8.2.2) implies that $\{a_{jk}\}$ is a null sequence while (8.2.3) implies that $\{a_{jk}\}$ is a sequence of bounded variation.

Definition [35]. A null sequence $\{a_{jk}\}$ belongs to class $\mathcal{C}_2$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $0 \leq m \leq M$ and $0 \leq n \leq N$, we have

$$C(m, M; n, N; \delta) := \int\int_{D_\delta} \left| \sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x) D_k(y) \triangle_{11} a_{jk} \right| dx \ \ dy \leq \epsilon \tag{8.2.4}$$

where

$$D_\delta := Q - (\delta, \pi] \times (\delta, \pi] = \{(x, y) : 0 \leq x, y \leq \pi \ \ \& \ \ \min(x, y) \leq \delta\}$$

or

$$\int\int_{D_\delta} \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} D_j(x) D_k(y) \triangle_{11} a_{jk} \right| dx \ \ dy \leq \epsilon \quad \forall \ m, n \geq 0.$$

Definition [35]. A double sequence $\{a_{jk}\}$ is said to be quasi-convex if

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (j+1)(k+1) |\triangle_{22} a_{jk}| < \infty \tag{8.2.5}$$

Motivated by (8.1.1), we introduce the modified cosine trigonometric sums u_{mn} of the series (8.2.1) as:

$$u_{mn}(x, y) = \frac{a_{00}}{2} + \sum_{j=1}^{m} \sum_{k=1}^{n} \left\{ \sum_{r=j}^{m} \sum_{l=k}^{n} \triangle_{11}(a_{rl} \cos rx \cos ly) \right\} \tag{8.2.6}$$

It will turn out that the u_{mn} approximate f better than S_{mn} since they converge to f in L^1-norm when S_{mn} may not.

Our first main result is the following:

Theorem 1. If a double sequence $\{a_{jk}\}$ belongs to class J_d, then

$$||u_{mn} - f|| \to 0 \ \ as \ \ \min(m, n) \to \infty.$$

Here $||.||$ denotes the two-dimensional $L^1(Q)$-norm.

We draw the following corollaries from Theorem 1.

Corollary 1. Under the conditions of theorem 1, the sum f of the series (8.2.1) is integrable and (8.2.1) is a Fourier series of f.

Corollary 2. If a double sequence $\{a_{jk}\}$ belongs to class J_d, then

$$||S_{mn} - f|| \to 0 \quad as \quad \min(m,n) \to \infty.$$

We note that theorem 1 and corollaries 1 and 2 can be considered as analogous results to the corresponding results of chapter IV from one dimensional to two dimensional cosine series.

Remarks. (a) We notice, from (8.1.6) that

(8.2.7) $$|\triangle_{11}a_{jk}| \leq |\triangle_{10}a_{jk}| + |\triangle_{10}a_{j,k+1}| \leq A_{jk} + A_{j,k+1} \leq 2A_{jk}$$

Moreover, it follows from (8.2.7) and condition (8.1.5) that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\triangle_{11}a_{jk}| < \infty,$$

therefore, $\{a_{jk}\} \in J_d$ implies that $\{a_{jk}\} \in BV_2$.

Next we show that $\{a_{jk}\} \in J_d \Rightarrow \{a_{jk}\} \in \mathcal{C}_2$. For that, consider

$$\begin{aligned}\int_0^\delta \int_0^\delta \left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} \triangle_{11}a_{jk}D_j(x)D_k(y)\right| dx \ \ dy &\leq \int_0^\pi \int_0^\pi \left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} \triangle_{11}a_{jk}D_j(x)D_k(y)\right| dx \ \ dy \\ &\leq 2\int_0^\pi \int_0^\pi \left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} A_{jk}D_j(x)D_k(y)\right| dx \ \ dy\end{aligned}$$

where $D_n(x)$ represent the Dirichlet kernel. Since, $\int_0^\pi |D_n(x)|dx \approx \log n$, thus from (8.1.5), we get $\{a_{jk}\} \in \mathcal{C}_2$. Hence, the Theorem 1.1 and Corollaries 1.1 and 1.2 of Móricz [35] holds true in case $\{a_{jk}\}$ belongs to class J_d.

(b) Further, by setting $A_{jk} = |\triangle_{22}a_{jk}|$ in (8.1.5), we notice that class J_d contains all quasi-convex null sequences. Therefore, corollary 3 of Móricz [37] holds true in case $\{a_{jk}\}$ belongs to class J_d.

8.3 Sine series We consider double sine series

$$(8.3.1) \qquad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky$$

on the positive quadrant $Q = [0,\pi] \times [0,\pi]$ of the two dimensional torus and $\{a_{jk}\}$ is a double sequence of real numbers.

We denote by

$$S_{mn}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk} \sin jx \sin ky \quad (m,n \geq 1)$$

the rectangular partial sum of the series (8.3.1) and $g(x,y) = \lim_{m,n\to\infty} S_{mn}$.

Definition [35]. We say that $\{a_{jk}\} \in \widetilde{BV}_2$ if condition (8.2.2) is satisfied and

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk|\triangle_{11} b_{jk}| < \infty,$$

where

$$b_{jk} := \frac{a_{jk}}{jk} \ (j,k = 1,2....).$$

In the case of double sine series, $\widetilde{BV}_2$ is a more appropriate class of coefficient sequences than BV_2.

Definition [35]. We say that $\{a_{jk}\} \in \widetilde{\mathcal{C}}_2$ if condition (8.2.2) is satisfied and for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $1 \leq m \leq M$ and $1 \leq n \leq N$, we have

$$C(m,M;n,N;\delta) := \int\int_{D_\delta} \left| \sum_{j=m}^{M}\sum_{k=n}^{N} [D_j(x)D_k(y)]_{xy} \triangle_{11} b_{jk} \right| dx \ \ dy \leq \epsilon$$

where

$$D_\delta := Q - (\delta,\pi] \times (\delta,\pi] = \{(x,y) : 0 \leq x,y \leq \pi \ \& \ \min(x,y) \leq \delta\}$$

or

$$\int\int_{D_\delta} \left| \sum_{j=m}^{\infty}\sum_{k=n}^{\infty} [D_j(x)D_k(y)]_{xy} \triangle_{11} b_{jk} \right| dx \ \ dy \leq \epsilon, \quad \forall \ m,n \geq 1.$$

where $[\ .\ ]_{xy}$ means the partial derivative $\dfrac{\partial^2[\ .\ ]}{\partial x \partial y}$.

Motivated by (8.1.2), we introduce the modified sine trigonometric rectangular sums v_{mn} of the series (8.3.1) as:

$$(8.3.2) \qquad v_{mn}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\sin rx \sin ly)\right\}$$

Our second main result of this chapter is as follows:

Theorem 2. If a double sequence $\{a_{jk}\}$ belongs to class J_d, then

$$||v_{mn} - g|| \to 0 \ \ as \ \ \min(m,n) \to \infty.$$

We draw two corollaries from Theorem 2.

Corollary 3. Under the conditions of Theorem 2, the sum g of the series (8.3.1) is integrable and (8.3.1) is a Fourier series of g.

Corollary 4. If a double sequence $\{a_{jk}\}$ belongs to class J_d, then

$$||S_{mn} - g|| \to 0 \ \ as \ \ \min(m,n) \to \infty.$$

We note that Theorem 2 and corollaries 3 and 4 can be considered as analogous results of chapter IV from one dimensional to two dimensional sine series.

Remark. We note that,

$$(8.3.3) \qquad |\triangle_{11}b_{jk}| \leq \frac{A_{jk}}{jk}$$

Moreover, it follows from (8.3.3) and condition (8.1.5) that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk|\triangle_{11}b_{jk}| < \infty,$$

therefore, $\{a_{jk}\} \in J_d$ implies that $\{a_{jk}\} \in \widetilde{BV}_2$.

Next we show that $\{a_{jk}\} \in J_d \Rightarrow \{a_{jk}\} \in \widetilde{\mathcal{C}}_2$. For this, Consider

$$\begin{aligned}
&\int_0^\delta \int_0^\delta \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} \triangle_{11} b_{jk} [D_j(x) D_k(y)]_{xy} \right| dx \ \ dy \\
&\leq \int_0^\pi \int_0^\pi \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} \triangle_{11} b_{jk} [D_j(x) D_k(y)]_{xy} \right| dx \ \ dy \\
&\leq \int_0^\pi \int_0^\pi \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} \frac{A_{jk}}{jk} [D_j(x) D_k(y)]_{xy} \right| dx \ \ dy
\end{aligned}$$

Since, $\displaystyle\int_0^\pi [D_n(x)]_x dx \approx n \log n$, thus from (8.1.5), we get $\{a_{jk}\} \in \widetilde{\mathcal{C}}_2$. Hence, if $\{a_{jk}\}$ belongs to class J_d, then $\{a_{jk}\} \in \widetilde{BV}_2 \bigcap \widetilde{C}_2$. Thus, the Theorem 2.1 and corollaries 2.1 and 2.2 of Móricz [35] holds true in case $\{a_{jk}\}$ belongs to class J_d.

8.4 Mixed series. We consider mixed series

$$\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} a_{jk} \sin jx \cos ky \tag{8.4.1}$$

on the positive quadrant $Q = [0, \pi] \times [0, \pi]$ of the two dimensional torus and $\{a_{jk}\}$ is a double sequence of real numbers.

We denote by

$$S_{mn}(x, y) = \sum_{j=1}^{m} \sum_{k=0}^{n} a_{jk} \sin jx \cos ky \quad (m \geq 1, \quad n \geq 0)$$

the rectangular partial sum of the series (8.4.1) and $t(x, y) = \lim\limits_{m,n \to \infty} S_{mn}$.

Definition [35]. Let $\{a_{jk}\} \in BV_2^*$ if condition (8.2.2) is satisfied and

$$\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} j |\triangle_{11} c_{jk}| < \infty,$$

where

$$c_{jk} := \frac{a_{jk}}{j} \ \ (j, k = 1, 2....).$$

Definition [35]. We say that $\{a_{jk}\} \in \mathcal{C}_2^*$ if condition (8.2.2) is satisfied and for every $\epsilon > 0$

there exists $\delta > 0$ such that for all $1 \leq m \leq M$ and $0 \leq n \leq N$, we have

$$C(m, M; n, N; \delta) := \int\int_{D_\delta} \left| \sum_{j=m}^{M} \sum_{k=n}^{N} [D_j(x)]_x D_k(y) \triangle_{11} c_{jk} \right| dx \quad dy \leq \epsilon$$

where

$$D_\delta := Q - (\delta, \pi] \times (\delta, \pi] = \{(x, y) : 0 \leq x, y \leq \pi \ \& \ \min(x, y) \leq \delta\}$$

or

$$\int\int_{D_\delta} \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} [D_j(x)]_x D_k(y) \triangle_{11} c_{jk} \right| dx \quad dy \leq \epsilon \quad \forall \ m \geq 1, \ n \geq 0.$$

where $[\ .\ ]_x$ means the partial derivative $\dfrac{\partial [\ .\ ]}{\partial x}$.

Motivated by (8.1.1) and (8.1.2), we introduce the modified mixed trigonometric sums w_{mn} of the series (8.4.1) as:

(8.4.2) $$w_{mn}(x, y) = \sum_{j=1}^{m} \sum_{k=0}^{n} \left\{ \sum_{r=j}^{m} \sum_{l=k}^{n} \triangle_{11}(a_{rl} \sin rx \cos ly) \right\}$$

Our third main result is as follows:

Theorem 3. If a double sequence $\{a_{jk}\}$ belongs to class J_d, then

$$||w_{mn} - t|| \to 0 \quad as \quad \min(m, n) \to \infty.$$

We draw two corollaries from Theorem 3.

Corollary 5. Under the conditions of theorem 3, the sum t of the series (8.4.1) is integrable and (8.4.1) is a Fourier series of t.

Corollary 6. If a double sequence $\{a_{jk}\}$ belongs to class J_d, then

$$||S_{mn} - t|| \to 0 \quad as \quad \min(m, n) \to \infty.$$

Remark. We note that,

(8.4.3) $$|\triangle_{11} c_{jk}| \leq 2 \frac{A_{jk}}{j}$$

Moreover, it follows from (8.4.3) and condition (8.1.5) that

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} j|\triangle_{11}c_{jk}| < \infty,$$

therefore, $\{a_{jk}\} \in J_d$ implies that $\{a_{jk}\} \in BV_2^*$.

Next we show that $\{a_{jk}\} \in J_d \Rightarrow \{a_{jk}\} \in \mathcal{C^*}_2$. Therefore, consider

$$\begin{aligned}
&\int_0^{\delta}\int_0^{\delta}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\triangle_{11}c_{jk}[D_j(x)]_x D_k(y)\right| dx \quad dy\\
&\leq \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\triangle_{11}c_{jk}[D_j(x)]_x D_k(y)\right| dx \quad dy\\
&\leq 2\int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\frac{A_{jk}}{j}[D_j(x)]_x D_k(y)\right| dx \quad dy
\end{aligned}$$

Since, $\int_0^{\pi}[D_n(x)]_x dx \approx n\log n$, thus from (8.1.5), we get $\{a_{jk}\} \in \mathcal{C^*}_2$. Hence, if $\{a_{jk}\}$ belongs to class J_d, then $\{a_{jk}\} \in BV_2^* \bigcap \mathcal{C}_2^*$. Thus, the theorem 3.1 and corollaries 3.1 and 3.2 of Móricz [35] holds true in case $\{a_{jk}\}$ belongs to class J_d.

8.5 Proofs of the Theorems

Proof of Theorem 1. We shall first show that the point wise limit f of

$$u_{mn}(x,y) \quad = \quad \frac{a_{00}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\cos rx\cos ly)\right\}$$

exists in Q and that f is a Fourier series i.e. $f \in L^1(Q)$.

$$\begin{aligned}
u_{mn}(x,y) \quad = \quad & \frac{a_{00}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}\{\sum_{r=j}^{m}[a_{rk}\cos rx\cos ky - a_{r,k+1}\cos rx\cos(k+1)y\\
& -a_{r+1,k}\cos(r+1)x\cos ky + a_{r+1,k+1}\cos(r+1)x\cos(k+1)y\\
& +a_{r,k+1}\cos rx\cos(k+1)y - a_{r,k+2}\cos rx\cos(k+2)y\\
& -a_{r+1,k+1}\cos(r+1)x\cos(k+1)y + a_{r+1,k+2}\cos(r+1)x\cos(k+2)y\\
& +... + a_{rn}\cos rx\cos ny - a_{r,n+1}\cos rx\cos(n+1)y\\
& -a_{r+1,n}\cos(r+1)x\cos ny + a_{r+1,n+1}\cos(r+1)x\cos(n+1)y]\}
\end{aligned}$$

$$
\begin{aligned}
u_{mn}(x,y) &= \frac{a_{00}}{2} + \sum_{j=1}^{m}\sum_{k=1}^{n}[a_{jk}\cos jx\cos ky - a_{j+1,k}\cos(j+1)x\cos ky \\
&\quad -a_{j,n+1}\cos jx\cos(n+1)y + a_{j+1,n+1}\cos(j+1)x\cos(n+1)y \\
&\quad +a_{j+1,k}\cos(j+1)x\cos ky - a_{j+2,k}\cos(j+2)x\cos ky \\
&\quad -a_{j+1,n+1}\cos(j+1)x\cos(n+1)y + a_{j+2,n+1}\cos(j+2)x\cos(n+1)y \\
&\quad +\ldots + a_{mn}\cos mx\cos ny - a_{m+1,k}\cos(m+1)x\cos ky \\
&\quad -a_{m,n+1}\cos mx\cos(n+1)y + a_{m+1,n+1}\cos(m+1)x\cos(n+1)y] \\
&= \frac{a_{00}}{2} + \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\cos jx\cos ky - \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,n+1}\cos jx\cos(n+1)y \\
&\quad -\sum_{j=1}^{m}\sum_{k=1}^{n} a_{m+1,k}\cos(m+1)x\cos ky + mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y \\
&= S_{mn}(x,y) - \sum_{j=1}^{m} na_{j,n+1}\cos jx\cos(n+1)y - \sum_{k=1}^{n} ma_{m+1,k}\cos(m+1)x\cos ky \\
&\quad +mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y \qquad (8.5.1)
\end{aligned}
$$

Using double summation by parts and given hypothesis in (8.5.1), we get

$$
\begin{aligned}
u_{mn}(x,y) &= \frac{a_{00}}{2} + \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \triangle_{11}\left(\frac{a_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y) \\
&\quad +\sum_{j=1}^{m} \triangle_{10}\left(\frac{a_{jn}}{jn}\right)\tilde{D}'_j(x)\tilde{D}'_n(y) \\
&\quad +\sum_{k=1}^{n} \triangle_{01}\left(\frac{a_{mk}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y) \\
&\quad -\frac{a_{mn}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y) - \sum_{j=1}^{m} na_{j,n+1}\cos jx\cos(n+1)y \\
&\quad -\sum_{k=1}^{n} ma_{m+1,k}\cos(m+1)x\cos ky \\
&\quad +mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y
\end{aligned}
$$

It is known from Sheng [48] that

$$
|\tilde{D}'_n(x)| = O(n) \quad \text{for} \quad 0 < x \leq \pi \qquad (8.5.2)
$$

By (8.1.5), (8.1.6) we note that

$$
\sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \triangle_{11}\left(\frac{a_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y) \leq \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left(\frac{A_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y) < \infty,
$$

for all x and y such that $0 < x, y \leq \pi$.

By (8.1.5), (8.1.6) and (8.5.2) we have

$$\sum_{j=1}^{m} \triangle_{10}\left(\frac{a_{jn}}{jn}\right)\tilde{D}'_j(x)\tilde{D}'_n(y) \leq \sum_{j=1}^{m}\sum_{k=n}^{\infty}\left(\frac{A_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y) \to 0 \quad as \quad n \to \infty$$

uniformly in m, for all $0 < x, y \leq \pi$.

Similarly,

$$\sum_{k=1}^{n} \triangle_{01}\left(\frac{a_{mk}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y) \to 0 \quad as \quad n \to \infty$$

uniformly in n, for all $0 < x, y \leq \pi$.

Since $\{a_{jk}\}$ is double null sequence and by the use of the equation (8.5.2), we get

$$\frac{a_{mn}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y) \to 0 \quad as \quad m+n \to \infty$$

for all $0 < x, y \leq \pi$.

Further, we know that $\cos nx$ is bounded in $(0, \pi]$.

Therefore, by (8.1.5) and (8.1.6), we have

$$\begin{aligned}\sum_{j=1}^{m} n a_{j,n+1} &= \sum_{j=1}^{m} jn(n+1)\sum_{k=n+1}^{\infty}\triangle_{01}\left(\frac{a_{jk}}{jk}\right) \\ &\leq \sum_{j=1}^{m}\sum_{k=n+1}^{\infty} jk^2\left(\frac{A_{jk}}{jk}\right) \to 0 \quad as \quad n \to \infty\end{aligned}$$

This implies that

$$\sum_{j=1}^{m} n a_{j,n+1}\cos jx\cos(n+1)y \to 0 \quad as \quad n \to \infty$$

uniformly in m, for all $0 < x, y \leq \pi$.

Similarly,

$$\sum_{k=1}^{n} m a_{m+1,k}\cos(m+1)x\cos ky \to 0 \quad as \quad m \to \infty$$

uniformly in n, for all $0 < x, y \leq \pi$.

Also, by (8.1.5) and (8.1.6), we have

$$\begin{aligned}(8.5.3) \qquad mna_{m+1,n+1} &\leq \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} j^2k^2\triangle_{11}\left(\frac{a_{jk}}{jk}\right) \\ &\leq \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} j^2k^2\frac{A_{jk}}{jk} \to 0 \quad as \quad m+n \to \infty.\end{aligned}$$

Consequently, we get $\lim_{m+n\to\infty} u_{mn} = f(x,y)$ exists in $L^1(Q)$.

Next, we consider

$$
\begin{aligned}
||f-u_{mn}|| \le & \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \triangle_{11}\left(\frac{a_{jk}}{jk}\right) \tilde{D}'_j(x)\tilde{D}'_k(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} \triangle_{10}\left(\frac{a_{jn}}{jn}\right) \tilde{D}'_j(x)\tilde{D}'_n(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{k=1}^{n} \triangle_{01}\left(\frac{a_{mk}}{mk}\right) \tilde{D}'_m(x)\tilde{D}'_k(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \frac{a_{mn}}{mn} \tilde{D}'_m(x)\tilde{D}'_n(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} n a_{j,n+1} \cos jx \cos(n+1)y \right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{k=1}^{n} m a_{m+1,k} \cos(m+1)x \cos ky \right| dxdy \\
& + mn|a_{m+1,n+1}| \int_0^\pi \int_0^\pi |\cos(m+1)x\cos(n+1)y| dxdy
\end{aligned}
$$

$$
\begin{aligned}
(8.5.4) \qquad \le & \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \left(\frac{A_{jk}}{jk}\right) \tilde{D}'_j(x)\tilde{D}'_k(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} \left(\frac{A_{jn}}{jn}\right) \tilde{D}'_j(x)\tilde{D}'_n(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{k=1}^{n} \left(\frac{A_{mk}}{mk}\right) \tilde{D}'_m(x)\tilde{D}'_k(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \frac{a_{mn}}{mn} \tilde{D}'_m(x)\tilde{D}'_n(y)\right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} \sum_{k=n+1}^{\infty} jk^2 \left(\frac{A_{jk}}{jk}\right) \cos jx \cos(n+1)y \right| dxdy \\
& + \int_0^\pi \int_0^\pi \left| \sum_{k=1}^{n} \sum_{j=m+1}^{\infty} kj^2 \left(\frac{A_{jk}}{jk}\right) \cos(m+1)x \cos ky \right| dxdy \\
& + mn|a_{m+1,n+1}| \int_0^\pi \int_0^\pi |\cos(m+1)x\cos(n+1)y| dxdy
\end{aligned}
$$

We note that from Sheng [48], $\left|\left|\frac{\tilde{D}'_n(x)}{n^2}\right|\right| = O(1)$. Further, by (8.1.5) and (8.1.6), we get

$$
\int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} jk \left(\frac{A_{jk}}{j^2k^2}\right) \tilde{D}'_j(x)\tilde{D}'_k(y)\right| dxdy \to 0 \qquad as \qquad m+n\to\infty
$$

Thus by using equation (8.5.3) and the given hypothesis all the terms on the right hand side of the inequality (8.5.4) tends to zero as $m+n \to \infty$. Hence, the conclusion of the theorem 1 holds. □

Remark. Since proof of theorems 2 and 3 are on the same lines as that of theorem 1. But for ready reference we are giving the proof of theorems 2 and 3 in brief.

Proof of Theorem 2. First we shall show that the point wise limit g of

$$\begin{aligned} v_{mn}(x,y) &= \sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\sin rx \sin ly)\right\} \\ &= S_{mn}(x,y) - \sum_{j=1}^{m} na_{j,n+1}\sin jx \sin(n+1)y - \sum_{k=1}^{n} ma_{m+1,k}\sin(m+1)x\sin ky \\ &\quad +mna_{m+1,n+1}\sin(m+1)x\sin(n+1)y \end{aligned} \tag{8.5.5}$$

exists in Q and that g is a Fourier series i.e. $g \in L^1(Q)$.

Using double summation by parts and given hypothesis in (8.5.5), we get

$$\begin{aligned} v_{mn}(x,y) &= \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\triangle_{11}\left(\frac{a_{jk}}{jk}\right)D'_j(x)D'_k(y) \\ &\quad +\sum_{j=1}^{m}\triangle_{10}\left(\frac{a_{jn}}{jn}\right)D'_j(x)D'_n(y) \\ &\quad +\sum_{k=1}^{n}\triangle_{01}\left(\frac{a_{mk}}{mk}\right)D'_m(x)D'_k(y) \\ &\quad -\frac{a_{mn}}{mn}D'_m(x)D'_n(y) - \sum_{j=1}^{m} na_{j,n+1}\sin jx\sin(n+1)y \\ &\quad -\sum_{k=1}^{n} ma_{m+1,k}\sin(m+1)x\sin ky \\ &\quad +mna_{m+1,n+1}\sin(m+1)x\sin(n+1)y \end{aligned}$$

It is known from Sheng [48] that

$$|D'_n(x)| = O(n) \quad \text{for} \quad 0 < x \le \pi \tag{8.5.6}$$

By (8.1.5), (8.1.6) we note that

$$\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\triangle_{11}\left(\frac{a_{jk}}{jk}\right)D'_j(x)D'_k(y) \le \sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left(\frac{A_{jk}}{jk}\right)D'_j(x)D'_k(y) < \infty,$$

for all x and y such that $0 < x, y \leq \pi$.

By (8.1.5), (8.1.6) and (8.5.6) we have

$$\sum_{j=1}^{m} \triangle_{10}\left(\frac{a_{jn}}{jn}\right) D_j'(x)D_n'(y) \leq \sum_{j=1}^{m}\sum_{k=n}^{\infty}\left(\frac{A_{jk}}{jk}\right) D_j'(x)D_k'(y) \to 0 \quad as \quad n \to \infty$$

uniformly in m, for all $0 < x, y \leq \pi$.

Similarly,

$$\sum_{k=1}^{n} \triangle_{01}\left(\frac{a_{mk}}{mk}\right) D_m'(x)D_k'(y) \to 0 \quad as \quad n \to \infty$$

uniformly in n, for all $0 < x, y \leq \pi$.

Since $\{a_{jk}\}$ is double null sequence and by the use of the equation (8.5.6), we get

$$\frac{a_{mn}}{mn} D_m'(x)D_n'(y) \to 0 \quad as \quad m+n \to \infty$$

for all $0 < x, y \leq \pi$.

Further, we know that $\sin nx$ is bounded in $(0, \pi]$.

Therefore, by (8.1.5) and (8.1.6), we have

$$\begin{aligned}\sum_{j=1}^{m} na_{j,n+1} &= \sum_{j=1}^{m} jn(n+1)\sum_{k=n+1}^{\infty}\triangle_{01}\left(\frac{a_{jk}}{jk}\right)\\ &\leq \sum_{j=1}^{m}\sum_{k=n+1}^{\infty} jk^2\left(\frac{A_{jk}}{jk}\right) \to 0 \quad as \quad n \to \infty\end{aligned}$$

This implies that

$$\sum_{j=1}^{m} na_{j,n+1}\sin jx \sin(n+1)y \to 0 \quad as \quad n \to \infty$$

uniformly in m, for all $0 < x, y \leq \pi$.

Similarly,

$$\sum_{k=1}^{n} ma_{m+1,k}\sin(m+1)x \sin ky \to 0 \quad as \quad m \to \infty$$

uniformly in n, for all $0 < x, y \leq \pi$.

Also, by (8.1.5) and (8.1.6), we have

$$\begin{aligned}(8.5.7) \qquad mna_{m+1,n+1} &\leq \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} j^2k^2\triangle_{11}\left(\frac{a_{jk}}{jk}\right)\\ &\leq \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} j^2k^2\frac{A_{jk}}{jk} \to 0 \quad as \quad m+n \to \infty.\end{aligned}$$

Consequently, we get $\lim_{m+n\to\infty} v_{mn} = g(x,y)$ exists in $L^1(Q)$.

Next, we consider

$$||g - v_{mn}|| \leq \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^\infty \sum_{k=n+1}^\infty \triangle_{11} \left(\frac{a_{jk}}{jk}\right) D_j'(x) D_k'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^m \triangle_{10} \left(\frac{a_{jn}}{jn}\right) D_j'(x) D_n'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=1}^n \triangle_{01} \left(\frac{a_{mk}}{mk}\right) D_m'(x) D_k'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \frac{a_{mn}}{mn} D_m'(x) D_n'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^m n a_{j,n+1} \sin jx \sin(n+1)y \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=1}^n m a_{m+1,k} \sin(m+1)x \sin ky \right| dxdy$$
$$+ mn|a_{m+1,n+1}| \int_0^\pi \int_0^\pi |\sin(m+1)x \sin(n+1)y| dxdy$$

$$(8.5.8) \qquad \leq \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^\infty \sum_{k=n+1}^\infty \left(\frac{A_{jk}}{jk}\right) D_j'(x) D_k'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^m \left(\frac{A_{jn}}{jn}\right) D_j'(x) D_n'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=1}^n \left(\frac{A_{mk}}{mk}\right) D_m'(x) D_k'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \frac{a_{mn}}{mn} D_m'(x) D_n'(y) \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^m \sum_{k=n+1}^\infty jk^2 \left(\frac{A_{jk}}{jk}\right) \sin jx \sin(n+1)y \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=1}^n \sum_{j=m+1}^\infty kj^2 \left(\frac{A_{jk}}{jk}\right) \sin(m+1)x \sin ky \right| dxdy$$
$$+ mn|a_{m+1,n+1}| \int_0^\pi \int_0^\pi |\sin(m+1)x \sin(n+1)y| dxdy$$

We note that from Sheng [48], $\left|\left|\frac{D_n'(x)}{n^2}\right|\right| = O(1)$. Further, by (8.1.5) and (8.1.6), we get

$$\int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^\infty \sum_{k=n+1}^\infty jk \left(\frac{A_{jk}}{j^2k^2}\right) D_j'(x) D_k'(y) \right| dxdy \to 0 \quad as \quad m+n \to \infty$$

Thus by using equation (8.5.7) and the given hypothesis all the terms on the right hand side of the inequality (8.5.8) tends to zero as $m+n \to \infty$. Hence, the conclusion of the theorem 2 holds. □

Proof of Theorem 3. First we shall show that the point wise limit t of

$$\begin{aligned} w_{mn}(x,y) &= \sum_{j=1}^{m}\sum_{k=0}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\sin rx\cos ly)\right\} \\ &= S_{mn}(x,y) - \sum_{j=1}^{m} na_{j,n+1}\sin jx\cos(n+1)y - \sum_{k=1}^{n} ma_{m+1,k}\sin(m+1)x\cos ky \\ &\quad +mna_{m+1,n+1}\sin(m+1)x\cos(n+1)y \end{aligned} \tag{8.5.9}$$

exists in Q and that t is a Fourier series i.e. $t \in L^1(Q)$.

Using double summation by parts and given hypothesis in (8.5.9), we get

$$\begin{aligned} w_{mn}(x,y) &= \sum_{j=1}^{m-1}\sum_{k=0}^{n-1}\triangle_{11}\left(\frac{a_{jk}}{jk}\right) -D'_j(x)\tilde{D}'_k(y) \\ &\quad +\sum_{j=1}^{m}\triangle_{10}\left(\frac{a_{jn}}{jn}\right) -D'_j(x)\tilde{D}'_n(y) \\ &\quad +\sum_{k=0}^{n}\triangle_{01}\left(\frac{a_{mk}}{mk}\right) -D'_m(x)\tilde{D}'_k(y) \\ &\quad -\frac{a_{mn}}{mn} -D'_m(x)\tilde{D}'_n(y) - \sum_{j=1}^{m} na_{j,n+1}\sin jx\cos(n+1)y \\ &\quad -\sum_{k=1}^{n} ma_{m+1,k}\sin(m+1)x\cos ky \\ &\quad +mna_{m+1,n+1}\sin(m+1)x\cos(n+1)y \end{aligned}$$

By (8.1.5), (8.1.6) we note that

$$\sum_{j=1}^{m-1}\sum_{k=0}^{n-1}\triangle_{11}\left(\frac{a_{jk}}{jk}\right)D'_j(x)\tilde{D}'_k(y) \le \sum_{j=1}^{m-1}\sum_{k=0}^{n-1}\left(\frac{A_{jk}}{jk}\right)D'_j(x)\tilde{D}'_k(y) < \infty,$$

for all x and y such that $0 < x, y \le \pi$.

By (8.1.5), (8.1.6), (8.5.2) and (8.5.6) we have

$$\sum_{j=1}^{m}\triangle_{10}\left(\frac{a_{jn}}{jn}\right)D'_j(x)\tilde{D}'_n(y) \le \sum_{j=1}^{m}\sum_{k=n}^{\infty}\left(\frac{A_{jk}}{jk}\right)D'_j(x)\tilde{D}'_k(y) \to 0 \quad as \quad n \to \infty$$

uniformly in m, for all $0 < x, y \le \pi$.

Similarly,

$$\sum_{k=0}^{n} \triangle_{01}\left(\frac{a_{mk}}{mk}\right) D'_m(x)\tilde{D}'_k(y) \to 0 \quad as \quad n \to \infty$$

uniformly in n, for all $0 < x, y \leq \pi$.

Since $\{a_{jk}\}$ is double null sequence and by the use of (8.5.2) and (8.5.6), we get

$$\frac{a_{mn}}{mn} D'_m(x)\tilde{D}'_n(y) \to 0 \quad as \quad m+n \to \infty$$

for all $0 < x, y \leq \pi$.

Further, we know that $\cos nx$ and $\sin nx$ are bounded in $(0, \pi]$.

Therefore, by (8.1.5) and (8.1.6), we have

$$\begin{aligned} \sum_{j=1}^{m} n a_{j,n+1} &= \sum_{j=1}^{m} jn(n+1) \sum_{k=n+1}^{\infty} \triangle_{01}\left(\frac{a_{jk}}{jk}\right) \\ &\leq \sum_{j=1}^{m} \sum_{k=n+1}^{\infty} jk^2 \left(\frac{A_{jk}}{jk}\right) \to 0 \quad as \quad n \to \infty \end{aligned}$$

This implies that

$$\sum_{j=1}^{m} n a_{j,n+1} \sin jx \cos(n+1)y \to 0 \quad as \quad n \to \infty$$

uniformly in m, for all $0 < x, y \leq \pi$.

Similarly,

$$\sum_{k=0}^{n} m a_{m+1,k} \sin(m+1)x \cos ky \to 0 \quad as \quad m \to \infty$$

uniformly in n, for all $0 < x, y \leq \pi$.

Also, by (8.1.5) and (8.1.6), we have

$$\begin{aligned} mn a_{m+1,n+1} &\leq \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} j^2k^2 \triangle_{11}\left(\frac{a_{jk}}{jk}\right) \\ &\leq \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} j^2k^2 \frac{A_{jk}}{jk} \to 0 \quad as \quad m+n \to \infty. \end{aligned} \tag{8.5.10}$$

Consequently, we get $\lim_{m+n\to\infty} w_{mn} = t(x,y)$ exists in $L^1(Q)$.

Next, we consider

$$||t-w_{mn}|| \leq \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \triangle_{11} \left(\frac{a_{jk}}{jk}\right) - D_j'(x)\tilde{D}_k'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} \triangle_{10} \left(\frac{a_{jn}}{jn}\right) - D_j'(x)\tilde{D}_n'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=0}^{n} \triangle_{01} \left(\frac{a_{mk}}{mk}\right) - D_m'(x)\tilde{D}_k'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \frac{a_{mn}}{mn} D_m'(x)\tilde{D}_n'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} n a_{j,n+1} \sin jx \cos(n+1)y \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=1}^{n} m a_{m+1,k} \sin(m+1)x \cos ky \right| dxdy$$
$$+ mn|a_{m+1,n+1}| \int_0^\pi \int_0^\pi |\sin(m+1)x\cos(n+1)y| dxdy$$

$$\leq \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \left(\frac{A_{jk}}{jk}\right) D_j'(x)\tilde{D}_k'(y)\right| dxdy \tag{8.5.11}$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} \left(\frac{A_{jn}}{jn}\right) D_j'(x)\tilde{D}_n'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=0}^{n} \left(\frac{A_{mk}}{mk}\right) D_m'(x)\tilde{D}_k'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \frac{a_{mn}}{mn} D_m'(x)\tilde{D}_n'(y)\right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{j=1}^{m} \sum_{k=n+1}^{\infty} jk^2 \left(\frac{A_{jk}}{jk}\right) \sin jx \cos(n+1)y \right| dxdy$$
$$+ \int_0^\pi \int_0^\pi \left| \sum_{k=0}^{n} \sum_{j=m+1}^{\infty} kj^2 \left(\frac{A_{jk}}{jk}\right) \sin(m+1)x \cos ky \right| dxdy$$
$$+mn|a_{m+1,n+1}| \int_0^\pi \int_0^\pi |\sin(m+1)x\cos(n+1)y| dxdy$$

We note that from Sheng [48], $\left|\left|\frac{D_n'(x)}{n^2}\right|\right| = O(1)$ and $\left|\left|\frac{\tilde{D}_n'(x)}{n^2}\right|\right| = O(1)$. Further, by (8.1.5) and (8.1.6), we get

$$\int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} jk \left(\frac{A_{jk}}{j^2k^2}\right) D_j'(x)\tilde{D}_k'(y)\right| dxdy \to 0 \quad as \quad m+n \to \infty$$

Thus by using equation (8.5.10) and the given hypothesis all the terms on the right hand side of the inequality (8.5.11) tends to zero as $m + n \to \infty$. Hence, the conclusion of the theorem 3 holds. □

8.6 Proofs of the Corollaries

Proof of Corollary 1. It follows from theorem 1, that $f \in L^1(Q)$. Further-more it is known that convergence in $L^1 - norm$ (the so-called strong convergence) implies weak convergence.

Now, consider

$$\begin{aligned} u_{mn}(x,y) &= \frac{a_{00}}{2} + \sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\cos rx\cos ly)\right\} \\ &= S_{mn}(x,y) - \sum_{j=1}^{m}\sum_{k=1}^{n}\{a_{j,n+1}\cos jx\cos(n+1)y + a_{m+1,k}\cos(m+1)x\cos ky\} \\ &\quad + mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y \end{aligned}$$

for fixed $r, l \geq 1$, we get

$$\begin{aligned} &\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cos rx\cos ly dxdy \\ &= \lim_{m+n\to\infty}\frac{4}{\pi^2}\int_0^\pi\int_0^\pi u_{mn}(x,y)\cos rx\cos ly dxdy \\ &= a_{rl} - \lim_{m+n\to\infty}\left\{\sum_{j=1}^{m} na_{j,n+1} + \sum_{k=1}^{n} ma_{m+1,k} + mna_{m+1,n+1}\right\} \\ &= a_{rl} \end{aligned}$$

Since the limit of each term in braces is zero (as already shown in the proof of theorem 1). This proves that the (8.2.1) is a Fourier series of f. □

Proof of Corollary 2. Consider

$$\begin{aligned} ||f - S_{mn}|| &= ||f - u_{mn} + u_{mn} - S_{mn}|| \\ &\leq ||f - u_{mn}|| + ||u_{mn} - S_{mn}|| \\ &\leq ||f - u_{mn}|| + \int_0^\pi\int_0^\pi\left|\sum_{j=1}^{m} na_{j,n+1}\cos jx\cos(n+1)y\right| dxdy \\ &\quad + \int_0^\pi\int_0^\pi\left|\sum_{k=1}^{n} ma_{m+1,k}\cos(m+1)x\cos ky\right| dxdy \\ &\quad + mn|a_{m+1,n+1}|\int_0^\pi\int_0^\pi|\cos(m+1)x\cos(n+1)y|dxdy \end{aligned}$$

Using theorem 1 the conclusion of the corollary 2 follows. □

Proof of Corollary 3. It follows from theorem 2, that $g \in L^1(Q)$. Now, consider

$$\begin{aligned} v_{mn}(x,y) &= \sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\sin rx \sin ly)\right\} \\ &= S_{mn}(x,y) - \sum_{j=1}^{m}\sum_{k=1}^{n}\{a_{j,n+1}\sin jx\sin(n+1)y + a_{m+1,k}\sin(m+1)x\sin ky\} \\ &\quad + mna_{m+1,n+1}\sin(m+1)x\sin(n+1)y \end{aligned}$$

for fixed $r, l \geq 1$, we get

$$\begin{aligned} &\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} g(x,y)\sin rx \sin ly dxdy \\ &\qquad = \lim_{m+n\to\infty}\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} v_{mn}(x,y)\sin rx\sin ly dxdy \\ &\qquad = a_{rl} - \lim_{m+n\to\infty}\left\{\sum_{j=1}^{m} na_{j,n+1} + \sum_{k=1}^{n} ma_{m+1,k} + mna_{m+1,n+1}\right\} \\ &\qquad = a_{rl} \end{aligned}$$

Since the limit of each term in braces is zero (as already shown in the proof of theorem 2). This proves that the (8.3.1) is a Fourier series of g. □

Proof of Corollary 4. Consider

$$\begin{aligned} ||g - S_{mn}|| &= ||g - v_{mn} + v_{mn} - S_{mn}|| \\ &\leq ||g - v_{mn}|| + ||v_{mn} - S_{mn}|| \\ &\leq ||g - v_{mn}|| + \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=1}^{m} na_{j,n+1}\sin jx\sin(n+1)y\right| dxdy \\ &\quad + \int_0^{\pi}\int_0^{\pi}\left|\sum_{k=1}^{n} ma_{m+1,k}\sin(m+1)x\sin ky\right| dxdy \\ &\quad + mn|a_{m+1,n+1}|\int_0^{\pi}\int_0^{\pi}|\sin(m+1)x\sin(n+1)y|dxdy \end{aligned}$$

Using theorem 2 the conclusion of the corollary 4 follows. □

Proof of Corollary 5. It follows from theorem 3, that $t \in L^1(Q)$.

Now, consider

$$w_{mn}(x,y) = \sum_{j=1}^{m}\sum_{k=0}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\sin rx\cos ly)\right\}$$

$$
\begin{aligned}
w_{mn}(x,y) &= S_{mn}(x,y) - \sum_{j=1}^{m}\sum_{k=0}^{n}\{a_{j,n+1}\sin jx\cos(n+1)y + a_{m+1,k}\sin(m+1)x\cos ky\} \\
&\quad + mna_{m+1,n+1}\sin(m+1)x\cos(n+1)y
\end{aligned}
$$

for fixed $r, l \geq 1$, we get

$$
\begin{aligned}
&\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} t(x,y)\sin rx\cos ly dxdy \\
&\quad = \lim_{m+n\to\infty}\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} w_{mn}(x,y)\sin rx\cos ly dxdy \\
&\quad = a_{rl} - \lim_{m+n\to\infty}\left\{\sum_{j=1}^{m} na_{j,n+1} + \sum_{k=1}^{n} ma_{m+1,k} + mna_{m+1,n+1}\right\} \\
&\quad = a_{rl}
\end{aligned}
$$

Since the limit of each term in braces is zero (as already shown in the proof of theorem 3). This proves that the (8.4.1) is a Fourier series of t. □

Proof of Corollary 6. Consider

$$
\begin{aligned}
||t - S_{mn}|| &= ||t - w_{mn} + w_{mn} - S_{mn}|| \\
&\leq ||t - w_{mn}|| + ||w_{mn} - S_{mn}|| \\
&\leq ||t - w_{mn}|| + \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=1}^{m} na_{j,n+1}\sin jx\cos(n+1)y\right| dxdy \\
&\quad + \int_0^{\pi}\int_0^{\pi}\left|\sum_{k=0}^{n} ma_{m+1,k}\sin(m+1)x\cos ky\right| dxdy \\
&\quad + mn|a_{m+1,n+1}|\int_0^{\pi}\int_0^{\pi}|\sin(m+1)x\cos(n+1)y|dxdy
\end{aligned}
$$

Using theorem 3, the conclusion of the corollary 6 follows. □

CHAPTER IX

THE EXTENSION OF THE THEOREM OF J.W. GARRETT, C.S. REES AND C.V. STANOJEVIĆ FROM ONE DIMENSION TO TWO DIMENSION

9.1 Introduction. Consider the double cosine series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k\ a_{jk}\cos jx\cos ky \tag{9.1.1}$$

on the positive quadrant $Q = [0, \pi] \times [0, \pi]$ of the two dimensional torus, where $\lambda_0 = \frac{1}{2}$ and $\lambda_j = 1$ for $j = 1, 2, 3, ...$ and $\{a_{jk}\}$ is a double sequence of real numbers.

We denote the rectangular partial sum of the series (9.1.1) by S_{mn} i.e.

$$S_{mn}(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k a_{jk}\cos jx\cos ky \qquad (m, n \geq 0)$$

and let $f(x, y) = \lim\limits_{m,n\to\infty} S_{mn}$.

Concerning the L^1-convergence of the double cosine series Moricz [35] proved the following result:

Theorem A [35]. If $\{a_{jk}\} \in BV_2 \cap \mathcal{C}_2$, then the sum $f(x, y)$ of series (9.1.1) belongs to $L^1(Q)$ and (9.1.1) is a Fourier series of $f(x, y)$.

Garrett, Rees and Stanojević [20] introduced an equivalent class S^2 of the class S of Teljakovskii [68] for one dimensional coefficient sequence.

In this chapter, we have extended the class S^2 of coefficient sequences introduced by Garrett, Rees and Stanojević [20] from one dimensional to a new class S^2_d of two dimensional coefficient sequence $\{a_{jk}\}$, defined as:

The results obtained in this chapter have been published in **International Journal of Mathematical Analysis, 3(26), (2009), 1251 - 1257**

Definition. A double null sequence $\{a_{jk}\}$ belongs to S_d^2 if there exists a null sequence $\{A_{jk}\}$ of non-negative numbers such that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} jk|\triangle_{11}A_{jk}| < \infty, \tag{9.1.2}$$

$$|\triangle_{11}a_{jk}| \le A_{jk} \ \ \forall \ j,k. \tag{9.1.3}$$

The aim of this chapter is to extend the corresponding result of Garrett, Rees and Stanojević [20] from one dimensional to two-dimensional series and to obtain necessary and sufficient condition of L^1-convergence of double cosine series.

9.2 Result. We prove the following result:

Theorem. Let $\{a_{jk}\} \in S_d^2$. Then $f \in L^1(Q)$ and

$$||S_{mn} - f|| = o(1), \ \ as \ \ m,n \to \infty$$

if and only if

$$a_{mn}\ln(m{+}2)\ln(n{+}2) \to 0 \ \ as \ \ m{+}n \to \infty. \tag{9.2.1}$$

Proof. First we shall show that the point wise limit f of

$$S_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_j\lambda_k a_{jk}\cos jx\cos ky \tag{9.2.2}$$

exists in Q and that f is a Fourier series i.e. $f \in L^1(Q)$.

From $\{a_{jk}\} \in S_d^2$ it follows that $\{a_{jk}\} \in BV_2$. Indeed

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\triangle_{11}A_{jk}| \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} jk|\triangle_{11}A_{jk}| < \infty.$$

On the other hand

$$mnA_{mn} \le mn\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} |\triangle_{11}A_{jk}| \le \sum_{j=m}^{\infty}\sum_{k=n}^{\infty} jk|\triangle_{11}A_{jk}| = o(1). \tag{9.2.3}$$

Performing double summation by parts, we have

$$\sum_{j=0}^{m}\sum_{k=0}^{n} A_{jk} = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} jk\triangle_{11}A_{jk} - \sum_{j=0}^{m-1} jn\triangle_{10}A_{jn} - \sum_{k=0}^{n-1} km\triangle_{01}A_{mk} + mnA_{mn}$$

but

$$n\triangle_{10}A_{jn} = \sum_{k=n}^{\infty} k\triangle_{11}A_{jk}$$

and

$$m\triangle_{01}A_{mk} = \sum_{j=m}^{\infty} j\triangle_{11}A_{jk}$$

So, we get that $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{jk} < \infty$. Since $\{a_{jk}\} \in S_d^2$ implies that $|\triangle_{11}a_{jk}| \leq A_{jk}\ \forall\ j,k$, it follows that$\{a_{jk}\} \in BV_2$, and hence (9.2.2) converges in Q to the point wise limit f.

Now from theorem A we have that if $\{a_{jk}\} \in BV_2 \cap \mathcal{C}_2$, then $f \in L^1(Q)$.

Thus it suffices to prove that

$$\{a_{jk}\} \in S_d^2 \Rightarrow \{a_{jk}\} \in \mathcal{C}_2.$$

Therefore, consider

$$\int_0^{\delta}\int_0^{\delta}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\triangle_{11}a_{jk}D_j(x)D_k(y)\right| dx\ \ dy \leq \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\triangle_{11}a_{jk}D_j(x)D_k(y)\right| dx\ \ dy$$

$$\leq \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}A_{jk}D_j(x)D_k(y)\right| dx\ \ dy$$

Using double summation by parts, we get

$$(9.2.4)\qquad \leq \int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\triangle_{11}A_{jk}(j+1)F_j(x)(k+1)F_k(y)\right| dx\ \ dy$$

$$+ \lim_{N\to\infty}\int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\triangle_{10}A_{jN}(j+1)F_j(x)(N+1)F_N(y)\right| dx\ \ dy$$

$$+ \lim_{M\to\infty}\int_0^{\pi}\int_0^{\pi}\left|\sum_{k=n}^{\infty}\triangle_{01}A_{Mk}(M+1)F_M(x)(k+1)F_k(y)\right| dx\ \ dy$$

$$+ \lim_{M,N\to\infty}\int_0^{\pi}\int_0^{\pi}|MNA_{MN}F_M(x)F_N(y)|\, dx\ \ dy$$

where $D_n(x)$ and $F_n(x)$ represent the Dirichlet and Fejér kernel respectively.

Since $\{a_{jk}\} \in S_d^2$ and $\int_0^{\pi}|F_n(x)|dx = \pi$, we have

$$\int_0^{\pi}\int_0^{\pi}\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\triangle_{11}A_{jk}(j+1)F_j(x)(k+1)F_k(y)\right| dxdy$$

$$= \pi^2\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(j+1)(k+1)\triangle_{11}A_{jk}\right| = o(1),$$

$$\int_0^\pi \int_0^\pi \left| \sum_{j=m}^{\infty} \triangle_{10} A_{jN} (j+1) F_j(x) (N+1) F_N(y) \right| dxdy$$
$$= \pi^2 \left| \sum_{j=m}^{\infty} \sum_{k=N}^{\infty} (j+1)(k+1) \triangle_{11} A_{jk} \right| = o(1),$$

Similarly,

$$\int_0^\pi \int_0^\pi \left| \sum_{k=n}^{\infty} \triangle_{01} A_{Mk} (M+1) F_M(x) (k+1) F_k(y) \right| dxdy$$
$$= \pi^2 \left| \sum_{j=M}^{\infty} \sum_{k=n}^{\infty} (j+1)(k+1) \triangle_{11} A_{jk} \right| = o(1).$$

Further, from (9.2.3), the last term on the right hand side of the (9.2.4) is of $o(1)$. Hence, $f \in L^1(Q)$.

Now, it remains to show that $||S_{mn} - f|| = o(1), \;\; as \;\; m, n \to \infty$. Therefore, consider

$$||f - S_{mn}|| = \int_0^\pi \int_0^\pi \left| \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} a_{jk} \cos jx \cos ky \right| dx \; dy$$

Applying double summation by parts, we get

$$(9.2.5) \qquad ||f - S_{mn}|| \;\; \leq \;\; \int_0^\pi \int_0^\pi \left| \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} \triangle_{11} a_{jk} D_j(x) D_k(y) \right| dx \;\; dy$$
$$+ \lim_{N \to \infty} \int_0^\pi \int_0^\pi \left| \sum_{j=m}^{\infty} \triangle_{10} a_{jN} D_j(x) D_N(y) \right| dx \;\; dy$$
$$+ \lim_{M \to \infty} \int_0^\pi \int_0^\pi \left| \sum_{k=n}^{\infty} \triangle_{01} a_{Mk} D_M(x) D_k(y) \right| dx \;\; dy$$
$$+ \lim_{M,N \to \infty} \int_0^\pi \int_0^\pi |a_{MN} D_M(x) D_N(y)| \, dx \;\; dy$$

Since $\{a_{jk}\} \in S_d^2$ and $\int_0^\pi |D_n(x)| dx \approx \log n$, the 1st, 2nd and 3rd terms on the right hand side of inequality (9.2.5) are $o(1) \;\; as \;\; m, n \to \infty$.

Thus the conclusion of the theorem follows if and only if (9.2.1) holds. □

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