SCALAR AND VECTOR VALUED RADIAL BASIS FUNCTIONS

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CERTIFICATE

I hereby certify that the dissertation entitled, **"Scalar and Vector Valued Radial Basis Functions"**, which is being submitted by **Miss. Jaswinder Kaur** (Roll No. 301403006), in the partial fulfillment of the requirement for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Kavita** during the period from January 2016 to July 2016.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/ Institute for the award of any degree.

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ABSTRACT

Approximations of the differential operators such as laplacian operator, gradient operator, divergence operator, curl operator are important in numerical analysis. For example we can use these approximations for solving partial differential equations. Many times, their approximations on the sphere are required.

In this thesis Wendland's radial basis functions are used for the approximations of differential operators. Firstly we study about the scalar radial basis functions and its interpolation method. We explained differential operators in terms of cartesian and spherical polar coordinates and scalar valued Wendland's radial basis functions which are used for the approximations of these operators on the sphere. Based of these radial basis functions, the approximations are computed.

Next we considered divergence-free matrix valued radial basis functions generated by compactly supported scalar Wendland's radial basis functions.

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1

Introduction

In the present age, when computers are applied almost everywhere in science, engineering and all around us in day-to-day life, it becomes more and more important to implement mathematical functions for efficient evaluation in computer programs. It is usually necessary for this purpose to use all kinds of approximations rather than their exact mathematical form.

The reason for approximating the function is that in many instances it is not possible to implement the function exactly, for instance when they are only represented by an infinite expansion. Furthermore, the function we want to use may not be completely known to us, or may be too expensive or demanding of computer time and memory to compute in advance. There are many techniques for approximation of functions such as least square approximation, regression etc. Interpolation is one of the technique used for approximation.

Interpolation

Interpolation is a method of constructing new data points within a range of a discrete set of known data points. In engineering and science, one often has a number of data points, obtained by experimentation, which represent the value of the independent variable. It is often required to interpolate the value of that function for an intermediate value of the independent variable. This may be achieved by interpolation.

Interpolation is done by generating a function which best fits the known points. Interpolation can be done by using different interpolants. For instance, rational interpolation is interpolation by rational functions, and trigonometric interpolation is interpolation by trigonometric polynomials, interpolation by wavelets etc. Following interpolating methods are the most popular:

1. Lagrange's Interpolation
2. Newton's Interpolation
3. Spline Interpolation

Langrange's Interpolation

Suppose that our data pairs are $(x_0, f(x_0)), (x_1, f(x_1)), ..., (x_n, f(x_n))$. The Langrange's polynomial is given by:

$$\begin{aligned} P(x) &= \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} f(x_1) \\ &+ ... + \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n_1})} f(x_n). \end{aligned}$$

In short, it can be written as

$$P(x) = \sum_{j=1}^{n} P_j(x)$$

where

$$P_j(x) = y_j \prod_{\substack{k=1, \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

where $y_j = f(x_j)$.

Hence, by putting the value of x_j and y_j we can calculate the value of x at any unknown point.

Newton's Interpolation

Another popular method for interpolation is Newton's method. There are three types of Newton's method:

1. Forward difference interpolation formula
2. Backward difference interpolation formula
3. Divided difference interpolation formula

Forward difference interpolation formula is:

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + ... + \frac{p(p-1)...(p-n+1)}{n!}\Delta^n y_0,$$

where Δ is the forward difference operator.

$$\begin{aligned}\Delta y_i &= y_{i+1} - y_i,\\ \Delta^k y_i &= \Delta^{k-1}(y_{i+1} - y_i), \qquad i = 0, 1, ..., n-1.\end{aligned}$$

and

$$p = \frac{x - x_0}{h}.$$

Backward difference interpolation formula is:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + ... + \frac{p(p+1)...(p+n-1)}{n!}\nabla^n y_n,$$

where ∇ is backward difference operator.

$$\begin{aligned}\nabla y_i &= y_i - y_{i-1},\\ \nabla^k y_i &= \nabla(y_i - y_{i-1}), \qquad i = 0, 1, ..., n-1.\end{aligned}$$

and

$$p = \frac{x - x_n}{h}.$$

Divided difference interpolation formula is:
A divided difference is defined as the difference in the function values at two points, divided by the difference in the values of the corresponding independent variable.
In general,

$$f[x_0, x_1 ... x_n] = \frac{f[x_1, x_2, ..., x_n] - f[x_0, x_1, ..., x_{n-1}]}{x_n - x_0},$$

and the corresponding interpolation formula is

$$\begin{aligned} P(x) &= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &+ ... + (x - x_0)(x - x_1)...(x - x_{n-1})f[x_0, ..., x_n]. \end{aligned}$$

Spline Interpolation:
Spline interpolation uses a number of polynomial functions to interpolate a set of data points with each polynomial for two adjacent data points. The Spline method is necessary because sometimes when the order of the polynomial become large, polynomial interpolation shows oscillatory behavior.
Spline method is not another method for finding polynomial interpolation of a discrete function, but instead it results in a piecewise polynomial (splines) in order to avoid the oscillatory behavior. The most common spline interpolations are linear, quadratic and cubic splines.

There are many methods for interpolation in more than one dimension. For instance, polynomial interpolation, piecewise polynomial splines i.e., piecewise polynomial method, and some other non-polynomial methods etc. The main disadvantages of these methods is that they normally do not give an explicit analytic expression for one approximant for all the provided data at once. In conclusion, we can say that there are many approximation methods, however, these methods require much set-up work especially in more than two dimensions, and this is a strong argument in favour of use of radial basis functions(RBFs) which are used in this thesis.
RBF method is one of the primary tools for interpolating multidimensional scattered

data. The method's ability to handle arbitrarily scattered data, to easily generalize to several space dimensions, and to provide spectral accuracy have made it popular in several different types of applications. Some of the recent of these applications include neural networks, medical imaging and the solution of partial differential equations (PDEs).
The main feature of RBF method is the fact that a unique interpolant is often guranteed under rather mild conditions on data points. In several cases, the only restrictions are that there are atleast two data points and they all are distinct.
Among all the RBFs, the multiquadric is the best known one and best understood, and very often used. One reason for this is its versatility due to an adjustable parameter c which may sometimes be used to improve accuracy or stability of approximations with multiquadric functions.

The RBFs were introduced in 1971 by R. Hardy [6] where he introduced multiquadric (MQ) RBF interpolation method. In 1982, R. Franke made a report on 32 most commonly used interpolation methods [3], and concluded that MQ interpolation method is best one. This made MQ interpolation method very popular. R. Franke also conjectured the non singularity of the interpolation matrices associated with MQ RBFs and this conjecture was proved by C.A. Micchelli in 1986 [9]. After that the interpolation methods based on other RBFs such as thin plate spline, the Gaussian, the cubic etc. were proposed. In 1990s, the RBF method once again came into light when E.J. Kansa used this method to solve PDEs [7][8].
RBFs are means to approximate multivariate functions by linear combinations of terms based on a single univariate function. They are usually applied to approximate functions or data which are only known at a finite number of points.
The advantage in using RBFs is generally that they provide interpolants irrespective of the geometry of the centres x_j and for any dimension n, whereas for instance polynomial interpolation or even piecewise polynomials are no longer available or feasible in practical applications when n is very large or the geometry of x_j is complicated.
The main virtue of RBFs, namely being readily available for interpolation even in high dimensional settings, is important, for instance, in neural network applications, where

the dimension of the underlying approximation problem is often very large. Another attractive feature is that many of them are related to elliptic differential operators, in that they are Green's functions to, say, Laplace or other radially invariant operators. The thin-plate spline $\phi(r) = r^2 logr$ is a prime example for that, as it is a fundamental solution of the two-dimensional biharmonic operator.

Other currently used choices of ϕ that give good, accurate approximations if the centres are close enough together include the multiquadric radial function $\phi(r) = \sqrt{r^2 + c^2}$, c being a nonnegative parameter, and Gaussians $\phi(r) = e^{-c^2 r^2}$, c a positive parameter. Since the Gaussian RBF is so well localised in space, the parameter c in it should normally be dependent on the distances of points within x_j. As multiquadrics give invertible matrices A for all dimensions, all sets of distinct centres and all parameters, as do indeed the Gaussians, but the latter have the additional strong advantage that they give a positive definite, essentially banded interpolation matrix. In fact, the banded structure of the matrix A becomes more dominant if the parameter c in the Gaussian radial function is large, but this parameter pits locality against the accuracy of the approximation.
Some of the RBFs are given by:-

Distance

$$\phi(r) = r$$

Cubic power

$$\phi(r) = r^3$$

Gaussian

$$\phi(r) = \phi(\|x - c\|) = \exp^{-} \frac{r^2}{a^2}$$

Multiquadrics

$$\phi(r) = \phi(\|x - c\|) = \sqrt{r^2 + a^2}$$

Inverse Multiqudrics

$$\phi(r) = \phi(\|x - c\|) = \frac{1}{\sqrt{r^2 + a^2}}$$

For a given function $f \in C(\mathbb{R}^d)$ and a set of points $\tilde{X} = x_1, x_2, ..., x_N \subset \mathbb{R}^d$, the RBF interpolant $I_{\tilde{X}} f$ is given by

$$I_{\tilde{X}} f(x) = \sum_{i=1}^{N} \tilde{f^j} \phi_j(x),$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a RBF $\Phi(x) = \phi(\|x\|_2)$. The coefficients $\tilde{f^j} s$ are computed using the interpolation condition

$$I_{\tilde{X}} f(x_j) = f(x_j)$$

The large size of set $\tilde{X}$ makes it desirable to use compactly supported RBFs of the simplest possible form. If RBFs has compact support, the interpolation matrices involved are sparse and the corresponding algorithms are efficient [2]. Another advantage of comapctly supported RBFs is the principle of locality, which means the changes of one center x_j causes only a local change in the interpolant. Both the properties of the compactly supported RBFs are desirable for numerical treatment of PDEs.

One more reason for the requirement of compact support may be that there are such masses of data, which need to be interpolated, that even exponential or quick algebraic decay is not sufficient to localise the method well enough so as to provide stable and fast computations.

Compactly supported RBFs are useful especially when the number of data or evaluations of the interpolant is massive so that any basis functions of global support incur prohibitive cost for evaluation. Many of these RBFs are piecewise polynomials, and not all of them have similar nonsingularity properties for the interpolation problems. Moreover, RBFs with compact support are suitable for, and are used in solving linear PDEs by Galerkin methods. They provide a suitable replacement for the standard piecewise polynomial finite elements. It turns out that they can be just as good as means for approximation, while not requiring any triangulation or mesh, so they allow meshless approximation which is easier when the amount of data has to be continuously enlarged or made smaller. That is often the case when PDEs are solved numerically. By contrast, finite elements can be difficult to compute in three or more dimensions for scattered data due to the necessity of triangulating the domain before using finite elements and due to the complicated spaces of piecewise polynomials in more than two dimensions. While many such powerful theoretical results exist for RBFs.

Recently, radial functions of compact support are now being proposed that also give positive definite matrices A and have genuinely banded interpolation matrices. Early examples of radial functions with compact support that have a simple piecewise polynomial structure are due to [14], one of the first papers on compactly supported radial functions, further discussed in [15], and in Wendland [12][13], who established several of their special properties, such as certain optimality facts about their degree and smoothness.

Therefore, we focus on compactly supported and in particular Wendland's compactly supported RBFs in this thesis.

1.1 Radial Basis Functions

Let X be a manifold and $\tilde{X} = \{x_1, x_2, ..., x_n\}$ be a set of scattered distinct points on X. The spherical basis functions are derived from bizonal kernels $\Phi : X \times X \to \mathbb{R}$ of the form $\Phi(x, y) = \varphi(x.y), \ \ x, y \in X$ here φ is a univariate function defined on [-1,1] and $x.y$ is Euclidean dot product of position vectors of the points $x, y \in X$.

For a fixed value of x the value of $\Phi(x, y)$ depends on the geodesic distance from x to y, hence the function $\Phi(x, .)$ is radially symmetric function with respect to the point x and is called RBF [4]. For every point $x_j \in \tilde{X}$, we have a RBF defined as

$$\Phi_j(x) = \Phi(x, x_j) = \varphi(x.x_j) = \psi(|x - x_j|) \tag{1.1}$$

One special type of bizonal kernels is

$$\Phi(x, y) = \varphi(x.y) = \sum_{l=0}^{\infty} a_l P_l(n + 1; x.y), \quad a_l > 0, \quad \sum_{l=0}^{\infty} a_l < \infty \tag{1.2}$$

where $\{P_l(n+1; t)\}_{l=0}^{\infty}$ is a sequence of $(n+1)$-dimensional Legendre polynomials. Such a Φ is positive definite on X [10], i.e., the matrix $A = [\Phi_j(x_i)]_{i,j=1}^{N}$ is positive semi definite for every set of distinct points $\{x_1, x_2, ..., x_N\}$ on X and for every positive integer N.

When the coefficients a_l in the equation (1.2) are positive for every l, Φ is said to be positive definite [16]. In this case the matrix A becomes positive definite for every set of distinct points $\{x_1, x_2, ..., x_N\}$ on X and for every positive integer N. These bizonal functions are used to derive Wendland's RBFs.

1.1.1 Basic RBF interpolation method

Given a function f on X, there uniquely exists a sequence of numbers $\{\tilde{f^j}\}_{j=1}^{m}$ such that the function

$$I_{\tilde{X}}f(x) = \sum_{i=1}^{N} \tilde{f^j}\Phi_j(x), \tag{1.3}$$

satisfies the interpolating condition

$$I_{\tilde{X}}f(x_k) = f(x_k),$$

Put $x = x_1, x_2, ..., x_N$ in equation (1.3) we get

$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix} = \begin{pmatrix} \Phi_1(x_1) & \Phi_2(x_1) & \dots & \Phi_N(x_1) \\ \Phi_1(x_2) & \Phi_2(x_2) & \dots & \Phi_N(x_1) \\ \vdots & \vdots & & \vdots \\ \Phi_1(x_N) & \Phi_2(x_2) & \dots & \Phi_N(x_1) \end{pmatrix} \begin{pmatrix} \tilde{f}^1 \\ \tilde{f}^2 \\ \vdots \\ \tilde{f}^N \end{pmatrix}, \tag{1.4}$$

which can be written in the form as

$$f = A\tilde{f}, \tag{1.5}$$

Now since the matrix A is invertible, we have

$$\tilde{f} = A^{-1}f,$$

where $f = [f(x_1), f(x_2), ..., f(x_N)]'$ etc. Hence we have

$$f(x) \approx I_{\tilde{X}}f(x) = \sum_{i=1}^{N} \tilde{f^j}\Phi_j(x). \tag{1.6}$$

There are plenty of choices for choosing the function φ and hence the function ψ of the equation (1.1) and we have chosen it as Wendland's functions.

2

Differential Operators

An operator is a transformation that transforms a function into another function and a differential operator is one in which derivatives are involved. The differential operator given in Cartesian coordinates $\{x, y, z\}$ on three-dimensional Euclidean space as

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

was introduced in 1837 by the Irish mathematician and physicist William Rowan Hamilton which is denoted with the symbol of nabla. The nabla is a triangular symbol like an inverted Greek delta: ∇. The name comes, by reason of the symbol's shape.
Del, or nabla, is an operator used in mathematics, in particular, in vector calculus, as a vector differential operator. When applied to a function defined on a one-dimensional domain, it denotes its standard derivative as defined in calculus. When applied to a

field (a function defined on a multi-dimensional domain), del may denote the gradient (locally steepest slope) of a scalar field (or sometimes of a vector field, as in the Naiver Stokes equations), the divergence of a vector field, or the curl (rotation) of a vector field, depending on the way it is applied.

Del is not a specific operator, but rather a convenient mathematical notation for those three operators, that makes many equations easier to write and remember. The del symbol can be interpreted as a vector of partial derivative operators, and its three possible meanings: gradient, divergence, and curl can be formally viewed as the product with a scalar, dot product, and cross product, respectively, of the del operator with the field.

We will discuss some differential operators such as laplacian, gradient, divergence and curl.

2.1 Laplacian Operator

In terms of cartesian coordinates laplacian operator is defined as,

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}.$$

In terms of spherical polar coordinates this is given by,

$$\nabla^2 A = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A}{\partial \phi^2}.$$

2.1.1 Laplacian in terms of spherical polar coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \end{aligned} \tag{2.1}$$

where

$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = arc\cos\frac{z}{r}, \qquad \phi = arc\tan\frac{y}{x}. \tag{2.2}$$

Laplacian in polar coordinates is defined as

$$
\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, &(2.3)\\
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}.\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}.\frac{\partial \phi}{\partial x}, &(2.4)\\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}.\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}.\frac{\partial \phi}{\partial y}, &(2.5)\\
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}.\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}.\frac{\partial \phi}{\partial z}. &(2.6)
\end{aligned}
$$

The partial derivatives of r, θ and ϕ is given by

$$
\begin{aligned}
\frac{\partial r}{\partial x} &= \sin\theta\cos\phi,\\
\frac{\partial \theta}{\partial x} &= \frac{1}{r}\cos\theta\cos\phi,\\
\frac{\partial \phi}{\partial x} &= -\frac{1}{r}\frac{\sin\phi}{\sin\theta},\\
\frac{\partial r}{\partial y} &= \sin\theta\sin\phi,\\
\frac{\partial \theta}{\partial y} &= \frac{1}{r}\cos\theta\sin\phi,\\
\frac{\partial \phi}{\partial y} &= \frac{1}{r}\frac{\cos\phi}{\sin\theta},\\
\frac{\partial r}{\partial z} &= \cos\theta,\\
\frac{\partial \theta}{\partial z} &= -\frac{1}{r}\sin\theta,\\
\frac{\partial \phi}{\partial z} &= 0.
\end{aligned} \tag{2.7}
$$

Using equation (2.4) and equation (2.7), we get

$$
\frac{\partial f}{\partial x} = \sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial f}{\partial \theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial f}{\partial \phi}.
$$

Replace f by $\frac{\partial f}{\partial x}$ in above equation

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial\theta}\right) - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial\phi}\right),\\
\frac{\partial^2 f}{\partial x^2} &= \sin\theta\cos\phi\frac{\partial}{\partial r}\left[\sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial f}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial f}{\partial\phi}\right]\\
&+ \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}\left[\sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial f}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial f}{\partial\phi}\right]\\
&- \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\left[\sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial f}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial f}{\partial\phi}\right].
\end{aligned}$$

On solving above equation, we get

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \Bigg[\sin^2\theta\cos^2\phi\frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\theta\cos^2\phi\frac{\partial f}{\partial\theta} + \frac{1}{r}\cos\theta\sin\theta\cos^2\phi\frac{\partial^2 f}{\partial r\partial\theta} +\\
&\frac{1}{r^2}\sin\phi\cos\phi\frac{\partial f}{\partial\phi} - \frac{1}{r}\sin\phi\cos\phi\frac{\partial^2 f}{\partial\phi\partial r}\Bigg]\\
&+ \Bigg[\frac{1}{r}\sin\theta\cos\theta\cos^2\phi\frac{\partial^2 f}{\partial r\partial\theta} - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\phi\frac{\partial f}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\cos^2\phi\frac{\partial^2 f}{\partial\theta^2} +\\
&\frac{1}{r^2}\frac{\sin\phi\cos\phi\cos^2\theta}{\sin^2\theta}\frac{\partial f}{\partial\phi} - \frac{1}{r^2}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\frac{\partial^2 f}{\partial\theta\partial\phi}\Bigg]\\
&+ \Bigg[-\frac{1}{r}\sin\phi\cos\phi\frac{\partial^2 f}{\partial r\partial\phi} + \frac{1}{r^2}\frac{\cos\theta\sin^2\phi}{\sin\theta}\frac{\partial f}{\partial\theta} - \frac{1}{r^2}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\frac{\partial^2 f}{\partial\theta\partial\phi} +\\
&\frac{1}{r^2}\frac{\sin\phi\cos\phi}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi} + \frac{1}{r^2}\frac{\sin^2\phi}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\Bigg]. \qquad (2.8)
\end{aligned}$$

Similarly we do for $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial z^2}$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial f}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial f}{\partial\phi},\\
\frac{\partial^2 f}{\partial y^2} &= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)\\
&+ \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial y}\right),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \Bigg[\sin^2\theta\sin^2\phi\frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\theta\sin^2\phi\frac{\partial f}{\partial\theta} + \frac{1}{r}\cos\theta\sin\theta\sin^2\phi\frac{\partial^2 f}{\partial r\partial\theta} - \\
&\quad \frac{1}{r^2}\sin\phi\cos\phi\frac{\partial f}{\partial\phi} + \frac{1}{r}\sin\phi\cos\phi\frac{\partial^2 f}{\partial r\partial\phi}\Bigg] \\
&+ \Bigg[\frac{1}{r}\cos^2\theta\sin^2\phi\frac{\partial f}{\partial r} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\phi\frac{\partial f}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\sin^2\phi\frac{\partial^2 f}{\partial\theta^2} - \\
&\quad \frac{1}{r^2}\frac{\sin\phi\cos\phi\cos^2\theta}{\sin^2\theta}\frac{\partial f}{\partial\phi} + \frac{1}{r^2}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\frac{\partial^2 f}{\partial\theta\partial\phi}\Bigg] \\
&+ \Bigg[\frac{1}{r}\cos\phi\sin\phi\frac{\partial^2 f}{\partial r\partial\phi} + \frac{1}{r^2}\frac{\cos\theta\cos^2\phi}{\sin\theta}\frac{\partial f}{\partial\theta} + \frac{1}{r^2}\frac{\cos\theta\cos\phi\sin\phi}{\sin\theta}\frac{\partial^2 f}{\partial\theta\partial\phi} - \\
&\quad \frac{1}{r^2}\frac{\sin\phi\cos\phi}{\sin^2\theta}\frac{\partial f}{\partial\phi} + \frac{1}{r^2}\frac{\cos^2\phi}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\Bigg]. \qquad (2.9)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{\partial f}{\partial z} &= \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial\theta}, \\
\frac{\partial^2 f}{\partial z^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right) - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial z}\right), \\
\frac{\partial^2 f}{\partial z^2} &= \cos\theta\frac{\partial}{\partial r}\left[\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial\theta}\right] - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left[\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial\theta}\right], \\
\frac{\partial^2 f}{\partial z^2} &= \left[\cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial f}{\partial\theta} - \frac{1}{r}\cos\theta\sin\theta\frac{\partial^2 f}{\partial r\partial\theta}\right] \\
&+ \left[\frac{1}{r}\sin^2\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2 f}{\partial r\partial\theta} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial f}{\partial\theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2 f}{\partial\theta^2}\right]. \quad (2.10)
\end{aligned}
$$

On substituting the value of $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial z^2}$ in equation (2.3) and after simplifications, we get

$$
\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial f}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial f}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.
$$

2.2 Gradient Operator

Gradient is a generalization of the usual concept of derivative of a function in one dimension to a function in several dimensions. In terms of cartesian coordinates, this is given by

$$
\vec{\nabla}A = \frac{\partial A}{\partial x}\hat{\imath} + \frac{\partial A}{\partial y}\hat{\jmath} + \frac{\partial A}{\partial z}\hat{k},
$$

where $A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$.

In terms of spherical polar coordinates this is given by

$$\vec{\nabla}A = \frac{\partial A}{\partial r}\hat{r} + \frac{\partial A}{\partial \theta}\hat{\theta} + \frac{\partial A}{\partial \phi}\hat{\phi},$$

where $A = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$.

2.2.1 Gradient in terms of spherical polar coordinates

The value of unit vectors in terms of spherical polar coordinates is given by

$$\begin{aligned}
\hat{i} &= \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}, \\
\hat{j} &= \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}, \qquad (2.11) \\
\hat{k} &= \cos\theta\hat{r} - \sin\theta\hat{\theta}.
\end{aligned}$$

Gradient in spherical polar coordinates is defined as

$$\vec{\nabla}A = \frac{\partial A}{\partial r}\hat{r} + \frac{\partial A}{\partial \theta}\hat{\theta} + \frac{\partial A}{\partial \phi}\hat{\phi}. \qquad (2.12)$$

Gradient in terms of cartesian coordinates is defined as

$$\vec{\nabla}A = \frac{\partial A}{\partial x}\hat{i} + \frac{\partial A}{\partial y}\hat{j} + \frac{\partial A}{\partial z}\hat{k}, \qquad (2.13)$$

or

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}. \qquad (2.14)$$

Now on changing equation (2.14) into spherical polar coordinates, we can write it as

$$\begin{aligned}
\vec{\nabla} &= \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\right)\hat{i} + \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right)\hat{j} \\
&+ \left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}\right)\hat{k}. \qquad (2.15)
\end{aligned}$$

By substituting the values of partial derivatives of r, θ and ϕ, unit vectors in terms of spherical polar coordinates and using equations (2.1), (2.2), (2.7) defined in section(2.1) in equation (2.15), we get

$$
\begin{aligned}
\vec{\nabla} &= \left[(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi})(\sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi})\right] \\
&+ \left[(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi})(\sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi})\right] \\
&+ \left[(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta})(\cos\theta\hat{r} - \sin\theta\hat{\theta})\right], \\
&= \Big[\sin^2\theta\cos^2\phi\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\phi\cos\phi\frac{\partial}{\partial\phi} + \sin^2\theta\sin^2\phi\frac{\partial}{\partial r} \\
&\quad + \frac{1}{r}\sin\theta\cos\theta\sin^2\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\sin\phi\cos\phi\frac{\partial}{\partial\phi} + \cos^2\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial}{\partial\theta}\Big]\hat{r} \\
&+ \Big[(\sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos^2\theta\cos^2\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi\cos\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\theta\cos\theta\sin^2\phi\frac{\partial}{\partial r} \\
&\quad + \frac{1}{r}\cos^2\theta\sin^2\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\sin\phi\cos\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}) - \sin\theta\cos\theta\frac{\partial}{\partial r} + \frac{1}{r}\sin^2\theta\frac{\partial}{\partial\theta}\Big]\hat{\theta} \\
&+ \Big[-\sin\theta\sin\phi\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\cos\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\sin^2\phi}{\sin\theta}\frac{\partial}{\partial\phi} \\
&\quad + \sin\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\sin\phi\cos\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos^2\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big]\hat{\phi}, \\
&= \left[\left(\frac{\partial}{\partial r}\right)\hat{r} + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\hat{\theta} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\hat{\phi}\right].
\end{aligned}
\tag{2.16}
$$

2.3 Divergence Operator

In terms of cartesian coordinates divergence operator is defined as,

$$
\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.
$$

where $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$.

In terms of spherical polar coordinates this is given by,

$$
\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}.
$$

where $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$.

2.3.1 Divergence in terms of spherical polar coordinates

In terms of spherical polar coordinates,

$$\begin{aligned}\vec{\nabla}.\vec{A} &= \left[\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)\hat{i}+\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)\hat{j}\right.\\ &\left.+\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right)\hat{k}\right].\left[A_r\hat{r}+A_\theta\hat{\theta}+A_\phi\hat{\phi}\right],\end{aligned}$$

By using equation (2.16) in equation, we get

$$\begin{aligned}\vec{\nabla}.\vec{A} &= \left[\left(\frac{\partial}{\partial r}\right)\hat{r}+\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\hat{\theta}+\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\hat{\phi}\right].\left[A_r\hat{r}+A_\theta\hat{\theta}+A_\phi\hat{\phi}\right],\\ &= \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}. \qquad (2.17)\end{aligned}$$

2.4 Curl Operator

In terms of cartesian coordinates curl operator is defined as,

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k}.$$

In terms of spherical polar coordinates this is given by,

$$\begin{aligned}\vec{\nabla}\times\vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial\theta}-\frac{\partial A_\theta}{\partial\phi}\right)\hat{r}+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial(rA_\phi)}{\partial r}\right)\hat{\theta}\\ &+ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)\hat{\phi}.\end{aligned}$$

2.4.1 Curl in terms of spherical polar coordinates

In terms of spherical polar coordinates,

$$
\begin{aligned}
\vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ A_r & A_\theta & A_\phi \end{vmatrix}, \\
\vec{\nabla} \times \vec{A} &= \left(\frac{1}{r}\frac{\partial(A_\phi)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\hat{r} - \left(\frac{\partial(A_\phi)}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}\right)\hat{\theta} \\
&+ \left(\frac{\partial(A_\theta)}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\theta}\right)\hat{\phi}, \\
\vec{\nabla} \times \vec{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\theta} \\
&+ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}. \qquad (2.18)
\end{aligned}
$$

3

Approximations of Operators using Scalar Valued Wendland's RBFs

Wendland RBFs are compactly supported. This means that these RBFs can lead to a sparse linear system of equations for solving the interpolation coefficients.
Wendland's RBF(ψ) depends upon two parameters d and k, where d is the dimension of the space and $2k$ is the order of the smoothness of the function ψ. Since ψ depends on d and k, we will use the notation $\psi_{d,k}$ to denote it. It is defined as follows

$$\psi_{d,k}(r) = \begin{cases} p_{d,k} & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if } r > 1 \end{cases}$$

where

$$p_{d,k}(r) = \sum_{j=0}^{l+2k} d_{j,k}^{(l)} r^j$$

with $l = \lfloor \frac{d}{2} \rfloor + k + 1$. The coefficients $d_{j,k}^{(l)}$ can be computed recursively for $0 \leq s \leq k-1$ as follows:

$$\begin{aligned}
d_{j,0}^{(l)} &= (-1)^j \\
d_{0,s+1}^{(l)} &= \sum_{j=0}^{l+2s} \frac{d_{j,s}^{(l)}}{j+2}, \qquad s \geq 0. \\
d_{1,s+1}^{(l)} &= 0, \qquad s \geq 0. \\
d_{j,s+1}^{(l)} &= -\frac{d_{j-2,s}}{j}, \qquad s \geq 0, \quad 2 \leq j \leq l+2s+2.
\end{aligned}$$

3.1 Other methods for approximating differential operators

3.1.1 Finite Differences

Finite-difference methods (FDM) are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. FDMs are thus discretization methods. The principle of finite difference methods is close to the numerical schemes used to solve ordinary differential equations. It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space and in time or time points. The error between the numerical solution and exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The truncation reflects the fact that a finite part of Taylor series is used in the approximation.

3.1.2 Finite Element

The finite element method (FEM) is a numerical technique for finding approximate solutions to boundary value problems for PDEs. It is also referred to as finite element analysis (FEA). FEM subdivides a large problem into smaller, simpler, parts, called finite elements. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. FEM then uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function.

3.1.3 Finite Volume

The finite-volume method (FVM) is a method for representing and evaluating PDEs in the form of algebraic equations. Similar to the finite difference method or finite element method, values are calculated at discrete places on a meshed geometry. Finite volume refers to the small volume surrounding each node point on a mesh. In the finite volume method, volume integrals in a PDE that contain a divergence term are converted to surface integrals, using the divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, therefore these methods are conservative.

3.2 Approximation of Laplace-Beltrami Operator on the Sphere

We have the equation of the form,

$$f(x) \approx I_{\tilde{X}} f(x) = \sum_{j=1}^{N} \tilde{f}^{j} \Phi_j(x).$$

Applying Laplace-Beltrami operator on both sides of above equation, we obtain

$$\begin{aligned}
\nabla^2 f(x) &\approx \sum_{j=1}^{N} \tilde{f}^j \nabla^2 \Phi_j(x) \\
&= B\tilde{f}, \\
&= BA^{-1}f, \\
&= D^2 f,
\end{aligned}$$

where $B = [\nabla^2 \Phi_j(x_i)]_{i,j=1}^{N}$. It is clear that the matrix $D^2 = BA^{-1}$ is an approximation of the Laplace-Beltrami operator ∇^2 on the sphere.

3.2.1 Computation of the entries of the matrix B

The ij^{th} entry of the matrix B is

$$\nabla^2 \phi_j(x_i) = \nabla^2 \Phi(x_i, x_j) = \nabla^2 \varphi(x_i.x_j) = \nabla^2 \psi(|x_i - x_j|) = \nabla^2 \psi(\sqrt{2 - 2x.x_j}).$$

For a fixed $x_j = (a, b, c)$ with $a^2 + b^2 + c^2 = 1$, we calculate $\nabla^2 \psi \sqrt{2 - 2x.x_j}$. If $x = (y_1, y_2, y_3)$ then

$$\nabla^2 \psi(\sqrt{2 - 2x.x_j}) = \nabla^2 \psi(\sqrt{2 - 2(ay_1 + by_2 + cy_3)}).$$

In spherical polar coordinates with radial distance R, polar angle θ and azimuthal angle ϕ, we have

$$\begin{aligned}
y_1 &= R \sin\theta \cos\phi, \\
y_2 &= R \sin\theta \sin\phi, \\
y_3 &= R \cos\theta.
\end{aligned}$$

On a unit sphere $R = 1$, hence $y_1 = \sin\theta\cos\phi$, $y_2 = \sin\theta\sin\phi$, $y_3 = \cos\theta$. Therefore, we need to compute $\nabla^2 \psi(\sqrt{2 - 2(a\sin\theta\cos\phi + b\sin\theta\sin\phi + c\cos\theta)})$. Laplace-Beltrami operator $\nabla^2 \psi$ in spherical polar coordinates on a unit sphere is given by

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta} \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}. \tag{3.1}$$

Let us start with $\psi = \psi_{3,1}$ and with this choice

$$\begin{aligned}
\psi(\theta,\phi) &= (1-r)_+^4(4r+1), \\
r &= \sqrt{2-2t}, \\
t &= a\sin\theta\cos\phi + b\sin\theta\sin\phi + c\cos\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial\psi}{\partial r} &= -\begin{cases} -20r(1-r)^3 & \text{if } r<1 \\ 0 & otherwise \end{cases}, \\
\frac{\partial r}{\partial t} &= \frac{-2}{2\sqrt{2-2t}} = \frac{-1}{r},
\end{aligned}$$

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial t}\frac{\partial t}{\partial\theta} = \begin{cases} 20r(1-r)^3\frac{\partial t}{\partial\theta} & \text{if } r<1 \\ 0 & otherwise \end{cases}, \tag{3.2}$$

$$\frac{\partial^2\psi}{\partial\theta^2} = \begin{cases} 60\frac{(1-r)^2}{r} & \text{if } r<1 \\ 0 & otherwise \end{cases}, \tag{3.3}$$

$$\frac{\partial\psi}{\partial\phi} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial t}\frac{\partial t}{\partial\phi} = \begin{cases} 20r(1-r)^3\frac{\partial t}{\partial\phi} & \text{if } r<1 \\ 0 & otherwise \end{cases}, \tag{3.4}$$

$$\frac{\partial^2\psi}{\partial\phi^2} = \begin{cases} 60\frac{(1-r)^2}{r}(\frac{\partial t}{\partial\phi})^2 + 20(1-r)^3\frac{\partial^2 t}{\partial\phi^2} & \text{if } r<1 \\ 0 & otherwise \end{cases}. \tag{3.5}$$

Insert the values from equation (3.2), equation (3.3) and equation (3.5) into the equation (3.1) to obtain

$$\nabla^2\psi = \begin{cases} 20(1-r)^3(\frac{\partial^2 t}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial t}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}) + 60\frac{(1-r)^2}{r}((\frac{\partial t}{\partial\theta})^2 + \frac{1}{\sin^2\theta}(\frac{\partial t}{\partial\phi})^2) & \text{if } r<1 \\ 0 & otherwise \end{cases} \tag{3.6}$$

Next

$$\begin{aligned}
\frac{\partial t}{\partial \theta} &= a\cos\theta\cos\phi + b\cos\theta\sin\phi - c\sin\theta, && (3.7)\\
\frac{\partial^2 t}{\partial^2 \theta} &= -a\sin\theta\cos\phi - b\sin\theta\sin\phi - c\cos\theta = -t, &&\\
\frac{\partial t}{\partial \phi} &= -a\sin\theta\sin\phi + b\sin\theta\cos\phi, && (3.8)\\
\frac{\partial^2 t}{\partial^2 \phi} &= -a\sin\theta\cos\phi - b\sin\theta\sin\phi, &&
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial^2 t}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial t}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2} &= -t + \frac{\cos\theta}{\sin\theta}(a\cos\theta\cos\phi + b\cos\theta\sin\phi - c\sin\theta)\\
&+ \frac{1}{\sin^2\theta}(-a\sin\theta\cos\phi - b\sin\theta\sin\phi),\\
&= -t - a\cos\phi\sin\theta - a\sin\phi\sin\theta - \cos\theta,\\
&= -t - t,\\
&= -2t,\\
&= r^2 - 2, \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial t}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial t}{\partial \phi}\right)^2 &= (a\cos\theta\cos\phi + b\cos\theta\sin\phi - c\sin\theta)^2\\
&+ \frac{1}{\sin^2\theta}(-a\sin\theta\sin\phi + b\sin\theta\cos\phi)^2,\\
&= -t^2 + a^2 + b^2 + c^2,\\
&= 1 - t^2,\\
&= \frac{r^2(4-r^2)}{4}. \qquad (3.10)
\end{aligned}$$

Insert the equation (3.9) and equation (3.10) into the equation (3.6) to obtain, for $r < 1$,

$$\begin{aligned}
\nabla^2\psi &= 20(1-r)^3(r^2-2) + 60\frac{(1-r)^2}{r}\frac{r^2(4-r^2)}{4},\\
&= 20(1-r)^3(r^2-2) + 15r(1-r)^2(4-r^2),\\
&= 5(1-r^2)(-7r^3 + 4r^2 + 20r - 8),
\end{aligned}$$

and 0 otherwise. Therefore we have

$$\nabla^2\psi_{3,1} = \begin{cases} 5(1-r)^2(-7r^3+4r^2+20r-8) & \text{if } r<1 \\ 0 & otherwise \end{cases}. \tag{3.11}$$

Moving on similar lines, we can obtain

$$\nabla^2\psi_{3,2} = \begin{cases} 28(1-r)^4(-25r^4+8r^3+82r^2-16r-4) & \text{if } r<1 \\ 0 & otherwise \end{cases}, \tag{3.12}$$

and

$$\nabla^2\psi_{3,3} = \begin{cases} 5(1-r)^6(-52r^5+3r^4+182r^3+7r^2-12r-2) & \text{if } r<1 \\ 0 & otherwise \end{cases}. \tag{3.13}$$

3.3 Approximation of Gradient Operator on the Sphere

We have the equation of the form,

$$f(x) \approx I_{\tilde{X}}f(x) = \sum_{j=1}^{N} \tilde{f}^j \Phi_j(x),$$

Applying gradient operator on both sides of above equation, we obtain

$$\begin{aligned} \vec{\nabla} f(x) &\approx \sum_{j=1}^{N} \tilde{f}^j \vec{\nabla}\Phi_j(x), \\ &= C\tilde{f}, \\ &= CA^{-1}f, \end{aligned}$$

where $C = [\vec{\nabla}\Phi_j(x_i)]_{i,j=1}^{N}$. It is clear that the matrix $\vec{\nabla}_{app} = CA^{-1}$ approximates the gradient operator $\vec{\nabla}$ on the sphere.

3.3.1 Computation of the entries of the matrix C

The ij^{th} entry of the matrix C is

$$\begin{aligned} \vec{\nabla}\phi_j(x_i) &= \vec{\nabla}\varphi(x_i.x_j), \\ &= \vec{\nabla}\psi(|x_i - x_j|), \\ &= \vec{\nabla}\psi(\sqrt{2-2x_i.x_j}). \end{aligned}$$

For a fixed $x_j = (a, b, c)$ with $a^2+b^2+c^2 = 1$, we calculate $\vec{\nabla}\psi\sqrt{2-2x.x_j}$=$\vec{\nabla}\psi(\sqrt{2-2(ay_1+by_2+cy_3)})$ =$\vec{\nabla}\psi(\sqrt{2-2(a\sin\theta\cos\phi+b\sin\theta\sin\phi+c\cos\theta)})$. The gradient of a function ψ in spherical polar coordinates on the unit sphere is given by

$$\vec{\nabla}\psi \;=\; \frac{\partial\psi}{\partial\theta}\hat{\theta}+\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}, \tag{3.14}$$

where $\hat{\theta}$ and $\hat{\phi}$ are unit vectors in θ and ϕ directions respectively. We will start with $\psi = \psi_{3,1}$. Using equation (3.2) and equation (3.4), we get

$$\vec{\nabla}\psi \;=\; \begin{cases} 20(1-r)^3(\frac{\partial t}{\partial\theta}\hat{\theta}+\frac{1}{\sin\theta}\frac{\partial t}{\partial\phi}\hat{\phi}) & \text{if } r<1 \\ 0 & otherwise \end{cases}. \tag{3.15}$$

Now using equation (3.7) and equation (3.8) in equation (3.15), we get

$$\vec{\nabla}\psi \;=\; \begin{cases} 20(1-r)^3\left((a\cos\theta\cos\phi+b\cos\theta\sin\phi-c\sin\theta)\hat{\theta}+\frac{1}{\sin\theta}(-a\sin\theta\sin\phi+b\sin\theta\cos\phi)\hat{\phi}\right) & \text{if } r<1 \\ 0 & otherwise \end{cases}$$

or

$$\vec{\nabla}\psi \;=\; \begin{cases} 20(1-r)^3\left(a(\cos\theta\cos\phi\hat{\theta}-\sin\phi\hat{\phi})+b(\cos\theta\sin\phi+\cos\phi\hat{\phi})-c\sin\theta\hat{\theta}\right) & \text{if } r<1 \\ 0 & otherwise \end{cases}.$$

The relations between the unit vectors in Cartesian coordinates, i.e., $\hat{\imath},\hat{\jmath},\hat{k}$ and the unit vectors in the spherical polar coordinates i.e., $\hat{r},\hat{\theta},\hat{\phi}$ are

$$\begin{aligned} \hat{\imath} &= \sin\theta\cos\phi\hat{r}+\cos\theta\cos\phi\hat{\theta}-\sin\phi\hat{\phi}, \\ \hat{\jmath} &= \sin\theta\sin\phi\hat{r}+\cos\theta\sin\phi\hat{\theta}+\cos\phi\hat{\phi}, \\ \hat{k} &= \cos\theta\hat{r}-\sin\theta\hat{\theta}, \\ \hat{r} &= \sin\theta\cos\phi\hat{\imath}+\sin\theta\sin\phi\hat{\jmath}+\cos\theta\hat{k}, \\ \hat{\theta} &= \cos\theta\cos\phi\hat{\imath}+\cos\theta\sin\phi\hat{\jmath}-\sin\theta\hat{k}, \\ \hat{\phi} &= -\sin\phi\hat{\imath}+\cos\phi\hat{\jmath}. \end{aligned}$$

Using these relations we obtain, for $r < 1$

$$\begin{aligned}
\vec{\nabla}\psi(\sqrt{2-2x.x_j}) &= 20(1-r)^3(a(\hat{x}-\sin\theta\cos\phi\hat{r}) + b(\hat{y}-\sin\theta\sin\phi\hat{r}) \\
&- c(\hat{z}-\cos\theta\hat{r})), \\
&= 20(1-r)^3(a\hat{i}+b\hat{j}+c\hat{k}-t\hat{r}), \\
&= 20(1-r)^3(\vec{x_j}-t(\sin\theta\cos\phi\hat{i}+\sin\theta\sin\phi\hat{j} \\
&+ \cos\theta\hat{k})), \\
&= 20(1-r)^3(\vec{x_j}-t\vec{x}), \qquad (3.16)
\end{aligned}$$

where $\vec{x_j} = a\hat{i}+b\hat{j}+c\hat{k}$ and $\vec{x} = y_1\hat{i}+y_2\hat{j}+y_3\hat{k}$ and 0 otherwise. Therefore, we have

$$\begin{aligned}
\vec{\nabla}\psi_{3,1} &= \begin{cases} 20(1-r)^3(\vec{x_j}-t\vec{x}) & \text{if } r<1 \\ 0 & otherwise \end{cases}, \\
\vec{\nabla}\psi_{3,2} &= \begin{cases} 56(1-r)^5(5r+1)(\vec{x_j}-t\vec{x}) & \text{if } r<1 \\ 0 & otherwise \end{cases}, \\
\vec{\nabla}\psi_{3,3} &= \begin{cases} 22(1-r)^7(16r^2+7r+1)(\vec{x_j}-t\vec{x}) & \text{if } r<1 \\ 0 & otherwise \end{cases}.
\end{aligned}$$

3.4 Approximation of Divergence Operator on the Sphere

Suppose we are given a vector valued function $\vec{f}(x) = f_R(x)\hat{R} + f_\theta(x)\hat{\theta} + f_\phi(x)\hat{\phi}$, where $\hat{R}$, $\hat{\theta}$ and $\hat{\phi}$ are the unit vectors along the directions of axes of the spherical polar coordinate system. The divergence operator is defined as

$$\vec{\nabla}.\vec{f} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 f_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}(f_\phi).$$

When we are on the surface of the unit sphere, then $R = 1$, and therefore the divergence operator becomes

$$\vec{\nabla}.\vec{f} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(f_\phi),$$

or

$$\vec{\nabla}.\vec{f} \;=\; \frac{\partial}{\partial\theta}(f_\theta) + \frac{\cos\theta}{\sin\theta}(f_\theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(f_\phi).$$

Now if $f_\theta = \sum_{i=1}^{N} \tilde{f}_\theta^j \phi_j(x)$ and $f_\phi = \sum_{i=1}^{N} \tilde{f}_\phi^j \phi_j(x)$, then

$$\begin{aligned}
\vec{\nabla}.\vec{f} &= \frac{\partial}{\partial\theta}(\sum_{i=1}^{N} \tilde{f}_\theta^j \phi_j(x)) + \frac{\cos\theta}{\sin\theta}(\sum_{i=1}^{N} \tilde{f}_\theta^j \phi_j(x)) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\sum_{i=1}^{N} \tilde{f}_\phi^j \phi_j(x)), \\
\vec{\nabla}.\vec{f} &= \sum_{i=1}^{N} \tilde{f}_\theta^j \frac{\partial}{\partial\theta}(\phi_j(x)) + \frac{\cos\theta}{\sin\theta}(\sum_{i=1}^{N} \tilde{f}_\theta^j \phi_j(x)) + \frac{1}{\sin\theta}\sum_{i=1}^{N} \tilde{f}_\phi^j \frac{\partial}{\partial\phi}(\phi_j(x)). \qquad (3.17)
\end{aligned}$$

Put $x = x_1, x_2, ..., x_N$ in equation (3.17) , we get

$$\vec{\nabla}.\vec{f} \;=\; D_\theta A^{-1} f_\theta + I1_\theta f_\theta + I2_\theta D_\phi A^{-1} f_\phi, \qquad (3.18)$$

where $D_\theta = [\frac{\partial}{\partial\theta}(\phi_j(x_i))]_{i,j=1}^{N}$, $D_\phi = [\frac{\partial}{\partial\phi}(\phi_j(x_i))]_{i,j=1}^{N}$, and $I1_\theta$ and $I2_\theta$ are the identity matrices of size N with entries $\frac{\cos\theta_i}{\sin\theta_i}$ and $\frac{1}{\sin\theta_i}$ respectively. Therefore, we have $(\vec{\nabla}.\vec{f})_{app} = D_\theta A^{-1} f_\theta + I1_\theta f_\theta + I2 D_\phi A^{-1} f_\phi$.

3.4.1 Computation of the entries of matrices D_θ and D_ϕ

The ij^{th} entry of the matrix D_θ is

$$\begin{aligned}
\frac{\partial}{\partial\theta}\phi_j(x_i) &= \frac{\partial}{\partial\theta}\phi_j(x_i, x_j), \\
&= \frac{\partial}{\partial\theta}\varphi(x_i, x_j), \\
&= \frac{\partial}{\partial\theta}\psi(|x_i - x_j|), \\
&= \frac{\partial}{\partial\theta}\psi(\sqrt{2 - 2x_i.x_j}).
\end{aligned}$$

For a fixed $x_j = (a, b, c)$ with $a^2 + b^2 + c^2 = 1$, we calculate $\frac{\partial}{\partial\theta}\psi(\sqrt{2 - 2x_i.x_j}) = \frac{\partial}{\partial\theta}\psi(\sqrt{2 - 2(ay_1 + by_2 + cy_3)}) = \frac{\partial}{\partial\theta}\psi(\sqrt{2 - 2(a\sin\theta\cos\phi + b\sin\theta\sin\phi + c\cos\theta)})$. Let us start with $\psi = \psi_{3,1}$ and with this choice equation (3.2) and equation (3.7) give us

$$\frac{\partial\psi_{3,1}}{\partial\theta} \;=\; \frac{\partial\psi_{3,1}}{\partial r}\frac{\partial r}{\partial t}\frac{\partial t}{\partial\theta} = \begin{cases} 20(1-r)^3(a\cos\theta\cos\phi + b\cos\theta\sin\phi - c\sin\theta) & \text{if } r < 1 \\ 0 & otherwise \end{cases} \qquad (3.19)$$

Moving on similar lines, we can obtain

$$\frac{\partial\psi_{3,2}}{\partial\theta} = \begin{cases} 56(1-r)^5(5r+1)(a\cos\theta\cos\phi+b\cos\theta\sin\phi-c\sin\theta) & \text{if } r<1 \\ 0 & otherwise \end{cases}, \quad (3.20)$$

and

$$\frac{\partial\psi_{3,3}}{\partial\theta} = \begin{cases} 22(1-r)^7(16r^2+7r+1)(a\cos\theta\cos\phi+b\cos\theta\sin\phi-c\sin\theta) & \text{if } r<1 \\ 0 & otherwise \end{cases} \quad (3.21)$$

The ij^{th} entry of the matrix D_ϕ is

$$\frac{\partial}{\partial\phi}\phi_j(x_i) = \frac{\partial}{\partial\phi}\phi(x_i,x_j) = \frac{\partial}{\partial\phi}\varphi(x_i,x_j) = \frac{\partial}{\partial\phi}\psi(|x_i-x_j|) = \frac{\partial}{\partial\theta}\psi(\sqrt{2-2x_i.x_j}).$$

For a fixed $x_j = (a,b,c)$ with $a^2+b^2+c^2 = 1$, we calculate $\frac{\partial}{\partial\phi}\psi(\sqrt{2-2x_i.x_j}) = \frac{\partial}{\partial\phi}\psi(\sqrt{2-2(ay_1+by_2+cy_3)}) = \frac{\partial}{\partial\phi}\psi(\sqrt{2-2(a\sin\theta\cos\phi+b\sin\theta\sin\phi+c\cos\theta)})$. Let us start with $\psi = \psi_{3,1}$ and with this choice equation (3.4) and equation (3.8) give us

$$\frac{\partial\psi_{3,1}}{\partial\phi} = \frac{\partial\psi_{3,1}}{\partial r}\frac{\partial r}{\partial t}\frac{\partial t}{\partial\phi} = \begin{cases} 20(1-r)^3(-a\sin\theta\sin\phi+b\sin\theta\cos\phi) & \text{if } r<1 \\ 0 & otherwise \end{cases}, \quad (3.22)$$

Moving on similar lines, we can obtain

$$\frac{\partial\psi_{3,2}}{\partial\phi} = \begin{cases} 56(1-r)^5(5r+1)(-a\sin\theta\sin\phi+b\sin\theta\cos\phi) & \text{if } r<1 \\ 0 & otherwise \end{cases}, \quad (3.23)$$

and

$$\frac{\partial\psi_{3,3}}{\partial\phi} = \begin{cases} 22(1-r)^7(16r^2+7r+1)(-a\sin\theta\sin\phi+b\sin\theta\cos\phi) & \text{if } r<1 \\ 0 & otherwise \end{cases}. \quad (3.24)$$

3.5 Approximation of the Curl Operator on the Sphere

Suppose we are given a vector valued function $\vec{f}(x) = f_R(x)\hat{R} + f_\theta(x)\hat{\theta} + f_\phi(x)\hat{\phi}$, where $\hat{R}$, $\hat{\theta}$ and $\hat{\phi}$ are the unit vectors along the directions of axes of the spherical polar coordinate system. The curl operator is defined as

$$\begin{aligned}\vec{\nabla}\times\vec{f} &= \frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta f_\phi)-\frac{\partial}{\partial\phi}(f_\theta)\right)\hat{R}\\ &+ \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(f_R)-\frac{\partial}{\partial R}(Rf_\phi)\right)\hat{\theta}\\ &+ \frac{1}{R}\left(\frac{\partial}{\partial R}(Rf_\theta)-\frac{\partial}{\partial\theta}(f_R)\right)\hat{\phi}.\end{aligned}$$

When we are on the surface of of the unit sphere, then $R=1$, and therefore the curl operator becomes

$$\vec{\nabla}\times\vec{f} = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta f_\phi)-\frac{\partial}{\partial\phi}(f_\theta)\right)\hat{R}+\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(f_R)\right)\hat{\theta}+\left(-\frac{\partial}{\partial\theta}(f_r)\right)\hat{\phi},$$

or

$$\vec{\nabla}\times\vec{f} = \left(\frac{\partial}{\partial\theta}(f_\phi)+\frac{\cos\theta}{\sin\theta}(f_\phi)-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(f_\theta)\right)\hat{R}+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(f_r)\hat{\theta}-\frac{\partial}{\partial\theta}(f_r)\hat{\phi}.$$

Now if $f_r=\sum_{j=1}^{N}\tilde{f}_r^j\phi_j(x)$, $f_\theta=\sum_{j=1}^{N}\tilde{f}_\theta^j\phi_j(x)$, $f_\phi=\sum_{j=1}^{N}\tilde{f}_\phi^j\phi_j(x)$, then

$$\begin{aligned}\vec{\nabla}\times\vec{f} &= \left(\frac{\partial}{\partial\theta}(\sum_{j=1}^{N}\tilde{f}_\phi^j\phi_j(x))+\frac{\cos\theta}{\sin\theta}(\sum_{j=1}^{N}\tilde{f}_\phi^j\phi_j(x))-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\sum_{j=1}^{N}\tilde{f}_\theta^j\phi_j(x))\right)\hat{R}\\ &+ \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\sum_{j=1}^{N}\tilde{f}_r^j\phi_j(x))\hat{\theta}-\frac{\partial}{\partial\theta}(\sum_{j=1}^{N}\tilde{f}_r^j\phi_j(x))\hat{\phi}.\end{aligned}$$

or

$$\begin{aligned}\vec{\nabla}\times\vec{f} &= \left(\sum_{j=1}^{N}\tilde{f}_\phi^j\frac{\partial}{\partial\theta}(\phi_j(x))+\frac{\cos\theta}{\sin\theta}(\sum_{j=1}^{N}\tilde{f}_\phi^j\phi_j(x))-\frac{1}{\sin\theta}\sum_{j=1}^{N}\tilde{f}_\theta^j\frac{\partial}{\partial\phi}(\phi_j(x))\right)\hat{R}\\ &+ \frac{1}{\sin\theta}\sum_{j=1}^{N}\tilde{f}_r^j\frac{\partial}{\partial\phi}(\phi_j(x))\hat{\theta}-\sum_{j=1}^{N}\tilde{f}_r^j\frac{\partial}{\partial\theta}(\phi_j(x))\hat{\phi}. \qquad (3.25)\end{aligned}$$

Put $x=x_1,x_2,...,x_N$ in equation (3.25) , we get

$$\vec{\nabla}\times\vec{f} = (D_\theta A^{-1}f_\phi+I1_\theta f_\phi-I2_\theta D_\phi A^{-1}f_\theta)\hat{R})+(I2_\theta D_\phi A^{-1}f_r)\hat{\theta}-(D_\theta A^{-1}f_r)\hat{\phi} \qquad (3.26)$$

where D_θ, D_ϕ, $I1_\theta$ and $I2_\theta$ are the matrices defined in section(3.4). Therefore we have $(\vec{\nabla}\times\vec{f})_{app}=(D_\theta A^{-1}f_\phi+I1_\theta f_\phi-I2_\theta D_\phi A^{-1}f_\theta)\hat{R})+(I2_\theta D_\phi A^{-1}f_r)\hat{\theta}-(D_\theta A^{-1}f_r)\hat{\phi}$.

4

Divergence-free Matrix Valued Wendland's RBFs

In the last chapter we have used scalar RBFs for interpolation. But it is advantageous to use vector valued RBFs for interpolation of vector valued functions.

In some situations, the function to be approximated is such that $\nabla . f = 0$, for example, in case of flow of incompressible fluid. Therefore we look for an interpolant which satisfies the condition that its divergence is zero. But interpolants resulting from classical RBFs do not have this property. In 1994, Narowich and Ward constructed a matrix-valued RBF whose columns are divergence-free fields, which then give rise to divergence-free interpolants. These functions are generated by smooth RBFs of unbounded support. In this chapter we discuss such RBFs.

4.1 Divergence-free matrix valued RBFs

A divergence-free, $s \times s$ matrix-valued RBF is given by

$$\Phi(x) = (-\Delta I + \nabla\nabla^T)\psi(x), \tag{4.1}$$

where ψ is a scalar-valued RBF, Δ is the laplacian operator and is defined by $\Delta = \sum_{i=1}^{s} \partial^2{}_{x(i)}$, ∇ is the gradient operator and is given by $\nabla = (\partial_{x(1)}, ..., \partial_{x(s)})^T$ the I is the s-dimensional identity matrix. Therefore the components of $\Phi_{d,k}(x)$ are combinations of the second order derivatives of $\psi_{d,k}$ [9].

If we take $s = 2$ and then we compute the entries of $\Phi(x)$, we get

$$\begin{aligned}
\Phi(x) &= (-\Delta I + \nabla\nabla^T)\psi(x), \\
&= \left[-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix}\right]\psi(x), \\
&= \left[\begin{pmatrix} -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) & 0 \\ 0 & -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \end{pmatrix} + \begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} & \frac{\partial^2\psi}{\partial x\partial y} \\ \frac{\partial^2\psi}{\partial y\partial x} & \frac{\partial^2\psi}{\partial y^2} \end{pmatrix}\right], \\
&= \begin{pmatrix} -\frac{\partial^2\psi}{\partial y^2} & \frac{\partial^2\psi}{\partial x\partial y} \\ \frac{\partial^2\psi}{\partial y\partial x} & -\frac{\partial^2\psi}{\partial x^2} \end{pmatrix}.
\end{aligned}$$

4.1.1 Computation of divergence-free matrix valued Wendland's RBFs

We started with compactly supported Wendland's RBFs $\psi_{d,k}$, where $\psi_{d,k}$ is given by

$$\psi_{d,k}(r) = \begin{cases} p_{d,k} & \text{if } 0 \le r \le 1 \\ 0 & \text{if } r > 1 \end{cases}.$$

and the divergence-free matrix valued RBF $\Phi_{d,k} = \Phi_{6,4}$ is of the form

$$\Phi_{d,k} = \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix} = \begin{pmatrix} -\partial_y^2 & \partial_x\partial_y \\ \partial_x\partial_y & -\partial_x^2 \end{pmatrix}\psi_{d,k}. \tag{4.2}$$

Here we use two-dimensional scalar valued Wendland function $\psi_{d,k} = \psi_{6,4} = (1-r)_+^{10}(85.8r^4 + 90r^3 + 42r^2 + 10r + 1)$, where $r = \sqrt{x^2+y^2}$ and compute the entries of matrix valued RBF. Now

$$\Phi_{1,1} = \frac{\partial^2 \psi_{6,4}}{\partial y^2}$$

For the computation of the same we follow the following path

$$\begin{aligned}
\frac{\partial \psi}{\partial y} &= \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial y}, \\
&= \Bigg[10(1-r)_+^9(-85.8r^4 - 90r^3 - 42r^2 - 10r - 1) + (1-r)_+^{10}(343.2r^3 + 270r^2 + \\
&\quad 84r + 10)\Bigg]\left(\frac{y}{r}\right), \\
&= \Bigg[(1-r)_+^9(-1201.2r^4 - 826.8r^3 - 234r^2 - 26r)\Bigg]\left(\frac{y}{r}\right),
\end{aligned}$$

$$\frac{\partial \psi}{\partial y} = y\left[(1-r)_+^9(-1201.2r^3 - 826.8r^2 - 234r - 26)\right], \tag{4.3}$$

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial y^2} &= \Bigg[(1-r)_+^9(-1201.2r^3 - 826.8r^2 - 234r - 26)\Bigg] + y\Bigg[-9(1-r)_+^8(-1201.2r^3 \\
&\quad -826.8r^2 - 234r - 26) + (1-r)_+^9(-3603.6r^2 - 1653.6r - 234)\Bigg]\left(\frac{y}{r}\right), \\
&= \Bigg[(1-r)_+^9(-1201.2r^3 - 826.8r^2 - 234r - 26)\Bigg] + y^2\Bigg[(1-r)_+^8(14414.4r^3 + \\
&\quad 5491.2r^2 + 686.4r)\Bigg]\left(\frac{1}{r}\right), \\
&= \Bigg[(1-r)_+^9(-1201.2r^3 - 826.8r^2 - 234r - 26)\Bigg] + y^2\Bigg[(1-r)_+^8(14414.4r^2 + 5491.2r \\
&\quad +686.4)\Bigg], \qquad (4.4) \\
&= \Bigg[(1-r)_+^8(1201.2r^4 - 374.4r^3 - 592.8r^2 - 208r - 26 + 14414.4r^2y^2 + 5491.2ry^2 \\
&\quad +686.4y^2)\Bigg],
\end{aligned}$$

On rearranging the terms, we get

$$\begin{aligned}
\frac{\partial^2\psi}{\partial y^2} &= \Big[(1-r)_+^8(1201.2r^2x^2+15615.6r^2y^2-374.4rx^2+5116.8ry^2-592.8x^2 \\
&\quad +93.6y^2-208r-26)\Big], \\
\Phi_{1,1} &= \Big[(1-r)_+^8(-1201.2r^2x^2-15615.6r^2y^2+374.4rx^2-5116.8ry^2+592.8x^2 \\
&\quad -93.6y^2+208r+26)\Big].
\end{aligned}$$

or we can write it as,

$$\frac{\partial^2\psi}{\partial y^2}=\frac{26}{5}(1-r)_+^8(5+40r+114x^2-18y^2+24r(3x^2-41y^2)-231r^2(x^2+13y^2)). \tag{4.5}$$

Moving on similar lines, we get

$$\begin{aligned}
\Phi_{2,2} &= \frac{\partial^2\psi_{6,4}}{\partial x^2}, \\
\frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x}, \\
&= \Big[10(1-r)_+^9(-85.8r^4-90r^3-42r^2-10r-1)+(1-r)_+^{10}(343.2r^3+270r^2+84r \\
&\quad +10)\Big]\left(\frac{x}{r}\right), \\
&= \Big[(1-r)_+^9(-1201.2r^4-826.8r^3-234r^2-26r)\Big]\left(\frac{x}{r}\right), \\
\frac{\partial\psi}{\partial x} &= x\left[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\right], \\
\frac{\partial^2\psi}{\partial x^2} &= \Big[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\Big]+x\Big[-9(1-r)_+^8(-1201.2r^3-826.8r^2 \\
&\quad -234r-26)+(1-r)_+^9(-3603.6r^2-1653.6r-234)\Big]\left(\frac{x}{r}\right), \\
&= \Big[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\Big]+x^2\Big[(1-r)_+^8(14414.4r^3+5491.2r^2 \\
&\quad +686.4r)\Big]\left(\frac{1}{r}\right), \\
&= \Big[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\Big]+x^2\Big[(1-r)_+^8(14414.4r^2+5491.2r \\
&\quad +686.4)\Big], \\
&= \Big[(1-r)_+^8(1201.2r^4-374.4r^3-592.8r^2-208r-26+14414.4r^2x^2+5491.2rx^2 \\
&\quad +686.4x^2)\Big],
\end{aligned}$$

On rearranging the terms, we get

$$\begin{aligned}
\frac{\partial^2\psi}{\partial x^2} &= \Big[(1-r)_+^8(1201.2r^2y^2+15615.6r^2x^2-374.4ry^2+5116.8rx^2-592.8y^2 \\
&\quad +93.6x^2-208r-26)\Big], \\
\Phi_{2,2} &= -\frac{\partial^2\psi}{\partial x^2}, \\
\Phi_{2,2} &= \Big[(1-r)_+^8(-1201.2r^2y^2-15615.6r^2x^2+374.4ry^2-5116.8rx^2+592.8y^2 \\
&\quad -93.6x^2+208r+26)\Big],
\end{aligned}$$

or we can write it as,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{26}{5}(1-r)_+^8(5+40r+114y^2-18x^2+24r(3y^2-41x^2)-231r^2(y^2+13x^2)). \tag{4.6}$$

and

$$\begin{aligned}
\Phi_{1,2} &= \Phi_{2,1} = \frac{\partial^2\psi_{6,4}}{\partial x\partial y}, \qquad (4.7) \\
\frac{\partial\psi}{\partial y} &= \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y}, \\
&= \Big[10(1-r)_+^9(-85.8r^4-90r^3-42r^2-10r-1)+(1-r)_+^{10}(343.2r^3+270r^2 \\
&\quad +84r+10)\Big]\left(\frac{y}{r}\right), \\
&= \Big[(1-r)_+^9(-1201.2r^4-826.8r^3-234r^2-26r)\Big]\left(\frac{y}{r}\right), \\
\frac{\partial\psi}{\partial y} &= y\left[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\right].
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial x\partial y} &= y\frac{\partial}{\partial x}\Bigg[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\Bigg],\\
&= y\Bigg[-9(1-r)_+^8(-1201.2r^3-826.8r^2-234r-26)+(1-r)_+^9(-3603.6r^2\\
&\quad -1653.6r-234)\Bigg]\left(\frac{x}{r}\right),\\
&= \left(\frac{xy}{r}\right)\Bigg[(1-r)_+^8(10810.8r^3+7441.2r^2+2106r+234-3603.6r^2-1653.6r\\
&\quad -234+3603.6r^3+1653.6r^2+234r)\Bigg],\\
&= \left(\frac{xy}{r}\right)\Bigg[(1-r)_+^8(14414.4r^3+5491.2r^2+686.4r)\Bigg],\\
&= xy\left[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\right],\\
&= \frac{3432}{5}xy\Bigg[(1-r)_+^8(21r^2+8r+1)\Bigg]. \qquad (4.8)
\end{aligned}
$$

Lemma: The columns of the functions $\Phi_{d,k}=\{-\Delta I+\nabla\nabla^T\}\psi_{d,k}(x)$, where $\psi_{d,k}(x)$ is a scalar-valued Wendland function for $k\geq 2$, are divergence-free.

Proof. A vector-valued function f is divergence-free if and only if $\nabla.f(x)\equiv 0$ holds for all $x\in\mathbb{R}^s$. Let $\Phi^i_{d,k}$ be an arbitrary column of $\Phi_{d,k}(x)$, for $1\leq i\leq s$. Then, for $k\geq 2$,

$$
\begin{aligned}
\nabla.\Phi^i_{d,k}(x) &= \nabla.\{-\Delta I+\nabla\nabla^T\}_i\psi_{d,k}(x),\\
&= \nabla.\left(-\delta_{ij}\sum_{l=1}^{s}\partial^2_{x(l)}+\partial_{x(i)}\partial_{x(j)}\right)_{j=1}^{s}\psi_{d,k}(x),\\
&= -\sum_{l\neq i}\partial^2_{x(l)}\partial_{x(i)}+\sum_{r\neq i}\partial_{x(i)}\partial^2_{x(r)}\psi_{d,k},\\
&= 0.
\end{aligned}
$$

4.1.2 Computation of divergence-free columns

Now we will verify that the columns of $\Phi_{6,4}$ are divergence-free.

By taking first column of $\Phi_{6,4}$, we will check that $\nabla.(\Phi_{1,1}\hat{\imath}+\Phi_{2,1}\hat{\jmath})=0$ i.e., $\frac{\partial\Phi_{1,1}}{\partial x}+\frac{\partial\Phi_{2,1}}{\partial y}=0$.

By taking equation (4.4)

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial y^2} &= \Bigg[(1-r)_+^9(-1201.2r^3-826.8r^2-234r-26)\Bigg]+y^2\Bigg[(1-r)_+^8(14414.4r^2+\\
&\quad 5491.2r+686.4)\Bigg],\\
\Phi_{1,1} &= -\frac{\partial^2\psi}{\partial y^2},\\
\Phi_{1,1} &= \Bigg[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Bigg]-y^2\Bigg[(1-r)_+^8(14414.4r^2+\\
&\quad 5491.2r+686.4)\Bigg],\\
\frac{\partial\Phi_{1,1}}{\partial x} &= \Bigg[-9(1-r)_+^8(1201.2r^3+826.8r^2+234r+26)+(1-r)_+^9(3603.6r^2+\\
&\quad 1653.6r+234)\Bigg]\left(\frac{x}{r}\right)-y^2\Bigg[-8(1-r)_+^7(14414.4r^2+5491.2r+686.4)\\
&\quad +(1-r)_+^8(28828.8r+5491.2)\Bigg]\left(\frac{x}{r}\right),
\end{aligned}
$$

$$
\begin{aligned}
= &\Bigg[(1-r)_+^8(-10810.8r^3-7441.2r^2-2106r-234+3603.6r^2+1653.6r\\
&-43929.6r-5491.2+234-3603.6r^3-1653.6r^2-234r)\Bigg]\left(\frac{x}{r}\right)-y^2\\
&\Bigg[(1-r)_+^7(-115315.2r^2+28828.8r+5491.2-28828.8r^2-5491.2r)\Bigg]\left(\frac{x}{r}\right),
\end{aligned}
$$

$$
\begin{aligned}
= &\Bigg[(1-r)_+^8(-14414.4r^3-5491.2r^2-686.4r)\Bigg]\left(\frac{x}{r}\right)-y^2\Bigg[(1-r)_+^7\\
&(-144144r^2-20592r)\Bigg]\left(\frac{x}{r}\right),
\end{aligned}
$$

$$
\begin{aligned}
= &-x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]+y^2x\Bigg[(1-r)_+^7\\
&(144144r+20592)\Bigg].
\end{aligned}
$$

Now

$$
\begin{aligned}
\Phi_{2,1} &= \frac{\partial^2 \psi_{6,4}}{\partial x \partial y}, \\
\Phi_{2,1} &= xy\left[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\right], \\
\frac{\partial \Phi_{2,1}}{\partial y} &= x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]+xy\Bigg[-8(1-r)_+^7(14414.4r^2 \\
&\quad +5491.2r+686.4)+(1-r)_+^8(28828.8r+5491.2)\Bigg]\left(\frac{y}{r}\right), \\
&= x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]+xy\Bigg[(1-r)_+^7(-115315.2r^2 \\
&\quad -43929.6r-5491.2+28828.8r+5491.2-28828.8r^2-5491.2r)\Bigg]\left(\frac{y}{r}\right), \\
&= x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]+xy\Bigg[(1-r)_+^7 \\
&\quad (-144144r^2-20592r)\Bigg]\left(\frac{y}{r}\right), \\
&= x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]-xy^2\Bigg[(1-r)_+^7(144144r \\
&\quad +20592)\Bigg]\left(\frac{y}{r}\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\nabla.(\Phi_{1,1}\hat{\imath}+\Phi_{2,1}\hat{\jmath}) &= \frac{\partial \Phi_{1,1}}{\partial x}+\frac{\partial \Phi_{2,1}}{\partial y}, \\
&= -x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]+xy^2\Bigg[(1-r)_+^7 \\
&\quad (144144r+20592)\Bigg]+x\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg] \\
&\quad -xy^2\Bigg[(1-r)_+^7(144144r+20592)\Bigg]\left(\frac{y}{r}\right), \\
&= 0.
\end{aligned}
$$

4.2 Properties of $\Phi_{d,k}$

1. $\Phi_{d,k}(x)c$ is in $H^1(\mathbb{R}^s,\mathbb{C}^s)$ for $c\in\mathbb{C}^s$ for $k\geq 2$.
2. $\nabla.\Phi_{d,k}(x)c\equiv 0$ i.e., columns of $\Phi_{d,k}(x)c$ are divergence-free.

3. $\hat{\Phi}_{d,k}(\xi) = (\|\xi\|^2 I - \xi\xi^T)\hat{\psi}_{d,k}$ derived by fourier transform arguments.

Theorem: Let $\Phi_{d,k}(x) = (-\Delta I + \nabla\nabla^T)\psi(x)$ and define the spaces
$V := \{\sum_{j=1}^{N} \Phi_{d,k}(. - x_j)c_j;\ x_j \in \mathbb{R}^s,\ c_j \in C^s\ for\ 1 \leq j \leq N,\ N \in \mathbb{N}\}$,
$L := \{f \in H^1(\mathbb{R}^s, \mathbb{C}^s)\ such\ that\ \nabla.f \equiv 0\ a.e.\ on\ \mathbb{R}^s\}$.
Then V is dense in L, i.e., any divergence-free vector-valued function $f : \mathbb{R}^s \to \mathbb{C}^s$ is in Sobolev space H^1 can be approximated by a linear combination of divergence-free matrix-valued RBFs generated by a Wendland function.
Therefore f such that $\nabla.f \equiv 0$ can be written as $f = \sum_{i=1}^{N} c_i f_i$, where $f_i = (f_1, f_2, ..., f_N)$ and given by

$$\begin{aligned} f_1 &= \Phi_{d,k}(. - x_1), \\ f_2 &= \Phi_{d,k}(. - x_2), \\ &\vdots \\ f_N &= \Phi_{d,k}(. - x_N). \end{aligned}$$

where $x_1, x_2, ..., x_N$ are the N^{th} columns of the matrix valued divergence-free RBFs.
For $N = 1$

$$\begin{aligned} f_1 &= \sum_{j=1}^{N} \Phi_{d,k}(. - x_j)c_j, \\ &= \Phi_{d,k}(. - x_1)c_1, \\ &= \Phi_{d,k}(. - x_1). \end{aligned}$$

for $N = 2$

$$\begin{aligned} f_2 &= \sum_{j=1}^{N} \Phi_{d,k}(. - x_j)c_j, \\ &= \Phi_{d,k}(. - x_1)c_1 + \Phi_{d,k}(. - x_2)c_2. \end{aligned}$$

and for $N = 3$

$$\begin{aligned} f_3 &= \sum_{j=1}^{N} \Phi_{d,k}(. - x_j)c_j, \\ &= \Phi_{d,k}(. - x_1)c_1 + \Phi_{d,k}(. - x_2)c_2 + \Phi_{d,k}(. - x_3)c_3. \end{aligned}$$

so on. In this way we get the value of f_i.

4.3 Laplacian of divergence-free matrix valued RBFs

Laplacian of divergence-free matrix valued RBF is given by

$$\Delta\Phi = \begin{pmatrix} \Delta\Phi_{1,1} & \Delta\Phi_{1,2} \\ \Delta\Phi_{2,1} & \Delta\Phi_{2,2} \end{pmatrix} = \begin{pmatrix} \partial_x^2(\Phi_{1,1}) + \partial_y^2(\Phi_{1,1}) & \partial_x^2(\Phi_{1,2}) + \partial_x^2(\Phi_{1,1}) \\ \partial_x^2(\Phi_{2,1}) + \partial_y^2(\Phi_{2,1}) & \partial_x^2(\Phi_{2,2}) + \partial_y^2(\Phi_{2,2}) \end{pmatrix}. \tag{4.9}$$

4.3.1 Computation of the entries of laplacian matrix

Now we will compute the entries of Laplacian matrix.

$$\Delta\Phi_{1,1} = \partial_x^2(\Phi_{1,1}) + \partial_y^2(\Phi_{1,1}) \tag{4.10}$$

$$\begin{aligned}
\Phi_{1,1} &= \left[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\right] - y^2\Big[(1-r)_+^8(14414.4r^2+5491.2r \\
&\quad +686.4)\Big], \\
\partial_x^2(\Phi_{1,1}) &= \frac{\partial^2\Phi_{1,1}}{\partial x^2}, \\
\frac{\partial\Phi_{1,1}}{\partial x} &= \frac{\partial}{\partial x}\bigg(\Big[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Big] - y^2\Big[(1-r)_+^8(14414.4r^2 \\
&\quad +5491.2r+686.4)\Big]\bigg), \\
&= \Big[-9(1-r)_+^8(1201.2r^3+826.8r^2+234r+26) + (1-r)_+^9(3603.6r^2+1653.6r \\
&\quad +234)\Big]\left(\frac{x}{r}\right) - y^2\Big[-8(1-r)_+^7(14414.4r^2+5491.2r+686.4) + (1-r)_+^8(28828.8r \\
&\quad +5491.2)\Big]\left(\frac{x}{r}\right),
\end{aligned}$$

$$\begin{aligned}
&= \Big[(1-r)_+^8(-10810.8r^3 - 7441.2r^2 - 2106 - 234 + 3603.6r^2 + 1653.6r + 234 - \\
&\quad 3603.6r^3 - 1653.6r^2 - 234r)\Big]\left(\frac{x}{r}\right) - \Big[(1-r)_+^7(-115315.2r^2 - 43929.6r - 5491.2 \\
&\quad +28828.8r + 5491.2 - 28828.8r^2 - 5491.2r)\Big], \\
\frac{\partial\Phi_{1,1}}{\partial x} &= x\left[(1-r)_+^8(-14414.4r^2 - 5491.2r - 686.4)\right] - xy^2\left[(1-r)_+^7(-144144r - 20592)\right],
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \Phi_{1,1}}{\partial x^2} &= \Big[(1-r)_+^8(-14414.4r^2-5491.2r-686.4)\Big] + x\Big[-8(1-r)_+^7(14414.4r^2+5491.2r \\
&\quad +686.4)+(1-r)_+^8(28828.8r+5491.2)\Big]\left(\frac{x}{r}\right) - y^2\Big[(1-r)_+^7(-144144r \\
&\quad -20592)\Big] - xy^2\Big[-7(1-r)_+^6(-144144r-20592)+(1-r)^7 \\
&\quad (-144144)\Big]\left(\frac{x}{r}\right),
\end{aligned}
$$

$$
\begin{aligned}
&= \big[(1-r)_+^8(-14414.4r^2-5491.2r-686.4)\big] + x^2\left[(1-r)_+^7(144144r+20592)\right] \\
&+ \quad y^2\left[(1-r)_+^7(144144r+20592)\right] - x^2y^2\left[(1-r)_+^6(1153152)\right], \\
&= \big[(1-r)_+^8(-14414.4r^2-5491.2r-686.4)\big] + (x^2+y^2)\left[(1-r)_+^7(144144r+20592)\right] \\
&- \quad x^2y^2\left[(1-r)_+^6(1153152)\right],
\end{aligned}
$$

$$
\begin{aligned}
&= (686.4)\left[(1-r)_+^8(-21r^2-8r-1)\right] + (x^2+y^2)\left[(1-r)_+^7(210r+30)\right] \\
&- \quad x^2y^2\left[(1-r)_+^6(1680)\right], \\
&= (686.4)\Big[(1-r)_+^6(-21r^4+34r^3-6r^2-6r-1+210rx^2+210ry^2+30x^2+30y^2 \\
&\quad -210r^2x^2-210r^2y^2-30rx^2-30ry^2-1680x^2y^2)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&= (686.4)\Big[(1-r)_+^6(-21r^2(x^2+y^2)+34r(x^2+y^2)-6(x^2+y^2)-6r-1+210rx^2 \\
&\quad +210ry^2+30x^2+30y^2-210r^2x^2-210r^2y^2-30rx^2-30ry^2-1680x^2y^2)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\partial_x^2(\Phi_{1,1}) &= (686.4)\Big[(1-r)_+^6(-231r^2(x^2+y^2)+214r(x^2+y^2)+24(x^2+y^2)-6r-1 \\
&\quad -1680x^2y^2\Big].
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_{1,1} &= \Big[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Big] - y^2\Big[(1-r)_+^8(14414.4r^2+5491.2r \\
&\quad +686.4)\Big], \\
\frac{\partial \Phi_{1,1}}{\partial y} &= \frac{\partial}{\partial y}\Bigg(\Big[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Big] - y^2\Big[(1-r)_+^8(14414.4r^2 \\
&\quad +5491.2r+686.4)\Big]\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[-9(1-r)^{8}(1201.2r^{3}+826.8r^{2}+234r+26)+(1-r)_{+}^{9}(3603.6r^{2}+1653.6r\\
&\quad +234)\Big]\left(\frac{y}{r}\right)-2y\Big[(1-r)_{+}^{8}(14414.4r^{2}+5491.2r+686.4)\Big]-y^{2}\Big[-8(1-r)_{+}^{7}\\
&\quad (14414.4r^{2}+5491.2r+686.4)+(1-r)_{+}^{8}(28828.8r+5491.2)\Big]\left(\frac{y}{r}\right),\\
&= \Big[(1-r)_{+}^{8}(-10810.8r^{3}-7441.2r^{2}-2106-234+3603.6r^{2}+1653.6r+234-\\
&\quad 3603.6r^{3}-1653.6r^{2}-234r)\Big]\left(\frac{y}{r}\right)-2y\Big[(1-r)_{+}^{8}(14414.4r^{2}+5491.2r+686.4)\Big]\\
&\quad -\left(\frac{y^{3}}{r}\right)\Big[(1-r)_{+}^{7}(-115315.2r^{2}-43929.6r-5491.2+28828.8r+5491.2\\
&\quad -28828.8r^{2}-5491.2r)\Big],\\
\frac{\partial \Phi_{1,1}}{\partial y} &= (686.4)\{y\left[(1-r)_{+}^{8}(-63r^{2}-24r-3)\right]+y^{3}\left[(1-r)_{+}^{7}(210r+30)\right]\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^{2} \Phi_{1,1}}{\partial y^{2}} &= (686.4)\Bigg(\Big[(1-r)_{+}^{8}(-63r^{2}-24r-3)\Big]+y\Big[-8(1-r)_{+}^{7}(-63r^{2}-24r-3)\\
&\quad +(1-r)_{+}^{8}(-126r-24)\Big]\left(\frac{y}{r}\right)+3y^{2}\Big[(1-r)_{+}^{7}(210r+30)\Big]\\
&\quad +y^{3}\Big[-7(1-r)_{+}^{6}(210r+30)+(1-r)^{7}(210)\Big]\left(\frac{y}{r}\right)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&= (686.4)\Bigg(\Big[(1-r)_{+}^{8}(-63r^{2}-24r-3)\Big]+\left(\frac{y^{2}}{r}\right)\Big[(1-r)_{+}^{7}(630r^{2}+90r)\Big]\\
&+ y^{2}\Big[(1-r)_{+}^{7}(630r^{2}+90r)\Big]+\left(\frac{y^{4}}{r}\right)\Big[(1-r)_{+}^{6}(-1680r)\Big]\\
&= (686.4)\Bigg(\left[(1-r)_{+}^{8}(-63r^{2}-24r-3)\right]+2y^{2}\left[(1-r)_{+}^{7}(630r+90)\right]\\
&- y^{4}\left[(1-r)_{+}^{6}(1680)\right]\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&= (686.4)\Bigg(\left[(1-r)_{+}^{8}(-63r^{2}-24r-3)\right]+y^{2}\left[(1-r)_{+}^{7}(1260r+180)\right]\\
&- y^{4}\left[(1-r)_{+}^{6}(1680)\right]\Bigg),\\
&= (686.4)\Big[(1-r)_{+}^{6}(-63r^{4}+102r^{3}-18r^{2}-18r-3+1080ry^{2}+180y^{2}\\
&\quad -1260r^{2}y^{2}-1680y^{4})\Big],
\end{aligned}
$$

$$\begin{aligned}
&= (686.4)\Bigg[(1-r)_+^6(-63r^2(x^2+y^2)+102r(x^2+y^2)-18(x^2+y^2)-18r\\
&\quad -3+1080ry^2+180y^2-1260r^2y^2-1680y^4)\Bigg],\\
\frac{\partial^2\Phi_{1,1}}{\partial y^2} &= (686.4)\Bigg[(1-r)_+^6(-63r^2x^2-1323r^2y^2+102rx^2+1182ry^2-18x^2+162y^2\\
&\quad -18r-3-1680y^4)\Bigg].
\end{aligned}$$

By putting the value of $\partial_x^2(\Phi_{1,1})$ and $\partial_y^2(\Phi_{1,1})$ in equation (4.10), we get

$$\begin{aligned}
\Delta\Phi_{1,1} &= (686.4)\Bigg[(1-r)_+^6(-231r^2(x^2+y^2)+214r(x^2+y^2)+24(x^2+y^2)-6r-1\\
&\quad -1680x^2y^2\Bigg]+(686.4)\Bigg[(1-r)_+^6(-63r^2x^2-1323r^2y^2+102rx^2+1182ry^2\\
&\quad -18x^2+162y^2-18r-3-1680y^4)\Bigg],
\end{aligned}$$

$$\begin{aligned}
&= (686.4)\Bigg[(1-r)_+^6(-231r^2x^2-231r^2y^2+214rx^2+214ry^2+24x^2+24y^2-6r\\
&\quad -1-1680x^2y^2-63x^2y^2-1323r^2y^2+102rx^2+1182ry^2-18x^2+162y^2-18r\\
&\quad -3-1680y^4)\Bigg],
\end{aligned}$$

$$\begin{aligned}
&= (686.4)\Bigg[(1-r)_+^6(-294r^2x^2-1554r^2y^2+316rx^2+1396ry^2-39x^2-1299y^2\\
&\quad +6x^2+186y^2-1680x^2y^2-1680y^4-24r-4)\Bigg],\\
&= (686.4)\Bigg[(1-r)_+^6(-294r^2x^2-3234r^2y^2+316rx^2+1396ry^2-39x^2-1299y^2\\
&\quad +6x^2+186y^2-24r-4)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\Delta\Phi_{1,1} &= (-1372.8)\Bigg[(1-r)_+^6(147(x^2+11y^2)r^2-2(79x^2+349y^2)r+12r-3x^2\\
&\quad -93y^2+2)\Bigg]. \qquad (4.11)
\end{aligned}$$

Similarily, we can find the other entries of laplacian matrix.

$$\begin{aligned}
\Delta\Phi_{1,2} &= \Delta\Phi_{2,1}=(41184)\left[(1-r)_+^6(49r^2-18r-3)xy\right], \qquad (4.12)\\
\Delta\Phi_{2,2} &= (-1372.8)\Bigg[(1-r)_+^6(147(11x^2+y^2)r^2-2(349x^2+79y^2)r+12r-93x^2\\
&\quad -3y^2+2)\Bigg]. \qquad (4.13)
\end{aligned}$$

4.4 Curl of divergence-free matrix valued RBFs

Curl of divergence-free matrix valued RBF is given by

Curl of first column is$=\frac{\partial \Phi_{2,1}}{\partial x} - \frac{\partial \Phi_{1,1}}{\partial y}$ (A)

Now

$$\begin{aligned}
\Phi_{2,1} &= (686.4)\left[(1-r)_+^8(21r^2+8r+1)xy\right], \\
\frac{\partial \Phi_{2,1}}{\partial x} &= (686.4)\Bigg(y\Bigg[(1-r)_+^8(21r^2+8r+1)\Bigg] \\
&+ \ xy\Bigg[-8(1-r)_+^7(21r^2+8r+1)+(1-r)_+^8(42r+8)\Bigg]\left(\frac{x}{r}\right)\Bigg), \\
&= (686.4)\Bigg(\Bigg[y(1-r)_+^8(21r^2+8r+1)\Bigg] \\
&+ \ \left(\frac{x^2y}{r}\right)\Bigg[(1-r)_+^7(-210r^2+30r)\Bigg]\Bigg), \\
&= (686.4)\left(\left[y(1-r)_+^8(21r^2+8r+1)\right]-x^2y\left[(1-r)_+^7(210r+30)\right]\right). \qquad (4.14)
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{1,1} &= \Bigg[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Bigg]-y^2\Bigg[(1-r)_+^8(14414.4r^2+5491.2r \\
&\quad +686.4)\Bigg], \\
\frac{\partial \Phi_{1,1}}{\partial y} &= \frac{\partial}{\partial y}\Bigg(\Bigg[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Bigg] \\
&\quad -y^2\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]\Bigg),
\end{aligned}$$

$$\begin{aligned}
= \ &\Bigg[-9(1-r)_+^8(1201.2r^3+826.8r^2+234r+26)+(1-r)^9(3603.6r^2+1653.6r \\
&+234)\Bigg]\left(\frac{y}{r}\right)-2y\Bigg[(1-r)_+^8(14414.4r^2+5491.2r+686.4)\Bigg]-y^2\Bigg[-8(1-r)_+^7 \\
&(14414.4r^2+5491.2r+686.4)+(1-r)_+^8(28828.8r+5491.2)\Bigg]\left(\frac{y}{r}\right),
\end{aligned}$$

$$\begin{aligned}
&= \Bigg[(1-r)_+^8(-10810.8r^3 - 7441.2r^2 - 2106 - 234 + 3603.6r^2 + 1653.6r + 234 \\
&\quad -3603.6r^3 - 1653.6r^2 - 234r)\Bigg]\left(\frac{y}{r}\right) - 2y\Bigg[(1-r)_+^8(14414.4r^2 + 5491.2r \\
&\quad +686.4)\Bigg] - \left(\frac{y^3}{r}\right)\Bigg[(1-r)_+^7(-115315.2r^2 - 43929.6r - 5491.2 + 28828.8r \\
&\quad +5491.2 - 28828.8r^2 - 5491.2r)\Bigg], \\
&= (686.4)\left\{y\left[(1-r)_+^8(-63r^2 - 24r - 3)\right] + y^3\left[(1-r)_+^7(210r+30)\right]\right\}. \qquad (4.15)
\end{aligned}$$

By putting the values of derivatives $\Phi_{2,1}$ and $\Phi_{1,1}$ in equation (A), we get

$$\begin{aligned}
\frac{\partial \Phi_{2,1}}{\partial x} - \frac{\partial \Phi_{1,1}}{\partial y} &= (686.4)\left(\left[y(1-r)_+^8(21r^2+8r+1)\right] - x^2y\left[(1-r)_+^7(210r+30)\right]\right) \\
&- (686.4)\left(\left[y(1-r)_+^8(-63r^2-24r-3)\right] + y^3\left[(1-r)_+^7(210r+30)\right]\right),
\end{aligned}$$

$$\begin{aligned}
&= (686.4)y\Bigg[(1-r)_+^8(21r^2+8r+1) - x^2(1-r)^7(210r+30) + (1-r)_+^8 \\
&\quad (63r^2+24r+3) - y^2(1-r)_+^7(210r+30)\Bigg], \\
&= (686.4)y\Bigg[(1-r)_+^7(-84r^3+52r^2+28r-210rx^2-210ry^2-30x^2 \\
&\quad -30y^2+4)\Bigg],
\end{aligned}$$

$$\begin{aligned}
&= (686.4)y\left[(1-r)_+^7(-294r(x^2+y^2)+22(x^2+y^2)+28r+4)\right], \\
&= (1372.8)y\left[(1-r)_+^7(-147r(x^2+y^2)+11(x^2+y^2)+14r+2)\right]. \qquad (4.16)
\end{aligned}$$

Curl of second column is$=\frac{\partial \Phi_{2,2}}{\partial x} - \frac{\partial \Phi_{1,2}}{\partial y}$ (B)

$$\begin{aligned}
\Phi_{2,2} &= \Bigg[(1-r)_+^9(1201.2r^3+826.8r^2+234r+26)\Bigg] - y^2\Bigg[(1-r)_+^8(14414.4r^2+5491.2r \\
&\quad +686.4)\Bigg],
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \Phi_{2,2}}{\partial x} &= \frac{\partial}{\partial x}\Bigg(\bigg[(1-r)^9_+(1201.2r^3+826.8r^2+234r+26)\bigg] \\
&- x^2\bigg[(1-r)^8_+(14414.4r^2+5491.2r+686.4)\bigg]\Bigg), \\
&= \bigg[-9(1-r)^8_+(1201.2r^3+826.8r^2+234r+26)+(1-r)^9_+(3603.6r^2+1653.6r \\
&\quad +234)\bigg]\left(\frac{x}{r}\right)-2y\bigg[(1-r)^8_+(14414.4r^2+5491.2r+686.4)\bigg]-y^2 \\
&\quad \bigg[-8(1-r)^7_+(14414.4r^2+5491.2r+686.4)+(1-r)^8(28828.8r+5491.2)\bigg]\left(\frac{x}{r}\right),
\end{aligned}
$$

$$
\begin{aligned}
= {} & \bigg[(1-r)^8_+(-10810.8r^3-7441.2r^2-2106-234+3603.6r^2+1653.6r+234 \\
& -3603.6r^3-1653.6r^2-234r)\bigg]\left(\frac{x}{r}\right)-2y\bigg[(1-r)^8_+(14414.4r^2+5491.2r \\
& +686.4)\bigg]-\left(\frac{x^3}{r}\right)\bigg[(1-r)^7_+(-115315.2r^2-43929.6r-5491.2+28828.8r \\
& +5491.2-28828.8r^2-5491.2r)\bigg],
\end{aligned}
$$

$$
= (686.4)\left\{x\left[(1-r)^8_+(-63r^2-24r-3)\right]+x^3\left[(1-r)^7_+(210r+30)\right]\right\}. \quad (4.17)
$$

and

$$
\begin{aligned}
\Phi_{1,2} &= (686.4)\left[(1-r)^8_+(21r^2+8r+1)xy\right], \\
\frac{\partial \Phi_{1,2}}{\partial y} &= (686.4)\Bigg(y\bigg[(1-r)^8_+(21r^2+8r+1)\bigg] \\
&+ xy\bigg[-8(1-r)^7_+(21r^2+8r+1)+(1-r)^8_+(42r+8)\bigg]\left(\frac{y}{r}\right)\Bigg), \\
&= (686.4)\bigg[x(1-r)^8_+(21r^2+8r+1)\bigg]+\left(\frac{xy^2}{r}\right)\bigg[(1-r)^7_+(-210r^2+30r)\bigg], \\
&= (686.4)\{\left[x(1-r)^8_+(21r^2+8r+1)\right]-xy^2\left[(1-r)^7_+(210r+30)\right]\}. \qquad (4.18)
\end{aligned}
$$

By putting the value of derivatives of $\Phi_{2,2}$ and $\Phi_{1,2}$ in equation (B), we get

$$\begin{aligned}
\frac{\partial \Phi_{2,2}}{\partial x} - \frac{\partial \Phi_{1,2}}{\partial y} &= (686.4)\left(\left[x(1-r)_+^8(-63r^2-24r-3)\right] + x^3\left[(1-r)_+^7(210r+30)\right]\right) \\
&\quad -(686.4)\left(\left[x(1-r)_+^8(21r^2+8r+1)\right] - xy^2\left[(1-r)_+^7(210r+30)\right]\right), \\
&= (686.4)x\Bigg[(1-r)_+^8(-63r^2-24r-3) + x^2(1-r)_+^7(210r+30) \\
&\quad -(1-r)^8(21r^2+8r+1) + y^2(1-r)_+^7(210r+30)\Bigg], \\
&= (686.4)x\Bigg[(1-r)_+^7(84r^3-52r^2-28r+210rx^2+210ry^2+30x^2 \\
&\quad +30y^2-4)\Bigg], \\
&= (686.4)x\left[(1-r)_+^7(294r(x^2+y^2)-22(x^2+y^2)-28r-4)\right], \\
&= (1372.8)y\left[(1-r)_+^7(147r(x^2+y^2)-11(x^2+y^2)-14r-2)\right]. \qquad (4.19)
\end{aligned}$$

Conclusion

In this thesis we have studied about scalar Wendland's radial basis functions and used these functions to approximate laplacian, gradient, divergence and curl operators on the sphere. Next we have learnt about divergence-free matrix valued Wendland's RBFs and computed their divergence, laplacian and curl.

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