ITERATIVE SOLUTIONS FOR NON-LINEAR SYSTEMS

A Thesis

submitted for partial fulfillment of the requirements for the award of the degree of

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by

Sonia
Reg no: 901411004

under the supervison of

Dr. Sanjeev Kumar
Assistant Professor
School of Mathematics
T.I.E.T, Patiala 147004, INDIA

Dr. Jatinderdeep Kaur
Assistant Professor
School of Mathematics
T.I.E.T, Patiala 147004, INDIA

THAPAR INSTITUTE OF ENGINEERING AND TECHNOLOGY
(Deemed to be University)
Patiala-147004, Punjab, India
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Certificate

I hereby certify that the work, which is being exhibited in the thesis, entitled **ITERATIVE SOLUTIONS FOR NON-LINEAR SYSTEMS**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work did amid the period the period **July 2014** to **February 2019** under the supervision of **Dr. Sanjeev Kumar**, Assistant Professor, School of Mathematics and **Dr. Jatinderdeep Kaur**, Assistant Professor, School of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from any institution.

Date: 21-02-2019

Sonia

(**Sonia**)
Candidate

It is certified that the above assertion made by the applicant is correct to the best of our insight.

(**Dr. Sanjeev Kumar**)
Assistant Professor
School of Mathematics
T.I.E.T, Patiala, 147004, India

(**Dr. Jatinderdeep Kaur**)
Assistant Professor
School of Mathematics
T.I.E.T, Patiala, 147004, India

DEDICATED

TO

GOD, MY PARENTS

AND

SUPERVISORS

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To my life-coach, my husband Nitin Bhalla: because I owe it all to you. Many Thanks!

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Abstract

The research work presented in this thesis deals with the study of the **"Iterative solutions for non-linear systems"**.

A non-linear system is a set of simultaneous 'n' non-linear equations in which each equation is a function in 'n' unknown variables. Such type of non-linear systems have perceived several significant contributions in mathematics and allied engineering areas, for example, electrical circuits, chemical reactions, physical law, biological phenomena, computational economics etc. Due to the wider variety of behavior, finding solutions of non-linear systems is much more tedious than a scalar case. Moreover it is extremely hard to solve non-linear systems analytically. In this context, numerical techniques provide a fruitful way to solve non-linear systems. On account of this reality, one needs to rely on numerical techniques for solving non-linear systems. Therefore, a reliably expanding extent of present-day numerical research is focused on the analysis of the approximate solutions of non-linear systems. Among all numerical techniques, Newton's technique is the most basic and outstanding iterative method for solving non-linear systems. In literature, numerous adjustments have been incorporated in Newton's technique (known as Newton's variants), which have either equivalent or better efficiency over Newton's technique. In the present thesis, an endeavor has been made to the construct more variants of Newton's method to solve non-linear systems. The essential standard of numerical algorithms has been followed to attain computational efficiency, which is always proportional to the quality of an algorithm and inversely proportional to its computational cost. The quality of an algorithm concerns with the convergence speed of algorithm along with its structure. Computational cost concerns with the amount of calculation work required to evaluate functions, derivatives, matrix inversions during the entire process.

The primary focus of present research work is to address the construction of iterative schemes to propose solutions for systems of non-linear equations arising in different disciplines of science and engineering. The present work also sheds light on the development of iterative schemes for systems of non-linear equations associated with ordinary and partial differential equations. The development of iterative schemes consist of two parts: the first one is the 'construction part' and the second part establishes the proof of local convergence in Banach settings. Finally, a variety of problems involving non-linear systems have been numerically tested in order to demonstrate the exactness and the computational efficiency of the proposed iterative algorithms.

List of Publications

International Journal

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2. S. Bhalla, S. Kumar, I. K. Argyros, Ramandeep Behl, *A family of higher order derivative free methods for nonlinear systems with local convergence analysis*, Computational and Applied Mathematics, Vol. 37(5), pp. 5807–5828, 2018. (2017 Imapct Factor: 0.863)

Communicated Paper

1. S. Bhalla, S. Kumar, I. K. Argyros, Ramandeep Behl, *Semilocal analysis and Real life applications of higher-order iterative schemes for nonlinear systems*, communicated in SCI journal.

International Conference

1. Attended and presented a paper entitled *"An iterative family for solution of system of nonlinear equations"* in "2nd National Level Conference on Recent Developments in Mathematical Modelling & Fuzziology-II [M2FZY-II]" held at Chandigarh University, Gharuan (Mohali) from 27-28 March, 2018.

2. Attended and presented a paper entitled *"A local convergence analysis of non-linear systems"* in "International conference on Innovative trends in Mathematics, Computer & Information Sciences" held at Maharishi Markandeshwar University, Mullana from 09-10 March, 2017.

3. Attended and presented a paper entitled *"A new class for solving non-linear systems with local convergence"* in "International conference on Recent Advances on Theoretical and Computational Partial Differential Equations" held at Panjab University, Chandigarh from 05-09 December, 2016.

List of Notations

$\mathbb{R}^n$	a set of n-tuple real numbers
D	convex set
ρ	order of convergence
e^k	$(e_1^k, e_2^k, \ldots, e_n^k)^T$ error at k^{th} iteration
X^0	an initial guess
X^*	exact solution of system
$ACOC(\rho_c)$	approximated computational order of convergence
J_F	Jacobian matrix of function
B	Banach space
C	computational cost per iteration
CEI	computational efficiency index
$[X, Y; F]$	divided difference
$U(X, r)$	open ball with center X and radius r
$\bar{U}(X, r)$	closed ball with center X and radius r

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Chapter 1

Preliminaries

1.1 Introduction

Numerical Analysis is performing an imperative part in the field of engineering, medical sciences, applied sciences and allied areas. It is a study of algorithms for solving the problems which involve real or complex variables. It is a mathematical subject of approximation theory in which many problems can be solved by producing a sequence of approximations by an iterative procedure. It enables us to solve large, complicated and complex problems, which are impossible or difficult to handle analytically. Numerical methods are an effective tool for learning to use computers. Since numerical methods are essentially designed for implementation on computers so they are ideal for illustrating their powers and limitations. Also, computers helps in recognizing and controlling the errors of approximation that are part and parcel of large-scale numerical calculations. Thus, with the advancement of the computers, numerical analysis gained more popularity than before.

Non-linear systems appear very frequently in real life problems such as electrical circuits, chemical reactions, physical law, biological phenomenon, computational economics etc. The modeling of such scientific problems in many cases leads to the formulation of non-linear systems naturally or construction of non-linear systems according to given conditions. For example, central finite-difference discretization of a second order non-linear boundary value problem (see reference [1, pp.360]) defined as:

$$y'' = y - y^2,\ y(0) = 1,\ y(1) = 4, \tag{1.1.1}$$

produces the following non-linear systems of $n-1$ equations in $n-1$ variables:

$$y_{i-1} - (2 + h_1^2)y_i + y_{i+1} + h_1^2 y_i^2 = 0,\ i = 1,\ 2,\ \ldots,\ n-1, \tag{1.1.2}$$

here the following partition of the interval $[0, 1]$ has been assumed

$$\begin{aligned} x_0 =& 0 < x_1 < x_2 < x_3 < \cdots < x_n = 1, \text{ where } x_i = x_0 + ih_1,\ h_1 = \frac{1}{n} \text{ and} \\ y_0 =& y(x_0) = 1,\ y_1 = y(x_1),\ \ldots,\ y_{n-1} = y(x_{n-1}),\ y_n = y(x_n) = 4. \end{aligned} \tag{1.1.3}$$

Similarly, the problem of the determination of the preliminary orbit of a celestial body [2], the problem of the characterization of chemical reactions [3], global positioning problem [4], load flow problem [5] and many more problems in control theory can be transformed into non-linear systems of equations. Due to the wider variety of behaviour in different fields, finding solutions of non-linear systems are much more harder than a scalar case and analytical methods for their solution are almost non-existent. Therefore, numerical techniques based on the iterative procedure are an alternate approach for solving non-linear systems. All the iterative methods support the concept of successive approximation, i.e., starting from an arbitrary point (initial approximation) which is the closest possible point to the solution sought and involves arriving at solution gradually through successive tests by substituting the initial approximations into some formulae involving the equations to get a sequence of successive approximations that within the limit converges to the solution. Normally, these iterative methods do not strive for the exactness. So, researchers tried to construct a method which is able to yield an approximate root different from the exact root within a particular tolerance limit, or by an amount that has less than a specified probability of exceeding that tolerance. In the author aspect, this is the primary reason for the vast collection of research papers on iterative methods and the other is that no iterative method applies to all types of problems. Therefore, construction and analysis of some higher-order numerical methods for evaluating the approximate solutions of the systems of non-linear equations for the present context is the main focus of the present thesis. Several computer algebra software systems such as Mathematica, Matlab and Maple etc are available that have turned the role and applications of non-linear systems in many scientific branches. With the help of these software, lengthy and complicated calculations have become easier and simpler as compared to the calculation done with a manual approach. Through these computational packages, graphical visualization of the accuracy of various numerical methods has become more crystal and clear.

The author motivation for writing this thesis is to put on record the work done through the years developing high-quality algorithms having the higher order of convergence, minimal computational cost and time, high precision and simple body structure of iterative methods for solving non-linear systems of equations.

1.2 Some essential concepts

This section incorporates some essential definitions and concepts that are foundation for the evaluation of numerical techniques. Most of the symbols and notations used in the thesis are global and maintain their meaning.

1.2.1 Mathematical form of non-linear system

Let $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a sufficiently differentiable vector function defined as:

$$F(X) = 0, \tag{1.2.1}$$

where D is a convex set and $F(X) = (f_1(X), f_2(X), \ldots, f_n(X))^T, X = (x_1, x_2, \cdots, x_n)^T$. The vector

$$X^* = (x_1^*, x_2^*, \cdots, x_n^*)^T, \tag{1.2.2}$$

which satisfies $f_1(X^*) = 0; f_2(X^*) = 0; \cdots; f_n(X^*) = 0$; is called the solution of systems of non-linear equations $F(X) = 0$.

1.2.2 Error equation

Let $e^k = X^k - X^*$ be the error of approximation in the k^{th} iterative step. Then the following relation

$$e^{k+1} = L(e^k)^\rho + O((e^k)^{\rho+1}), \tag{1.2.3}$$

is called an error equation. If one can obtain the error equation of any iterative method from convergence analysis, then the smallest exponent ρ of e^k represents the convergence order and L represents a $\rho - linear$ function. i.e.$L \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n \times \overset{\rho}{\ldots\ldots} \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{L}$ is the set of bounded linear functions.

1.2.3 Convergence analysis

In order to explore the convergence properties, let's recall the following results of Taylor's series expression on vector function (see [6]). Let $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be d-times Fréchet differentiable in a convex set $D \subseteq \mathbb{R}^n$ then for any $X, h \in \mathbb{R}^n$, the following expression holds:

$$\begin{aligned} F(X+h) =& F(X) + F'(X)h + \frac{1}{2!}F''(X)h^2 + \frac{1}{3!}F'''(X)h^3 + \ldots.. + \frac{1}{(d-1)!}F^{(d-1)}(X)h^{(d-1)} \\ &+ R_d. \end{aligned}$$

where

$$||R_d|| \leq \sup_{0 \leq u \leq 1} \frac{1}{d!}||F^{(d)}(x+uh)||||h||^d \text{ and } h^d = (h, h, \overset{d}{\ldots\ldots}, h).$$

For any $X, X^k \in D$ the Taylor's series of $F(X)$ about X^k as follows:

$$F(X) = F(X^k) + F'(X^k)(X - X^k) + \frac{1}{2!}F''(X^k)(X - X^k)^2 + \frac{1}{3!}F'''(X^k)(X - X^k)^3 + \dots + \frac{1}{(d-1)!}F^{(d-1)}(X^k)(X - X^k)^{(d-1)} + O(||X - X^k||^d).$$

1.2.4 Fréchet derivative

Let X, Y be normed linear spaces. The Fréchet derivative of an operator $F : X \to Y$ is the bounded linear operator $DF(a) : X \to Y$ which satisfies the following relation,

$$\lim_{\tilde{h} \to 0} \frac{||F(a + \tilde{h}) - F(a) - DF(a)\tilde{h}||}{||\tilde{h}||} = 0 \tag{1.2.4}$$

For functions $F : \mathbb{R}^n \to \mathbb{R}^n$ the Fréchet derivative $DF(a)$ is the Jacobian of F, a linear operator which is represented by an $n \times n$ matrix. The Jacobian matrix representation of $F'(X)$ is given by the matrix below:

$$J_F(X) = \begin{pmatrix} \frac{\partial f_1(X)}{\partial x_1} & \frac{\partial f_1(X)}{\partial x_2} & \dots & \frac{\partial f_1(X)}{\partial x_n} \\ \frac{\partial f_2(X)}{\partial x_1} & \frac{\partial f_2(X)}{\partial x_2} & \dots & \frac{\partial f_2(X)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(X)}{\partial x_1} & \frac{\partial f_n(X)}{\partial x_2} & \dots & \frac{\partial f_n(X)}{\partial x_n} \end{pmatrix}. \tag{1.2.5}$$

1.2.5 Divided difference

The divided difference $[X, Y; F]$ of F is a $n \times n$ matrix with elements [7] defined as:

$$[X, Y; F]_{ij} = \frac{f_i(x_1, ..., x_j, y_{j+1}, ..., y_n) - f_i(x_1, ..., x_{j-1}, y_j, ..., y_n)}{x_j - y_j} \tag{1.2.6}$$

where
$X = (x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n), Y = (y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_n)$ and $1 \leq i, j \leq n$.
Or
The divided difference defined in [8] as:

$$[W, X; F] = [X + F, X; F] = \Big(F(X + He_1) - F(X), \dots, F(X + He_n) - F(X)\Big)\{H\}^{-1},$$

with $H = diag(f_1(X), f_2(X), \dots, f_n(X))$ and $e_1 = (1, 0, \dots, 0)^T, e_2 = (0, 1, \dots, 0)^T, \dots,$ $e_n = (0, 0, \dots, 1)^T$.

1.2.6 Computational efficiency

For any iterative method the following computational efficiency index is defined in [9]:

$$CEI = \rho^{1/C}, \tag{1.2.7}$$

where ρ is called the order of convergence and C is the computational cost per iteration. For a system of 'n' non-linear equations with 'n' variables, the computational cost per iteration introduced by [10] is given as:

$$C(\nu, n, \ell) = A(n)\nu + P(n, \ell), \tag{1.2.8}$$

where $A(n)$ denotes the number of evaluations of scalar functions used in the evaluation of F, $[X, Y; F]$ and $P(n, \ell)$ denotes the number of products needed per iteration. Here, $\nu > 0$ is a ratio between products and evaluations of scalar functions and $\ell \geq 1$ is ratio between products and quotients.

1.2.6.1 Cost of elementary functions

To judge the values of ν and ℓ, the cost of evaluations of elementary functions is expressed in terms of products which depends upon the software and the arithmetics (see [11, 12]) used in computer. In the following Table 1.1, an estimation of the cost of the elementary functions in number of equivalent products is shown, where running time of one product is measured in milliseconds (ms). For the hardware and software used in the numerical work, the computational cost of quotient with respect to product is $\ell = 3$.

Table 1.1: Estimation of computational cost of elementary functions computed with Mathematica 10 in a processor Intel(R) Core (TM) i5-3210M CPU @ 2.50 GHz (64-bit machine) Microsoft Windows 8 , where $x = \sqrt{3} - 1$ and $y = \sqrt{5}$.

$Digits(2048)/Functions$	$x * y$	x/y	$\sqrt{x}$	$exp(x)$	$ln(x)$	$sin(x)$	$cos(x)$	$arccos(x)$	$arctan(x)$
$CPU\ time\ (ms)$	0.02313	0.0725	0.03203	2.734	3.063	1.937	1.922	3.469	3.359
$Cost$	1	3.13	1.38	118.20	132.42	83.74	83.09	149.98	145.22

1.2.6.2 Operation counts

In view of computational efficiency index for any iterative scheme, 'n' scalar functions for $F(X)$ and $n(n-1)$ scalar functions for forward divided difference $[X, Y; F]$ and $n(n+1)$ scalar functions for central divided difference $[X + F, X - F; F]$, where $F(X)$ and $F(Y)$ are computed separately, are considered. Also n^2 quotient for any divided difference and

n^2 products for multiplication of a vector by a scalar are added. In order to calculate an inverse linear operator (LU decomposition) following operational counts have been taken.

1.2.6.3 LU decomposition method

$$\text{Total number of products} = \frac{n}{6}(n-1)(2n-1)$$
$$\text{Total number of quotients} = \frac{n}{2}(n-1).$$

1.2.6.4 Resolution of two triangular linear systems

$$\text{Total number of products} = n(n-1)$$
$$\text{Total number of quotients} = n.$$

1.2.6.5 Total counts for inverse matrix

$$\text{Total number of products} = \frac{n}{6}(n-1)(2n+5)$$
$$\text{Total number of quotients} = \frac{n}{2}(n+1).$$

For the simplicity, the author has evaluated the following computational efficiency cost of variant of Ostrowski's method (see [7]) and shown in Table 1.2

$$\begin{aligned} Y^k &= X^k - \{[X^k+F, X^k-F; F]\}^{-1}F(X^k), \\ X^{k+1} &= Y^k - \{2[Y^k, X^k; F] - [X^k+F, X^k-F; F]\}^{-1}F(Y^k). \end{aligned}$$

This scheme involves two function evaluations $F(X^k)$, $F(Y^k)$ two divided difference operators: one central divided difference $[X^k+F, X^k-F; F]$ and other forward divided difference operator $[Y^k, X^k; F]$ and two inversion matrices $\{[X^k+F, X^k-F; F]\}^{-1}$ and $\{2[Y^k, X^k; F] - [X^k+F, X^k-F; F]\}^{-1}$.
Note that, the proposed schemes throughout this work utilizes central divided difference operator for first step and forward divided difference for remaining steps of iterative schemes.

Table 1.2: Computational efficiency of a variant of Ostrowski's method

Function evaluations	$A(n)$	*Product counts*	*Quotient counts*	$C(\nu,n,\ell)=A(n)\nu+P(n,\ell)$
$F(X^k)$	n			
$F(Y^k)$	n			
$[X^k+F,X^k-F;F]$	$n(n+1)$		n^2	
$[Y^k,X^k;F]$	$n(n-1)$		n^2	
$\{[X^k+F,X^k-F;F]\}^{-1}$		$\frac{n}{6}(n-1)(2n+5)$	$\frac{n}{2}(n+1)$	
$\{2[Y^k,X^k;F]-[X^k+F,X^k-F;F]\}^{-1}$		$\frac{n}{6}(n-1)(2n+5)$	$\frac{n}{2}(n+1)$	
Computational efficiency				$\frac{n}{3}(2n^2+3n-5)+$ $(2n^2+2n)\nu+\ell n(3n+1)$

1.2.7 Approximate order of convergence

The approximated computational order of convergence (ACOC) [13, 14] is defined by

$$\rho_c \approx \frac{\ln \frac{\|e^{k+1}\|}{\|e^k\|}}{\ln \frac{\|e^k\|}{\|e^{k-1}\|}}, \qquad \text{for each } k=1,\ 2,\ldots, \tag{1.2.9}$$

where e^{k-1}, e^k, and e^{k+1} are errors at three consecutive iterations.
or

$$\rho_c \approx \frac{\ln \frac{\|X^{k+1}-X^k\|}{\|X^k-X^{k-1}\|}}{\ln \frac{\|X^k-X^{k-1}\|}{\|X^{k-1}-X^{k-2}\|}}, \qquad \text{for each } k=2,\ 3,\ldots, \tag{1.2.10}$$

where X^{k-2}, X^{k-1}, X^k, and X^{k+1} are four consecutive approximations in the iterations process.

1.2.8 Convex set

Let X be a vector space. A set $S \subset X$ is called convex if for every $x, y \in S$ and every $a \in \mathbb{R}$, with $0 \le a \le 1$, the element $ax + (1-a)y \in S$. In other words, the line segment joining x and y is contained in S. For example, a solid cube is a convex set, but anything that is hollow or has an indent, for example, a crescent shape, is not convex.

1.2.9 Open ball

Let $\mathbb{R}^n$ be any n dimensional space, $X \in \mathbb{R}^n$ and $r > 0$ be any positive real number, then the open ball $U(X, r)$ center at X with radius r is defined as:

$$U(X,r) = \{Y \in \mathbb{R}^n : ||Y - X|| < r\}. \tag{1.2.11}$$

1.2.10 Closed ball

An closed ball $\bar{U}(X,r)$ with center at $X \in \mathbb{R}^n$ and radius $r > 0$ is defined as:

$$\bar{U}(X,r) = \{Y \in \mathbb{R}^n : ||Y - X|| \leq r\}. \tag{1.2.12}$$

1.2.11 Norm

A norm on a vector space X is a non negative-valued scalar function $F : X \to [0,\infty)$, which satisfies the following properties. For all $a \in \mathbb{R}$ and all $x, y \in X$,

- $x \neq 0 \Rightarrow ||x|| > 0$,
- $||ax|| = |a|||x||$,
- $||x + y|| \leq ||x|| + ||y||$.

1.2.12 Banach space

A complete normed vector space X is called a Banach space. The complete space means the space in which every Cauchy sequence of vectors always converges to a well defined limit that is within the space X. In other words, a Banach space is a vector space X over the field $\mathbb{R}$ of real numbers, or over the field $\mathbb{C}$ of complex numbers, which is complete with respect to the norm $\|\cdot\|_X$ that is, for every Cauchy sequence x_n in X, there exists an element x in X such that $\lim\limits_{n\to\infty} x_n = x$. For instance,

- The vector space $C[0,1]$ with the norm $||x|| = \max\{|x(t)| : t \in [0,1]\}$ is a Banach space.
- The vector space $P[0,1]$ of polynomials on $[0,1]$ with the norm $||x|| = \max\{|x(t)| : t \in [0,1]\}$ is a normed vector space which is not complete so not a Banach space.

1.2.13 Banach lemma

Banach lemma on invertible operators $[15, 16]$ is defined as:
If T is a bounded linear operator in X, T^{-1} exists if and only if there is a bounded linear operator P in X such that P^{-1} exists and $||I - PT|| < 1$. If T^{-1} exists, then

$$T^{-1} = \sum_{n=0}^{\infty} (I - PT)^n P \tag{1.2.13}$$

and

$$||T^{-1}|| \leq \frac{||P||}{1 - ||I - PT||}. \tag{1.2.14}$$

1.3 Classification of iterative methods

Traub [17, pp. 8-9] had classified iterative methods into two categories namely, one-point iterative methods and multipoint iterative methods. These categories are further divided into two sub-categories: one-point iterative methods with and without memory and multipoint iterative methods with and without memory. The details of these classifications are as follows:

1.3.1 One-point iterative methods without memory

If the estimation of a solution is determined only by using new information at one point and no old information is required, then the method is called a one-point iterative method without memory. Thus, if X^{k+1} is determined by new information at X^k and no old information is used, one can write

$$X^{k+1} = G(X^k), \quad k = 0, 1, 2, \ldots,$$

then G is called a one-point iteration method without memory. According to the Traub [17, Theorem 5.3], to construct a one-point iteration method of order ρ, one has to use all derivatives up to order $\rho - 1$. The most common and popular one-point iterative method without memory shown below

$$X^{k+1} = X^k - \{F'(X^k)\}^{-1} F(X^k), \quad k = 0, 1, 2, \ldots,$$

is called Newton method. Other one-point iterative method without memory have been developed as Chebyshev method, Halley method, Super-Halley method.

1.3.2 One-point iterative methods with memory

An iterative method is called a one-point iterative method with memory, if estimation to a solution is determined by using new information at one point and by using previous information at either one or more than one points. Thus, if X^{k+1} is determined by new information at X^k and reused information at $X^{k-1}, \ldots, X^{k-s}$, one can have

$$X^{k+1} = G(X^k; X^{k-1}, \ldots, X^{k-s}), s \geq 1$$

then G is called a one-point iterative function with memory. The semicolon separates the point at which new data are used from the points at which old data are reused. One of the well-known one-point iterative methods with memory is the secant method.

1.3.3 Multipoint iterative methods without memory

An iterative method is called a multipoint iterative method without memory, if the estimation of a solution is determined only by new information at a number of points, no old information is used. Thus, if X^{k+1} is determined by new information at $X^k, W_1(X^k), W_2(X^k), \ldots, W_i(X^k), \quad i \geq 1$, no old information is reused, one can have

$$X^{k+1} = G(X^k, W_1(X^k), W_2(X^k), \ldots, W_i(X^k)),$$

then G is called a multipoint iterative function without memory. Newton-secant method, Traub-Ostrowski's method, Jarratt's method etc. are all the famous multipoint iterative methods without memory.

1.3.4 Multipoint iterative methods with memory

If estimation to a solution is determined by new information at a number of points with reusing the old information at some other points, the method is called a multipoint iterative method with memory. Consider Z^j represents the $i+1$ quantities $X^j, W_1(X^j), W_2(X^j), \ldots, W_i(X^k), \quad i \geq 1$, and

$$X^{k+1} = G(Z^k; Z^{k-1}, \ldots, Z^{k-s}), s \geq 1.$$

Then G is called a multipoint iterative function with memory. The semicolon separates the points at which new data are used from the points at which old data is reused. Traub [17] was the first mathematician who introduced a multipoint iterative method with memory.

1.4 Exploration of literature

One of the most basic and earliest problems of numerical analysis concerns with finding efficiently and accurately the approximate solution of a non-linear system (1.2.1). Several iterative techniques can be found in the literature to solve non-linear systems. Among all these techniques the most popular, effective and basic iterative technique is Newton's method [17] and given by

$$X^{k+1} = X^k - \{F'(X^k)\}^{-1} F(X^k), \tag{1.4.1}$$

where $F'(X^k)$ is Jacobian matrix of F and $F'(X^k) = \frac{\partial}{\partial x_j} f_i(X^k)$. Although this technique has quadratic convergence and is easy to apply, still, it has two drawbacks. Firstly, it is very sensitive to the choice of initial guess and secondly, this method fails, if at any stage of the computation the Jacobian of the function is zero or very small in the vicinity of required solutions.

Kou *et al.* [18] have proposed a modified Newton's method given as

$$X^{k+1} = X^k - \{F'(X^k) + diag(\alpha F(X^k))\}^{-1} F(X^k), \alpha \in \mathbb{R} \text{ and } 0 < |\alpha| < \infty. \tag{1.4.2}$$

This method works even if Jacobian is zero or very small. Clearly, this method preserves same computational cost and order of convergence as Newton's method requires. With the passage of time, a lot of work has been carried to increase the order of convergence of Newton's Method.

In this direction, the following third order family of iterative methods have been introduced by Chebyshev and Halley [19, 20]

$$X^{k+1} = X^k - \left[I + \frac{1}{2} L_F(X^k)\right] \{F'(X^k)\}^{-1} F(X^k), \tag{1.4.3}$$

$$X^{k+1} = X^k - \left[I + \frac{1}{2} L_F(X^k) \left\{I - \frac{1}{2} L_F(X^k)\right\}^{-1}\right] \{F'(X^k)\}^{-1} F(X^k), \tag{1.4.4}$$

respectively, where I is an identity operator on a Banach space X and $L_F(X^k)$ the linear operator defined by

$$L_F(X^k) = \{F'(X^k)\}^{-1} F''(X^k) \{F'(X^k)\}^{-1} F(X^k),$$

provided that $\{F'(X^k)\}^{-1}$ exists. Unfortunately, in these techniques, the second-order derivative is to be evaluated at each iteration, which is very difficult and time-consuming. In fact, for the $n \times n$ system, the first order Fréchet derivative or Jacobian F' is a matrix with n^2 evaluations while the second-order Fréchet derivative F'' is a matrix with $\frac{n^2(n+1)}{2}$ evaluations. Thus from the computational point of view, iterative schemes having second-order Fréchet derivative are less efficient. For this reason, several third-order methods like Halley's method [19, 20] and Chebyshev's method [19, 20] etc. are not preferred to solve systems of non-linear equations.

In quest of constructing schemes without the involvement of second-order Fréchet derivative, a variety of third and higher -order multipoint methods have been available in the literature. For instance, Hernàndez [21] scheme defined as:

$$\begin{aligned} Y^k =& X^k - \{F'(X^k)\}^{-1}F(X^k), \\ Z^k =& X^k + \frac{1}{2}(Y^k - X^k), \\ X^{k+1} =& Y^k - \{F'(X^k)\}^{-1}\left[F'(Z^k) - F'(X^k)\right](Y^k - X^k). \end{aligned} \tag{1.4.5}$$

and the following cubically convergent technique introduced by Cordero and Torregrosa [22] which are variants of Newton's method based upon trapezoidal and midpoint rules of quadrature:

$$X^{k+1} = X^k - 2\left\{F'(X^k) + F'(Z^k)\right\}^{-1} F(X^k), \tag{1.4.6}$$

and

$$X^{k+1} = X^k - \left\{F'\left(\frac{X^k + Z^k}{2}\right)\right\}^{-1} F(X^k), \tag{1.4.7}$$

respectively, where $Z^k = X^k - \left\{F'(X^k)\right\}^{-1} F(X^k)$ and $F'(X^k)$ is a Jacobian matrix of function F evaluated at X^k. These methods are more efficient than classical Newton's method.

In [23], Darvishi and Barati have presented a third order Newton's type method based on Adomian decomposition technique to solve the system of non-linear equations. They have also constructed two super-cubic iterative methods [24] to solve non-linear systems. Later on, Darvishi and Barati [25] have developed a fourth-order scheme free from second order derivative and given by

$$X^{k+1} = X^k - \left\{\frac{1}{6}F'(X^k) + \frac{2}{3}F'\left(\frac{X^k + G(X^k)}{2}\right) + \frac{1}{6}F'\left(G(X^k)\right)\right\}^{-1} F(X^k), \tag{1.4.8}$$

where

$$G(X^k) = X^k - \{F'(X^k)\}^{-1}\Big(F(X^k) + F\Big(X^k - \{F'(X^k)\}^{-1}F(X^k)\Big)\Big). \tag{1.4.9}$$

Now researchers started contributing in the development of iterative schemes having more than third-order of convergence to solve systems of non-linear equations. Like, Nedzhibov [26] has constructed a family of multi-point iterative methods for solving systems of non-linear equations, which is given below:

$$\phi(X^k) = X^k - \left(I + \frac{1}{2\beta}\{I - \frac{\zeta}{\beta}G(X^k)\}^{-1}G(X^k)\right) K(X^k), \tag{1.4.10}$$

where $G(X^k) = I - H(X^k)F'(Y^k),\ Y^k = X^k - \beta h(X^k),\ K = H(X^k)F(X^k)$ and $H(X^k) = \{F'(X^k)\}^{-1}$. This family depends upon two parameters, one evaluation of the vector function F, the solution of two linear systems and two Jacobian functions per iteration. For parameters $\zeta \neq 1$ and $\beta \neq \frac{2}{3}$ it reduces to third order of convergence while for $\zeta = 1$ and $\beta = \frac{2}{3}$ it becomes fourth order of convergence scheme.

Wang [27] in 2009 has introduced a cubically convergent method that does not require the evaluation of second or higher order derivative and applicable for those problems, where Jacobian is singular at some points. This method per iteration needs two evaluations of function vector $F(X^k)$ and solution of one linear system. In the similar year, Noor and Wassem [28] have introduced a third order two-step iterative method based upon quadrature formulae. Cordero *et al.* [29] have constructed a family of methods with order four and five based on the Adomian decomposition technique. They concluded that this iterative family is more stable, efficient and uses fewer iterations than the classical Newton's and Traub's method [17]. Babajee *et al.* [30] have considered two Chebyshev-like third order methods which are free from second-order derivative. They have used different approximations to the second-order derivative present in the Chebyshev method and used point of attraction theory to prove the local convergence.

Cordero *et al.* [31] have derived new iterative methods with the order of convergence four or higher for solving non-linear systems. They have composed these iterative methods with modified Newton's method and golden ratio method. Several new families of order four have been presented by Kanwar *et al.* [32]. These families are based on two parameters and require the evaluation of one function and two first order Fréchet derivatives per iteration. They have analyzed a wide general class of Jarratt's method for solving non-linear equations in the multivariate case.

Since the evaluation of first order Fréchet derivative is very costly or time consuming at each step of iteration which leads to high computation cost. So, to reduce the computational cost, some researchers utilize the first-order divided operator instead of first-order Fréchet derivative. Grau *et al.* [7] have proposed a derivative free algorithms from Ostrowski's method [33] by using divided difference operator and given by

$$\begin{aligned} Y^k =& X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k), \\ Z^k =& Y^k - \{2[Y^k, X^k; F] - [X^k + F, X^k - F; F]\}^{-1} F(Y^k), \\ W^k =& Z^k - \{2[Y^k, X^k; F] - [X^k + F, X^k - F; F]\}^{-1} F(Z^k). \end{aligned} \tag{1.4.11}$$

In the similar way, Sharma and Arora [34] have also presented the multi-step iterative schemes by using divided difference operator based upon weight functions. Further, Wang and Zhang [8] have introduced a seventh order scheme as shown below:

$$
\begin{aligned}
Y^k =& X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k), \\
Z^k =& Y^k - \{[Y^k, X^k; F] + [Y^k, X^k + F; F] - [X^k + F, X^k; F]\}^{-1} F(Y^k), \\
W^k =& Z^k - \{[Y^k, X^k; F] + [X^k + F, Y^k; F] - [Y^k, X^k : F]\}^{-1} F(Z^k).
\end{aligned}
\tag{1.4.12}
$$

They have used the minimum number of function evaluations to reduce the computational cost. Later on, Sharma *et al.* [35] presented a derivative free two-step family of fourth-order methods for solving systems of non-linear equations using the well-known Traub-Steffensen method in the first step. This method is a three-parameter family and totally derivative free.

In 2014, Soleymani *et al.* [36] have introduced a class of m step methods with the order of convergence $m+1$ by considering a frozen Jacobian. Similarly, Sharma and Gupta [37,38] and Grau *et al.* [39] have introduced a derivative free multistep family of higher order to solve the systems of non-linear equations.

Recently researchers have developed higher order, derivative free and minimum computational cost iterative techniques. In 2015, Malik Zaka Ullah *et al.* [40] have presented a class of multi-step iterative methods for finding approximate real and complex solutions of non-linear systems. The point of attraction theory was used to prove the convergence behavior. Ezquerro *et al.* [41] have extended the family of fourth-order Steffensen-type methods which was proposed by Zheng *et al.* [42]. In this extension, they have used multi-dimensional divided differences of first and second orders. Wang *et al.* [43] have developed a multi-step iterative method with seventh order convergence and derivative free technique for solving systems of non-linear equations. The advantage of this method is less computational time for the large non-linear system. Based on Newton's method Xiao and Yin [44] have proposed an $m+1$ step iterative scheme with $m+2$ convergence order to solve the system of non-linear equations. Even, in this same year, Cordero *et al.* [45, 46] not only constructed the iterative methods for the solution of systems of non-linear equations but also they have shown the applicability of these schemes in Banach spaces. Santiago *et al.* [47] designed the iterative methods of order four and five by using weight function. In present days, the iterative schemes [48–53] for solution of non-linear systems have developed in parameters of efficiency, dynamical stability, local and semi-local convergence analysis etc.

It is observed from the literature that most of iterative schemes have been developed by extension of one-dimensional equations to multidimensional case, by the construction of a new step, by the involvement of some parameters etc. In general, every method for scalar non-linear equations cannot be extended to solve non-linear systems. Furthermore, the concept of optimality regarding functional evaluations and order of convergence is meaningless in the construction of iterative techniques for the solution of systems of non-linear equations. For instance, the well-known Kung-Traub conjecture [54] does not hold for systems of non-linear equations. The reason behind this fact is, function evaluation and its derivative evaluation affects the computational cost whereas there are some other factors like evaluation of inverse matrix, number of products, number of quotents etc. involved in computation cost of systems of non-linear equations. In multidimensional case, an iterative scheme is said to be more efficient if it involves the minimum number of matrix inversions, functional evaluations, fewer matrix multiplications, CPU time per iteration and simple structure of iterative schemes to achieve the desired degree of accuracy.

1.4.1 Stopping criteria and computational aspects

In literature, there are many iterative methods for approximating the solution of non-linear systems. Several measures have been utilized to compare different iterative schemes and to claim the superiority and inferiority of some of them over others. The measures for comparison of iterative methods are usually in terms of convergence order, the numerical stability, computational costs, asymptotic error constants, the dependence of convergence on the choice of initial guesses, the simple body structures, CPU time and a number of iterations required for convergence until a given tolerance is achieved. In essence, one can apply the more simple procedure for checking which method is superior than others in the following way: method A is superior than method B if A attains more accuracy in terms of significant figures gained by each method by utilizing same total numbers of function evaluation with same initial approximations. For better and fair comparison of iterative methods, the author has displayed the residual errors at first three iterations, a number of iterations required to achieve the desired tolerance, the computational cost of convergence to verify the theoretical convergence order and CPU time.

It is well-known that most of the one-point and higher-order multipoint methods have been developed by considering Newton's method as the base step. Thus, for the good convergence of Newton-like methods an initial approximation must be close enough to the desired solution. But, finding a reasonably good approximation that guarantees the convergence is very difficult and non-trivial task. So, one of the way to choose initial approximation is Broyden's method [55, 56] and other way is to find initial guess using mathematical modeling. Throughout the thesis work, all computations of nonlinear systems are performed in the software Wolfram Mathematica-11 [57] using multiple-precision arithmetic. Using Mathematica command $SetAccuracy[expr, 2000]$, 2000 digits floating point arithmetic have been considered to minimize the round-off errors. An approximate solution up to a specified degree rather than exact root has been accepted.

Thus, the following stopping criterion in computer programs is considered:

$$||X^{k+1} - X^k|| + ||F(X^k)|| < \epsilon, \tag{1.4.13}$$

where $\epsilon = 10^{-1000}$, for finding the number of iterations for each method. If the stopping criteria are satisfied, then X^{k+1} is considered as the required solution.

1.5 Summary of the thesis

In this thesis, the author will present many new computationally effective root-finding methods for solving non-linear systems of equations. The main goal and motivation is the development of new equally competitive methods to achieve the highest computational efficiency with a fixed number of function evaluations per iteration. In addition to this, this work offers a collection of recently used iterative strategies of solving systems of nonlinear equations for future researchers who are interested in this topic. The present thesis has been prepared in six chapters where the introduction is enclosed within the first chapter. A brief summary of the contents of each chapter is as follows:

In **the second chapter,** several families of Ostrowski's method for solving non-linear systems have been proposed. The new families are completely derivative free, higher order and suited to those problems in which derivatives require lengthy computations. Grau's methods are seen as special cases of these families. Numerical experiments demonstrate that proposed multipoint strategies are viable and equivalent to the outstanding iterative techniques.

In **chapter third,** the local convergence of proposed family discussed in **chapter second** has been proved. Further some practical differential problems like Bratu's problem in one and two dimensional, Fisher's problem, Frank-Kamenetskii problem etc. has been solved. The results indicate better performance in terms of CPU time.

In **chapter fourth**, the author has constructed a new higher-order family of iterative methods in such a way that it eradicates the drawback of evaluating the first and higher order derivatives at each iteration. The proposed iterative methods are the modification of Steffensen's type method. Some particular cases of the proposed family have been presented. The proposed innovative family has assured the theoretical convergence using Taylor's series expansion. Computational efficiencies of the developed scheme is considered and compared with their closest competitors. Moreover, by the way of numerical experiments which are tested on several large and complex non-linear systems, it is shown that the results are effective and comparable to other existing methods and also support the theoretical results.

The fifth chapter presents the applicability of convergence of numerical techniques in Banach space. The local convergence analysis of the proposed family of methods presented in **chapter fourth** has been done which is based on Lipschitz constants and hypotheses on the divided difference of order one in the more general settings of a Banach space.

In last **sixth Chapter,** the author has explored the future scope of work.

Towards the finish of the thesis, a bibliography has been given which by no means is a comprehensive one but lists only those research papers and books which have been referred to in the main context of the thesis.

Chapter 2

A family of higher order derivative free methods for non-linear systems

The main focus of this chapter is to find the numerical solution of non-linear system (1.2.1). The author starts with the well-known general class of Ostrowski's families without memory proposed by Behl *et al.* (International Journal of Computer Mathematics, 90(2), (2013), 408-422) and extended to solve systems of non-linear equations. This extension uses multidimensional divided differences of first-order. Many more new derivative free iterative families with higher-order local convergence are presented. Also the proposed iterative family for $\alpha_1 \in \mathbb{R}\backslash\{0\}$ and $\alpha_2 = 0$ are special cases of Grau *et al.* (Journal of Computational and Applied Mathematics, 237, (2013), 363–372) for iterative schemes of fourth and sixth order. The computational efficiency is compared with some known methods. It is proved that the proposed methods are equally competent with their existing counter parts. Numerical experiments are performed which support the theoretical results.

2.1 Introduction

The important attention of this work is to provide a incredibly efficient class of higher-order derivative free family for solving non-linear systems (1.2.1). Many iterative methods [4, 7, 8, 10, 13, 18, 22–44, 51–53, 58–63] have been appeared in the literature for solving such type of non-linear systems. Almost all of these iterative procedures has used Newton's method [17] as a base step and refine it further to construct the new iterative schemes. Though the Newton's method is most popular and quadratic convergence iterative method, defined as follows:

$$Y^k = X^k - \left\{F'(X^k)\right\}^{-1} F(X^k), \quad k = 0, 1, 2, ... \tag{2.1.1}$$

where $\left\{F'(X^k)\right\}^{-1}$ is the inverse of first Fréchet derivative $F'(X^k)$ of the involved function $F(X^k)$. But, this method has two well-known drawbacks. One is its dependence on a sufficiently close initial guess and other one is case of failure when computation of derivative of a function is very small or zero. Moreover, computation of first-order Fréchet derivative at

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each step is expensive in terms of time and difficulty in large complex systems of equations. It is quite obvious that all the modifications or variants of Newton's method (see [22, 23]) have also same drawbacks. Traub [17] introduced a different approach to construct the iterative scheme for solving non-linear systems of equations that overcome the above drawbacks. This technique replaces first Fréchet derivative with first-order divided difference operator and maintaining the same Newton's order of convergence, which is defined as follows:

$$Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \tag{2.1.2}$$

where $W^k = X^k + \beta_1 F(X^k)$; $\beta_1 \in \mathbb{R}\backslash\{0\}$ and $[W^k, X^k; F]$ is defined in (1.2.6) as first order divided difference of F in n dimensional space as a $n \times n$ matrix. For $\beta_1 = 1$, the Traub's scheme reduces to Steffensen's method [64]. Motivated and inspired from this work, many scholars from worldwide have constructed several variants of Traub scheme (2.1.2) in their research articles [7, 8, 10, 33–37, 39, 43, 58–62, 65] where the local order of convergence was preserved. In this regards, the author has extended the following family of Ostrowski's families without memory proposed by Behl *et al.* [66] in multi-dimensional case which is given as:

$$\begin{aligned} y^k &= x^k - \frac{f(x^k)}{f'(x^k)}, \\ x^{k+1} &= x^k - \frac{f(x^k)}{f'(x^k)} \left[\frac{(\alpha_1^2 + \alpha_1\alpha_2 - \alpha_2^2) f(x^k) f(y^k) - \alpha_1(\alpha_1 - \alpha_2) f(x^k)^2}{(\alpha_1 f(x^k) - \alpha_2 f(y^k))((2\alpha_1 - \alpha_2) f(y^k) - (\alpha_1 - \alpha_2) f(x^k))} \right], \end{aligned} \tag{2.1.3}$$

where $\alpha_1,\ \alpha_2 \in \mathbb{R}$ but choose α_1 and α_2 such that neither $\alpha_1 = 0$ nor $\alpha_1 = \alpha_2$.
Using the concept of first-order divided difference operator introduced by Traub [17] in place of first Fréchet derivative $f'(X^k)$ of the function of $f(X^k)$ in scheme (2.1.3), the scholar has constructed higher-order generalization for several variables of given families of Ostrowski's methods. The proposed iterative family for $\alpha_1 \in \mathbb{R}\backslash\{0\}$ and $\alpha_2 = 0$ are special cases of Grau *et al.* [7] for iterative schemes of fourth and sixth order.

2.2 Construction of an iterative family

For the construction of new iterative family some modifications are considered over scheme (2.1.3). The first step of (2.1.3) can be used as:

$$\begin{cases} y^k = x^k - \dfrac{f(x^k)}{f'(x^k)} \\ \text{or} \\ y^k - x^k = -\dfrac{f(x^k)}{f'(x^k)}. \end{cases} \tag{2.2.1}$$

Now, the second step of iterative scheme (2.1.3) rewritten as:

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)} - \frac{f(x^k)}{f'(x^k)}\Big[-1+ \\ \frac{(\alpha_1^2+\alpha_1\alpha_2-\alpha_2^2)f(x^k)f(y^k)-\alpha_1(\alpha_1-\alpha_2)(f(x^k))^2}{\alpha_1^2\left(2f(x^k)f(y^k)-(f(x^k))^2\right)+\alpha_2\alpha_1\left((f(x^k))^2-2(f(y^k))^2\right)+\alpha_2^2\left((f(y^k))^2-f(x^k)f(y^k)\right)}\Big]. \tag{2.2.2}$$

Using the expression (2.2.1) and after simplification the above expression shown as:

$$x^{k+1} = y^k - \frac{\eta(x^k,y^k)}{\tau(x^k,y^k)}f(y^k), \tag{2.2.3}$$

where

$$\begin{cases} \eta(x^k,y^k) = (\alpha_1\alpha_2-\alpha_1^2)f(x^k)+(2\alpha_1\alpha_2-\alpha_2^2)f(y^k), \\ \tau(x^k,y^k) = \dfrac{1}{y^k-x^k}\Big[\left(\alpha_1\alpha_2-\alpha_1^2\right)(f(x^k))^2+\left(2\alpha_1^2-\alpha_1\alpha_2\right)f(x^k)f(y^k) \\ \qquad +\left(\alpha_1\alpha_2-\alpha_2^2\right)f(y^k)f(x^k)+\left(\alpha_2^2-2\alpha_1\alpha_2\right)(f(y^k))^2\Big]. \end{cases} \tag{2.2.4}$$

Note that for simplicity $\eta(x^k,y^k)$ and $\tau(x^k,y^k)$ are consider as η and τ respectively. After some simplifications, (2.2.4) can be written as:

$$\begin{cases} \eta = (2\alpha_1\alpha_2-\alpha_2^2)\left(f(y^k)-f(x^k)\right)+(3\alpha_1\alpha_2-\alpha_1^2-\alpha_2^2)f(x^k), \\ \tau = \dfrac{1}{y^k-x^k}\Big[\left(-\alpha_1\alpha_2+\alpha_1^2\right)(f(x^k))^2+\left(2\alpha_1^2+\alpha_2^2-3\alpha_1\alpha_2\right)f(x^k)\left(f(y^k)-f(x^k)\right) \\ \qquad -\alpha_1\alpha_2\left(f(y^k)-f(x^k)\right)f(x^k)+\left(\alpha_2^2-2\alpha_1\alpha_2\right)\left(f(y^k)-f(x^k)\right)^2\Big]. \end{cases} \tag{2.2.5}$$

Dividing the η and τ of the above expression (2.2.5) by $y_k - x_k$ and with the help of (2.2.1), one obtain

$$\begin{cases} \eta = (2\alpha_1\alpha_2-\alpha_2^2)\dfrac{f\left(y^k\right)-f\left(x^k\right)}{y_k-x_k}-(3\alpha_1\alpha_2-\alpha_1^2-\alpha_2^2)f'\left(x^k\right), \\ \tau = -\left(-\alpha_1\alpha_2+\alpha_1^2\right)(f'(x^k))^2+\left(2\alpha_1^2+\alpha_2^2-3\alpha_1\alpha_2\right)f'\left(x^k\right)\dfrac{f\left(y^k\right)-f\left(x^k\right)}{y_k-x_k} \\ \qquad -\alpha_1\alpha_2\dfrac{f\left(y^k\right)-f\left(x^k\right)}{y_k-x_k}f'\left(x^k\right)-\left(\alpha_2^2-2\alpha_1\alpha_2\right)\left(\dfrac{f\left(y^k\right)-f\left(x^k\right)}{y_k-x_k}\right)^2. \end{cases} \tag{2.2.6}$$

Further using the concept of divided difference operator $[y^k, x^k; F] = \frac{f(y^k)-f(x^k)}{y_k-x_k}$ and central divided operator $[x^k - f, x^k + f; f]$ in place of $f'(x^k)$ in (2.2.6) reduces to the following expressions

$$\begin{cases} \eta =(2\alpha_1\alpha_2 - \alpha_2^2)[y^k, x^k; F] - (3\alpha_1\alpha_2 - \alpha_1^2 - \alpha_2^2)[x^k - f, x^k + f; f], \\ \tau = - \left(-\alpha_1\alpha_2 + \alpha_1^2\right) [x^k - f, x^k + f; f]^2 + \left(2\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2\right) [x^k - f, x^k + f; f] \\ \quad \times [y^k, x^k; F] - \alpha_1\alpha_2[y^k, x^k; F][x^k - f, x^k + f; f] - \left(\alpha_2^2 - 2\alpha_1\alpha_2\right) [y^k, x^k; F]^2. \end{cases} \tag{2.2.7}$$

So, final modified iterative scheme (2.1.3) with expressions (2.2.3) and (2.2.7) are given as:

$$\begin{aligned} y^k =& x^k - \frac{f(x^k)}{f'(x^k)}, \\ x^{k+1} =& y^k - \frac{\eta}{\tau} f(y^k). \end{aligned} \tag{2.2.8}$$

The author has applied this modified scheme (2.2.8) in multivariable non-linear equations and further extended to higher order schemes shown as:

$$\begin{cases} \psi_1^k =X^{k+1} = X^k - [X^k + F, X^k - F; F]^{-1} F(X^k), \\ \psi_2^k =\psi_1^k - \eta\ F(\psi_1^k), \\ \psi_3^k =\psi_2^k - \eta\ F(\psi_2^k), \\ \psi_4^k =\psi_3^k - \eta\ F(\psi_3^k), \\ \quad \vdots \\ \psi_{i-1}^k =\psi_{i-2}^k - \eta\ F(\psi_{i-2}^k), \\ \psi_i^k =\psi_{i-1}^k - \eta\ F(\psi_{i-1}^k). \\ \quad \text{This relation is true for } i = 2, 3, 4 \ldots m. \end{cases} \tag{2.2.9}$$

Here,

$$\begin{cases} \eta =\tau^{-1}\Big(- (\alpha_2^2 - 2\alpha_1\alpha_2)[\psi_1^k, X^k; F] + (\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2)[X^k + F, X^k - F; F]\Big), \\ \tau =(2\alpha_1\alpha_2 - \alpha_2^2)[\psi_1^k, X^k; F][\psi_1^k, X^k; F] - \alpha_1\alpha_2[\psi_1^k, X^k; F][X^k + F, X^k - F; F] \\ \quad + (2\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2)[X^k + F, X^k - F; F][\psi_1^k, X^k; F] \\ \quad - (\alpha_1^2 - \alpha_1\alpha_2)[X^k + F, X^k - F; F][X^k + F, X^k - F; F]. \end{cases} \tag{2.2.10}$$

Here α_1 and α_2 are real parameters. From equation (2.2.9), the various multi-step methods can be proposed by taking different values of α_1 and α_2.

2.2.1 Special cases

(i) For $\alpha_1 \in \mathbb{R}\backslash\{0\}$ and $\alpha_2 = 0$, first two-steps ($i = 2$) of family (2.2.9) reduces as follows:

$$\begin{aligned}\psi_1^k =& X^{k+1} = X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k),\\ \psi_2^k =& \psi_1^k - \left\{2[\psi_1^k, X^k; F] - [X^k + F, X^k - F; F]\right\}^{-1} F(\psi_1^k).\end{aligned}$$

This is a fourth-order iterative scheme derived by Grau *et al.* [7].

(ii) For $\alpha_1 \in \mathbb{R}\backslash\{0\}$ and $\alpha_2 = 0$, first three-steps ($i = 3$) of family (2.2.9) reads as:

$$\begin{aligned}\psi_1^k =& X^{k+1} = X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k),\\ \psi_2^k =& \psi_1^k - \left\{2[\psi_1^k, X^k; F] - [X^k + F, X^k - F; F]\right\}^{-1} F(\psi_1^k),\\ \psi_3^k =& \psi_2^k - \left\{2[\psi_1^k, X^k; F] - [X^k + F, X^k - F; F]\right\}^{-1} F(\psi_2^k).\end{aligned}$$

This is a sixth-order iterative scheme derived by Grau *et al.* [7].

(iii) For $\alpha_1 \in \mathbb{R}\backslash\{0\}$ and $\alpha_2 = 0$, first four-steps ($i = 4$) of family (2.2.9) reduces as follows:

$$\begin{aligned}\psi_1^k =& X^{k+1} = X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k),\\ \psi_2^k =& \psi_1^k - \left\{2[\psi_1^k, X^k; F] - [X^k + F, X^k - F; F]\right\}^{-1} F(\psi_1^k),\\ \psi_3^k =& \psi_2^k - \left\{2[\psi_1^k, X^k; F] - [X^k + F, X^k - F; F]\right\}^{-1} F(\psi_2^k),\\ \psi_4^k =& \psi_3^k - \left\{2[\psi_1^k, X^k; F] - [X^k + F, X^k - F; F]\right\}^{-1} F(\psi_3^k).\end{aligned}$$

This is a new eighth-order iterative scheme.

2.3 Convergence analysis

Consider the first-order divided difference operator of F on $\mathbb{R}^n$ as a mapping $[\cdot, \cdot : F] : D \times D \subseteq \mathbb{R}^n \times \mathbb{R}^n \to L(\mathbb{R}^n)$, which is defined by [7, 10]

$$\left[X^k + h, X^k; F\right] = \int_0^1 F'(X^k + uh)du, \ \forall (X^k, h) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2.3.1}$$

Developing $F'(X^k + uh)$ in Taylor's series at X^k and after integrating, one can obtain

$$\int_0^1 F'(X^k + uh)du = F'(X^k) + \frac{1}{2}F''(X^k)h + \frac{1}{6}F'''(X^k)h^2 + O(h^3). \tag{2.3.2}$$

Taking into account $e^k = X^k - X^*$, one develop $F(X^k)$ and its derivatives in a neighbourhood of X^*, where $X^* \in \mathbb{R}^n$ is the solution of system $F(X) = 0$. Assuming that

$\Gamma = \left\{F'(X^*)\right\}^{-1}$ exists, one can have

$$F(X^k) = F'(X^*)\Big[e^k + A_2\big(e^k\big)^2 + A_3\big(e^k\big)^3 + A_4\big(e^k\big)^4 + A_5\big(e^k\big)^5 + O\left((e^k)^6\right)\Big], \quad (2.3.3)$$

where $A_j = \frac{1}{j!}\Gamma F^{(j)}(X^*) \in L_j(\mathbb{R}^n, \mathbb{R}^n),\ j = 2, 3, \ldots$.
From equation (2.3.3), the derivatives of $F(X^k)$ can be written as

$$F'(X^k) = F'(X^*)\Big[I + 2A_2\big(e^k\big) + 3A_3\big(e^k\big)^2 + 4A_4\big(e^k\big)^3 + 5A_5\big(e^k\big)^4 + O\left((e^k)^5\right)\Big], \quad (2.3.4)$$

$$F''(X^k) = F'(X^*)\Big[2A_2 + 6A_3\big(e^k\big) + 12A_4\big(e^k\big)^2 + 20A_5\big(e^k\big)^3 + O\left((e^k)^4\right)\Big], \quad (2.3.5)$$

$$\text{and} \quad F'''(X^k) = F'(X^*)\Big[6A_3 + 24A_4\big(e^k\big) + O\left((e^k)^2\right)\Big], \quad (2.3.6)$$

where I is an identity matrix of order n.
Setting $\psi_1^k = X^k + h$ & $\epsilon_1^k = \psi_1^k - X^*$, one can have $h = \psi_1^k - X^k = \epsilon_1^k - e^k$.
By substituting equations (2.3.4)-(2.3.6) into equation (2.3.2), one gets

$$[\psi_1^k, X^k; F] = F'(X^*)\Big[I + A_2\big(\epsilon_1^k + e^k\big) + A_3\big(\big(\epsilon_1^k\big)^2 + \big(e^k\big)^2 + \epsilon_1^k e^k\big) + O\left((e^k)^3\right)\Big]. \quad (2.3.7)$$

Further, the center difference operator

$$[X^k + F, X^k - F; F] = F'(X^*)\Big[I + 2A_2\big(e^k\big) + A_3\big(3 + (G'(r))^2\big)(e^k)^2 + O\left((e^k)^3\right)\Big], \quad (2.3.8)$$

which is obtained after replacing ϵ_1^k by $e^k + F(X^k)$ and e^k by $e^k - F(X^k)$ in equation (2.3.7). The convergence of iterative schemes (2.2.9) can be proved through the following theorem.

Theorem 2.3.1 *Let $X^* \in \mathbb{R}^n$ be solution of the system $F(X) = 0$ and $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood D of X^* at which $F'(X^*)$ is nonsingular. Then, for an initial approximation sufficiently close to X^*, iterative scheme* (2.2.9) *will have $2 \times i$ local order of convergence with error equation*

$$\epsilon_i^k = (-1)^{i+1}\frac{\big(\lambda A_3 - PA_2^2\big)^{i-2}\big(\lambda A_2 A_3 - QA_2^3\big)}{\alpha_1^{i-1}(\alpha_1 - \alpha_2)^{i-1}}(e^k)^{2i} + O\left((e^k)^{2i+1}\right),$$

provided that $\alpha_1 \in \mathbb{R}\backslash\{0\}$, $\alpha_2 \in \mathbb{R}$ and $\alpha_1 \neq \alpha_2$, where
$\lambda = \alpha_1(\alpha_1 - \alpha_2)(1 + \gamma^2)$, $Q = (\alpha_1^2 - 3\alpha_1\alpha_2 + \alpha_2^2)$, $P = (2\alpha_1^2 - 4\alpha_1\alpha_2 + \alpha_2^2)$ & $\gamma = F'(X^*)$.

Proof The theorem will be proved by induction method.

For $i = 2$ the relation (2.2.9) reduces as follows:

$$\left.\begin{aligned} \psi_1^k =& X^{k+1} = X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k), \\ \psi_2^k =& \psi_1^k - \eta \ F(\psi_1^k). \end{aligned}\right\}. \tag{2.3.9}$$

The inverse operator of equation (2.3.8) is

$$\begin{aligned} \{[X^k + F, X^k - F; F]\}^{-1} =& \gamma^{-1}\Big[I - 2A_2 e^k + \big(4A_2^2 - A_3(3+\gamma^2)\big)(e^k)^2 \\ & + 2\big((6+\gamma^2)A_2A_3 - 2(1+\gamma^2)A_4 - 4A_4^3\big)(e^k)^3 \\ & + O\left((e^k)^4\right)\Big]. \end{aligned} \tag{2.3.10}$$

Using (2.3.3) and (2.3.10) in the first-step of equation (2.2.9), one can get the following error equation

$$\begin{aligned} \epsilon_1^k = \psi_1^k - X^* =& A_2(e^k)^2 + (-2A_2^2 + (2+\gamma^2)A_3)(e^k)^3 \\ & + \big(-(7+\gamma^2)A_2A_3 + (3+4\gamma^2)A_4 + 4A_2^3\big)(e^k)^4 + O\left((e^k)^5\right). \end{aligned} \tag{2.3.11}$$

Expanding $F(\psi_1^k)$ by Taylor's series expansion around the solution X^* by using (2.3.11), one gets

$$F(\psi_1^k) = \gamma\Big[\epsilon_1^k + A_2\big(\epsilon_1^k\big)^2 + A_3\big(\epsilon_1^k\big)^3 + O\left((\epsilon_1^k)^4\right)\Big]. \tag{2.3.12}$$

By substituting equations (2.3.7) and (2.3.8) in equation (2.2.10), one can obtain,

$$\begin{cases} \tau =& \gamma^2\alpha_1(\alpha_1 - \alpha_2) + PA_2\gamma^2\big(e^k\big) \\ & + \gamma^2\Big[2\alpha_1^2(A_2^2 + A_3) - 2\alpha_1\alpha_2(A_2^2 + (3+\gamma^2)A_3) + (2+\gamma^2)\alpha_2^2A_3\Big](e^k)^2 \\ & + O((e^k)^3), \\ \eta =& \dfrac{1}{\gamma} + \dfrac{(1+\gamma^2)A_3\alpha_1(\alpha_1-\alpha_2) - A_2^2P}{\gamma\alpha_1(\alpha_1-\alpha_2)}(e^k)^2 \\ & + \dfrac{1}{\gamma\alpha_1^2(\alpha_1-\alpha_2)^2}\Big[2(1+2\gamma^2)A_4\alpha_1^2(\alpha_1-\alpha_2)^2 + A_2^3P^2 \\ & + 2A_2A_3\alpha_1\Big(-3\alpha_1^3 + 2(\gamma^2+5)\alpha_1^2\alpha_2 - 3(\gamma^2+3)\alpha_1\alpha_2^2 + (2+\gamma 2)\alpha_2^3\Big)\Big](e^k)^3 \\ & + O\left((e^k)^4\right), \end{cases} \tag{2.3.13}$$

The second step of equation (2.3.9) can be rewritten as

$$\psi_2^k - X^* = \psi_1^k - X^* - \eta \ F(\psi_1^k),$$

$$\Rightarrow \epsilon_2^k = \epsilon_1^k - \eta\ F(\psi_1^k). \tag{2.3.14}$$

Putting the values of $\epsilon_1^k,\ \ F(\psi_1^k),$ and η from equations (2.3.11)-(2.3.13) respectively, the equation (2.3.14) yields,

$$\begin{aligned}
\epsilon_2^k = \psi_2^k - X^* = &-\frac{\lambda A_2 A_3 - Q A_2^3}{\alpha_1(\alpha_1-\alpha_2)}(e^k)^4 \\
&- \frac{1}{(\lambda A_3 - P A_2^2)\alpha_1(\alpha_1-\alpha_2)}\Big[- P(\alpha_2^4 - 10\alpha_2^3\alpha_1 + 26\alpha_1^2\alpha_2^2 - 20\alpha_1^3\alpha_2 + 4\alpha_1^4)A_2^6 \\
&+ 2\alpha_1^3(\alpha_1-\alpha_2)^3(2\gamma^2+1)(\gamma^2+1)A_2A_3A_4 + \alpha_1^3(\alpha_1-\alpha_2)^3(\gamma^2+2)(\gamma^2+1)^2A_3^3 \\
&- 2\alpha_1^2(\alpha_1-\alpha_2)^2(2\gamma^2+1)(2\alpha_1^2 - 4\alpha_1\alpha_2 + \alpha_2^2)A_2^3A_4 - \alpha_1^2(\alpha_1-\alpha_2)^2(1+\gamma^2)(4\alpha_1^2\gamma^2 \\
&- 12\alpha_1\alpha_2\gamma^2 + 4\alpha_2^2\gamma^2 + 12\alpha_1^2 - 28\alpha_1\alpha_2 + 8\alpha_2^2)A_2^2A_3^2 + \alpha_1(\alpha_1-\alpha_2)\big(20\alpha_1^4 - 92\alpha_1^3\alpha_2 \\
&- 44\alpha_1^3\alpha_2\gamma^2 + 126\alpha_1^2\alpha_2^2 + 74\alpha_1^2\alpha_2^2\gamma^2 - 30\alpha_1\alpha_2^3\gamma^2 - 54\alpha_1\alpha_2^3 + 7\alpha_2^4 + 4\alpha_2^4\gamma^2 \\
&+ 8\alpha_1^4\gamma^2\big)A_2^4A_3\Big](e^k)^5 + O\left((e^k)^6\right).
\end{aligned} \tag{2.3.15}$$

Thus the iterative family (2.2.9) has fourth-order of convergence for first two-steps. For $i = m$, the iterative scheme (2.2.9) is written as:

$$\begin{cases}
\psi_1^k = X^{k+1} = X^k - \{[X^k + F, X^k - F; F]\}^{-1}F(X^k), \\
\psi_2^k = \psi_1^k - \eta\ F(\psi_1^k), \\
\psi_3^k = \psi_2^k - \eta\ F(\psi_2^k), \\
\psi_4^k = \psi_3^k - \eta\ F(\psi_3^k), \\
\quad \vdots \\
\psi_{m-1}^k = \psi_{m-2}^k - \eta\ F(\psi_{m-2}^k), \\
\psi_m^k = \psi_{m-1}^k - \eta\ F(\psi_{m-1}^k).
\end{cases} \tag{2.3.16}$$

Let us assume the scheme (2.3.16) has order of convergence $2*m$ for first m steps, with error equation

$$\begin{aligned}
\epsilon_m^k = \psi_m - X^* = &(-1)^{m+1}\frac{\big(\lambda A_3 - P A_2^2\big)^{m-2}\big(\lambda A_2 A_3 - Q A_2^3\big)}{\alpha_1^{m-1}(\alpha_1-\alpha_2)^{m-1}}(e^k)^{2m} \\
&+ (-1)^{m+1}\frac{(\lambda A_3 - P A_2^2)^{m-3}}{\alpha_1^m(\alpha_1-\alpha_2)^m}\Big[- P\big((m-1)\alpha_2^4 + (4-7m)\alpha_2^3\alpha_1 \\
&+ (15m-4)\alpha_1^2\alpha_2^2 - 10m\alpha_1^3\alpha_2 + 2m\alpha_1^4\big)A_2^6 + (2m-2)\alpha_1^3(\alpha_1-\alpha_2)^3 \\
&\times (2\gamma^2+1)(\gamma^2+1)A_2A_3A_4 + \alpha_1^3(\alpha_1-\alpha_2)^3(\gamma^2+2)(\gamma^2+1)^2A_3^3 \\
&- 2\alpha_1^2(\alpha_1-\alpha_2)^2(2\gamma^2+1)(m\alpha_1^2 + (2-3m)\alpha_1\alpha_2 + (m-1)\alpha_2^2)A_2^3A_4 \\
&- \alpha_1^2(\alpha_1-\alpha_2)^2(1+\gamma^2)(4\alpha_1^2\gamma^2 - (4m+4)\alpha_1\alpha_2\gamma^2 + 2m\alpha_2^2\gamma^2 + 6m\alpha_1^2
\end{aligned}$$

$$
\begin{aligned}
&- 14m\alpha_1\alpha_2 + 4m\alpha_2^2)A_2^2A_3^2 + \alpha_1(\alpha_1-\alpha_2)\big(10m\alpha_1^4 + (4-48m)\alpha_1^3\alpha_2 \\
&- (20m+4)\alpha_1^3\alpha_2\gamma^2 + (72m-18)\alpha_1^2\alpha_2^2 + (34m+2)\alpha_1^2\alpha_2^2\gamma^2 \\
&+ (6-18m)\alpha_1\alpha_2^3\gamma^2 + (14-34m)\alpha_1\alpha_2^3 + (5m-3)\alpha_2^4 + (3m-2)\alpha_2^4\gamma^2 \\
&+ 4m\alpha_1^4\gamma^2\big)A_2^4A_3\Big](e^k)^{2m+1} + O\left((e^k)^{2m+2}\right).
\end{aligned} \tag{2.3.17}
$$

Further, for $i = m+1$, the iterative family (2.2.9) is represented as:

$$
\begin{cases}
\psi_1^k = X^{k+1} = X^k - \{[X^k+F, X^k-F; F]\}^{-1}F(X^k), \\
\psi_2^k = \psi_1^k - \eta\ F(\psi_1^k), \\
\psi_3^k = \psi_2^k - \eta\ F(\psi_2^k), \\
\psi_4^k = \psi_3^k - \eta\ F(\psi_3^k), \\
\quad\vdots \\
\psi_{m-1}^k = \psi_{m-2}^k - \eta\ F(\psi_{m-2}^k), \\
\psi_m^k = \psi_{m-1}^k - \eta\ F(\psi_{m-1}^k), \\
\psi_{m+1}^k = \psi_m^k - \eta\ F(\psi_m^k).
\end{cases} \tag{2.3.18}
$$

Now we shall show that the result is true for $i = m+1$, i.e. we have to prove that iterative method (2.3.18) has $2(m+1)$ order of convergence for first $m+1$ steps.
Since we have assumed that the result is true for first m steps, therefore expanding $F(\psi_m^k)$ by Taylor's series around the solution X^* one gets

$$
\begin{aligned}
F(\psi_m^k) =& \gamma\Big[(-1)^{m+1}\frac{\big(\lambda A_3 - PA_2^2\big)^{m-2}\big(\lambda A_2A_3 - QA_2^3\big)}{\alpha_1^{m-1}(\alpha_1-\alpha_2)^{m-1}}(e^k)^{2m} \\
&+ (-1)^{m+1}\frac{(\lambda A_3 - PA_2^2)^{m-3}}{\alpha_1^m(\alpha_1-\alpha_2)^m}\Big[-P\big((m-1)\alpha_2^4 + (4-7m)\alpha_2^3\alpha_1 + (15m-4)\alpha_1^2\alpha_2^2 \\
&- 10m\alpha_1^3\alpha_2 + 2m\alpha_1^4\big)A_2^6 + (2m-2)\alpha_1^3(\alpha_1-\alpha_2)^3(2\gamma^2+1)(\gamma^2+1)A_2A_3A_4 \\
&+ \alpha_1^3(\alpha_1-\alpha_2)^3(\gamma^2+2)(\gamma^2+1)^2A_3^3 - 2\alpha_1^2(\alpha_1-\alpha_2)^2(2\gamma^2+1)(m\alpha_1^2 + (2-3m)\alpha_1\alpha_2 \\
&+ (m-1)\alpha_2^2)A_2^3A_4 - \alpha_1^2(\alpha_1-\alpha_2)^2(1+\gamma^2)(4\alpha_1^2\gamma^2 - (4m+4)\alpha_1\alpha_2\gamma^2 + 2m\alpha_2^2\gamma^2 \\
&+ 6m\alpha_1^2 - 14m\alpha_1\alpha_2 + 4m\alpha_2^2)A_2^2A_3^2 + \alpha_1(\alpha_1-\alpha_2)\big(10m\alpha_1^4 + (4-48m)\alpha_1^3\alpha_2 \\
&- (20m+4)\alpha_1^3\alpha_2\gamma^2 + (72m-18)\alpha_1^2\alpha_2^2 + (34m+2)\alpha_1^2\alpha_2^2\gamma^2 + (6-18m)\alpha_1\alpha_2^3\gamma^2 \\
&+ (14-34m)\alpha_1\alpha_2^3 + (5m-3)\alpha_2^4 + (3m-2)\alpha_2^4\gamma^2 + 4m\alpha_1^4\gamma^2\big)A_2^4A_3\Big](e^k)^{2m+1} \\
&+ O\left((e^k)^{2m+2}\right)\Big].
\end{aligned} \tag{2.3.19}
$$

Now, last step of equation (2.3.18) rewritten as

$$\psi_{m+1}^{k} - X^{*} = \psi_{m}^{k} - X^{*} - \eta\ F(\psi_{m}^{k}),$$

$$\Rightarrow \epsilon_{m+1}^{k} = \epsilon_{m}^{k} - \eta\ F(\psi_{m}^{k}). \tag{2.3.20}$$

Substituting the values of η, ϵ_m^k, and $F(\psi_m^k)$ from equation (2.3.13), (2.3.17), and (2.3.19) respectively in equation (2.3.20) and after some simplifications, one can get the error equation

$$\epsilon_{m+1}^{k} = (-1)^{m+2}\frac{\left(\lambda A_3 - PA_2^2\right)^{m-1}\left(\lambda A_2 A_3 - QA_2^3\right)}{\alpha_1^m(\alpha_1-\alpha_2)^m}(e^k)^{2m+2} + O\left((e^k)^{2m+3}\right), \tag{2.3.21}$$

which shows that the iterative scheme (2.3.18) has $2(m+1)$ order of convergence for first $m+1$ steps. That is the result is true for $i = m+1$. Hence, by induction method one deduces that the result is true $\forall\ i = 2, 3, 4 \ldots m$. □

2.4 Computational efficiency

For the estimation of efficiency of proposed families, the efficiency index (2.2.9) has been used. Also for the efficiency index of proposed family $n(n+1)$ scalar functions for central divided difference operator $[X+F, X-F; F]$ and $5n^2$ products for multiplication of a vector by a scalar are evaluated. For comparison of computational efficiencies of proposed schemes $\psi_1^k, \psi_2^k, \psi_3^k$, and ψ_4^k order of convergence in two, four, six, and eight respectively, the efficiency indices and computational cost are denoted by CEI_i and C_i respectively. Taking into account the above considerations, one can have

$$C_1 = \frac{n}{6}(2n^2 + 6n\nu + 3n + 9ln + 12\nu + 3\ell - 5) \text{ and } CEI_1 = 2^{1/C_1}. \tag{2.4.1}$$

$$C_2 = \frac{n}{3}(2n^2 + 6n\nu + 18n + 9ln + 6\nu + 3\ell - 5) \text{ and } CEI_2 = 4^{1/C_2}. \tag{2.4.2}$$

$$C_3 = \frac{n}{3}(2n^2 + 6n\nu + 21n + 9ln + 9\nu + 6\ell - 8) \text{ and } CEI_3 = 6^{1/C_3}. \tag{2.4.3}$$

$$C_4 = \frac{n}{3}(2n^2 + 6n\nu + 24n + 9ln + 12\nu + 9\ell - 11) \text{ and } CEI_4 = 8^{1/C_4}. \tag{2.4.4}$$

2.4.1 Comparison between efficiencies

In order to compare the iterative families $\psi_i, 1 \le i \le 4$, the following ratio can be defined as

$$R_{i,j} = \frac{log\ CEI_i}{log\ CEI_j} = \frac{log(\rho_i)C_j}{log(\rho_j)C_i}.$$

It is clear that if $R_{i,j} > 1$, the iterative method ψ_i is more efficient than ψ_j. Taking into account that the border between two computational efficiencies is given by $R_{i,j} = 1$, this boundary is given by the equation of ν written as a function of ℓ and n, that is $\nu = M_{i,j}(\ell, n)$. Here $\nu > 0,\ \ell \ge 1$ and n is a positive integer $n \ge 2$.

Case 1 : Iterative method $\boldsymbol{\psi_1}$ verses iterative family $\boldsymbol{\psi_3}$

The boundary $R_{3,1} = 1$ expressed by ν written as a function of ℓ and n is

$$M_{3,1} = \frac{(4n^2 + 42n + 18n\ell + 12\ell - 16)log2 - (2n^2 + 3n + 9n\ell + 3\ell - 5)log6}{(6n + 12)log6 - (12n - 48)log2}. \quad (2.4.5)$$

This function has the vertical asymptote for $n = -3.70951$. Note that the numerator of equation (2.4.5) is negative for $n \ge 25$ and the denominator of equation (2.4.5) is positive for $n \ge 2$. Consequently, It shows that ν is always positive for $2 \le n < 25$ and for all $\ell \ge 1$.

So, one can have $CEI_3 > CEI_1,\ \forall\ \nu > 0,\ \ell \ge 1\ \&\ 2 \le n < 25$.

Case 2 : Iterative method $\boldsymbol{\psi_1}$ verses iterative family $\boldsymbol{\psi_4}$

The boundary $R_{4,1} = 1$ expressed by ν written as a function of ℓ and n is

$$M_{4,1} = \frac{-2n^2 + 39n - 9n\ell + 9\ell - 7}{6n + 12}. \quad (2.4.6)$$

This function has the vertical asymptote for $n = -2$. Note that the numerator of equation (2.4.6) is negative for $n > 20$ and the denominator of equation (2.4.6) is positive for $n \ge 2$. Consequently, It shows that ν is positive for $2 \le n < 20$ and for all $\ell \ge 1$.

So, one gets $CEI_4 > CEI_1,\ \forall\ \nu > 0,\ \ell \ge 1\ \&\ 2 \le n < 20$.

Case 3 : Iterative family $\boldsymbol{\psi_2}$ verses iterative family $\boldsymbol{\psi_3}$

The boundary $R_{3,2} = 1$ expressed by ν written as a function of ℓ and n is

$$M_{3,2} = \frac{(2n^2 + 21n + 9n\ell + 6\ell - 8)log4 - (2n^2 + 18n + 9n\ell + 3\ell - 5)log6}{(6n + 6)log6 - (6n + 9)log4}. \quad (2.4.7)$$

This function has the vertical asymptote for $n = 0.7095$. Note that the numerator of equation (2.4.7) is negative for $n \geq 0$ and the denominator of equation (2.4.7) is positive for $n > 0$. Consequently, It shows that ν is always negative for $n \geq 2$ and for all $\ell \geq 1$.
So, one can get $CEI_3 < CEI_2,\ \forall\ \nu > 0,\ \ell \geq 1\ \&\ \ n \geq 2$.

Case 4 : Iterative family ψ_2 verses iterative family ψ_4
The boundary $R_{4,2} = 1$ expressed by ν written as a function of ℓ and n is

$$M_{4,2} = \frac{(2n^2 + 24n + 9n\ell + 9\ell - 11)log4 - (2n^2 + 18n + 9n\ell + 3\ell - 5)log8}{(6n + 6)log8 - (6n + 12)log4}. \qquad (2.4.8)$$

This function has the vertical asymptote for $n = 1$. Note that the numerator of equation (2.4.8) is negative for $n \geq 0$ and the denominator of equation (2.4.8) is positive for $n > 1$. Consequently, It shows that ν is always negative for $n \geq 2$ and for all $\ell \geq 1$.
So, one can obtain $CEI_4 < CEI_2,\ \forall\ \nu > 0,\ \ell \geq 1\ \&\ \ n \geq 2$.

Case 5 : Iterative family ψ_3 verses iterative family ψ_4
The boundary $R_{4,3} = 1$ expressed by ν written as a function of ℓ and n is

$$M_{4,3} = \frac{(2n^2 + 24n + 9n\ell + 9\ell - 11)log6 - (2n^2 + 21n + 9n\ell + 6\ell - 8)log8}{(6n + 9)log8 - (6n + 12)log6}. \qquad (2.4.9)$$

This function has the vertical asymptote for $n = 1.61413$. Note that the numerator of equation (2.4.9) is positive for $n \geq 1$ and the denominator of equation (2.4.9) is negative for $n > 1$. Consequently, It shows that ν is always negative for $n \geq 2$ and for all $\ell \geq 1$.
So, one can have $CEI_4 < CEI_3,\ \forall\ \nu > 0,\ \ell \geq 1\ \&\ \ n \geq 2$.

Theorem 4.1: For all $\nu > 0$ and $\ell \geq 1$, one obtains:
(i) $CEI_3 > CEI_1, \quad for \ \ 2 \leq n < 25$ (see Figure 2.1).
(ii) $CEI_4 > CEI_1, \quad for \ \ 2 \leq n < 20$ (see Figure 2.2).
(iii) $CEI_3 < CEI_2, \quad for \ \ n \geq 2$ (see Figure 2.3).
(iv) $CEI_4 < CEI_2, \quad for \ \ n \geq 2$ (see Figure 2.4).
(v) $CEI_4 < CEI_3, \quad for \ \ n \geq 2$ (see Figure 2.5).
Otherwise the comparisons depend upon the value of n, ν and ℓ. To verify the results of above theorem the graphs are plotted for the set $(\nu,\ell)=(1,1)$. These graphs in (n, CEI) variables are shown in Figures 2.1–2.5.

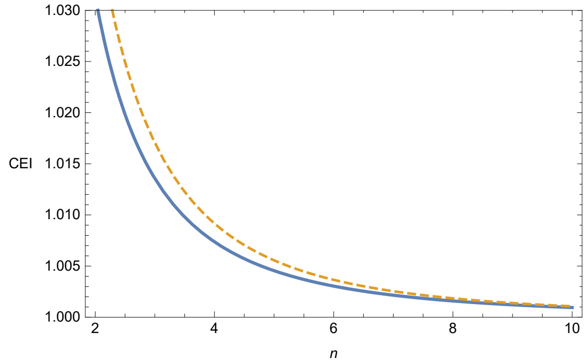

Figure 2.1.
CEI_1 $(dashedline)$, CEI_3 $(thickline)$

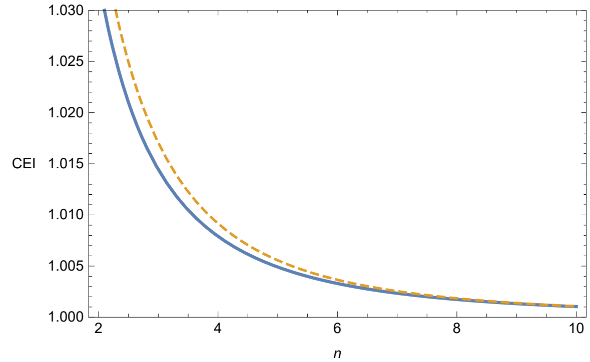

Figure 2.2.
CEI_1 $(dashedline)$, CEI_4 $(thickline)$

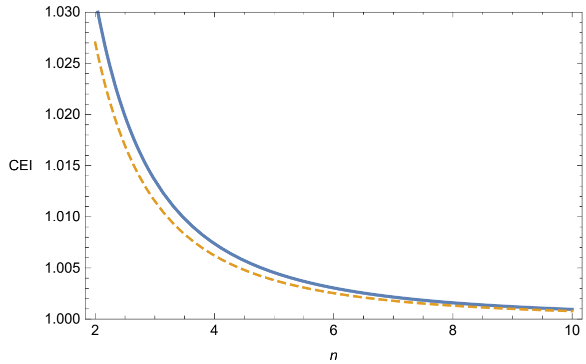

Figure 2.3.
CEI_2 $(dashedline)$, CEI_3 $(thickline)$

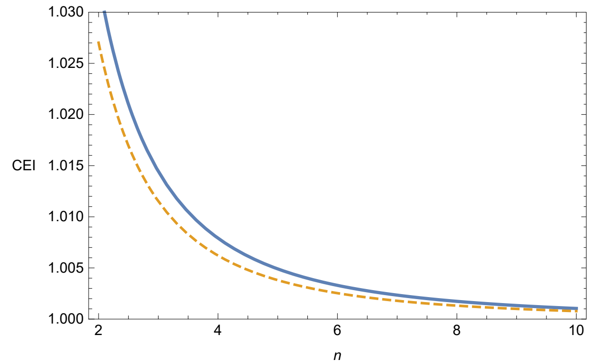

Figure 2.4.
CEI_2 $(dashedline)$, CEI_4 $(thickline)$

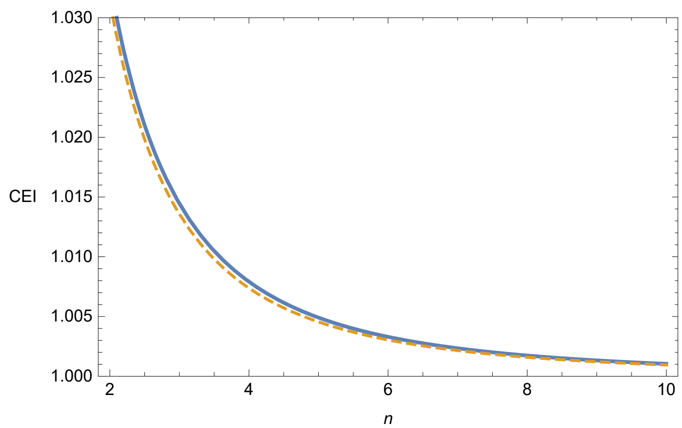

Figure 2.5. CEI_3 $(dashedline)$, CEI_4 $(thickline)$

2.5 Numerical reports

In this section, some numerical problems are considered to illustrate the convergence behavior and computational efficiency of the proposed methods. For comparison of the computational efficiencies of the proposed schemes (2.2.9) $\psi_{2,1}, \psi_{3,1}$ which are special cases of Grau *et al.* [7] and $\psi_{4,1}$ for $\alpha_1 \in \mathbb{R}\backslash\{0\}$ & $\alpha_2 = 0$ are considered. In same manner the proposed schemes (2.2.9) $\psi_{2,2}, \psi_{3,2}, \psi_{4,2}$ for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = \pm 10^{-1000}$ and $\psi_{2,3}, \psi_{3,3}, \psi_{4,3}$ for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$ are denoted and compared with existing schemes of fourth order namely, $M_{4,1}, M_{4,2}$ for Sharma and Arora [34] and seventh order S_7 Sharma and Arora [62]. To verify the theoretical order of convergence, author has used the computational order of convergence (ρ_c) using the formula (1.2.10) and an estimation of the factors ν and $\ell = 3$ evaluated according to the Table 1.1 for all numerical tests.

Example 2.5.1 *Considering mixed Hammerstein integral equation (see [6, pp. 19-20]).* $x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)(x(t))^3 dt$ *where* $x \in C[0,1]; s,t \in [0,1]$ *and the kernel* G *is*

$$G(s,t) = \begin{cases} (1-s)t, t \le s, \\ s(1-t), s \le t. \end{cases}$$

To transform the above equation into a finite-dimensional problem by using Gauss Legendre quadrature formula given as $\int_0^1 f(t)dt \simeq \sum_{j=1}^{8} w_j f(t_j)$*, where the abscissas* t_j *and the weights* w_j *are determined for* $t = 8$ *by Gauss Legendre quadrature formula. Denoting the approximations of* $x(t_i)$ *by* x_i $(i = 1, 2, ..., 8)$*, one gets the systems of nonlinear equations*

$$F = 5x_i - 5 - \sum_{j=1}^{8} a_{ij} x_j^3 = 0, \text{ where } i = 1, 2, ..., 8 \text{ and}$$

$$a_{ij} = \begin{cases} w_j t_j (1-t_i), j \le i, \\ w_j t_i (1-t_j), i < j. \end{cases}$$

Where the abscissas t_j *and the weights* w_j *are known and given in following table for* $n = 8$.

Abscissas and weights of Gauss Legendre quadrature formula for $n = 8$

j	t_j	w_j
1	0.01985507175123188415821957...	0.05061426814518812957626567...
2	0.10166676129318663020422303...	0.11119051722668723527217800...
3	0.23723379504183550709113047...	0.15685332293894364366898110...
4	0.40828267875217509753026193...	0.18134189168918099148257522...
5	0.59171732124782490246973807...	0.18134189168918099148257522...
6	0.76276620495816449290886952...	0.15685332293894364366898110...
7	0.89833323870681336979577696...	0.11119051722668723527217800...
8	0.98014492824876811584178043...	0.05061426814518812957626567...

Also $(n, \nu) = (8, 11)$ *are the values used in equations* $(2.4.1) - (2.4.4)$. *The convergence of the methods towards the solution* $X^* = (1.00209624503115679\ldots, 1.00990031618748877\ldots,$ $1.01972696099317687\ldots, 1.02643574303062052\ldots, 1.02643574303062052\ldots,$ $1.01972696099317687\ldots, 1.00990031618748877\ldots, 1.00209624503115679\ldots)^T$ *is tested with* *following computational terms and results display in the Table 2.1.*

$$[X+F,X;F]=$$

$$\Big\{\{-\frac{-5x_1+x_1^3a_{11}-5\Big(5-6x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)+a_{11}\Big(5-6x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)^3}{5-5x_1+\sum_{i=1}^{8}a_{1i}x_i^3},$$

$$-\frac{a_{12}\Big(x_2^3+\Big(5-6x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)^3\Big)}{5-5x_2+\sum_{i=1}^{8}a_{2i}x_i^3},\ldots,-\frac{a_{18}\Big(x_8^3+\Big(5-6x_8+\sum_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{5-5x_8+\sum_{i=1}^{8}a_{8i}x_i^3}\},$$

$$\{-\frac{a_{21}\Big(x_1^3+\Big(5-6x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)^3\Big)}{5-5x_1+\sum_{i=1}^{8}a_{1i}x_i^3},$$

$$-\frac{-5x_2+x_2^3a_{22}-5\Big(5-6x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)+a_{22}\Big(5-6x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)^3}{5-5x_2+\sum_{i=1}^{8}a_{2i}x_i^3},\ldots,$$

$$-\frac{a_{28}\Big(x_8^3+\Big(5-6x_8+\sum_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{5-5x_8+\sum_{i=1}^{8}a_{8i}x_i^3}\},\ldots,\{-\frac{a_{81}\Big(x_1^3+\Big(5-6x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)^3\Big)}{5-5x_1+\sum_{i=1}^{8}a_{1i}x_i^3},$$

$$-\frac{a_{82}\Big(x_2^3+\Big(5-6x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)^3\Big)}{5-5x_2+\sum_{i=1}^{8}a_{2i}x_i^3},\ldots,$$

$$-\frac{-5x_8+x_8^3a_{88}-5\Big(5-6x_8+\sum_{i=1}^{8}a_{8i}x_i^3\Big)+a_{88}\Big(5-6x_8+\sum_{i=1}^{8}a_{8i}x_i^3\Big)^3}{5-5x_8+\sum_{i=1}^{8}a_{8i}x_i^3}\}\Big\},$$

$$[X+F,X-F;F]=$$

$$\Big\{\{-\frac{-50+50x_1-10\sum_{i=1}^{8}a_{1i}x_i^3+a_{11}\Big(\Big(5-6x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)^3-\Big(5-4x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)^3\Big)}{2\Big(5-5x_1+\sum_{i=1}^{8}a_{1i}x_i^3\Big)},$$

$$-\frac{a_{12}\Big(\Big(5-6x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)^3+\Big(5-4x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)^3\Big)}{2\Big(5-5x_2+\sum_{i=1}^{8}a_{2i}x_i^3\Big)},\ldots,$$

$$-\frac{a_{18}\Big(\Big(5-6x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3+\Big(5-4x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{2\Big(5-5x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)}\},$$

$$\{-\frac{a_{21}\Big(\Big(5-6x_1+\sum\limits_{i=1}^{8}a_{1i}x_i^3\Big)^3+\Big(5-4x_1+\sum\limits_{i=1}^{8}a_{1i}x_i^3\Big)^3\Big)}{2\Big(5-5x_1+\sum\limits_{i=1}^{8}a_{1i}x_i^3\Big)},$$

$$-\frac{-50+50x_2-10\sum\limits_{i=1}^{8}a_{2i}x_i^3+a_{22}\Big(\Big(5-6x_2+\sum\limits_{i=1}^{8}a_{2i}x_i^3\Big)^3-\Big(5-4x_2+\sum\limits_{i=1}^{8}a_{2i}x_i^3\Big)^3\Big)}{2\Big(5-5x_2+\sum\limits_{i=1}^{8}a_{2i}x_i^3\Big)},\dots,$$

$$-\frac{a_{28}\Big(\Big(5-6x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3+\Big(5-4x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{2\Big(5-5x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)}\},\dots,$$

$$\{-\frac{a_{81}\Big(\Big(5-6x_1+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3+\Big(5-4x_1+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{2\Big(5-5x_1+\sum\limits_{i=1}^{8}a_{1i}x_i^3\Big)},$$

$$-\frac{a_{82}\Big(\Big(5-6x_2+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3+\Big(5-4x_2+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{2\Big(5-5x_2+\sum\limits_{i=1}^{8}a_{2i}x_i^3\Big)},\dots,$$

$$-\frac{-50+50x_8-10\sum\limits_{i=1}^{8}a_{8i}x_i^3+a_{88}\Big(\Big(5-6x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3-\Big(5-4x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)^3\Big)}{2\Big(5-5x_8+\sum\limits_{i=1}^{8}a_{8i}x_i^3\Big)}\}\Big\},$$

$[Y, X; F] =$

$$\begin{bmatrix} 5-(x_1^2+x_1y_1+y_1^2)a_{11} & -(x_2^2+x_2y_2+y_2^2)a_{12} & \dots & -(x_8^2+x_8y_8+y_8^2)a_{18} \\ -(x_1^2+x_1y_1+y_1^2)a_{21} & 5-(x_2^2+x_2y_2+y_2^2)a_{22} & \dots & -(x_8^2+x_8y_8+y_8^2)a_{28} \\ \vdots & \vdots & \ddots & \vdots \\ -(x_1^2+x_1y_1+y_1^2)a_{81} & -(x_2^2+x_2y_2+y_2^2)a_{82} & \dots & 5-(x_8^2+x_8y_8+y_8^2)a_{88} \end{bmatrix}.$$

Table 2.1: Performance of various iterative schemes at initial value $(0.85, 0.85, \ldots, 0.85)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ_c	C	CEI	CPU $Time\ (sec)$
$M_{4,1}$	6.15(−5)	1.07(−19)	1.03(−78)	4.000	3296	1.000420	0.333
$M_{4,2}$	6.23(−5)	1.15(−19)	1.44(−78)	4.000	3376	1.000410	0.236
$\psi_{2,1}$	2.46(−5)	1.62(−21)	3.28(−86)	4.000	2896	1.000478	0.344
$\psi_{2,2}$	2.46(−5)	1.62(−21)	3.28(−86)	4.000	2896	1.000478	0.326
$\psi_{2,3}$	2.46(−5)	1.62(−21)	3.28(−86)	4.000	2896	1.000478	0.293
$\psi_{3,1}$	3.74(−7)	1.64(−42)	1.28(−254)	6.000	3064	1.000584	0.601
$\psi_{3,2}$	3.74(−7)	1.64(−42)	1.28(−254)	6.000	3064	1.000584	0.636
$\psi_{3,3}$	3.74(−7)	1.64(−42)	1.28(−254)	6.000	3064	1.000584	0.615
S_7	5.27(−11)	1.48(−79)	2.14(−559)	7.000	8328	1.000233	2.042
$\psi_{4,1}$	5.69(−9)	8.85(−71)	3.50(−565)	8.000	3232	1.000643	0.592
$\psi_{4,2}$	5.69(−9)	8.85(−71)	3.50(−565)	8.000	3232	1.000643	0.628
$\psi_{4,3}$	5.69(−9)	8.85(−71)	3.50(−565)	8.000	3232	1.000643	0.617

Example 2.5.2 *Consider the following example taken from the research paper [34]*

$$F(x_1, x_2) = \begin{cases} (x_1 - 1)^4 + e^{-x_2} - x_2^2 + 3x_2 + 1, \\ 4sin(x_1 - 1) - \ln(x_1^2 - x_1 + 1) - x_2^2. \end{cases}$$

In order to calculate computational cost and efficiency indices the values $(n, \nu) = (2, 120)$ *are used in equations* (2.4.1) − (2.4.4). *The convergence of the methods towards the solution* $X^* = (1.271384307950131633\ldots,\ 0.880819073102661 02\ldots)^T$ *is tested with following computational terms and shown in the Table 2.2.*

$$[X + F, X; F] =$$

$$\Bigg\{\Big\{\frac{\Big((x_1 - 1)^4 + e^{-x_2^2} + x_1 + 3x_2\Big)^4 - (x_1 - 1)^4}{(x_1 - 1)^4 + e^{-x_2^2} + 3x_2 + 1},$$

$$\frac{3x_2^2 + e^{-x_2^2} - e^{-\left(x_2^2 - x_2 - 4\sin(x_1 - 1) + \ln(x_1^2 - x_2 + 1)\right)^2} + 3\ln(x_1^2 - x_2 + 1) - 12\sin(x_1 - 1)}{x_2^2 - 4\sin(x_1 - 1) + \ln(x_1^2 - x_2 + 1)}\Big\},$$

$$\{\frac{1}{(x_1-1)^4+e^{-x_2^2}+3x_2+1}\Big[-4\sin(x_1-1)+\ln(x_1^2-x_2+1)$$
$$+4\sin\Big((x_1-1)^4+e^{-x_2^2}+x_1+3x_2\Big)-\ln\Big(\Big((x_1-1)^4+e^{-x_2^2}+x_1+3x_2+1\Big)^2-x_2+1\Big)\Big],$$
$$+\frac{1}{x_2^2+4\sin(1-x_1)+\ln(x_1^2-x_2+1)}\Big[-x_2^2+\Big(x_2^2-x_2+4\sin(1-x_1)+\ln(x_1^2-x_2+1)\Big)^2$$
$$-\ln\Big(\Big((x_1-1)^4+e^{-x_2^2}+x_1+3x_2+1\Big)^2-x_2+1\Big)$$
$$+\ln\Big(x_2^2-x_2+\Big((x_1-1)^4+e^{-x_2^2}+x_1+3x_2+1\Big)^2+4\sin(1-x_1)+\ln(x_1^2-x_2+1)+1\Big)\Big]\}\Big\},$$

$$[X+F,X-F;F]=$$
$$\Big\{\{\frac{\Big((x_1-1)^4+e^{-x_2^2}+x_1+3x_2)\Big)^4-\Big(-(x_1-1)^4-e^{-x_2^2}+x_1-3x_2-2)\Big)^4}{2(x_1-1)^4+2e^{-x_2^2}+6x_2+2},$$

$$\frac{1}{-2x_2^2-2\ln(x_1^2-x_2+1)-8\sin(1-x_1)}\Big[e^{-\Big(-x_2^2+x_2-\ln(x_1^2-x_2+1)-4\sin(1-x_1)\Big)^2}$$

$$-e^{-\Big(x_2^2+x_2+\ln(x_1^2-x_2+1)+4\sin(1-x_1)\Big)^2}-6\Big(x_2^2+\ln(x_1^2-x_2+1)+4\sin(1-x_1)\Big)\Big]\},$$

$$\{\frac{1}{2(x_1-1)^4+2e^{-x_2^2}+6x_2+2}\Big[4\sin\Big(\Big(x_1-1\Big)^4+e^{-x_2^2}-x_1+3x_2+2\Big)$$
$$+4\sin\Big(\Big(x_1-1\Big)^4+e^{-x_2^2}+x_1+3x_2\Big)+\log\Big(\Big(-\Big(x_1-1\Big)^4-e^{-x_2^2}+x_1-3x_2-1\Big)^2$$
$$-x_2^2-x_2-\log\Big(x_1^2-x_2+1\Big)-4\sin\Big(1-x_1\Big)+1\Big)-\log\Big(-x_2^2-x_2$$
$$+\Big(\Big(x_1-1\Big)^4+e^{-x_2^2}+x_1+3x_2+1\Big)^2-\log\Big(x_1^2-x_2+1\Big)-4\sin\Big(1-x_1\Big)+1\Big)\Big],$$
$$\frac{1}{-2x_2^2-2\ln(x_1^2-x_2+1)-8\sin(1-x_1)}\Big[-\Big(-x_2^2+x_2-\log\Big(x_1^2-x_2+1\Big)-4\sin\Big(1-x_1\Big)\Big)^2$$
$$+\Big(x_2^2+x_2+\log\Big(x_1^2-x_2+1\Big)+4\sin\Big(1-x_1\Big)\Big)^2+\log\Big(-x_2^2-x_2+\Big(\Big(x_1-1\Big)^4+e^{-x_2^2}$$
$$+x_1+3x_2+1\Big)^2-\log\Big(x_1^2-x_2+1\Big)-4\sin\Big(1-x_1\Big)+1\Big)-\log\Big(x_2^2-x_2+\Big(\Big(x_1-1\Big)^4$$
$$+e^{-x_2^2}+x_1+3x_2+1\Big)^2+\log\Big(x_1^2-x_2+1\Big)+4\sin\Big(1-x_1\Big)+1\Big)\Big]\}\Big\},$$

$$[Y,X;F] = \begin{bmatrix} \frac{(y_1-1)^4-(x_1-1)^4}{y_1-x_1} & \frac{-3x_2-e^{-x_2^2}+e^{-y_2^2}+3y_2}{y_2-x_2} \\ \frac{4\sin(1-x_1)+\ln\left(x_1^2-x_2+1\right)-4\sin(1-y_1)-\ln\left(y_1^2-x_2+1\right)}{y_1-x_1} & \frac{x_2^2-y_2^2+\ln\left(y_1^2-x_2+1\right)-\ln\left(y_1^2-y_2+1\right)}{y_2-x_2} \end{bmatrix}.$$

Table 2.2: Performance of various iterative schemes at initial guess $(1.2,\ -1.2)^T$

Scheme	$\|X^{(1)}-X^*\|$	$\|X^{(2)}-X^*\|$	$\|X^{(3)}-X^*\|$	ρ_c	C	CEI	CPU $Time$ (sec)
$M_{4,1}$	1.09(−18)	1.71(−73)	1.04(−292)	4.000	1500	1.00092	0.282
$M_{4,2}$	1.34(−17)	1.19(−69)	4.49(−278)	4.001	1508	1.00091	0.283
$\psi_{2,1}$	4.18(−21)	2.81(−82)	5.79(−327)	4.000	1508	1.00091	0.319
$\psi_{2,2}$	4.18(−21)	2.81(−82)	5.79(−327)	4.000	1508	1.00091	0.296
$\psi_{2,3}$	4.18(−21)	2.81(−82)	5.79(−327)	4.000	1508	1.00091	0.298
$\psi_{3,1}$	5.04(−11)	2.56(−62)	4.33(−370)	6.000	1756	1.00102	1.398
$\psi_{3,2}$	5.04(−11)	2.56(−62)	4.33(−370)	6.000	1756	1.00102	1.434
$\psi_{3,3}$	5.04(−11)	2.56(−62)	4.33(−370)	6.000	1756	1.00102	1.442
S_7	2.27(−6)	3.23(−41)	5.41(−285)	6.999	3470	1.00056	1.486
$\psi_{4,1}$	3.70(−18)	9.94(−140)	2.69(−1112)	8.000	2004	1.00103	1.388
$\psi_{4,2}$	3.70(−18)	9.94(−140)	2.69(−1112)	8.000	2004	1.00103	1.445
$\psi_{4,3}$	3.70(−18)	9.94(−140)	2.69(−1112)	8.000	2004	1.00103	1.428

Example 2.5.3 *Consider the following boundary value problem (see in [63])*

$$y'' + y^3 = 0,\ y(0) = 0, y(1) = 1.$$

Further, assume the partition of the interval $[0,1]$*, which is defined as follows*

$$x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_n,\ where\ x_i = x_0 + ih_1,\ h_1 = \frac{1}{n}.$$

Define $y_0 = y(x_0) = 0,\ y_1 = y(x_1),\ \ldots,\ y_{n-1} = y(x_{n-1}),\ y_n = y(x_n) = 1.$
The following discretization for the second derivative is used

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h_1^2},\ i = 1,\ 2,\ \ldots,\ n-1,$$

which reduces to a system of nonlinear equations of order $n-1$

$$y_{i-1}-2y_i+y_{i+1}+h_1^2y_i^3=0,\ i=1,\ 2,\ \ldots,\ n-1.$$

The solution of this system $X^*=(0.0207113\ldots,0.0414227\ldots,0.0621341\ldots,$
$0.0828453\ldots,0.1035564\ldots,0.1242670\ldots,0.1449769\ldots,0.1656856\ldots,0.1863926\ldots,$
$0.2070970\ldots,0.2277981\ldots0.2484946\ldots,0.2691852\ldots,0.2898683\ldots,0.3105421\ldots,$
$0.3312043\ldots,0.3518526\ldots,0.3724841\ldots,0.3930958\ldots,0.4136841\ldots,0.4342452\ldots,$
$0.4547747\ldots,0.4752682\ldots,0.4957203\ldots,0.5161257\ldots,0.5364781\ldots,0.5567712\ldots,$
$0.5769980\ldots,0.5971509\ldots,0.6172219\ldots,0.6372025\ldots,0.6570837\ldots,0.6768557\ldots,$
$0.6965086\ldots,0.7160316\ldots,0.7354134\ldots,0.7546422\ldots,0.7737059\ldots,0.7925915\ldots,$
$0.8112857\ldots,0.8297745\ldots,0.8480437\ldots,0.8660785\ldots,0.8838634\ldots,0.9013829\ldots,$
$0.9186208\ldots,0.9355607\ldots,0.9521858\ldots,0.9684789\ldots,0.9844228\ldots)^T$ *by taking* $n=51$ *and the values* $(n,\nu)=(50,4)$ *used in equations* $(2.4.1)-(2.4.4)$ *is tested with the following computational terms in the Table 2.3.*

$$[X-F,X;F]=\Big\{\{\frac{-3\frac{x_1^3}{2601}+4x_1+\frac{\left(\frac{x_1^3}{2601}-x_1+x_2\right)^3}{2601}-2x_2}{\frac{x_1^3}{2601}-2x_1+x_2},1,0,0,\ldots,0\},$$
$$\{1,\frac{-3\frac{x_2^3}{2601}+4x_2+\frac{\left(\frac{x_2^3}{2601}-x_2+x_1+x_3\right)^3}{2601}-2(x_1+x_3)}{\frac{x_2^3}{2601}-2x_2+x_1+x_3},1,0,\ldots,0\},$$
$$\ldots,\{0,0,\ldots,1,\frac{-3\frac{x_{50}^3}{2601}+4x_{50}+\frac{\left(\frac{x_{50}^3}{2601}-x_{50}+x_{49}+1\right)^3}{2601}-2(x_{49}+1)}{\frac{x_{50}^3}{2601}-2x_{50}+x_{49}+1}\}\Big\},$$

$$[X-F,X+F;F]=\Big\{\{\frac{(x_1^4-10404x_1^2+47356407)x_1^2+5202(x_1^2-5202)x_2x_1+6765201x_2^2}{17596287801}$$
$$-2,1,0,\ldots,0\},$$
$$\{1,\frac{6765201x_1^2+5202(x_2^3-5202x_2+2601x_3)x_1+6765201x_3^2+x_2^2(x_2^4-10404x_2^2+47356407)}{17596287801}$$
$$+\frac{5202x_2(x_2^2-5202)x_3}{17596287801}-2,1,0,\ldots,0\},\ldots,\{0,0,0,\ldots,1,$$
$$\frac{6765201(x_{48}^2+x_{50}^2)+5202(x_{49}^3-5202x_{49}+2601x_{50})x_{48}+x_{49}^2(x_{49}^4-10404x_{49}^2+47356407)}{17596287801}$$
$$+\frac{5202x_{49}(x_{49}^2-5202)x_{50}}{17596287801}-2,1\},\{0,0,0,\ldots,1,$$

$$\frac{x_{50}^6 - 10404x_{50}^4 + 5202(x_{49}+1)x_{50}^3 + 47356407x_{50}^2 - 27060804(x_{49}+1)x_{50}}{17596287801}$$
$$+ \frac{6765201(x_{49}(x_{49}+2) - 5201)}{17596287801}\}\Big\},$$

$[Y, X; F] =$

$$\begin{bmatrix} \dfrac{\frac{y_1^3 - x_1^3}{2601} + 2(x_1 - y_1)}{y_1 - x_1} & 1 & \dots & 0 \\ 1 & \dfrac{\frac{y_2^3 - x_2^3}{2601} + 2(x_2 - y_2)}{y_2 - x_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dfrac{\frac{y_{50}^3 - x_{50}^3}{2601} + 2(x_{50} - y_{50})}{y_{50} - x_{50}} \end{bmatrix}.$$

Table 2.3: Results of various iterative schemes at initial value $(0.6, 0.6, \dots, 0.6)^T$

Scheme	$\Vert X^{(1)} - X^*\Vert$	$\Vert X^{(2)} - X^*\Vert$	$\Vert X^{(3)} - X^*\Vert$	ρ_c	C	CEI	CPU $Time\ (sec)$
$M_{4,1}$	4.67(−4)	4.68(−18)	6.16(−74)	4.000	944	1.00146	5.217
$M_{4,2}$	4.99(−4)	7.35(−18)	4.59(−73)	4.000	992	1.00139	9.672
$\psi_{2,1}$	4.89(−4)	5.33(−18)	1.04(−73)	4.000	980	1.00141	9.421
$\psi_{2,2}$	4.89(−4)	5.33(−18)	1.04(−73)	4.000	980	1.00141	9.544
$\psi_{2,3}$	4.89(−4)	5.33(−18)	1.04(−73)	4.000	980	1.00141	7.255
$\psi_{3,1}$	4.27(−6)	3.03(−39)	5.31(−238)	5.975	1064	1.00168	11.486
$\psi_{3,2}$	4.27(−6)	3.03(−39)	5.31(−238)	5.975	1064	1.00168	13.934
$\psi_{3,3}$	4.27(−6)	3.03(−39)	5.31(−238)	5.975	1064	1.00168	14.113
S_7	1.09(−7)	3.22(−57)	1.94(−297)	7.000	2375	1.00000	39.167
ψ_4	3.73(−8)	9.06(−69)	1.68(−553)	7.983	1058	1.00196	11.836
ψ_4	3.73(−8)	9.06(−69)	1.68(−553)	7.983	1058	1.00196	15.429
ψ_4	3.73(−8)	9.06(−69)	1.68(−553)	7.983	1058	1.00196	13.394

Example 2.5.4 *The following problem has been considered from the paper [7]*

$$cos^{-1}(x_i) - \sum_{i=1}^{20} x_i + 2x_i = 0, i = 1, 2, \ldots, 20,$$

where $(n, \nu) = (20, 119)$ *are the values used in equations* $(2.4.1) - (2.4.4)$. *Solution of this problem is* $X^* = (0.08266851975958913\ldots, 0.08266851975958913\ldots, \ldots,$ $0.08266851975958913\ldots)^T$ *with the following computational terms and comparisons of the method is display in the Table 2.4.*

$$[X+F, X; F] =$$

$$\Bigg\{\{\frac{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}\left(2x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)\right)}{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)}, \frac{x_1 - x_2 + \sum_{i=3}^{20} x_i - \cos^{-1}(x_2)}{-x_1 + x_2 - \sum_{i=3}^{20} x_i + \cos^{-1}(x_2)},$$

$$\ldots, \frac{\sum_{i=1}^{19} -x_{20} - \cos^{-1}(x_{20})}{-\sum_{i=1}^{19} x_{20} + \cos^{-1}(x_{20})}\}, \{\frac{-x_1 + \sum_{i=2}^{20} x_i - \cos^{-1}(x_1)}{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)},$$

$$\frac{x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}\left(-x_1 + 2x_2 - \sum_{i=3}^{20} x_i + \cos^{-1}(x_2)\right)}{x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}(x_2)}, \ldots, \frac{\sum_{i=1}^{19} x_i - x_{20} - \cos^{-1}(x_{20})}{-\sum_{i=1}^{19} x_i + x_{20} + \cos^{-1}(x_{20})}\},$$

$$\ldots, \{\frac{-x_1 + \sum_{i=2}^{20} x_i - \cos^{-1}(x_1)}{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)}, \frac{-x_2 + \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - \cos^{-1}(x_2)}{x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}(x_2)}, \ldots,$$

$$\frac{-\sum_{i=1}^{19} x_i + x_{20} + \cos^{-1}\left(-\sum_{i=1}^{19} x_i + 2x_{20} + \cos^{-1}(x_{20})\right)}{-\sum_{i=1}^{19} x_i + x_{20} + \cos^{-1}(x_{20})}\}\Bigg\},$$

$$[X+F, X-F; F] =$$

$$\Bigg\{\{\frac{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1) + \cos^{-1}\left(2x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)\right) - \cos^{-1}\left(\sum_{i=2}^{20} x_i - \cos^{-1}(x_1)\right)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)},$$

$$\frac{2x_1 - 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_2)}{2x_2 - 2\sum_{i=1}^{20} x_i + 2\cos^{-1}(x_2)}, \dots, \frac{2\sum_{i=1}^{19} x_i - 2x_{20} - 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\}, \{\frac{-2x_1 + 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_1)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)},$$

$$\frac{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2) + \cos^{-1}\left(2x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}(x_2)\right) - \cos^{-1}\left(\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - \cos^{-1}(x_2)\right)}{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2)},$$

$$\dots, \frac{2\sum_{i=1}^{19} x_i - 2x_{20} - 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\}, \dots, \{\frac{-2x_1 + 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_1)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)},$$

$$\frac{-2x_2 + 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - 2\cos^{-1}(x_2)}{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2)}, \dots, \frac{2\sum_{i=1}^{19} x_i - 2x_{20} - 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\},$$

$$\{\frac{-2x_1 + 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_1)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)}, \frac{-2x_2 + 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - 2\cos^{-1}(x_2)}{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2)}, \dots,$$

$$\frac{-2\sum_{i=1}^{19} x_i + 2x_{20} - \cos^{-1}\left(\sum_{i=1}^{19} x_i - \cos^{-1}(x_{20})\right) + 2\cos^{-1}(x_{20}) + \cos^{-1}\left(-\sum_{i=1}^{19} x_i + 2x_{20} + \cos^{-1}(x_{20})\right)}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\}\Big\},$$

$$[Y, X; F] =$$

$$\begin{bmatrix} \frac{-x_1-\cos^{-1}(x_1)+y_1+\cos^{-1}(y_1)}{y_1-x_1} & \frac{x_2-y_2}{y_2-x_2} & \dots & \frac{x_{20}-y_{20}}{y_{20}-x_{20}} \\ \frac{x_1-y_1}{y_1-x_1} & \frac{-x_2-\cos^{-1}(x_2)+y_2+\cos^{-1}(y_2)}{y_2-x_2} & \dots & \frac{x_{20}-y_{20}}{y_{20}-x_{20}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1-y_1}{y_1-x_1} & \frac{x_2-y_2}{y_2-x_2} & \dots & \frac{-x_{20}-\cos^{-1}(x_{20})+y_{20}+\cos^{-1}(y_{20})}{y_{20}-x_{20}} \end{bmatrix}.$$

Table 2.4: Performance of various iterative schemes at initial guess $(0.083, 0.083, \ldots, 0.083)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ_c	C	CEI	CPU $Time\ (sec)$
$M_{4,1}$	3.39(−16)	9.53(−67)	5.95(−269)	4.000	15336	1.00000	2.589
$M_{4,2}$	3.39(−16)	9.53(−67)	5.95(−269)	4.000	15380	1.00000	2.852
$\psi_{2,1}$	5.81(−16)	9.19(−67)	5.74(−265)	3.999	11132	1.00001	2.053
$\psi_{2,2}$	5.81(−16)	9.19(−67)	5.74(−265)	3.999	11132	1.00001	2.464
$\psi_{2,3}$	5.81(−16)	9.19(−67)	5.74(−265)	3.999	11132	1.00001	2.217
$\psi_{3,1}$	2.09(−22)	1.11(−135)	2.42(−815)	6.000	11414	1.00001	7.354
$\psi_{3,2}$	2.09(−22)	1.11(−135)	2.42(−815)	6.000	11414	1.00001	7.476
$\psi_{3,3}$	2.09(−22)	1.11(−135)	2.42(−815)	6.000	11414	1.00001	7.191
S_7	4.13(−34)	5.39(−248)	3.46(−1745)	7.000	47602	1.00000	37.715
$\psi_{4,1}$	7.53(−29)	2.34(−231)	1.35(−1851)	8.000	11696	1.00001	13.906
$\psi_{4,2}$	7.53(−29)	2.34(−231)	1.35(−1851)	8.000	11696	1.00001	16.121
$\psi_{4,3}$	7.53(−29)	2.34(−231)	1.35(−1851)	8.000	11696	1.00001	15.264

Example 2.5.5 *Considering the gravity-flow discharge chute problem (see [56, pp. 646]).*

$$G_i = \begin{cases} \dfrac{sin\ x_{i+1}}{v_{i+1}}(1 - \mu w_{i+1}) - \dfrac{sin\ x_i}{v_i}(1 - \mu w_i) = 0, & 1 \leq i \leq 19, \\ \Delta y \Sigma_{i=1}^{20} tan\ x_i - X = 0, & i = 20, \end{cases}$$

where $v_i^2 = v_0^2 + 2gi\Delta y - 2\mu \Delta y \Sigma_{j=1}^{20} \frac{1}{cos\ x_j}$, $1 \leq i \leq 20$ *and* $w_i = -\Delta y v_i \Sigma_{j=1}^{20} \frac{1}{v_j^3 cos\ x_j}$, $1 \leq i \leq 20$.

In this example $v_0 = 0$ *initial velocity of the granular material,* $X = 2$ *the x-coordinate the end of the chute,* $\mu = 0$ *the friction force,* $g = 32.17 ft/sec^2$ *gravitational force and* $\Delta y = 0.2$ *has been considered.*

The solution of this system $X^* = (0.14062\ldots, 0.19954\ldots, 0.24522\ldots, 0.28413\ldots, 0.31878\ldots, 0.35045\ldots, 0.37990\ldots, 0.40763\ldots, 0.43398\ldots, 0.45920\ldots, 0.48348\ldots, 0.50697\ldots, 0.52980\ldots, 0.55205\ldots, 0.57382\ldots, 0.59516\ldots, 0.61615\ldots, 0.63683\ldots, 0.65726\ldots, 0.67746\ldots)^T$ *and the values* $(n, \nu) = (20, 84.8)$ *used in equations* (2.4.1) − (2.4.4) *is tested with following computational terms and displays in the Table 2.5.*

$$[X+F,X;F]=$$

$$\Big\{\{\frac{0.278769\sin\Big(-x_1+0.278769\sin(x_1)-0.197119\sin(x_2)\Big)+0.278769\sin(x_1)}{0.197119\sin(x_2)-0.278769\sin(x_1)},$$

$$\frac{-0.197119\sin\Big(-x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)-0.197119\sin(x_2)}{0.160947\sin(x_3)-0.197119\sin(x_2)},0,\ldots,0\},$$

$$\{0,\frac{0.197119\sin\Big(-x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)+0.197119\sin(x_2)}{0.160947\sin(x_3)-0.197119\sin(x_2)},$$

$$\frac{-0.160947\sin\Big(-x_3+0.160947\sin(x_3)-0.139384\sin(x_4)\Big)-0.160947\sin(x_3)}{0.139384\sin(x_4)-0.160947\sin(x_3)},\ldots,0\},\ldots,$$

$$\{0,\ldots,\frac{0.063954\sin\Big(-x_{19}+0.063954\sin(x_{19})-0.0623346\sin(x_{20})\Big)+0.063954\sin(x_{19})}{0.0623346\sin(x_{20})-0.063954\sin(x_{19})},$$

$$\frac{-0.0623346\sin(x_{20})-0.0623346\sin\Big(-x_{20}-0.2\sum_{i=1}^{20}\tan(x_i)+2\Big)}{0.2\sum_{i=1}^{20}\tan(x_i)-2}\},$$

$$\{\frac{-0.2\tan(x_1)-0.2\tan\Big(-x_1+0.278769\sin(x_1)-0.197119\sin(x_2)\Big)}{0.197119\sin(x_2)-0.278769\sin(x_1)},$$

$$\frac{-0.2\tan(x_2)-0.2\tan\Big(-x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.160947\sin(x_3)-0.197119\sin(x_2)},$$

$$\frac{0.2\tan\Big(-x_3+0.160947\sin(x_3)-0.139384\sin(x_4)\Big)-0.2\tan(x_3)}{0.139384\sin(x_4)-0.160947\sin(x_3)},\ldots,$$

$$\frac{-0.2\tan(x_{20})-0.2\tan\Big(-x_{20}-0.2\Big(\sum_{i=1}^{20}\tan(x_i)\Big)+2\Big)}{0.2\sum_{i=1}^{20}\tan(x_i)-2}\}\Big\},$$

$$[X+F,X-F;F]=$$

$$\Big\{\{\frac{0.278769\sin\Big(-x_1+0.278769\sin(x_1)-0.197119\sin(x_2)\Big)}{0.394239\sin(x_2)-0.557538\sin(x_1)}+$$

$$\frac{0.278769\sin\Big(x_1+0.278769\sin(x_1)-0.197119\sin(x_2)\Big)}{0.394239\sin(x_2)-0.557538\sin(x_1)},$$

$$\frac{-0.197119\sin\Big(-x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.321895\sin(x_3)-0.394239\sin(x_2)}+$$

$$\frac{-0.197119\sin\Big(x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.321895\sin(x_3)-0.394239\sin(x_2)},0,\ldots,0\},$$

$$\{0,\frac{0.197119\sin\Big(-x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.321895\sin(x_3)-0.394239\sin(x_2)}+$$

$$\frac{0.197119\sin\Big(x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.321895\sin(x_3)-0.394239\sin(x_2)},$$

$$\frac{-0.160947\sin\Big(-x_3+0.160947\sin(x_3)-0.139384\sin(x_4)\Big)}{0.278769\sin(x_4)-0.321895\sin(x_3)}+$$

$$\frac{-0.160947\sin\Big(x_3+0.160947\sin(x_3)-0.139384\sin(x_4)\Big)}{0.278769\sin(x_4)-0.321895\sin(x_3)},\ldots,0\},\ldots,\{0,\ldots,$$

$$\frac{0.063954\sin\Big(-x_{19}+0.063954\sin(x_{19})-0.0623346\sin(x_{20})\Big)}{0.124669\sin(x_{20})-0.127908\sin(x_{19})}+$$

$$\frac{0.063954\sin\Big(x_{19}+0.063954\sin(x_{19})-0.0623346\sin(x_{20})\Big)}{0.124669\sin(x_{20})-0.127908\sin(x_{19})},$$

$$-\frac{0.0623346\sin\Big(-x_{20}-0.2\sum_{i=1}^{20}\tan(x_i)+2\Big)}{0.4\sum_{i=1}^{20}\tan(x_i)-4}-\frac{0.0623346\sin\Big(x_{20}-0.2\sum_{i=1}^{20}\tan(x_i)+2\Big)}{0.4\sum_{i=1}^{20}\tan(x_i)-4}\},$$

$$\{-\frac{0.2\tan\Big(-x_1+0.278769\sin(x_1)-0.197119\sin(x_2)\Big)}{0.394239\sin(x_2)-0.557538\sin(x_1)}+$$

$$-\frac{0.2\tan\Big(x_1+0.278769\sin(x_1)-0.197119\sin(x_2)\Big)}{0.394239\sin(x_2)-0.557538\sin(x_1)},$$

$$-\frac{0.2\tan\Big(-x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.321895\sin(x_3)-0.394239\sin(x_2)}+$$

$$-\frac{0.2\tan\Big(x_2+0.197119\sin(x_2)-0.160947\sin(x_3)\Big)}{0.321895\sin(x_3)-0.394239\sin(x_2)},\ldots,$$

$$-\frac{0.2\tan\Big(-x_{20}-0.2\sum_{i=1}^{20}\tan(x_i)+2\Big)}{0.4\sum_{i=1}^{20}\tan(x_i)-4}-\frac{0.2\tan\Big(x_{20}-0.2\sum_{i=1}^{19}\tan(x_i)+\tan(x_{20})+2\Big)}{0.4\sum_{i=1}^{20}\tan(x_i)-4}\}\Big\},$$

$$[Y,X;F]=\Big\{\frac{0.278769\Big(\sin(x_1)-\sin(y_1)\Big)}{y_1-x_1},\frac{0.197119\Big(\sin(y_2)-\sin(x_2)\Big)}{y_2-x_2},$$
$$0,\dots,0\},\{0,\frac{0.197119\Big(\sin(x_2)-\sin(y_2)\Big)}{y_2-x_2},\frac{0.160947\Big(\sin(y_3)-\sin(x_3)\Big)}{y_3-x_3},$$
$$\dots,0\},\dots,\{0,\dots,\frac{0.063954\Big(\sin(x_{19})-\sin(y_{19})\Big)}{y_{19}-x_{19}},$$
$$\frac{0.0623346\Big(\sin(y_{20})-\sin(x_{20})\Big)}{y_{20}-x_{20}}\},\{\frac{0.2\Big(\tan(x_1)-\tan(y_1)\Big)}{x_1-y_1},$$
$$\frac{0.2\Big(\tan(x_2)-\tan(y_2)\Big)}{x_2-y_2},\dots,\frac{0.2\Big(\tan(x_{19})-\tan(y_{19})\Big)}{x_{19}-y_{19}},\frac{0.2\Big(\tan(x_{20})-\tan(y_{20})\Big)}{x_{20}-y_{20}}\}\Big\}.$$

Table 2.5: Results of various iterative schemes at initial value $(0.75, 0.75, \dots, 0.75)^T$

Scheme	$\|X^{(1)}-X^*\|$	$\|X^{(2)}-X^*\|$	$\|X^{(3)}-X^*\|$	ρ_c	C	CEI	CPU $Time\ (sec)$
$M_{4,1}$	2.36(−4)	2.95(−17)	6.13(−69)	4.001	11232	1.00001	7.864
$M_{4,2}$	2.32(−4)	2.70(−16)	1.24(−63)	4.000	11276	1.00001	7.924
$\psi_{2,1}$	1.33(−5)	8.85(−22)	1.69(−86)	3.999	82592	1.00001	7.875
$\psi_{2,2}$	1.33(−5)	8.85(−22)	1.69(−86)	3.999	82592	1.00001	7.878
$\psi_{2,3}$	1.33(−5)	8.85(−22)	1.69(−86)	3.999	82592	1.00001	7.823
$\psi_{3,1}$	9.14(−12)	2.76(−70)	1.39(−421)	5.997	84728	1.00002	25.334
$\psi_{3,2}$	9.14(−12)	2.76(−70)	1.39(−421)	5.997	84728	1.00002	24.632
$\psi_{3,3}$	9.14(−12)	2.76(−70)	1.39(−421)	5.997	84728	1.00002	19.553
S_7	4.42(−28)	5.85(−197)	3.43(−804)	6.989	34332	1.00000	47.809
$\psi_{4,1}$	1.77(−20)	6.52(−163)	2.71(−1303)	8.041	86804	1.00002	24.259
$\psi_{4,2}$	1.77(−20)	6.52(−163)	2.71(−1303)	8.041	86804	1.00002	30.806
$\psi_{4,3}$	1.77(−20)	6.52(−163)	2.71(−1303)	8.041	86804	1.00002	28.747

In above Tables 2.1–2.5, $\|X^{(k)}-X^*\|$ shows the errors of approximations to the corresponding solutions of examples 2.5.1 – 2.5.5, (ρ_c) the computational order of convergence and C_i the computational costs given by equations (2.4.1) – (2.4.4) in terms of products and the computational efficiencies CEI, where $\widetilde{b}(-a)$ denoted by $\widetilde{b}\times 10^{-a}$. The numerical results in above Tables 2.1–2.5 demonstrates that proposed methods works more efficiently with less error as compared to existing methods namely, $M_{4,1}$, $M_{4,2}$ and S_7. Also the higher order methods not only works on simple experiment, it also works on application oriented problems as shown in examples 2.5.3 and 2.5.5.

2.6 Conclusions

In this chapter, several techniques of Ostrowski's method have been proposed for solving nonlinear systems. The new methods are completely derivative free and therefore, suited to those problems in which derivatives require lengthy computations. A development of an inverse first-order divided difference operator for multi variable function is applied to prove the convergence order of proposed methods. Moreover, the fourth and sixth-order methods proposed by Grau *et al.* [7] have been recovered as the special cases of the presented families. Further, the computational efficiency index is used to compare the efficiency of these different proposed families. Computational results have conformed robust and efficient character of the proposed families. Some numerical experimentations have also being carried out for a number of problems and results are found to be at a par with those presented here. Thus, the new methods are very suitable and applicable to solve non-linear systems.

Chapter 3

Local Convergence Analysis and Applications of non-linear systems

The aim of this chapter is to extend the applicability of the family (2.2.9) proposed in the pervious chapter, in local convergence analysis and applications of non-linear systems. The local convergence is based on Lipschitz constants and hypotheses on the divided difference of order one, in contrast to other works requiring higher order derivatives not appearing in these schemes. Hence, the applicability of these methods is expanded. Further, a variety of applications of non-linear systems namely, Frank-Kamenetskii, Bratu's problem in one and two dimensional and Fisher's problems are applied in order to check the performance of family and to verify the theoretical results. On the account of these examples, it is concluded that the family (2.2.9) is more efficient and show better performance as compared to the existing one.

3.1 Local convergence

The convergence study of an iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points. In the local convergence, a unique locally zero X^* of operator F is approximated, where F is a non-linear operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. So, the interesting part of local convergence of an iterative scheme is to find the radius of convergence for the scheme. An open ball $U(X^*, r) \subset X$ is called a convergence ball of an iterative method, if the sequence generated by iterative technique starting from any initial values in the ball converges. Though, a small radius of convergence is obtained with local analysis, but it ensures the convergence of an iterative scheme.

In the pervious chapter, higher order of convergence of the family (2.2.9) is shown using

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Taylor expansions and hypotheses reaching up to the higher derivatives of the function F, although such derivatives do not appear in method (2.2.9). Therefore, the author presents the local convergence analysis of the proposed family of methods (2.2.9) which is based on Lipschitz constants and hypotheses on the divided difference of order one. In this way, the applicability of the proposed methods has been expanded.

For this aim, consider the iterative expression (2.2.9) as:

$$\begin{cases} \psi_1^k = X^{k+1} = X^k - \{[X^k + F, X^k - F; F]\}^{-1} F(X^k), \\ \psi_i^k = \psi_{i-1}^k - \dfrac{\eta^k}{\tau^k} F(\psi_{i-1}^k), \quad i = 2, 3, 4 \ldots p. \end{cases} \tag{3.1.1}$$

Where,

$$\eta^k = b_4\{[X^k + F(X^k),\ X^k - F(X^k); F]\}^{-1}[\psi_1^k,\ X^k;\ F] + b_2 I, \tag{3.1.2}$$

$$\begin{aligned} \tau^k &= b_1[X^k + F(X^k),\ X^k - F(X^k);\ F] + b_2[\psi_1^k,\ X^k;\ F] \\ &+ b_3\{[X^k + F(X^k),\ X^k - F(X^k);\ F]\}^{-1}[\psi_1^k,\ X^k;\ F][X^k + F(X^k),\ X^k - F(X^k);\ F] \\ &+ b_4\{[X^k + F(X^k),\ X^k - F(X^k); F]\}^{-1}([\psi_1^k,\ X^k;\ F])^2, \end{aligned} \tag{3.1.3}$$

and $b_i,\ i = 1,\ 2,\ 3,\ 4,\ 5$ by $b_1 = -\alpha_1^2 - \alpha_1\alpha_2,\ b_2 = 2\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2,$ $b_3 = -\alpha_1\alpha_2,\ b_4 = 2\alpha_1\alpha_2 - \alpha_2^2,\ b_5 = \alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2$ and $b = b_1 + b_2$. Let α_1 and α_2 be real parameters. The local convergence analysis of method (3.1.1) is based on some scalar functions and parameters. This analysis is given for $F : D \subseteq B \to B$, in a more general setting where B is a Banach space. Let $K_0 > 0,\ K > 0,\ c_0 > 0,\ c > 0,\ c_1 > 0$ be parameters.

Define function g_1 on the interval $[0,\ r_0^-)$ by

$$g_1(s) = \frac{(1 + 2c_0)Ks}{1 - r_0 s}, \tag{3.1.4}$$

where

$$r_0 = 2(1 + c_0)K_0, \tag{3.1.5}$$

and parameter r_1 by

$$r_1 = \frac{1}{(1 + 2c_0)K + 2(1 + c_0)K_0}. \tag{3.1.6}$$

Then, one can have that $g_1(r_1) = 1,\ 0 < r_1 < r_0^-$ and for each $s \in [0,\ r_1)$, $0 \le g_1(s) < 1$. Define functions q and h_q in the following way

$$q(s) = |b|^{-1}\left(|b_1| r_0 s + |b_2| K_0 (1 + g_1(s))s + \frac{(|b_3| + |b_4|)c_0 c_1 c}{1 - r_0 s}\right),\ b_0 \neq 0 \tag{3.1.7}$$

and

$$h_q(s) = q(s) - 1. \tag{3.1.8}$$

Suppose that

$$(|b_3| + |b_4|)c_0 c_1 c < |b|. \tag{3.1.9}$$

Using equations (3.1.7), (3.1.8), and (3.1.9), one gets $h_q(0) = -1 < 0$ and $h_q(s) \to +\infty$ as $s \to \frac{1^-}{r_0}$. It then follows from the intermediate value theorem that function h_q has zeros in the interval $(0,\ r_0^-)$. Let us consider that r_q be the smallest zero among such zero. Moreover, define some functions g_i and h_i on the interval $[0,\ r_q)$ for $i = 2,\ 3,\ \ldots,\ p$ in the following way

$$\begin{aligned} g_i(s) &= \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|(1-q(s))(1-r_0 s)}\right) g_{i-1}(s) \\ &= \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|(1-q(s))(1-r_0 s)}\right)^{i-1} g_1(s), \end{aligned}$$

$$h_i(s) = g_i(s) - 1.$$

Then, one can obtain that $h_i(0) = -1 < 0$ and $h_i(s) \to +\infty$ as $s \to r_q^-$. Denote by $r_i,\ i = 2,\ 3,\ \ldots,\ p$ the smallest zeros of functions g_i on the interval zeros of functions g_i on the interval $(0,\ r_q)$. Notice that $h_i(r_{i-1}) = c\left(|b_2| + \frac{|b_4|c}{|b|(1-r_0 r_{i-1})(1-q(r_{i-1}))}\right) > 0$, which imply that

$$r_p < r_{p-1} < \cdots < r_2. \tag{3.1.10}$$

Define

$$r^* = \min\{r_p,\ r_1\}. \tag{3.1.11}$$

Then, for each $s \in [0,\ r^*)$, one gets

$$0 \leq g_i(s) < 1 \text{ and } 0 \leq q(t) < 1,\ i = 1,\ 2,\ \ldots,\ p.$$

Next, the local convergence analysis of method (3.1.1) using the preceding notation has been presented.

Theorem 3.1.1 *Let $F : D \subseteq B \to B$ be a continuous operator. Suppose that there exist divided difference of order one for operator F, $[\cdot,\ \cdot;\ F] : D \times D \to L(B)$, $X^* \in D$, for which $\{F'(X^*)\}^{-1}$ exists, $\alpha_1,\ \alpha_2 \in \mathbb{R}$, $K_0 > 0$, $K > 0, c_0 > 0$, $c > 0$, $c_1 > 0$ and $p = 1,\ 2,\ 3,\ \ldots$ such that (3.1.9) holds and $b \neq 0$ for each $X,\ Y,\ Z \in D$ and $F(X^*) = 0$, $\{F'(X^*)\}^{-1} \in L(X)$, $\|F'(X^*)^{-1}\| \leq c_1,\ b \neq 0$*

$$\left\|\{F'(X^*)\}^{-1}([X,\ Y;\ F] - F'(X^*))\right\| \leq K_0(\|X - X^*\| + \|Y - X^*\|) \tag{3.1.12}$$

$$\left\|\{F'(X^*)\}^{-1}\left([X,\ Y;\ F]-[Z,\ X^*;\ F]\right)\right\|\leq K(\|X-Z\|+\|Y-X^*\|) \tag{3.1.13}$$

$$\|[X,\ Y;\ F]\|\leq c_0 \tag{3.1.14}$$

$$\left\|\{F'(X^*)\}^{-1}[X,\ Y;\ F]\right\|\leq c \tag{3.1.15}$$

and

$$\bar{U}(X^*,\ (1+c_0)K_0)\subset D. \tag{3.1.16}$$

Then, the sequence generated by method (3.1.1) *for* $X^0\in U(X^*,\ r^*)-\{X^*\}$ *is well defined, remains in* $U(X^*,\ r^*)$ *and converges to* X^**. Moreover, the following estimates hold*

$$\|\psi_i^k-X^*\|\leq g_i(\|X^k-X^*\|)\|X^k-X^*\|<\|X^k-X^*\|<r^*, \tag{3.1.17}$$

for each $i=1,\ 2,\ \ldots,p$*, where the 'g' functions are defined previously. Furthermore, for* $T\in\left[r^*,\ \frac{1}{K_0}\right)$*, the limit point* X^* *is the only solution of equation* $F(X)=0$ *in* $\bar{U}(X^*,\ T)\cap D$*.*

Proof We shall show estimate (3.1.17) holds with the help of mathematical induction. By hypotheses $X^0\in U(X^*,\ r^*)-\{X^*\}$, (3.1.10), (3.1.11), and (3.1.12), one gets that

$$\begin{aligned}
&\left\|\{F'(X^*)\}^{-1}\left([X^0+F,\ X^0-F;F]-F'(X^*)\right)\right\|\\
&\leq K_0\left(\|X^0-X^*+F(X^0)\|+\|X^0-X^*-F(X^0)\|\right)\\
&\leq K_0\left(\|X^0-X^*\|+\|F(X^0)-F(X^*)\|+\|X^0-X^*\|+\|F(X^0)-F(X^*)\|\right)\\
&=2K_0\left(\|X^0-X^*\|+c_0\|X^0-X^*\|\right)\\
&=2K_0(1+c_0)\|X^0-X^*\|\\
&=r_0\|X^0-X^*\|<r_0r^*<1.
\end{aligned} \tag{3.1.18}$$

Notice that $\|X^0+F-X^*\|\leq\|X^0-X^*\|+\|F(X^0)-F(X^*)\|\leq(1+c_0)\|X^0-X^*\|<(1+c_0)r^*$, so $X^0+F\in\bar{U}(X^*,\ (1+c_0r^*))\subset D$. Similarly, one can get that $X^0-F(X^0)\in D$. Then, it follows from (3.1.18) and the Banach Lemma (1.2.14) on invertible operators (3.1.1) that ψ_1^0 is well defined by the first sub step of method (3.1.1) and

$$\left\|\{[X^0+F(X^0),\ X^0-F(X^0);F]\}^{-1}F'(X^*)\right\|\leq\frac{1}{1-r_0\|X^0-X^*\|}. \tag{3.1.19}$$

One can write by (3.1.11) and the first sub step of method (3.1.1) that

$$
\begin{aligned}
&\psi_1^0 - X^* \\
&= X^0 - X^* - \{[X^0 + F(X^0),\ X^0 - F(X^0); F]\}^{-1} F(X^0) \\
&= \{[X^0 + F(X^0),\ X^0 - F(X^0);\ F]\}^{-1} \left([X^0 + F(X^0),\ X^0 - F(X^0); F](X^0 - X^*) - F(X^0) \right) \\
&= \Big(\{[X^0 + F(X^0),\ X^0 - F(X^0);\ F]\}^{-1} F'(X^*) \Big) \Big(\{F'(X^*)\}^{-1} [X^0 + F(X^0),\ X^0 \\
&\quad - F(X^0); F] - [X^0,\ X^0; F] \Big) (X^0 - X^*).
\end{aligned} \tag{3.1.20}
$$

Using (3.1.10), (3.1.11) (for $i = 1$), (3.1.13) and (3.1.19), one obtain in turn that

$$
\begin{aligned}
&\|\psi_1^0 - X^*\| \le \left\| \{[X^0 + F(X^0),\ X^0 - F(X^0); F]\}^{-1} F(X^*) \right\| \\
&= \left\| \{F'(X^*)\}^{-1} \left([X^0 + F(X^0),\ X^0 - F(X^0); F] - [X^0,\ X^*; F] \right) \right\| \|X^0 - X^*\| \\
&\le \frac{(1 + 2c_0)K}{1 - 2K_0(1 + c_0)} \|X^0 - X^*\| \\
&\le g_1(\|X^0 - X^*\|) \|X^0 - X^*\| < \|X^0 - X^*\| < r^*,
\end{aligned} \tag{3.1.21}
$$

which shows (3.1.17) for $k = 0,\ i = 1$ and $\psi_1^0 \in U(X^*,\ r^*)$. Next, we shall show that $(\tau^0)^{-1} \in L(X)$. Using (3.1.10), (3.1.11), (3.1.12), (3.1.15), (3.1.21), and (3.1.3), one gets in turn that since $b \ne 0$

$$
\begin{aligned}
&\left\| \{bF'(X^*)\}^{-1} [\tau^0 - (b_1 + b_2 + b_3) F'(x^*)] \right\| \\
&\le |b|^{-1} \left[|b_1| \| \{F'(X^*)\}^{-1} \left([X^0 + F(X^0),\ X^0 - F(X^0);\ F] - F'(X^*) \right) \| \right] \\
&\quad + |b_2| \left\| \{F'(X^*)\}^{-1} \left([\psi_1^0,\ X^0;\ F] - F'(X^*) \right) \right\| \\
&\quad + |b_3| \left\| \{F'(X^*)\}^{-1} \right\| \left\| [X^0 + F(X^0),\ X^0 - F(X^0); F]^{-1} F'(X^*) \right\| \\
&\quad \times \left\| \{F'(X^*)\}^{-1} [\psi_1^0,\ X^0;\ F] \right\| \left\| [X^0 + F(X^0),\ X^0 - F(X^0);\ F] \right\| \\
&\quad + |b_4| \left\| \{F'(X^*)\}^{-1} \right\| \left\| [X^0 + F(X^0),\ X^0 - F(X^0); F]^{-1} F'(X^*) \right\| \left\| F'(X^*)^{-1} [\psi_1^0,\ X^0;\ F] \right\| \\
&\quad \times \left\| [\psi_1^0,\ X^0;\ F] \right\| \\
&\le |b|^{-1} \left[|b_1| r_0 \|X^0 - X^*\| + |b_2| K_0 \left(\|\psi_1^0 - X^*\| + \|X^0 - X^*\| \right) + \frac{(|b_3| + |b_4|) c_0 c_1 c}{1 - r_0 \|X^0 - X^*\|} \right] \\
&\le |b|^{-1} \left[|b_1| r_0 \|X^0 - X^*\| + |b_2| K_0 \left(1 + g_1(\|X^0 - X^*\|) \right) \|X^0 - X^*\| + \frac{(|b_3| + |b_4|) c_0 c_1 c}{1 - r_0 \|X^0 - X^*\|} \right] \\
&\le q(\|X^0 - X^*\|) < q(r^*) < 1.
\end{aligned} \tag{3.1.22}
$$

Then, it follows from (3.1.22) that

$$
\left\| (\tau^0)^{-1} F'(X^*) \right| \le \frac{1}{|b| \left(1 - q(\|X^0 - X^*\|) \right)}. \tag{3.1.23}
$$

By (3.1.15), (3.1.19) and (3.1.2), one can get,

$$\begin{aligned}\|\eta^0\| &\le |b_4| \left\|\{[X^0+F(X^0),\ X^0-F(X^0);\ F]\}^{-1}F'(X^*)\right\| \|\{F'(X^*)\}^{-1}[\psi_1^0,\ X^0;\ F]\| + |b_2| \\ &\le \frac{|b_4|c}{1-r_0\|X^0-X^*\|} + |b_2|.\end{aligned} \tag{3.1.24}$$

Then, using (3.1.10), (3.1.11), (3.1.21), (3.1.23), (3.1.24) and the definition of the 'g_i' functions, one obtain from the second sub step of method (3.1.1) that

$$\begin{aligned}\|\psi_2^0 - X^*\| &\le \|\psi_1^0 - X^*\| + \|\eta^0\|\|(\tau^0)^{-1}F'(X^*)\|\|\{F'(X^*)\}^{-1}F(\psi_1^0)\| \\ &\le \|\psi_1^0 - X^*\| + \|\eta^0\|\|(\tau^0)^{-1}F'(X^*)\|\|\{F'(X^*)\}^{-1}\left(F(\psi_1^0) - F(X^*)\right)\| \\ &\le \|\psi_1^0 - X^*\| + \|\eta^0\|\|(\tau^0)^{-1}F'(X^*)\|\|\{F'(X^*)\}^{-1}[\psi_1^0,\ X^0;\ F]\|\|\psi_1^0 - X^*\| \\ &\le \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|\left(1-q(\|X^0-X^*\|)\right)\left(1-r_0(\|X^0-X^*\|)\right)}\right) g_1(\|X^0-X^*\|)\|X^0-X^*\| \\ &= g_2(\|X^0-X^*\|)\|X^0-X^*\| < \|X^0-X^*\| < r^*,\end{aligned} \tag{3.1.25}$$

which shows (3.1.17) for $i=2,\ k=0$ and $\psi_2^0 \in U(X^*,\ r^*)$. Similarly, one can show that

$$\begin{aligned}\|\psi_3^0 - X^*\| &\le \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|\left(1-q(\|X^0-X^*\|)\right)\left(1-r_0(\|X^0-X^*\|)\right)}\right) g_2(\|X^0-X^*\|)\|X^0-X^*\| \\ &= g_3(\|X^0-X^*\|)\|X^0-X^*\| < \|X^0-X^*\| < r^*,\end{aligned}$$

until

$$\begin{aligned}\|X^1 - X^*\| = \|\psi_n^k - X^*\| &\le g_n(\|X^0-X^*\|)\|X^0-X^*\| \\ &\le \mu\|X^0-X^*\| < \|X^0-X^*\| < r^*,\end{aligned} \tag{3.1.26}$$

where $\mu = g_p(r^*) \in (0,\ 1)$. By simply replacing $\psi_1^0, \psi_2^0,\ \ldots, \psi_p^0,\ X^0$ by $\psi_1^m, \psi_2^m,\ \ldots, \psi_p^m,\ X^m$ in the preceding estimates one completes the induction for (3.1.17). Then, in view of the estimates $\|X^{m+1} - X^*\| \le \mu\|X^m - X^*\|$(see (3.1.26)), one deduces that $\{X^m\}$ converges to X^* and $X^m \in U(X^*,\ r^*)$ for each $m = 0,\ 1,\ 2,\ \ldots$. Finally, to show the uniqueness part, let $H = [X^*, Y^*; F]$ where $F(Y^*) = 0$ and $Y^* \in \bar{U}(X^*, T)$. Then, using (3.1.12), one gets that

$$\left|\{F'(X^*)\}^{-1}([X^*, Y^*; F] - F'(X^*))\right| \le K_0\left(\|X^* - Y^*\|\right) < 1. \tag{3.1.27}$$

Hence, $H^{-1} \in L(B)$. Then, from the identity $0 = F(X^*) - F(Y^*) = H(X^* - Y^*)$, one conclude that $X^* = Y^*$.

Remark

(a) If $X = \mathbb{R}^n$ then Theorem 3.1.1 specializes in the case studied in the earlier sections.

(b) The convergence of method (3.1.1) in the previous chapter was shown using hypothesis limit the applicability of method (3.1.1). In [67], some examples where the third or higher derivatives do not exist have been presented. Therefore, in Example 3.1.1, an another such a case for such equations where method (3.1.1) is not applicable has been presented. However, in Theorem 3.1.1 only hypothesis on the divided difference of order one and on $F'(X^*)$, which actually appear in method (3.1.1) has been used. This way the applicability of method (3.1.1) has been expanded. Moreover, computable radius of convergence and error radius of convergence and error bounds on the distances involved (see (3.1.17)) using only Lipschitz constants has been showed.

Example 3.1.1 *As a motivational example, define function* F *on* $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = [-\frac{1}{\pi}, \frac{2}{\pi}]$ *by*

$$F(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, we have that

$$F'(x) = 2x^2 - x^3 \cos\left(\frac{1}{x}\right) + 3x^2 \log(\pi^2 x^2) + 5x^4 \sin\left(\frac{1}{x}\right),$$

$$F''(x) = -8x^2 \cos\left(\frac{1}{x}\right) + 2x(5 + 3\log(\pi^2 x^2)) + x(20x^2 - 1)\sin\left(\frac{1}{x}\right)$$

and

$$F'''(x) = \frac{1}{x}\left[(1 - 36x^2)\cos\left(\frac{1}{x}\right) + x\left(22 + 6\log(\pi^2 x^2) + (60x^2 - 9)\sin\left(\frac{1}{x}\right)\right)\right].$$

One can easily find that the function $F'''(x)$ *is unbounded on* $\mathbb{D}$ *at the point* $x = 0$*. Therefore, the convergence theorem proved in pervious chapter cannot apply to show the convergence of method* (3.1.1)*. In particular, hypotheses on the third derivative of function* F *or even higher are assumed to prove convergence of method* (3.1.1)*. But, according to this section, one just need the hypotheses on first order. Moreover, one can have*

$$K = K_0 = \frac{80 + 16\pi + (\pi + 12\log 2)\pi^2}{2\pi + 1}, \quad c_1 = \frac{\pi^3}{2\pi + 1},$$

$$c = \frac{8}{\pi(2\pi + 1)(10 + \pi + (1 + 3\log 2)\pi^2)}, \quad c_0 = \frac{8}{[10 + \pi + (1 + 3\log 2)\pi^2]\pi^4},$$

and the required zero is $X^* = \frac{1}{\pi}$. *The obtained different radius of convergence, COC* (ρ_c) *and CPU time are shown in the following Table 3.1.*

Table 3.1: Different radius of convergence for example 3.1.1

Schemes	Different values of parameters which satisfy Theorem 3.1.1					
	r_1	r_n	r^*	x^0	ρ_c	CPU Time (sec)
ψ_2 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = 10^{-1000}$	0.009951	1.8×10^{-40}	1.8×10^{-40}	0.3183	4.000	0.108
ψ_2 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.009951	0.009793	0.009793	0.3181	4.000	0.088
ψ_3 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = 10^{-1000}$	0.009951	1.2×10^{-78}	1.2×10^{-78}	0.3183	6.000	1.033
ψ_3 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.009951	0.009633	0.009633	0.3183	6.000	1.088
ψ_4 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = 10^{-1000}$	0.009951	7.2×10^{-117}	7.2×10^{-117}	0.3183	8.000	1.003
ψ_4 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.009951	0.009471	0.009471	0.3183	8.000	1.039

Example 3.1.2 *Define function* F *on* $D = U(0,1)$ *for* $v = (x, y, z)^T$ *by* $F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T$. *With the values of parameters*

$$K = K_0 = e - 1,\ K_1 = e$$

$$c = 2,\ c_1 = 1,\ c_0 = 2$$

and the required zero is $X^* = (0,0,0)^T$. *The different radius of convergence, COC* (ρ_c) *and CPU time with initial value* $X^0 = (0.1, 0.1, 0.1)^T$ *are shown in the following Table 3.2.*

Table 3.2: Different radius of convergence for example 3.1.2

Schemes	Different values of parameters which satisfy Theorem 3.1.1				
	r_1	r_n	r^*	ρ_c	CPU Time (sec)
ψ_2 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = 10^{-1000}$	0.041839	1.8×10^{-42}	1.8×10^{-42}	3.999	0.022
ψ_2 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.041839	0.005347	0.005347	3.999	0.029
ψ_3 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = 10^{-1000}$	0.041839	4.5×10^{-83}	4.5×10^{-83}	6.000	0.020
ψ_3 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.041839	0.000433	0.000433	6.000	0.021
ψ_4 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = 10^{-1000}$	0.041839	1.1×10^{-123}	1.1×10^{-123}	8.000	0.021
ψ_4 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.041839	0.000033	0.000033	8.000	0.023

3.2 Applications in differential equations

Example 3.2.1 Frank-Kamenetskii Problem:

Consider the Frank-Kamenetskii Problem [68] describes the following differential equation

$$y'' + \frac{1}{x}y' + e^{y} = 0,\ y'(0) = y(1) = 0. \tag{3.2.1}$$

To convert the above boundary value problem (3.2.1) *into non-linear system of size* 150×150 *with step size* $h_1 = 1/151$, *the finite difference discretization is used. For second derivative central difference has been used which is as follows:*

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h_1^2},\ i = 1,\ 2,\ \ldots,\ 150,$$

The computational comparison of solution of this problem with initial value $(0.05, 0.05, \ldots, 0.05)^T$ *is shown in Table 3.3 and graphical solution in Figure 3.1.*

Table 3.3: Performance of various iterative schemes for Frank-Kamenetskii problem.

Schemes	$SA1_4$	$SA2_4$	$GM1_4$	$OM1_4$	$OM2_4$	$GM2_6$	$OM3_6$	$OM4_6$	$WZ1_7$	$WZ2_7$	$OM5_8$	$OM6_8$	$OM7_8$
CPU Time (sec)	324.63	410.53	531.28	15.29	15.92	15.90	15.34	16.15	29.50	29.56	15.64	15.37	16.20

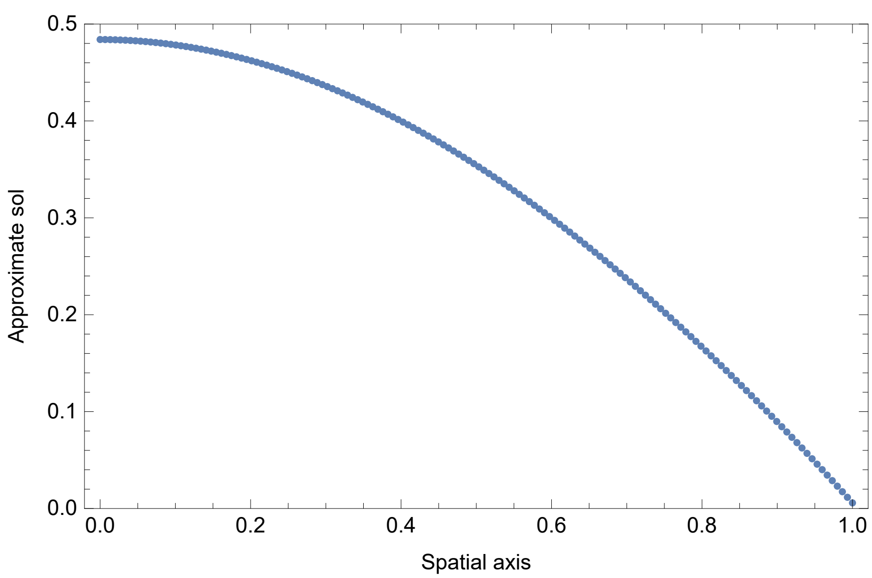

Figure 3.1. Approximated solution of Frank-Kamenetskii problem.

Example 3.2.2 Bratu Problem:

Consider the Bratu Problem [69] that has large variety of application areas such as the fuel ignition model of thermal combustion, radioactive heat transfer, thermal reaction, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology [70–73]. The problem is defined as:

$$y'' + C_1 e^y = 0,\ y(0) = y(1) = 0. \tag{3.2.2}$$

The finite difference discretization is used convert the above boundary value problem (3.2.2) *into non-linear system of size* 40×40 *with step size* $h_1 = 1/41$*. For second derivative central difference has been used which is as follows:*

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h_1^2},\ i = 1,\ 2,\ \ldots,\ 40.$$

The computational comparison of solution of this problem with initial value $\Big(sin(\pi h_1), sin(2\pi h_1), \ldots, sin(40\pi h_1)\Big)^T$ *is shown in Table 3.4 and graphical solution in Figure 3.2.*

Table 3.4: Performance of various iterative schemes for Bratu problem.

Schemes	$SA1_4$	$SA2_4$	$GM1_4$	$OM1_4$	$OM2_4$	$GM2_6$	$OM3_6$	$OM4_6$	$WZ1_7$	$WZ2_7$	$OM5_8$	$OM6_8$	$OM7_8$
CPU Time (sec)	17.59	21.36	15.70	15.29	15.92	15.90	15.34	16.15	29.50	29.56	15.64	15.37	16.20

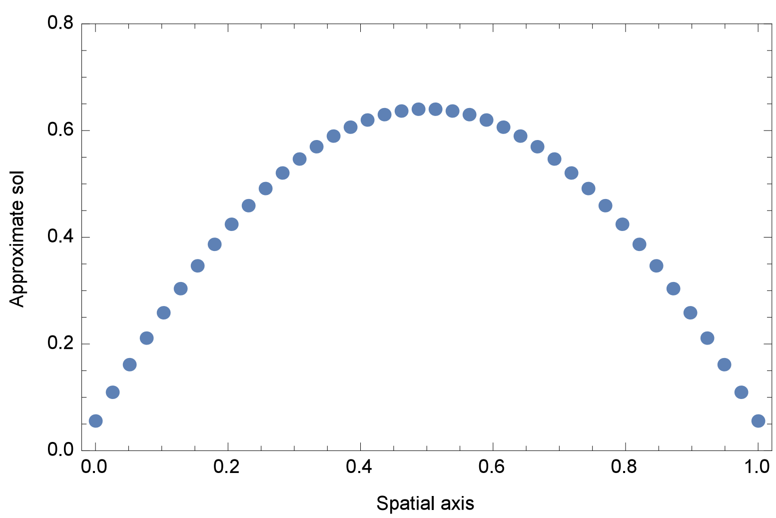

Figure 3.2. Approximated solution of Bratu problem with $C_1 = 3$ for $t \in [0, 1]$.

Example 3.2.3 Bratu Problem in 2D:

The Bratu problem in two dimensional [74, 75] is defined as:

$$\begin{aligned} &u_{xx} + u_{tt} + C_1 e^u = 0, \text{ on} \\ &\Omega : (x,t) \in 0 \le x \le 1, 0 \le t \le 1, \\ &\textit{with boundary conditions } u = 0 \textit{ on } \Omega. \end{aligned} \tag{3.2.3}$$

The solution of a nonlinear partial differential equation can be found using finite difference discreitization which reduces to solving a system of nonlinear equations. Let $w_{i,j} = u(x_i, t_j)$ *be its approximate solution at the grid points of the mesh. Let M and N be the number of steps in x and t directions, and* h_1 *and* h_2 *be the respective step size. To solve the given PDE, apply central difference to* u_{xx} *and* u_{tt} *i.e* $u_{xx}(x_i, t_j) = (w_{i+1,j} - 2w_{i,j} + w_{i-1,j})/h_1^2$. *We have consider the solution of the system for M = 11 and N = 11 of size 100, with the initial vector* $0.1\Big(sin(\pi h_1)sin(\pi h_2), sin(2\pi h_1)sin(2\pi h_2), \ldots, sin(10\pi h_1)sin(10\pi h_2)\Big)^T$ *has been evaluated using different methods and depicted in Table 3.5. Also the approximate solution found has been plotted in Figure 3.3.*

Table 3.5: Performance of various iterative schemes for Bratu's problem in 2D.

Scheme	$SA1_4$	$SA2_4$	$GM1_4$	$OM1_4$	$OM2_4$	$GM2_6$	$OM3_6$	$OM4_6$	$WZ1_7$	$WZ2_7$	$OM5_8$	$OM6_8$	$OM7_8$
CPU Time (sec)	241.63	303.07	224.73	218.45	238.30	217.65	210.22	220.96	1343.25	1310.85	395.73	431.84	441.66

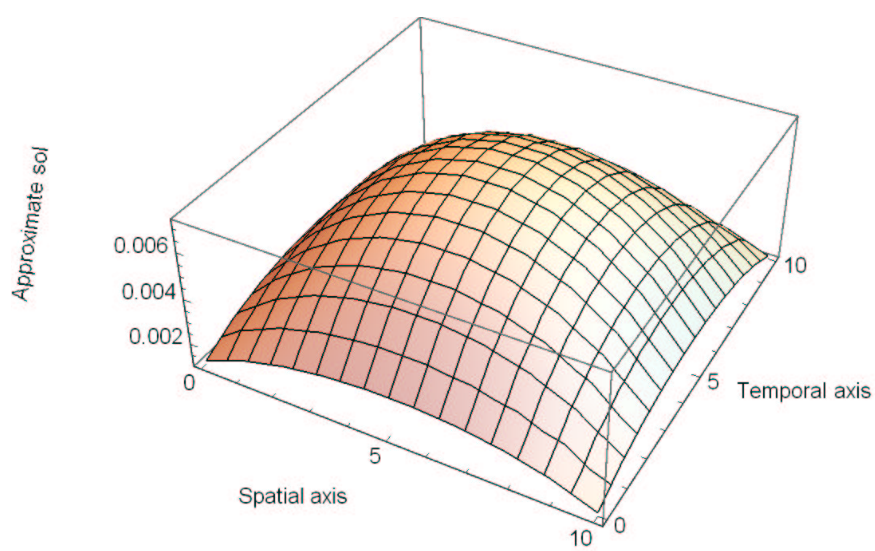

Figure 3.3. Approximated solution for Bratu's problem in 2D with $C_1 = 0.1$ for $t \in [0, 1]$.

Example 3.2.4 Fisher's Equation: *Consider another typical non linear problem that is Fisher's equation [1] with homogeneous Neumann's boundary conditions and the diffusion coefficient D is*

$$\begin{aligned} &u_t = Du_{xx} + u(1-u) = 0, \\ &u(x,0) = 1.5 + 0.5cos(\pi x), 0 \leq x \leq 1, \\ &u_x(0,t) = 0, \forall t \geq 0, \\ &u_x(1,t) = 0, \forall t \geq 0. \end{aligned} \tag{3.2.4}$$

Again using finite difference discreitization the equation (3.2.4) *reduces to a system of nonlinear equations. Consider* $w_{i,j} = u(x_i, t_j)$ *be its approximate solution at the grid points of the mesh. Let* M *and* N *be the number of steps in* x *and* t *directions, and* h_1 *and* h_2 *be the respective step size. Apply central difference to* $u_{xx}(x_i, t_j) = (w_{i+1,j} - 2w_{i,j} + w_{i-1,j})/h_1^2$ *, backward difference for* $u_t(x_i, t_j) = (w_{i,j} - w_{i,j-1})/h_2$ *and forward difference for* $u_x(x_i, t_j) = (w_{i+1,j} - w_{i,j})/(h_1)$*. For the solution of the system we have consider M = 21 and N = 21 which reduces to nonlinear system of size 400, with the initial vector* $i/(M-1)^2,\ i = 1, 2, \ldots, M-1$ *has been evaluated using different methods and shown in Table 3.6. The approximate solution has been plotted in Figure 3.4.*

Table 3.6: Performance of various iterative schemes for Fisher's Equation.

Scheme	$SA1_4$	$SA2_4$	$GM1_4$	$OM1_4$	$OM2_4$	$GM2_6$	$OM3_6$	$OM4_6$	$WZ1_7$	$WZ2_7$	$OM5_8$	$OM6_8$	$OM7_8$
CPU Time (sec)	*div.*	*div.*	1127.58	1409.30	1405.62	1376.54	1383.58	1189.68	*div.*	*div.*	1093.62	1356.34	1359.12

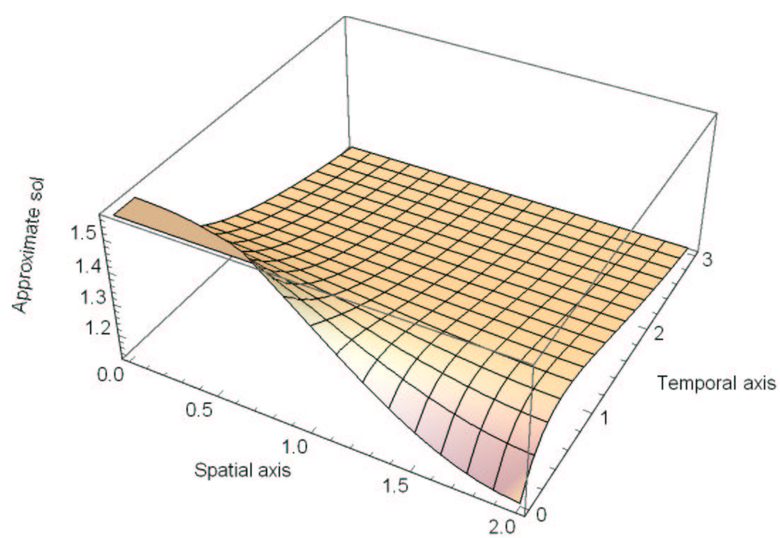

Figure 3.4. Approximated solution for Fisher's Equation $t \in [0, 1]$.

3.3 Conclusions

In this work, local convergence analysis of family (3.1.1) proposed in chapter two has been proved based on divided differences of order one and Lipschitz constants on a Banach space setting under weak conditions to expand the applicability of scheme (3.1.1). The use of this family on real life problems namely Frank's problem, Bratu's in one and two dimensional case and Fisher's problems also confirms the applicability of this family and shows less CPU time as compared to existing counter parts.

Chapter 4

Higher-order modification of Steffensen's method for solving systems of non-linear equations

This chapter describes the class of higher-order derivative free family by considering the modifications on Steffensen's type method for solving non-linear systems numerically. The order of convergence of the constructed family is at least ten which has been determined by Taylor's series expansion as well as computationally. Computational efficiencies of the developed scheme are considered and compared with their closest competitors. Additionally, numerical trials are performed on a few expensive and complex non-linear systems. The outcomes are observed to be powerful and similar to other existing strategies which in like manner support the theoretical results.

4.1 Introduction

In iterative schemes for non-linear systems, the idea of using first-order divided difference in place of Fréchet derivative is significant as well as reasonable, as it reduces the computational cost per iteration. Keeping this fact in mind, a higher order derivative free iterative family for the solution of the non-linear systems has been developed. It is a well-known fact that there are methods without derivatives in the literature, unlike higher order methods with derivatives for solving systems of non-linear equations. In fact, designing efficient, higher- order derivative free methods for non-linear systems is a difficult task and only a few methods have been developed. For instance, Wang and Zhang [8] have constructed the two multi-point seventh-order families of Steffensen's-type methods. These families are based on Ren *et al.* [61] and Liu *et al.* [60] fourth-order methods, which are given as follows:

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k - \big\{[Y^k, X^k; F] + [Y^k, W^k; F] - [W^k, X^k; F]\big\}^{-1} F(Y^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + [Z^k, X^k; F] - [Y^k, X^k; F]\big\}^{-1} F(Z^k), \end{cases} \tag{4.1.1}$$

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and

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1}F(X^k), \\ Z^k = Y^k - \Big(\{[Y^k, X^k; F]\}^{-1}\Big([Y^k, X^k; F] - [Y^k, W^k; F] + [W^k, X^k; F]\Big)\Big) \\ \qquad \times \{[Y^k, X^k; F]\}^{-1}F(Y^k), \\ X^{k+1} = Z^k - \{[Z^k, Y^k; F] + [Z^k, X^k; F] - [Y^k, X^k; F]\}^{-1}F(Z^k), \end{cases} \tag{4.1.2}$$

respectively. In addition, Sharma and Arora [62] have presented a multi-point seventh-order scheme, which is given as follows:

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1}F(X^k), \\ Z^k = Y^k - \Big(3I - \{[W^k, X^k; F]\}^{-1}\Big([Y^k, X^k; F] + [Y^k, W^k; F]\Big)\Big)\{[W^k, X^k; F]\}^{-1}F(Y^k), \\ X^{k+1} = Z^k - \{[Z^k, Y^k; F]\}^{-1}\Big([W^k, X^k; F] + [Y^k, X^k; F] - [Z^k, X^k; F]\Big) \\ \qquad \times \{[W^k, X^k; F]\}^{-1}F(Z^k). \end{cases} \tag{4.1.3}$$

It is worth mentioning that above mentioned methods use five divided difference operators namely, $[W^k, X^k; F], [Y^k, X^k; F], [Y^k, W^k; F], [Z^k, Y^k; F], [Z^k, X^k; F]$, three inverse matrices and three function evaluations having seventh-order convergence. Going a step further, Wang *et al.* [43] developed a seventh-order derivative free iterative scheme using three divided difference operators, function evaluations and only one inverse matrix in the following way:

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1}F(X^k), \\ Z^k = Y^k - \Big(\Big(3I - 2\{[W^k, X^k; F]\}^{-1}[Y^k, X^k; F]\Big)\Big)\{[W^k, X^k; F]\}^{-1}F(Y^k), \\ X^{k+1} = Z^k - \Big[\dfrac{13}{4} - \{[W^k, X^k; F]\}^{-1}[Z^k, Y^k; F]\Big(\dfrac{7}{2} - \dfrac{5}{2}\{[W^k, X^k; F]\}^{-1}[Z^k, Y^k; F]\Big)\Big] \\ \qquad \times \{[W^k, X^k; F]\}^{-1}F(Z^k). \end{cases} \tag{4.1.4}$$

To this end, the main goal of the author is to develop a higher-order derivative free family with higher computational efficiency based first-order divided difference operator. The developed four-step higher-order family is the modification of Coredero *et al.* scheme for scalar case with minimum inverse matrices. It is found by way of illustrations that the proposed family is highly efficient in multi-precision computing environment.

4.2 Construction of an iterative family

This section presents new higher-order modification of Steffensen's type method. The derivation of the proposed scheme is based on the first two-step of Chebyshev-Halley type methods with optimal eighth order convergence scheme for solving scalar equations which was proposed by Coredero *et al.* [76], defined as follows:

$$\begin{cases} y^k = x^k - \dfrac{f(x^k)}{f'(x^k)}, \\ z^k = x^k - \dfrac{f(x^k)\left[2(f(x^k))^2 + (4\alpha-4)(1-2\alpha)(f(y^k))^2 + 2(2\alpha-3)f(y^k)f(x^k)\right]}{f'(x^k)\left[2(f(x^k))^2 + 4(\alpha-2)f(x^k)f(y^k) - 8\alpha(\alpha-1)f(y^k)^2\right]}, \\ x^{k+1} = z^k - \dfrac{f(z^k)}{f[z^k,y^k] + (z^k-y^k)f[z^k,y^k,x^k] + (z^k-y^k)(z^k-x^k)f[z^k,y^k,x^k,x^k]}, \end{cases} \tag{4.2.1}$$

where $\alpha \in \mathbb{R}$.
Next, consider the following modification over iterative scheme (4.2.1):

$$\begin{cases} y^k = x^k - \dfrac{f(x^k)}{f'(x^k)}, \\ z^k = x^k - \dfrac{f(x^k)\left[2(f(x^k))^2 + (4\alpha-4)(1-2\alpha)(f(y^k))^2 + 2(2\alpha-3)f(y^k)f(x^k)\right]}{f'(x^k)\left[2(f(x^k))^2 + 4(\alpha-2)f(x^k)f(y^k) - 8\alpha(\alpha-1)(f(y^k))^2\right]}, \\ u^k = z^k - \dfrac{f(z^k)}{f'(z^k)}, \\ x^{k+1} = z^k - \dfrac{f(z^k)+f(u^k)}{f'(z^k)}. \end{cases} \tag{4.2.2}$$

The first step of (4.2.2) can be read as:

$$\begin{cases} y^k = x^k - \dfrac{f(x^k)}{f'(x^k)} \\ \text{or} \\ y^k - x^k = -\dfrac{f(x^k)}{f'(x^k)}. \end{cases} \tag{4.2.3}$$

The second step of (4.2.2) can be written as:

$$z^k = x^k - \frac{f(x^k)}{f'(x^k)} - \frac{f(x^k)}{f'(x^k)}\left[-1 + \frac{2(f(x^k))^2 + (4\alpha-4)(1-2\alpha)(f(y^k))^2 + 2(2\alpha-3)f(y^k)f(x^k)}{2(f(x^k))^2 + 4(\alpha-2)f(x^k)f(y^k) - 8\alpha(\alpha-1)(f(y^k))^2}\right]. \tag{4.2.4}$$

Using the expression (4.2.3) and performing some simplifications, the above expression may be shown as:

$$z^k = y^k - \frac{\eta(x^k, y^k)}{\tau(x^k, y^k)} f(y^k), \tag{4.2.5}$$

where

$$\begin{cases} \eta(x^k, y^k) = \left[(4\alpha - 4)f(y^k) + 2f(x^k)\right] f(y^k), \\ \tau(x^k, y^k) = \dfrac{1}{y^k - x^k}\Big[2(f(x^k))^2 + 4(\alpha - 2)f(x^k)f(y^k) - 8\alpha(\alpha - 1)(f(y^k))^2\Big]. \end{cases} \tag{4.2.6}$$

Note that for simplicity $\eta(x^k, y^k)$ and $\tau(x^k, y^k)$ are consider as η and τ respectively. After some simplifications, (4.2.6) can be rewritten as:

$$\begin{aligned} \eta =& \left[(4\alpha - 4)\left(f(y^k) - f(x^k)\right) + (4\alpha - 2)f(x^k)\right] f(y^k), \\ \tau =& \frac{1}{y^k - x^k}\Big[(-8\alpha^2 + 12\alpha - 6)(f(x^k))^2 + (-16\alpha^2 + 20\alpha - 8)f(x^k)\left(f(y^k) - f(x^k)\right) \\ & - 8\alpha(\alpha - 1)\left(f(y^k) - f(x^k)\right)^2\Big]. \end{aligned} \tag{4.2.7}$$

Dividing the η and τ of the above expression (4.2.7) by $y_k - x_k$ and with the help of (4.2.3), one obtains

$$\begin{aligned} \eta =& \left[(4\alpha - 4)\left(\frac{f(y^k) - f(x^k)}{y^k - x^k}\right) - (4\alpha - 2)f'(x^k)\right] f(y^k), \\ \tau =& \Big[-(-8\alpha^2 + 12\alpha - 6)\left(\frac{f(x^k)}{y^k - x^k}\right)^2 - (-16\alpha^2 + 20\alpha - 8)\frac{f(x^k)}{y^k - x^k}\left(\frac{f(y^k) - f(x^k)}{y^k - x^k}\right) \\ & + 8\alpha(\alpha - 1)\left(\frac{f(y^k) - f(x^k)}{y^k - x^k}\right)^2\Big]. \end{aligned} \tag{4.2.8}$$

Which further implies,

$$\begin{aligned} \eta =& \left((4\alpha - 4)[y^k, x^k; f] - (4\alpha - 2)f'(x^k)\right) f(y^k), \\ \tau =& (8\alpha^2 - 12\alpha + 6)\left(f'(x^k)\right)^2 + (-16\alpha^2 + 20\alpha - 8)f'(x^k)[y^k, x^k; f] + 8\alpha(\alpha - 1)[y^k, x^k; f]^2. \end{aligned} \tag{4.2.9}$$

Further, we approximate the first derivative $f'(z^k)$ involved in the third and fourth steps of iterative scheme (4.2.2) by Newton interpolating polynomial of degree two with available approximations z^k, y^k, x^k and is given by
$N_3(p) = f(z^k) + [z^k, y^k; f](v - z^k) + (v - z^k)(v - y^k)[z^k, y^k, x^k; f]$.
Therefore

$$f'(z^k) \approx N_3'(v) = [z^k, y^k; f] + (z^k - y^k)[z^k, y^k, x^k; f]. \tag{4.2.10}$$

Consequently, the modified scheme (4.2.2), is presented as follows:

$$\begin{cases} y^k =x^k - \dfrac{f(x^k)}{f'(x^k)}, \\ z^k =y^k - \left[\dfrac{\eta}{\tau}\right] f(y^k), \\ u^k =z^k - \dfrac{f(z^k)}{[z^k, y^k; f] + (z^k - y^k)[z^k, y^k, x^k; f]}, \\ x^{k+1} =z^k - \dfrac{f(z^k)+f(u^k)}{[z^k, y^k; f] + (z^k - y^k)[z^k, y^k, x^k; f]}. \end{cases} \tag{4.2.11}$$

By replacing $f'(x^k)$ with first order central divided difference operator in equations (4.2.11) and (4.2.9), one get final modified iterative scheme given by

$$\begin{cases} y^k = x^k - \dfrac{f(x^k)}{[x^k + f, x^k - f; f]}, \\ z^k = y^k - \left[\dfrac{\eta}{\tau}\right] f(y^k), \\ u^k = z^k - \dfrac{f(z^k)}{[z^k, y^k; f] + (z^k - y^k)[z^k, y^k, x^k; f]}, \\ x^{k+1} = z^k - \dfrac{f(z^k)+f(u^k)}{[z^k, y^k; f] + (z^k - y^k)[z^k, y^k, x^k; f]}. \end{cases} \tag{4.2.12}$$

This is a new iterative scheme for solving scalar non-linear equations which can be easily extended to system of non-linear equations as follows.
The following derivative free iterative scheme has been proposed:

$$\begin{cases} Y^k =X^k - [W^k, X^k; F]^{-1} F(X^k), \\ Z^k =Y^k - \big\{4(\alpha-1)[Y^k, X^k; F] - 2(2\alpha-1)[W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \\ U^k =Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} =Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} \big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.13}$$

where $\tau = 2(4\alpha^2 - 6\alpha + 3)[W^k, X^k; F]^2 + 4(-4\alpha^2 + 5\alpha - 2)[Y^k, X^k; F][W^k, X^k; F] + 8\alpha(\alpha - 1)[Y^k, X^k; F]^2$ and $[W^k, X^k; F] = [X^k + F, X^k - F; F]$. With the help of free disposable parameter α, the various iterative methods can be deduced from the proposed scheme (4.2.13).

4.2.1 Special cases

1. For $\alpha = 1$, the proposed family (4.2.13) reduces to

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k + 2[W^k, X^k; F]\tau^{-1} F(Y^k), \\ U^k = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} \big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.14}$$

where $\tau = 2[W^k, X^k; F]^2 - 4[Y^k, X^k; F][W^k, X^k; F]$.

2. For $\alpha = -\frac{1}{2}$, the proposed family (4.2.13) deduces to

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k - \big\{ - 6[Y^k, X^k; F] + 4[W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \\ U^k = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} \big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.15}$$

where $\tau = 14[W^k, X^k; F]^2 - 22[Y^k, X^k; F][W^k, X^k; F] + 6[Y^k, X^k; F]^2$.

3. By putting $\alpha = -\frac{1}{4}$, in the proposed family (4.2.13) one gets

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k - \big\{ - 5[Y^k, X^k; F] + 6[W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \\ U^k = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} \big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.16}$$

where $\tau = 11[W^k, X^k; F]^2 - 14[Y^k, X^k; F][W^k, X^k; F] + \frac{5}{2}[Y^k, X^k; F]^2$.

4. By substituting $\alpha = \frac{3}{4}$, in the proposed family (4.2.13) one can have

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k - \big\{ - [Y^k, X^k; F] - [W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \\ U^k = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} \big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.17}$$

where $\tau = \frac{3}{2}[W^k, X^k; F]^2 + 6[Y^k, X^k; F][W^k, X^k; F] - \frac{3}{2}[Y^k, X^k; F]^2$.

5. For $\alpha = -1000$, the proposed family (4.2.13) represents as:

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k - \big\{ -4004[Y^k, X^k; F] + 4002[W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \\ U^k = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1}\big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.18}$$

where $\tau = 8012006[W^k, X^k; F]^2 + 15980008[Y^k, X^k; F][W^k, X^k; F] + 8008000[Y^k, X^k; F]^2$.

6. For $\alpha = \frac{1}{\sqrt{6}}$ the proposed family (4.2.13) considered as:

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k + \big\{2.367006838[Y^k, X^k; F] + 0.3670068382[W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \\ U^k = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1}\big(F(Z^k) + F(U^k)\big), \end{cases} \tag{4.2.19}$$

where $\tau = (2.4343538478)[W^k, X^k; F]^2 - (2.5017008572)[Y^k, X^k; F][W^k, X^k; F]$
$-(1.9326529904)[Y^k, X^k; F]^2$.

4.2.2 Convergence analysis

For the analysis of convergence, one consider the first-order divided difference operator of F on $\mathbb{R}^n$ as a mapping $[\cdot, \cdot : F] : D \times D \subseteq \mathbb{R}^n \times \mathbb{R}^n \to L(\mathbb{R}^n)$, which is defined by [10, 33]

$$\big[X^k + h, X^k; F\big] = \int_0^1 F'(X^k + uh)du, \ \forall (X^k, h) \ \in \ \mathbb{R}^n \times \mathbb{R}^n. \tag{4.2.20}$$

Developing $F'(X^k + uh)$ in Taylor's series expansion at X^k and after integrating, one can obtain

$$\int_0^1 F'(X^k + uh)du = F'(X^k) + \frac{1}{2}F''(X^k)h + \frac{1}{6}F'''(X^k)h^2 + O(h^3). \tag{4.2.21}$$

Taking into account

$$e^k = X^k - X^*, \tag{4.2.22}$$

one can develop $F(X^k)$ and its derivatives in a neighborhood of X^*. Assuming that $\big\{F'(X^*)\big\}^{-1}$ exists, one can have

$$F(X^k) = F'(X^*)\Big[e^k + A_2\big(e^k\big)^2 + A_3\big(e^k\big)^3 + A_4\big(e^k\big)^4 + A_5\big(e^k\big)^5 + O\left((e^k)^6\right)\Big], \tag{4.2.23}$$

where $A_j = \frac{1}{j!}\{F'(X^*)\}^{-1}F^{(j)}(X^*) \in L_j(\mathbb{R}^n, \mathbb{R}^n),\ j = 2, 3, \ldots$.
From equation (4.2.23), the derivative of $F(X^k)$ can be written as

$$F'(X^k) = \Gamma\Big[I + 2A_2e^k + 3A_3\big(e^k\big)^2 + 4A_4\big(e^k\big)^3 + 5A_5\big(e^k\big)^4 + O\left((e^k)^5\right)\Big], \tag{4.2.24}$$

$$F''(X^k) = \Gamma\Big[2A_2 + 6A_3e^k + 12A_4\big(e^k\big)^2 + 20A_5\big(e^k\big)^3 + O\left((e^k)^4\right)\Big], \tag{4.2.25}$$

$$\text{and} \quad F'''(X^k) = \Gamma\Big[6A_3 + 24A_4e^k + O\left((e^k)^2\right)\Big], \tag{4.2.26}$$

where I is an identity matrix of order n and $\Gamma = F'(X^*)$.
Setting $Y^k = X^k + h$ and

$$e_1^k = Y^k - X^*, \tag{4.2.27}$$

one obtain $h = Y^k - X^k = e_1^k - e^k$.
By substituting equations (4.2.24)-(4.2.26) into equation (4.2.21), one gets
$[Y^k, X^k; F] = [X^k + h, X^k; F]$

$$= \Gamma\Big[I + A_2\big(e_1^k + e^k\big) + A_3\big((e_1^k)^2 + (e^k)^2 + e_1^ke^k\big) + O\left((e^k)^3\right)\Big]. \tag{4.2.28}$$

In this analysis, the center difference operator has been considered as follows:

$$[W^k, X^k; F] = [X^k+F, X^k-F; F] = \Gamma\Big[I+2A_2e^k+A_3\big(3+\Gamma^2\big)(e^k)^2+O\left((e^k)^3\right)\Big], \tag{4.2.29}$$

which has been obtained after replacing e_1^k by $e^k + F(X)$ and e^k by $e^k - F(X)$ in equation (4.2.28).
The convergence of developed iterative scheme (4.2.13) can be proved through the following theorem:

Theorem 4.2.1 *Let $X^* \in \mathbb{R}^n$ be a solution of the system $F(X) = 0$ and $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood D of X^* at which $\Gamma(= F'(X^*))$ is nonsingular. Then for an initial approximation sufficiently close to X^*, the iterative family* (4.2.13) *will have local order of convergence at least ten.*

Proof The first step of scheme (4.2.13) can be rewritten as:

$$Y^k - X^* = X^k - X^* - \{[W^k, X^k; F]\}^{-1}F(X^k). \tag{4.2.30}$$

In view of (4.2.22) and (4.2.27), the equation (4.2.30) becomes

$$e_1^k = e^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \tag{4.2.31}$$

by using (4.2.29), one can obtain

$$\begin{aligned}\{[W^k, X^k; F]\}^{-1} = \Big[&1 - 2A_2 e^k - ((3+\Gamma^2)A_3 + 4A_2^2)(e^k)^2 + 2(-4A_2^3 + (6+\Gamma^2)A_2A_3 \\ &- 2(1+\Gamma^2)A_4)(e^k)^3\Big]\Gamma^{-1} + O\left((e^k)^4\right).\end{aligned} \tag{4.2.32}$$

Substituting equations (4.2.23) and (4.2.32) in (4.2.31), and after simplifications, one gets

$$\begin{aligned}e_1^k =& A_2(e^k)^2 + \big[-2A_2^2 + (2+\Gamma^2)A_3\big](e^k)^3 + \\ &[4A_2^3 - (7+\Gamma^2)A_2A_3 + (3+4\Gamma^2)A_4](e^k)^4 + O\left((e^k)^5\right).\end{aligned} \tag{4.2.33}$$

The second step of scheme (4.2.13) can be rewritten as:

$$Z^k - X^* = Y^k - X^* - \big\{(4\alpha-4)[Y^k, X^k; F] - (4\alpha-2)[W^k, X^k; F]\big\}\tau^{-1} F(Y^k), \tag{4.2.34}$$

where

$$F(Y^k) = \Gamma\Big[e_1^k + A_2\big(e_1^k\big)^2 + A_3\big(e_1^k\big)^3 + O\left((e_1^k)^4\right)\Big], \tag{4.2.35}$$

$$\tau^{-1} = \Gamma^{-2}\Big[\frac{-1}{2} + \alpha A_2 e^k + \big(-A_3(1+\Gamma^2) + (3\alpha - 4\alpha^2)A_2^2 + (2\alpha + \alpha\Gamma^2)A_3\big)(e^k)^2\Big]. \tag{4.2.36}$$

Setting $e_2^k = Z^k - X^*$ and using (4.2.27), the equation (4.2.34) reads as

$$e_2^k = e_1^k - \big\{(4\alpha-4)[Y^k, X^k; F] - (4\alpha-2)[W^k, X^k; F]\big\}\tau^{-1} F(Y^k). \tag{4.2.37}$$

Simplifying equation (4.2.37) using equations (4.2.28),(4.2.29),(4.2.33), (4.2.35) and (4.2.36), one can get

$$e_2^k = Z^k - X^* = -A_2\big[(1+\Gamma^2)A_3 + A_2^2(3 + 4(-2+\alpha)\alpha)\big](e^k)^4 + O((e^k)^5). \tag{4.2.38}$$

The third step of scheme (4.2.13) can be rewritten as:

$$U^k - X^* = Z^k - X^* - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k). \tag{4.2.39}$$

Setting $e_3^k = U^k - X^*$,one can have

$$e_3^k = e_2^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1} F(Z^k), \tag{4.2.40}$$

where

$$
\begin{aligned}
Z^k - Y^k =& - A_2(e^k)^2 + (2A_2^2 - (2+\Gamma^2)A_3)(e^k)^3 + (6A_2A_3 - (3+4\Gamma^2)A_4 \\
&+ A_2^3(-7-4(-2+\alpha)\alpha))(e^k)^4 + O((e^k)^5),
\end{aligned} \tag{4.2.41}
$$

$$
F(Z^k) = \Gamma\Big[e_2^k + A_2\big(e_2^k\big)^2 + O\left((e_2^k)^3\right)\Big], \tag{4.2.42}
$$

$$
[Z^k, Y^k; F] = F'(Y^k) + \frac{F''(Y^k)}{2!}(Z^k - Y^k) + \frac{F'''(Y^k)}{3!}(Z^k - Y^k)^2 + O\left((Z^k - Y^k)^3\right), \tag{4.2.43}
$$

and

$$
[Z^k, Y^k, X^k; F] = \frac{[Z^k, Y^k; F] - [Y^k, X^k; F]}{Z^k - X^k}. \tag{4.2.44}
$$

Further, in view of (4.2.23)-(4.2.26), one can manipulate

$$
F'(Y^k) = \Gamma\Big[I + 2A_2e_1^k + 3A_3\big(e_1^k\big)^2 + 4A_4\big(e_1^k\big)^3 + 5A_5\big(e_1^k\big)^4 + O\left((e_1^k)^5\right)\Big], \tag{4.2.45}
$$

$$
F''(Y^k) = \Gamma\Big[2A_2 + 6A_3e_1^k + 12A_4\big(e_1^k\big)^2 + 20A_5\big(e_1^k\big)^3 + O\left((e_1^k)^4\right)\Big], \tag{4.2.46}
$$

$$
\text{and} \quad F'''(Y^k) = \Gamma\Big[6A_3 + 24A_4e_1^k + O\left((e_1^k)^2\right)\Big]. \tag{4.2.47}
$$

By using equations (4.2.41),(4.2.45)-(4.2.47) in (4.2.43), one can have

$$
\begin{aligned}
[Z^k, Y^k; F] =& \Big[1 + A_2^2(e^k)^2 + A_2(-2A_2^2 + (2+\Gamma^2)A_3)(e^k)^3 + A_2(-(7+2\Gamma^2)A_2A_3 \\
&+ (3+4\Gamma^2)A_4 + A_2^3(1-4(-2+\alpha)\alpha))(e^k)^4\Big]\Gamma + O\left((e^k)^5\right).
\end{aligned} \tag{4.2.48}
$$

Now, from first two steps of scheme (4.2.13),

$$
\begin{aligned}
Z^k - X^k =& - e^k - A_2((1+\Gamma^2)A_3 + A_2^2(3+4(-2+\alpha)\alpha))(e^k)^4 + (-(2+3\Gamma^2+\Gamma^4)A_3^2 \\
&- 2(1+2\Gamma^2)A_2A_4 - 2A_2^2A_3(8+5\Gamma^2+6(2+\Gamma^2)(-2+\alpha)\alpha) \\
&+ 4A_2^4(3-6\alpha+2\alpha^3))(e^k)^5 + O\left((e^k)^5\right).
\end{aligned} \tag{4.2.49}
$$

Making making use of equations (4.2.28), (4.2.48) and (4.2.49) in equation (4.2.44), one gets

$$
\begin{aligned}
[Z^k, Y^k, X^k; F] =& \Gamma[A_2 + A_3e^k + (A_2A_3 + A_4)(e^k)^2 \\
&+ (-2A_2^2A_3 + (2+\Gamma^2)A_3^2 + A_2A_4 + A_5)(e^k)^3] + O\left((e^k)^4\right).
\end{aligned} \tag{4.2.50}
$$

After simplifying by using equations (4.2.38), (4.2.41), (4.2.42), (4.2.48) and (4.2.50), the equation(4.2.40) reads as

$$e_3^k = A_2^2 A_3\big[(1+\Gamma^2)A_3 + A_2^2(3+4(-2+\alpha)\alpha)\big](e^k)^7 + O((e^k)^8). \tag{4.2.51}$$

Finally, the last step of scheme (4.2.13) written as:

$$X^{k+1} - X^* = Z^k - X^* - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1}\big(F(Z^k) + F(U^k)\big), \tag{4.2.52}$$

where

$$F(U^k) = \Gamma\Big[e_3^k + O\left((e_3^k)^2\right)\Big]. \tag{4.2.53}$$

Again by setting $e_4^k = X^{k+1} - X^*$,

$$e_4^k = e_2^k - \big\{[Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]\big\}^{-1}\big(F(Z^k) + F(U^k)\big). \tag{4.2.54}$$

Substituting equations (4.2.42), (4.2.48), (4.2.50), (4.2.51) and (4.2.53) in (4.2.54) and after full simplification, one can get the final error equation as

$$e_4^k = -A_2^3 A_3^2\big[(1+\Gamma^2)A_3 + A_2^2(3+4(-2+\alpha)\alpha)\big](e^k)^{10} + O((e^k)^{11}). \tag{4.2.55}$$

□

4.3 Computational efficiency

Bearing the definition of computational efficiency (1.2.8) in mind, the efficiency of proposed methods has been estimated. In addition, $5n^2$ products for multiplication of a vector by a scalar are evaluated. For comparison of the computational efficiencies of the proposed scheme (4.2.13) PS_{10} with existing schemes of seventh order namely, Wang and Zhang (4.1.1) WA_7, Wang and Zhang (4.1.2) WB_7, Sharma and Arora (4.1.3) S_7 and Wang at el.(4.1.4) WD_7, the efficiency indices are denoted by $C_{PS_{10}}$, C_{WA_7}, C_{WB_7}, C_{S_7}, C_{WD_7} and computational cost (calculated according to (1.2.1)) are denoted by $CEI_{PS_{10}}$, CEI_{WA_7}, CEI_{WB_7}, CEI_{S_7}, and CEI_{WD_7} respectively.
Taking into account the above considerations, one can have

$$C_{S_7} = \frac{n}{3}(2n^2 + 30n\nu + 12n + 18ln - 18\nu + 12\ell - 11) \text{ and } CEI_{S_7} = 7^{1/C_{S_7}}. \tag{4.3.1}$$

$$C_{WA_7} = \frac{n}{2}(2n^2 + 10n\nu + 3n + 13ln - 2\nu + 3\ell - 5) \text{ and } CEI_{WA_7} = 7^{1/C_{WA_7}}. \tag{4.3.2}$$

$$C_{WB_7} = \frac{n}{2}(2n^2 + 10n\nu + 5n + 13ln - 2\nu + 5\ell - 7) \text{ and } CEI_{WB_7} = 7^{1/C_{WB_7}}. \tag{4.3.3}$$

$$C_{WD_7} = \frac{n}{6}(2n^2 + 18n\nu + 51n + 21ln + 12\nu + 33\ell - 5) \text{ and } CEI_{WD_7} = 7^{1/C_{WD_7}}. \tag{4.3.4}$$

$$C_{PS_{10}} = \frac{n}{2}(2n^2 + 6n\nu + 11n + 9ln + 6\nu + 5\ell + 4) \text{ and } CEI_{PS_{10}} = 10^{1/C_{PS_{10}}}. \tag{4.3.5}$$

4.3.1 Comparison between efficiencies

In order to compare the iterative schemes (4.1.1) WA_7, (4.1.2) WB_7, (4.1.4) WD_7, (4.1.3) S_7 and (4.2.13) PS_{10} the following ratio can be defined as:

$$R_{i,j} = \frac{log\ CEI_i}{log\ CEI_j} = \frac{log(\rho_i)C_j}{log(\rho_j)C_i}.$$

It is clear that if $R_{i,j} > 1$, the iterative method i is more efficient than method j. Taking into account that the border between two computational efficiencies is given by $R_{i,j} = 1$, this boundary is given by the equation of ν written as a function of ℓ and n, that is $\nu = M_{i,j}(\ell, n)$. Here $\nu > 0,\ \ell \geq 1$ and n is a positive integer $n \geq 2$.

Case 1 : **Constructed iterative family** (4.2.13) $\mathbf{PS_{10}}$ **verses iterative scheme** (4.1.1) $\mathbf{WA_7}$
The boundary $R_{PS_{10},WA_7} = 1$ expressed by ν written as a function of ℓ and n is

$$\nu = \frac{9.64828n + 7.24863n^2 - 0.356675n^3 + 0.972955\ell n(5. + 9.n) - 1.15129\ell n(3. + 13.n)}{-8.14032n + 5.6752n^2}. \tag{4.3.6}$$

This function has the vertical asymptote for $n = 1.43437$. Note that, the numerator of equation (4.3.6) is negative for $n > 7$ and the denominator of equation (4.3.6) is negative for $n \geq 2$. Consequently, It shows that ν is always positive for $2 \leq n \leq 7$ and for all $\ell \geq 1$. So, one can have $CEI_{PS_{10}} > CEI_{WA_7},\ \forall\ \nu > 0,\ \ell \geq 1\ \&\ 2 \leq n \leq 7$.

Case 2 : **Developed iterative method** (4.2.13) $\mathbf{PS_{10}}$ **verses iterative method** (4.1.2) $\mathbf{WB_7}$
The boundary $R_{PS_{10},WB_7} = 1$ expressed by ν written as a function of ℓ and n is

$$\nu = \frac{16.1181 - 7.62111n + 6.09734n^2 + 1.94591n^3 + 0.972955\ell n(5. + 9.n) - 2.30259\ell(5. + 13.n)}{-4.60517 + 17.1881n - 5.83773n^2}. \tag{4.3.7}$$

This function has the vertical asymptote for $n = 2.6462$. Note that, the numerator of equation (4.3.7) is positive for $n \geq 2$ and the denominator of equation (4.3.7) is negative

for $n \geq 3$. Consequently, It shows that ν is negative for $n \geq 3$ and for all $\ell \geq 1$.
So, one gets $CEI_{PS_{10}} < CEI_{WB_7}, \ \forall\ \nu > 0,\ \ell \geq 1\ \&\ \ n \geq 2$ but for $n = 2,\ CEI_{PS_{10}} > CEI_{WB_7}$.

Case 3 : **Presented iterative family** (4.2.13) $\mathbf{PS_{10}}$ **verses iterative method** (4.1.3) $\mathbf{S_7}$
The boundary $R_{PS_{10},S_7} = 1$ expressed by ν written as a function of ℓ and n is

$$\nu = \frac{12.3346n + 1.49217n^2 + 0.410853n^3 - 4.60517\ell n(2. + 3.n) + 0.972955\ell n(5. + 9.n)}{-19.6532n + 17.1881n^2}. \tag{4.3.8}$$

This function has vertical asymptote at $n = 1.14342$. Note that, the numerator of equation (4.3.8) is always positive for $n \geq 2$ and the denominator of equation (4.3.8) is positive for $n \geq 2$. Consequently, It shows that $CEI_{PS_{10}} > CEI_{S_7}, \ \forall\ \nu > 0,\ \ell \geq 1\ \&\ n \geq 2$.

Case 4 : **Proposed iterative family** (4.2.13) $\mathbf{PS_{10}}$ **verses iterative scheme** (4.1.4) $\mathbf{WD_7}$
The boundary $R_{PS_{10},WD_7} = 1$ expressed by ν written as a function of ℓ and n is

$$\nu = \frac{4.72515n + 2.20251n^2 + 1.61258n^3 - 0.5\ell n(11. + 7.n) + 0.972955\ell n(5. + 9.n)}{-1.23256n + 1.07002n^2}. \tag{4.3.9}$$

This function has the vertical asymptote for $n = 1.1519$. Note that, the numerator of equation (4.3.9) is positive for $n \geq 2$ and the denominator of equation (4.3.9) is positive for $n \geq 2$. Consequently, It shows that ν is always positive for $n \geq 2$ and for all $\ell \geq 1$.
So, one can get $CEI_{PS_{10}} > CEI_{WD_7}, \ \forall\ \nu > 0,\ \ell \geq 1\ \&\ \ n \geq 2$.

Theorem 4.3.1 *For all $\nu > 0$ and $\ell \geq 1$, one can have:*
(i) $CEI_{PS_{10}} > CEI_{WA_7}, \quad for \ \ 2 \leq n \leq 7.$
(ii) $CEI_{PS_{10}} > CEI_{WB_7}, \quad for \ \ n = 2.$
(iii) $CEI_{PS_{10}} > CEI_{S_7}, \quad for \ \ n \geq 2.$
(iv) $CEI_{PS_{10}} > CEI_{WD_7}, \quad for \ \ n \geq 2.$

Otherwise the comparisons depend upon the value of n, ν and ℓ. To verify the results of above theorem the graphs are plotted for the set $(\nu,\ell)=(1,1)$. These graphs in (n, CEI) variables are shown in Figures 4.1–4.4.

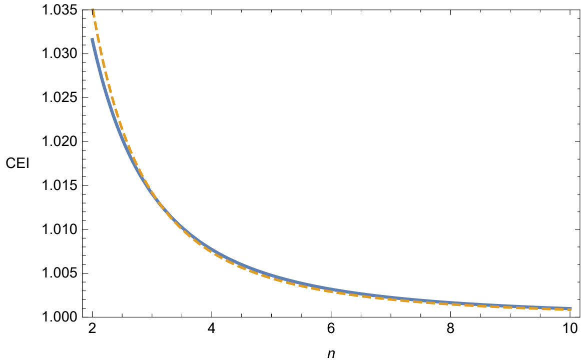

Figure 4.1.
CEI_{WA_7} *(dashedline)*, $CEI_{PS_{10}}$ *(thickline)*

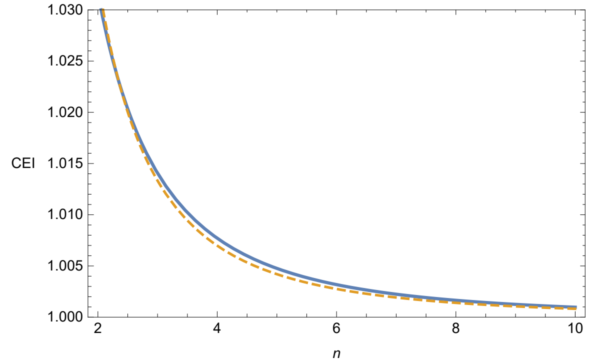

Figure 4.2.
CEI_{WB_7} *(dashedline)*, $CEI_{PS_{10}}$ *(thickline)*

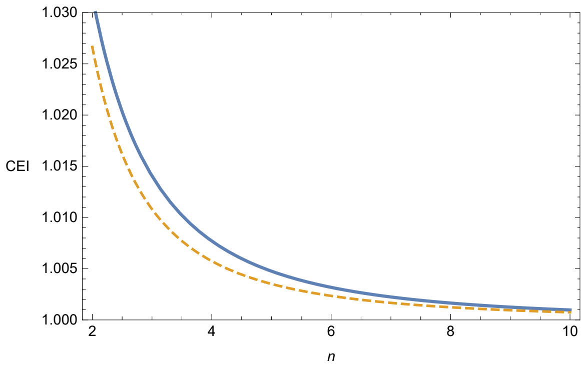

Figure 4.3.
CEI_{S_7} *(dashedline)*, $CEI_{PS_{10}}$ *(thickline)*

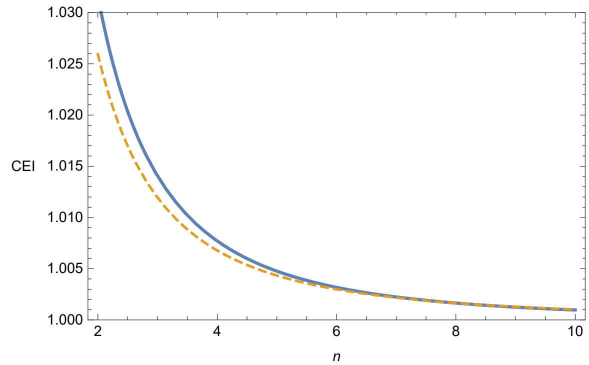

Figure 4.4.
CEI_{WD_7} *(dashedline)*, $CEI_{PS_{10}}$ *(thickline)*

4.4 Numerical reports

In this section, several standard numerical problems are considered to illustrate the convergence behavior and computational efficiency of the proposed methods. To verify the theoretical order of convergence, authors have used the computational order of convergence (ρ_c) using the formula (1.2.10) and an estimation of the factors ν and $\ell = 3$ evaluated according to the Table 1.1 for all numerical tests.

Example 4.4.1 *Consider the following large system of hundred non-linear equations*

$$F_i = \begin{cases} x_i^2 x_{i+1} - 1 = 0, \ 1 \leq i \leq 99, \\ x_i^2 x_1 - 1 = 0, \ i = 100. \end{cases}$$

where $(n, \nu) = (100, 200)$ *are the values used in equations* $(4.3.1) - (4.3.5)$*. The convergence of the methods towards the solution* $X^* = (1, 1, \ldots, 1)^T$ *is tested with the following computational terms and results are shown in Table 4.1.*

$[X+F, X; F] =$

$$\begin{bmatrix} x_2\left(x_1\left(x_1x_2+2\right)-1\right) & \left(x_2x_1^2+x_1-1\right)^2 & \dots & 0 \\ 0 & x_3\left(x_2\left(x_2x_3+2\right)-1\right) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{100}^2 & 0 & \dots & \left(x_2x_1^2+x_1-1\right)\left(x_{100}\left(x_1x_{100}+2\right)-1\right) \end{bmatrix},$$

$[X+F, X-F; F] =$

$$\begin{bmatrix} 2x_1\left(-x_3x_2^2+x_2+1\right) & \left(x_2x_1^2+x_1-1\right)^2 & \dots & 0 \\ 0 & 2x_2\left(-x_4x_3^2+x_3+1\right) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(-x_1x_{100}^2+x_{100}+1\right)^2 & 0 & \dots & 2\left(x_2x_1^2+x_1-1\right)x_{100} \end{bmatrix},$$

$$[Y, X; F] = \begin{bmatrix} x_2\left(x_1+y_1\right) & y_1^2 & 0 & \dots & 0 & 0 \\ 0 & x_3\left(x_2+y_2\right) & y_2^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{100}\left(x_{99}+y_{99}\right) & y_{99}^2 \\ x_{100}^2 & 0 & 0 & \dots & 0 & y_1\left(x_{100}+y_{100}\right) \end{bmatrix}.$$

Table 4.1: Performance of various iterative techniques at initial approximation $(1.6, 1.6, \ldots, 1.6)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ_c	C	CEI	CPU $Time\ (sec)$
S_7	7.5(−2)	1.1(−13)	1.5(−96)	6.998	1086300	1.0000017913	15.18
WA_7	9.5(−2)	8.2(−13)	4.0(−90)	6.994	1310000	1.0000148544	13.41
WB_7	1.0(−1)	2.0(−12)	2.3(−87)	6.991	1320200	1.000001474	14.01
WD_7	9.4(−1)	1.5(−5)	3.0(−39)	7.004	585300	1.0000033246	11.17
$PS_{10}(\alpha = 1)$	3.5(−1)	3.2(−14)	1.3(−144)	10.001	1251500	1.0000018399	11.79
$PS_{10}(\alpha = -1/2)$	4.4(−1)	3.4(−13)	1.0(−133)	9.953	1251500	1.0000018399	15.07
$PS_{10}(\alpha = -1/4)$	4.4(−1)	1.9(−13)	2.9(−136)	9.940	1251500	1.0000018399	14.85
$PS_{10}(\alpha = 3/4)$	1.1(−1)	4.3(−19)	2.7(−191)	9.999	1251500	1.0000018399	14.09
$PS_{10}(\alpha = -1000)$	5.0(−1)	3.9(−7)	5.4(−68)	9.954	1251500	1.0000018399	12.01
$PS_{10}(\alpha = \frac{1}{\sqrt{6}})$	6.8(−1)	1.6(−11)	1.4(−117)	9.964	1251500	1.0000018399	15.03

Example 4.4.2 *Take another system of non-linear equations*

$$F_i = \sum_{i \neq j} x_i - e^{-x_j} = 0,\ 1 \leq i, j \leq 13.$$

where $(n, \nu) = (13, 118.20)$ *are the values used in equations* $(4.3.1) - (4.3.5)$*. The convergence of the methods towards the solution* $X^* = (0.077146207613064638\ldots,$ $0.077146207613064638\ldots, \ldots, 0.077146207613064638\ldots)^T$ *is tested with following computational terms and given in following Table 4.2.*

$[X + F, X; F] =$

$$\begin{bmatrix} \frac{e^{-x_1} - e^{\left(-x_1 + e^{-x_1} - \sum_{i=2}^{13} x_i\right)}}{x_2 - e^{-x_1} + \sum_{i=3}^{13} x_i} & 1 & \cdots & 1 \\ 1 & \frac{e^{-x_2} - e^{\left(-x_1 + e^{-x_2} - \sum_{i=2}^{13} x_i\right)}}{x_1 - e^{-x_2} + \sum_{i=3}^{13} x_i} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{-e^{\left(-x_1 + e^{-x_{13}} - \sum_{i=2}^{13} x_i\right)} + e^{-x_{13}}}{x_1 - e^{-x_{13}} + \sum_{i=2}^{12} x_i} \end{bmatrix},$$

$$[X+F,X-F;F]=\Big\{\{\frac{-e^{\left(e^{-x_1}-\sum\limits_{i=1}^{13}x_i\right)}+e^{\left(-x_1-e^{-x_1}+\sum\limits_{i=2}^{13}x_i\right)}}{-2e^{-x_1}+2\sum\limits_{i=2}^{13}x_i},1,\ldots,1\},$$

$$\{1,\frac{-e^{\left(e^{-x_2}-\sum\limits_{i=1}^{13}x_i\right)}+e^{\left(-e^{-x_2}-x_2+\sum\limits_{\substack{i=1,\\ i\neq 2}}^{13}x_i\right)}}{-2e^{-x_2}+2\sum\limits_{\substack{i=1,\\ i\neq 2}}^{13}x_i},\ldots,1\},\ldots,$$

$$\{1,\ldots,1,\frac{-e^{\left(e^{-x_{13}}-\sum\limits_{i=1}^{13}x_i\right)}+e^{\left(-e^{-x_{13}}+\sum\limits_{\substack{i=1,\\ i\neq 13}}^{12}x_i-x_{13}\right)}}{-2e^{-x_{13}}+2\sum\limits_{i=1}^{12}x_i}\}\Big\},$$

$$[Y,X;F]=\begin{bmatrix}\frac{e^{-x_1}-e^{-y_1}}{y_1-x_1} & 1 & \ldots & 1\\ 1 & \frac{e^{-x_2}-e^{-y_2}}{y_2-x_2} & \ldots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \ldots & \frac{e^{-x_{13}}-e^{-y_{13}}}{y_{13}-x_{13}}\end{bmatrix}.$$

Table 4.2: Performance of various iterative techniques at initial value $(0.45, 0.45, \ldots, 0.45)^T$

Scheme	$\|X^{(1)}-X^*\|$	$\|X^{(2)}-X^*\|$	$\|X^{(3)}-X^*\|$	ρ_c	C	CEI	CPU *Time (sec)*
S_7	2.1(−10)	7.7(−79)	6.8(−558)	7.000	195829.4	1.0000099368	1.18
WA_7	4.4(−8)	5.9(−59)	5.2(−415)	7.000	104114.4	1.0000186903	0.90
WB_7	4.3(−8)	5.8(−59)	4.4(−415)	7.000	104309.4	1.0000186553	1.10
WD_7	1.0(−2)	4.2(−30)	1.3(−214)	6.992	67147.6	1.0000289800	1.25
$PS_{10}(\alpha=1)$	1.8(−6)	4.3(−71)	2.2(−717)	10.000	70062.2	1.0000328654	1.25
$PS_{10}(\alpha=-1/2)$	1.7(−5)	1.9(−61)	5.6(−621)	10.000	70062.2	1.0000328654	1.21
$PS_{10}(\alpha=-1/4)$	2.4(−5)	6.3(−60)	9.7(−606)	10.000	70062.2	1.0000328654	1.25
$PS_{10}(\alpha=3/4)$	1.4(−6)	3.1(−72)	7.1(−729)	10.000	70062.2	1.0000328654	1.29
$PS_{10}(\alpha=-1000)$	1.9(−5)	1.1(−57)	7.6(−580)	9.997	70062.2	1.0000328654	1.26
$PS_{10}(\alpha=\frac{1}{\sqrt{6}})$	8.1(−4)	1.4(−44)	2.2(−452)	10.000	70062.2	1.0000328654	1.42

Example 4.4.3 *Consider the following system*

$$F_i = arctan(x_i) + 1 - 2\sum_{i\neq j} x_j^2 = 0,\ \ 1 \leq i,j \leq 18.$$

where $(n,\nu) = (18, 162.22)$ *are the values used in equations* (4.3.1) − (4.3.5). *The convergence of the methods towards the solution* $X^* = (0.186652313388246062\ldots, 0.186652313388246062\ldots, \ldots, 0.186652313388246062\ldots)^T$ *is tested with following computational terms and shown in following Table 4.3.*

$[X+F, X; F] =$

$$\Big\{\{\frac{\tan^{-1}\left(\tan^{-1}(x_1)+x_1-2\sum\limits_{i=2}^{i=18}x_i^2+1\right)-\tan^{-1}(x_1)}{\tan^{-1}(x_1)-2\sum\limits_{i=2}^{i=18}x_i^2+1}, 4\sum_{\substack{i=1,\\ i\neq 2}}^{i=18}x_i^2-4x_2-2\tan^{-1}(x_2),$$

$$\ldots, 4\sum_{\substack{i=1,\\ i\neq 18}}^{i=17}x_i^2-4x_{18}-2\tan^{-1}(x_{18})-2\}, \{4\sum_{i=2}^{i=18}x_i^2-4x_1-2\tan^{-1}(x_1)-2,$$

$$\frac{\tan^{-1}\left(x_2-2\sum\limits_{\substack{i=1,\\ i\neq 2}}^{i=18}x_i^2+\tan^{-1}(x_2)+1\right)-\tan^{-1}(x_2)}{-2\sum\limits_{\substack{i=1,\\ i\neq 2}}^{i=18}x_i^2+\tan^{-1}(x_2)+1}, \ldots, 4\sum_{\substack{i=1,\\ i\neq 18}}^{i=17}x_i^2-4x_{18}-2\tan^{-1}(x_{18})-2\},$$

$$\ldots, \{4\sum_{i=2}^{i=18}x_i^2-4x_1-2\tan^{-1}(x_1)-2, 4\sum_{\substack{i=1,\\ i\neq 2}}^{i=18}x_i^2-4x_2-2\tan^{-1}(x_2)-2, \ldots,$$

$$\frac{\tan^{-1}\left(\tan^{-1}(x_{18})+x_{18}-2\sum\limits_{i=1}^{i=17}x_i^2+1\right)-\tan^{-1}(x_{18})}{\tan^{-1}(x_{18})-2\sum\limits_{i=1}^{i=17}x_i^2+1}\}\Big\},$$

$[X+F, X-F; F] =$

$$\Big\{\{\frac{\tan^{-1}\left(-x_1-2\sum\limits_{i=2}^{i=18}x_i^2+\tan^{-1}(x_1)+1\right)+\tan^{-1}\left(x_1-2\sum\limits_{i=2}^{i=18}x_i^2+\tan^{-1}(x_1)+1\right)}{-4\sum\limits_{i=2}^{i=18}x_i^2+2\tan^{-1}(x_1)+2}, -4x_2,$$

$$\ldots, -4x_{18}\}, \{-4x_1,$$

$$\frac{\tan^{-1}\left(-x_2-2\sum_{i=1,i\neq 2}^{i=18}x_i^2+\tan^{-1}(x_2)+1\right)+\tan^{-1}\left(x_2-2\sum_{i=1,i\neq 2}^{i=18}x_i^2+\tan^{-1}(x_2)+1\right)}{-4\sum_{i=1,i\neq 2}^{i=18}x_i^2+2\tan^{-1}(x_2)+2},$$

$$\ldots,-4x_{18}\},\ldots,\{-4x_1,-4x_2,\ldots,-4x_{18}\},\{-4x_1,-4x_2,\ldots,$$

$$\frac{\tan^{-1}\left(-2\sum_{i=1}^{i=17}x_i^2-x_{18}+\tan^{-1}(x_{18})+1\right)+\tan^{-1}\left(-2\sum_{i=1}^{i=17}x_i^2+x_{18}+\tan^{-1}(x_{18})+1\right)}{-4\sum_{i=1}^{i=17}x_i^2+2\tan^{-1}(x_{18})+2}\}\Big\},$$

$$[Y,X;F]=\begin{bmatrix} \frac{\tan^{-1}(x_1)-\tan^{-1}(y_1)}{x_1-y_1} & -2\left(x_2+y_2\right) & \ldots & -2\left(x_{18}+y_{18}\right) \\ -2\left(x_1+y_1\right) & \frac{\tan^{-1}(x_2)-\tan^{-1}(y_2)}{x_2-y_2} & \ldots & -2\left(x_{18}+y_{18}\right) \\ \vdots & \vdots & \ddots & \vdots \\ -2\left(x_1+y_1\right) & -2\left(x_2+y_2\right) & \ldots & \frac{\tan^{-1}(x_{18})-\tan^{-1}(y_{18})}{x_{18}-y_{18}} \end{bmatrix}.$$

Table 4.3: Performance of various iterative techniques at initial value $(0.15, 0.15, \ldots, 0.15)^T$

Scheme	$\|X^{(1)}-X^*\|$	$\|X^{(2)}-X^*\|$	$\|X^{(3)}-X^*\|$	ρ_c	C	*CEI*	*CPU Time (sec)*
S_7	3.0(−6)	1.6(−39)	1.6(−272)	7.000	519239.04	1.0000037476	9.34
WA_7	1.9(−5)	3.2(−34)	1.2(−235)	7.000	272548.44	1.0000071397	8.07
WB_7	1.7(−5)	1.2(−34)	1.3(−234)	7.000	272908.44	1.0000071303	9.29
WD_7	6.9(−5)	4.9(−29)	4.6(−598)	7.000	171899.76	1.0000113201	7.01
$PS_{10}(\alpha=1)$	5.1(−10)	2.8(−95)	5.5(−948)	10.000	178587.72	1.0000128934	8.67
$PS_{10}(\alpha=-1/2)$	4.4(−8)	6.9(−75)	7.1(−743)	10.000	178587.72	1.0000128934	8.65
$PS_{10}(\alpha=-1/4)$	2.0(−8)	1.5(−78)	1.4(−779)	10.000	178587.72	1.0000128934	9.09
$PS_{10}(\alpha=3/4)$	3.6(−10)	4.5(−97)	4.5(−966)	10.000	178587.72	1.0000128934	8.84
$PS_{10}(\alpha=-1000)$	7.6(−6)	7.0(−44)	3.8(−427)	10.077	178587.72	1.0000128934	8.92
$PS_{10}(\alpha=\frac{1}{\sqrt{6}})$	9.0(−10)	9.0(−93)	9.1(−923)	10.000	178587.72	1.0000128934	8.90

Example 4.4.4

$$F_i = cos^{-1}(x_i) - \sum_{i=1}^{20} x_i + 2x_i = 0, i = 1, 2, \ldots, 20,$$

where $(n, \nu) = (20, 149.98)$ *are the values used in equations* $(4.3.1) - (4.3.5)$*. Solution of this problem is* $X^* = (0.08266851975958913\ldots, 0.08266851975958913\ldots, \ldots, 0.08266851975958913\ldots)^T$ *and results with following computational terms are shown in Table 4.4.*

$$[X + F, X; F] =$$

$$\Big\{\{\frac{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}\left(2x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)\right)}{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)}, \frac{x_1 - x_2 + \sum_{i=3}^{20} x_i - \cos^{-1}(x_2)}{-x_1 + x_2 - \sum_{i=3}^{20} x_i + \cos^{-1}(x_2)}, \ldots,$$

$$\frac{\sum_{i=1}^{19} -x_{20} - \cos^{-1}(x_{20})}{-\sum_{i=1}^{19} x_{20} + \cos^{-1}(x_{20})}\}, \{\frac{-x_1 + \sum_{i=2}^{20} x_i - \cos^{-1}(x_1)}{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)},$$

$$\frac{x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}\left(-x_1 + 2x_2 - \sum_{i=3}^{20} x_i + \cos^{-1}(x_2)\right)}{x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}(x_2)}, \ldots, \frac{\sum_{i=1}^{19} x_i - x_{20} - \cos^{-1}(x_{20})}{-\sum_{i=1}^{19} x_i + x_{20} + \cos^{-1}(x_{20})}\},$$

$$\ldots, \{\frac{-x_1 + \sum_{i=2}^{20} x_i - \cos^{-1}(x_1)}{x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)}, \frac{-x_2 + \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - \cos^{-1}(x_2)}{x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}(x_2)}, \ldots,$$

$$\frac{-\sum_{i=1}^{19} x_i + x_{20} + \cos^{-1}\left(-\sum_{i=1}^{19} x_i + 2x_{20} + \cos^{-1}(x_{20})\right)}{-\sum_{i=1}^{19} x_i + x_{20} + \cos^{-1}(x_{20})}\}\Big\},$$

$$[X+F, X-F; F] =$$

$$\Bigg\{\{\frac{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1) + \cos^{-1}\left(2x_1 - \sum_{i=2}^{20} x_i + \cos^{-1}(x_1)\right) - \cos^{-1}\left(\sum_{i=2}^{20} x_i - \cos^{-1}(x_1)\right)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)},$$

$$\frac{2x_1 - 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_2)}{2x_2 - 2\sum_{i=1}^{20} x_i + 2\cos^{-1}(x_2)}, \dots, \frac{2\sum_{i=1}^{19} x_i - 2x_{20} - 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\}, \{\frac{-2x_1 + 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_1)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)},$$

$$\frac{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2) + \cos^{-1}\left(2x_2 - \sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + \cos^{-1}(x_2)\right) - \cos^{-1}\left(\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - \cos^{-1}(x_2)\right)}{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2)},$$

$$\dots, \frac{2\sum_{i=1}^{19} x_i - 2x_{20} - 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\}, \dots, \{\frac{-2x_1 + 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_1)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)},$$

$$\frac{-2x_2 + 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - 2\cos^{-1}(x_2)}{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2)}, \dots, \frac{2\sum_{i=1}^{19} x_i - 2x_{20} - 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\},$$

$$\{\frac{-2x_1 + 2\sum_{i=2}^{20} x_i - 2\cos^{-1}(x_1)}{2x_1 - 2\sum_{i=2}^{20} x_i + 2\cos^{-1}(x_1)}, \frac{-2x_2 + 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i - 2\cos^{-1}(x_2)}{2x_2 - 2\sum_{\substack{i=1,\\ i\neq 2}}^{20} x_i + 2\cos^{-1}(x_2)}, \dots,$$

$$\frac{-2\sum_{i=1}^{19} x_i + 2x_{20} - \cos^{-1}\left(\sum_{i=1}^{19} x_i - \cos^{-1}(x_{20})\right) + 2\cos^{-1}(x_{20})}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}$$

$$+ \frac{\cos^{-1}\left(-\sum_{i=1}^{19} x_i + 2x_{20} + \cos^{-1}(x_{20})\right)}{-2\sum_{i=1}^{19} x_i + 2x_{20} + 2\cos^{-1}(x_{20})}\}\Bigg\},$$

$[Y, X; F] =$

$$\begin{bmatrix} \frac{-x_1-\cos^{-1}(x_1)+y_1+\cos^{-1}(y_1)}{y_1-x_1} & \frac{x_2-y_2}{y_2-x_2} & \cdots & \frac{x_{20}-y_{20}}{y_{20}-x_{20}} \\ \frac{x_1-y_1}{y_1-x_1} & \frac{-x_2-\cos^{-1}(x_2)+y_2+\cos^{-1}(y_2)}{y_2-x_2} & \cdots & \frac{x_{20}-y_{20}}{y_{20}-x_{20}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1-y_1}{y_1-x_1} & \frac{x_2-y_2}{y_2-x_2} & \cdots & \frac{-x_{20}-\cos^{-1}(x_{20})+y_{20}+\cos^{-1}(y_{20})}{y_{20}-x_{20}} \end{bmatrix}.$$

Table 4.4: Performance of various iterative techniques at initial value $(0.1, 0.1, \ldots, 0.1)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ_c	C	CEI	CPU $Time\ (sec)$
S_7	5.7(−22)	4.8(−163)	1.6(−1150)	7.0054	596222.4	1.0000032637	14.82
WA_7	1.8(−19)	2.1(−141)	4.4(−994)	6.9194	313400.4	1.000006209	11.81
WB_7	1.8(−19)	2.1(−141)	4.4(−995)	6.9194	313840.4	1.0000062003	13.99
WD_7	2.5(−14)	1.0(−106)	2.2(−752)	7.393	196555.2	1.0000099001	15.79
$PS_{10}(\alpha = 1)$	8.1(−25)	4.6(−259)	1.9(−2601)	10.192	204754.8	1.0000112456	15.48
$PS_{10}(\alpha = -1/2)$	8.2(−25)	5.2(−259)	5.4(−2601)	10.000	204754.8	1.0000112456	15.90
$PS_{10}(\alpha = -1/4)$	8.1(−25)	5.0(−259)	3.9(−2601)	10.000	204754.8	1.0000112456	15.40
$PS_{10}(\alpha = 3/4)$	8.1(−25)	4.7(−259)	1.9(−2601)	10.000	204754.8	1.0000112456	14.53
$PS_{10}(\alpha = -1000)$	1.5(−19)	1.3(−205)	5.4(−2066)	10.499	204754.8	1.0000112456	14.96
$PS_{10}(\alpha = \frac{1}{\sqrt{6}})$	8.1(−25)	4.7(−259)	2.2(−2601)	10.192	204754.8	1.0000112456	15.04

Example 4.4.5 *Consider the following system of equations*

$$F_i = \begin{cases} x_{i+1}^2 + e^{x_i} - 1 = 0, \ 1 \le i \le 49, \\ x_1^2 + e^{x_i} - 1 = 0, \ i = 50. \end{cases}$$

where $(n, \nu) = (50, 119.20)$ *are the values used in equations* $(4.3.1) - (4.3.5)$. *The convergence of the methods towards the solution* $X^* = (0, 0, \ldots, 0)^T$ *is tested with following computational terms and given in following in Table 4.5.*

$$[X+F,X;F]=\begin{bmatrix} \frac{e^{(x_1-1)}\left(e^{(x_2^2+e^{x_1})}-e\right)}{x_2^2+e^{x_1}-1} & x_3^2+e^{x_2}+2x_2-1 & \dots & 0 \\ 0 & \frac{e^{(x_2-1)}\left(e^{(x_3^2+e^{x_2})}-e\right)}{x_3^2+e^{x_2}-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^2+e^{x_1}+2x_1-1 & 0 & \dots & \frac{e^{(x_1^2+e^{x_{50}})}-e}{e^{(1-x_{50})}\left(x_1^2-1\right)+e} \end{bmatrix},$$

$$[X+F,X-F;F]=$$
$$\begin{bmatrix} \frac{e^{\left(x_2^2+e^{x_1}+x_1-1\right)}-e^{\left(-x_2^2-e^{x_1}+x_1+1\right)}}{2\left(x_2^2+e^{x_1}-1\right)} & 2x_2 & \dots & 0 \\ 0 & \frac{e^{\left(x_3^2+e^{x_2}+x_2-1\right)}-e^{\left(-x_3^2-e^{x_2}+x_2+1\right)}}{2\left(x_3^2+e^{x_2}-1\right)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2x_1 & 0 & \dots & \frac{e^{\left(-x_1^2-e^{x_{50}}+x_{50}-1\right)}\left(e^{2\left(x_1^2+e^{x_{50}}\right)}-e^2\right)}{2\left(x_1^2+e^{x_{50}}-1\right)} \end{bmatrix},$$

$$[Y,X;F]=\begin{bmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1} & x_2+y_2 & \dots & 0 \\ 0 & \frac{e^{x_2}-e^{y_2}}{x_2-y_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1+y_1 & 0 & \dots & \frac{e^{x_{50}}-e^{y_{50}}}{x_{50}-y_{50}} \end{bmatrix}.$$

Table 4.5: Performance of various iterative techniques at initial value $(0.5, 0.5, \ldots, 0.5)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ_c	C	CEI	CPU $Time\ (sec)$
S_7	4.9(−2)	5.9(−13)	2.3(−89)	6.996	3082990	1.0000006312	4.09
WA_7	3.3(−2)	1.6(−15)	1.3(−108)	6.992	1661640	1.0000011711	3.452
WB_7	4.1(−2)	9.6(−15)	5.7(−103)	6.984	1664240	1.0000011692	3.81
WD_7	1.2(−1)	1.7(−9)	2.7(−64)	6.981	995870	1.000001954	3.34
$PS_{10}(\alpha = 1)$	3.6(−4)	2.2(−41)	1.4(−413)	10.000	1084830	1.0000021225	3.76
$PS_{10}(\alpha = -1/2)$	1.2(−1)	9.3(−16)	2.9(−156)	10.000	1084830	1.0000021225	3.90
$PS_{10}(\alpha = -1/4)$	2.2(−1)	2.6(−13)	5.5(−132)	9.943	1084830	1.0000021225	3.67
$PS_{10}(\alpha = 3/4)$	1.5(−4)	2.7(−45)	7.7(−453)	10.000	1084830	1.0000021225	3.67
$PS_{10}(\alpha = -1000)$	8.3(−2)	2.3(−10)	1.2(−96)	10.083	1084830	1.0000021225	3.93
$PS_{10}(\alpha = \frac{1}{\sqrt{6}})$	2.1(−3)	8.3(−34)	6.0(−338)	9.999	1084830	1.0000021225	3.62

4.5 Conclusions

A new multi-point tenth-order variants of Steffensen's method for solving non-linear systems have been proposed in this study. The proposed methods are completely derivative free. Subsequently, it is very evident that the proposed scheme suite to those problems where derivatives require lengthy numerical computations. A development of an inverse first-order divided difference operator for multi variable function demonstrates the convergence order of constructed scheme. In addition, the computational efficiency index is used to compare the efficiency of the proposed methods. Computational results have affirmed powerful and efficient character of the developed scheme. A few standard numerical experimentations have being completed for various problems and results are observed to be superior to the current ones. In this manner, the new scheme is exceptionally reasonable and applicable to solve non-linear systems. Besides this, one can easily obtain several new tenth-order variants of Steffensen's method by considering the value of disposable parameter α.

Chapter 5

Local convergence analysis in Banach space

5.1 Introduction

The purpose of this chapter is to expand the applicability of proposed scheme (4.2.13) that has been presented in pervious chapter. The convergence analysis of this technique shown in the Section 4.2.2 requires hypotheses reaching up to the tenth derivative of operator F, but actually only the divided difference of order one and two appear in this method. So, the local convergence analysis of this scheme has been presented in the more general setting of a Banach space using hypotheses only on the first and second divided differences. Moreover, the radius of convergence and a uniqueness result are provided based on Lipschitz type constants. This way the applicability of the method is expanded.

5.2 Local convergence

For the local convergence, consider the proposed scheme (4.2.13) as:

$$\begin{cases} Y^k = X^k - \{[W^k, X^k; F]\}^{-1} F(X^k), \\ Z^k = Y^k - \Big(4(\alpha-1)[Y^k, X^k; F] - 2(2\alpha-1)[W^k, X^k; F]\Big)(\tau^k)^{-1} F(Y^k), \\ U^k = Z^k - \{A^k\}^{-1} F(Z^k), \\ X^{k+1} = Z^k - \{A^k\}^{-1}\big(F(Z^k) + F(U^k)\big), \end{cases} \tag{5.2.1}$$

where $\tau^k = 2(4\alpha^2 - 6\alpha + 3)[W^k, X^k; F]^2 + 4(-4\alpha^2 + 5\alpha - 2)[Y^k, X^k; F][W^k, X^k; F]$ $+8\alpha(\alpha-1)[Y^k, X^k; F]^2$ and $A^k = [Z^k, Y^k; F] + (Z^k - Y^k)[Z^k, Y^k, X^k; F]$.

The local convergence analysis of method (5.2.1) is based on some scalar functions and parameters. Let $\kappa > 0$ and $\alpha \in \mathbb{R}$ be parameters. Let's also assume that the functions $\varphi_0,\ \bar{\varphi}_0,\ \varphi_1$ defined on $[0,\ \infty)$, $\varphi_2,\ \bar{\varphi}_2$, defined on $[0,\ \infty) \times [0,\ \infty)$ and $\varphi_3,\ \varphi_4,\ \bar{\varphi}_4$ defined on $[0,\ \infty) \times [0,\ \infty) \times [0,\ \infty)$ are continuous and nondecreasing functions with

$$\varphi_0(0) = \bar{\varphi}_0(0) = \varphi_2(0,\ 0) = \bar{\varphi}_2(0,\ 0) = 0. \tag{5.2.2}$$

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Define parameters

$$r_0 = sup\{t \geq 0 : \varphi_0(t) < 1\},$$
$$\bar{r_0} = sup\{t \geq 0 : \varphi_2(t,\ t) < 1\} \tag{5.2.3}$$

and

$$\bar{r} = \min\{r_0,\ \bar{r_0}\}. \tag{5.2.4}$$

Define functions $\varphi_5,\ g_1$ and h_1 on the interval $[0,\ \bar{r})$ by

$$\varphi_5(t) = (1 + \bar{\varphi}_0(t))t, \tag{5.2.5}$$

$$g_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1-\varphi_0(t)} + \frac{\int_0^1 \varphi_1(t\theta)d\theta}{(1-\varphi_0(t))(1-\varphi_2(t,\ t))}[\varphi_2(\varphi_5(t),\ t) + \varphi_0(t)],$$

and

$$h_1(t) = g_1(t) - 1. \tag{5.2.6}$$

Using equations (5.2.2), (5.2.5) and (5.2.6) one can have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to \bar{r}^-$. It follows from the intermediate value theorem that the function h_1 has zero in the interval $(0,\ \bar{r})$. Denote by r_1 the the smallest such zero. Let's define $d_1,\ d_2$ and d_3 by

$$d_1 = 4(-4\alpha^2 + 5\alpha - 2),$$
$$d_2 = 2(4\alpha^2 - 6\alpha + 3)$$
and
$$d_3 = 8\alpha(\alpha - 1). \tag{5.2.7}$$

Notice that $d_1 \neq 0$, since the discriminant of the quadratic polynomial is $-7 < 0$.
Define

$$\bar{\bar{r_2}} = sup\{t \geq 0 : \kappa\bar{\varphi}_2(\varphi_5(t),\ t) < 1\} \tag{5.2.8}$$

and

$$\bar{r_2} = \min\{r_1,\ \bar{\bar{r_2}}\}. \tag{5.2.9}$$

Then, define functions p and h_p on the interval $[0,\ \bar{\bar{r_2}}]$ by

$$p(t) = \frac{1}{|d_1|}\left[|d_1|\varphi_2(g_1(t)t,\ t) + \frac{\kappa|d_2|\varphi_4(\varphi_5(t),\ t)\bar{\varphi}_4(\varphi_5(t),\ t)}{1-\kappa\bar{\varphi}_2(\varphi_5(t),\ t)} + \frac{\kappa|d_3|\varphi_4(g_1(t)t,\ t)\bar{\varphi}_4(\varphi_5(t),\ t)}{1-\kappa\bar{\varphi}_2(\varphi_5(t),\ t)}\right] \tag{5.2.10}$$

and

$$h_p(t) = p(t) - 1. \tag{5.2.11}$$

Suppose that

$$\frac{2\kappa(|d_2|+|d_3|)\varphi_4(0,0)\bar{\varphi}_4(0,0)}{|d_1|} < 1 \tag{5.2.12}$$

In view of equations (5.2.2), (5.2.5), (5.2.6), (5.2.10) and (5.2.12), one gets that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \bar{\bar{r}}_2^-$. Denote by r_p the the smallest such zero of the function h_p on the interval $[0,\ \bar{\bar{r}}_2)$. Define functions g_2 and h_2 on the interval $[0,\ r_p)$ by

$$g_2(t) = \left(1 + \frac{\Big[4|\alpha-1|\varphi_4(g_1(t)t,\ t) + 2|2\alpha-1|\bar{\varphi}_4(\varphi_5(t),\ t)\Big]\kappa\varphi_1(g_1(t)t)}{|d_1|(1-\kappa\bar{\varphi}_2(\varphi_5(t),\ t))(1-p(t))}\right) g_1(t). \tag{5.2.13}$$

and

$$h_2(t) = g_2(t) - 1. \tag{5.2.14}$$

Clearly, $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2 the the smallest such zero of the function h_2 on the interval $[0,\ r_p)$. Define functions q and h_q on the interval $[0,\ r_p)$ by

$$q(t) = \varphi_2(g_2(t)t,\ g_1(t)t) + \varphi_3(g_2(t)t,\ g_1(t)t,\ t)(g_2(t)+g_1(t))t \tag{5.2.15}$$

and

$$h_q(t) = q(t) - 1. \tag{5.2.16}$$

Considering equations (5.2.13) and (5.2.15), one obtains that $h_q(0) = -1 < 0$ and $h_q(t) \to +\infty$ as $t \to r_p^-$. Denote by r_q the the smallest such zero of the function h_q on the interval $[0,\ r_p)$. Finally, define functions $g_3,\ h_3,\ g_4$ and h_4 on the interval $[0,\ r_q)$ by

$$\begin{aligned} g_3(t) &= \left(1 + \frac{\int_0^1 \varphi_1(\theta g_2(t))d\theta}{1-q(t)}\right) g_2(t), \\ h_3(t) &= g_3(t) - 1, \\ g_4(t) &= g_3(t) + \frac{\int_0^1 \varphi_1(\theta g_2(t)t)d\theta g_2(t) + \int_0^1 \varphi_1(\theta g_3(t)t)g_3(t)d\theta}{1-q(t)}, \end{aligned} \tag{5.2.17}$$

and

$$h_4(t) = g_4(t) - 1. \tag{5.2.18}$$

Again, one must have that $h_3(0) = h_4(0) = -1 < 0$ and $h_3(t) \to +\infty,\ h_4(t) \to +\infty$ as $t \to r_q^-$. Denote by r_3 and r_4 the the smallest such zeros of the functions h_3 and h_4, respectively on the interval $[0,\ r_q)$. Define the radius of convergence r by

$$r = \min\{r_i\},\ i = 1,2,3,4. \tag{5.2.19}$$

Then, one can have that

$$0 < r \leq \bar{r} \tag{5.2.20}$$

and for each $t \in [0,\ r)$

$$0 \leq g_i(t) < 1, \tag{5.2.21}$$

$$0 \leq p(t) < 1, \tag{5.2.22}$$

and

$$0 \leq q(t) < 1. \tag{5.2.23}$$

The local convergence analysis of method (5.2.1) will be showed using the conditions (H):

(H_1) $F : D \subseteq B_1 \to B_2$ is a Fréchet differentiable operator, where B_1 and B_2 are Banach spaces and D is an open and convex subset. The operators $[\cdot,\ \cdot;\ F]$ and $[\cdot,\ \cdot,\ \cdot;\ F]$ are divided differences of order one and two, respectively for operator F.

(H_2) There exist $X^* \in D$ and $\kappa > 0$ such that

$$F(X^*) = 0 \quad \text{and} \quad \|\{F'(X^*)\}^{-1}\| \leq \kappa.$$

(H_3) There exist $\varphi_0,\ \bar{\varphi}_0 : [0,\ +\infty) \to [0,\ +\infty),\ \varphi_2,\ \bar{\varphi}_2 : [0,\ +\infty) \times [0,\ +\infty) \to [0,\ +\infty)$, continuous and nondecreasing functions with $\varphi_0(0) = \bar{\varphi}_0(0) = \varphi_2(0,\ 0) = \bar{\varphi}_2(0,\ 0) = 0$, such that for each $X,\ Y \in D$

$$\|\{F'(X^*)\}^{-1}(F'(X) - F'(X^*))\| \leq \varphi_0(\|X - X^*\|),$$
$$\|F'(X) - F'(X^*)\| \leq \bar{\varphi}_0(\|X - X^*\|),$$
$$\|\{F'(X^*)\}^{-1}([X,\ Y;\ F] - F'(X^*))\| \leq \varphi_2(\|X - X^*\|,\ \|Y - X^*\|)$$

and

$$\|[X,\ Y;\ F] - F'(X^*)\| \leq \bar{\varphi}_2(\|X - X^*\|,\ \|Y - X^*\|).$$

(H_4) There exist functions $\varphi,\ \varphi_1 : [0,\ \infty) \to [0,\ +\infty),\ \varphi_3 : [0,\ \infty) \times [0,\ \infty) \times [0,\ \infty) \to [0,\ +\infty),\ \varphi_4,\ \bar{\varphi}_4 : [0,\ \infty) \times [0,\ \infty) \to [0,\ +\infty)$ continuous and nondecreasing with $\varphi(0) = 0$ such that for each $X,\ Y,\ Z \in D_0 = D \cap U(X^*,\ \bar{r})$

$$\|\{F'(X^*)\}^{-1}\big(F'(X) - F'(Y)\big)\| \leq \varphi(\|X - Y\|),$$
$$\|\{F'(X^*)\}^{-1}F'(X)\| \leq \varphi_1(\|X - X^*\|),$$
$$\|\{F'(X^*)\}^{-1}[Z,\ Y, X;\ F\| \leq \varphi_3(\|Z - X^*\|, \|Y - X^*\|, \|X - X^*\|),$$

$$\|\{F'(X^*)\}^{-1}[X,\ Y;\ F]\| \leq \varphi_4(\|X - X^*\|,\ \|Y - X^*\|)$$

and

$$\|[X,\ Y;\ F]\| \leq \bar{\varphi}_4(\|X - X^*\|,\ \|Y - X^*\|).$$

(H_5) $\bar{U}(X^*,\ R_0) \subseteq D$ and (5.2.12) is satisfied where

$$R_0 = (1 + \bar{\varphi}_0(r))r,$$

and r is defined by (5.2.19).

(H_6) There exists $R \geq r$ such that

$$\int_0^1 \varphi_0(\theta R)d\theta < 1.$$

Next, the local convergence analysis of method (5.2.1) under the (H) conditions and using the preceding notations has been presented below.

Theorem 5.2.1 *Suppose that the conditions* (H) *hold. Then, the sequence* $\{x_n\}$ *generated for* $X^0 \in U(X^*,\ r) - \{X^*\}$ *by method* (5.2.1) *is well defined in* $U(X^*,\ r)$*, remains in* $U(X^*,\ r)$ *for each* $k = 0, 1, 2, \ldots$ *and converges to* X^**. Moreover, the following estimates hold*

$$\|Y^k - X^*\| \leq g_1(\|X^k - X^*\|)\|X^k - X^*\| \leq \|X^k - X^*\| < r, \tag{5.2.24}$$

$$\|Z^k - X^*\| \leq g_2(\|X^k - X^*\|)\|X^k - X^*\| \leq \|X^k - X^*\| < r, \tag{5.2.25}$$

$$\|U^k - X^*\| \leq g_3(\|X^k - X^*\|)\|X^k - X^*\| \leq \|X^k - X^*\| < r \tag{5.2.26}$$

and

$$\|X^{k+1} - X^*\| \leq g_4(\|X^k - X^*\|)\|X^k - X^*\| \leq \|X^k - X^*\| < r, \tag{5.2.27}$$

where the radius of convergence r *is defined by* (5.2.19) *and the functions* $g_i,\ i = 1, 2, 3, 4$ *are defined previously. Furthermore, the point* X^* *is the only solution of equation* $F(X) = 0$ *in* $D_1 := D \cap \bar{U}(X^*,\ R)$.

Proof We shall show using mathematical induction that the sequences $\{X^k\}$ is well defined, converges to X^* and estimates (5.2.24) – (5.2.27) hold. One can have by (H_2), (H_3)

and (H_5) that

$$\begin{aligned}\|W^0 - X^*\| &= \|X^0 - X^* + F(X^0)\| \le \|X^0 - X^*\| + \|F(X^0)\| \\ &\le \|X^0 - X^*\| + \|F(X^0) - F(X^*)\| \\ &\le (1 + \bar{\varphi}_0(\|X^0 - X^*\|)\|X^0 - X^*\| \le \varphi_5(r) = R_0,\end{aligned}$$

so $W^0 \in U(X^*,\ r)$. Similarly, one have $\|X^* + \theta(X^0 - X^*) - X^*\| \le \theta\|X^0 - X^*\| < r$, so $X^* + \theta(X^0 - X^*) \in U(X^*,\ r)$. Using (5.2.19), (5.2.20) and (H_3), we get

$$\|\{F'(X^*)\}^{-1}(F'(X^0) - F'(X^*))\| \le \varphi_0(\|X - X^*\|) \le \varphi_0(r) < 1. \tag{5.2.28}$$

It follows from (5.2.28) and the Banach Lemma on invertible operators (1.2.14) that $\{F'(X^0)\}^{-1}$ exists and

$$\|\{F'(X^0)\}^{-1}F'(X^*)\| \le \frac{1}{1 - \varphi_0(\|X^0 - X^*\|)}. \tag{5.2.29}$$

In view of (H_3) one gets,

$$\begin{aligned}\|\{F'(X^*)\}^{-1}([W^0,\ X^0;\ F] - F'(X^*))\| &\le \varphi_2(\|W^0 - X^*\|,\ \|X^0 - X^*\|) \\ &\le \varphi_2(\varphi_5(r),\ r) < 1,\end{aligned} \tag{5.2.30}$$

so, $\{[W^0,\ X^0;\ F]\}^{-1}$ exists and

$$\|\{[W^0,\ X^0; F]\}^{-1}F'(X^*)\| \le \frac{1}{1 - \varphi_2(\|W^0 - X^*\|,\ \|X^0 - X^*\|)}. \tag{5.2.31}$$

Hence, Y^0 is well defined by the first sub step of method (5.2.19) for $k = 0$. One can write

$$\begin{aligned}Y^0 - X^* =& X^0 - X^* - \{F'(X^0)\}^{-1}F(X^0) + \{F'(X^0)\}^{-1}\Big([W^0,\ X^0;\ F] \\ &- F'(X^0)\Big)\{[W^0,\ X^0;\ F]\}^{-1}F(X^0).\end{aligned} \tag{5.2.32}$$

Then, by (5.2.19), (5.2.20), (5.2.21) (for $i = 1$), $(H_2) - (H_4)$ and (5.2.32), one obtain in turn that

$$\begin{aligned}\|Y^0 - X^*\| = & \Big\|\{F'(X^0)\}^{-1}F'(X^*)\Big\| \\ &\times \Big\| \int_0^1 \{F'(X^*)\}^{-1}\Big(F'(X^* + \theta(X^0 - X^*)) - F'(X^0)\Big)(X^0 - X^*)d\theta\Big\| \\ &+ \Big\|\{F'(X^0)\}^{-1}F'(X^*)\Big\| \Big(\Big\|\{F'(X^*)\}^{-1}([W^0,\ X^0;\ F] - F'(X^*))\Big\|\end{aligned}$$

$$
\begin{aligned}
&+ \big\|\{F'(X^*)\}^{-1}(F'(X^0) - F'(X^*))\big\| \Big) \\
&\times \big\|[W^0,\ X^0;\ F]^{-1}F'(X^*)\big\| \, \big\|\{F'(X^*)\}^{-1}F(X^0)\big\| \\
&\leq \frac{\int_0^1 \varphi((1-\theta)\|X^0 - X^*\|)d\theta\|X^0 - X^*\|}{1 - \varphi_0(\|X^0 - X^*\|)} \\
&+ \frac{1}{1 - \varphi_0(\|X^0 - X^*\|)}\Big[\varphi_2(\|W^0 - X^*\|,\ \|X^0 - X^*\|) + \varphi_0(\|X^0 - X^*\|)\Big] \\
&\times \frac{\int_0^1 \varphi_1(\theta\|X^0 - X^*\|)\|X^0 - X^*\|d\theta}{1 - \varphi_2(\|W^0 - X^*\|,\ \|X^0 - X^*\|)}
\end{aligned}
$$

$$
\|Y^0 - X^*\| = g_1(\|X^0 - X^*\|)\|X^0 - X^*\| \leq \|X^0 - X^*\| < r, \tag{5.2.33}
$$

which shows (5.2.24) for $k = 0$ and $Y^0 \in U(X^*,\ r)$. Let

$$
D^0 = d_1[Y^0,\ X^0;\ F] + d_2[W^0,\ X^0;\ F]^2\{[W^0,\ X^0;\ F]\}^{-1} + d_3[Y^0,\ X^0;\ F]^2\{[W^0,\ X^0;\ F]\}^{-1}. \tag{5.2.34}
$$

Then, one can obtain

$$
\tau^0 = D^0[W^0,\ X^0;\ F]. \tag{5.2.35}
$$

One must show that $\{D^0\}^{-1}$ exists. In view of (5.2.19), (5.2.20), (5.2.22), (H_3), (H_4), (5.2.33) and (5.2.34), one get in turn that

$$
\begin{aligned}
\|\{d_1F'(X^*)\}^{-1}(D^0 - d_1F'(X^*))\| &\leq \frac{1}{|d_1|}\big[|d_1|\|\{F'(X^*)\}^{-1}([Y^0,\ X^0;\ F] - F'(X^*))\| \\
&+ |d_2|\|\{F'(X^*)\}^{-1}[W^0,\ X^0;\ F]\|\|[W^0,\ X^0;\ F]\|\|\{[W^0,\ X^0;\ F]\}^{-1}\| \\
&+ |d_3|\|\{F'(X^*)\}^{-1}[Y^0,\ X^0;\ F]\|\|[Y^0,\ X^0;\ F]\|\|\{[W^0,\ X^0;\ F]\}^{-1}\|\big] \\
&\leq \frac{1}{|d_1|}\Big[|d_1|\varphi_2(\|Y^0 - X^*\|,\ \|X^0 - X^*\|) \\
&\quad + |d_2|\varphi_4(\|W^0 - X^*\|,\ \|X^0 - X^*\|) \\
&\quad \times \frac{\bar{\varphi}_4(\|W^0 - X^*\|,\ \|X^0 - X^*\|)\kappa}{1 - \kappa\bar{\varphi}_2(\|W^0 - X^*\|,\ \|X^0 - X^*\|)} \\
&\quad + \frac{|d_3|\varphi_4(\|Y^0 - X^*\|,\ \|X^0 - X^*\|)\bar{\varphi}_4(\|W^0 - X^*\|,\ \|X^0 - X^*\|)\kappa}{1 - \kappa\bar{\varphi}_2(\|Y^0 - X^*\|,\ \|X^0 - X^*\|)}\Big] \\
&\leq p(\|X^0 - X^*\|) \leq p(r) < 1,
\end{aligned} \tag{5.2.36}
$$

so,

$$
\|\{D^0\}^{-1}F'(X^*)\| \leq \frac{1}{|d_1|(1 - p(\|X^0 - X^*\|))}. \tag{5.2.37}
$$

It follows from (5.2.35) and (5.2.37) that $\{D^0\}^{-1}$ exists. By (5.2.28) and (5.2.37), one obtain that

$$\{\tau^0\}^{-1} = \{[W^0,\ X^0;\ F]\}^{-1}\{D^0\}^{-1} \tag{5.2.38}$$

and Z^0 is well defined by the second sub step of method (4.2.13) for $k = 0$. Taking in account of equations (5.2.19), (5.2.20), (5.2.21) (for $i = 2$), (H_3), (H_4), (5.2.37) and (5.2.38) one can achieve,

$$\begin{aligned}\|Z^0 - X^*\| \leq \|Y^0 - X^*\| + \Big[&4|\alpha - 1|\bar{\varphi}_4(\|Y^0 - X^*\|,\ \|X^0 - X^*\|)\\ &+ 2|2\alpha - 1|\bar{\varphi}_4(\|W^0 - X^*\|,\ \|X^0 - X^*\|)\Big]\\ &\times \Big(\kappa \int_0^1 \varphi_1(\theta\|Y^0 - X^*\|)\|Y^0 - X^*\| d\theta\\ &+ |d_1|(1 - p(\|X^0 - X^*\|))(1 - \kappa\bar{\varphi}_2(\|W^0 - X^*\|,\ \|X^0 - X^*\|))\Big)\\ &= g_2(\|X^0 - X^*\|)\|X^0 - X^*\| \leq \|X^0 - X^*\|,\end{aligned} \tag{5.2.39}$$

which implies (5.2.25) for $k = 0$ and $Z^0 \in U(X^*,\ r)$.
Let

$$A^0 = [Z^0,\ Y^0;\ F] + [Z^0,\ Y^0,\ X^0;\ F](Z^0 - Y^0). \tag{5.2.40}$$

We need to show that $\{A^0\}^{-1}$ exists. By (5.2.19), (5.2.20), (5.2.23), (H_3), (H_4), (5.2.33), (5.2.39) and (5.2.40), one get in turn that

$$\begin{aligned}\|\{F'(X^*)\}^{-1}(A^0 - F'(X^*))\| &= \|\{F'(X^*)\}^{-1}([Z^0,\ Y^0;\ F] - F'(X^*))\\ &\quad + \{F'(X^*)\}^{-1}([Z^0,\ Y^0,\ X^0; F]((Z^0 - X^*) + (X^* - Y^0)))\|\\ &\leq \varphi_2(\|Z^0 - X^*\|,\ \|Y^0 - X^*\|)\\ &\quad + \varphi_3(\|Z^0 - X^*\|,\ \|Y^0 - X^*\|,\ \|X^0 - X^*\|)\\ &\quad \times (\|Z^0 - X^*\| + \|Y^0 - X^*\|)\\ &= q(\|X^0 - X^*\|) \leq q(r) < 1,\end{aligned} \tag{5.2.41}$$

so,

$$\|\{A^0\}^{-1}F'(X^*)\| \leq \frac{1}{1 - q(\|X^0 - X^*\|)} \tag{5.2.42}$$

and $U^0,\ X^1$ are well defined by last two sub steps of method (5.2.1). Then, by (5.2.19), (5.2.20), (5.2.21) (for $i = 3, 4$), (H_3) and (H_4), (5.2.33), (5.2.39) and (5.2.42), one obtains,

$$\begin{aligned}\|U^0 - X^*\| &\leq \|Z^0 - X^*\| + \|\{A^0\}^{-1}F'(X^*)\|\|\{F'(X^*)\}^{-1}F(Z^0)\| \\ &\leq \|Z^0 - X^*\| + \frac{\int_0^1 \varphi_1(\theta\|Z^0 - X^*\|)\|Z^0 - X^*\|d\theta}{1 - q(\|X^0 - X^*\|)} \\ &\leq \left(1 + \frac{\int_0^1 \varphi_1(\theta g_2(\|X^0 - X^*\|)\|Z^0 - X^*\|)d\theta}{1 - q(\|X^0 - X^*\|)}\right) g_2(\|X^0 - X^*\|)\|X^0 - X^*\| \\ &= g_3(\|X^0 - X^*\|)\|X^0 - X^*\| \leq \|X^0 - X^*\| < r,\end{aligned} \tag{5.2.43}$$

and

$$\begin{aligned}&\|X^1 - X^*\| = \|U^0 - X^*\| + \|\{A^0\}^{-1}F'(X^*)\| \left[\|\{F'(X^*)\}^{-1}F(Z^0)\| + \|\{F'(X^*)\}^{-1}F(U^0)\|\right] \\ &\leq g_3(\|X^0 - X^*\|)\|X^0 - X^*\| \\ &+ \frac{\varphi_1(g_2(\|X^0 - X^*\|)\|X^0 - X^*\|)g_2(\|X^0 - X^*\|)\|X^0 - X^*\| + \int_0^1 \varphi_1(\theta g_3(\|X^0 - X^*\|)\|X^0 - X^*\|)d\theta}{1 - q(\|X^0 - X^*\|)} \\ &\times g_3(\|X^0 - X^*\|)\|X^0 - X^*\| \\ &= g_4(\|X^0 - X^*\|)\|X^0 - X^*\| \leq \|X^0 - X^*\| < r,\end{aligned} \tag{5.2.44}$$

which implies (5.2.26), (5.2.27) hold and $U^0,\ X^1 \in U(X^*,\ r)$. By simply replacing X^0, $Y^0,\ Z^0,\ U^0, X^1$ by $X^k, Y^k,\ Z^k,\ U^k, X^{k+1}$ in the preceding estimates we arrive at (5.2.24)–(5.2.27). Using the estimate

$$\|X^{k+1} - X^*\| \leq c\|X^k - X^*\| < r,\ c = g_4(\|X^0 - X^*\|) \in [0,\ 1), \tag{5.2.45}$$

one deduce that $\lim_{k\to\infty} X^k = X^*$ and $X^{k+1} \in U(X^*,\ r)$. Finally, to show the uniqueness part, let $Y^* \in D_1$ with $F(Y^*) = 0$.
Define $Q = \int_0^1 F'(X^* + \theta(X^* - Y^*))d\theta$. Using (H_3), one gets

$$\begin{aligned}\|F'(X^*)^{-1}(Q - F'(X^*))\| &\leq \| \int_0^1 \varphi_0(\theta\|Y^* - X^*\|)d\theta \\ &\leq \int_0^1 \varphi_0(\theta R)d\theta < 1.\end{aligned} \tag{5.2.46}$$

It follows from (5.2.46) that Q is invertible. Then, in view of the identity

$$0 = F(X^*) - F(Y^*) = Q(X^* - Y^*), \tag{5.2.47}$$

one can conclude that $X^* = Y^*$. □

The application of of Theorem 5.2.1 in a Banach space setting is shown through the following example.

Example 5.2.1 *Let $X = Y = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$ equipped with the max norm. Let $D = \bar{U}(0,1)$. Define*

$$F(\mu)(x) = \delta\mu(x) - \int_0^1 x\theta\mu(\theta)^3 d\theta, \text{ for some } \delta \in \mathbb{R} - \{0\}. \tag{5.2.48}$$

One can have that

$$F'(\mu(\lambda))(x) = \delta\lambda(x) - 3\int_0^1 x\theta\mu(\theta)^2\lambda(\theta)d\theta \text{ for each } \lambda \in D. \tag{5.2.49}$$

Then, we have that $x^* = 0$. Using the divided difference as defined by $[X, Y, ; F] = \frac{1}{2}\big(F'(X) + F'(Y)\big)$, (H_3), (H_4) and (5.2.49), we can choose: $\kappa = \frac{1}{\delta}$, $\varphi_0(t) = \frac{3t}{2\delta}$, $\bar{\varphi}_0(t) = \frac{3t}{2}$, $\varphi_2(s,t) = \frac{\varphi_0(t)+\varphi_0(s)}{2\delta}$, $\bar{\varphi}_2(s,t) = \frac{\varphi_0(t)+\varphi_0(s)}{2}$, $\varphi(t) = \frac{3t}{\delta}$, $\varphi_1(t) = 1+\varphi_0(t)$, $\varphi_3(s_1,s_2,s_3) = \frac{3}{\delta}$ and $\varphi_4(s,t) = \frac{\varphi_1(t)+\varphi_1(s)}{2\delta}$, $\bar{\varphi}_4(s,t) = \frac{\varphi_1(t)+\varphi_1(s)}{2}$. Notice that we can choose any value of δ, α and β, provided (5.2.12) should satisfy. Here, by choosing $\delta = 2, \alpha = \beta = 1$ in the expression (5.2.12), we obtain $0.5 < 1$. This mean that the expression (5.2.12) is satisfied. Then, the parameters using method (4.2.13) are:

$$r_1 = 0.297317,\ r_2 = 0.252807,\ r_3 = 0.145856,\ r_4 = 0.0768941,\ r_q = 0.2736,$$

so,

$$r = 0.0768941.$$

5.3 Conclusions

A local convergence of a family of higher order iterative methods (5.2.1) for solving non-linear equations in Banach spaces is established under the assumption that the Fréchet derivative satisfies the Lipschitz continuity condition. This method is of order ten for any real value of parameter α. The convergence results of the proposed scheme is established from the existence and uniqueness theorem. Moreover, the initial approximation for solving non-linear systems can be obtained from the convergence ball of the prescribed radii as discussed in numerical section. In this way, local convergence analysis of the proposed technique expands the applicability of method in Banach setting.

Chapter 6

Future Scope

A lot of problems related to the solutions of non-linear systems are brought forward in many applied sciences and engineering applications. Generally, the solutions of a non-linear system cannot be found through analytical methods, thus iterative schemes for approximating solutions of systems of nonlinear equations are the most frequently used techniques. Therefore, this topic has always been of paramount importance in computational mathematics to approximate the solutions of non-linear system.

During the progress of the present study on the construction of iterative schemes, several interesting topics have come up to our attention. However, it is not possible to pursue all these within the research framework of this thesis. Hence, in the concluding chapter, some of them will be proposed as a focus for further research. In the present work, mainly the iterative schemes without memory with Traub approach have been constructed for solution of non-linear systems. It will be worthwhile to extend the present methods with memory, so that methods become more efficient with same computational cost. In the present investigation, proposed methods are applied on some differential problems like Bratu in one dimensional and two dimensional case, Fisher's problem etc. In future work, iterative methods for systems of non-linear equations may be constructed that can be applied to many engineering problem like electrical engineering problem Load flow analysis (or Power flow). It has been noted that the some iterative schemes has been constructing by introducing some scalar parameters. With different values of these parameters different iterative methods are introduced to find the efficient approximate solution of non-linear systems. In the published articles, the researcher has also achieved efficient higher order multi-step iterative methods with scalar parameters for systems of non-linear equations. In author's opinion, another interesting task is to find the range of these scalar parameters for which iterative schemes will be more efficient. Since, most of the practical non-linear systems does not have exact solution, so how to approximate their solution is another difficult task. Also the initial guess plays a major role for the convergence of iterative schemes, but there is no way to choose the initial value when a problem has no exact solution. Some other issues that can be considered for future researchers for the construction of iterative methods for non-linear systems are stability, applicability of techniques in those practical problems which have discontinuity at some points and multiple solutions of systems of non-linear equations.

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