Approximation by Certain Linear Convergence Techniques

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in

MATHEMATICS

by

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under the supervision of

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to

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Certificate

This is to certify that the thesis titled *"Approximation by Certain Linear Convergence Techniques"* which is being submitted by Ms. Jaspreet Kaur, in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is a record of the candidate's own independent and original research work carried out under my supervision. The matter embodied in this thesis has not been submitted in part or full to any university or institute for the award of a degree.

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Declaration

I hereby declare that thesis titled *"Approximation by Certain Linear Convergence Techniques"* submitted for the award of the degree of Doctor of Philosophy in the Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Meenu Rani, Assistant Professor at Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or abroad and that the ideas and references cited herein have been duly acknowledged.

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List of Publications

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Abstract

The present thesis titled as "Approximation by Certain Linear Convergence Techniques" involves the construction of new positive linear operators and examines their approximation properties as well as their applications in multidisciplinary fields. The thesis is divided into eight chapters that majorly covers three important aspects of approximation theory.

Firstly, we give the brief introduction of approximation theory and the basic results that are the inspiration for the development of this theory. We also provide basic definitions as well as approximation tools such as moduli of continuity, modulus of smoothness, Peetre's K-functional for the univariate operators, complete and partial modulus of continuities for function of two variables to check the convergence of positive linear operators for the given function. Also, we list the brief account of the related work of various authors for the positive linear operators and found some gaps according which the research work in this thesis has been carried out.

The first direction of the thesis is to focus on the order of approximation of existing positive linear operators. We know the convergence and the order of approximation of the positive linear operators are two important attributes in approximation theory to approximate any function. Using the well-known Korovkin theorem, we can easily verify the convergence of these operators but improving the order of approximation is a critical attribute. We describe a recursive method designed to improve the approximation results of known operators, which results in increased accuracy and efficiency. We presented three modifications of α-Bernstein Păltănea operators with linear, quadratic and cubic order of approximation and study some approximation results concerning the rate of convergence, error estimation and Voronovskaja type formulas for the new modifications.

The next aspect for the thesis is to introduce new positive linear operators that outperform classical operators in terms of approximation properties. These new operators are designed to approximate functions effectively on both finite and infinite domains by using certain parameters to introduce the flexibility in these positive linear operators. It helps to increase the applicability and utility of positive linear operators in various mathematical and engineering contexts.

We define the bivariate operators as well as Generalized Boolean Sum (GBS) operators associated with these operators for the first order modification of α-Bernstein Păltănea operators. In order to approximate Lebesgue integrable functions, we introduce the Durrmeyer-variant of Lupaş type operators by using Pochhammer k-symbol in one as well as two dimensional space. Also, we give the univariate and bivariate versions of the Bernstein-Lototsky operators that are able to preserve any polynomial with some certain conditions by introducing a real parameter $\rho > 0$. We define the new operators to approximate integrable functions by using α-Baskakov operators and a non-negative parameter defined on infinite domain. We study the approximation results of these operators by using well-known tools of the approximation theory including convergence and error estimates in terms of moduli of continuity. The results of all these positive linear operators have been verified by graphical illustrations for certain examples. Also, the introduced bivariate versions of the operators can be extended to approximate the functions of several variables. This feature is especially beneficial in real-world situations when functions rely on more than one variable.

In the last part of the thesis, we introduce the applications of positive linear operators in the realm of Bézier curves. With the widespread use of computers in all industries, these curves have become critical to study. These curves utilize positive linear operators such as Bernstein and their generalizations. We can include some parameters to obtain greater control over these curves. These factors assist in reducing time and expense while improving the curves' accuracy and adaptability.

We generalize the Bézier curves by using two parameters to get the better control on shape of the curves. Firstly, we construct the generalized Bézier curves and Bézier surfaces depending upon the parameter α. Secondly, we explore the applications of q-calculus in polynomial basis functions and curve modeling. We define the q-variant of Bernstein-Chlodowsky basis polynomials and introduce generalization of Bézier curves by utilizing these basis polynomials. We study the properties of these curves and surfaces and show that the introduced parameter provides us the flexibility to modify the curves as well as surfaces by giving some numerical examples with the help of MATLAB. Also, we provide an exact approach to calculate the control points for the given α-Bézier curve.

In summary, the thesis makes substantial contributions to the field of approximation theory by improving existing approximation techniques, introducing new operators with superior properties, and extending the applications of these operators to the construction and modification of Bézier curves and surfaces. These advancements hold promise for enhancing various practical applications in computational mathematics, computer graphics, and related areas.

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Notations

$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{R}$	the set of real numbers,
$\mathbb{C}$	the set of complex numbers,
$\mathbb{R}_+$	the set of positive real numbers,
$[a,b]$	a closed interval,
(a,b)	an open interval,
$C[a,b]$	the set of all real-valued and continuous functions defined on compact interval $[a,b]$,
$C^r[a,b]$	the set of all real-valued, r-times continuously differentiable functions on $[a,b]$ $(r \in \mathbb{N})$,
$Lip^{\tau}_{\mathscr{M}}$	the set of all $C[a,b]$ functions that verify the Lipschitz condition $\lvert f(x_2)-f(x_1)\rvert \leq \mathscr{M}\lvert x_2-x_1\rvert^{\tau}$, for all $x_1, x_2 \in [a,b], 0 < r \leq 1, \mathscr{M} > 0$,
e_n	denotes the nth order monomial with $e_n : [a,b] \to \mathbb{R}, e_n(t) = t^n, n \in \mathbb{N}_0$,
$\phi^r_{\mathrm{t}}(x)$	denotes the rth order central moment where $\phi^r_{\mathrm{t}}(x) = (\mathrm{t}-x)^r, r \in \mathbb{N}_0$,
$(x)_n$	the rising factorial $(x)_n := x(x+1)\cdots(x+n-1)$ and $(x)_0 := 1$,
$(v)_{n,k}$	k-Pochhammar symbol $(v)_{n,k} = v(v+k)\cdots(v+k(n-1))$,
$C[0,\infty)$	the set of all continuous functions defined on $[0,\infty)$,
$C_B[0,\infty)$	the set of all continuous bounded functions on $[0,\infty)$,
$\exists$	there exists,
$\forall$	for all,
i.e.	that is,
resp.	respectively,

w.r.t.	with respect to,
$\| . \|$	norm,
$A \backslash B$	$\{x : x \in A, x \notin B\}$,
g'_x	first order derivative of g w.r.t. x,
g''_{xy}	second order derivative of g w.r.t. x and y.

Contents

Chapter 1

Introduction

1.1 Introduction

Approximation theory is an important bridge between pure and applied mathematics since it consists of a theoretical study of methods that use numerical approximation to solve problems of mathematical analysis by means of computational algorithms and computer simulation. After more than a century from the seminal works of Bernstein and Chebyshev (among others), approximation theory now becomes a very extensive branch of Mathematics, interlacing with various other scientific fields. In fact, it plays a central role in the analysis of numerical methods for Mathematical, Physical, Medical, Engineering, and Social Sciences and provides directions for future research. For example, polynomial approximation is the basis for the study of Gaussian rules, splines and radial basis functions. Interpolation is an important tool in the geometric design of automotive and aerospatial vessels, while wavelets and their generalizations are used for the compression of large digital images and videos. Recently, approximation methods have been applied to the construction of numerical methods for integral equations, partial differential equations, fractional calculus, signal theory and deep learning. In general, the problem we solve, is called input information and the corresponding result is output information. The process of transforming input into output, is called an algorithm. To find the output information, we generally have two concepts, first is interpolation and the second is approximation. Now, we give a brief description of the difference between both.

By interpolation, we come to functions, that pass exactly through all given points, and we use it for a small amount of input data. We choose the function f according to the nature of the model, but so that it is relatively simple to calculate. These are most often polynomials, trigonometric functions, exponential functions, and more recently, rational functions. In practice, it has been shown that it is not wise to use polynomials of degree greater than three for interpolation, because for some functions an increase in the degree of an interpolation polynomial can lead to an increase in errors.

By approximation, we arrive at functions that pass through a group of data in the best possible way, without the obligation to pass exactly through the given points. The approximation is suitable for small and large groups of scattered data. Approximation problems occur in two forms. One in which we know the function f, but its form is complicated to compute. The error of the obtained approximation can be estimated with respect to the true value of the function. Secondly, the function f is unknown to us, but only some information about f is known. For example, values at a set of points are known. The substitution function ξ is determined from the available information, which, in addition to the data itself, includes the expected form of data behavior of the function ξ. In this case, we cannot make an error estimate without additional information about the unknown function f. In practice, we often get the second variant that the function f is not given to us.

The first significant results of approximation theory were proved by Karl Weierstrass (1815-1897), in 1885, which are the density of algebraic polynomials in the class of real-valued continuous functions on a compact interval, and the density of trigonometric polynomials in the class of 2π-periodic continuous real-valued functions. Such results were a counterbalance to Weierstrass' famous example of 1861 on the existence of a continuous nowhere differentiable function. While on the one hand, the set of continuous functions contains non-smooth functions. On the other hand, every continuous function can be approximated arbitrarily well by smooth functions, which are polynomials. This result is stated below:

Theorem 1.1.1. *Weierstrass' First Approximation Theorem: For any arbitrarily* *assumed* $f(x) \in C[a,b]$*, where* $\epsilon > 0$*, a polynomial* $P(x)$ *exists of such type that for every* $x \in [a,b]$*, the following inequality holds*

$$\mid P(x) - f(x) \mid < \epsilon.$$

This theorem is proved by many researchers, we list some of them: Picard [121], Carl Runge [136, 137], Henri Lebesgue [99], Edmund Landau [98], Charles de la Vallée-Pousin [144], Lipot Fejér, Mittag Leffler [109] but one of them is most famous, known as S. N. Bernstein. He proved the theorem that is stated below:

Theorem 1.1.2. *[36] If* $f(x)$ *is continuous in the segment* $[0,1]$*, then in relation to* x

$$\lim_{n \to \infty} B_n(f;x) = f(x)$$

holds uniformly in this segment, where

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)\, f\left(\frac{k}{n}\right),$$

and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ *are commonly known as Bernstein basis polynomials or B-basis polynomials.*

Bernstein has not only proved the Weierstrass theorem, but gave us the idea to construct the polynomials to approximate a given continuous function.

By Weierstrass theorem, we observe that polynomials can uniformly approximate any function that is merely continuous over a closed interval. This represents a significant advance over using the Taylor's series expansion to generate polynomial approximations of a function in two important aspects: (a) the function need not be analytic (not differentiable); and (b) the interval $[a,b]$ can be freely specified, whereas the Taylor's series must be confined within its radius of convergence, which can be difficult to compute.

The properties, that make Bernstein basis polynomials useful, are defined as:

1. **Basis for polynomial space:** The B-basis polynomials of degree n forms a

basis for P_n, where P_n denotes the space of all polynomials having at most n degree.

2. **Symmetrical:** The B-basis polynomials satisfy symmetry property, which is given by
$$p_{n,k}(x) = p_{n,n-k}(1-x), \quad k = 0, 1, ..., n, \forall\, x \in [0,1].$$
3. **Positivity:** The B-basis polynomials are all positive over $[0,1]$, *i.e.*
$$p_{n,k}(x) \geq 0, \quad \forall\, x \in [0,1].$$
4. **Partition of unity:** The B-basis polynomials form a partition of unity
$$\sum_{k=0}^{n} p_{n,k}(x) = 1, \quad \forall\, x \in [0,1].$$
5. **Recursion formula:** The B-basis polynomials satisfy the following recursion relation:
$$p_{n,k}(x) = (1-x)\, p_{n-1,k}(x) + x\, p_{n-1,k-1}(x).$$

Since the change of variables specified by $t = \dfrac{x-a}{b-a}$ maps the interval $[a,b]$ to $[0,1]$ without changing the max norm of any function, we can restrict our attention to continuous functions f on $[0,1]$ without loss of generality. It should be noted that, in general, the polynomial approximant does not interpolate the sampled values, that is $B_n(f;x_k) \neq f\left(x_k\right), k = 1, 2, \cdots, n-1$, but they satisfy end point interpolation property that is $B_n(f;x_0) = f(x_0)$ and $B_n(f;x_n) = f(x_n)$. Due to the above properties of B-basis polynomials, Bernstein operators $B_n(f;x)$ have many interesting properties. Their ability to exactly reproduce linear (or constant) functions is called the linear precision property of the Bernstein approximation.

Simultaneous approximation: The Bernstein polynomial approximant to a given function $f(x)$ is always at least as smooth as $f(x)$. If $f(x) \in C^r$ rather than just C^0 continuity, all derivatives of $B_n(f;x)$ up to order r converge uniformly to the corresponding derivatives of $f(x)$. If bounds on the derivatives of $f(x)$ of each order over $[0,1]$ exist, then the corresponding derivatives of $B_n(f;x)$ also satisfy those bounds. For example, if $f(x)$ is monotone or convex, $B_n(f;x)$ is correspondingly monotone or convex.

From the convergence point of view, the major contribution in the field of approximation theory is given by Korovkin [17]. He proved a simple theorem to verify the uniform convergence of positive linear operators to a continuous function, which is given as:

Theorem 1.1.3. Bohman-Korovkin theorem

Let $(L_n)_{n\geq 1}$ *be a sequence of positive linear operators such that for every* $g \in \{e_0, e_1, e_2\}$

$$\lim_{n\to\infty} L_n(g) = g \qquad \textit{uniformly on} \qquad [a,b].$$

Then for any function $h \in C[a,b]$, *we have*

$$\lim_{n\to\infty} L_n(h) = h \qquad \textit{uniformly on} \qquad [a,b].$$

This theorem gives the importance of the monomials $e_\imath$ for $\imath = 0,1,2$ in the space of real valued continuous functions. These functions are called Korovkin test functions. This theorem inspired many researchers to work in this direction for different spaces. This study becomes so extensive such that it is called Korovkin type approximation theory. A detailed discussion of this topic can be seen in [18].

Polynomials are widely used in computational models of scientific or engineering problems, because of their properties such as finite evaluation schemes, closure under addition, multiplication, differentiation, integration, composition, and their ability to approximate functions that have no closed-form expressions. The ability of polynomials to approximate any continuous function to any desired accuracy over a prescribed interval motivated the introduction of the B-basis polynomials. Their slow convergence rate and the lack of digital computers to construct them efficiently, caused the Bernstein polynomials to lie dormant in the theory rather than practice of approximation for the better part of a century. With the advent of computers, Bernstein basis polynomials found its true vocation not in approximation of functions by polynomials, but also in exploiting computers to interactively design (vector-valued) polynomial functions like parametric curves and surfaces. In this context, it became apparent that the Bernstein coefficients of a polynomial provide valuable insight into its behavior over a given finite interval, yielding many useful properties and elegant algorithms that are now being increasingly adopted in other application domains.

1.2 Literature Survey

Since the Bernstein operators enjoy remarkable properties, but they are defined only on finite interval $[0,1]$. Chlodowsky [51] introduced the classical Bernstein-Chlodowsky operators, which extended the domain from $[0,1]$ to $[0,d_n]$, such that $d_n > 0, \forall n \in \mathbb{N}$ satisfying $\lim\limits_{n\to\infty} d_n = \infty$ and $\lim\limits_{n\to\infty} \frac{d_n}{n} = 0$. To approximate a continuous bounded function on positive real line, in 1950, Szász [143] generalized the Bernstein polynomials for bounded and continuous functions on $[0,\infty)$, denoted by $f \in C_B[0,\infty)$, as:

$$\begin{aligned} S_n(f;x) &= \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty) \\ \text{where} \quad s_{n,k}(x) &= e^{-nx}\frac{(nx)^k}{k!}. \end{aligned}$$

These operators follow the well known Poisson distribution, whereas Bernstein operators follow the binomial distribution.

Similarly, in 1957, Baskakov [34] introduced positive linear operators in the following form:

$$V_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0,\infty). \tag{1.1}$$

These operators had expanded the domain of positive linear operators from the functions defined on $[0,1]$ to functions defined on an unbounded interval that enables us to approximate a wide range of functions by using these operators. Thus, a variety of research is done on these operators which is still going on and can be seen from the references [11, 15, 66, 69, 118, 153]. In 1930, Kantorovich [86] modified the Bernstein operators by replacing $f\left(\frac{k}{n}\right)$ with the integral mean of $f(x)$ from the interval $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$. In 1967, Durrmeyer [58] proposed the generalization of Bernstein operators to approximate Lebesgue integrable functions, as

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(\mathrm{t})\, f(\mathrm{t})\mathrm{dt}, \quad x \in [0,1]. \tag{1.2}$$

Derriennic [53] found the direct results for these operators (1.2) in ordinary and simultaneous approximation. She studied the convergence of these operators, Voronovskaja type asymptotic results as well as the convergence of its derivatives. Thus, these operators have enlarged the class of functions that can be approximated by positive linear operators. The usefulness of the Durrmeyer variant of Bernstein operators attracted the attention of a significant number of authors (see [2, 3, 9, 53, 70, 82, 110, 139]). Acar *et al.* [4] defined the Durrmeyer variant for the mobile interval of $[0,1]$ and presented its local and global approximation properties. Gal and Gupta [63] defined the Durrmeyer operators to approximate analytic functions. Păltănea [130] generalized the Durrmeyer type operators with the help of a parameter $\rho > 0$. Ansari *et al.* [21] studied the approximation and error estimation properties by modified Păltănea operators for Gould-Hopper polynomials. From the applications of parametric generalizations, many other researchers have worked in this direction (see [7, 22, 112]). In 1985, Kasana *et al.* [90] and Mazhar and Totik [105] independently obtained the Durrmeyer-type modification of Szász-Mirakyan operators to approximate Lebesgue integrable functions. Similarly, Gupta *et al.* [72] modified Szász operators by using weights of beta basis function and studied their approximation results also. In 1987, Lupaş [100] introduced the q-analogue of the Bernstein operators and derived its approximation properties. But Lupaş q-Bernstein operators are rational functions rather than polynomials. In 1997, Phillips [120] presented another generalization of Bernstein polynomials in q-analogue for $0 \leq q \leq 1$ and studied its approximation results including convergence and Voronovskaja type formula. In 2010, Mahmudov

[102] introduced the q-Szász-Mirakjan operators where $0 < q < 1$ and studied their approximation properties. In 2012, he [103] also presented another generalization of Szász-Mirakjan operators based on q parameter where $q > 1$, in which he showed that this generalization has better approximation properties than the classical Szász-Mirakjan operators as well as q-Szász-Mirakjan operators for $0 < q < 1$. From [87–89] and references in these articles, we can get the importance of parameter q in the generalizations of positive linear operators.

In 2017, Chen *et al.* [50] generalized the Bernstein polynomials depending on a real parameter $\alpha \in [0, 1]$, as below:

$$T_{n,\alpha}(f; x) = \sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\, f\left(\frac{k}{n}\right), \tag{1.3}$$

$$\text{where } p_{n,k}^{\alpha}(x) = \left\{\binom{n-2}{k}(1-\alpha)\, x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha\, x(1-x)\right\} \times x^{k-1}(1-x)^{n-k-1}. \tag{1.4}$$

The authors studied elementary properties as linearity, positivity and end-point interpolation and its approximation results. They observed that convergence of these operators is independent of α, whereas the upper bound of the error of approximation is dependent on the parameter. In 2019, Deo and Pratap [52] presented the Kantorovich variant of α-Bernstein operators and studied the direct approximation theorem, asymptotic results for these operators. Recently, Kajla and Acar [81] introduced the Durrmeyer modification of the summation operators (1.3) and studied the rate of convergence and some approximation properties. Kajla and Goyal [85] modified the Durrmeyer variant of these operators by using Păltănea basis function in an integral depending on a parameter $\rho > 0$, as

$$Q_{n,\rho}^{\alpha}(g; x) = \sum_{j=0}^{n} p_{n,j}^{\alpha}(x) \int_0^1 \mu_{n,j}^{\rho}(\mathrm{t}) g(\mathrm{t})\mathrm{dt}, \tag{1.5}$$

where

$$\mu_{n,j}^{\rho}(\mathrm{t}) = \frac{\mathrm{t}^{j\rho}(1-\mathrm{t})^{(n-j)\rho}}{B(j\rho+1,(n-j)\rho+1)}, \tag{1.6}$$

and $B(i,j)$ is beta function. They studied the approximation properties, asymptotic behavior and the order of convergence of these operators. Mihesan [108] introduced the following generalized Baskakov operators with a constant $a > 0$ restrained from n as

$$V_n^a(f;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x)\, f\left(\frac{k}{n}\right), \tag{1.7}$$

$$\text{where} \qquad W_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \quad \text{such that} \quad \sum_{k=0}^{\infty} W_{n,k}^a(x) = 1$$

$$\text{and} \qquad P_k(n,a) = \sum_{\imath=0}^{n} \binom{k}{\imath} (n)_\imath a^{k-\imath},$$

where $(n)_\imath$ denotes the rising factorial. He obtained a pointwise estimate and proved that these operators converge uniformly on $[0,b]$ for functions having exponential growth.

In 2018, Erençin [60] modified the operators (1.7) to approximate the Lebesgue integrable functions $f \in C_B[0,\infty)$ and derived some direct results. In [25], Aral and Gupta investigated the generalization of these operators based on parameter q. The authors proved its shape preserving properties and estimated its convergence rate in the weighted norm. In 2019, Aral and Erbay [23] generalized the Baskakov operators depending on a real parameter $\alpha \in [0,1]$ as

$$L_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} q_{n,k}^{\alpha}(x)\, f\left(\frac{k}{n}\right), \quad n \geq 1, \quad x \in [0,\infty) \tag{1.8}$$

where $q_{n,k}^{\alpha}(x)$ is given by

$$\begin{aligned} q_{n,k}^{\alpha}(x) = & \frac{x^{k-1}}{(1+x)^{n+k-1}} \left\{ \frac{\alpha x}{1+x} \binom{n+k-1}{k} - (1-\alpha)(1+x) \binom{n+k-3}{k-2} \right. \\ & \left. + (1-\alpha)x \binom{n+k-1}{k} \right\}. \end{aligned} \tag{1.9}$$

The authors studied convergence, error of approximation and Voronovskaja type result for these operators. They verified that the parameter α does not affect the convergence but it effects the error of approximation for these operators. Recently, Nasiruzzaman *et al.* [114] defined the Durrmeyer variant of the operators (1.8) to extend the results for Lebesgue integral functions, called α-Baskakov Durrmeyer operators

$$S_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} q_{n,k}^{\alpha}(x) \int_0^{\infty} b_{n,k}(\mathrm{t}) f(\mathrm{t})\, \mathrm{dt}, \tag{1.10}$$
$$\text{and } b_{n,k}(\mathrm{t}) = \frac{\mathrm{t}^k}{B(k+1,n)(1+\mathrm{t})^{n+k+1}}.$$

The authors studied the order of approximation, rate of convergence, Korovkin-type and weighted Korovkin-type approximation theorems for these operators.

The convergence of positive linear operators is an important aspect. Bernstein operators as well as other positive linear operators converge to the given function. For practical purposes, we do not only require the uniform convergence, but the speed of convergence is more important. The first example of saturation of Bernstein operators is given by Voronovskaja. He proved that if for any bounded function f on $[0,1]$ such that $f''(x)$ exists at $(0,1)$, we have

$$\lim_{n\to\infty} n[B_n(f;x) - f(x)] = \frac{x(1-x)}{2} f''(x).$$

It states that if $f''(x) \neq 0$, then order of difference $B_n(f;x) - f(x)$ is $\dfrac{1}{n}$. We can conclude that the existence of higher order derivatives of $f(x)$ does not contribute to the order of approximation. This statement is true for some other positive linear operators also. The rate of convergence of these generalizations is slow, it motivates the researchers to find the new operators whose order is better than the Bernstein type operators. Different approaches are available to improve their order of convergence. Bernstein [37] defined the new positive linear operators as

$$Q_n(f;x) = \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - \frac{x(1-x)}{2n} f''\left(\frac{k}{n}\right) \right] p_{n,k}(x).$$

He proved that if $f^{(4)}(x)$ exists at $x \in (0,1)$, then

$$\lim_{n\to\infty} n^2 \left[Q_n(f;x) - f(x)\right] = \frac{1}{6}x(1-2x)(1-x)f^{(3)}(x) - \frac{1}{8}x^2(1-x)^2 f^{(4)}(x).$$

From the above estimate, it is clear that order of the approximation is improved from $\frac{1}{n}$ to $\frac{1}{n^2}$. To obtain this order, we need the values of the $f''(x)$ at discrete points $\frac{k}{n}$, which is a difficult task.

In order to get rid of this situation, Butzer [43] introduced another approach to improve the order of approximation in which he used the linear combinations of Bernstein operators. The recursive equation is given by

$$(2^r - 1)B_n(f,r;x) = 2^r B_{2n}(f,r-1;x) - B_n(f,r-1;x),$$

where $B_n(f,0;x) = B_n(f;x)$. The explicit form of these operators is given as

$$B_n(f,r;x) = \sum_{\imath=0}^{r} \prod_{k\neq\imath} \frac{2^\imath}{2^\imath - 2^k} B_{2^\imath n}(f;x).$$

The author proved that for smooth functions, the term $B_n(f,r;x) - f(x)$ tends to zero faster than $B_n(f;x) - f(x)$.

Let $f(x)$ be bounded such that $f^{(6)}(x)$ exists at $x \in (0,1)$ and $r = 2$, then the rate of convergence for these operators is $\frac{1}{n^3}$ which is much better than $\frac{1}{n}$. In this technique, the number of sample points and the degree of corresponding polynomial increase exponentially as r increases.

Micchelli [106] presented another procedure in which he used iterative combinations of Bernstein operators. Recently, Khosravian-Arab *et al.* [94] propounded another process for improving the order of approximation by perturbing the recurrence formula satisfied by Bernstein polynomials. Using this new approach, many operators have been modified in a very short period of time as we can see [2, 8, 74, 83].

In order to approximate the functions of two and several variables, the initialization of new positive linear operators defined in two as well as several dimensions has begun in the approximation theory. Kingsley [97] introduced the bivariate Bernstein polynomials and studied its approximation properties. Some properties of these operators are studied by Butzer [44]. Stancu [142] defined the positive linear operators to approximate the functions of two as well as several variables defined on a triangle. In [28], Bărbosu obtained the results of bivariate extension of Stancu generalization of q-Bernstein operators. Following this, a lot of articles can be seen on the bivariate extensions of Bernstein type operators [67, 84, 134] and modified Szász-Mirakyan and Baskakov operators [19, 49, 61, 75, 135, 140]. Extending this research, Wafi and Khatoon [147] modified the generalized Baskakov operators for the functions of two variables in polynomial as well as exponential weighted spaces. Agrawal and Goyal [10] presented the bivariate extensions of the different operators in their work, which contains discrete and summation integral type operators.

With the bivariate operators, there are other associated operators called Generalized Boolean Sum (GBS) operators. Bögel [40, 41] in 1934 and 1935 introduced the concept of Bögel continuous and Bögel differentiable functions. The approximation results concerning these functions were firstly introduced by Dobrescu and Matei [57]. Badea and Cottin [27] proved the Korovkin theorem for B-continuous functions which is also famous as test function theorem. In 2013, Miclăuş [107] studied the approximation results of GBS Bernstein-Stancu operators. Agrawal and İspir [12] studied bivariate Chlodowsky-Szász-Charlier type operators and its degree of approximation. Similarly, Bărbosu *et al.* [32] proposed the q-variant of GBS Durrmeyer operators and studied convergence and degree of approximation of these variants. The importance of these operators can be seen by the ongoing research as [13, 29–31, 33, 65, 122–128, 146].

One of the key feature of the Bernstein operators and its generalizations is that these operators replicate the affine functions. But for specific activities, we need the reproduction of some additional functions too. King [96] was the first to generalize Bernstein operators that preserve 1 and x^2 in a non-trivial way. Additionally, he demonstrated that these operators provide a more accurate estimate for $0 \leq x \leq \frac{1}{3}$. The survey report by Acar *et al.* [5] shows that a number of researchers have worked in this regard.

Generalizing King's work, Aldaz *et al.* [16] defined the operators that preserve constant and x^j for $j > 1$. The authors proved their shape preserving properties. Many researchers have studied these operators. Morales *et al.* [47] defined the Voronovskaja type result using a different strategy for $j = 2, 3$ for Aldaz operators. Additionally, he offered a conjecture of Voronovskaja-type outcomes for any chosen j, which has been demonstrated by Gavrea and Ivan [64].

1.2.1 Applications

The current subsection displays the importance of positive linear operators in our daily life. Bézier curves, defined by using B-basis polynomials, are elegant and fundamental mathematical constructions used in computer graphics, Computer-Aided Design (CAD), animation, and various other fields such as shipbuilding, architectural design, computer animation, medical imaging, textile business, letter writing, and many more to describe and model smooth curves and shapes. It might be interesting to note here that Donald Knuth has used Bézier curves for the design of TEX-fonts. Classical Bézier curve was introduced by P. Bézier [38] in 1960, while working in an automobile industry. Bézier curves have become an essential tool in the creation of digital imagery and design. The curve was determined by using the points, popularly known as control points. He defined the curve as follows:

$$P(x) = \sum_{\imath=0}^{n} p_{n,\imath}(x)\, P_\imath,$$

where $p_{n,\imath}(x)$ are Bernstein basis polynomials and $P_\imath$, $\imath = 0, 1, 2, \cdots, n$ are the control points.

Classical Bézier curve possess many interesting properties as the B-basis polynomial. To find the Bézier curve, a simplified method was given by Paul de Casteljau, known as De Casteljau algorithm. In this algorithm, he used the linear interpolation repeatedly to generate the curves. At their core, Bézier curves are a way to represent mathematically and interpolate points within a curve's path. They are defined by a set of control points that influence the curve's shape. What makes Bézier curves especially powerful, is their ability to generate curves of varying complexity, from simple straight lines to intricate and smooth curves. This flexibility along with their intuitive control mechanisms has made Bézier curves a cornerstone in the world of computer graphics. Due to the importance of Bézier curves, they are widely studied by many researchers as one can see [35, 39, 78, 111] and the references there in.

In [116], P*á*le*š* and R*é*dl gave the real life applications of Bézier curves. But these curves are not enough to produce all the shapes, for example, circle and ellipse like shapes can not be drawn by using classical Bézier curves. Therefore, rational Bézier curves are defined, which are similar to classical Bézier curves but have certain weights on the control points, given as below:

$$P(x,w)=\frac{\sum\limits_{\imath=0}^{n} p_{n,\imath}(x)\,P_\imath\,w_\imath}{\sum\limits_{\imath=0}^{n} p_{n,\imath}(x)\,w_\imath},$$

where $w_\imath \geq 0, \imath = 0, 1, ..., n$.

In [79], Ismail and Ali presented the applications of rational Bézier curves by producing many shapes such as leaf, spoon with the help of blending technique. In order to modify the shape of curves without changing the control points, parametric generalizations of the Bézier curves are initiated. In 2008, Han *et al.* [77] extended the Bézier curves with generalized Bernstein basis by taking n shape parameters say $\lambda_\imath, \imath = 1, 2, \cdots, n$, known as Q-Bézier curves. The authors observed that by altering only the values of shape parameters, the shape of the curves can be changed instead of changing the control points. In [76], Han *et al.* modified the Bézier curves with Lupaş q-analogue to control the shape of curves. For $q = 1$, the generalization becomes classical Bézier curves. Similarly, Khan *et al.* [92] generalized Bézier curves based on shifted nodes having parameters α and β. The authors showed that the curves generated over any sub interval of $[0, 1]$ with the shifted nodes are similar to classical Bézier curves.

1.3 Definitions

In the current section, we discuss the basic definitions that will be helpful in the subsequent chapters.

Definition 1.3.1. Normed space

Let X be a vector space over a field K. A real valued function defined as $\| \, . \, \|: X \to \mathbb{R}$ is called a norm, if the following properties holds:

1. $\| x \| = 0 \Leftrightarrow x = 0$;
2. $\| x \| \geq 0, \qquad$ *for all* $x \in X$;
3. $\| \alpha x \| = | \alpha | \| x \|, \qquad$ *for* $\alpha \in K, x \in X$;
4. $\| x + y \| \leq \| x \| + \| y \|, \qquad$ *for* $x, y \in X$,

then $(X, \| \, . \, \|)$ *is called the normed space.*

Definition 1.3.2. Positive Linear Operator

Let X and Y be two normed linear spaces over a field K. For $X \subset Y$ define an operator $L : X \to Y$, which is a called linear operator if it satisfies

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \quad \textit{for every} \;\; \alpha, \beta \in K, f, g \in X.$$

The operator L is called positive operator if

$$L(f; x) \geq 0, \; \textit{for every } f \in X \;\; \textit{such that} \;\; f(x) \geq 0, \; \forall \; x \in X.$$

The properties of positive linear operator are

1. a positive linear operator is monotone.
2. if L is a positive linear operator, then for every $f \in X$ we have $| Lf | \leq L(| f |)$.

The norm of L, denoted by $\| L \|$, is defined as follows:

$$\| L \| := \sup_{\substack{f \in X \\ \|f\| \leq 1}} \| Lf \| \, .$$

The following are two important inequalities related to positive linear operators:

Definition 1.3.3. Hölder inequality

Consider the positive linear operator $L : X \to Y$ *and the numbers* $p, q > 1$ *satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, the following inequality holds:

$$L(\mid f.g \mid) \leq (L(\mid f \mid^{p}))^{\frac{1}{p}} \, (L(\mid g \mid^{q}))^{\frac{1}{q}}, \quad \textit{for every} \qquad f, g \in X.$$

Definition 1.3.4. Cauchy-Schwarz inequality

For $p = q = 2$ *in the above inequality, it comes out to be well-known Cauchy-Schwarz inequality for positive linear operator* L *as below:*

$$\mid L(f.g; x) \mid \leq \sqrt{L(f^2; x)} \, \sqrt{L(g^2; x)}.$$

Now, we discuss some tools that are used to measure the smoothness of the function. The very well-known and simple example is first order modulus of continuity. The modulus of continuity is a mathematical concept used to measure the degree of continuity of a function over a given interval.

Definition 1.3.5. First order modulus of continuity

Let $f(x)$ *be bounded on* $[a, b]$, *the modulus of continuity of* $f(x)$ *on* $[a, b]$, *is denoted by* $\omega(f; \delta)$, *and is defined for* $\delta > 0$ *as*

$$\omega(f; \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|,$$

with the properties listed below:

1. $\omega(f; \delta) \to 0 \quad$ *as* $\quad \delta \to 0$;

2. $\omega(f; \delta)$ *is non-negative and non-decreasing;*

3. *$\omega(f;\delta)$ is continuous;*

4. *ω is sub-additive,* i.e.

$$\omega(f;\delta_1+\delta_2) \leq \omega(f;\delta_1)+\omega(f;\delta_2);$$

5. *for $\lambda > 0$, $\omega(f;\lambda\delta) \leq (1+\lambda)\,\omega(f;\delta)$.*

Graphically, the modulus of continuity can be interpreted as the maximum vertical distance between two points on the graph of the function that are no more than a distance of δ apart horizontally.

The modulus of continuity provides a measure of how "wiggly" or "oscillatory" a function is over a given interval. If a function is very smooth *i.e.* it has no sudden jumps or oscillations, then its modulus of continuity will be small. On the contrary, for a function which is very "wiggly" and contains many sudden jumps or oscillations, its modulus of continuity will be large.

Definition 1.3.6. Second order modulus of smoothness

The second order modulus of smoothness for any $f \in C[a,b]$, is defined by

$$\omega_2(f;\delta) = \sup\{|f(x+2h)-2f(x+h)+f(x)| : 0 \leq h \leq \delta, x, x+2h \in [a,b]\}.$$

Definition 1.3.7. Lipschitz continuity

A function f satisfies the Lipschitz condition of order τ with the constant $\mathscr{M}$ on $[a,b]$, if

$$|f(x_1)-f(x_2)| \leq \mathscr{M}\,|x_1-x_2|^{\tau}, \qquad 0<\tau\leq 1, \quad \mathscr{M}>0, \quad \textit{for} \quad x_1,x_2 \in [a,b].$$

It is denoted by $f \in Lip_{\mathscr{M}}(\tau)$.

There is a relation between modulus of continuity and Lipschitz continuity, i.e. *$f \in Lip_{\mathscr{M}}(\tau)$ iff $\omega(f;\delta) \leq \mathscr{M}\,\delta^{\tau}$.*

K-functional is an another tool to measure the smoothness of a positive linear operator and how well it is approximated by smooth functions. It is also known as Peetre's K-functional named after J. Peetre who introduced it.

Definition 1.3.8. K_2-Functional

Peetre [117] introduced the second order K-functional for $f \in C[0,1]$

$$K_2(f;\delta) = \inf\{\|f-g\|_{C[0,1]} + \delta\|g''\|_{C^2[0,1]} : g \in C^2[0,1]\},$$

where

$$C^2[0,1] := \{g \in C[0,1] : g', g'' \in C[0,1]\},$$

having the norm:

$$\begin{aligned} \|g\|_{C^2[0,1]} &= \|g''\|_{C[0,1]} + \|g'\|_{C[0,1]} + \|g\|_{C[0,1]}, \qquad (1.11) \\ \text{where} \quad \|g\|_{C[0,1]} &= \sup_{x\in[0,1]} |g(x)|. \end{aligned}$$

Now, we list some properties of K_2-functional:

1. For fixed $f \in C[0,1]$, the K_2-functional is concave and monotonically increasing function.

2. $K_2(f;\delta) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous at $\delta = 0$, it means

$$\lim_{\delta\to 0^+} K_2(f;\delta) = 0 = K_2(f;0).$$

3. For $f_1, f_2 \in C[0,1]$, we get

$$K_2(f_1+f_2;\delta) \le K_2(f_1;\delta) + K_2(f_2;\delta), \quad \delta \ge 0.$$

4. For fixed $f \in C[0,1]$ and any $\lambda, \delta \ge 0$, we have

$$K_2(f;\lambda\,\delta) \le \max\{1,\lambda\} K_2(f;\delta).$$

There is a close relationship between moduli of smoothness and K_2-functional [45]. For all $f \in C[0,1]$ and $\delta > 0$, there exists constant C free from δ and x such that

$$K_2(f;\delta) \leq C\left[\omega_2(f;\sqrt{\delta}) + min(1,\delta)\|f\|\right]. \tag{1.12}$$

Now, we give some instruments to measure the approximation properties of positive linear operators of two variables.

Definition 1.3.9. ***Complete Modulus of Continuity***
([20], p.80) For any $f \in C(I^2)$, where $I^2 = [0,1] \times [0,1]$, the complete modulus of continuity is defined as

$$\begin{aligned} \omega(f;\delta_1,\delta_2) &= \sup\{|f(u,v)-f(x,y)| : (u,v),(x,y) \in I^2 \,\&\, |u-x| \leq \delta_1,\ |v-y| \leq \delta_2\}, \\ \textit{or}\quad \omega(f;\delta) &= \sup\{|f(u,v)-f(x,y)| : \sqrt{(u-x)^2+(v-y)^2} \leq \delta,\ (u,v),(x,y) \in I^2\}, \end{aligned}$$

with the following properties:

i) $\omega(f;\delta_1,\delta_2) \to 0 \quad \text{as} \quad \delta_1,\delta_2 \to 0.$

ii)

$$| f(u,v) - f(x,y) | \leq \left(1 + \frac{| u-x |}{\delta_1}\right)\left(1 + \frac{| v-y |}{\delta_2}\right)\omega(f;\delta_1,\delta_2). \tag{1.13}$$

Definition 1.3.10. ***Partial Modulus of Continuity***
([20], p.81) The partial modulus of continuity for any $f \in C(I^2)$ is defined by

$$\begin{aligned} \omega^1(f;\delta_1) &= \sup\{|f(x_1,v)-f(x_2,v)| : v \in I, |x_1-x_2| \leq \delta_1\}, \\ \textit{and}\quad \omega^2(f;\delta_2) &= \sup\{|f(u,y_1)-f(u,y_2)| : u \in I, |y_1-y_2| \leq \delta_2\}. \end{aligned}$$

Definition 1.3.11. ***Lipschitz Condition of two variables***
A function f is said to satisfy Lipschitz condition, denoted by $f \in Lip_M(\zeta,\eta)$, if it follows:

$$| f(t,s) - f(x,y) | < M \mid t-x \mid^{\zeta} \mid s-y \mid^{\eta},$$

where $\zeta, \eta \in (0,1]$.

Let $C^2(I^2)$ denotes the space of all functions $f \in C(I^2)$ such that $\dfrac{\partial^i f}{\partial x^i}, \dfrac{\partial^i f}{\partial y^i} \in C(I^2), i = 1, 2$.

The norm on the space $C^2(I^2)$ is defined as

$$\|f\|_{C^2(I^2)} = \|f\| + \sum_{i=1}^{2} \left(\left\| \frac{\partial^i f}{\partial x^i} \right\| + \left\| \frac{\partial^i f}{\partial y^i} \right\| \right),$$

where the other norms are defined on $C(I^2)$.

Definition 1.3.12. Second order modulus of smoothness

([6], p.5558) For $f \in C(I^2)$*, the second order modulus of smoothness is defined by*

$$\omega_2(f;\delta) = \sup\{|f(x+2u, y+2v) - 2f(x+u, y+v) + f(x,y)| : (x,y), (x+2u, y+2v) \in I^2\}.$$

Let X and Y be compact subsets of real numbers.

Definition 1.3.13. Bögel Continuous

Consider a real-valued function defined on $X \times Y$ *i.e.* $f : X \times Y \to \mathbb{R}$*. If it satisfies*

$$\lim_{(t,s)\to(x,y)} \Delta f[(t,s),(x,y)] = 0,$$

with $\Delta f[(t,s),(x,y)] = f(t,s) - f(x,s) - f(t,y) + f(x,y)$*, then it is said to be Bögel continuous or B-continuous at a point* $(x,y) \in X \times Y$*.*

Definition 1.3.14. B-Differentiable

For a real-valued function $f : X \times Y \to \mathbb{R}$*, if*

$$\lim_{(t,s)\to(x,y)} \frac{\Delta f[(t,s),(x,y)]}{(t-x)(s-y)}$$

exists and finite, then it is called B-differentiable at $(x,y) \in X \times Y$*. It is denoted by* $D_B f(x,y)$*.*

The space of all B-differentiable functions is denoted by $D_b(X \times Y)$.

Definition 1.3.15. B-bounded

If for a real-valued function $f : X \times Y \to \mathbb{R}, \exists$ *a constant* $M > 0$ *such that*

$$\mid \Delta f[(t,s),(x,y)] \mid \leq M,$$

for every $(t,s),(x,y) \in X \times Y$*, then it is called B-bounded on* $X \times Y$*.*

$B_b(X \times Y)$ denotes the space of all B-bounded functions, where

$$\|f\|_B = \sup_{(t,s),(x,y) \in X \times Y} \mid \Delta f[(t,s),(x,y)] \mid$$

is the required norm.

Definition 1.3.16. Mixed modulus of continuity

The mixed modulus of continuity for $f \in C_b(I^2)$*, denoted by* $\omega_{mixed} : [0,\infty) \times [0,\infty) \to \mathbb{R}$*, ([20], p.81) is given as*

$$\omega_{mixed}(f;\sigma_1,\sigma_2) = \sup\{|\Delta f[(t,s),(x,y)]| : |t-x| < \sigma_1, |s-y| < \sigma_2\},$$

for all $(t,s),(x,y) \in I^2$*. It satisfies the property*

$$\omega_{mixed}(f;\lambda_1\sigma_1,\lambda_2\sigma_2) \leq (1+\lambda_1)(1+\lambda_2)\,\omega_{mixed}(f;\sigma_1,\sigma_2), \tag{1.14}$$

where $\lambda_1, \lambda_2 > 0$.

When approximating a univariate function on positive real line, the defined Bohman Korovkin theorem does not work there. Therefore, we apply weighted Korovkin theorem in this case. Thus, we put some restrictions on the functions.

A positive continuous function w is said to be weight function if it is defined on the whole real line $\mathbb{R}$. In our study, we consider only the polynomial growth. So, we choose $w(x) = 1 + x^2$, $x \in \mathbb{R}$. Suppose

$$D_2[0,\infty) := \{f : |f(x)| \leq A_f(1+x^2)\},$$

such that $A_f > 0$, a constant number depends only on f.

$C_2[0,\infty)$ is the space of all continuous functions in $D_2[0,\infty)$ having the norm

$$\|f\|_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Define $C_2^*[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty \right\}.$

1.4 Gaps in Literature

(i) As we have seen from the literature that many generalizations of positive linear operators are of linear order. Therefore, the researchers used different techniques to improve the order of approximation of positive linear operators but that require more information about the given function. The new perturbed scheme does not need much data but it slackens the condition of positivity. Till now, order of some operators has been improved. But still there are many more useful operators whose order can be improved without affecting its useful properties.

(ii) From the literature, we are able to see that generalizations of positive linear operators are useful in order to get better results and it provides the flexibility to choose certain parameters in order to achieve desired accuracy. Although many generalizations have been carried out but still positive linear operators can be generalized by introducing some more general parameters that does not effect the convergence but helps to reduce the error of approximation

(iii) We see that, due to the importance of Bézier curves, a variety of research is going on. So, there is some possibility to generalize these curves to get better flexibility by introducing shape parameters to modify the shape of the curve without changing the initial data, which helps to reduce the time and cost for these generalized Bézier curves.

1.5 Objectives of the Study

After the comprehensive review of existing literature and identifying key gaps, we proposed the following objectives for our Ph.D. research:

1. To improve the order of approximation for some positive linear operators.
2. To study the approximation properties of certain positive linear operators.
3. To find the generalization of Bézier curves for more flexibility in its shape.

1.6 Contents of Thesis

The present thesis is assembled into eight chapters including the present one which is about the brief introduction of approximation theory and its applications in different branches and in daily life. Also, we provide the supplementary material such as some definitions, preliminary results that are useful for upcoming chapters. It also includes the literature survey of positive linear operators, by which we able to find some gaps in literature. Thus, we get the motivation for our work that is presented in chapter 2 to chapter 8 described below:

Chapter 2 deals with the sequence of operators which are used to improve the order of approximation of well defined α-Bernstein Păltănea operators. We have provided three modifications of these operators of order one, two and three. We study the convergence and some approximation results of these operators. Also, the error of approximation is calculated for these operators using modulus of continuity and K-functional. The order of approximation is shown by using Voronovskaja type result. Similar procedure can be applied to get the higher order sequence of operators. Also, we justify our theoretical result by giving some graphical figures with the help of Maple software.

The results of this chapter are published in the journal “International Journal of Non-linear Analysis and Applications”.

Chapter 3 is the extension of our work done in the previous chapter. We defined the bivariate version of the first order modification of α- Bernstein Păltănea operators defined in chapter 2. We prove the convergence of these operators by using Volkov theorem, asymptotic formula, and find the error of approximation in terms of complete and partial modulus of continuities. Also, we define the Generalized Boolean Sum operators associated with bivariate operators and study its approximation results for the space of Bögel continuous functions.
The results in this chapter are introduced in the journal "Filomat".

In Chapter 4, we introduce the Durrmeyer-variant of Lupaş type operators and discuss its approximation in Lipschitz type spaces and using approximation tools. Also, we construct their generalization in two dimensional space and study its properties in terms of K_2-functional and complete as well as partial modulus of continuities. Lastly, we present some numerical results by using graphs to show the convergence and approximation error of the defined operators for certain functions in both single as well as two dimensional case.
The results of this chapter are published in the journal "Annali dell' Università di Ferrara".

Chapter 5 is devoted for Bernstein-Lototsky operators based on a real parameter $\rho > 0$ for the approximation of Lebesgue integrable functions. These operators are able to preserve any polynomial with some certain conditions. We present the univariate and bivariate version of these operators. We discuss some approximation characteristics for these operators including convergence and error estimates, using Peetre's K_2-functional and modulus of continuities of first and second order.
The results of these chapters are submitted.

In Chapter 6, we define the new operators to approximate integrable functions. We use α-Baskakov basis functions and a non-negative parameter for this construction, which will be useful to get the flexibility in the properties of these operators. We study the convergence as well as weighted convergence of these operators. We also study pointwise convergence of these operators for Lipschitz functions. Graphical representations of certain functions are given to show the effect of our introduced

parameter.
The results in this chapter are published in the journal “Rocky Mountain journal of Mathematics”.

Chapter 7 is based on the applications of positive linear operators in the form of Bézier curves. We present the generalized Bézier curves depending upon the parameter α, named as α-Bézier curves. Also, we introduce tensor product α-Bézier surfaces. We study some properties of these curves and surfaces and show that the parameter provides us the flexibility to modify the curves as well as surfaces by giving some numerical examples with the help of Matlab. In the last part of this chapter, we give an exact approach to calculate the control points for the given α-Bézier curve.
Some of the results in this chapter are published in the journal “Bollettino dell'Unione Matematica Italiana” and others are submitted.

Chapter 8 explores the applications of q-calculus in polynomial basis functions and curve modeling. We define the q-variant of Bernstein-Chlodowsky basis polynomials. Then, by utilizing these basis polynomials, we introduce the generalization of Bézier curves to enjoy the shape-preserving properties. The properties of these newly defined Bézier curves are studied. Additionally, we present degree elevation and De Casteljau algorithms tailored for these curves. At last, we present some numerical illustrations to show the flexibility achieved by the introduced parameter.
The results of this chapter are accepted in the journal “Iranian Journal of Science”.

Chapter 2

Order improvement for the sequence of α-Bernstein-Păltănea operators

2.1 Introduction

The order of approximation of the positive linear operators is an important tool to measure how fastly an operator converges to the given function. As discussed in Chapter 1, there are many techniques to improve the order of convergence of the operators. Here, our main motive is to improve approximation behavior and order of convergence for the operators $Q^{\alpha}_{n,\rho}(f;x)$ (1.5) by a new method.

In [94], Khosravian-Arab *et al.* modified the well known Bernstein operators by using a perturbed technique to improve their degree of approximation. Following this, Acu *et al.* [8] have applied this approach on the Bernstein-Durrmeyer operators. In another paper [74], same authors have put it on the Bernstein-Kantorovich operators too. Similarly, Kajla and Acar [83] have modified the α-Bernstein summation operators. The inspiration of getting better approximation results for positive linear operators leads us to modify the α-Bernstein-Păltănea operators which are defined in (1.5). In the present chapter, we apply an approach by using recurrence relation of these operators for their modifications, which gives better convergence rate than the classical one.

This chapter is organized as follows: In section 2.2, we define α-Bernstein-Păltănea operators of first order and study their convergence. In section 2.3 and 2.4, we introduce α-Bernstein-Păltănea operators of second and third order resp., which possess higher order of approximation than the operators (1.5). In section 2.5, we verify the theoretical results numerically using Maple algorithms.

Through out this chapter, we denote $\mathcal{J}^{\alpha,\jmath}_{n,\rho}(g;x)$, $\jmath = 1,2,3$ as three modifications of $\jmath$th order of approximation of the operators $Q^{\alpha}_{n,\rho}(f;x)$.

2.2 α-Bernstein-Păltănea Operators of First Order

In the present section, we define α-Bernstein-Păltănea operators of first order as

$$\begin{aligned} \mathcal{J}^{\alpha,1}_{n,\rho}(g;x) =& \sum_{\imath=0}^{n} p^{\alpha,1}_{n,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t})\, g(\mathrm{t})\,\mathrm{dt}, \quad x \in [0,1] \qquad (2.1)\\ \text{where} \quad p^{\alpha,1}_{n,\imath}(x) =& a(x,n)\, p^{\alpha}_{n-1,\imath}(x) + a(1-x,n)\, p^{\alpha}_{n-1,\imath-1}(x), \quad 0 \le \imath \le n-1, \end{aligned}$$

$\alpha \in [0,1], \rho > 0$, and $a(x,n) = a_1(n)x + a_0(n)$, such that $a_0(n)$ and $a_1(n)$ are two unknown sequences, which can be determined to satisfy our purposes. For $a_0(n) = 1$ and $a_1(n) = -1$, (2.1) reduce to the operators (1.5).

Now, we compute some preliminary results which will be useful to study the uniform convergence and asymptotic results.

Lemma 2.2.1. *For the operators (2.1), we have*

$$\begin{aligned} \mathcal{J}^{\alpha,1}_{n,\rho}(e_0;x) =& (2a_0(n) + a_1(n));\\ \mathcal{J}^{\alpha,1}_{n,\rho}(e_1;x) =& (2a_0(n) + a_1(n))x + \frac{1}{n\rho+2}\left[(1-2x)(a_0(n)(\rho+2) + a_1(n)(\rho+1))\right];\\ \mathcal{J}^{\alpha,1}_{n,\rho}(e_2;x) =& (2a_0(n) + a_1(n))x^2 + \frac{1}{(n\rho+2)(n\rho+3)}\Big[n\left\{\rho x(3-5x)(2a_0(n) + a_1(n))\right.\\ &\left. + \rho^2 x(a_0(n)(4-6x) + a_1(n)(3-5x))\right\}\\ &+ (-6x^2 + 2\alpha\rho^2 x^2 - 2\alpha\rho^2 x + 2)(2a_0(n) + a_1(n))\\ &+ \rho(\rho+3-6x)a_0(n) + \rho(2\rho x^2 - 6x - 2\rho x + \rho + 3)a_1(n)\Big]. \end{aligned}$$

Lemma 2.2.2. *For the operators (2.1), we have the central moments as*

$$\begin{aligned}
\mathcal{J}_{n,\rho}^{\alpha,1}(\phi_x^1(\mathrm{t});x) =& \frac{1}{n\rho+2}\left[(1-2x)(a_0(n)(\rho+2)+a_1(n)(\rho+1))\right];\\
\mathcal{J}_{n,\rho}^{\alpha,1}(\phi_x^2(\mathrm{t});x) =& \frac{1}{(n\rho+2)(n\rho+3)}\left[x(1-x)\rho(1+\rho)(2a_0(n)+a_1(n))n\right.\\
&-x(1-x)\left(4a_0(n)(3+3\rho+\alpha\rho^2)+2a_1(n)(3+6\rho+\rho^2(1+\alpha))\right)\\
&\left.+a_0(n)(4+3\rho+\rho^2)+a_1(n)(2+3\rho+\rho^2)\right];\\
\mathcal{J}_{n,\rho}^{\alpha,1}(\phi_x^4(\mathrm{t});x) =& \frac{1}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)}\left[3\rho^2(1+\rho)^2x^2(1-x)^2(2a_0(n)\right.\\
&\left.+a_1(n))n^2\right]+O\left(\frac{1}{n^3}\right).
\end{aligned}$$

To obtain uniform convergence of the operators (2.1), we assume that the sequences $a_\imath(n),\ \imath=0,1$ satisfy the condition

$$2a_0(n)+a_1(n)=1. \tag{2.2}$$

Depending on the choices of the sequences, we get two cases which are given as follows:
Case 1. Let

$$a_0(n)\geq 0 \quad \text{and} \quad a_0(n)+a_1(n)\geq 0. \tag{2.3}$$

From this, we get $0\leq a_0(n)\leq 1$ and $-1\leq a_1(n)\leq 1$. So, both $a_0(n)$ and $a_1(n)$ are the bounded sequences. The operators (2.1) are positive for this case.
Case 2. Let

$$a_0(n)<0 \quad \text{or} \quad a_0(n)+a_1(n)<0. \tag{2.4}$$

If $a_0(n)<0$, then $a_0(n)+a_1(n)>1$ and if $a_0(n)+a_1(n)<0$, then $a_0(n)>1$. In these cases the operators (2.1) are not positive.
Firstly, we prove the basic convergence and asymptotic results for case 1.

Theorem 2.2.3. *Let $g\in C[0,1]$. If $a_0(n),a_1(n)$ satisfy both the equations (2.2) and*

(2.3), then

$$\lim_{n\to\infty} J^{\alpha,1}_{n,\rho}(g;x) = g(x),$$

uniformly on [0,1].

Proof. From the conditions on $a_\imath(n), \imath = 0,1$ the operators (2.1) are positive. So, by Korovkin theorem 1.1.3 and Lemma 2.2.1, we can find the uniform convergence of the operators. □

Theorem 2.2.4. *Let $a_\imath(n), \imath = 0,1$ be convergent sequences satisfying the conditions (2.2)-(2.3) and $l_\imath = \lim_{n\to\infty} a_\imath(n)$. If $g'' \in C[0,1]$, then*

$$\begin{aligned}\lim_{n\to\infty} n(J^{\alpha,1}_{n,\rho}(g;x) - g(x)) =& \frac{(1-2x)((\rho+2)l_0 + (\rho+1)l_1)}{\rho} g'(x)\\ &+ \frac{x(1-x)(1+\rho)(2l_0+l_1)}{2\rho} g''(x),\end{aligned}$$

uniformly on [0,1].

Proof. By the Taylor's formula, we have

$$g(\mathrm{t}) = g(x) + \phi^1_x(\mathrm{t})g'(x) + \frac{1}{2!}\phi^2_x(\mathrm{t})g''(x) + \Theta(\mathrm{t},x)\phi^2_x(\mathrm{t}),$$

where $\Theta(\mathrm{t},x) \in C[0,1]$ with $\lim_{\mathrm{t}\to x} \Theta(\mathrm{t},x) = 0$.
Apply the operators $J^{\alpha,1}_{n,\rho}(.;x)$ on Taylor's formula, we get

$$n(J^{\alpha,1}_{n,\rho}(g;x) - g(x)) = nJ^{\alpha,1}_{n,\rho}(\phi^1_x(\mathrm{t});x)g'(x) + \frac{n}{2}J^{\alpha,1}_{n,\rho}(\phi^2_x(\mathrm{t});x)g''(x) + nJ^{\alpha,1}_{n,\rho}(\Theta(\mathrm{t},x)\phi^2_x(\mathrm{t});x).$$

Using Cauchy-Schwarz inequality on the last term of the above equation, we obtain

$$nJ^{\alpha,1}_{n,\rho}(\Theta(\mathrm{t},x)\phi^2_x(\mathrm{t});x) \le n\sqrt{J^{\alpha,1}_{n,\rho}(\Theta^2(\mathrm{t},x);x)}\sqrt{J^{\alpha,1}_{n,\rho}(\phi^4_x(\mathrm{t});x)}. \tag{2.5}$$

Since $\Theta^2(x,x) = 0$, $\Theta^2(\mathrm{t},x) \in C[0,1]$, and $J^{\alpha,1}_{n,\rho}(g;x) \to g(x)$, then we have
$\lim_{n\to\infty} J^{\alpha,1}_{n,\rho}(\Theta^2(\mathrm{t},x);x) = 0$ uniformly on [0,1].

Hence, from Lemma 2.2.2, the above inequality (2.5) reduces to

$$\lim_{n\to\infty} nJ^{\alpha,1}_{n,\rho}(\Theta(\mathrm{t},x)\phi^2_x(\mathrm{t});x)=0,$$

which gives the required result. □

Now, we study the convergence and asymptotic results for the case 2.

Theorem 2.2.5. *Let $g\in C[0,1]$ and $a_\imath(n),\imath=0,1$ be convergent sequences which satisfy the conditions (2.2) and (2.4). Then*

$$\lim_{n\to\infty} J^{\alpha,1}_{n,\rho}(g;x)=g(x),$$

uniformly on [0,1].

Proof. We can rewrite the operators (2.1) as

$$J^{\alpha,1}_{n,\rho}(g;x)=K^{\alpha,1}_{n,\rho}(g;x)-L^{\alpha,1}_{n,\rho}(g;x), \tag{2.6}$$

where

$$\begin{aligned}
K^{\alpha,1}_{n,\rho}(g;x)=&\sum_{\imath=0}^{n}[a_1(n)\ x\ p^{\alpha}_{n-1,\imath}(x)+a_1(n)\ p^{\alpha}_{n-1,\imath-1}(x)]\int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t})\,g(\mathrm{t})\,\mathrm{dt},\\
L^{\alpha,1}_{n,\rho}(g;x)=&\sum_{\imath=0}^{n}[-a_0(n)\ p^{\alpha}_{n-1,\imath}(x)+(a_1(n)\ x-a_0(n))\ p^{\alpha}_{n-1,\imath-1}(x)]\int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t})\,g(\mathrm{t})\,\mathrm{dt}.
\end{aligned}$$

As both the operators $K^{\alpha,1}_{n,\rho}(g;x)$ and $L^{\alpha,1}_{n,\rho}(g;x)$ are positive, so we can apply extended Korovkin theorem ([94], see page 122) on these operators. For this, we give the moments of these operators as below:

$$\begin{aligned}
K^{\alpha,1}_{n,\rho}(e_0;x)=&a_1(n)(1+x);\\
K^{\alpha,1}_{n,\rho}(e_1;x)=&a_1(n)(1+x)\left[\frac{(n-1)\rho x}{n\rho+2}\right]+\frac{a_1(n)x}{n\rho+2}+\frac{a_1(n)(\rho+1)}{n\rho+2};\\
K^{\alpha,1}_{n,\rho}(e_2;x)=&\frac{a_1(n)(1+x)}{(n\rho+2)(n\rho+3)}\left[\rho^2(n-1)^2\left(x^2+\frac{n-1+2(1-\alpha)}{(n-1)^2}x(1-x)\right)\right.\\
&\left.+3\rho(n-1)x+2\right]+\frac{a_1(n)}{(n\rho+2)(n\rho+3)}\left[2\rho^2(n-1)x+\rho^2+3\rho\right],
\end{aligned}$$

$$\begin{aligned} L^{\alpha,1}_{n,\rho}(e_0;x) =& a_1(n)x - 2a_0(n);\\ L^{\alpha,1}_{n,\rho}(e_1;x) =& (a_1(n)x - 2a_0(n))\left[\frac{(n-1)\rho x}{n\rho+2}\right] - \frac{a_0(n)(\rho+2)}{n\rho+2} + \frac{a_1(n)x(\rho+1)}{n\rho+2};\\ L^{\alpha,1}_{n,\rho}(e_2;x) =& \frac{a_1(n)x-2a_0(n)}{(n\rho+2)(n\rho+3)}\Big[\rho^2(n-1)^2x^2 + \rho^2(n-1+2(1-\alpha))x(1-x)\\ &+3\rho(n-1)x+2\Big] + \frac{a_1(n)x-a_0(n)}{(n\rho+2)(n\rho+3)}[2\rho^2(n-1)x+\rho^2+3\rho]. \end{aligned}$$

Since $a_1(n)$ is convergent, then $\lim_{n\to\infty} a_1(n) = l_1(say)$, by using the definition of $K^{\alpha,1}_{n,\rho}(g;x)$ and $L^{\alpha,1}_{n,\rho}(g;x)$, we obtain

$$\lim_{n\to\infty} K^{\alpha,1}_{n,\rho}(g;x) = l_1(1+x)g(x) \text{ uniformly on } [0,1],$$

$$\lim_{n\to\infty} L^{\alpha,1}_{n,\rho}(g;x) = [l_1(1+x)-1]g(x) \text{ uniformly on } [0,1].$$

By using both the above limits and equation (2.6), we get the required result. □

Theorem 2.2.6. *Let $a_\imath(n), \imath = 0,1$ be convergent sequences satisfying the conditions (2.2), (2.4) and $l_\imath = \lim_{n\to\infty} a_\imath(n)$. If $g'' \in C[0,1]$, then*

$$\begin{aligned} \lim_{n\to\infty} n(J^{\alpha,1}_{n,\rho}(g;x) - g(x)) =& \frac{(1-2x)((\rho+2)l_0+(\rho+1)l_1)}{\rho}g'(x)\\ &+ \frac{x(1-x)(1+\rho)(2l_0+l_1)}{2\rho}g''(x), \end{aligned}$$

uniformly on [0,1].

Proof. Similar to the proof of Theorem 2.2.4

$$n(J^{\alpha,1}_{n,\rho}(g;x) - g(x)) = nJ^{\alpha,1}_{n,\rho}(\phi^1_x(\mathrm{t});x)g'(x) + \frac{n}{2}J^{\alpha,1}_{n,\rho}(\phi^2_x(\mathrm{t});x)g''(x) + nJ^{\alpha,1}_{n,\rho}(\Theta(\mathrm{t},x)\phi^2_x(\mathrm{t});x).$$

It is enough to prove that

$$\lim_{n\to\infty} nJ^{\alpha,1}_{n,\rho}(\Theta(\mathrm{t},x)\phi^2_x(\mathrm{t});x) = 0.$$

We can rewrite operators (2.1) in the following way:

$$J_{n,\rho}^{\alpha,1}(g;x)=\sum_{\imath=0}^{n-1}p_{n-1,\imath}^{\alpha}(x)\left(a(x,n)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})g(\mathrm{t})\ \mathrm{dt}+a(1-x,n)\int_0^1\mu_{n,\imath+1}^{\rho}(\mathrm{t})g(\mathrm{t})\ \mathrm{dt}\right). \tag{2.7}$$

For $\epsilon>0$, $\exists\ \delta>0$ such that $|\mathrm{t}-x|<\delta$, then $|\Theta(\mathrm{t},x)|<\epsilon$.

Divide the interval $[0,1]$ into two parts as below:

$$I_1=(x-\delta,x+\delta)\cap[0,1],\qquad\qquad I_2=[0,1]\setminus(x-\delta,x+\delta).$$

Since $a_\imath(n),\imath=0,1$ are convergent, then they are bounded. Thus, $\exists\ C>0$ such that $|a(x,n)|<C$.

Now, $n|J_{n,\rho}^{\alpha,1}(\Theta(\mathrm{t},x)\phi_x^2(\mathrm{t});x)|$

$$\begin{aligned}
&\leq nC\sum_{\imath=0}^{n-1}p_{n-1,\imath}^{\alpha}(x)\left(\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})|\Theta(\mathrm{t},x)|\phi_x^2(\mathrm{t})\ \mathrm{dt}+\int_0^1\mu_{n,\imath+1}^{\rho}(\mathrm{t})|\Theta(\mathrm{t},x)|\phi_x^2(\mathrm{t})\ \mathrm{dt}\right)\\
&<nC\sum_{\imath=0}^{n-1}p_{n-1,\imath}^{\alpha}(x)\left[\epsilon\left(\int_{I_1}\mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\ \mathrm{dt}+\int_{I_1}\mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\ \mathrm{dt}\right)\right.\\
&\quad\left.+\frac{M}{\delta^2}\left(\int_{I_2}\mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\ \mathrm{dt}+\int_{I_2}\mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\ \mathrm{dt}\right)\right],\ \text{where } M=\sup_{0\leq \mathrm{t}\leq 1}|\Theta(\mathrm{t},x)|\\
&\leq n\epsilon C\sum_{\imath=0}^{n-1}p_{n-1,\imath}^{\alpha}(x)\left[\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\ \mathrm{dt}+\int_0^1\mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\ \mathrm{dt}\right]\\
&\quad+\frac{nMC}{\delta^2}\sum_{\imath=0}^{n-1}p_{n-1,\imath}^{\alpha}(x)\left[\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\ \mathrm{dt}+\int_0^1\mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\ \mathrm{dt}\right]\\
&\leq\epsilon C_1(x,\rho,\alpha)+O\left(\frac{1}{n}\right).
\end{aligned}\tag{2.8}$$

Since $\epsilon>0$ is arbitrary, therefore from equation (2.8), we get the proof. □

Theorem 2.2.7. *Let g be bounded for $x\in[0,1]$, $a_0(n)$ be a bounded sequence and $a_\imath(n),\imath=0,1$ satisfy the condition (2.2), then*

$$||J_{n,\rho}^{\alpha,1}g-g||\leq(1+3|a_0(n)|)C_2\,\omega\left(g;\frac{1}{\sqrt{n}}\right),$$

where $||.||$ is the uniform norm over $[0,1]$, $\omega(g;\sigma)$ is the first order modulus of conti-

nuity and $C_2 > 0$ is a constant.

Proof. From the definition of our operators (2.7) and using relation $\omega(g;\lambda\sigma) \leq (1+\lambda)\,\omega(g;\sigma)$ for $\lambda > 0$, $\left(\text{by taking } \lambda = \sqrt{n}\,|\mathrm{t}-x|, \sigma = \frac{1}{\sqrt{n}}\right)$, we get

$$\begin{aligned}
|J^{\alpha,1}_{n,\rho}(g;x) - g(x)| \leq & |a(x,n)| \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) |g(\mathrm{t}) - g(x)| \,\mathrm{dt} \\
& + |a(1-x,n)| \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath+1}(\mathrm{t}) |g(\mathrm{t}) - g(x)| \,\mathrm{dt} \\
\leq & |a(x,n)| \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t})\, \omega(g; |\phi^1_x(\mathrm{t})|) \,\mathrm{dt} \\
& + |a(1-x,n)| \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath+1}(\mathrm{t})\, \omega(g; |\phi^1_x(\mathrm{t})|) \,\mathrm{dt} \\
\leq & |a(x,n)|\, \omega\left(g; \frac{1}{\sqrt{n}}\right) \left[1 + \sqrt{n} \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) |\phi^1_x(\mathrm{t})| \,\mathrm{dt}\right] \\
& + |a(1-x,n)|\, \omega\left(g; \frac{1}{\sqrt{n}}\right) \left[1 + \sqrt{n} \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath+1}(\mathrm{t}) |\phi^1_x(\mathrm{t})| \,\mathrm{dt}\right].
\end{aligned} \tag{2.9}$$

From Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) |\phi^1_x(\mathrm{t})| \,\mathrm{dt} \leq & \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \left[\int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) \,\mathrm{dt}\right]^{\frac{1}{2}} \left[\int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) (\mathrm{t}-x)^2 \,\mathrm{dt}\right]^{\frac{1}{2}} \\
= & \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \left[\int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) (\mathrm{t}-x)^2 \,\mathrm{dt}\right]^{\frac{1}{2}} \\
\leq & \left[\sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x)\right]^{\frac{1}{2}} \left[\sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) \phi^2_x(\mathrm{t}) \,\mathrm{dt}\right]^{\frac{1}{2}} \\
= & \sqrt{\frac{n(-\rho^2x^2 - \rho x^2 + \rho^2 + 3\rho x) + (6x^2 + 6\rho x^2 + \rho^2 x - 3\rho x + 2)}{(n\rho+2)(n\rho+3)}}.
\end{aligned}$$

Therefore,

$$\sqrt{n} \sum_{\imath=0}^{n-1} p^{\alpha}_{n-1,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t}) |\phi^1_x(\mathrm{t})| \,\mathrm{dt} \leq \frac{C_3\sqrt{n}}{\sqrt{n\rho+2}} \leq C_3. \tag{2.10}$$

Similarly,

$$\sqrt{n}\sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})|\phi_x^1(\mathrm{t})|\,\mathrm{dt} \le \frac{C_4\sqrt{n}}{\sqrt{n\rho+2}} \le C_4. \tag{2.11}$$

Using the inequalities (2.10) and (2.11) in (2.9), we get the following relation:

$$|\mathcal{J}_{n,\rho}^{\alpha,1}(g;x)-g(x)| \le \omega\left(g;\frac{1}{\sqrt{n}}\right)\left[|a(x,n)|(1+C_3)+|a(1-x,n)|(1+C_4)\right]. \tag{2.12}$$

From equation (2.2), we find

$$|a(x,n)| = |a_1(n)x+a_0(n)| \le 1+3|a_0(n)| \quad \text{and} \quad |a(1-x,n)| \le 1+3|a_0(n)|.$$

Now, using above inequalities and equation (2.12), the proof is completed. □

Corollary 2.2.8. *(i) If we assume $g \in C[0,1]$ in Theorem 2.2.7, then $\lim_{n\to\infty}\omega\left(g;\frac{1}{\sqrt{n}}\right) = 0$, which gives another proof of the Theorems 2.2.3 and 2.2.5.*
(ii) If $g \in Lip_{\mathscr{M}}(\tau)$ on $[0,1]$, then result obtained in Theorem 2.2.7 reduces to

$$|\mathcal{J}_{n,\rho}^{\alpha,1}(g;x)-g(x)| \le \mathscr{M}C_5\left(1+3|a_0(n)|\right)n^{-\tau/2},$$

where C_5 is same constant as in Theorem 2.2.7.

Now, we consider $\mathcal{J}_{n,\rho}^{\alpha,1}(e_\imath;x) = e_\imath, \imath = 0,1$. Thus, we have the conditions

$$2a_0(n)+a_1(n) = 1, a_0(n)(\rho+2)+a_1(n)(\rho+1) = 0.$$

By solving these equations, we get

$$a_0(n) = \frac{\rho+1}{\rho}, a_1(n) = -\frac{\rho+2}{\rho}. \tag{2.13}$$

In the following result, we find the error of approximation in terms of modulus of continuity for the operators (2.1) satisfying (2.13).

Theorem 2.2.9. *Let $g \in C^2[0,1]$, $x \in [0,1]$ be fixed. Then*

$$\left| \mathcal{J}_{n,\rho}^{\alpha,1}(g;x) - g(x) - \frac{1}{2}\mathcal{J}_{n,\rho}^{\alpha,1}(\phi_x^2(\mathrm{t});x)g''(x) \right| \le C_6 \frac{1}{n}\omega\left(g'';\frac{1}{n}\right),$$

where $C_6 > 0$ is a constant restrained from n, x.

Proof. For $g \in C^2[0,1]$. By using the Taylor's formula and applying $\mathcal{J}_{n,\rho}^{\alpha,1}(.;x)$, we get

$$\left| \mathcal{J}_{n,\rho}^{\alpha,1}(g;x) - g(x) - \frac{1}{2}\mathcal{J}_{n,\rho}^{\alpha,1}(\phi_x^2(\mathrm{t});x)g''(x) \right| = \frac{1}{2}|\mathcal{J}_{n,\rho}^{\alpha,1}(\Theta(\mathrm{t},x)\phi_x^2(\mathrm{t});x)|,$$

where $\Theta(\mathrm{t},x) = g''(\xi_x) - g''(x)$ and ξ_x lies between t and x. From modulus of continuity, we have

$$|\Theta(\mathrm{t},x)| = |g''(\xi_x) - g''(x)| \le \omega(g''; |\phi_x^1(\mathrm{t})|) \le (1 + \sqrt{n}\,|\phi_x^1(\mathrm{t})|)\,\omega\left(g''; \frac{1}{\sqrt{n}}\right).$$

Also, $|a(x,n)| = |a_1(n)x + a_0(n)| \le \dfrac{\rho+1}{\rho}$ and $|a(1-x,n)| \le \dfrac{\rho+1}{\rho}$, for $x \in [0,1]$ and using the operators (2.7), we obtain

$$|\mathcal{J}_{n,\rho}^{\alpha,1}(\Theta(\mathrm{t},x)\phi_x^2(\mathrm{t});x)| \le \frac{\rho+1}{\rho}\omega\left(g''; \frac{1}{\sqrt{n}}\right)(J_1 + \sqrt{n}J_2), \tag{2.14}$$

where
$$\begin{aligned}
J_1 =& \sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\left[\int_0^1 \mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\,\mathrm{dt} + \int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\,\mathrm{dt}\right] \\
=& \frac{1}{(n\rho+2)(n\rho+3)}\Big[n(-2\rho^2x^2 - 2\rho x^2 + 2\rho^2 x + 4\rho x) \\
&+ (12x^2 + 12\rho x^2 - 12\rho x + \rho^2 + 3\rho + 4 - 4\alpha\rho^2 x(1-x))\Big] \le \frac{(\rho+2)^2(n+2)}{(n\rho+2)(n\rho+3)}. \\
J_2 =& \sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\left[\int_0^1 \mu_{n,\imath}^{\rho}(\mathrm{t})|\phi_x^1(\mathrm{t})|\phi_x^2(\mathrm{t})\,\mathrm{dt} + \int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})|\phi_x^1(\mathrm{t})|\phi_x^2(\mathrm{t})\,\mathrm{dt}\right] \\
\le& \sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\left[\int_0^1 \mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}}\left[\int_0^1 \mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}} \\
&+ \sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\left[\int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}}\left[\int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}}
\end{aligned}$$

$$\leq \left[\sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\int_0^1 \mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}} \left[\sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\int_0^1 \mu_{n,\imath}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}}$$
$$+ \left[\sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^2(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}} \left[\sum_{\imath=0}^{n-1} p_{n-1,\imath}^{\alpha}(x)\int_0^1 \mu_{n,\imath+1}^{\rho}(\mathrm{t})\phi_x^4(\mathrm{t})\,\mathrm{dt}\right]^{\frac{1}{2}}.$$

Using the results of Lemma 2.2.2, we have

$$J_2 \leq \frac{P}{n\sqrt{n}}, \text{ where } P(>0) \text{ is unrestrained from } n.$$

Now replacing the values of J_1 and J_2 in the relation (2.14), we can find a constant $C_6 > 0$ which is independent from n and x such that

$$|J_{n,\rho}^{\alpha,1}(\Theta(\mathrm{t},x)\phi_x^2(\mathrm{t});x)| \leq C_6\frac{1}{n}\omega\left(g'';\frac{1}{\sqrt{n}}\right). \tag{2.15}$$

Hence, we get the required result. $\square$

Now, we find error estimation of the operators $J_{n,\rho}^{\alpha,1}(g;x)$ with regard to second order modulus of continuity which gives better results than the results found in Theorem 2.2.7.

Theorem 2.2.10. *Let* $g \in C[0,1]$, $a_0(n) = \dfrac{\rho+1}{\rho}, a_1(n) = -\dfrac{\rho+2}{\rho}$, *then*

$$||J_{n,\rho}^{\alpha,1}(g;.) - g|| \leq C_7\,\omega_2\left(g;\frac{1}{\sqrt{n}}\right).$$

Proof. From the definition of our operators (2.1), we can write

$$||J_{n,\rho}^{\alpha,1}\,g|| \leq (|a_0(n)| + |a_1(n)|)||g||. \tag{2.16}$$

Let $f \in C^2[0,1]$. Now, using Taylor's theorem, we get

$$\left|J_{n,\rho}^{\alpha,1}(f;x) - f(x)\right| = \left|\frac{1}{2}J_{n,\rho}^{\alpha,1}(\phi_x^2(\mathrm{t});x)f''(x) + \frac{1}{2}J_{n,\rho}^{\alpha,1}(\Theta(\mathrm{t},x)\phi_x^2(\mathrm{t});x)\right|.$$

From Theorem 2.2.9, we write

$$\left|\mathcal{J}_{n,\rho}^{\alpha,1}(f;x)-f(x)\right| \leq \frac{1}{2}\mathcal{J}_{n,\rho}^{\alpha,1}(\phi_x^2(\mathrm{t});x)|f''(x)|+\frac{C_8}{n}\omega\left(f'';\frac{1}{\sqrt{n}}\right),$$

where $C_8>0$ is a constant unrestrained from n and x.

Using Lemma 2.2.2 and the property of modulus of continuity, i.e. $\omega(h,\delta)\leq 2\|h\|$, we get

$$||\mathcal{J}_{n,\rho}^{\alpha,1}(f;.)-f|| \leq \frac{C_9}{n}||f''||, \tag{2.17}$$

where $C_9>0$ is a constant unrestrained from n and x.

Thus, for $g\in C[0,1]$, and using (2.16) and (2.17), we get

$$\begin{aligned}||\mathcal{J}_{n,\rho}^{\alpha,1}(g;.)-g|| \leq& ||\mathcal{J}_{n,\rho}^{\alpha,1}(g-f)||+||(g-f)||+||\mathcal{J}_{n,\rho}^{\alpha,1}(f)-f||\\ \leq& C_{10}||g-f||+\frac{C_9}{n}||f''|| \leq C_{11}\left[||g-f||+\frac{1}{n}||f''||\right],\end{aligned} \tag{2.18}$$

where C_{10} and C_{11} are some positive constants unrestrained from n and x.

Now, keeping in mind the equivalence of second order modulus of continuity $\omega_2(g,\delta)$ and K-functional (1.12) and by taking the infimum over all $f\in C^2[0,1]$ to (2.18), we get the required result. □

2.3 α-Bernstein Păltănea Operators of Second Order

Similarly, we can define the second order modification of the operators $\mathcal{Q}_{n,\rho}^{\alpha}(g;x)$ which is given by

$$\mathcal{J}_{n,\rho}^{\alpha,2}(g;x)=\sum_{\imath=0}^{n}p_{n,\imath}^{\alpha,2}(x)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})\,g(\mathrm{t})\,\mathrm{dt}, \tag{2.19}$$

$$\text{where}\qquad p_{n,\imath}^{\alpha,2}(x)=a(x,n)p_{n-2,\imath}^{\alpha}(x)+b(x,n)p_{n-2,\imath-1}^{\alpha}(x)+a(1-x,n)p_{n-2,\imath-2}^{\alpha}(x), \tag{2.20}$$

$\mu_{n,\imath}^{\rho}(\mathrm{t})$ is defined as (1.6), and $a(x,n) = a_2(n)x^2 + a_1(n)x + a_0(n)$, $b(x,n) = b_0(n)x(1-x)$.

If $a_2(n) = 1, a_1(n) = -2, a_0(n) = 1$, and $b_0(n) = 2$, then we will get our original operators (2.1).

Lemma 2.3.1. *For the operators (2.19), we have*

$$\begin{aligned}
J_{n,\rho}^{\alpha,2}(e_0;x) =& x^2(2a_2(n) - b_0(n)) + x(b_0(n) - 2a_2(n)) + (2a_0(n) + a_1(n) + a_2(n));\\
J_{n,\rho}^{\alpha,2}(e_1;x) =& \frac{1}{(n\rho+2)}\Big[n\{x^3(2a_2(n) - b_0(n))\rho + x^2(-2a_2(n) + b_0(n))\rho\\
&+ x(2a_0(n) + a_1(n) + a_2(n))\rho\} + \Big\{x^3(-4a_2(n) + 2b_0(n))\rho\\
&+ x^2(2a_2(n)(3\rho+1) - b_0(n)(3\rho+1))\\
&+ x(-2a_2(n)(3\rho+1) - 4\rho a_1(n) - 4\rho a_0(n) + b_0(n)(\rho+1))\\
&+ \ (a_2(n)(1+2\rho) + a_1(n)(1+2\rho) + 2a_0(n)(1+\rho))\}\Big];\\
J_{n,\rho}^{\alpha,2}(e_2;x) =& \frac{1}{(n\rho+2)(n\rho+3)}\Big[n^2\rho^2\Big\{x^4(2a_2(n) - b_0(n)) + x^3(-2a_2(n) + b_0(n))\\
&+ x^2(2a_0(n) + a_1(n) + a_2(n))\Big\} + n\Big\{-5x^4\rho^2(2a_2(n) - b_0(n))\\
&+ x^3\rho((16\rho+6)a_2(n) - (7\rho+4)b_0(n)) + x^2\rho(-3(5\rho+2)a_2(n) - 9\rho a_1(n)\\
&- 10\rho a_0(n) + 3(\rho+1)b_0(n)) + x\rho(6(\rho+1)a_0(n) + (3+5\rho)(a_1(n) + a_2(n)))\Big\}\\
&+ \Big\{2x^4\rho^2(2+\alpha)(2a_2(n) - b_0(n)) - 2x^3\rho(2\alpha\rho + 4\rho + 3)(2a_2(n) - b_0(n))\\
&+ x^2\Big(\rho^2(6(4+\alpha)a_2(n) + 2(6+\alpha)a_1(n) + 4(2+\alpha)a_0(n) - (5+2\alpha)b_0(n))\\
&+ (9\rho+2)(2a_2(n) - b_0(n))) + x\,(-2(2a_2(n) - b_0(n)) - 3\rho(6a_2(n)\\
&+ 4(a_1(n) + a_0(n)) - b_0(n)) - \rho^2(2(8+\alpha)a_2(n) + 2(6+\alpha)a_1(n)\\
&+4(2+\alpha)a_0(n) - b_0(n))) + 4\rho^2(a_2(n) + a_1(n) + a_0(n))\\
&+ 6\rho(a_2(n) + a_1(n) + a_0(n)) + 2(a_2(n) + a_1(n) + 2a_0(n))\Big\}\Big].
\end{aligned}$$

To study the uniform convergence of these operators, we set $J_{n,\rho}^{\alpha,2}(e_0;x) = 1$, *which gives the following conditions:*

$$2a_2(n) - b_0(n) = 0, \quad 2a_0(n) + a_1(n) + a_2(n) = 1.$$

With both of these conditions, other moments reduce to

$$\begin{aligned}
\mathcal{J}_{n,\rho}^{\alpha,2}(e_1;x) =& x+\frac{(1-2x)(1+2\rho-2\rho a_0(n)}{n\rho+2};\\
\mathcal{J}_{n,\rho}^{\alpha,2}(e_2;x) =& x^2-\frac{1}{(n\rho+2)(n\rho+3)}\left[\rho^2(2a_2(n)+2\alpha-n-4)-\rho(n+8)-6)x(1-x)\right.\\
&\left.+(1+2\rho)\right].
\end{aligned}$$

In order to have $\lim\limits_{n\to\infty}\mathcal{J}_{n,\rho}^{\alpha,2}(e_\imath;x)=x^\imath, \imath=0,1,2$, we choose undetermined coefficients as

$$a_0(n)=\frac{1+2\rho}{2\rho}, a_2(n)=\frac{n(\rho+1)}{2\rho},\ b_0(n)=\frac{n(\rho+1)}{\rho}, a_1(n)=\frac{-(n+2)(\rho+1)}{2\rho}.$$

Thus, our operators become

$$\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(g;x)=\sum_{\imath=0}^{n}\overline{p}_{n,\imath}^{\alpha,2}(x)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})\,g(\mathrm{t})\,\mathrm{dt}, \tag{2.21}$$

where

$$\begin{aligned}
\overline{p}_{n,\imath}^{\alpha,2}(x)=&\left(\frac{n(\rho+1)}{2\rho}x^2-\frac{(n+2)(\rho+1)}{2\rho}x+\frac{1+2\rho}{2\rho}\right)p_{n-2,\imath}^{\alpha}(x)\\
&+\frac{n(\rho+1)}{\rho}x(1-x)p_{n-2,\imath-1}^{\alpha}(x)\\
&+\left(\frac{n(\rho+1)}{2\rho}x^2-\frac{(n-2)(\rho+1)}{2\rho}x-\frac{1}{2\rho}\right)p_{n-2,\imath-2}^{\alpha}(x).
\end{aligned}$$

Lemma 2.3.2. *For the operators (2.21), we get*

$$\begin{aligned}
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(e_0;x) =& 1;\\
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(e_1;x) =& x;\\
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(e_2;x) =& x^2+\frac{1}{(n\rho+2)(n\rho+3)}\left[-1-2\rho+(6+8\rho+4\rho^2-2\alpha\rho^2)x(1-x)\right].
\end{aligned}$$

Lemma 2.3.3. *For the operators (2.21), we get the central moments as*

$$\begin{aligned}
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(\phi_x^2(\mathrm{t});x) =& \frac{1}{(n\rho+2)(n\rho+3)}\left[-1-2\rho+(6+8\rho+4\rho^2-2\alpha\rho^2)x(1-x)\right];\\
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(\phi_x^3(\mathrm{t});x) =& \frac{2x(1-x)(2x-1)\rho(1+\rho)(2+\rho)n}{(n\rho+2)(n\rho+3)(n\rho+4)}+O\left(\frac{1}{n^3}\right);\\
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(\phi_x^4(\mathrm{t});x) =& -\frac{3x^2(1-x)^2\rho^2(1+\rho)^2n^2}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)}+O\left(\frac{1}{n^3}\right);\\
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(\phi_x^5(\mathrm{t});x) =& \frac{30x^2(1-x)^2(2x-1)\rho^2(1+\rho)^2(2+\rho)n^2}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)(n\rho+6)}+O\left(\frac{1}{n^4}\right);\\
\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(\phi_x^6(\mathrm{t});x) =& -\frac{30x^3(1-x)^3\rho^3(1+\rho)^3n^3}{(n\rho+2)(n\rho+3)(n\rho+4)(n\rho+5)(n\rho+6)(n\rho+7)}+O\left(\frac{1}{n^4}\right).
\end{aligned}$$

Theorem 2.3.4. *If* $g\in C^6[0,1]$ *and* $x\in[0,1]$, *then for sufficiently large* n, *we have*

$$\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(g;x)-g(x)=O\left(\frac{1}{n^2}\right).$$

Proof. Consider the Taylor's formula and apply the operators $\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(.;x)$, we get

$$\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}(g;x)=g(x)+\sum_{\imath=1}^{6}\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}\left(\phi_x^{\imath}(\mathrm{t});x\right)\frac{g^{(\imath)}(x)}{\imath!}+\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}\left(\Theta(\mathrm{t},x)\phi_x^6(\mathrm{t});x\right),$$

where $\lim\limits_{\mathrm{t}\to x}\Theta(\mathrm{t},x)=0$.

We can easily see from Lemma 2.3.3, that it is enough to prove

$$\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}\left(\Theta(\mathrm{t},x)\phi_x^6(\mathrm{t});x\right)=0.$$

Now, $|\overline{\mathcal{J}}_{n,\rho}^{\alpha,2}\left(\Theta(\mathrm{t},x)\phi_x^6(\mathrm{t});x\right)|$

$$\begin{aligned}
\leq& \left|-\frac{n(\rho+1)}{2\rho}x(1-x)-\frac{2(\rho+1)}{2\rho}x+\frac{1+2\rho}{2\rho}\right|\sum_{\imath=0}^{n-2}p_{n-2,\imath}^{\alpha}(x)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})|\Theta(\mathrm{t},x)|\phi_x^6(\mathrm{t})\,\mathrm{dt}\\
&+\left|\frac{n(\rho+1)}{\rho}x(1-x)\right|\sum_{\imath=1}^{n-1}p_{n-2,\imath-1}^{\alpha}(x)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})|\Theta(\mathrm{t},x)|\phi_x^6(\mathrm{t})\,\mathrm{dt}\\
&+\left|-\frac{n(\rho+1)}{2\rho}x(1-x)+\frac{2(\rho+1)}{2\rho}x-\frac{1}{2\rho}\right|\sum_{\imath=2}^{n}p_{n-2,\imath-2}^{\alpha}(x)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})|\Theta(\mathrm{t},x)|\phi_x^6(\mathrm{t})\,\mathrm{dt}.
\end{aligned}$$

Let $M=\sup\limits_{\mathrm{t}\in[0,1]}|\Theta(\mathrm{t},x)|$, then we have

$$
\begin{aligned}
&|\overline{\mathcal{J}}^{\alpha,2}_{n,\rho}(\Theta(\mathrm{t},x)\phi_x^6(\mathrm{t});x)|\\
&< M\left[\frac{n(\rho+1)}{8\rho}+\frac{1+2\rho}{2\rho}\right]\sum_{\imath=0}^{n-2}p^{\alpha}_{n-2,\imath}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\\
&\quad + M\left[\frac{n(\rho+1)}{4\rho}\right]\sum_{\imath=1}^{n-1}p^{\alpha}_{n-2,\imath-1}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\\
&\quad + M\left[\frac{n(\rho+1)}{8\rho}+\frac{1+2\rho}{2\rho}\right]\sum_{\imath=2}^{n}p^{\alpha}_{n-2,\imath-2}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\\
&= \frac{Mn(\rho+1)}{8\rho}\left[\sum_{\imath=0}^{n-2}p^{\alpha}_{n-2,\imath}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}+2\sum_{\imath=1}^{n-1}p^{\alpha}_{n-2,\imath-1}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\right.\\
&\quad \left.+\sum_{\imath=2}^{n}p^{\alpha}_{n-2,\imath-2}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\right]+\frac{1+2\rho}{2\rho}M\left[\sum_{\imath=0}^{n-2}p^{\alpha}_{n-2,\imath}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\right.\\
&\quad \left.+\sum_{\imath=2}^{n}p^{\alpha}_{n-2,\imath-2}(x)\int_0^1\mu^{\rho}_{n,\imath}(\mathrm{t})\phi_x^6(\mathrm{t})\mathrm{dt}\right]\\
&= \frac{Mn(\rho+1)}{8\rho}\left\{\frac{15x^3\rho^3(4(1+\rho)^3(1-3x+3x^2)-x^3(4+4\rho(3+3\rho(3+\rho))))n^3}{\prod_{k=2}^{7}(n\rho+k)}\right.\\
&\quad \left.+O\left(\frac{1}{n^4}\right)\right\}+\frac{1+2\rho}{2\rho}M\left\{\frac{15x^3\rho^3((1+\rho)^3(1-3x+3x^2)-x^3(1+\rho(3+\rho(3+\rho))))n^3}{\prod_{k=2}^{7}(n\rho+k)}\right.\\
&\quad \left.+\frac{15x^3(1-x)^3\rho^3(1+\rho)^3n^3}{\prod_{k=2}^{7}(n\rho+k)}+O\left(\frac{1}{n^4}\right)\right\}=O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Hence, the proof is completed. □

Let us recall the following asymptotic formula for $Q^{\alpha}_{n,\rho}(g;x)$ proved in [85], in order to compare it with our operators:

Theorem 2.3.5. *Let $g \in C[0,1]$. If g'' exists at a point $x \in [0,1]$, then we have*

$$\lim_{n\to\infty} n(Q^{\alpha}_{n,\rho}(g;x)-g(x)) = \frac{1-2x}{\rho}g'(x)+\frac{(1+\rho)x(1-x)}{2\rho}g''(x).$$

The following result gives the comparison between $Q^{\alpha}_{n,\rho}(g;x)$ and $\overline{\mathcal{J}}^{\alpha,2}_{n,\rho}(g;x)$.

Theorem 2.3.6. *Let* $g \in C^6[0,1]$. *If there exists an* $n_0 \in \mathbb{N}$ *such that*

$$g(x) \leq \overline{J}^{\alpha,2}_{n,\rho}(g;x) \leq Q^{\alpha}_{n,\rho}(g;x), \quad \forall\, n \geq n_0,$$

then $2(1-2x)g'(x) + (1+\rho)x(1-x)g''(x) \geq 0$, $x \in [0,1]$.

Proof. Let us consider

$$\begin{aligned} g(x) \leq& \overline{J}^{\alpha,2}_{n,\rho}(g;x) \leq Q^{\alpha}_{n,\rho}(g;x),\\ \text{then} \quad 0 \leq& n(\overline{J}^{\alpha,2}_{n,\rho}(g;x) - g(x)) \leq n(Q^{\alpha}_{n,\rho}(g;x) - g(x)). \end{aligned}$$

From Theorems 2.3.4 and 2.3.5, we get the result. □

2.4 α-Bernstein Păltănea Operators of Third Order

Continuing in the same way as above, we can modify the operators $Q^{\alpha}_{n,\rho}(g;x)$ to obtain third order approximation operators, given as

$$J^{\alpha,3}_{n,\rho}(g;x) = \sum_{\imath=0}^{n} p^{\alpha,3}_{n,\imath}(x) \int_0^1 \mu^{\rho}_{n,\imath}(\mathrm{t})\, g(\mathrm{t})\, \mathrm{dt}, \tag{2.22}$$

where

$$\begin{aligned} p^{\alpha,3}_{n,\imath}(x) =& a(x,n)p^{\alpha}_{n-4,\imath}(x) + b(x,n)p^{\alpha}_{n-4,\imath-1}(x) + d(x,n)p^{\alpha}_{n-4,\imath-2}(x)\\ &+ b(1-x,n)p^{\alpha}_{n-4,\imath-3}(x) + a(1-x,n)p^{\alpha}_{n-4,\imath-4}(x), \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} a(x,n) =& a_4(n)x^4 + a_3(n)x^3 + a_2(n)x^2 + a_1(n)x + a_0(n),\\ b(x,n) =& b_4(n)x^4 + b_3(n)x^3 + b_2(n)x^2 + b_1(n)x + b_0(n),\\ d(x,n) =& d_0(n)x^2(1-x)^2. \end{aligned}$$

Here $a_\imath, b_\imath, \imath = 0,1,2,3,4$ and $d_0(n)$ are the sequences to be determined in such a way that the operators (2.22) reduce to new operators $\tilde{J}^{\alpha,3}_{n,\rho}(g;x)$ (*say*) with third order approximation. In order to get $J^{\alpha,3}_{n,\rho}(e_\imath, x) = e_\imath, \imath = 0, 1, 2, 3$, we find unknown sequences which are given by

$$\begin{aligned}
\tilde{a}_0(n) =& \frac{12\rho^3 + 19\rho^2 + 8\rho + 1}{12\rho^3},\\
\tilde{a_1}(n) =& -\frac{7\rho^2 + 11\rho + 4}{12\rho^2}n - \frac{(29 + 5\alpha)\rho^3 + (48 + 3\alpha)\rho^2 + 30\rho + 7}{6\rho^3},\\
\tilde{a}_2(n) =& \frac{(1 + \rho)^2}{8\rho^2}n^2 + \frac{17\rho^2 + 29\rho + 12}{12\rho^2}n + \frac{(41 - 11\alpha)\rho^3 + (71 - 9\alpha)\rho^2 + 60\rho + 18}{6\rho^3},\\
\tilde{a}_3(n) =& -\frac{(1 + \rho)^2}{4\rho^2}n^2 - \frac{5\rho^2 + 9\rho + 4}{6\rho^2}n - \frac{(3 - \alpha)\rho^3 + (7 - \alpha)\rho^2 + 6\rho + 2}{\rho^3},\\
\tilde{a}_4(n) =& \frac{(1 + \rho)^2}{8\rho^2}n^2, \qquad \tilde{b_0}(n) = -\frac{12\rho^2 + 7\rho + 1}{6\rho^3},\\
\tilde{b}_1(n) =& \frac{(3\rho^2 + 5\rho + 2)}{3\rho^2}n + \frac{(16 - 4\alpha)\rho^3 + (37 - 3\alpha)\rho^2 + 27\rho + 7}{3\rho^3},\\
\tilde{b}_2(n) =& -\frac{(1 + \rho)^2}{2\rho^2}n^2 - \frac{8\rho^2 + 14\rho + 6}{3\rho^2}n - \frac{(34 - 10\alpha)\rho^3 + (71 - 9\alpha)\rho^2 + 57\rho + 18}{3\rho^3},\\
\tilde{b}_3(n) =& \frac{(1 + \rho)^2}{\rho^2}n^2 + \frac{5\rho^2 + 9\rho + 4}{3\rho}n + \frac{2((3 - \alpha)\rho^3 + (7 - \alpha)\rho^2 + 6\rho + 2)}{\rho^3},\\
\tilde{b}_4(n) =& -\frac{(1 + \rho)^2}{2\rho^2}n^2, \qquad \tilde{d_0}(n) = \frac{3(1 + \rho)^2}{4\rho^2}n^2.
\end{aligned}$$

Lemma 2.4.1. *For the operators $\tilde{J}^{\alpha,3}_{n,\rho}(g; x)$, we get the following central moments*

$$\begin{aligned}
\tilde{J}^{\alpha,3}_{n,\rho}(\phi^1_x(\mathrm{t}); x) =& \tilde{J}^{\alpha,3}_{n,\rho}(\phi^2_x(\mathrm{t}); x) = \tilde{J}^{\alpha,3}_{n,\rho}(\phi^3_x(\mathrm{t}); x) = 0;\\
\tilde{J}^{\alpha,3}_{n,\rho}(\phi^4_x(\mathrm{t}); x) =& \frac{x(1 - x)\rho(1 + \rho)[11 + 27\rho + 12\rho^2 - x(1 - x)\{2\rho^2(6\alpha - 29) + 58(1 + 2\rho)\}]n}{\prod\limits_{k=2}^{5}(n\rho + k)}\\
&+ O\left(\frac{1}{n^4}\right);\\
\tilde{J}^{\alpha,3}_{n,\rho}(\phi^5_x(\mathrm{t}); x) =& \frac{5x^2(1 - x)^2(2x - 1)\rho^2(1 + \rho)^2(5 + 4\rho)n^2}{\prod\limits_{k=2}^{6}(n\rho + k)} + O\left(\frac{1}{n^4}\right);\\
\tilde{J}^{\alpha,3}_{n,\rho}(\phi^6_x(\mathrm{t}); x) =& \frac{15x^3(1 - x)^3\rho^3(1 + \rho)^3 n^3}{\prod\limits_{k=2}^{7}(n\rho + k)} + O\left(\frac{1}{n^4}\right);\\
\tilde{J}^{\alpha,3}_{n,\rho}(\phi^7_x(\mathrm{t}); x) =& O\left(\frac{1}{n^4}\right); \tilde{J}^{\alpha,3}_{n,\rho}(\phi^8_x(\mathrm{t}); x) = O\left(\frac{1}{n^4}\right);
\end{aligned}$$

$\tilde{J}^{\alpha,3}_{n,\rho}(\phi^9_x(\mathrm{t});x) = O\left(\frac{1}{n^5}\right); \tilde{J}^{\alpha,3}_{n,\rho}(\phi^{10}_x(\mathrm{t});x) = O\left(\frac{1}{n^4}\right).$

To prove the asymptotic order of approximation of the operators $\tilde{J}^{\alpha,3}_{n,\rho}(g;x)$, we require $g \in C^{10}[0,1]$ in a similar way, as in Theorem 2.3.4, which is given as follows:

Theorem 2.4.2. *If $g \in C^{10}[0,1]$ and $x \in [0,1]$, then for sufficiently large n, we have*

$$\tilde{J}^{\alpha,3}_{n,\rho}(g;x) - g(x) = O\left(\frac{1}{n^3}\right).$$

2.5 Numerical Results

In the present section, we give the numerical examples to validate our theoretical results and error estimation by using maple algorithms.

Example 2.5.1. *Consider $g(x) = 2sin\left(\frac{\pi x}{2}\right) + sin(2\pi x)$, $n = 10, \rho = 5, \alpha = 0.2, a_0(n) = \frac{n-1}{2n}$, and $a_1(n) = \frac{1}{n}$. The comparison of convergence of the α-Bernstein Păltănea operators and its above modifications of orders one, two and three to $g(x)$ is given in Fig. 2.1.*

Let $E^{\alpha}_{n,\rho}(g;x) = |g(x) - Q^{\alpha}_{n,\rho}(g;x)|$ and $E^{\alpha,\jmath}_{n,\rho}(g;x) = |g(x) - J^{\alpha,\jmath}_{n,\rho}(g;x)|$, $\jmath = 1,2,3$ be

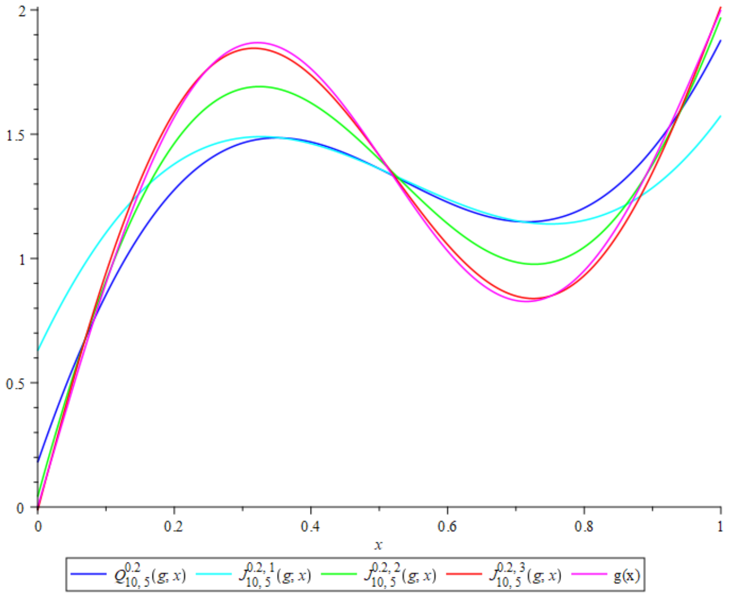


Figure 2.1: Approximation process for $n = 10, \alpha = 0.2$ and $\rho = 5$

error function of classical operators and its modifications respectively. The error of approximation of these operators is given in Fig. 2.2.

From both the figures, we conclude that our modified operators are converging faster than original α-Bernstein Păltănea operators. Also, we have given error of approximation at some certain points in Table 2.1.

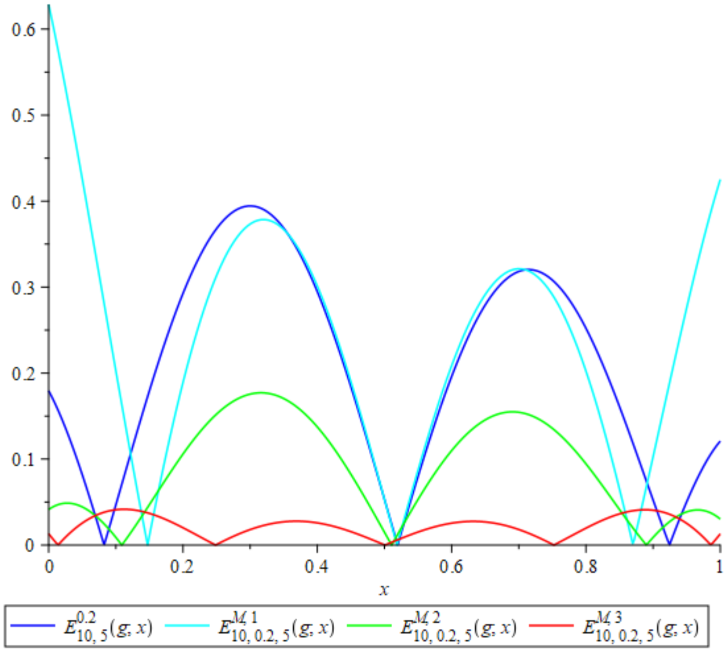


Figure 2.2: Error estimation for $n = 10, \alpha = 0.2$ and $\rho = 5$

Table 2.1: Error of Approximation $E^{\alpha}_{n,\rho}$ and $E^{\alpha,\jmath}_{n,\rho}, \jmath = 1, 2, 3, n = 10, \rho = 5, \alpha = 0.2$

x	$E^{0.2}_{10,5}(g;x)$	$E^{0.2,1}_{10,5}(g;x)$	$E^{0.2,2}_{10,5}(g;x)$	$E^{0.2,3}_{10,5}(g;x)$
0.1	0.0465927917	0.2029191714	0.0090258508	0.0410711480
0.2	0.292607380	0.189033401	0.105857578	0.020163736
0.3	0.394424647	0.373376025	0.175570300	0.018015963
0.4	0.294804927	0.301231910	0.137423887	0.025812675
0.5	0.053236320	0.053236321	0.014926178	0.000239280
0.6	0.196044423	0.208534327	0.109839626	0.025493047
0.7	0.3182317026	0.3213021846	0.1544639662	0.0180914623
0.8	0.2521506342	0.2023484272	0.0944769302	0.0196894259
0.9	0.052294816	0.102854286	0.008980412	0.040540562

Example 2.5.2. *Let us choose* $g(x) = x\cos(2\pi x)$, $\rho = 4, \alpha = 0.3, a_0(n) = \dfrac{n-1}{2n}$,

and $a_1(n) = \dfrac{1}{n}$. *The behavior of* α*-Bernstein-Păltănea operators* $Q^{\alpha}_{n,\rho}(g;x)$ *and its three modifications* $J^{\alpha,\jmath}_{n,\rho}(g;x)$ *where* $\jmath = 1,2,3$ *to* $g(x)$ *for* $n = 10, 20$ *is given in Figs. 2.3, 2.5. We can observe from figures that our modifications are converging to the function as we increase the value of* n*. Also, it give better convergence than original operators* $Q^{\alpha}_{n,\rho}(g;x)$*.*

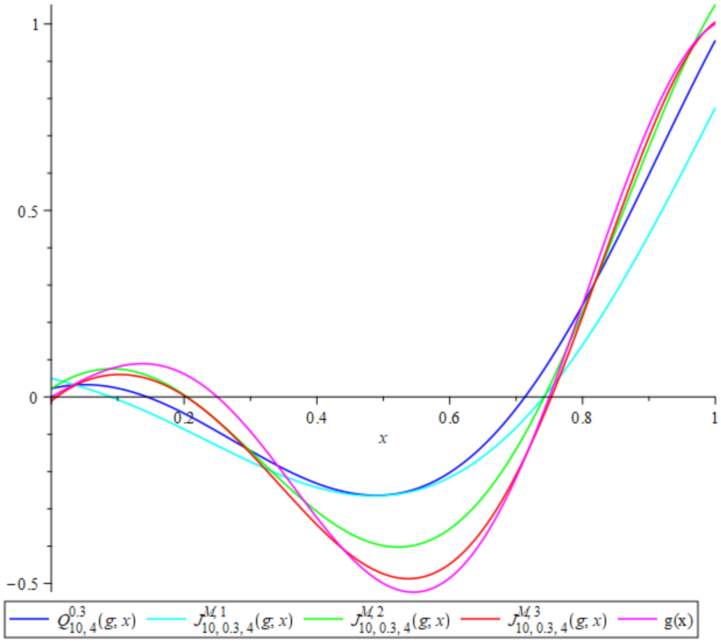


Figure 2.3: Approximation process for $n = 10, \alpha = 0.3$ and $\rho = 4$

The error of approximation of α*-Bernstein Păltănea operators and its modifications of order* $1,2,3$ *are given in Figs. 2.4, 2.6 for* $n = 10, 20$ *resp. It can be easily seen that error estimation by our modifications are less than original operators* $Q^{\alpha}_{n,\rho}(g;x)$*. Also, the error of operators and its modifications at some points are given in Tables 2.2, 2.3 at the values* $n = 10, 20$ *resp.*

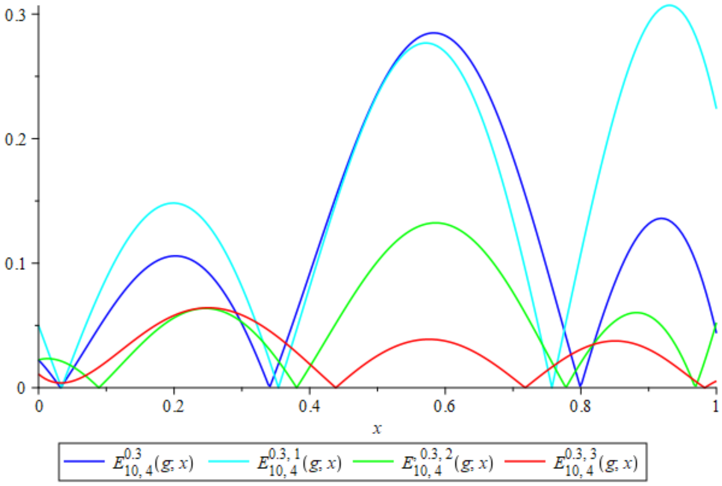


Figure 2.4: Error estimation for $n = 10, \alpha = 0.3$ and $\rho = 4$

Table 2.2: Error of Approximation $E^{\alpha}_{n,\rho}$ and $E^{\alpha,\jmath}_{n,\rho}, \jmath = 1, 2, 3, n = 10, \rho = 4, \alpha = 0.3$

x	$E^{0.3}_{10,4}(g;x)$	$E^{0.3,1}_{10,4}(g;x)$	$E^{0.3,2}_{10,4}(g;x)$	$E^{0.3,3}_{10,4}(g;x)$
0.1	0.05755638690	0.07198182417	0.00593343135	0.02211119337
0.2	0.1058403144	0.1347020007	0.05618431396	0.00086323684
0.3	0.05131069544	0.07328503894	0.05274231364	0.01900023794
0.4	0.0916546188	0.0844657778	0.0166755487	0.0079394543
0.5	0.2366464156	0.2366464156	0.1013426174	0.0193312761
0.6	0.2827508318	0.2698804172	0.1314792584	0.0277669661
0.7	0.1851049397	0.1344774017	0.0765221240	0.0019799015
0.8	0.0010362418	0.1078472481	0.0202148003	0.0324610925
0.9	0.1316309131	0.2930201701	0.0579303733	0.0309826354

Table 2.3: Error of Approximation $E^{\alpha}_{n,\rho}$ and $E^{\alpha,\jmath}_{n,\rho}, \jmath = 1, 2, 3, n = 20, \rho = 4, \alpha = 0.3$

x	$E^{0.3}_{20,4}(g;x)$	$E^{0.3,1}_{20,4}(g;x)$	$E^{0.3,2}_{20,4}(g;x)$	$E^{0.3,3}_{20,4}(g;x)$
0.1	0.02861514285	0.02978527157	0.00135692531	0.00493671473
0.2	0.06515132595	0.08083293178	0.0r1810232856	0.00184414784
0.3	0.04171602804	0.05754420964	0.02246595754	0.00339046874
0.4	0.0423780236	0.0356126371	0.0005113770	0.0023993240
0.5	0.1336280335	0.1336280335	0.0330134666	0.003147654
0.6	0.1637928626	0.1572063097	0.0448962830	0.0048952830
0.7	0.1024392756	0.0721778840	0.0230362411	0.0006129461
0.8	0.0113308695	0.0741312889	0.0121952799	0.0062328086
0.9	0.0822517165	0.1657103520	0.0201291953	0.0033482141

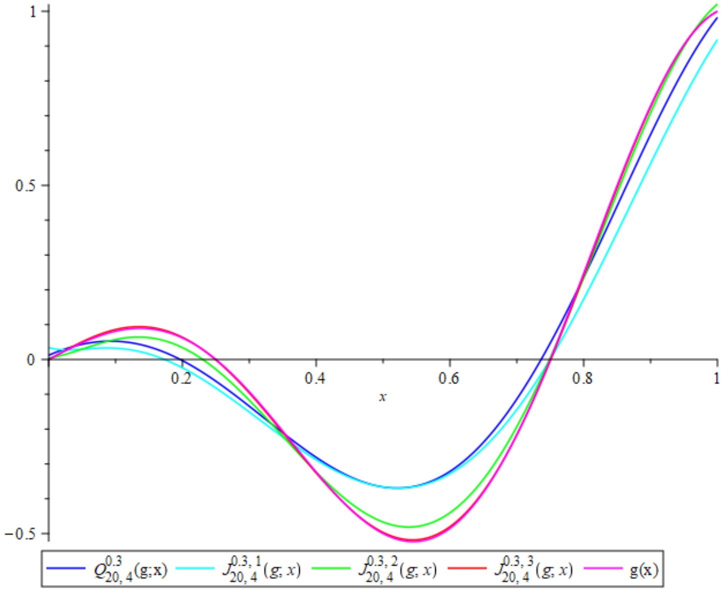


Figure 2.5: Approximation process for $n = 20, \alpha = 0.3$ and $\rho = 4$

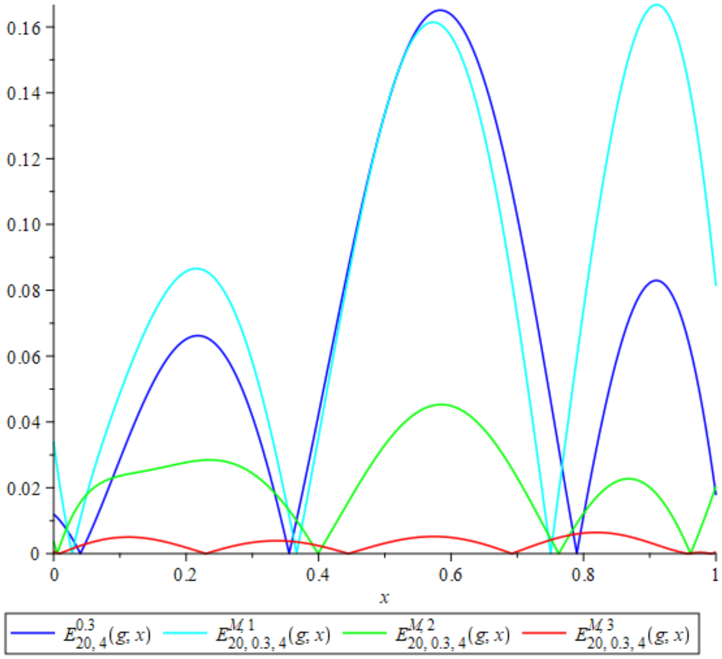


Figure 2.6: Error estimation for $n = 20, \alpha = 0.3$ and $\rho = 4$

2.6 Conclusion

As the rate of convergence of classical α-Bernstein Durrmeyer type operators $Q_{n,\rho}^{\alpha}(g;x)$ is slow, so we have improved the rate of convergence of these operators by introducing three modifications which have better order of approximation. Moreover the convergence of these operators is independent of parameters involved in it. In order to validate our theoretical results, we have presented some numerical examples and their graphics by using MAPLE algorithms. With the similar procedure, we can also get higher order of approximation of these positive linear operators.

In this chapter, we discuss the operators of a single variable. But, in different aspects of life, we do not only deal with the functions of one variable, we also encounter the functions of two as well as several variables. Thus, it is important to study the positive linear operators that can approximate functions of more than one variables. In the next chapter, we will study this behavior for the operators (2.1).

Chapter 3

Approximation properties of bivariate extension of blending type operators

3.1 Introduction

In addressing various real-world challenges, where considerations involve multiple factors like product costs, store profits or price of shares, the application of mathematical modeling becomes essential. These models often generate functions dependent on two or more variables. The utilization of such functions and their approximations extends across diverse fields, spanning Economics, Continuum Mechanics, Thermodynamics, Fluid Dynamics, and others. This highlights the importance of multi-variable functions in providing insights and solutions to complex problems across various domains. Following Kingsley [97], who initiate the idea to approximate functions of two or more variables by positive linear operators, a lot of work done on approximation of functions of several variables. Motivated by this literature and applications of two dimensional operators, we provide the bivariate extension of the operators $J_{n,\rho}^{\alpha,1}(g;x)$ defined by (2.1). The format of the current chapter is as follows:

We start by defining the bivariate extension of the operators (2.1) followed by some preliminary findings like moments and central moments. In section 3.3, we present the results such as convergence of operators, Voronovskaja type asymptotic results, error of operators for the Lipschitz continuous functions and in form of approximation tools such as complete and partial modulus of continuities, second order modulus of smoothness. The next section 3.4 contains some graphs of certain functions to verify our theoretical results. In the last section, Generalized Boolean Sum (GBS) operators associated with the operators $J^{\alpha,1}_{n,\rho}(g;x)$ (2.1) are defined and studied their approximation results.

3.2 Construction of the Operators

For any $g \in C(I^2)$, the bivariate extension of the operators (2.1) is defined as

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y) = \sum_{\imath=0}^{n}\sum_{\jmath=0}^{m} p^{\alpha_1,1}_{n,\imath}(x)\, p^{\alpha_2,1}_{m,\jmath}(y) \int_0^1\int_0^1 \mu^{\rho_1}_{n,\imath}(\mathrm{t})\, \mu^{\rho_2}_{m,\jmath}(\mathrm{s})\, g(\mathrm{t},\mathrm{s})\,\mathrm{ds}\,\mathrm{dt}, \qquad (3.1)$$

where $\mu^{\rho_1}_{n,\imath}(\mathrm{t})$, $\mu^{\rho_2}_{m,\jmath}(\mathrm{s})$ are same as in (1.6) and

$$p^{\alpha_1,1}_{n,\imath}(x) = a(x,n)\, p^{\alpha_1}_{n-1,\imath}(x) + a(1-x,n)\, p^{\alpha_1}_{n-1,\imath-1}(x),$$

$$p^{\alpha_2,1}_{m,\jmath}(y) = b(y,m)\, p^{\alpha_2}_{m-1,\jmath}(y) + b(1-y,m)\, p^{\alpha_2}_{m-1,\jmath-1}(y).$$

Let $f(x), g(y) \in C(I)$, then our operators (3.1) satisfy the following relationship:

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(f(x)g(y);x,y) = J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(f(x);x,y)\, J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g(y);x,y).$$

Lemma 3.2.1. *For the operators (3.1) and test functions* $e_{\imath\jmath} = \mathrm{t}^{\imath}\mathrm{s}^{\jmath}$, *we have the following:*

$$\begin{aligned}
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{00};x,y) =& 1;\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{10};x,y) =& x + \frac{(1-2x)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{01};x,y) =& y + \frac{(1-2y)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{20};x,y) =& x^2 + \frac{1}{(n\rho_1+2)(n\rho_1+3)}\left[n\rho_1 x\{(1+\rho_1)(3-5x) - 2a_0(n)\rho_1(1-2x)\}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ {\rho_1}^2(1-2x(1-x)(1+\alpha_1)) + 3\rho_1(1-2x) + 2(1-3x^2)\\
&-a_0(n)\rho_1(\rho_1(4x^2-4x+1)+3(1-2x))\Big];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{02};x,y) =& y^2 + \frac{1}{(m\rho_2+2)(m\rho_2+3)}\left[m\rho_2 y\{(1+\rho_2)(3-5y) - 2b_0(m)\rho_2(1-2y)\}\right.\\
&+ {\rho_2}^2(1-2y(1-y)(1+\alpha_2)) + 3\rho_2(1-2y) + 2(1-3y^2)\\
&-b_0(m)\rho_2(\rho_2(4y^2-4y+1)+3(1-2y))\Big].
\end{aligned}
$$

Lemma 3.2.2. *For the operators (3.1), we have the following central moments:*

$$
\begin{aligned}
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^1_x(\mathrm{t});x,y) =& \frac{(1-2x)(\rho_1+1-\rho_1 a_0(n))}{n\rho_1+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^1_y(\mathrm{s});x,y) =& \frac{(1-2y)(\rho_2+1-\rho_2 b_0(m))}{m\rho_2+2};\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y) =& \frac{1}{(n\rho_1+2)(n\rho_1+3)}\left[n\rho_1(1+\rho_1)x(1-x) - x(1-x)\{6(1+2\rho_1)\right.\\
&\left.+ 2{\rho_1}^2(1+\alpha_1)\} + (\rho_1+1)(\rho_1+2) + a_0(n)\rho_1(\rho_1+3)(4x(1-x)-1)\right];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y) =& \frac{1}{(m\rho_2+2)(m\rho_2+3)}\left[m\rho_2(1+\rho_2)y(1-y) - y(1-y)\{6(1+2\rho_2)\right.\\
&\left.+ 2{\rho_2}^2(1+\alpha_2)\} + (\rho_2+1)(\rho_2+2) + b_0(m)\rho_2(\rho_2+3)(4y(1-y)-1)\right];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^4_x(\mathrm{t});x,y) =& \frac{3{\rho_1}^2(1+\rho_1)^2x^2(1-x)^2n^2}{(n\rho_1+2)(n\rho_1+3)(n\rho_1+4)(n\rho_1+5)} + O\left(\frac{1}{n^3}\right);\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^4_y(\mathrm{s});x,y) =& \frac{3{\rho_2}^2(1+\rho_2)^2y^2(1-y)^2m^2}{(m\rho_2+2)(m\rho_2+3)(m\rho_2+4)(m\rho_2+5)} + O\left(\frac{1}{m^3}\right).
\end{aligned}
$$

Corollary 3.2.3. *From the Lemma 3.2.2, we can easily get*

$$
\begin{aligned}
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y) \leq& \frac{1+\rho_1}{n\rho_1+2}\left[x(1-x) + \frac{2+\rho_1}{n\rho_1+2}\right];\\
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y) \leq& \frac{1+\rho_2}{m\rho_2+2}\left[y(1-y) + \frac{2+\rho_2}{m\rho_2+2}\right].
\end{aligned}
$$

3.3 Approximation Results

Theorem 3.3.1. *Let $g \in C(I^2)$, the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ converge uniformly to $g(x,y)$.*

Proof. By Lemma 3.2.1, we can calculate

$$\begin{aligned}\lim_{n,m\to\infty} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{10};x,y) &= x;\\ \lim_{n,m\to\infty} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{01};x,y) &= y;\\ \lim_{n,m\to\infty} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(e_{20}+e_{02};x,y) &= x^2+y^2.\end{aligned}$$

Now, by using Volkov's result [145] for bivariate operators, we get the uniform convergence of the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ to $g(x,y)$. ☐

Theorem 3.3.2. *For any $g \in C^2(I^2)$ and $(x,y) \in I^2$, we obtain*

$$\begin{aligned}\lim_{n\to\infty} n(J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(g;x,y)-g(x,y)) =& \frac{(\rho_1+1-\rho_1 a_0(n))}{\rho_1}\left[g_x(x,y)(1-2x)+g_y(x,y)(1-2y)\right]\\ &+\frac{(1+\rho_1)}{2\rho_1}\left[g_{xx}(x,y)x(1-x)+g_{yy}(x,y)y(1-y)\right].\end{aligned}$$

Proof. Using the Taylor's formula for a fixed point $(x,y) \in I^2$, we get

$$\begin{aligned}g(\mathrm{t},\mathrm{s}) =& g(x,y)+g_x(x,y)\phi^1_x(\mathrm{t})+g_y(x,y)\phi^1_y(\mathrm{s})+\frac{1}{2}\Big[g_{xx}(x,y)\phi^2_x(\mathrm{t})\\ &+2g_{xy}(x,y)\phi^1_x(\mathrm{t})\phi^1_y(\mathrm{s})+g_{yy}(x,y)\phi^2_y(\mathrm{s})\Big]+\Theta(\mathrm{t},\mathrm{s})\{\phi^2_x(\mathrm{t})+\phi^2_y(\mathrm{s})\}, \qquad (3.2)\end{aligned}$$

where $\lim\limits_{(\mathrm{t},\mathrm{s})\to(x,y)} \Theta(\mathrm{t},\mathrm{s}) = 0$ and $(x,y) \in I^2$.
By applying the operators to (3.2) and by its linearity property, we obtain
$n\left(J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(g;x,y)-g(x,y)\right)$

$$\begin{aligned}=& n\Bigg(J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^1_x(\mathrm{t});x,y)g_x(x,y)+J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^1_y(\mathrm{s});x,y)g_y(x,y)\\ &+\frac{1}{2}\Big\{g_{xx}(x,y)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^2_x(\mathrm{t});x,y)+2g_{xy}(x,y)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^1_x(\mathrm{t})\phi^1_y(\mathrm{s});x,y)\\ &+g_{yy}(x,y)J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^2_y(\mathrm{s});x,y)\Big\}+J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta(\mathrm{t},\mathrm{s})\{\phi^2_x(\mathrm{t})+\phi^2_y(\mathrm{s})\};x,y)\Bigg). \qquad (3.3)\end{aligned}$$

Apply Hölder's inequality to the last term of right hand side of (3.3)

$$
\begin{aligned}
&nJ^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}\Big(\Theta(\mathrm{t},\mathrm{s})\{\phi^2_x(\mathrm{t})+\phi^2_y(\mathrm{s})\};x,y\Big)\\
&\qquad\leq n\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta^2(\mathrm{t},\mathrm{s});x,y)\right]^{\frac{1}{2}}\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}((\phi^2_x(\mathrm{t})+\phi^2_y(\mathrm{s}))^2;x,y)\right]^{\frac{1}{2}}\\
&\qquad\leq\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta^2(\mathrm{t},\mathrm{s});x,y)\right]^{\frac{1}{2}}\\
&\qquad\quad\times\sqrt{2}n\left[J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^4_x(\mathrm{t});x,y)+J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\phi^4_y(\mathrm{s});x,y)\right]^{\frac{1}{2}} \qquad (3.4)
\end{aligned}
$$

$$[\because (a+b)^2\leq 2(a^2+b^2)].$$

By using Theorem 3.3.1, we get

$$\lim_{n\to\infty} J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta^2(\mathrm{t},\mathrm{s});x,y)=0.$$

With the help of central moments of order 4 in Lemma 3.2.2, we obtain

$$\lim_{n\to\infty} n\,J^{\alpha_1,\alpha_1}_{n,n,\rho_1,\rho_1}(\Theta(\mathrm{t},\mathrm{s})\{\phi^2_x(\mathrm{t})+\phi^2_y(\mathrm{s})\};x,y)=0.$$

Thus, by combining (3.3), (3.4), and using Lemma 3.2.2, we get our required result.

□

Theorem 3.3.3. *Let $g\in C(I^2)$, the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ satisfy the following relation:*

$$|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\leq 4\,\omega(g;\delta_1(x),\delta_2(y)),$$

where $\delta_1(x):=\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y)}$ and $\delta_2(y):=\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y)}$.

Proof. By using linearity of the operators (3.1), we have

$$|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|g(\mathrm{t},\mathrm{s})-g(x,y)|;x,y).$$

With the help of the relation (1.13) of complete modulus of continuity

$$|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\leq\omega(g;\delta_1(x),\delta_2(y))\left(1+\frac{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_x(\mathrm{t})|;x,y)}{\delta_1(x)}\right)$$

$$\times\left(1+\frac{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_y(\mathrm{s})|;x,y)}{\delta_2(y)}\right). \tag{3.5}$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} &J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_x(\mathrm{t})|;x,y)\le\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y)\right]^{\frac12}:=\delta_1(x),\\ \text{and}\quad &J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_y(\mathrm{s})|;x,y)\le\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y)\right]^{\frac12}:=\delta_2(y).\end{aligned}$$

Now, substituting the above two inequalities in (3.5), we obtain the desired result. □

Theorem 3.3.4. *For $g\in C(I^2)$, the operators (3.1) have the following inequality:*

$$|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\le 2\left[\omega^1(g;\delta_1(x))+\omega^2(g;\delta_2(y))\right],$$

where $\delta_1(x)$ and $\delta_2(y)$ are same as in Theorem 3.3.3.

Proof. With the help of linearity and partial modulus of continuity, we get

$$\begin{aligned}|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\le&J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|g(\mathrm{t},\mathrm{s})-g(x,y)|;x,y)\\ \le&J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|g(\mathrm{t},y)-g(x,y)|;x,y)\\ &+J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|g(\mathrm{t},\mathrm{s})-g(\mathrm{t},y)|;x,y)\\ \le&J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\omega^1(g;|\phi^1_x(\mathrm{t})|);x,y)\\ &+J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\omega^2(g;|\phi^1_y(\mathrm{s})|);x,y).\end{aligned} \tag{3.6}$$

Since modulus of continuity satisfies the relation $\omega(\lambda\delta)\le(1+\lambda)\,\omega(\delta)$, for $\lambda>0$, then using this relation in (3.6), we obtain

$$\begin{aligned}|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\le&J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left(1+\frac{|\phi^1_x(\mathrm{t})|}{\delta_1(x)}\right)\omega^1\left(g;\delta_1(x)\right);x,y\right)\\ &+J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left(1+\frac{|\phi^1_y(\mathrm{s})|}{\delta_2(y)}\right)\omega^2\left(g;\delta_2(y)\right);x,y\right).\end{aligned}$$

Using Cauchy-Schwarz inequality, the above term reduces to

$$\begin{aligned}
\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid \leq & \left\{1+\frac{\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^2(\mathrm{t});x,y)}}{\delta_1(x)}\right\}\omega^1(g;\delta_1(x)) \\
& +\left\{1+\frac{\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_y^2(\mathrm{s});x,y)}}{\delta_2(y)}\right\}\omega^2(g;\delta_2(y)).
\end{aligned}$$

On choosing $\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^2(\mathrm{t});x,y)} := \delta_1(x)$ and $\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_y^2(\mathrm{s});x,y)} := \delta_2(y)$, the proof is done. □

Theorem 3.3.5. *Let $g \in Lip_{\mathscr{M}}(\zeta,\eta)$, then the following inequality holds:*

$$|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)| \leq \mathscr{M}(\delta_1(x))^{\zeta}\,(\delta_2(y))^{\eta},$$

where $\mathscr{M} > 0$ is a constant and $\delta_1(x)$ and $\delta_2(y)$ are as in Theorem 3.3.3.

Proof. As $g \in Lip_{\mathscr{M}}(\zeta,\eta)$, then it gives

$$|g(\mathrm{t},\mathrm{s})-g(x,y)| \leq \mathscr{M}|\phi_x^1(\mathrm{t})|^{\zeta}|\phi_y^1(\mathrm{s})|^{\eta}.$$

Now,

$$\begin{aligned}
|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)| \leq & J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|g(\mathrm{t},\mathrm{s})-g(x,y)|;x,y) \\
\leq & J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\mathscr{M}|\phi_x^1(\mathrm{t})\ |^{\zeta}|\ \phi_y^1(\mathrm{s})|^{\eta};x,y) \\
= & \mathscr{M}J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|^{\zeta};x,y).J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_y^1(\mathrm{s})|^{\eta};x,y).
\end{aligned} \tag{3.7}$$

Using Hölder's inequality

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|^{\zeta};x,y) \leq \left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^2(\mathrm{t});x,y)\right]^{\frac{\zeta}{2}} = (\delta_1(x))^{\zeta}.$$

Similarly,

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_y(\mathrm{s})|^{\eta};x,y) \leq \left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y)\right]^{\frac{\eta}{2}} = (\delta_2(y))^{\eta}.$$

By using these inequalities in (3.7), we attain the desired outcome. □

Theorem 3.3.6. *For $g \in C(I^2)$, we get the error estimation for the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ in form of first and second order modulus of continuities*

$$\begin{aligned}|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)| \leq& 4C\left[\omega_2\left(g;\frac{1}{2}\sqrt{V_{n,m}(x,y)}\right)+\min\left(1,\frac{1}{4}V_{n,m}(x,y)\right)\right]\\ &+\omega(g;\mu_{n,m}(x,y)),\\ \textit{where}\quad V_{n,m}(x,y)=&\frac{1}{2}\Bigg[(\delta_1(x))^2+\left(\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}\right)^2\\ &+(\delta_2(y))^2+\left(\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)^2\Bigg],\\ \textit{and}\quad \mu_{n,m}(x,y)=&\Bigg(\left(\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}\right)^2\\ &+\left(\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)^2\Bigg)^{\frac{1}{2}},\end{aligned}$$

where $C>0$ is a constant and $\delta_1(x)$ and $\delta_2(y)$ are given in Theorem 3.3.3.

Proof. Firstly, we define auxiliary operators for $(x,y)\in I^2$ as

$$\begin{aligned}\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)=&J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)+g(x,y)\\ &-g\left(x+\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2},y+\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right).\end{aligned}$$

By using this definition, we can get

$$\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^1_x(\mathrm{t});x,y)=0,\qquad \tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^1_y(\mathrm{s});x,y)=0.$$

For $h\in C^2(I^2)$, we consider

$$
\begin{aligned}
&|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)|\\
&\quad\leq \left|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)\right|+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;x,y)\right|\\
&\qquad+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;x,y)-h(x,y)\right|+|h(x,y)-g(x,y)|\\
&\quad=\left|g\left(x+\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2},y+\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)-g(x,y)\right|\\
&\qquad+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g-h;x,y)\right|+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;x,y)-h(x,y)\right|+|h(x,y)-g(x,y)|\,. \qquad (3.8)
\end{aligned}
$$

Now, by Taylor's polynomial for $h(\mathrm{t},\mathrm{s})\in C^2(I^2)$, we have

$$
\begin{aligned}
h(\mathrm{t},\mathrm{s})=&h(x,y)+\phi^1_x(\mathrm{t})\frac{\partial h(x,y)}{\partial x}+\int_x^{\mathrm{t}}\phi^1_u(\mathrm{t})\frac{\partial^2 h(u,y)}{\partial u^2}\,du+\phi^1_y(\mathrm{s})\frac{\partial h(x,y)}{\partial y}\\
&+\int_y^{\mathrm{s}}\phi^1_v(\mathrm{s})\frac{\partial^2 h(x,v)}{\partial v^2}\,dv.
\end{aligned}
$$

Applying the operators $\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(.;x,y)$, we get

$$
\begin{aligned}
&\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;x,y)-h(x,y)\right|\\
&\quad\leq\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\int_x^{\mathrm{t}}\phi^1_u(\mathrm{t})\frac{\partial^2 h(u,y)}{\partial u^2}\,du;x,y\right)\right|+\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\int_y^{\mathrm{s}}\phi^1_v(\mathrm{s})\frac{\partial^2 h(x,v)}{\partial v^2}\,dv;x,y\right)\right|\\
&\quad\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left|\int_x^{\mathrm{t}}\left|\phi^1_u(\mathrm{t})\right|\left|\frac{\partial^2 h(u,y)}{\partial u^2}\right|\,du\right|;x,y\right)\\
&\qquad+\left|\int_x^{x+\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}}\left|x+\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}-u\right|\left|\frac{\partial^2 h(u,y)}{\partial u^2}\right|\,du\right|\\
&\qquad+J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left|\int_y^{\mathrm{s}}\left|\phi^1_v(\mathrm{s})\right|\left|\frac{\partial^2 h(x,v)}{\partial v^2}\right|\,dv\right|;x,y\right)\\
&\qquad+\left|\int_y^{y+\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}}\left|y+\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}-v\right|\left|\frac{\partial^2 h(x,v)}{\partial v^2}\right|\,dv\right|\\
&\quad\leq\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y)+\left(\frac{(1-2x)(\rho_1+1-\rho_1a_0(n))}{n\rho_1+2}\right)^2\right.\\
&\qquad\left.+J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y)+\left(\frac{(1-2y)(\rho_2+1-\rho_2b_0(m))}{m\rho_2+2}\right)^2\right]\|h\|_{C^2(I^2)}.
\end{aligned}
$$

With the values of $\delta_1(x)$ and $\delta_2(y)$, we have

$$\left|\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(h;x,y)-h(x,y)\right|\leq V_{n,m}(x,y)\|h\|_{C^2(I^2)}.$$

Using the definition of operators $\tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$, we obtain

$$\mid \tilde{J}^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)\mid\leq 3\|g\|_{C(I^2)}.$$

Thus, by using above inequalities, the equation (3.8) becomes

$$\begin{aligned}\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid\leq&4\|g-h\|_{C(I^2)}+V_{n,m}(x,y)\|h\|_{C^2(I^2)}+\omega\left(g;\mu_{n,m}(x,y)\right)\\=&4\left\{\|g-h\|_{C(I^2)}+\frac{1}{4}V_{n,m}(x,y)\|h\|_{C^2(I^2)}\right\}+\omega(g;\mu_{n,m}(x,y)).\end{aligned}$$

Taking the infimum over $h\in C^2(I^2)$ and using the relation (1.12), we get the result.

□

3.4 Numerical Verification

In this section, we justify our previously proven theoretical findings by providing an example with various values of parameters and sequences $a_\imath(n), \imath=0,1$. For this purpose, we take into the account the function $g(x,y)=x^4y^2-2x^2y^3+xy^2$.

Firstly, we present the convergence of the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ with specific values of parameters to the function $g(x,y)$ in Fig. 3.1 having sequences $a_0(n)=\dfrac{n-1}{2n}, a_1(n)=\dfrac{1}{n}, b_0(m)=\dfrac{m-1}{2m}, b_1(m)=\dfrac{1}{m}$.

Also, we give their error estimation, which is defined as $E^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)=\mid J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid$ for the particular values of parameters in Fig. 3.2.

We can see in both the images that by increasing the values of n,m, the operators converge quickly to the specified function and the error term decrease rapidly.

In Fig. 3.3, we show the effect of the different sequences $a_\imath(n), \imath=0,1$ in the convergence of the operators with specific parameters $n=m=30, \alpha_1=\alpha_2=0.3, \rho_1=\rho_2=4$.

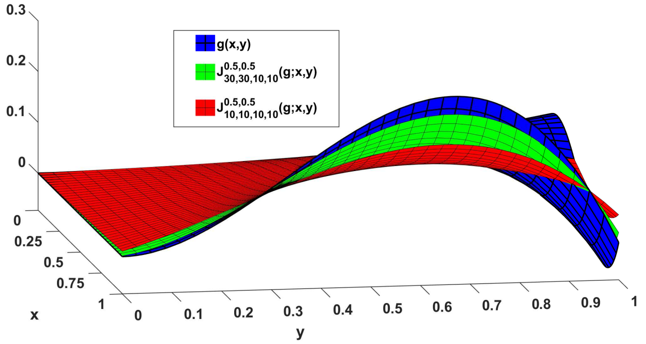


Figure 3.1: Approximation process for $n = m = 10, 30, \alpha_1 = \alpha_2 = 0.5$ and $\rho_1 = \rho_2 = 10$

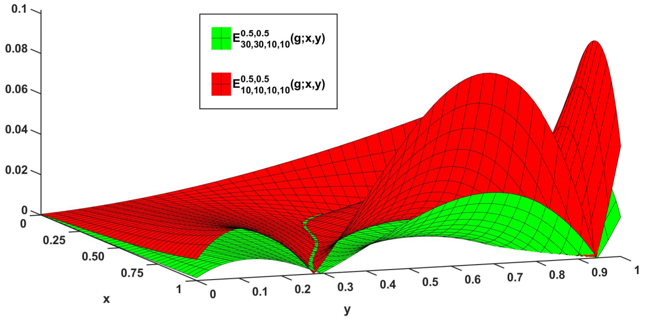


Figure 3.2: Error estimation for $n = m = 10, 30, \alpha_1 = \alpha_2 = 0.5$ and $\rho_1 = \rho_2 = 10$

For the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; x, y)$, we choose the different sets of sequences with different rate of convergence

$$a_0(n) = \frac{1}{2n}, a_1(n) = \frac{n-1}{n}, b_0(m) = \frac{1}{2m}, b_1(m) = \frac{m-1}{m};$$

$$a_0(n) = \frac{2n^2-1}{2n^2}, a_1(n) = \frac{1-n^2}{n^2}, b_0(m) = \frac{2m^2-1}{2m^2}, b_1(m) = \frac{1-m^2}{m^2};$$

$$a_0(n) = 1, a_1(n) = -1, b_0(m) = 1, b_1(m) = -1.$$

Also, we present the error of approximation of the operators in Fig. 3.4.

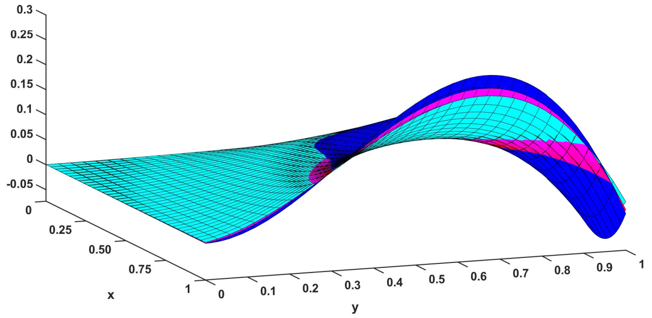


Figure 3.3: Approximation process for $n = m = 30, \alpha_1 = \alpha_2 = 0.3$ and $\rho_1 = \rho_2 = 4$

$$g(x, y)(Blue),\ [a_0(n) = -1, a_1(n) = 1, b_0(m) = -1, b_1(m) = 1](Red),$$
$$\left[a_0(n) = \frac{1}{2n}, a_1(n) = \frac{n-1}{n}, b_0(m) = \frac{1}{2m}, b_1(m) = \frac{m-1}{m}\right](Cyan)$$
$$\left[a_0(n) = \frac{2n^2-1}{2n^2}, a_1(n) = \frac{1-n^2}{n^2}, b_0(m) = \frac{2m^2-1}{2m^2}, b_1(m) = \frac{1-m^2}{m^2}\right](Magenta)$$

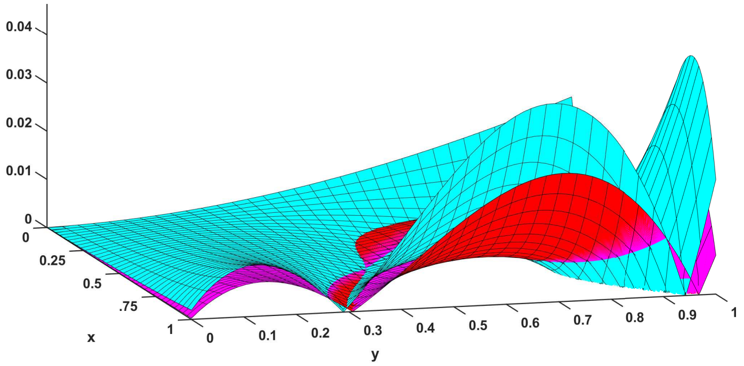


Figure 3.4: Error estimation for $n = m = 30, \alpha_1 = \alpha_2 = 0.3$ and $\rho_1 = \rho_2 = 4$

$$[a_0(n) = -1, a_1(n) = 1, b_0(m) = -1b_1(m) = 1](Red),$$
$$\left[a_0(n) = \frac{1}{2n}, a_1(n) = \frac{n-1}{n}, b_0(m) = \frac{1}{2m}, b_1(m) = \frac{m-1}{m}\right](Cyan)$$
$$\left[a_0(n) = \frac{2n^2-1}{2n^2}, a_1(n) = \frac{1-n^2}{n^2}, b_0(m) = \frac{2m^2-1}{2m^2}, b_1(m) = \frac{1-m^2}{m^2}\right](Magenta)$$

3.5 Generalized Boolean Sum (GBS) Operators

In the present section, we define the GBS operators that are associated with the operators (3.1).

3.5.1 Construction of GBS operators

Let $g \in C_b(I^2)$. The GBS operators $K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ associated to the operators $J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y)$ are given as

$$K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y) = J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}[g(\mathrm{t},y) + g(x,\mathrm{s}) - g(\mathrm{t},\mathrm{s});x,y], \quad (x,y) \in I^2. \tag{3.9}$$

Theorem 3.5.1. *For every $g \in C_b(I^2)$, the operators (3.9) satisfy the following inequality:*

$$|K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y)| \leq M\, \omega_{mixed}\left(g; \frac{1}{\sqrt{n\rho_1+2}}, \frac{1}{\sqrt{m\rho_2+2}}\right),$$

where $M > 0$ is a constant entirely determined by ρ_1 and ρ_2.

Proof. As we know

$$\Delta g[(\mathrm{t},\mathrm{s}),(x,y)] = g(x,y) - g(\mathrm{t},y) - g(x,\mathrm{s}) + g(\mathrm{t},\mathrm{s}).$$

Applying the operators (3.1) and using (1.14), we have
$|K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y)|$

$$\begin{aligned}
&\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\Delta g[(\mathrm{t},\mathrm{s}),(x,y)]|;x,y)\\
&\leq J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}\left(\left(1+\frac{|\phi^1_x(\mathrm{t})|}{\sigma_1}\right)\left(1+\frac{|\phi^1_y(\mathrm{s})|}{\sigma_2}\right)\omega_{mixed}(g;\sigma_1,\sigma_2);x,y\right)\\
&\leq \Big\{1 + \sigma_1^{-1}\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y)\right]^{\frac{1}{2}} + \sigma_2^{-1}\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y)\right]^{\frac{1}{2}}\\
&\quad + \sigma_1^{-1}\sigma_2^{-1}\left[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t});x,y).J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s});x,y)\right]^{\frac{1}{2}}\Big\}\omega_{mixed}(g;\sigma_1,\sigma_2).
\end{aligned}$$

By using Corollary 3.2.3, we get

$$J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^2(\mathrm{t});x,y) \leq \frac{2(1+\rho_1)}{n\rho_1+2} \quad \text{and} \quad J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_y^2(\mathrm{s});x,y) \leq \frac{2(1+\rho_2)}{m\rho_2+2}.$$

Also, by choosing $\sigma_1 = \dfrac{1}{\sqrt{n\rho_1+2}}$ and $\sigma_2 = \dfrac{1}{\sqrt{m\rho_2+2}}$, the proof is completed.

□

Theorem 3.5.2. *Let $g \in D_b(I^2)$ with $D_B(g) \in B(I^2)$. Then for every $(x,y) \in I^2$, we get*

$$|K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y)| \leq \frac{M}{\sqrt{n\rho_1+2}.\sqrt{m\rho_2+2}}\left[\|D_Bg\| + \omega_{mixed}\left(D_Bg;\frac{1}{\sqrt{n\rho_1+2}},\frac{1}{\sqrt{m\rho_2+2}}\right)\right].$$

Proof. For $g \in D_b(I^2)$,

$$\begin{aligned} \Delta g[(\mathrm{t},\mathrm{s}),(x,y)] =& \phi_x^1(\mathrm{t})\phi_y^1(\mathrm{s})D_Bg(\zeta,\eta) \quad \text{with} \quad x<\zeta<\mathrm{t},\ y<\eta<\mathrm{s}, \\ \text{where} \quad D_Bg(\zeta,\eta) =& \Delta D_Bg[(\zeta,\eta),(x,y)] + D_Bg(\zeta,y) + D_Bg(x,\eta) - D_Bg(x,y). \end{aligned}$$

By using this relation and $D_Bg \in B(I^2)$, we get

$$\begin{aligned} |K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y)| =& |J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\Delta g[(\mathrm{t},\mathrm{s}),(x,y)];x,y)| \\ =& |J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^1(\mathrm{t})\,\phi_y^1(\mathrm{s})D_Bg(\zeta,\eta);x,y)| \\ =& |J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^1(\mathrm{t})\,\phi_y^1(\mathrm{s})\{\Delta D_Bg[(\zeta,\eta),(x,y)] \\ &+ D_Bg(\zeta,y) + D_Bg(x,\eta) - D_Bg(x,y)\};x,y)| \\ \leq& J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|\,|\phi_y^1(\mathrm{s})|\,|\Delta D_Bg[(\zeta,\eta),(x,y)]|;x,y) \\ &+ J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|\,|\phi_y^1(\mathrm{s})|\{|D_Bg(\zeta,y)| + |D_Bg(x,\eta)| \\ &+ |D_Bg(x,y)|\};x,y) \\ \leq& J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|\,|\phi_y^1(\mathrm{s})|\omega_{mixed}(D_Bg;|\phi_x^1(\mathrm{t})|\,|\phi_y^1(\mathrm{s})|);x,y) \\ &+ 3\|D_Bg\|J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|\,|\phi_y^1(\mathrm{s})|;x,y) \\ \leq& \Big[J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi_x^1(\mathrm{t})|\,|\phi_y^1(\mathrm{s})|;x,y) \\ &+ {\sigma_1}^{-1}J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi_x^2(\mathrm{t})|\phi_y^1(\mathrm{s})|;x,y) \end{aligned}$$

$$
\begin{aligned}
&+ {\sigma_2}^{-1} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_x(\mathrm{t})|\phi^2_y(\mathrm{s}); x, y) \\
&+ {\sigma_1}^{-1}{\sigma_2}^{-1} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t})\phi^2_y(\mathrm{s}); x, y)\Big]\, \omega_{mixed}(D_B g; \sigma_1, \sigma_2) \\
&+ 3\|D_B g\| J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(|\phi^1_x(\mathrm{t})||\phi^1_y(\mathrm{s})|; x, y).
\end{aligned}
$$

With the help of Cauchy-Schwarz inequality, it turns out to be

$$
\begin{aligned}
|K^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(g; x, y) - g(x, y)| \leq &\Bigg[\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t})\,\phi^2_y(\mathrm{s}); x, y)} \\
&+ {\sigma_1}^{-1}\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^4_x(\mathrm{t})\,\phi^2_y(\mathrm{s}); x, y)} \\
&+ {\sigma_2}^{-1}\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t})\,\phi^4_y(\mathrm{s}); x, y)} \\
&+ {\sigma_1}^{-1}{\sigma_2}^{-1} J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t})\,\phi^2_y(\mathrm{s}); x, y)\Bigg]\omega_{mixed}(D_B g; \sigma_1, \sigma_2) \\
&+ 3\|D_B g\|\sqrt{J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t})\,\phi^2_y(\mathrm{s}); x, y)}.
\end{aligned}
$$

Using Corollary 3.2.3, we obtain

$$
J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_x(\mathrm{t}); x, y) \leq \frac{2(1+\rho_1)}{n\rho_1 + 2} \text{ and } J^{\alpha_1,\alpha_2}_{n,m,\rho_1,\rho_2}(\phi^2_y(\mathrm{s}); x, y) \leq \frac{2(1+\rho_2)}{m\rho_2 + 2}.
$$

Now, by choosing $\sigma_1 = \dfrac{1}{\sqrt{n\rho_1 + 2}}$ and $\sigma_2 = \dfrac{1}{\sqrt{m\rho_2 + 2}}$ and Lemma 3.2.2, we get the result. □

3.6 Conclusion

This chapter is dedicated to the approximation of functions with two variables, prevalent in various real-life scenarios, through the utilization of bivariate operators. Our exploration delved into the convergence and approximation characteristics of these operators, accompanied by illustrative numerical graphs that underscore their efficacy. Towards the conclusion, we introduced and studied GBS operators associated with bivariate operators. Also, we shed the light on the influence of parameters $a_\imath, \imath = 0, 1$ on the convergence of these operators.

This exploration has inspired the generalization of several positive linear operators through the incorporation of pertinent parameters. The information obtained in this chapter lays the groundwork for a deeper understanding of the fundamental dynamics, thus equipping the reader for the subsequent chapter that presents and examines the Durrmeyer-variant of Lupaş operators.

Chapter 4

Approximation properties of Durrmeyer-variant of Lupaş type operators

4.1 Introduction

The Weierstrass theorem is considered as the founding stone of approximation theory. The probabilistic proof of this theorem, given by S. N. Bernstein brought rapid growth in this subject that can be seen from the references [42, 55, 68, 152] and therein. Stancu [141] generalized these operators by a sequence of positive linear operators, depending on a non-negative parameter α, as

$$S_n^{\alpha}(g;x) = \sum_{\jmath=0}^{n} s_{n,\jmath}^{\alpha}(x) g\left(\frac{\jmath}{n}\right),$$

$$\text{where} \qquad s_{n,\jmath}^{\alpha}(x) = \frac{\prod_{\imath=0}^{\jmath-1}(x+\imath\alpha)\prod_{k=0}^{n-\jmath-1}(1-x+k\alpha)}{\prod_{k=0}^{n-1}(1+k\alpha)}\binom{n}{\jmath}, \ x\in[0,1], \ n\in\mathbb{N}.$$

For $\alpha = 0$, it reduces to Bernstein operators. A special case of these operators for $\alpha = \frac{1}{n}$ is given by Lupaş and Lupaş [101]. In order to generalize these operators for Lebesgue integrable functions, Gupta and Rassias [73] defined its Durrmeyer type modification. The authors studied their local and global results as well as its asymptotic approximation.

The generalization of factorial notation, that was given by Christian Kramp for positive integers, is defined by German Mathematician Leo Pochhammer as

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \qquad \alpha \in \mathbb{C},\ n \geq 1,$$

where $\mathbb{C}$ is set of complex numbers. This notation is known as shifted factorial. It is widely used in special functions like hyper geometric, Combinatorics, etc., which can be seen in [119, 129, 133, 138].

In the similar way, the Pochhammer k-symbol is defined by Diaz and Pariguan [54] as

$$(\nu)_{m,k} = \begin{cases} \nu(\nu+k)\cdots(\nu+(m-1)k), & m \geq 1 \\ 1, & m = 0,\ \nu \neq 0 \end{cases},$$

where $\nu \in \mathbb{C}$, $m \in \mathbb{N}$ and k is any non-negative real number. Recently, Yilmaz *et al.* [151] presented the Lupaş-type operators with the Pochhammer k-symbol as well as Kantorovich-Stancu modification of Lupaş type operators and their bivariate extension and examined the convergence, asymptotic result as well as the error of approximation for these variants. Motivated from this approach, we introduce the Lupaş type Durrmeyer operators having Pochhammer k-symbol and studied its approximation behavior. The study is formatted as follows:

In section 4.2, we construct the operators in one variable and calculate the moments and central moments for these operators. Section 4.3 contains the approximation properties of these operators as convergence, asymptotic result, error of approximation in terms of first order modulus of continuity. In the next section, we define the bivariate extension of the operators of section 4.2. We present the results for these operators in section 4.5. In the last section, we show some numerical illustrates that validate our results.

4.2 Construction of the Operators

For any $g \in C[0,1]$ and $k \geq 0$, we present the Durrmeyer variant of Lupaş type operators as

$$D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) = (n+1)\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n} \right\rangle}(x)\, \int_0^1 p_{n,\jmath}(t)\, g(\mathrm{t})\, \mathrm{dt}, \tag{4.1}$$

where $q_{n,\jmath}^{\left\langle \frac{k}{n} \right\rangle}(x) = \binom{n}{\jmath}\frac{(nx)_{\jmath,k}(n-nx)_{n-\jmath,k}}{(n)_{n,k}}$, $(n)_{n,k} = n(n+k)\cdots(n+(n-1)k)$, and $p_{n,\jmath}(t)$ are the B-basis polynomials.

Lemma 4.2.1. *The moments of the operators (4.1) are given as*

$$\begin{aligned}
D_n^{\left\langle \frac{k}{n} \right\rangle}(e_0;x) =&1;\\
D_n^{\left\langle \frac{k}{n} \right\rangle}(e_1;x) =&x+\frac{1-2x}{n+2};\\
D_n^{\left\langle \frac{k}{n} \right\rangle}(e_2;x) =&x^2+\frac{1}{(n+3)(n+2)}\left[2+3nx-x^2(5n+6)+\frac{n^2(k+1)x(1-x)}{n+k}\right];\\
D_n^{\left\langle \frac{k}{n} \right\rangle}(e_3;x) =&x^3+\frac{1}{(n+4)(n+3)(n+2)}\left[\frac{n^3(3n+2k-2)(k+1)x^2(1-x)}{(n+k)(n+2k)}\right.\\
&+\frac{n^3(2k+1)(k+1)x(1-x)}{(n+k)(n+2k)}+\frac{6n^2(k+1)x(1-x)}{n+k}-x^3(9n^2+26n+24)\\
&\left.+6x^2n^2+11nx+6\right];\\
D_n^{\left\langle \frac{k}{n} \right\rangle}(e_4;x) =&x^4+\frac{1}{(n+5)(n+4)(n+3)(n+2)}\left[\frac{n^4(k+1)\{(11n-6)(k-1)+6(n^2+k^2)\}x^3(1-x)}{(n+k)(n+2k)(n+3k)}\right.\\
&+\frac{n^4(k+1)(7n+11nk+6(k^2-k-1))x^2(1-x)}{(n+k)(n+2k)(n+3k)}\\
&+\frac{n^3(k+1)(n-k+6nk(k+1))x(1-x)}{(n+k)(n+2k)(n+3k)}\\
&+\frac{10n^3(3n+2k-2)(k+1)x(1-x)}{(n+k)(n+2k)}+\frac{10n^3(2k+1)(k+1)x(1-x)}{(n+k)(n+2k)}\\
&+\frac{35n^2(k+1)x(1-x)}{n+k}\\
&\left.-x^4(14n^3+71n^2+154n+120)+10n^3x^3+35n^2x^2+50nx+24\right].
\end{aligned}$$

Lemma 4.2.2. *The central moments of the operators (4.1) are given below:*

$$\begin{aligned}
D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^1(x);x) &= \frac{1-2x}{n+2};\\
D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^2(x);x) &= \frac{1}{(n+3)(n+2)}\left[\left(\frac{n^2(k+1)}{n+k}+n-6\right)x(1-x)+2\right];\\
D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^4(x);x) &= O\left(\frac{1}{n^2}\right).
\end{aligned}$$

4.3 Approximation Results

Here, we present the convergence of our operators (4.1) with the help of Korovkin theorem.

Theorem 4.3.1. *For every $g \in C[0,1]$*

$$\lim_{n\to\infty} D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) = g(x), \ \textit{uniformly on } [0,1],$$

where $n \in \mathbb{N}$, $k \geq 0$.

Proof. By using the moments of the operators given in Lemma 4.2.1, we obtain

$$\lim_{n\to\infty} D_n^{\left\langle \frac{k}{n} \right\rangle}(e_\imath;x) = e_\imath, \qquad \imath = 0,1,2$$

uniformly in $[0,1]$.

Thus, by Korovkin theorem, we get the uniform convergence of the operators (4.1). □

Now, we find the order of approximation for the operators (4.1).

Theorem 4.3.2. *Let $g \in C^2[0,1]$ such that $g''(x)$ exists for any $x \in (0,1)$. We obtain the following equation:*

$$\lim_{n\to\infty} n(D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x)) = (1-2x)g'(x) + \frac{(k+2)x(1-x)}{2!}g''(x).$$

Proof. Let $x \in (0,1)$. Using Taylor's polynomial for $g \in C^2[0,1]$, we have

$$g(\mathrm{t}) = g(x) + \phi_{\mathrm{t}}^1(x)g'(x) + \frac{\phi_{\mathrm{t}}^2(x)}{2!}g''(x) + \vartheta(\mathrm{t},x)\phi_{\mathrm{t}}^2(x), \tag{4.2}$$

where $\vartheta(\mathrm{t},x)$ is the remainder term with $\vartheta(\mathrm{t},x) \in C[0,1]$ such that $\lim\limits_{\mathrm{t}\to x} \vartheta(\mathrm{t},x) = 0$. Applying the operators $D_n^{\left\langle \frac{k}{n} \right\rangle}(.;x)$ on (4.2) and then taking limit as $n \to \infty$, we can see

$$\begin{aligned}\lim_{n\to\infty} n(D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x)) = &\lim_{n\to\infty} n \Big[D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^1(x);x)g'(x) + \frac{1}{2}D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^2(x);x)g''(x) \\ &\quad + D_n^{\left\langle \frac{k}{n} \right\rangle}(\vartheta(\mathrm{t},x)\phi_{\mathrm{t}}^2(x);x)\Big] \\ =&(1-2x)g'(x) + \frac{1}{2}(k+2)x(1-x)g''(x) \\ &\quad + \lim_{n\to\infty} n\, D_n^{\left\langle \frac{k}{n} \right\rangle}(\vartheta(\mathrm{t},x)\phi_{\mathrm{t}}^2(x);x). \end{aligned} \tag{4.3}$$

It is enough to prove that $\lim\limits_{n\to\infty} D_n^{\left\langle \frac{k}{n} \right\rangle}(\vartheta(\mathrm{t},x)\phi_{\mathrm{t}}^2(x);x) = 0$. Since the operators are positive, then we use Cauchy Schwarz inequality to get

$$n(D_n^{\left\langle \frac{k}{n} \right\rangle}(\vartheta(\mathrm{t},x)\phi_{\mathrm{t}}^2(x);x)) \leq n\sqrt{D_n^{\left\langle \frac{k}{n} \right\rangle}(\vartheta^2(\mathrm{t},x);x)}\sqrt{D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^4(x);x)}. \tag{4.4}$$

As $\vartheta(\mathrm{t},x) = 0$ as $\mathrm{t} \to x$ and using Theorem 4.3.1, we have

$$\lim_{n\to\infty} n\, D_n^{\left\langle \frac{k}{n} \right\rangle}(\vartheta^2(\mathrm{t},x);x) = 0.$$

Now, by using Lemma 4.2.2 in (4.4), we acquire the result. □

Next, we show the error of approximation for the operators (4.1) in terms of first order modulus of continuity.

Theorem 4.3.3. *For any* $k \geq 0$, $n \in \mathbb{N}$, *and every* $g \in C[0,1]$, *we get the inequality*

$$\mid D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x) \mid \leq 2\,\omega\left(g; \sqrt{D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^2(x);x)}\right).$$

Proof. From the definition of the operators (4.1), we have

$$\begin{aligned} \mid D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)-g(x)\mid =&\left|(n+1)\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\int_0^1 p_{n,\jmath}(\mathrm{t})\,g(\mathrm{t})\,\mathrm{dt}-g(x)\right|\\ \leq&(n+1)\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\int_0^1 p_{n,\jmath}(\mathrm{t})\,\omega(g;\mid \phi_{\mathrm{t}}^1(x)\mid)\,\mathrm{dt}\\ \leq&(n+1)\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\int_0^1 p_{n,\jmath}(\mathrm{t})\left(1+\frac{\mid \phi_{\mathrm{t}}^1(x)\mid}{\delta}\right)\omega(g;\delta)\,\mathrm{dt}\\ =&\omega(g;\delta)\left[1+\frac{(n+1)}{\delta}\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\int_0^1 p_{n,\jmath}(\mathrm{t})\mid \phi_{\mathrm{t}}^1(x)\mid\,\mathrm{dt}\right]. \quad (4.5) \end{aligned}$$

Now, applying Cauchy-Schwarz inequality

$$\begin{aligned} (n+1)\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\int_0^1 p_{n,\jmath}(\mathrm{t})\mid \phi_{\mathrm{t}}^1(x)\mid\,\mathrm{dt} \leq& \sqrt{(n+1)}\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\left[\int_0^1 p_{n,\jmath}(\mathrm{t})\phi_{\mathrm{t}}^2(x)\,\mathrm{dt}\right]^{\frac{1}{2}}\\ \leq&\left[(n+1)\sum_{\jmath=0}^{n} q_{n,\jmath}^{\left\langle \frac{k}{n}\right\rangle}(x)\int_0^1 p_{n,\jmath}(\mathrm{t})\phi_{\mathrm{t}}^2(x)\,\mathrm{dt}\right]^{\frac{1}{2}}\\ =&\left[D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)\right]^{\frac{1}{2}}. \end{aligned}$$

Thus, by using this inequality, (4.5) becomes

$$\mid D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)-g(x)\mid \leq \omega(g;\delta)\left[1+\frac{1}{\delta}\sqrt{D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)}\right].$$

By choosing $\delta=\sqrt{D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)}$, we come to our desired inequality. □

Theorem 4.3.4. *For $g\in C[0,1], k\geq 0, n\in\mathbb{N}$, we have the following inequality:*

$$\mid D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)-g(x)\mid \leq C\left[\omega_2\left(g;\frac{\Theta_{n,k}(x)}{2}\right)+\min\left(1,\frac{\Theta_{n,k}^2(x)}{4}\right)\parallel g\parallel\right]+\omega\left(g;\frac{1-2x}{n+2}\right),$$

where $\Theta_{n,k}(x)=D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)+\left(\frac{1-2x}{n+2}\right)^2.$

Proof. Consider the associated operators of $D_n^{\left\langle\frac{k}{n}\right\rangle}(g;x)$ defined as

$$\tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}(g;x) = D_n^{\left\langle\frac{k}{n}\right\rangle}(g;x) - g\left(x + \frac{1-2x}{n+2}\right) + g(x). \tag{4.6}$$

Clearly from the moments, we can find that $\tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}(\mathrm{t}-x;x) = 0$.
Also, $\mid \tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}(g;x) \mid \leq 3 \parallel g \parallel$.
Consider the Taylor's expansion for $f \in C^2[0,1]$

$$f(\mathrm{t}) = f(x) + \phi_{\mathrm{t}}^1(x)f'(x) + \int_x^{\mathrm{t}} \phi_{\mathrm{t}}^1(u)f''(u)\,du,\ \ u \in (x,\mathrm{t}), \mathrm{t}, x \in [0,1]. \tag{4.7}$$

Apply the operators $\tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}(.;x)$, we obtain

$$\begin{aligned}\mid \tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}(f;x) - f(x) \mid =& \left|\tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}\left(\int_x^{\mathrm{t}} \phi_{\mathrm{t}}^1(u)\,f''(u);x\right)\right| \\ =& \left|D_n^{\left\langle\frac{k}{n}\right\rangle}\left(\int_x^{\mathrm{t}} \phi_{\mathrm{t}}^1(u)f''(u);x\right)\right| + \left|\int_x^{x+\frac{1-2x}{n+2}}\left(x+\frac{1-2x}{n+2}-u\right)f''(u)\,du\right| \\ :=& A_1 + A_2 \ \ \text{(say)}. \end{aligned} \tag{4.8}$$

Now,

$$A_1 = \left|D_n^{\left\langle\frac{k}{n}\right\rangle}\left(\int_x^{\mathrm{t}} \phi_{\mathrm{t}}^1(u)f''(u)\,du;x\right)\right| \leq D_n^{\left\langle\frac{k}{n}\right\rangle}\left(\left|\int_x^{\mathrm{t}} \phi_{\mathrm{t}}^1(u)f''(u)\,du\right|;x\right) \leq \parallel f \parallel_{C^2(I^2)} D_n^{\left\langle\frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x).$$

$$A_2 = \left|\int_x^{x+\frac{1-2x}{n+2}}\left(x+\frac{1-2x}{n+2}-u\right)f''(u)\,du\right| \leq \parallel f \parallel_{C^2(I^2)} \left(\frac{1-2x}{n+2}\right)^2.$$

Using A_1, A_2 and (4.8), we get

$$\begin{aligned}\mid \tilde{D}_n^{\left\langle\frac{k}{n}\right\rangle}(f;x) - f(x) \mid \leq& \parallel f \parallel_{C^2(I^2)} \left[D_n^{\left\langle\frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x) + \left(\frac{1-2x}{n+2}\right)^2\right] \\ =& \parallel f \parallel_{C^2(I^2)} \Theta_{n,k}^2(x), \end{aligned} \tag{4.9}$$

where $\Theta_{n,k}^2(x) = D_n^{\left\langle\frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x) + \left(\frac{1-2x}{n+2}\right)^2$.

Now, for $g \in C[0,1]$

$$\begin{aligned}
\mid D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x) \mid \leq & \mid D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - \tilde{D}_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) \mid + \mid \tilde{D}_n^{\left\langle \frac{k}{n} \right\rangle}(f-g;x) \mid \\
& + \mid \tilde{D}_n^{\left\langle \frac{k}{n} \right\rangle}(f;x) - f(x) \mid + \mid f(x) - g(x) \mid \\
\leq & 4 \parallel g - f \parallel + \Theta_{n,k}^2(x) \parallel f \parallel_{C^2(I^2)} + \left| g\left(x + \frac{1-2x}{n+2}\right) - g(x) \right| \\
\leq & 4 \left[\parallel g - f \parallel + \frac{\Theta_{n,k}^2(x)}{4} \parallel f \parallel_{C^2(I^2)} \right] + \omega\left(g; \frac{1-2x}{n+2}\right) \\
\leq & 4K_2\left(g; \frac{\Theta_{n,k}^2(x)}{4}\right) + \omega\left(g; \frac{1-2x}{n+2}\right).
\end{aligned}$$

Using the relation of second order modulus of continuity and K functional (1.12), we get the desired result. □

Theorem 4.3.5. *For $g \in Lip_{\mathscr{M}}(\tau)$, we get the inequality*

$$\mid D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x) \mid \leq \mathscr{M} \left[D_n^{\left\langle \frac{k}{n} \right\rangle}(\phi_{\mathrm{t}}^2(x);x) \right]^{\frac{\tau}{2}}.$$

Proof. For $g \in Lip_{\mathscr{M}}(\tau)$, we have

$$\mid g(x) - g(y) \mid \leq \mathscr{M} \mid x - y \mid^{\tau}. \tag{4.10}$$

By the linearity of the operators and using (4.10), we obtain

$$\begin{aligned}
\mid D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x) \mid \leq & D_n^{\left\langle \frac{k}{n} \right\rangle}(\mid g(\mathrm{t}) - g(x) \mid; x) \\
\leq & D_n^{\left\langle \frac{k}{n} \right\rangle}(\mathscr{M} \mid \phi_{\mathrm{t}}^1(x) \mid^{\tau}; x).
\end{aligned} \tag{4.11}$$

By applying Hölder's inequality in (4.11), we get the result. □

Theorem 4.3.6. *Let $g, h \in C^2[0,1]$. Then for each $x \in [0,1]$, we have*

$$\lim_{n\to\infty} n \left[D_n^{\left\langle \frac{k}{n} \right\rangle}(gh;x) - D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) D_n^{\left\langle \frac{k}{n} \right\rangle}(h;x) \right] = (k+2)x(1-x)g'(x)h'(x).$$

Proof. Using the Taylor's series, we get the following identity:

$$
\begin{aligned}
&D_n^{\left\langle \frac{k}{n}\right\rangle}(gh;x) - D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)D_n^{\left\langle \frac{k}{n}\right\rangle}(h;x)\\
&\quad = D_n^{\left\langle \frac{k}{n}\right\rangle}(gh;x) - g(x)h(x) - (gh)'(x)D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^1(x);x) - \frac{(gh)''(x)}{2!}D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)\\
&\qquad - h(x)\left[D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x) - g(x) - g'(x)D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^1(x);x) - \frac{g''(x)}{2!}D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)\right]\\
&\qquad - D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)\left[D_n^{\left\langle \frac{k}{n}\right\rangle}(h;x) - h(x) - h'(x)D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^1(x);x) - \frac{h''(x)}{2!}D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)\right]\\
&\qquad + \frac{1}{2}D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)\left[g(x)h''(x) + 2g'(x)h'(x) - h''(x)D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)\right]\\
&\qquad + D_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^1(x);x)\left[g(x)h'(x) - h'(x)D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)\right]. \qquad (4.12)
\end{aligned}
$$

Using the convergence of the operators and central moments, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} n\left[D_n^{\left\langle \frac{k}{n}\right\rangle}(gh;x) - D_n^{\left\langle \frac{k}{n}\right\rangle}(g;x)D_n^{\left\langle \frac{k}{n}\right\rangle}(h;x)\right] &= \lim_{n\to\infty} nD_n^{\left\langle \frac{k}{n}\right\rangle}(\phi_{\mathrm{t}}^2(x);x)g'(x)h'(x)\\
&=(k+2)x(1-x)g'(x)h'(x).
\end{aligned}
$$

□

4.4 Bivariate Extension of Durrmeyer Operators

In the present section, we define the bivariate extension of (4.1) as

$$
D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y) =(n+1)(m+1)\sum_{\imath=0}^{n}\sum_{\jmath=0}^{m} q_{n,\imath}^{\langle l_1\rangle}(x)q_{m,\jmath}^{\langle l_2\rangle}(y)\int_0^1\int_0^1 p_{n,\imath}(\mathrm{t})p_{m,\jmath}(\mathrm{s})g(\mathrm{t},\mathrm{s})\,\mathrm{ds}\,\mathrm{dt}, \tag{4.13}
$$

where $l_1 = \dfrac{k_1}{n}$ and $l_2 = \dfrac{k_2}{m}, n, m \in \mathbb{N}$ and k_1, k_2 are non-negative real numbers.

Lemma 4.4.1. *For the operators (4.13) the moments are*

$$
\begin{aligned}
D_{n,m}^{\langle l_1,l_2\rangle}(e_{00};x,y) &=1;\\
D_{n,m}^{\langle l_1,l_2\rangle}(e_{10};x,y) &=\frac{nx+1}{n+2};
\end{aligned}
$$

$$\begin{aligned}
D_{n,m}^{\langle l_1,l_2\rangle}(e_{01};x,y) =& \frac{my+1}{m+2};\\
D_{n,m}^{\langle l_1,l_2\rangle}(e_{20};x,y) =& \frac{1}{(n+3)(n+2)}\left[n^2x^2+3nx+2+\frac{n^2(k_1+1)x(1-x)}{n+k_1}\right];\\
D_{n,m}^{\langle l_1,l_2\rangle}(e_{02};x,y) =& \frac{1}{(m+3)(m+2)}\left[m^2y^2+3my+2+\frac{m^2(k_2+1)y(1-y)}{m+k_2}\right].
\end{aligned}$$

Lemma 4.4.2. *For the operators (4.13) the central moments are*

$$\begin{aligned}
D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y) =& \frac{1-2x}{n+2};\\
D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^1(y);x,y) =& \frac{1-2y}{m+2};\\
D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y) =& \frac{1}{(n+3)(n+2)}\left[2+\left\{n-6+\frac{n^2(k_1+1)}{n+k_1}\right\}x(1-x)\right];\\
D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y) =& \frac{1}{(m+3)(m+2)}\left[2+\left\{m-6+\frac{m^2(k_2+1)}{m+k_2}\right\}y(1-y)\right];\\
D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^4(x);x,y) =& O\left(\frac{1}{n^2}\right);\\
D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^4(y);x,y) =& O\left(\frac{1}{m^2}\right).
\end{aligned}$$

In the next section, we find the approximation behavior for bivariate operators (4.13).

4.5 Main Results

Theorem 4.5.1. *For $n,m\in\mathbb{N}, k_1,k_2\geq 0$ and every $g\in C(I^2)$, we have*

$$\lim_{n,m\to\infty}\|D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\|=0.$$

Proof. With the help of the moments in Lemma 4.4.1 and Volkov's theorem [145], we get the result. □

Theorem 4.5.2. *For $n,m\in\mathbb{N}, k_1,k_2\geq 0,$ and every $g\in C(I^2)$, we have*

$$\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid\leq 4\,\omega(g;\delta_1(x),\delta_2(y)),$$

where $\omega(g;.,.)$ *is defined as* (1.3.9), $\delta_1(x) = \sqrt{D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)}$, *and* $\delta_2(y) = \sqrt{D_{n,m}^{\langle l_1,l_2\rangle}(\phi_s^2(y);x,y)}$.

Proof. By the linearity of the operators

$$\begin{aligned} \mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y) - g(x,y) \mid \leq & D_{n,m}^{\langle l_1,l_2\rangle}(\mid g(\mathrm{t},\mathrm{s}) - g(x,y) \mid;x,y) \\ \leq & \omega(g,\delta_1(x),\delta_2(y)) \left(1 + \frac{D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{t}}^1(x) \mid;x,y)}{\delta_1(x)}\right) \\ & \times \left(1 + \frac{D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{s}}^1(y) \mid;x,y)}{\delta_2(y)}\right). \end{aligned} \tag{4.14}$$

By using Cauchy-Schwarz inequality, we obtain

$$D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{t}}^1(x) \mid;x,y) \leq \sqrt{D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)},$$

$$D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{s}}^1(y) \mid;x,y) \leq \sqrt{D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)}.$$

By choosing $\delta_1(x) = \sqrt{D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)}$ and $\delta_2(y) = \sqrt{D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)}$, in (4.14), we attain the inequality. □

Theorem 4.5.3. *For* $n,m \in \mathbb{N}, k_1,k_2 \geq 0$ *and* $g \in C(I^2)$, *we have*

$$\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y) - g(x,y) \mid \leq 2[\omega^1(g;\delta_1(x)) + \omega^2(g;\delta_2(y))],$$

where $\omega^1(g;.)$, $\omega^2(g;.)$ *are partial modulus of continuities defined as* (1.3.10) *and* $\delta_1(x)$ *and* $\delta_2(y)$ *are as in Theorem 4.5.2.*

Proof. We know that

$$\mid g(\mathrm{t},\mathrm{s}) - g(x,y) \mid \leq \mid g(\mathrm{t},\mathrm{s}) - g(x,\mathrm{s}) \mid + \mid g(x,\mathrm{s}) - g(x,y) \mid.$$

By applying the operators $D_{n,m}^{\langle l_1,l_2\rangle}(.;x,y)$ and using the linearity, we get

$$\begin{aligned} \mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y) - g(x,y) \mid \leq & D_{n,m}^{\langle l_1,l_2\rangle}(\mid g(\mathrm{t},\mathrm{s}) - g(x,\mathrm{s}) \mid;x,y) + D_{n,m}^{\langle l_1,l_2\rangle}(\mid g(x,\mathrm{s}) - g(x,y) \mid;x,y) \\ \leq & D_{n,m}^{\langle l_1,l_2\rangle}(\omega^1(g;\mid \phi_{\mathrm{t}}^1(x) \mid);x,y) + D_{n,m}^{\langle l_1,l_2\rangle}(\omega^2(g;\mid \phi_{\mathrm{s}}^1(y) \mid);x,y) \end{aligned}$$

$$
\begin{aligned}
\leq&\omega^1(g;\delta_1(x))\left(1+\frac{D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{t}}^1(x)\mid;x,y)}{\delta_1(x)}\right)\\
&+\omega^2(g;\delta_2(y))\left(1+\frac{D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{s}}^1(y)\mid;x,y)}{\delta_2(y)}\right).
\end{aligned}
$$

Using Cauchy-Schwarz inequality and values of $\delta_1(x)$ and $\delta_2(y)$, we get the desired result. □

Theorem 4.5.4. *For $g \in Lip_{\mathscr{M}}(\tau_1,\tau_2)$ such that $\tau_1,\tau_2 \in (0,1], \mathscr{M} > 0$, we get the inequality*

$$
\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid \leq \mathscr{M}(\delta_1(x))^{\tau_1}(\delta_2(y))^{\tau_2},
$$

where $\delta_1(x)$ and $\delta_2(y)$ are same as in Theorem 4.5.2.

Proof. As $g \in Lip_{\mathscr{M}}(\tau_1,\tau_2)$, therefore

$$
\mid g(\mathrm{t},\mathrm{s})-g(x,y)\mid \leq \mathscr{M} \mid \phi_{\mathrm{t}}^1(x)\mid^{\tau_1}\mid \phi_{\mathrm{s}}^1(y)\mid^{\tau_2}.
$$

Now, consider

$$
\begin{aligned}
\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid \leq& D_{n,m}^{\langle l_1,l_2\rangle}(\mid g(\mathrm{t},\mathrm{s})-g(x,y)\mid;x,y)\\
\leq& D_{n,m}^{\langle l_1,l_2\rangle}(\mathscr{M}\mid \phi_{\mathrm{t}}^1(x)\mid^{\tau_1}\mid \phi_{\mathrm{s}}^1(y)\mid^{\tau_2};x,y)\\
=&\mathscr{M} D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{t}}^1(x)\mid^{\tau_1};x,y)\, D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{s}}^1(y)\mid^{\tau_2};x,y). \quad (4.15)
\end{aligned}
$$

By Holder's inequality, we have

$$
D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{t}}^1(x)\mid^{\tau_1};x,y) \leq [D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)]^{\frac{\tau_1}{2}} := (\delta_1(x))^{\tau_1},
$$

$$
D_{n,m}^{\langle l_1,l_2\rangle}(\mid \phi_{\mathrm{s}}^1(y)\mid^{\tau_2};x,y) \leq [D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)]^{\frac{\tau_2}{2}} := (\delta_2(y))^{\tau_2}.
$$

Using these inequalities in (4.15), we reach at the required result. □

Theorem 4.5.5. *For $g \in C^2(I^2)$ and $n \in \mathbb{N}$, k_1, $k_2 \geq 0$, we have the following equality:*

$$
\lim_{n\to\infty} n(D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)) =(1-2x)g_x(x,y)+(1-2y)g_y(x,y)
$$

$$+\frac{(k_1+2)x(1-x)}{2}g_{xx}(x,y)+\frac{(k_2+2)y(1-y)}{2}g_{yy}(x,y).$$

Proof. By Taylor's formula, we have

$$\begin{aligned} g(\mathrm{t},\mathrm{s}) =& g(x,y)+\phi_{\mathrm{t}}^1(x)g_x(x,y)+\phi_{\mathrm{s}}^1(y)g_y(x,y)+\frac{\phi_{\mathrm{t}}^2(x)}{2!}g_{xx}(x,y)+\phi_{\mathrm{t}}^1(x)\phi_{\mathrm{s}}^1(y)g_{xy}(x,y)\\ &+\frac{\phi_{\mathrm{s}}^2(y)}{2!}g_{yy}(x,y)+\Theta(\mathrm{t},\mathrm{s})(\phi_{\mathrm{t}}^2(x)+\phi_{\mathrm{s}}^2(y)), \end{aligned}$$

such that $\lim\limits_{(\mathrm{t},\mathrm{s})\to(x,y)}\Theta(\mathrm{t},\mathrm{s})=0$.
Applying the operators $D_{n,n}^{\langle l_1,l_2\rangle}(.;x,y)$ and taking limit as $n\to\infty$ on both sides, we obtain

$$\begin{aligned} &\lim_{n\to\infty} n(D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y))\\ =&\lim_{n\to\infty} n\Bigg[D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)g_x(x,y)+D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^1(y);x,y)g_y(x,y)+\frac{1}{2}D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)g_{xx}(x,y)\\ &+D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^1(y);x,y)g_{xy}(x,y)+\frac{1}{2}D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)g_{yy}(x,y)\\ &+D_{n,n}^{\langle l_1,l_2\rangle}(\Theta(\mathrm{t},\mathrm{s})(\phi_{\mathrm{t}}^2(x)+\phi_{\mathrm{s}}^2(y));x,y)\Bigg]\\ =&(1-2x)g_x(x,y)+(1-2y)g_y(x,y)+\frac{(k_1+2)x(1-x)}{2}g_{xx}(x,y)+\frac{(k_2+2)y(1-y)}{2}g_{yy}(x,y)\\ &+\lim_{n\to\infty} nD_{n,n}^{\langle l_1,l_2\rangle}(\Theta(\mathrm{t},\mathrm{s})(\phi_{\mathrm{t}}^2(x)+\phi_{s}^2(y));x,y). \end{aligned}$$

In order to reach at desired outcome, we are left to prove

$$\lim_{n\to\infty} n\left(D_{n,n}^{\langle l_1,l_2\rangle}(\Theta(\mathrm{t},s)(\phi_{\mathrm{t}}^2(x)+\phi_{\mathrm{s}}^2(y));x,y)\right)=0. \tag{4.16}$$

Using Cauchy Schwarz inequality

$$\begin{aligned} &n\left(D_{n,n}^{\langle l_1,l_2\rangle}(\Theta(\mathrm{t},\mathrm{s})(\phi_{\mathrm{t}}^2(x)+\phi_{\mathrm{s}}^2(y));x,y)\right)\\ &\le n\left[D_{n,n}^{\langle l_1,l_2\rangle}(\Theta^2(\mathrm{t},\mathrm{s});x,y)\right]^{\frac{1}{2}}\left[D_{n,n}^{\langle l_1,l_2\rangle}((\phi_{\mathrm{t}}^2(x)+\phi_{\mathrm{s}}^2(y))^2;x,y)\right]^{\frac{1}{2}}\\ &\le\sqrt{2}n\left[D_{n,n}^{\langle l_1,l_2\rangle}(\Theta^2(\mathrm{t},\mathrm{s});x,y)\right]^{\frac{1}{2}}\left[D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^4(x);x,y)+D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^4(y);x,y)\right]^{\frac{1}{2}}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\because\ (a+b)^2\le 2(a^2+b^2)]. \end{aligned}$$

Due to the convergence of the operators $D_{n,n}^{\langle l_1,l_2\rangle}(.;x,y)$ and $\lim\limits_{(\mathrm{t},\mathrm{s})\to(x,y)}\Theta(\mathrm{t},\mathrm{s})=0$, we get

$$\lim_{n\to\infty}D_{n,n}^{\langle l_1,l_2\rangle}(\Theta^2(\mathrm{t},\mathrm{s});x,y)=0.$$

Also, with the help of the central moments of order 4, which are given in Lemma 4.4.2, we get the equation (4.16).

Hence, the proof is completed. □

Theorem 4.5.6. *For $g\in C(I^2)$ and $n,m\in\mathbb{N}$, $k_1,k_2\geq 0$, we have the inequality*

$$\begin{aligned}\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid\leq C_1\left[\omega_2\left(g;\frac{1}{2}\sqrt{\sigma_{n,m}(x,y)}\right)+min\left(1,\frac{1}{4}\sigma_{n,m}(x,y)\right)\|g\|\right]\\+\omega(g;\mu_{n,m}(x,y)),\end{aligned}$$

where $\sigma_{n,m}(x,y)=D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)+D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)+\left(\frac{1-2x}{n+2}\right)^2+\left(\frac{1-2y}{m+2}\right)^2$ *and* $\mu_{n,m}(x,y)=\sqrt{\left(\frac{1-2x}{n+2}\right)^2+\left(\frac{1-2y}{m+2}\right)^2}$.

Proof. Consider the auxiliary operators

$$\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)=D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g\left(x+\frac{1-2x}{n+2},y+\frac{1-2y}{m+2}\right)+g(x,y).\tag{4.17}$$

Using this definition, we can easily see

$$\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)=0.$$

For any $f\in C^2(I^2)$

$$\begin{aligned}\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid\leq&\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)\mid\\&+\mid\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(f;x,y)\mid\\&+\mid\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(f;x,y)-f(x,y)\mid+\mid f(x,y)-g(x,y)\mid.\end{aligned}\tag{4.18}$$

By using (4.17), we have

$$\begin{aligned}
\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)\mid =&\left|g\left(x+\frac{1-2x}{n+2},y+\frac{1-2y}{m+2}\right)-g(x,y)\right| \\
\leq&\omega\left(g;\sqrt{\left(\frac{1-2x}{n+2}\right)^2+\left(\frac{1-2y}{m+2}\right)^2}\right):=\omega(g;\mu_{n,m}(x,y)).
\end{aligned} \tag{4.19}$$

Consider Taylor's polynomial for $f\in C^2(I^2)$

$$\begin{aligned}
f(\mathrm{t},\mathrm{s})=&f(x,y)+\phi_{\mathrm{t}}^1(x)\frac{\partial f(x,y)}{\partial x}+\phi_{\mathrm{s}}^1(y)\frac{\partial f(x,y)}{\partial y}+\int_x^{\mathrm{t}}\phi_{\mathrm{t}}^1(u)\frac{\partial^2 f(u,y)}{\partial u^2}\,du \\
&+\int_y^{\mathrm{s}}\phi_{\mathrm{s}}^1(v)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv.
\end{aligned}$$

Applying the operators $\hat{D}_{n,m}^{\langle l_1,l_2\rangle}(.;x,y)$, we get
$\mid \hat{D}_{n,m}^{\langle l_1,l_2\rangle}(f;x,y)-f(x,y)\mid$

$$\begin{aligned}
\leq&\left|\hat{D}_{n,m}^{\langle l_1,l_2\rangle}\left(\int_x^{\mathrm{t}}\phi_{\mathrm{t}}^1(u)\frac{\partial^2 f(u,y)}{\partial u^2}\,du;x,y\right)\right|+\left|\hat{D}_{n,m}^{\langle l_1,l_2\rangle}\left(\int_y^{\mathrm{s}}\phi_{\mathrm{s}}^1(v)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv;x,y\right)\right| \\
=&\left|D_{n,m}^{\langle l_1,l_2\rangle}\left(\int_x^{\mathrm{t}}\phi_{\mathrm{t}}^1(u)\frac{\partial^2 f(u,y)}{\partial u^2}\,du;x,y\right)-\int_x^{x+\frac{1-2x}{n+2}}\left(x+\frac{1-2x}{n+2}-u\right)\frac{\partial^2 f(u,y)}{\partial u^2}\,du\right| \\
&+\left|D_{n,m}^{\langle l_1,l_2\rangle}\left(\int_y^{\mathrm{s}}\phi_{\mathrm{s}}^1(v)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv;x,y\right)-\int_y^{y+\frac{1-2y}{m+2}}\left(y+\frac{1-2y}{m+2}-v\right)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv\right| \\
\leq&\Bigg[D_{n,m}^{\langle l_1,l_2\rangle}\left(\left|\int_x^{\mathrm{t}}\mid\phi_{\mathrm{t}}^1(u)\mid\,du\right|;x,y\right)+D_{n,m}^{\langle l_1,l_2\rangle}\left(\left|\int_y^{\mathrm{s}}\mid\phi_{\mathrm{s}}^1(v)\mid\,dv\right|;x,y\right) \\
&+\left|\int_x^{x+\frac{1-2x}{n+2}}\left|x+\frac{1-2x}{n+2}-u\right|du\right|+\left|\int_y^{y+\frac{1-2y}{m+2}}\left|y+\frac{1-2y}{m+2}-v\right|dv\right|\Bigg]\|f\|_{C^2(I^2)} \\
\leq&\left[D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)+D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)+\left(\frac{1-2x}{n+2}\right)^2+\left(\frac{1-2y}{m+2}\right)^2\right]\|f\|_{C^2(I^2)}.
\end{aligned} \tag{4.20}$$

With the help of (4.19) and (4.20), (4.18) becomes

$$\begin{aligned}
\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid\leq&\omega(g;\mu_{n,m}(x,y))+4\|g-f\|+\Big[D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y) \\
&+D_{n,m}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{s}}^2(y);x,y)+\left(\frac{1-2x}{n+2}\right)^2+\left(\frac{1-2y}{m+2}\right)^2\Big]\|f\|_{C^2(I^2)}
\end{aligned}$$

$$=4\left[\|g-f\|+\frac{1}{4}\sigma_{n,m}(x,y)\|f\|_{C^2(I^2)}\right]+\omega(g;\mu_{n,m}(x,y)).$$

Taking infimum over $f\in C^2(I^2)$ and using Definition 1.3.8, we get

$$\mid D_{n,m}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)\mid\leq 4K_2\left(g;\frac{1}{4}\sigma_{n,m}(x,y)\right)+\omega(g;\mu_{n,m}(x,y)).$$

By using the relation (1.12), we reach at the desired result. □

Theorem 4.5.7. *Let $g,h\in C^2(I^2)$, then we have the following identity:*

$$\begin{aligned}\lim_{n\to\infty} n\left[D_{n,n}^{\langle l_1,l_2\rangle}(gh;x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)D_{n,n}^{\langle l_1,l_2\rangle}(h;x,y)\right]=&(k_1+2)x(1-x)g'_x(x,y)h'_x(x,y)\\&+(k_2+2)y(1-y)g'_y(x,y)h'_y(x,y).\end{aligned}$$

Proof. By Taylor's series, we obtain

$$D_{n,n}^{\langle l_1,l_2\rangle}(gh;x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)\,D_{n,n}^{\langle l_1,l_2\rangle}(h;x,y)$$

$$\begin{aligned}=&D_{n,n}^{\langle l_1,l_2\rangle}(gh;x,y)-(gh)(x,y)-(g'_x(x,y)h(x,y)+g(x,y)h'_x(x,y))D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)\\&-(g'_y(x,y)h(x,y)+g(x,y)h'_y(x,y))D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^1(y);x,y)-\frac{1}{2}(g''_{xx}(x,y)h(x,y)+2g'_x(x,y)h'_x(x,y)\\&+g(x,y)h''_{xx}(x,y))D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)-\frac{1}{2}(g''_{yy}(x,y)h(x,y)+2g'_y(x,y)h'_y(x,y)\\&+g(x,y)h''_{yy}(x,y))D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^2(y);x,y)-(g(x,y)h''_{xy}(x,y)+g'_y(x,y)h'_x(x,y)\\&+g'_x(x,y)h'_y(x,y)+g''_{xy}(x,y)h(x,y))D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x)\phi_s^1(y);x,y)\\&-h(x,y)\Bigg[D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)-g(x,y)-g'_x(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)-g'_y(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^1(y);x,y)\\&-\frac{1}{2}g''_{xx}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)-\frac{1}{2}g''_{yy}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^2(y);x,y)\\&-g''_{xy}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x)\phi_s^1(y);x,y)\Bigg]-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)\Big[D_{n,n}^{\langle l_1,l_2\rangle}(h;x,y)-h(x,y)\\&-h'_x(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)-h'_y(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^1(y);x,y)-\frac{1}{2}h''_{xx}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)\\&-\frac{1}{2}h''_{yy}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^2(y);x,y)-h''_{xy}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x)\phi_s^1(y);x,y)\Big]\\&+h'_x(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x);x,y)[g(x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)]\\&+h'_y(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^1(y);x,y)[g(x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}h''_{xx}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)[g(x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)]\\
&+\frac{1}{2}h''_{yy}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^2(y);x,y)[g(x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)]\\
&+h''_{xy}(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x)\phi_s^1(y);x,y)[g(x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)]\\
&+g'_x(x,y)h'_x(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^2(x);x,y)+g'_x(x,y)h'_y(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x)\phi_s^1(y);x,y)\\
&+g'_y(x,y)h'_x(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_{\mathrm{t}}^1(x)\phi_s^1(y);x,y)+g'_y(x,y)h'_y(x,y)D_{n,n}^{\langle l_1,l_2\rangle}(\phi_s^2(y);x,y).
\end{aligned}
$$

Now, applying the operators and taking limit as $n \to \infty$, we attain

$$
\begin{aligned}
\lim_{n\to\infty} n[D_{n,n}^{\langle l_1,l_2\rangle}(gh;x,y)-D_{n,n}^{\langle l_1,l_2\rangle}(g;x,y)D_{n,n}^{\langle l_1,l_2\rangle}(h;x,y)] =&(k_1+2)x(1-x)g'_x(x,y)h'_x(x,y)\\
&+(k_2+2)y(1-y)g'_y(x,y)h'_y(x,y).
\end{aligned}
$$

□

4.6 Numerical Examples

In this section, we present some examples to verify our above proved results, with their graphs and approximation error for certain functions.

Example 4.6.1. *Let $g(x)=\sin\pi x$ (green). Firstly, we show the convergence of the operators $D_n^{\left\langle\frac{k}{n}\right\rangle}(g;x)$ with particular parameters, that are $n=30,40$ (red, blue resp.), $k=0.5$ in the Fig. 4.1.*

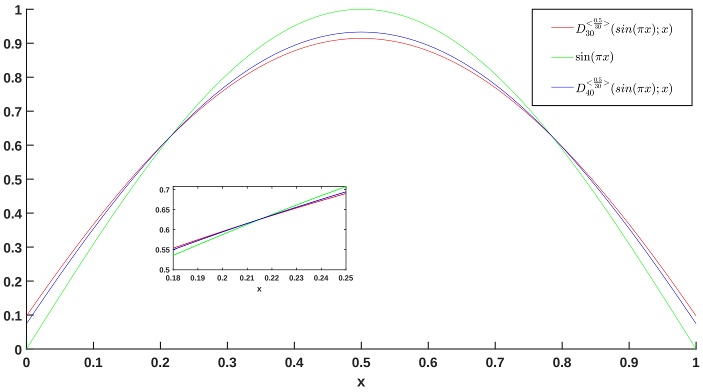


Figure 4.1: Convergence analysis for $n=30,40,k=0.5$

Also in Fig. 4.2, we display the error of approximation of the operators for these parametric values $n = 30, 40$ *(red, blue resp.), which is defined by* $E_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) = \mid D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x) - g(x) \mid$.

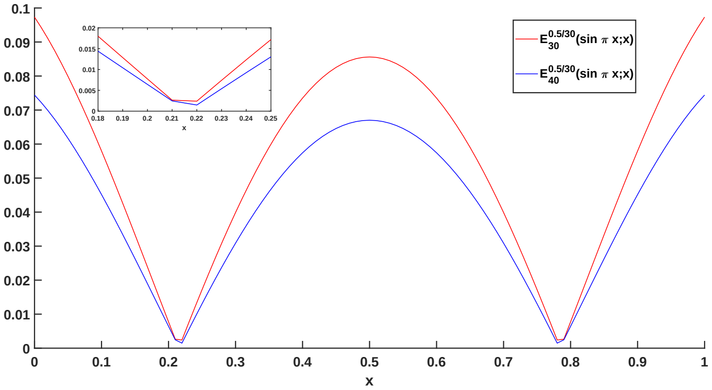


Figure 4.2: Error estimation for $n = 30, 40, k = 0.5$

Example 4.6.2. *Let* $g(x) = \cos 2\pi x$ *(green). We present the convergence of the operators* $D_n^{\left\langle \frac{k}{n} \right\rangle}(g;x)$ *by choosing* $n = 40$ *and* $k = 0.1, 1, 2$*(purple, red, cyan resp.) in the Fig. 4.3. In Fig. 4.4, the deviation of the operators from the function for parametric values* $n = 40$ *and* $k = 0.1, 1, 2$ *(purple, red, cyan resp.) is shown.*

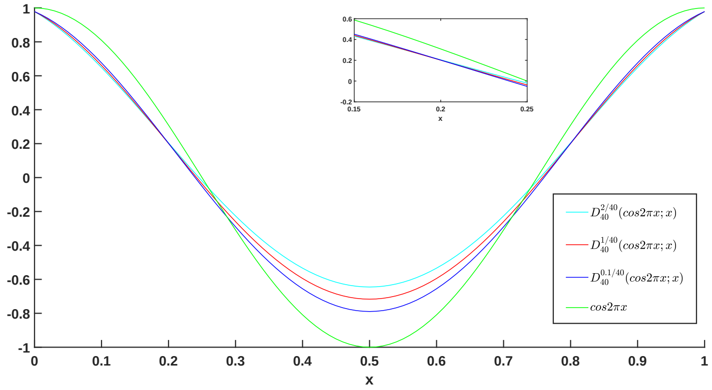


Figure 4.3: Convergence analysis for $n = 40, k = 0.1, 1, 2$

Example 4.6.3. *Let* $g(x, y) = x^2 y$ *(green). We present the convergence of the bivariate operators* $D_{n,m}^{\langle l_1,l_2 \rangle}(g;x,y)$ *by choosing* $n = m = 8$ *(cyan),* $n = m = 10$ *(red) and* $k_1 = k_2 = 3$ *in the Fig. 4.5. In Fig. 4.6, the error of approximation of the operators for the parametric values given above is shown.*

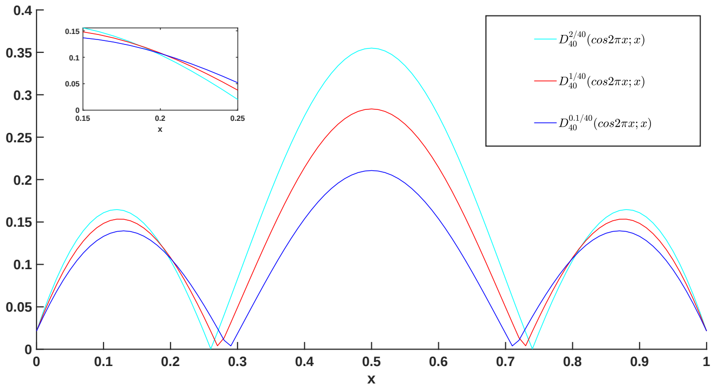


Figure 4.4: Error estimation for $n = 40, k = 0.1, 1, 2$

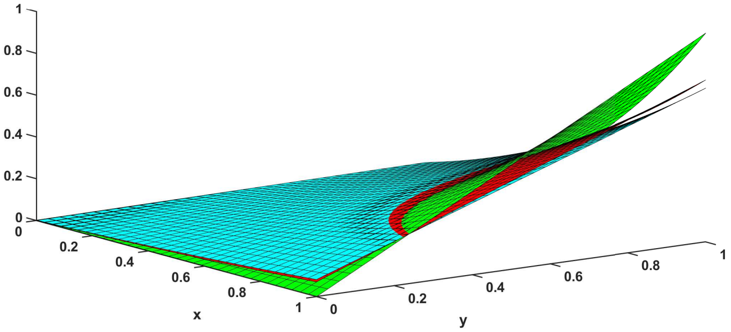


Figure 4.5: Convergence analysis for $n = m = 8, 10, k_1 = k_2 = 3$

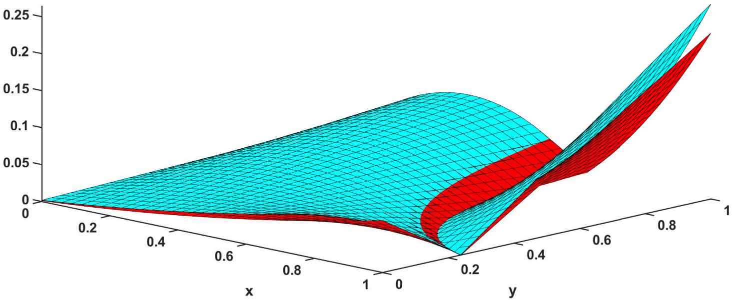


Figure 4.6: Error estimation for $n = m = 8, 10, k_1 = k_2 = 3$

4.7 Conclusion

In the current chapter, we have given the approximation results of the single and two-variable operators defined as (4.1) and (4.13) resp. We presented their order of approximation and error of approximation in terms of modulus of continuity. Finally from the given theoretical results and their verification with the help of figures, we can observe that the operators (4.1) are converging faster as we increase the value of n and vary the parameter k with suitable values.

Bernstein operators and most of its generalizations preserve linear functions. But in some situations, we need to preserve some other functions than linear. Therefore, King and many other researchers worked in this direction. In the next chapter, we will discuss this approach and study operators that can preserve any specified function with certain conditions.

Chapter 5

Integral type generalization of Bernstein-Lototsky operators in one and two variables

5.1 Introduction

In order to replicate the desired function, King [95] defined the operators named as Bernstein-Lototsky operators using Lototsky matrix in the following way:

$$\mathsf{K}_n(g;x) = \sum_{k=0}^{n} a_{n,k}(x)\, g\left(\frac{k}{n}\right),$$

where the basis function $a_{n,k}(x)$ is obtained by the given relation

$$\sum_{k=0}^{n} a_{n,k}(x) y^{\jmath} = \prod_{\jmath=1}^{n} \Big(p_{\jmath}(x)y + 1 - p_{\jmath}(x)\Big),$$

such that $p_{\jmath}(x) \in C[0,1]$, $1 \leq \jmath \leq n$, $0 < p_{\jmath}(x) < 1$ for $x \in (0,1)$ and $p_{\jmath}(0) = 0, p_{\jmath}(1) = 1$. By simple steps, we can get the value of $a_{n,k}(x)$ as follows:

$$a_{n,k}(x) = \sum_{\substack{I \cup J = \mathbb{N}_n \\ card(I)=k \\ card(J)=n-k}} \prod_{\jmath \in J} (1 - p_{\jmath}(x)) \prod_{\jmath \in I} p_{\jmath}(x), \tag{5.1}$$

where $\mathbb{N}_n = \{1, 2, \cdots, n\}$ and $a_{0,0}(x) = 1, a_{0,k}(x) = 0, k > 0$.

In particular, for $p_{\jmath}(x) = x$, these operators come back to classical Bernstein operators. In order to prove the uniform convergence of the operators $\mathsf{K}_n(g;x)$ to $g(x)$ on $[0,1]$, the author imposed the condition on $p_\jmath(x)$ that $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{\jmath=1}^{n}p_\jmath(x)=\lim\limits_{n\to\infty}s_n$ (say) converges uniformly to x on $[0,1]$. The shape-preserving characteristics of Bernstein-Lototsky operators were described by Xu *et al.* [150]. Eisenberg [59] constructed the complex variant of these operators to approximate the analytic functions. Xu and Goldman [149] also gave another approach to choose the sequence of functions $p_\jmath(x)$, where $p_1(x)$ is strictly increasing function and other functions can be determined by recursion such that the Bernstein-Lototsky operators preserve $p_1(x)$ and uniformly convergent for all continuous functions.

The Kantorovich type generalization of these operators is proposed by Khan *et al.* [91] to approximate the Lebesgue integrable functions and their approximation properties are studied. The Lototsky-Bernstein-Durrmeyer operators were developed by Abel and Agratini [1], who also introduced these operators in several dimensions.

In the present chapter, we proposed the generalization of Bernstein-Lototsky-Durrmeyer operators based on a real parameter $\rho > 0$. Section 5.2 is devoted to the construction of these operators whereas section 5.3 contains its approximation properties. In the next section, the generalization of these operators for two-variables is presented and their convergence properties and other properties are studied.

5.2 Construction of the Operators

For $g \in C(I)$ where $I = [0,1]$ and $\rho > 0$, the generalization of Benstein-Lototsky operators having Păltănea basis, is defined as

$$T_{n,\rho}(g;x)=\sum_{\imath=0}^{n}a_{n,\imath}(x)\int_0^1\mu_{n,\imath}^{\rho}(\mathrm{t})g(\mathrm{t})\,\mathrm{dt},\quad x\in[0,1],\tag{5.2}$$

where $\mu_{n,\imath}^{\rho}(\mathrm{t})$ is defined in (1.6).

Lemma 5.2.1. *The moments of the operators (5.2) are given by*

$T_{n,\rho}(e_0;x)=1;$

$$\begin{aligned}
T_{n,\rho}(e_1;x) =& \frac{n\rho}{n\rho+2}s_n(x)+\frac{1}{n\rho+2};\\
T_{n,\rho}(e_2;x) =& \frac{1}{(n\rho+3)(n\rho+2)}\left[n^2\rho^2 s_n^2(x)+\rho^2\sum_{\imath=1}^{n}p_\imath(x)(1-p_\imath(x))+3n\,\rho\,s_n(x)+2\right];\\
T_{n,\rho}(e_3;x) =& \frac{1}{(n\rho+4)(n\rho+3)(n\rho+2)}\left[n^3\rho^3\left\{s_n^3(x)+\frac{3}{n}s_n^2(x)+\frac{2}{n^2}s_n(x)+\frac{2}{n^3}\sum_{\imath=1}^{n}p_\imath^3(x)\right.\right.\\
&\left.-\frac{3}{n^3}\sum_{\imath=1}^{n}p_\imath^2(x)-\frac{3}{n^2}s_n(x)\sum_{\imath=1}^{n}p_\imath^2(x)\right\}+6n^2\rho^2\left\{s_n^2(x)+\frac{1}{n}s_n(x)-\frac{1}{n^2}\sum_{\imath=1}^{n}p_\imath^2(x)\right\}\\
&\left.+11n\,\rho+6\right];\\
T_{n,\rho}(e_4;x) =& \frac{1}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}\left[n^4\rho^4\left\{s_n^4(x)+\frac{6}{n}s_n^3(x)+\frac{7}{n^2}s_n^2(x)\right.\right.\\
&+\frac{7}{n^3}s_n(x)+\frac{8}{n^3}s_n(x)\sum_{\imath=1}^{n}p_\imath^3(x)-\frac{6}{n^2}s_n^2(x)\sum_{\imath=1}^{n}p_\imath^2(x)+\frac{3}{n^4}\left(\sum_{\imath=1}^{n}p_\imath^2(x)\right)^2\\
&\left.-\frac{6}{n^4}\sum_{\imath=1}^{n}p_\imath^4(x)+\frac{12}{n^4}\sum_{\imath=1}^{n}p_\imath^3(x)-\frac{18}{n^3}s_n(x)\sum_{\imath=1}^{n}p_\imath^2(x)-\frac{7}{n^4}\sum_{\imath=1}^{n}p_\imath^2(x)\right\}\\
&+10n^3\rho^3\left\{s_n^3(x)+\frac{3}{n}s_n^2(x)+\frac{2}{n^2}s_n(x)-\frac{3}{n^2}s_n(x)\sum_{\imath=1}^{n}p_\imath^2(x)+\frac{2}{n^3}\sum_{\imath=1}^{n}p_\imath^3(x)\right.\\
&\left.\left.-\frac{3}{n^3}\sum_{\imath=1}^{n}p_\imath^2(x)\right\}+35n^2\rho^2\left\{s_n^2(x)+\frac{1}{n}s_n(x)-\frac{1}{n^2}\sum_{\imath=1}^{n}p_\imath^2(x)\right\}+50n\,\rho+24\right].
\end{aligned}$$

Lemma 5.2.2. *The central moments of the operators (5.2) are given by*

$$\begin{aligned}
T_{n,\rho}(\phi_{\mathrm{t}}^1(x);x) =& s_n(x)-x+\frac{1-2s_n(x)}{n\rho+2};\\
T_{n,\rho}(\phi_{\mathrm{t}}^2(x);x) =& (s_n(x)-x)^2+\frac{1}{(n\rho+3)(n\rho+2)}\left[\rho^2\sum_{\imath=1}^{n}\rho_\imath(x)(1-\rho_\imath(x))+3n\rho s_n(x)+2\right.\\
&\left.-(5n\rho+6)s_n^2(x)-2x(1-2s_n(x))(n\rho+3)\right];\\
T_{n,\rho}(\phi_{\mathrm{t}}^4(x);x) =& (s_n(x)-x)^4+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

5.3 Approximation Results

Theorem 5.3.1. *Let $n\in\mathbb{N},\rho>0$. If $s_n(x)$ converges uniformly to x on I, then*

$$\lim_{n\to\infty}T_{n,\rho}(g;x)=g(x)$$

uniformly on I for every $g \in C(I)$.

Proof. We can achieve our required result with the help of Lemma 5.2.1 as well as Bohman-Korovkin theorem. □

Theorem 5.3.2. *For $g \in C(I)$, we have the inequality*

$$\mid T_{n,\rho}(g;x) - g(x) \mid \leq 2\,\omega(g;\delta_n(x)),$$

where $\delta_n(x) = \sqrt{T_{n,\rho}(\phi^2_{\mathrm{t}}(x);x)}$.

Proof. For every positive linear operators $L_n(.;x)$, Shisha and Mond [115] have given the following inequality:

$$\mid L_n(g;x) - g(x) \mid \leq \mid g(x) \mid\mid L_n(e_0;x) - 1 \mid + \left(L_n(e_0;x) + \frac{1}{\delta}\sqrt{L_n(e_0;x)L_n(\phi^2_{\mathrm{t}}(x);x)}\right)\omega(g;\delta),$$

where $\delta > 0$.

As $T_{n,\rho}(g;x)$ are positive linear operators, thus by using Lemma 5.2.1, we get

$$\mid T_{n,\rho}(g;x) - g(x) \mid \leq \left(1 + \frac{1}{\delta}\sqrt{T_{n,\rho}(\phi^2_{\mathrm{t}}(x);x)}\right)\omega(g;\delta).$$

Now, by choosing $\delta(x) = \delta_n(x) := \sqrt{T_{n,\rho}(\phi^2_{\mathrm{t}}(x);x)}$, we reach at our desired result. □

Theorem 5.3.3. *Let $n \in \mathbb{N}, \rho > 0$. If $g \in C(I)$, then*

$$\mid T_{n,\rho}(g;x) - g(x) \mid \leq C\left\{\omega_2\left(g;\frac{\Gamma_{n,\rho}(x)}{2}\right) + min\left(1, \frac{\Gamma^2_{n,\rho}(x)}{4}\|g\|\right)\right\} + \omega(g;\gamma_n(x)).$$

where $\gamma_n(x) = (T_{n,\rho}(e_1;x) - x)$ and $\Gamma_{n,\rho}(x) = \sqrt{T_{n,\rho}(\phi^2_{\mathrm{t}}(x);x) + \gamma^2_n(x)}$.

Proof. Firstly, we define

$$\tilde{T}_{n,\rho}(g;x) = T_{n,\rho}(g;x) - g\Big(T_{n,\rho}(e_1;x)\Big) + g(x). \tag{5.3}$$

We can easily check that $\tilde{T}_{n,\rho}(e_\imath;x) = x^\imath$ for $\imath = 0,1$. Therefore, the central moment $\tilde{T}_{n,\rho}(\phi^1_t(x);x) = 0$.

Also, by using (5.3), the operators $\tilde{T}_{n,\rho}(.;x)$ satisfy

$$| \tilde{T}_{n,\rho}(g;x) | \leq 3\|g\|. \tag{5.4}$$

Let $f \in C^2(I)$ be an arbitrary function. Then, by Taylor's expansion with integral expression of remainder, we obtain the following expression:

$$f(\mathrm{t}) = f(x) + \phi^1_{\mathrm{t}}(x) f'(x) + \int_x^{\mathrm{t}} \phi^1_{\mathrm{t}}(u) f''(u)\, du, \qquad \mathrm{t}, x \in [0,1], u \in (x, \mathrm{t}).$$

Apply the operators $\tilde{T}_{n,\rho}(.;x)$ on both sides

$$\begin{aligned}\tilde{T}_{n,\rho}(f;x) - f(x) =& \tilde{T}_{n,\rho}\left(\int_x^{e_1} (e_1 - u) f''(u)\, du; x\right)\\ =& T_{n,\rho}\left(\int_x^{e_1} (e_1 - u) f''(u)\, du; x\right) - \int_x^{T_{n,\rho}(e_1;x)} \left(T_{n,\rho}(e_1;x) - u\right) f''(u)\, du.\end{aligned} \tag{5.5}$$

Also, we can find

$$\begin{aligned}\left| T_{n,\rho}\left(\int_x^{e_1} (e_1 - u) f''(u)\, du; x\right)\right| \leq & T_{n,\rho}\left(\left|\int_x^{e_1} | e_1 - u || f''(u) | \, du\right|; x\right)\\ \leq & \| f \|_{C^2(I)} \, T_{n,\rho}(\phi^2_{\mathrm{t}}(x); x),\end{aligned}$$

and

$$\left|\int_x^{T_{n,\rho}(e_1;x)} \left(T_{n,\rho}(e_1;x) - u\right) f''(u)\, du\right| \leq \| f \|_{C^2(I)} \left(T_{n,\rho}(e_1;x) - x\right)^2 := \| f \|_{C^2(I)} \, \gamma^2_n(x).$$

Thus, with the help of above inequalities, (5.5) becomes

$$| \tilde{T}_{n,\rho}(f;x) - f(x) | \leq \left(T_{n,\rho}(\phi^2_{\mathrm{t}}(x); x) + \gamma^2_n(x)\right) \| f \|_{C^2(I)} := \Gamma^2_{n,\rho}(x) \|f\|_{C^2(I)}. \tag{5.6}$$

Now, for $g \in C(I)$, we have

$$| T_{n,\rho}(g;x) - g(x) | \leq | T_{n,\rho}(g;x) - \tilde{T}_{n,\rho}(g;x) | + | \tilde{T}_{n,\rho}(g - f;x) | + | \tilde{T}_{n,\rho}(f;x) - f(x) |$$

$$
\begin{aligned}
&+ \mid g(x) - f(x) \mid \\
&\leq \mid \tilde{T}_{n,\rho}(g-f;x) \mid + \mid \tilde{T}_{n,\rho}(f;x) - f(x) \mid + \mid g(x) - f(x) \mid \\
&+ \mid g(T_{n,\rho}(e_1;x)) - g(x) \mid .
\end{aligned}
$$

Using (5.4), (5.6), and Definition 1.3.5, we get

$$\mid T_{n,\rho}(g;x) - g(x) \mid \leq 4 \parallel g - f \parallel_{C(I)} + \parallel f \parallel_{C^2(I)} \Gamma^2_{n,\rho}(x) + \omega(g;\gamma_n(x)).$$

Taking the infimum *w.r.t.* $f \in C^2(I)$, we reach at

$$\mid T_{n,\rho}(g;x) - g(x) \mid \leq 4K_2\left(g;\frac{\Gamma^2_{n,\rho}(x)}{4}\right) + \omega(g;\gamma_n(x)).$$

Now, using (1.12), we get the required result. □

Theorem 5.3.4. *For $g \in Lip_{\mathscr{M}}(\tau)$, we have*

$$\mid T_{n,\rho}(g;x) - g(x) \mid \leq \mathscr{M}\left[T_{n,\rho}(\phi^2_{\mathrm{t}}(x);x)\right]^{\frac{\tau}{2}} .$$

Proof. Since for any $g \in Lip_{\mathscr{M}}(\tau)$, we get

$$\mid g(y) - g(x) \mid \leq \mathscr{M} \mid y - x \mid^{\tau} .$$

Now, using the linearity of the operators $T_{n,\rho}(.;x)$, we have

$$
\begin{aligned}
\mid T_{n,\rho}(g;x) - g(x) \mid &\leq T_{n,\rho}(\mid g(t) - g(x) \mid;x) \\
&\leq T_{n,\rho}(\mathscr{M} \mid \phi^1_{\mathrm{t}}(x) \mid^{\tau};x) \\
&\leq \mathscr{M} T_{n,\rho}(\mid \phi^1_{\mathrm{t}}(x) \mid^{\tau};x).
\end{aligned}
$$

By Hölder's inequality, we obtain the result. □

5.4 Bivariate of the Operators

Since in most of the cases, we get the functions of more than one variable. Thus it becomes essential to study bivariate operators to approximate these functions as well as univariate operators. For $g \in C(I^2)$, the operators are defined as

$$T_{n,m,\rho_1,\rho_2}(g;x,y) = \sum_{\imath=0}^{n}\sum_{\jmath=0}^{m} a_{n,\imath}(x)a_{m,\jmath}(y)\int_0^1\int_0^1 \mu_{n,\imath}^{\rho_1}(\mathrm{t})\mu_{m,\jmath}^{\rho_2}(\mathrm{s})\, g(\mathrm{t},\mathrm{s})\,\mathrm{ds}\,\mathrm{dt}, \tag{5.7}$$

where $a_{n,\imath}(x)$, $\mu_{n,\imath}^{\rho_1}(\mathrm{t})$ are defined as in (5.1) and (1.6) resp., $\rho_1, \rho_2 > 0$, and $n, m \in \mathbb{N}$.

Lemma 5.4.1. *The following are the moments of the bivariate operators (5.7):*

$$\begin{aligned}
T_{n,m,\rho_1,\rho_2}(e_{00};x,y) =& 1;\\
T_{n,m,\rho_1,\rho_2}(e_{10};x,y) =& \frac{n\rho_1\, s_n(x)+1}{n\rho_1+2};\\
T_{n,m,\rho_1,\rho_2}(e_{01};x,y) =& \frac{m\rho_2\, s_m(y)+1}{m\rho_2+2};\\
T_{n,m,\rho_1,\rho_2}(e_{20};x,y) =& \frac{1}{(n\rho_1+3)(n\rho_1+2)}\left[n^2{\rho_1}^2 s_n^2(x) + {\rho_1}^2\sum_{\imath=1}^{n} p_\imath(x)(1-p_\imath(x))\right.\\
&\left. +3\,n\,\rho_1\, s_n(x)+2\right];\\
T_{n,m,\rho_1,\rho_2}(e_{02};x,y) =& \frac{1}{(m\rho_2+3)(m\rho_2+2)}\left[m^2{\rho_2}^2 s_m^2(y) + {\rho_2}^2\sum_{\jmath=1}^{m} p_\imath(y)(1-p_\imath(y))\right.\\
&\left. +3\,m\,\rho_2\, s_m(y)+2\right].
\end{aligned}$$

Lemma 5.4.2. *The central moments of the bivariate operators (5.7) are as follows:*

$$\begin{aligned}
T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^1(x);x,y) =& s_n(x) - x + \frac{1-2s_n(x)}{n\rho_1+2};\\
T_{n,m,\rho_1,\rho_2}(\phi_{s}^1(y);x,y) =& s_n(y) - y + \frac{1-2s_m(y)}{m\rho_2+2};\\
T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y) =& (s_n(x)-x)^2 + \frac{1}{(n\rho_1+3)(n\rho_1+2)}\left[{\rho_1}^2\sum_{\imath=1}^{n}\rho_{1\imath}(x)(1-\rho_{1\imath}(x))\right.\\
&\left. +\,3n\rho_1 s_n(x) + 2 - (5n\rho_1+6)s_n^2(x) - 2x(1-2s_n(x))(n\rho_1+3)\right]\\
\leq& (s_n(x)-x)^2 + \frac{\rho_1+28}{4n\rho_1};
\end{aligned}$$

$$\begin{aligned}T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y) =&(s_m(y)-y)^2+\frac{1}{(m\rho_2+3)(m\rho_2+2)}\Bigg[{\rho_2}^2\sum_{\imath=1}^{m}\rho_{2_\imath}(y)(1-\rho_{2_\imath}(y))\\ &+3m\rho_2 s_m(y)+2-(5m\rho_2+6)s_m^2(y)-2y(1-2s_m(y))(m\rho_2+3)\Bigg]\\ \leq&(s_m(y)-y)^2+\frac{\rho_2+28}{4m\rho_2}.\end{aligned}$$

Theorem 5.4.3. *For $g\in C(I^2)$ the sequence of the operators $T_{n,m,\rho_1,\rho_2}(g;x,y)$ converges uniformly to $g(x,y)$ on I^2.*

Proof. Using the Lemma 5.4.1, we can get

$$\begin{aligned}\lim_{n,m\to\infty}T_{n,m,\rho_1,\rho_2}(e_{10};x,y)=&x;\\ \lim_{n,m\to\infty}T_{n,m,\rho_1,\rho_2}(e_{01};x,y)=&y;\\ \lim_{n,m\to\infty}T_{n,m,\rho_1,\rho_2}(e_{20}+e_{02};x,y)=&x^2+y^2.\end{aligned}$$

With the help of the generalization of Bohman-Korovkin theorem, given by Volkov [145] for operators of two variables, we obtain our desired result. □

Theorem 5.4.4. *For $g\in C(I^2)$, we have the inequality*

$$|\ T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\ |\leq 4\,\omega(g;\delta_1(x),\delta_2(y)),$$

where $\delta_1(x):=\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)}$ and $\delta_2(y):=\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)}$.

Proof. With the help of linearity of $T_{n,m,\rho_1,\rho_2}(g;x,y)$ and Definition 1.3.9, we have

$$\begin{aligned}|\ T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\ |\leq& T_{n,m,\rho_1,\rho_2}(|\ g(\mathrm{t},\mathrm{s})-g(x,y)\ |;x,y)\\ \leq& T_{n,m,\rho_1,\rho_2}(\omega(g;|\ \phi_{\mathrm{t}}^1(x)\ |,|\ \phi_{\mathrm{s}}^1(y)\ |);x,y).\end{aligned}$$

By using the properties of modulus of continuity, we get

$$\begin{aligned}|\ T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\ |\leq&\omega(g;\delta_1(x),\delta_2(y))\left(1+\frac{T_{n,m,\rho_1,\rho_2}(|\ \phi_{\mathrm{t}}^1(x)\ |;x,y)}{\delta_1(x)}\right)\\ &\times\left(1+\frac{T_{n,m,\rho_1,\rho_2}(|\ \phi_{\mathrm{s}}^1(y)\ |;x,y)}{\delta_2(y)}\right).\end{aligned}$$

With the help of Cauchy-Schwarz inequality, we attain

$$\begin{aligned}&\mid T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid\\ &\leq\omega(g;\delta_1(x),\delta_2(y))\left(1+\frac{\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)}}{\delta_1(x)}+\frac{\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)}}{\delta_2(y)}\right.\\ &\left.+\frac{\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)}}{\delta_1(x)\delta_2(y)}\right).\end{aligned}$$

On choosing $\delta_1(x):=\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)}$ and $\delta_2(y):=\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)}$, we reach at the required result. □

Theorem 5.4.5. *For $g\in C(I^2)$, we have*

$$\mid T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid\leq 2[\omega^1(g;\delta_1(x))+\omega^2(g;\delta_2(y))],$$

where $\delta_1(x)$ and $\delta_2(y)$ are as defined in Theorem 5.4.4.

Proof. By using the linearity of the operators $T_{n,m,\rho_1,\rho_2}(g;x,y)$ and Definition 1.3.10, we get

$$\begin{aligned}\mid T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid&\leq T_{n,m,\rho_1,\rho_2}(\mid g(\mathrm{t},\mathrm{s})-g(\mathrm{t},y)+g(\mathrm{t},y)-g(x,y)\mid;x,y)\\ &\leq T_{n,m,\rho_1,\rho_2}(\mid g(\mathrm{t},y)-g(x,y)\mid;x,y)+T_{n,m,\rho_1,\rho_2}(\mid g(\mathrm{t},\mathrm{s})-g(\mathrm{t},y)\mid;x,y)\\ &\leq T_{n,m,\rho_1,\rho_2}(\omega^1(g;\mid\phi_{\mathrm{t}}^1(x)\mid);x,y)+T_{n,m,\rho_1,\rho_2}(\omega^2(g;\mid\phi_{\mathrm{s}}^1(y)\mid);x,y)\\ &\leq\omega^1(g;\delta_1(x))\left(1+\frac{\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)}}{\delta_1(x)}\right)\\ &\quad+\omega^2(g;\delta_2(y))\left(1+\frac{\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)}}{\delta_2(y)}\right).\end{aligned}$$

By choosing $\delta_1(x):=\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)}$ and $\delta_2(y):=\sqrt{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)}$, we get the result. □

Theorem 5.4.6. *For any $g\in C(I^2)$, we have the following inequality:*

$$\mid T_{n,m,\rho_1,\rho_2}(g;x,y)-g(x,y)\mid\leq 4C_1\left\{\omega_2\left(g;\frac{1}{2}\sqrt{\Gamma_{n,m}(x,y)}\right)+min\left(1,\frac{1}{4}\Gamma_{n,m}(x,y)\right)\right\}$$

$$+\,\omega(g;\gamma_{n,m}(x,y)),$$

where

$$\begin{aligned}\Gamma_{n,m}(x,y)=&\Bigg\{T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y)+T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y)\\&+\frac{1}{2}\left(s_n(x)-x+\frac{1-2s_n(x)}{n\rho_1+2}\right)^2+\frac{1}{2}\left(s_m(y)-y+\frac{1-2s_m(y)}{m\rho_2+2}\right)^2\Bigg\},\end{aligned}$$

and

$$\gamma_{n,m}(x,y)=\sqrt{\left(s_n(x)-x+\frac{1-2s_n(x)}{n\rho_1+2}\right)^2+\left(s_m(y)-y+\frac{1-2s_m(y)}{m\rho_2+2}\right)^2}.$$

Proof. Define the associated operators $T^*_{n,m,\rho_1,\rho_2}(.;x,y)$ as follows:

$$\begin{aligned}T^*_{n,m,\rho_1,\rho_2}(g;x,y)=&T_{n,m,\rho_1,\rho_2}(g;x,y)-g\left(\frac{n\rho_1s_n(x)+1}{n\rho_1+2},\frac{m\rho_2s_m(y)+1}{m\rho_2+2}\right)\\&+g(x,y).\end{aligned}$$

For any $f\in C^2(I^2)$ and $\mathrm{t},\mathrm{s}\in I$. By the Taylor's Theorem, we have

$$\begin{aligned}f(\mathrm{t},\mathrm{s})=&f(x,y)+\frac{\partial f(x,y)}{\partial x}\phi_{\mathrm{t}}^1(x)+\frac{\partial f(x,y)}{\partial y}\phi_{\mathrm{s}}^1(y)+\int_x^{\mathrm{t}}\phi_{\mathrm{t}}^1(u)\frac{\partial^2 f(u,y)}{\partial u^2}\,du\\&+\int_y^{\mathrm{s}}\phi_{\mathrm{s}}^1(v)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv.\end{aligned}\tag{5.8}$$

Applying the operators $T^*_{n,m,\rho_1,\rho_2}(.;x,y)$ on (5.8) and using the fact $T^*_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^1(x);x,y)=T^*_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^1(y);x,y)=0$, we get
$\mid T^*_{n,m,\rho_1,\rho_2}(f;x,y)-f(x,y)\mid$

$$\begin{aligned}=&\left|T^*_{n,m,\rho_1,\rho_2}\left(\int_x^{\mathrm{t}}\phi_{\mathrm{t}}^1(u)\frac{\partial^2 f(u,y)}{\partial u^2}\,du;x,y\right)+T^*_{n,m,\rho_1,\rho_2}\left(\int_y^{\mathrm{s}}\phi_{\mathrm{s}}^1(v)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv;x,y\right)\right|\\=&\Bigg|T_{n,m,\rho_1,\rho_2}\left(\int_x^{\mathrm{t}}\phi_{\mathrm{t}}^1(u)\frac{\partial^2 f(u,y)}{\partial u^2}\,du;x,y\right)+T_{n,m,\rho_1,\rho_2}\left(\int_y^{\mathrm{s}}\phi_{\mathrm{s}}^1(v)\frac{\partial^2 f(x,v)}{\partial v^2}\,dv;x,y\right)\\&-\int_x^{\frac{n\rho_1s_n(x)+1}{n\rho_1+2}}\left(\frac{n\rho_1s_n(x)+1}{n\rho_1+2}-u\right)\frac{\partial^2 f(u,y)}{\partial u^2}\,du\end{aligned}$$

$$
\begin{aligned}
&- \int_{y}^{\frac{m\rho_2 s_m(y)+1}{m\rho_2+2}} \left(\frac{m\rho_2 s_m(y)+1}{m\rho_2+2} - v\right) \frac{\partial^2 f(x,v)}{\partial v^2}\, dv \Bigg| \\
\leq &\|f\|_{C^2(I^2)} \Bigg\{ T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{t}}^2(x);x,y) + T_{n,m,\rho_1,\rho_2}(\phi_{\mathrm{s}}^2(y);x,y) \\
&+ \frac{1}{2}\left(s_n(x) - x + \frac{1-2s_n(x)}{n\rho_1+2}\right)^2 + \frac{1}{2}\left(s_m(y) - y + \frac{1-2s_n(y)}{m\rho_2+2}\right)^2 \Bigg\} \\
:= &\|f\|_{C^2(I^2)} \Gamma_{n,m}(x,y) (say).
\end{aligned}
$$

Now, for $g \in C(I^2)$, we have

$$
\begin{aligned}
| T_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y) | \leq & \, | T^*_{n,m,\rho_1,\rho_2}(f-g;x,y) | + | T^*_{n,m,\rho_1,\rho_2}(f;x,y) - f(x,y) | \\
& + \left| g\left(s_n(x) - x + \frac{1-2s_n(x)}{n\rho_1+2}, s_m(y) - y + \frac{1-2s_n(y)}{m\rho_2+2}\right) - g(x,y) \right| \\
& + | f(x,y) - g(x,y) | \\
\leq & 4\|g-f\|_{C(I^2)} + \|f\|_{C^2(I^2)} \Gamma_{n,m}(x,y) + \omega(g;\gamma_{n,m}(x,y)) \\
= & 4K_2\left(g; \frac{1}{4}\Gamma_{n,m}(x,y)\right) + \omega(g;\gamma_{n,m}(x,y)).
\end{aligned}
$$

By using the relationship of K_2-functional and second order modulus of smoothness (1.12), we attain our result. □

Theorem 5.4.7. *For $g \in Lip_{\mathscr{M}}(\tau_1,\tau_2)$, we have the identity*

$$
| T_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y) | \leq \mathscr{M} \left[\delta_1(x)\right]^{\tau_1} \left[\delta_2(x)\right]^{\tau_2},
$$

where $\delta_1(x)$ and $\delta_2(x)$ are defined as in Theorem 5.4.4.

Proof. Let $g \in Lip_{\mathscr{M}}(\tau_1,\tau_2)$, then by definition

$$
| g(t,s) - g(x,y) | \leq \mathscr{M} \mid \phi_{\mathrm{t}}^1(x) \mid^{\tau_1} \mid \phi_{\mathrm{s}}^1(y) \mid^{\tau_2}.
$$

Now, let

$$
| T_{n,m,\rho_1,\rho_2}(g;x,y) - g(x,y) | \quad \leq \quad T_{n,m,\rho_1,\rho_2}(| g(t,s) - g(x,y) |;x,y)
$$

$$\begin{aligned} &\leq T_{n,m,\rho_1,\rho_2}(\mathscr{M} \mid \phi_{\mathrm{t}}^1(x) \mid^{\tau_1} \mid \phi_{\mathrm{s}}^1(y) \mid^{\tau_2}; x, y) \\ &\leq \mathscr{M} T_{n,m,\rho_1,\rho_2}(\mid \phi_{\mathrm{t}}^1(x) \mid^{\tau_1}; x, y)\, T_{n,m,\rho_1,\rho_2}(\mid \phi_{\mathrm{s}}^1(y) \mid^{\tau_2}; x, y). \end{aligned}$$

By Hölder's inequality, we get the required result. □

5.5 Conclusion

This chapter focuses on univariate and bivariate positive linear operators designed to preserve any polynomial, subject to specific endpoint conditions. We study the fundamental properties of these operators encompassing convergence, error of approximation measured by the modulus of continuities, and applicability to Lipschitz-type functions. The significance of generalizations is given, which allows the selection of specific parameter values to enhance accuracy for a given n.

All operators studied so far are defined on a finite interval. In order to obtain the approximation behavior of functions that are defined on infinite interval, we can see the operators such as Baskakov and Szász etc. The subsequent chapter introduces the new operators of Baskakov type and shows the impact of parameter on convergence behavior of these operators.

Chapter 6

A note on α-Baskakov Durrmeyer type operators

6.1 Introduction

Recently, Aral and Erbay [23] introduced the generalization of classical Baskakov operators based on a real parameter $\alpha \in [0,1]$, as

$$L_n^{\alpha}(g;x) = \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x)\, g\left(\frac{\imath}{n}\right), \qquad n \geq 1, \qquad x \in [0,\infty), \tag{6.1}$$

where $q_{n,\imath}^{\alpha}(x)$ is given by

$$\begin{aligned} q_{n,\imath}^{\alpha}(x) = &\left\{ \frac{\alpha x}{1+x}\binom{n+\imath-1}{\imath} - (1-\alpha)(1+x)\binom{n+\imath-3}{\imath-2} + (1-\alpha)x\binom{n+\imath-1}{\imath} \right\} \\ &\times \frac{x^{\imath-1}}{(1+x)^{n+\imath-1}}. \end{aligned} \tag{6.2}$$

For $\alpha = 1$, it reduces to the classical Baskakov operators.
The authors studied convergence, error approximation and Voronovskaja type result for these operators. They verified that the parameter α does not affect the convergence but it can reduce the error of approximation by choosing suitable values of α.

Recently, Nasiruzzaman *et al.* [114] defined the Durrmeyer variant of these operators to extend the results on Lebesgue measurable spaces as below:

$$S_n^{\alpha}(g;x) = \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} s_{n,\imath}(\mathrm{t}) g(\mathrm{t})\, \mathrm{dt}, \tag{6.3}$$

$$\text{where} \quad s_{n,\imath}(\mathrm{t}) = \frac{\mathrm{t}^{\imath}}{B(\imath+1,n)(1+\mathrm{t})^{n+\imath+1}}$$

and $B(a,b)$ is a beta function. The authors studied the approximation properties for these operators as Korovkin-type and weighted Korovkin-type theorems, order of approximation, and rate of convergence. Also, Nasiruzzaman *et al.* [113] defined the bivariate of these operators and studied their properties.

The generalization of positive linear operators is the subject of research in order to reduce approximation error with the help of different parameters (as [14, 23–25, 48, 108, 131, 132, 148]). Although, generalization does not give better approximation in all the cases. But, if we choose the specific value of the parameter wisely for the certain function, we can achieve the great results. By motivated from these benefits of the generalizations of positive linear operators, which reduces the error of approximation, we modify the operators (6.3) by using a parameter $\rho > 0$ in the next section 6.2.

6.2 Construction of the Operators

We define the generalization of α-Baskakov Durrmeyer operators based on a real parameter $\rho > 0$. For $\Lambda > 0$ and $C_{\Lambda}[0,\infty) := \{g \in C[0,\infty) : g(\mathrm{t}) = O(\mathrm{t}^{\Lambda}), \mathrm{t} > 0\}$, the operators are given as

$$A_n^{\alpha,\rho}(g;x) = \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t}) g(\mathrm{t})\, \mathrm{dt}, \tag{6.4}$$

where $\mu_{n,\imath}^{\rho}(\mathrm{t})$ is defined as (1.6).

The chapter is structured as: In section 6.3, we give the moments, central moments that are used in subsequent sections. In section 6.4, we show main results for the operators (6.4) which are divided in two sub-sections. In the first sub-section, we prove basic convergence theorems in Korovkin and weighted Korovkin spaces and in the second sub-section, we study pointwise approximation properties of these operators. In section 6.5, we verify our theoretical results by numerical examples with the use of Mathematica.

6.3 Basic Results

Lemma 6.3.1. *The moments of the operators* $A_n^{\alpha,\rho}(.;x)$ *are given as*

$$\begin{aligned}
A_n^{\alpha,\rho}(e_0;x) =&1;\\
A_n^{\alpha,\rho}(e_1;x) =&\frac{\rho n x-2\rho(1-\alpha)x+1}{n\rho-1};\\
A_n^{\alpha,\rho}(e_2;x) =&\frac{1}{(n\rho-1)(n\rho-2)}\Bigg[x^2\Big\{\rho^2n^2+(4\alpha-3)\rho^2 n\Big\}\\
&+x\Big\{\rho^2(n+4\alpha-4)+3\rho n-6\rho(1-\alpha)\Big\}+2\Bigg].
\end{aligned}$$

Lemma 6.3.2. *The central moments of the operators* $A_n^{\alpha,\rho}(.;x)$ *are given as:*

$$\begin{aligned}
A_n^{\alpha,\rho}(\phi_x^1(\mathrm{t});x) =&\frac{1}{n\rho-1}\Bigg[x\left\{1-2\rho(1-\alpha)\right\}+1\Bigg];\\
A_n^{\alpha,\rho}(\phi_x^2(\mathrm{t});x) =&\frac{1}{(n\rho-1)(n\rho-2)}\Bigg[x^2\Big\{n\rho(\rho+1)-8\rho(1-\alpha)+2\Big\}\\
&+x\Big\{n\rho(\rho+1)-4\rho^2(1-\alpha)-6\rho(1-\alpha)+4\Big\}+2\Bigg];\\
A_n^{\alpha,\rho}(\phi_x^4(\mathrm{t});x) =&\frac{\rho^2(1+\rho)^2x^2(1+x)^2-96(1-\alpha)\rho^3x^3+24\rho^3x^3}{(n\rho-1)(n\rho-2)(n\rho-3)(n\rho-4)}n^2+O\left(\frac{1}{n^3}\right).
\end{aligned}$$

6.4 Main Results

6.4.1 Approximation in Korovkin and weighted Korovkin spaces

Theorem 6.4.1. *For every* $g\in C_\Lambda[0,\infty)$*, the operators (6.4) are uniformly convergent to* g *on each compact subset of* $[0,A]$*, where* $A\in(0,\infty)$*.*

Proof. For the uniform convergence of the operators (6.4), we need to prove $A_n^{\alpha,\rho}(e_\imath;x) \to e_\imath$, for $\imath = 0,1,2$.

From Lemma 6.3.1, it is obvious that $A_n^{\alpha,\rho}(e_0;x) \to e_0$, as $n \to \infty$.

For $\imath = 1$, we can check from below

$$\begin{aligned}\lim_{n\to\infty} A_n^{\alpha,\rho}(e_1;x) &= \lim_{n\to\infty} \frac{n\rho x - 2\rho(1-\alpha)x + 1}{n\rho - 1}\\ &= \lim_{n\to\infty}\left(x + \frac{x - 2\rho(1-\alpha)x + 1}{n\rho - 1}\right) = x.\end{aligned}$$

Now, for $\imath = 2$, we have

$$\begin{aligned}\lim_{n\to\infty} A_n^{\alpha,\rho}(e_2;x) = \lim_{n\to\infty} &\frac{1}{(n\rho-1)(n\rho-2)}\Big[x^2\Big\{\rho^2n^2 + (4\alpha-3)\rho^2 n\Big\}\\ &+ x\Big\{\rho^2(n+4\alpha-4) + 3\rho n - 6\rho(1-\alpha)\Big\} + 2\Big]\\ =&x^2 + \lim_{n\to\infty}\frac{1}{(n\rho-1)(n\rho-2)}\Big[x^2\Big\{3\rho n - 2 + (4\alpha-3)\rho^2 n\Big\}\\ &+ x\Big\{\rho^2(n+4\alpha-4) + 3\rho n - 6\rho(1-\alpha)\Big\} + 2\Big] = x^2.\end{aligned}$$

Hence, by using Theorem 1.1.3, we get the uniform convergence of our operators. □

Theorem 6.4.2. *For $g \in C_2^*[0,\infty)$, the operators $A_n^{\alpha,\rho}(g;x)$ satisfy*

$$\lim_{n\to\infty} \|A_n^{\alpha,\rho}(g;.) - g\|_2 = 0.$$

Proof. By weighted Korovkin theorem, it is sufficient to prove

$$\lim_{n\to\infty} \|A_n^{\alpha,\rho}(e_\imath;.) - e_\imath\|_2 = 0, \qquad \text{for} \quad \imath = 0,1,2.$$

For $\imath = 0$, using Lemma 6.3.1, we have $A_n^{\alpha,\rho}(e_0;x) = 1$.

Therefore, we have $\|A_n^{\alpha,\rho}(e_0;.) - e_0\|_2 = 0$.

For $\imath = 1$,

$$\begin{aligned}\|A_n^{\alpha,\rho}(e_1;x) - e_1\|_2 &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{n\rho x - 2\rho(1-\alpha)x + 1}{n\rho - 1} - x\right| \\ &\le \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\left|\frac{1-2\rho(1-\alpha)}{n\rho-1}\right| + \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{1}{n\rho-1}\right|.\end{aligned}$$

Thus, we get

$$\|A_n^{\alpha,\rho}(e_1;x) - e_1\|_2 = 0 \,\text{as}\; n\to\infty.$$

Similarly, we can prove for $\imath = 2$

$$\begin{aligned}\|A_n^{\alpha,\rho}(e_2;x) - e_2\|_2 &= \sup_{x\in[0,\infty)} \frac{|A_n^{\alpha,\rho}(e_2;x) - x^2|}{1+x^2} \\ &\le \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\left|\frac{\rho^2 n^2 + (4\alpha-3)\rho^2 n}{(n\rho-1)(n\rho-2)}\right| \\ &\quad + \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\left|\frac{\rho^2(n+4\alpha-4) + 3\rho n - 6\rho(1-\alpha)}{(n\rho-1)(n\rho-2)}\right| \\ &\quad + \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{2}{(n\rho-1)(n\rho-2)}\right|,\end{aligned}$$

which gives us $\|A_n^{\alpha,\rho}(e_2;x) - e_2\|_2 = 0$ as $n\to\infty$.

This completes the proof. □

6.4.2 Pointwise approximation properties by $A_n^{\alpha,\rho}(g;x)$

Theorem 6.4.3. *Let $g \in C_\Lambda[0,\infty)$ and $x\in[0,b]$ such that $b>0$, then we have*

$$|A_n^{\alpha,\rho}(g;x) - g(x)| \le 2\omega\left(g; \sqrt{A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)}\right).$$

Proof. From Definition 1.3.5, we have

$$\begin{aligned}|A_n^{\alpha,\rho}(g;x) - g(x)| &= \left|\left(\sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x)\int_0^\infty \mu_{n,\imath}^{\rho}(\mathrm{t})\, g(\mathrm{t})\,\mathrm{dt}\right) - g(x)\right| \\ &\le \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x)\int_0^\infty \mu_{n,\imath}^{\rho}(\mathrm{t})\,|g(\mathrm{t}) - g(x)|\,\mathrm{dt}\end{aligned}$$

$$\leq \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t})\, \omega(g; |\phi_{\mathrm{t}}^1(x)|)\, \mathrm{dt}.$$

By using the property of modulus of continuity with $\lambda = \frac{1}{\delta}|\phi_{\mathrm{t}}^1(x)|$, we get

$$\begin{aligned}
|A_n^{\alpha,\rho}(g;x) - g(x)| \leq & \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t}) \left(1 + \frac{1}{\delta}\,|\phi_{\mathrm{t}}^1(x)|\right) \omega(g;\delta)\, \mathrm{dt} \\
= & \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t})\, \omega(g;\delta)\, \mathrm{dt} \\
& + \frac{1}{\delta} \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t})\, |\phi_{\mathrm{t}}^1(x)|\, \omega(g;\delta)\, \mathrm{dt} \\
= & \omega(g;\delta) + \frac{1}{\delta} \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t})\, |\phi_{\mathrm{t}}^1(x)|\, \omega(g;\delta)\, \mathrm{dt} \\
= & \left(1 + \frac{1}{\delta} \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t})\, |\phi_{\mathrm{t}}^1(x)|\, \mathrm{dt}\right) \omega(g;\delta).
\end{aligned} \tag{6.5}$$

With the help of Cauchy Schwarz inequality, we have

$$\begin{aligned}
\sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t})\, |\phi_{\mathrm{t}}^1(x)|\, \mathrm{dt} \leq & \sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \left[\int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t}) \phi_{\mathrm{t}}^2(x)\, \mathrm{dt}\right]^{\frac{1}{2}} \\
\leq & \left[\sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x) \int_0^{\infty} \mu_{n,\imath}^{\rho}(\mathrm{t}) \phi_{\mathrm{t}}^2(x)\, \mathrm{dt}\right]^{\frac{1}{2}} \\
= & [A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)]^{\frac{1}{2}}.
\end{aligned}$$

Thus (6.5) reduces to

$$|A_n^{\alpha,\rho}(g;x) - g(x)| \leq \left(1 + \frac{1}{\delta}\sqrt{A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)}\right) \omega(g;\delta).$$

By choosing $\delta = \sqrt{A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)}$, the proof is completed. □

Theorem 6.4.4. *For every* $g \in C_\Lambda[0,\infty)$

$$\lim_{n\to\infty} (n\rho - 1)\, [A_n^{\alpha,\rho}(g;x) - g(x)] = [1 - 2\rho(1-\alpha)x + 1] g'(x) + (\rho+1)x(x+1)\frac{g''(x)}{2}$$

uniformly for $0 \leq x \leq b,$ *where* $b > 0.$

Proof. By the Taylor's polynomial, we have

$$g(\mathrm{t}) = g(x) + \phi_{\mathrm{t}}^{1}(x)g'(x) + \frac{\phi_{\mathrm{t}}^{2}(x)}{2!}g''(x) + (\Theta(\mathrm{t},x)\phi_{\mathrm{t}}^{2}(x) \tag{6.6}$$

having the conditions $\lim\limits_{\mathrm{t}\to x} \epsilon(\mathrm{t},x) = 0$.

Now, applying $A_n^{\alpha,\rho}(.;x)$ and then multiplying with $(n\rho - 1)$ on both sides of (6.6) and using Lemma 6.3.2, we get

$$\begin{aligned}(n\rho-1)(A_n^{\alpha,\rho}(g;x)-g(x)) =&(n\rho-1)\Bigg[A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^{1}(x);x)g'(x)+\frac{1}{2}A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^{2}(x);x)g''(x)\\ &+A_n^{\alpha,\rho}(\epsilon(\mathrm{t},x)\phi_{\mathrm{t}}^{2}(x);x)\Bigg]\\ =&\Bigg[x(1-2\rho(1-\alpha))+1\Bigg]g'(x)+\Bigg[(\rho+1)x(x+1)+\frac{2(\rho+1)x(x+1)}{n\rho-2}\\ &+\frac{-8\rho(1-\alpha)x^2+2x^2-4\rho^2(1-\alpha)x-6\rho(1-\alpha)x+4x+2}{n\rho-2}\Bigg]\frac{g''(x)}{2}\\ &+(n\rho-1)A_n^{\alpha,\rho}(\epsilon(\mathrm{t},x)\phi_{\mathrm{t}}^{2}(x);x). \qquad (6.7)\end{aligned}$$

By Cauchy-Schwarz inequality on the last term of above equation, we get

$$(n\rho-1)A_n^{\alpha,\rho}(\epsilon(\mathrm{t},x)\phi_{\mathrm{t}}^{2}(x);x) \leq (n\rho-1)\sqrt{A_n^{\alpha,\rho}(\epsilon^2(\mathrm{t},x);x)}\sqrt{A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^{4}(x);x)}.$$

Hence, by using Theorem 6.4.1 and Lemma 6.3.2, we have

$$\lim_{n\to\infty}(n\rho-1)A_n^{\alpha,\rho}(\epsilon(\mathrm{t},x)\phi_{\mathrm{t}}^{2}(x);x) = 0.$$

Thus, by taking limit as $n \to \infty$ on equation (6.7), we obtain the required result. □

Theorem 6.4.5. *Let* $g \in Lip_{\mathscr{M}}^{(\tau)}$ *with* $\mathscr{M} > 0$ *and* $0 < \tau \leq 1$. *Then the operators* $A_n^{\alpha,\rho}(.;x)$ *satisfy*

$$|A_n^{\alpha,\rho}(g;x) - g(x)| \leq \mathscr{M}\left(A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^{2}(x);x)\right)^{\frac{\tau}{2}}.$$

Proof. By using the linearity of operators $A_n^{\alpha,\rho}(g;x)$, we obtain

$$|A_n^{\alpha,\rho}(g;x)-g(x)| \leq A_n^{\alpha,\rho}(|g(\mathrm{t})-g(x)|;x).$$

From Definition 1.3.7, we have

$$\begin{aligned}|A_n^{\alpha,\rho}(g;x)-g(x)| \leq& A_n^{\alpha,\rho}(\mathscr{M}|\phi_{\mathrm{t}}^1(x)|^\tau;x) = \mathscr{M}\sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x)\int_0^\infty \mu_{n,\imath}^{\rho}(\mathrm{t})|\phi_{\mathrm{t}}^1(x)|^\tau \mathrm{dt}\\ \leq& \mathscr{M}\left[\sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x)\right]^{\frac{2-\tau}{2}}\left[\sum_{\imath=0}^{\infty} q_{n,\imath}^{\alpha}(x)\int_0^\infty \mu_{n,\imath}^{\rho}(\mathrm{t})\phi_{\mathrm{t}}^2(x)\mathrm{dt}\right]^{\frac{\tau}{2}}\\ &\left[\text{By Hölder's inequality with } p=\frac{2}{2-\tau} \text{ and } q=\frac{2}{\tau}\right]\\ =&\mathscr{M}\left(A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)\right)^{\frac{\tau}{2}}.\end{aligned}$$

Hence, the proof is completed. □

Theorem 6.4.6. *For every $g \in C_B[0,\infty)$, we have*

$$|A_n^{\alpha,\rho}(g;x)-g(x)| \leq 2C\left\{\omega_2\left(g;\sqrt{\frac{\Gamma_n(x)}{2}+\frac{\Delta_n(x)}{4}}\right)+min\left(1,\frac{\Gamma_n(x)}{2}+\frac{\Delta_n(x)}{4}\right)\|g\|_{C_B[0,\infty)}\right\},$$

where $C>0$ is any constant, $\Gamma_n(x)=A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^1(x);x)$, and $\Delta_n(x)=A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)$.

Proof. The second order Taylor's polynomial for $f \in C_B^2[0,\infty)$ and $\theta \in (x,\mathrm{t})$, we have

$$f(\mathrm{t}) = f(x)+\phi_{\mathrm{t}}^1(x)f'(x)+\frac{\phi_{\mathrm{t}}^2(x)}{2!}f''(\theta).$$

By applying the operators $A_n^{\alpha,\rho}(.;x)$ on both sides, we get

$$\begin{aligned}|A_n^{\alpha,\rho}(f;x)-f(x)| \leq& |A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^1(x);x)|\|f'\|_{C_B[0,\infty)}+\frac{1}{2}|A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)|\|f''\|_{C_B[0,\infty)}\\ :=&\Gamma_n(x)\|f'\|_{C_B[0,\infty)}+\frac{\Delta_n(x)}{2}\|f''\|_{C_B[0,\infty)}. \qquad (6.8)\end{aligned}$$

Now from equation (1.11), we obtain the following inequalities:

$$\|f'\|_{C_B[0,\infty)} \leq \|f\|_{C_B^2[0,\infty)} \text{ and } \|f''\|_{C_B[0,\infty)} \leq \|f\|_{C_B^2[0,\infty)}.$$

Using these inequalities in (6.8), we get

$$|A_n^{\alpha,\rho}(f;x) - f(x)| \leq \left(\Gamma_n(x) + \frac{\Delta_n(x)}{2}\right) \|f\|_{C_B^2[0,\infty)}. \tag{6.9}$$

Now, for $g \in C_B[0,\infty)$, we have

$$|A_n^{\alpha,\rho}(g;x) - g(x)| \leq |A_n^{\alpha,\rho}(g-f;x)| + |f(x) - g(x)| + |A_n^{\alpha,\rho}(f;x) - f(x)|. \tag{6.10}$$

Since

$$A_n^{\alpha,\rho}(g;x) \to g(x), \qquad A_n^{\alpha,\rho}(f;x) \to f(x),$$

then

$$|A_n^{\alpha,\rho}(g-f;x)| \to |g(x) - f(x)| \leq \|g-f\|_{C_B[0,\infty)}. \tag{6.11}$$

By using equations (6.9), (6.10), and (6.11), we get

$$\begin{aligned} |A_n^{\alpha,\rho}(g;x) - g(x)| \leq& 2\|g-f\|_{C_B[0,\infty)} + \left(\Gamma_n(x) + \frac{\Delta_n(x)}{2}\right) \|f\|_{C_B^2[0,\infty)} \\ =& 2\left(\|g-f\|_{C_B[0,\infty)} + \left(\frac{\Gamma_n(x)}{2} + \frac{\Delta_n(x)}{4}\right) \|f\|_{C_B^2[0,\infty)}\right). \end{aligned}$$

By taking the infimum over all $f \in C_B^2[0,\infty)$ and using the Definition 1.3.8, we have

$$|A_n^{\alpha,\rho}(g;x) - g(x)| \leq 2K_2\left(g; \frac{\Gamma_n(x)}{2} + \frac{\Delta_n(x)}{4}\right).$$

Also, using the equivalence relation of second order K-functional and second order modulus of continuity (1.12), we obtain the result. □

Theorem 6.4.7. *Suppose* $f \in C_2^*[0,\infty)$ *and* $\beta > 0$, *then*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\mid A_n^{\alpha,\rho}(g;x) - g(x) \mid}{(1+x^2)^{1+\beta}} = 0.$$

Proof. Let $x_0 > 0$ be an arbitrary but fixed number. Then

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\mid A_n^{\alpha,\rho}(g;x) - g(x) \mid}{(1+x^2)^{1+\beta}} &\leq \sup_{x\leq x_0} \frac{\mid A_n^{\alpha,\rho}(g;x) - g(x) \mid}{(1+x^2)^{1+\beta}} + \sup_{x>x_0} \frac{\mid A_n^{\alpha,\rho}(g;x) - g(x) \mid}{(1+x^2)^{1+\beta}} \\ &\leq \sup_{x\leq x_0} \mid A_n^{\alpha,\rho}(g;x) - g(x) \mid + \sup_{x>x_0} \frac{\mid A_n^{\alpha,\rho}(g;x) \mid}{(1+x^2)^{1+\beta}} + \sup_{x>x_0} \frac{\mid g(x) \mid}{(1+x^2)^{1+\beta}}.\end{aligned}$$

Since $\mid f(t) \mid \leq \parallel f \parallel_\rho (1+t^2)$ for $t > 0$, then

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\mid A_n^{\alpha,\rho}(g;x) - g(x) \mid}{(1+x^2)^{1+\beta}} &\leq \parallel A_n^{\alpha,\rho}(g;.) - g \parallel_{C[0,x_0]} + \parallel g \parallel_\rho \sup_{x>x_0} \frac{\mid A_n^{\alpha,\rho}(1+t^2;x) \mid}{(1+x^2)^{1+\beta}} \\ &+ \sup_{x>x_0} \frac{\parallel g \parallel_\rho}{(1+x^2)^{1+\beta}} := A_1 + A_2 + A_3.\end{aligned}$$

By using the convergence theorem, for given $\epsilon > 0$, a positive integer $k_1 \in \mathbb{N}$ exists such that

$$\parallel A_n^{\alpha,\rho}(g;.) - g \parallel_{C[0,x_0]} < \frac{\epsilon}{3} \quad \text{for } n \geq k_1.$$

Since $\lim_{n\to\infty} \sup_{x>x_0} \frac{A_n^{\alpha,\rho}(1+t^2;x)}{1+x^2} = 1$. Therefore, there exists $k_2 \in \mathbb{N}$ such that

$$\sup_{x>x_0} \frac{A_n^{\alpha,\rho}(1+t^2;x)}{1+x^2} \leq \frac{(1+x_0^2)^\beta \epsilon}{3 \parallel g \parallel_\rho} + 1 \qquad \forall\, n \geq k_2.$$

Hence,

$$\begin{aligned}A_2 &= \parallel g \parallel_\rho \sup_{x>x_0} \frac{\mid A_n^{\alpha,\rho}(1+t^2;x) \mid}{(1+x^2)^{1+\beta}} \\ &\leq \frac{\parallel g \parallel_\rho}{(1+x_0^2)^\beta} \sup_{x>x_0} \frac{A_n^{\alpha,\rho}(1+t^2;x)}{1+x^2} \\ &\leq \frac{\parallel g \parallel_\rho}{(1+x_0^2)^\beta} \left\{ \frac{(1+x_0^2)^\beta \epsilon}{3 \parallel g \parallel_\rho} + 1 \right\} \\ &= \frac{\epsilon}{3} + \frac{\parallel g \parallel_\rho}{(1+x_0^2)^\beta} \qquad \forall\, n \geq k_2.\end{aligned}$$

Choose x_0 to be large enough that $\dfrac{\| g \|_\rho}{(1+x_0^2)^\beta} \leq \dfrac{\epsilon}{6}$.
Thus, $A_2 \leq \dfrac{\epsilon}{6} + \dfrac{\epsilon}{3} = \dfrac{\epsilon}{2}$.
Now, $A_3 = \sup\limits_{x > x_0} \dfrac{\| g \|_\rho}{(1+x_0^2)^\beta} \leq \dfrac{\epsilon}{6}$.
Let $k = \max\{k_1, k_2\}$, then from the above inequalities, we get

$$\sup_{x \in [0,\infty)} \frac{\mid A_n^{\alpha,\rho}(g;x) - g(x) \mid}{(1+x^2)^{1+\beta}} < \epsilon \qquad \forall\, n \geq k.$$

As $\epsilon > 0$ is an arbitrary number, therefore we attain our result. $\square$

In the next theorem, we give the Grüss Voronovskaja type result.

Theorem 6.4.8. *Suppose that $g, h \in C_2[0,\infty)$ such that $g', h', g'', h'', (gh)', (gh)'' \in C_2[0,\infty)$. Then for each $x \in [0,\infty)$, we get*

$$A_n^{\alpha,\rho}(gh;x) - A_n^{\alpha,\rho}(g;x)A_n^{\alpha,\rho}(h;x) = g'(x)h'(x)\frac{x(1+x)(\rho+1)}{\rho}.$$

Proof. We know the identity
$A_n^{\alpha,\rho}(gh;x) - A_n^{\alpha,\rho}(g;x)A_n^{\alpha,\rho}(h;x)$

$$\begin{aligned}
&= \left[A_n^{\alpha,\rho}(gh;x) - (gh)(x) - (gh)'(x)A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^1(x);x) - \frac{(gh)''(x)}{2!}A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)\right] \\
&\quad - h'(x)\left[A_n^{\alpha,\rho}(g;x) - g(x) - g'(x)A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^1(x);x) - \frac{g''(x)}{2!}A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)\right] \\
&\quad - A_n^{\alpha,\rho}(g;x)\left[A_n^{\alpha,\rho}(h;x) - h(x) - h'(x)A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^1(x);x) - \frac{h''(x)}{2!}A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)\right] \\
&\quad + \frac{1}{2!}A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^2(x);x)\left[g(x)h''(x) + 2g'(x)h'(x) - h''(x)A_n^{\alpha,\rho}(g;x)\right] \\
&\quad + A_n^{\alpha,\rho}(\phi_{\mathrm{t}}^1(x);x)\left[g(x)h'(x) - h'(x)A_n^{\alpha,\rho}(g;x)\right].
\end{aligned}$$

Now, with the help of central moments and Voronovskaja result, we get
$\lim\limits_{n\to\infty} n\left[A_n^{\alpha,\rho}(gh;x) - A_n^{\alpha,\rho}(g;x)A_n^{\alpha,\rho}(h;x)\right]$

$$= \lim_{n\to\infty} \frac{g'(x)h'(x)n}{(n\rho-1)(n\rho-2)}\Big[x^2\Big\{n\rho(\rho+1) - 8\rho(1-\alpha) + 2\Big\}$$

$$
\begin{aligned}
&+ x\Big\{n\rho(\rho+1) - 4\rho^2(1-\alpha) - 6\rho(1-\alpha) + 4\Big\} + 2\Big] \\
=&g'(x)h'(x)\frac{x(1+x)(\rho+1)}{\rho}.
\end{aligned}
$$

□

6.5 Numerical Verifications

In this section, we give the numerical examples to verify the theoretical results that we have proved in the previous sections. Here, we compare the operators (6.3) with our operators $A_n^{\alpha,\rho}(g;x)$ for different values of ρ.

Example 6.5.1. *Let $g(x) = \sqrt{x}$. To show the convergence of the operators $A_n^{\alpha,\rho}(g,x)$, we choose certain values of parameters such as $n = 20, \alpha = 0.1$ and $\rho = 1, 5, 0.5$ in Figure 6.1.*

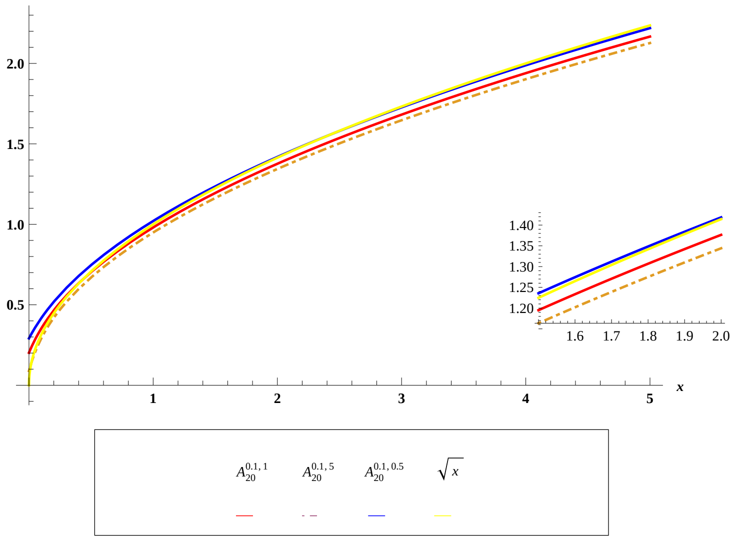


Figure 6.1: Approximation process for $n = 20, \alpha = 0.1$ and $\rho = 1, 5, 0.5$

Also, we have given the error of approximation of our operators $A_n^{\alpha,\rho}(g;x)$ from the function $g(x)$ for the same values, which is defined as $E_n^{\alpha,\rho}(g,x) = |g(x) - A_n^{\alpha,\rho}(g;x)|$

in Figure 6.2.

From the given figures, we can easily see that the operators $A_n^{\alpha,\rho}(g;x)$ *give the better*

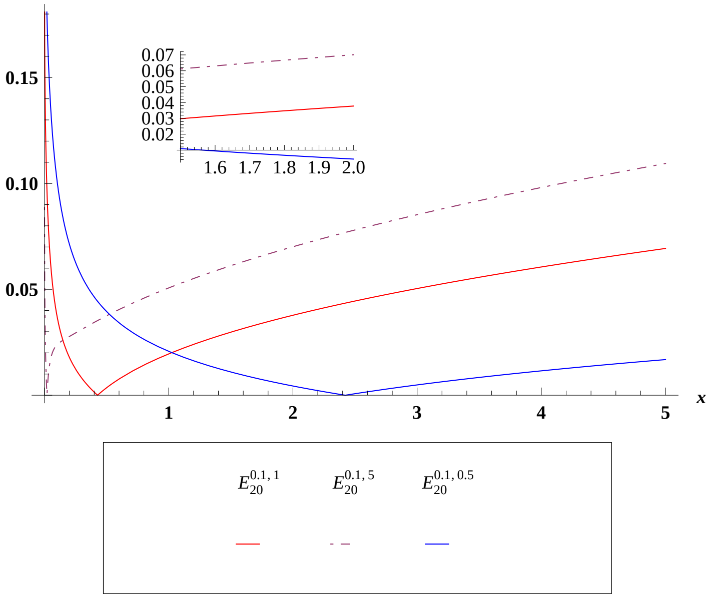


Figure 6.2: Error estimation for $n = 20, \alpha = 0.1$ and $\rho = 1, 5, 0.5$

approximation at $\rho = 0.5$.

Also, we give the convergence and the error approximation of the operators by choosing the parameters $\alpha = 1$, $n = 20$ *and different values of* $\rho = 1, 5, 0.5$, *which are shown in Figs. 6.3 and 6.4 resp. For these values, it is clear that approximation is better at* $\rho = 5$. *Therefore, the operators give better results by choosing suitable values of parameters.*

Example 6.5.2. *Let* $g(x) = x^2 + 5x + 2$. *Here, we give the convergence of our operators* $A_n^{\alpha,\rho}(g;x)$ *at* $n = 20, \alpha = 0.7$ *and certain values of* $\rho = 1, 5, 0.3$. *Also, we give the error of approximation* $E_n^{\alpha,\rho}(g;x)$ *of our operators* $A_n^{\alpha,\rho}(g;x)$ *from the function* $g(x)$ *at the chosen values.*

From the Figs. 6.5 and 6.6, we can easily see that the operators $A_n^{\alpha,\rho}(g;x)$ *give the better approximation at* $\rho = 5$.

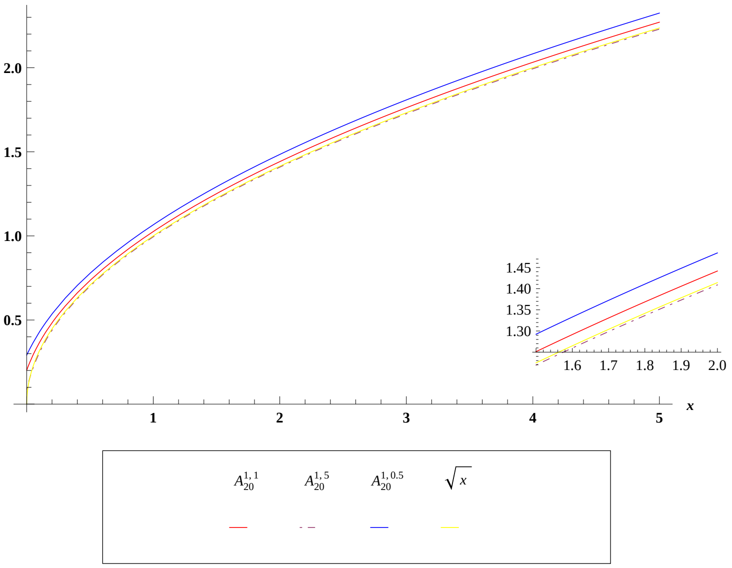


Figure 6.3: Approximation process for $n = 20, \alpha = 1$ and $\rho = 1, 5, 0.5$

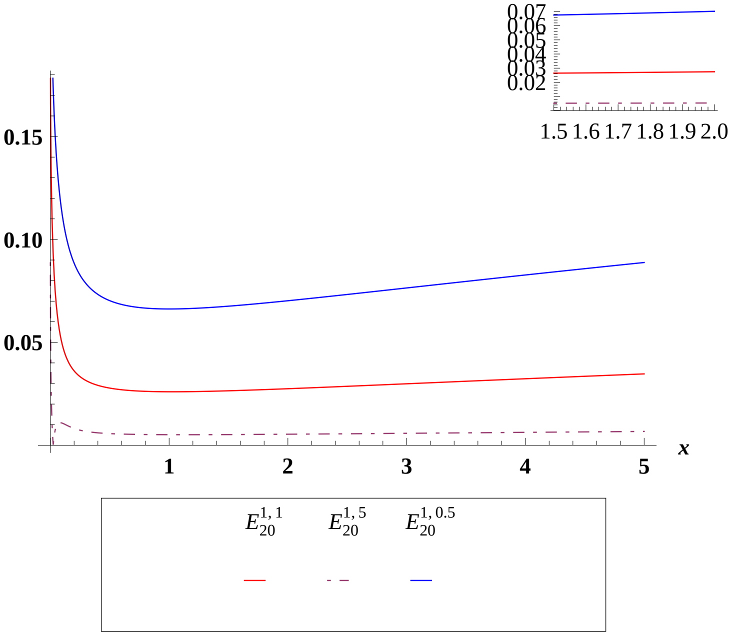


Figure 6.4: Error estimation for $n = 20, \alpha = 1$ and $\rho = 1, 5, 0.5$

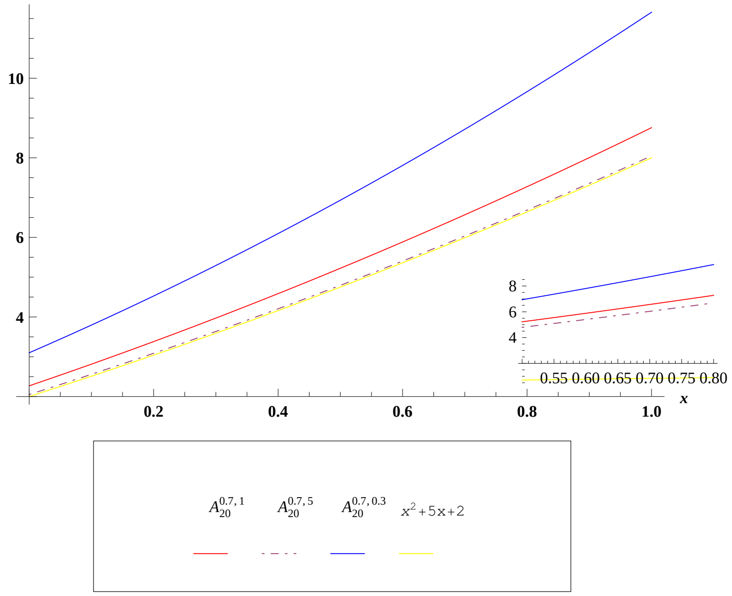


Figure 6.5: Approximation process for $n = 20, \alpha = 0.7$ and $\rho = 1, 5, 0.3$

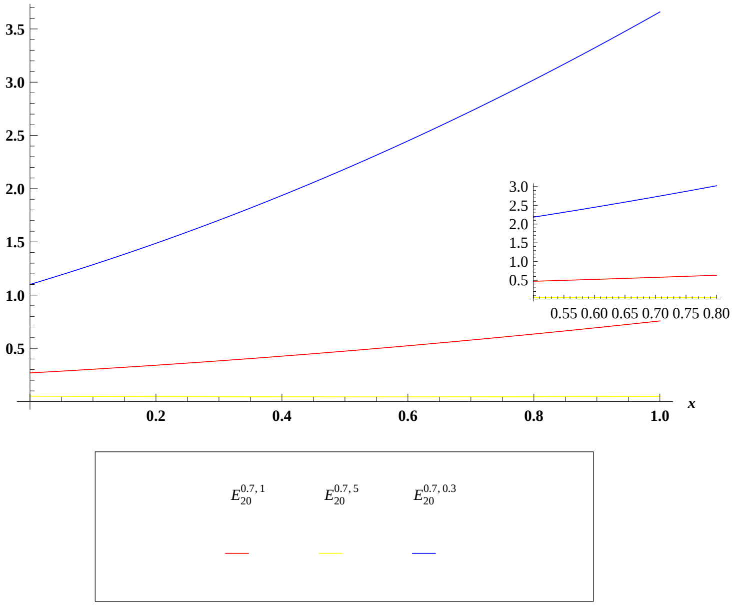


Figure 6.6: Error estimation for $n = 20, \alpha = 0.7$ and $\rho = 1, 5, 0.3$

6.6 Conclusion

The operators $A_n^{\alpha,\rho}(g;x)$ converge uniformly to a continuous bounded function $g(x)$ on infinite interval $[0,\infty)$. By these operators, we verified that generalizations of the positive linear operators can provide the better approximation than the classical ones, which is shown graphically through the examples. Here, we have seen that the convergence of the operators $A_n^{\alpha,\rho}(g;x)$ is independent of parameters α as well as ρ, whereas the error of approximation depends on these parameters.

The positive linear operators defined on finite interval are useful in many computer applications as Bernstein operators are useful to define Bézier curves. In the next chapter, we study the importance of Bézier curves and their generalizations to get flexible curves from the given data set. Also, we give an algorithm to find the control points for any given Bézier curve.

Chapter 7

A Note on α-Bézier curves and surfaces

7.1 Introduction

Bézier curves, as discussed in chapter 1, are essential in computer graphics, animation, and CAD applications because they can generate curves that are smooth, scalable, and easy to control. Bézier curves are a fundamental tool in the development of graphic designs, typefaces, and complicated forms in engineering and animation. They are defined by a collection of control points and enable designers to shape and modify curves in an understandable manner.

Parametric extensions of Bézier curves offer enhanced manipulation of the curve's form and characteristics. This extension enables a greater degree of flexibility and precision in manipulating the curves, proving these curves highly useful in complex design and animation projects. In the present chapter, we construct the generalization of Bézier curves by introducing a well known parameter.

Bernstein operators are the foundation stone of Bézier curves. They have many interesting properties, therefore an immense research is going on these operators [36, 43, 56, 71]. These operators are generalized by introducing some parameters see [46, 62, 104, 141]. Recently, Chen *et al.* [50] introduced the modification of Bernstein basis polynomials depending on a real parameter $\alpha \in [0,1]$, which are defined as

$$p^{\alpha}_{n,\imath}(x) = \left[\binom{n-2}{\imath}(1-\alpha)x + \binom{n-2}{\imath-2}(1-\alpha)(1-x) + \binom{n}{\imath}\alpha x(1-x)\right] \\ \times x^{\imath-1}(1-x)^{n-\imath-1},\ n \geq 2. \tag{7.1}$$

For $n = 1$, we have $p^{\alpha}_{1,0}(x) = 1 - x, \quad p^{\alpha}_{1,1}(x) = x$.

In the present chapter, we define the Bézier curves and surfaces with the polynomial basis *i.e.* $p^{\alpha}_{n,\imath}(x)$, which attain the flexibility to modify their shapes.

We organize our chapter as: In section 7.2, we give the properties of α-Bernstein basis polynomials. In section 7.3, we construct the α-Bézier curves and list its properties. Section 7.4 focuses on the study of tensor product α-Bézier surfaces and their properties. In the next section 7.5, we present the technique to find the control points from given α-Bézier curves. In the end, we display some numerical examples to show the impact of the parameter α on the classical Bézier curves as well as the significance of our algorithm to find the control points by numerical examples and their graphical illustrations.

7.2 Properties of α-Bernstein Basis Polynomials

Theorem 7.2.1. *The α-Bernstein basis polynomials i.e. $p^{\alpha}_{n,\imath}(x)$ preserve the following properties:*

1. *Non-Negativity: for each $x \in [0,1]$, we have $p^{\alpha}_{n,\imath}(x) \geq 0$ for $\imath = 0, 1, \cdots, n$.*
2. *End Point Interpolation:*

$$p^{\alpha}_{n,\imath}(0) = \begin{cases} 1, & \imath = 0 \\ 0, & \imath \neq 0 \end{cases} \quad \textit{and} \quad p^{\alpha}_{n,\imath}(1) = \begin{cases} 1, & \imath = n \\ 0, & \imath \neq n \end{cases}.$$

3. *Reducibility: for $\alpha = 1$, the α-Bernstein basis polynomials i.e. $p^{\alpha}_{n,\imath}(x)$ reduce to classical Bernstein basis polynomials i.e. $p_{n,\imath}(x)$.*

4. *Partition of Unity:* $\sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x) = 1.$

5. *Symmetric:* $p_{n,\imath}^{\alpha}(x) = p_{n,n-\imath}^{\alpha}(1-x).$

Proof. The proof of properties (1) - (3) are easy to verify and can be seen from [50]. Hence, we omit the details of these properties.

Property (4):

From (7.1), we have

$$\begin{aligned}\sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x) =& \sum_{\imath=0}^{n} \left\{ \binom{n-2}{\imath}(1-\alpha)x + \binom{n-2}{\imath-2}(1-\alpha)(1-x) + \binom{n}{\imath}\alpha x(1-x) \right\} x^{\imath-1} \\ &\times (1-x)^{n-\imath-1} \\ =&(1-\alpha)(1-x)\sum_{\imath=0}^{n-2}\binom{n-2}{\imath}x^{\imath}(1-x)^{n-\imath-2} + (1-\alpha)x\sum_{\imath=2}^{n}\binom{n-2}{\imath-2}x^{\imath-2} \\ &\times (1-x)^{n-\imath} + \alpha\sum_{\imath=0}^{n}\binom{n}{\imath}x^{\imath}(1-x)^{n-\imath} \\ =&(1-\alpha)(1-x) + (1-\alpha)x + \alpha = 1.\end{aligned}$$

Property (5):

With the help of the following property of binomial coefficients:

$$\binom{n}{\imath} = \binom{n}{n-\imath},$$

(7.1) becomes

$$\begin{aligned}p_{n,\imath}^{\alpha}(x) =& \left\{ \binom{n-2}{n-\imath-2}(1-\alpha)x + \binom{n-2}{n-\imath}(1-\alpha)(1-x) + \binom{n}{n-\imath}\alpha x(1-x) \right\} \\ &\times x^{\imath-1}(1-x)^{n-\imath-1} = p_{n,n-\imath}^{\alpha}(1-x).\end{aligned}$$

Hence, the proof is completed. □

From the graphical representation of $p_{n,\imath}^{\alpha}(x)$ at $\alpha = 0.1$ and $n = 4$ in Fig. 7.1, we can observe that all the basis polynomials are positive for $x \in [0,1]$ as well as they follow the end point interpolation property.

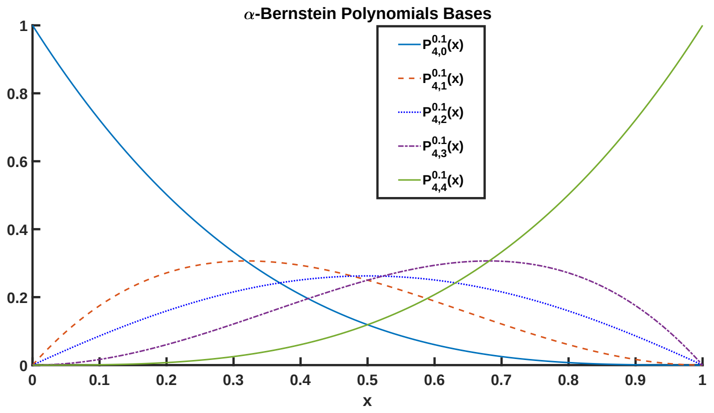


Figure 7.1: α-Bernstein Basis polynomials for $n = 4$ and $\alpha = 0.1$

Also these basis polynomials satisfy the reducibility for $\alpha = 1$, which can be observed from Fig. 7.2.

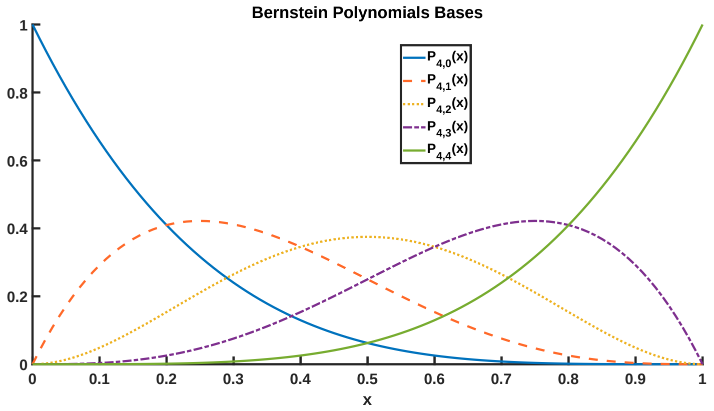


Figure 7.2: α-Bernstein Basis polynomials for $n = 4$ and $\alpha = 1$

Theorem 7.2.2. *Each nth degree α-Bernstein basis polynomial can be written as sum of two $(n-1)$th degree α-Bernstein basis polynomials, i.e.*

$$p^{\alpha}_{n,\imath}(x) = (1-x)p^{\alpha}_{n-1,\imath}(x) + xp^{\alpha}_{n-1,\imath-1}(x).$$

Proof. We know the Pascal's rule of binomial coefficients, which is as follows:

$$\binom{n}{\imath} = \binom{n-1}{\imath} + \binom{n-1}{\imath-1}.$$

By using this relation and (7.1), we obtain

$$\begin{aligned}
p_{n,\imath}^{\alpha}(x) =& \left\{ \left[\binom{n-3}{\imath} + \binom{n-3}{\imath-1} \right] (1-\alpha)x + \left[\binom{n-3}{\imath-2} + \binom{n-3}{\imath-3} \right] (1-\alpha)(1-x) \right. \\
& \left. + \left[\binom{n-1}{\imath} + \binom{n-1}{\imath-1} \right] \alpha x(1-x) \right\} x^{\imath-1}(1-x)^{n-\imath-1} \\
=& (1-x) \left\{ \binom{n-3}{\imath}(1-\alpha)x + \binom{n-3}{\imath-2}(1-\alpha)(1-x) + \binom{n-1}{\imath}\alpha x(1-x) \right\} \\
& \times x^{\imath-1}(1-x)^{n-\imath-2} \\
& + x \left\{ \binom{n-3}{\imath-1}(1-\alpha)x + \binom{n-3}{\imath-3}(1-\alpha)(1-x) + \binom{n-1}{\imath-1}\alpha x(1-x) \right\} \\
& \times x^{\imath-2}(1-x)^{n-\imath-1} \\
=& (1-x)\, p_{n-1,\imath}^{\alpha}(x) + x\, p_{n-1,\imath-1}^{\alpha}(x).
\end{aligned}$$

Thus, we get desired result. □

7.3 Construction of α-Bézier Curves

7.3.1 Definition

In this subsection, we define the generalization of Bézier curves by using α-Bernstein basis polynomials, in the following form:

$$P(x,\alpha) = \sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x)\, P_\imath, \tag{7.2}$$

where $\alpha \in [0,1]$ and $P_\imath, \imath = 0, 1, ..., n$ are the points on planes, known as control points.

7.3.2 Properties of α-Bézier curves

In the following result, we provide some properties of α-Bézier curves.

Theorem 7.3.1. *By using Definition (7.2), we obtain the following properties:*

1. *α-Bézier curves lie inside the convex hull of control polygon determined by the points $P_\imath$.*
2. *End Point Interpolation: $P(0,\alpha) = P_0$ and $P(1,\alpha) = P_n$.*
3. *Reducibility: By choosing $\alpha = 1$, we get the classical Bézier curve.*
4. *Symmetry: If $P = \{P_0, P_1, ..., P_n\}$ and $P^* = \{P_0^*, P_1^*, ..., P_n^*\} = \{P_n, P_{n-1}, ..., P_0\}$, then we have the relation*

   $$P^*(x,\alpha) = P(1-x,\alpha),$$

   where $P^*(x,\alpha)$ are the α-Bézier curves determined by P^*. From this property, we observe that if the order of control points is reversed, then we get the same α-Bézier curves.

Proof. The properties (1)-(3) are easy to prove by using the properties in Theorem 7.2.1 of α-Bernstein basis polynomials. Therefore, we omit the details.
Now, to prove property (4), we have

$$\begin{aligned} P^*(x,\alpha) =& \sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x)\, P_\imath^* = \sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x)\, P_{n-\imath}. \\ =& \sum_{\imath=0}^{n} p_{n,n-\imath}^{\alpha}(x)\, P_\imath. \end{aligned}$$

Now, by using the property (5) of Theorem 7.2.1, we get

$$P^*(x,\alpha) = P(1-x,\alpha).$$

$\square$

Theorem 7.3.2. *The α-Bézier curves have the end point derivative property as*

$$P'(0,\alpha) =(1-\alpha)[P_2 + (n-2)P_1 - (n-1)P_0] + \alpha\, n(P_1 - P_0),$$

$$P'(1,\alpha) = (1-\alpha)[(n-1)P_n - (n-2)P_{n-1} - P_{n-2}] + \alpha\, n(P_n - P_{n-1}).$$

Proof. From (7.2) and using [50] (Theorem 4.1), we have

$$\begin{aligned} P'(x,\alpha) =& (1-\alpha)\sum_{\imath=0}^{n-2}\Big[(\imath+1)\Delta P_{\imath+1} + (n-\imath-1)\Delta P_{\imath}\Big]\binom{n-2}{\imath}x^{\imath}(1-x)^{n-\imath-2} \\ &+ \alpha\, n\sum_{\imath=0}^{n-1}\Delta P_{\imath}\binom{n-1}{\imath}x^{\imath}(1-x)^{n-\imath-1}. \end{aligned}$$

In order to prove our results, we assign $x = 0$ for the first part and $x = 1$ for the second part. □

Remark 7.3.1. *The above property given in Theorem 7.3.2 represents that tangent at the point $x = 0$ is resultant of two vectors from P_0 to P_1 and P_1 to P_2. Similarly, the tangent at the end point $x = 1$ is also resultant of two vectors from P_{n-1} to P_n and P_{n-2} to P_{n-1}.*

7.4 Tensor Product α-Bézier Surfaces on $[0,1]\times[0,1]$

Let $I^2 = [0,1]\times[0,1]$. We define two parameter family $P^{(\alpha_1,\alpha_2)}(x,y)$ of tensor product Bézier surfaces of degree $n_1 \times n_2$ as follows:

$$P^{(\alpha_1,\alpha_2)}(x,y) = \sum_{\imath=0}^{n_1}\sum_{\jmath=0}^{n_2} p^{\alpha_1}_{n_1,\imath}(x)\; p^{\alpha_2}_{n_2,\jmath}(y)\; P_{\imath,\jmath}, \quad (x,y)\in I^2$$

where $P_{\imath,\jmath}\in\mathbb{R}^3,\ \imath = 0,1,\cdots,n_1,\ \jmath = 0,1,\cdots,n_2$ are the control points and $p^{\alpha_1}_{n_1,\imath}(x)$, $p^{\alpha_2}_{n_2,\jmath}(y)$ are defined as (7.1). The net obtained by joining the adjacent control points in row or column, is called the control net of tensor product α-Bézier surfaces.
In the following subsection, we present the properties of α-Bézier surfaces.

7.4.1 Properties of α-Bézier surfaces

1. **Corner point interpolation property:** α-Bézier surfaces pass through all four corner control points of the control net, i.e. $P^{(\alpha_1,\alpha_2)}(0,0), P^{(\alpha_1,\alpha_2)}(0,1)$,

$P^{(\alpha_1,\alpha_2)}(1,0)$ and $P^{(\alpha_1,\alpha_2)}(1,1)$ denoted by $P_{0,0}, P_{0,n_2}, P_{n_1,0}$ and P_{n_1,n_2} resp.

2. **Reducibility:** For $\alpha_1 = \alpha_2 = 1$, we get the classical tensor product Bézier surfaces.

3. **Convex hull property:** Since $P^{(\alpha_1,\alpha_2)}(x,y)$ is the convex combination of $P_{\imath,\jmath}$, then it lies in the convex hull of its control net.

4. **Isoparametric curves property:** The isoparametric curves for $y = y_1$ of a tensor product α-Bézier surfaces are α-Bézier curves in the direction of x of degree n_1 represented as

$$\begin{aligned} P^{(\alpha_1,\alpha_2)}(x,y_1) &= \sum_{\imath=0}^{n_1} \left(\sum_{\jmath=0}^{n_2} p_{n_2,\jmath}^{\alpha_2}(y_1)\ P_{\imath,\jmath} \right)\ p_{n_1,\imath}^{\alpha_1}(x), \quad x \in [0,1] \\ &= \sum_{\imath=0}^{n_1} p_{n_1,\imath}^{\alpha_1}(x)\ q_\imath(y_1), \end{aligned}$$

where $q_\imath(y_1) = \sum_{\jmath=0}^{n_2} p_{n_2,\jmath}^{\alpha_2}(y_1)\ P_{\imath,\jmath}$. Thus $P^{(\alpha_1,\alpha_2)}(x,y_1)$ are α-Bézier curves having control points as $q_0(y_1), q_1(y_1), \cdots, q_{n_1}(y_1)$. In the similar way, we can define the isoparametric curves for $x = x_1$ of a tensor product α-Bézier surfaces, which are α-Bézier curves in the direction of y of degree n_2.
The four special isoparametric curves, known as boundary curves are evaluated at $P^{(\alpha_1,\alpha_2)}(0,y)$,$P^{(\alpha_1,\alpha_2)}(1,y)$, $P^{(\alpha_1,\alpha_2)}(x,0)$ and $P^{(\alpha_1,\alpha_2)}(x,1)$.

5. **Partition of unity:** The sum of $p_{n_1,\imath}^{\alpha_1}(x) *\ p_{n_2,\jmath}^{\alpha_2}(y)$ is one

$$i.e. \quad \sum_{\imath=0}^{n_1} \sum_{\jmath=0}^{n_2} p_{n_1,\imath}^{\alpha_1}(x)\ p_{n_2,\jmath}^{\alpha_2}(y) = 1.$$

7.5 Calculating Control Points

The process of defining Bézier curves with given control points is indeed straightforward and well-established. However, the reverse task, which involves finding the control points from given Bézier curves can be more challenging. In a paper referenced as [39], the authors presented an algorithm to determine the control points of cubic Bézier curves with minimal error. Later, in 2019, Baydas and Karakas [35] proposed a direct method to precisely find the control points of any given Bézier curve. Unlike the previous algorithm that focused on approximation, this new procedure aimed to identify the exact control points that generate the original Bézier curve. The direct method seems to be a significant advancement in the field of Bézier curves manipulation, allowing for precise control and reproduction of the original curves.

Inspired by the direct approach introduced in [35], we suggest a similar procedure to find the control points for our α-Bézier curves. This new method is likely to yield more accurate control points, providing better control over the curve's shape and behavior. Thus, we aim to achieve similar benefits for this particular class of curves. In order to find the control points from given α-Bézier curves, we need to write the basis polynomial of the curve in form of power basis $\{x^\imath\}$. Therefore, the given α-Bézier curves can be rewritten as

$$\begin{aligned}\sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x)\;P_\imath =&\sum_{\imath=0}^{n}\left\{\binom{n-2}{\imath}(1-\alpha)\,x+\binom{n-2}{\imath-2}(1-\alpha)\,(1-x)\,.+\binom{n}{\imath}\alpha\,x\,(1-x)\right\}\\ &\times x^{\imath-1}\,(1-x)^{n-\imath-1}\,P_\imath\\ =&\sum_{\imath=0}^{n}\binom{n-2}{\imath}(1-\alpha)\,x^\imath P_\imath\sum_{\imath=0}^{n-\imath-1}\binom{n-\imath-1}{\imath}\,(-x)^\imath\\ &+\sum_{\imath=0}^{n}\binom{n-2}{\imath-2}(1-\alpha)\,x^{\imath-1}P_\imath\sum_{\imath=0}^{n-\imath}\binom{n-\imath}{\imath}\,(-x)^\imath\\ &+\sum_{\imath=0}^{n}\binom{n}{\imath}\alpha\,x^\imath P_\imath\sum_{\imath=0}^{n-\imath}\binom{n-\imath}{\imath}\,(-x)^\imath\,.\end{aligned}$$

By calculating the coefficients of like powers of x, we get

$$\begin{aligned}\sum_{\imath=0}^{n} p_{n,\imath}^{\alpha}(x)\,P_\imath =&x^n\alpha\left\{\binom{n}{0}\binom{n}{n}(-1)^nP_0+\binom{n}{1}\binom{n-1}{n-1}(-1)^{n-1}P_1+\cdots\right.\\ &\left.+\binom{n}{n}\binom{n-n}{n-n}(-1)^{n-n}P_n\right\}+x^{n-1}\left\{(1-\alpha)\left(\binom{n-2}{0}\binom{n-1}{n-1}(-1)^{n-1}P_0\right.\right.\\ &\left.+\binom{n-2}{1}\binom{n-2}{n-2}(-1)^{n-2}P_1+\cdots\;+\binom{n-2}{n-2}\binom{n-(n-1)}{n-(n-1)}(-1)^{n-(n-1)}P_{n-2}\right)\end{aligned}$$

$$\begin{aligned}
&+(1-\alpha)\Bigg(\binom{n-2}{0}\binom{n-2}{n-2}(-1)^{n-2}P_2+\binom{n-2}{1}\binom{n-3}{n-3}(-1)^{n-3}P_3\\
&+\cdots+\binom{n-2}{n-2}\binom{n-n}{n-n}(-1)^{n-n}P_n\Bigg)+\alpha\Bigg(\binom{n}{0}\binom{n}{n-1}(-1)^{n-1}P_0\\
&+\binom{n}{1}\binom{n-1}{n-2}(-1)^{n-2}P_1+\cdots+\binom{n}{n-1}\binom{n-(n-1)}{n-n}(-1)^{n-n}P_{n-1}\Bigg)\Bigg\}\\
&\cdots\\
&+x^2\Bigg\{\Bigg((1-\alpha)\binom{n-2}{0}\binom{n-1}{2}(-1)^2P_0+\binom{n-2}{1}\binom{n-2}{1}(-1)^1P_1\\
&+\binom{n-2}{2}\binom{n-3}{0}(-1)^0P_2\Bigg)+(1-\alpha)\Bigg(\binom{n-2}{0}\binom{n-2}{1}(-1)P_2\\
&+\binom{n-2}{1}\binom{n-3}{0}(-1)^0P_3\Bigg)+\alpha\Bigg(\binom{n}{0}\binom{n}{2}(-1)^2P_0\\
&+\binom{n}{1}\binom{n-1}{1}(-1)P_1+\binom{n}{2}\binom{n-2}{0}P_2\Bigg)\Bigg\}\\
&+x\Bigg\{(1-\alpha)\Bigg(\binom{n-2}{0}\binom{n-1}{1}(-1)P_0+\binom{n-2}{1}\binom{n-2}{0}P_1\Bigg)\\
&+(1-\alpha)\binom{n-2}{0}\binom{n-2}{0}(-1)^0P_2+\alpha\Bigg(\binom{n}{0}\binom{n}{1}(-1)P_0\\
&+\binom{n}{1}\binom{n-1}{0}(-1)^0P_1\Bigg)\Bigg\}+x^0\Bigg\{(1-\alpha)\binom{n-2}{0}\binom{n-1}{0}(-1)^0P_0\\
&+\alpha\binom{n}{0}\binom{n}{0}(-1)^0P_0\Bigg\}.
\end{aligned}$$

The given α-Bézier curves can be written in the following matrix form:

$$P\left(x,\alpha\right)=Y.A.P,$$

where $Y=[x^{\imath}], \imath=0,1,\cdots,n;$

$$\begin{aligned}
A=[a_{\imath\jmath}]=\Bigg[&\binom{n-2}{\jmath-1}\binom{n-\jmath}{\imath-\jmath}(-1)^{\imath-\jmath}(1-\alpha)+\binom{n-2}{\jmath-3}\binom{n-\jmath+1}{\imath-\jmath+1}(-1)^{\imath-\jmath+1}(1-\alpha)\\
&+\binom{n}{\jmath-1}\binom{n-\jmath+1}{\imath-\jmath}(-1)^{\imath-\jmath}\alpha\Bigg],\qquad \imath,\jmath=1,2,\cdots,(n+1),
\end{aligned}$$

$P^t = [P_\imath]$, $\imath = 0, 1, \cdots, n$, and P^t denotes the transpose of vector P. By comparing the coefficients of like powers of x on both sides, the system reduces to

$$C = A\,P,$$

where C is the coefficient matrix of the given α-Bézier curve. Now, in order to compute P, which is the vector of control points of the α-Bézier curve, we solve this system by $P = A^{-1}\,C$.

7.6 Numerical Examples

In the present section, we give some numerical examples of α-Bézier curves and surfaces using Matlab software. From these examples, we can see the influence of α on the classical Bézier curves and surfaces. Also, we find the control points for the given α-Bézier curves.

Example 7.6.1. *Let $P_0 = (2,5), P_1 = (5,10), P_2 = (8,11)$ and $P_3 = (11,6)$ are the control points. We draw the α-Bézier curves for $\alpha = 1, 0.5, 0.1$. For $\alpha = 1$, the α-Bézier curves reduce to classical Bézier curve. From the Fig. 7.3, we observe that as the value of α moves towards 0, we get flatten curve, which is smooth like classical Bézier curve but more flexible than it.*

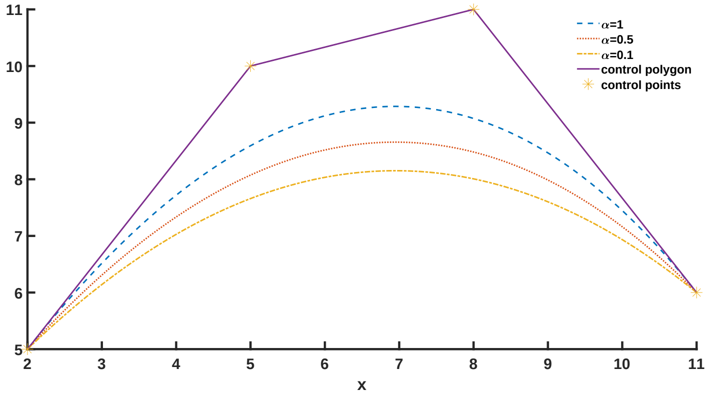


Figure 7.3: Shape modification for $\alpha = 1, 0.5, 0.1$

Example 7.6.2. *In this example, we show the α-Bézier curves for certain values of α, i.e. $1, 0.5, 0.1$. From the Fig. 7.4, we can see the bending of curve for the different values of the parameter α.*

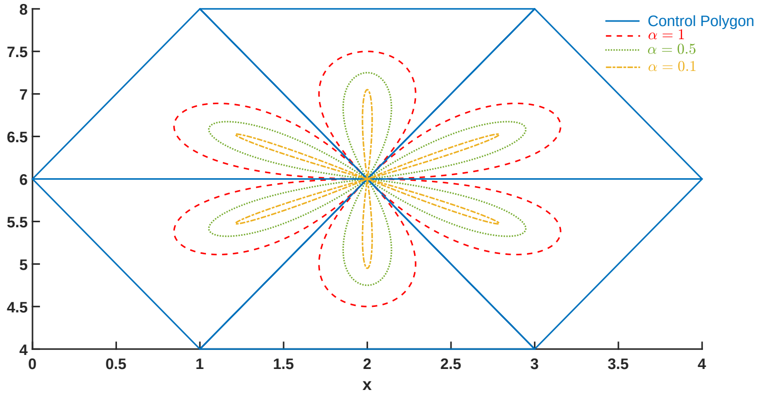


Figure 7.4: Shape modification for $\alpha = 1, 0.5, 0.1$

Example 7.6.3. *Here, we present two α-Bézier patches to show the impact of the parameters on the surfaces in Fig. 7.5. We have chosen two different pairs of parameters as $\alpha_1 = 0.5$, $\alpha_2 = 0.5$ and $\alpha_1 = 1$, $\alpha_2 = 1$.*

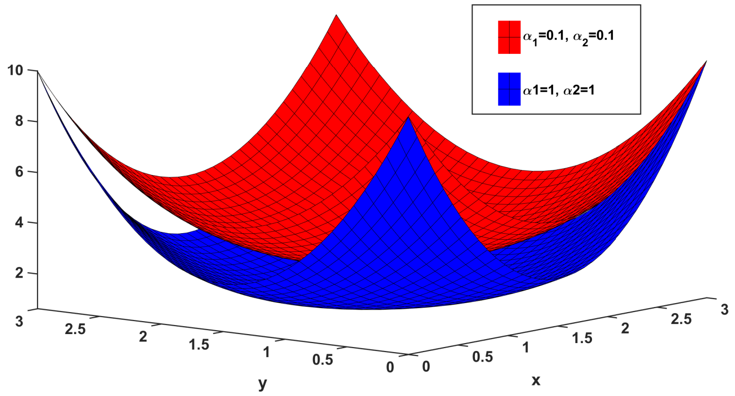
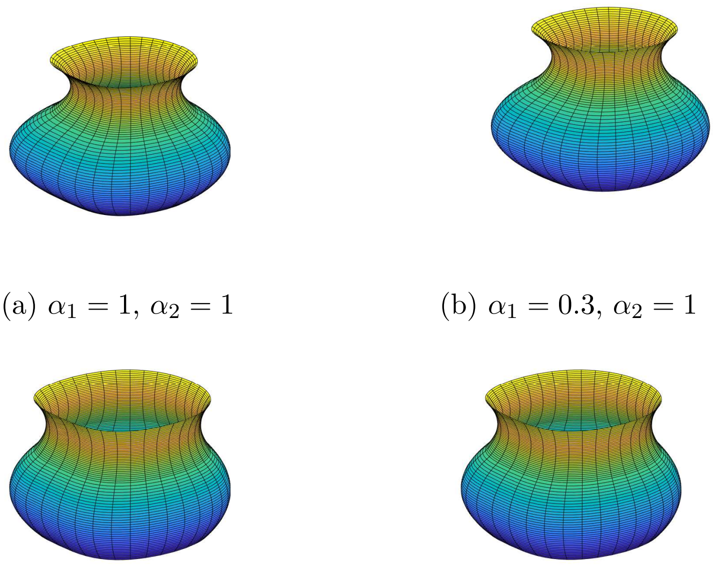


Figure 7.5: α-Bézier patches with $\alpha_1 = \alpha_2 = 0.1, \alpha_1 = \alpha_2 = 1$

Example 7.6.4. *In this example, we can easily see from Fig. 7.6 that by changing the values of parameters, we get different types of surfaces. So, α-Bézier surfaces are helpful to modify the shape of surfaces with the same control points.*



(a) $\alpha_1 = 1$, $\alpha_2 = 1$

(b) $\alpha_1 = 0.3$, $\alpha_2 = 1$

(c) $\alpha_1 = 1$, $\alpha_2 = 0.5$

(d) $\alpha_1 = 0.5$, $\alpha_2 = 0.5$

Figure 7.6: α-Bézier surfaces for different values of α_1, α_2

Example 7.6.5. *In the present example, we have represented two surfaces in a single Fig. 7.7 to show the comparison of both surfaces that are generated by parameters.*

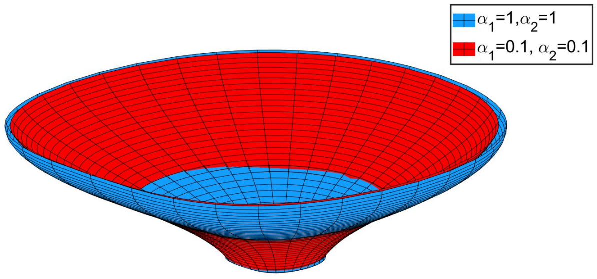


Figure 7.7: Comparison of α-Bézier surfaces for $\alpha_1 = \alpha_2 = 1$ and $\alpha_1 = \alpha_2 = 0.1$

In the next examples, we find the exact control points for the given α-Bézier curve by using our proposed method.

Example 7.6.6. *Consider the α-Bézier curves* $P(x, \alpha) = (P_1(x, \alpha), P_2(x, \alpha))$, *where* $P_1(x,\alpha) = -\dfrac{54x^5}{5} + 27x^4 - \dfrac{48x^3}{5} - \dfrac{63x^2}{5} + 15x + 2$ *and* $P_2(x,\alpha) = \dfrac{18x^5}{5} - 9x^4 + \dfrac{174x^3}{5} - \dfrac{216x^2}{5} + \dfrac{54x}{5} + 6$*. This curve can be seen in Fig. 7.8 with the parameter* $\alpha = 0.3$*. The matrix* A *for this curve is as follows:*

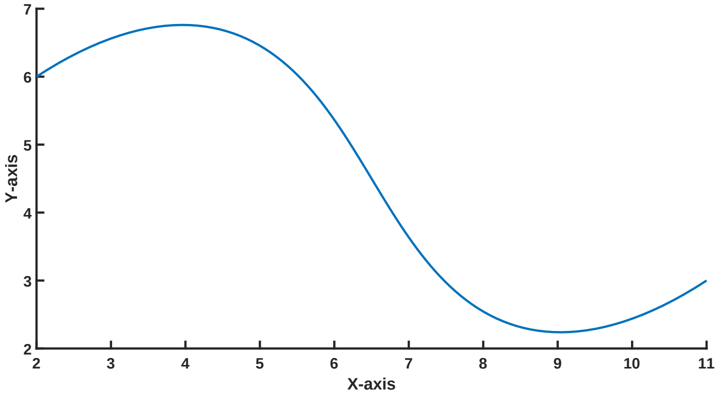


Figure 7.8: α-Bézier curve

$$A = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ -4.3000 & 3.6000 & 0.7000 & 0 & 0 & 0 \\ 7.2000 & -12.3000 & 3.0000 & 2.1000 & 0 & 0 \\ -5.8000 & 15.3000 & -11.1000 & -0.5000 & 2.1000 & 0 \\ 2.2000 & -8.1000 & 10.4000 & -4.6000 & -0.6000 & 0.7000 \\ -0.3000 & 1.5000 & -3.0000 & 3.0000 & -1.5000 & 0.3000 \end{bmatrix}.$$

We can find the inverse of this matrix A *as follows:*

$$A^{-1} = \begin{bmatrix} 1.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 1.0000 & 0.2000 & -0.0151 & 0.0040 & -0.0029 & 0.0068 \\ 1.0000 & 0.4000 & 0.0774 & -0.0206 & 0.0149 & -0.0348 \\ 1.0000 & 0.6000 & 0.2774 & 0.0528 & -0.0383 & 0.0893 \\ 1.0000 & 0.8000 & 0.5849 & 0.3508 & 0.0908 & -0.2118 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}.$$

Coefficient matrix of the curve is

$$C = \begin{bmatrix} -\frac{54}{5} & 27 & -\frac{48}{5} & -\frac{63}{5} & 15 & 2 \\ \frac{18}{5} & -9 & \frac{174}{5} & -\frac{216}{5} & \frac{54}{5} & 6 \end{bmatrix}^t.$$

Now, by calculating $A^{-1} * C$, we get the control points of the curve which are

$$Q = \begin{bmatrix} 2 & 5 & 8 & 5 & 8 & 11 \\ 6 & 9 & 6 & 3 & 0 & 3 \end{bmatrix}^t.$$

Fig. 7.9 represents the control polygon of calculated control points of the curve by using our technique, whereas Fig. 7.10 represents the control polygon of the points and the curve generated by those points which justify our claim that we get exact control points for the given curve.

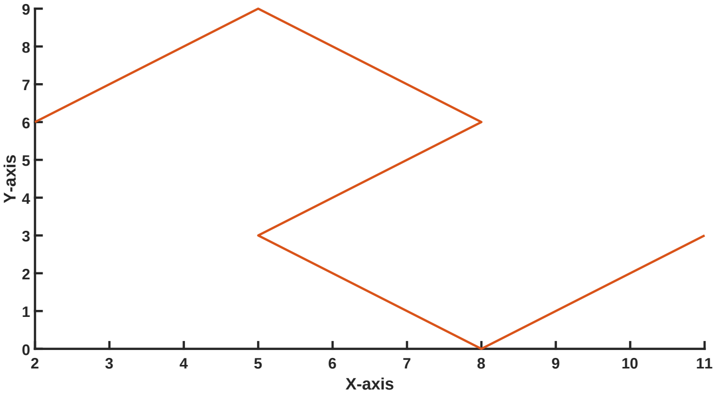


Figure 7.9: Control polygon of the α-Bézier curve

Example 7.6.7. *Consider the curve $P(x,\alpha) = (P_1(x,\alpha), P_2(x,\alpha), P_3(x,\alpha))$, where*

$$P_1(x,\alpha) = -\frac{955x^8}{2} + \frac{5095x^7}{2} - \frac{9695x^6}{2} + \frac{8005x^5}{2} - 1100x^4 - 300x^3 + 275x^2 - 115x + 25,$$

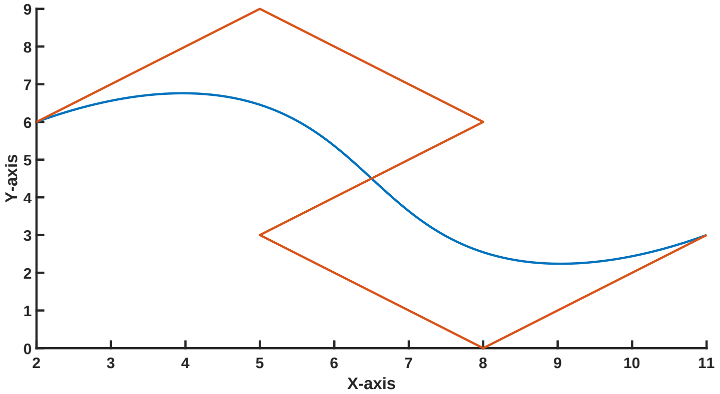


Figure 7.10: Control polygon and α-Bézier curve

$$
\begin{aligned}
P_2(x,\alpha) = &\frac{1525x^8}{2} - \frac{6625x^7}{2} + 6370x^6 - 7585x^5 + \frac{12175x^4}{2} - \frac{5925x^3}{2} + \frac{1475x^2}{2} - \frac{205x}{2} \\
&+ 25, \\
P_3(x,\alpha) = &\frac{431x^8}{2} - \frac{939x^7}{2} - x^6 + 1070x^5 - \frac{2725x^4}{2} + \frac{1745x^3}{2} - 490x^2 + 190x + 5,
\end{aligned}
$$

where $\alpha = 0.5$.

The matrix A for this curve is calculated below:

$$
A = \begin{bmatrix}
1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7.500 & 7.000 & 0.500 & 0 & 0 & 0 & 0 & 0 & 0 \\
24.500 & -46.000 & 18.500 & 3.000 & 0 & 0 & 0 & 0 & 0 \\
-45.500 & 129.000 & -114.000 & 23.000 & 7.500 & 0 & 0 & 0 & 0 \\
52.500 & -200.000 & 275.000 & -150.000 & 12.500 & 10.000 & 0 & 0 & 0 \\
-38.500 & 185.000 & -347.500 & 310.000 & -117.500 & 1.000 & 7.500 & 0 & 0 \\
17.500 & -102.000 & 244.500 & -305.000 & 202.500 & -60.000 & -0.500 & 3.000 & 0 \\
-4.500 & 31.000 & -91.000 & 147.000 & -140.000 & 77.000 & -21.000 & 1.000 & 0.500 \\
0.500 & -4.000 & 14.000 & -28.000 & 35.000 & -28.000 & 14.000 & -4.000 & 0.500
\end{bmatrix}.
$$

We can find the inverse of this matrix A as follows:

$$A^{-1} = \begin{bmatrix} 1.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 1.0000 & 0.1250 & -0.0023 & 0.0001 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 1.0000 & 0.2500 & 0.0318 & -0.0018 & 0.0002 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 1.0000 & 0.3750 & 0.1023 & 0.0131 & -0.0016 & 0.0003 & -0.0001 & 0.0001 & -0.0001 \\ 1.0000 & 0.5000 & 0.2091 & 0.0636 & 0.0084 & -0.0018 & 0.0007 & -0.0004 & 0.0004 \\ 1.0000 & 0.6250 & 0.3523 & 0.1687 & 0.0598 & 0.0087 & -0.0032 & 0.0019 & -0.0019 \\ 1.0000 & 0.7500 & 0.5318 & 0.3473 & 0.1983 & 0.0873 & 0.0168 & -0.0101 & 0.0101 \\ 1.0000 & 0.8750 & 0.7477 & 0.6181 & 0.4858 & 0.3509 & 0.2131 & 0.0722 & -0.0722 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}.$$

Coefficient matrix of the curve is

$$C = \begin{bmatrix} -\frac{955}{2} & \frac{5095}{2} & -\frac{9695}{2} & \frac{8005}{2} & -1100 & -300 & 275 & -115 & 25 \\ \frac{1525}{2} & -\frac{6625}{2} & 6370 & -7585 & \frac{12175}{2} & -\frac{5925}{2} & \frac{1475}{2} & -\frac{205}{2} & 25 \\ \frac{431}{2} & -\frac{939}{2} & -1 & 1070 & -\frac{2725}{2} & \frac{1745}{2} & -490 & 190 & 5 \end{bmatrix}.$$

Now, by calculating $A^{-1} * C$, we can find the control points of the given curve with specific values of parameters as

$$Q = \begin{bmatrix} 25 & 10 & 5 & 10 & -15 & -10 & 0 & 0 & 10 \\ 25 & 10 & 30 & 10 & 10 & -10 & 5 & 15 & 20 \\ 5 & 30 & 35 & 40 & 40 & 25 & 20 & 8 & 30 \end{bmatrix}^t.$$

Following the example 7.6.6 in Fig. 7.11, we present the given curve and in Fig. 7.12, we draw the control polygon of the calculated exact points by the proposed scheme. Fig. 7.13 represents the control polygon and the given curve simultaneously to justify our claim.

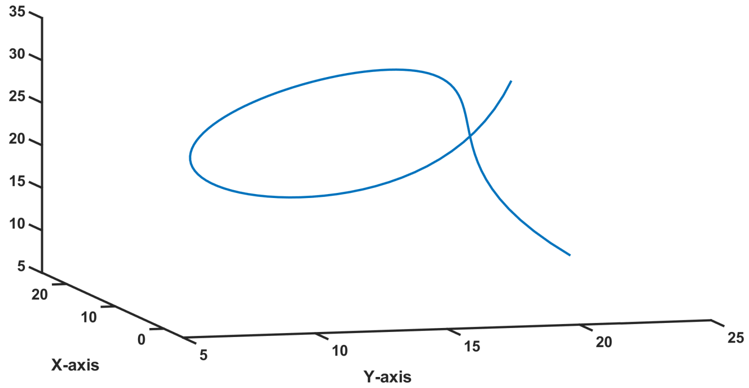


Figure 7.11: α-Bézier curve in three dimension

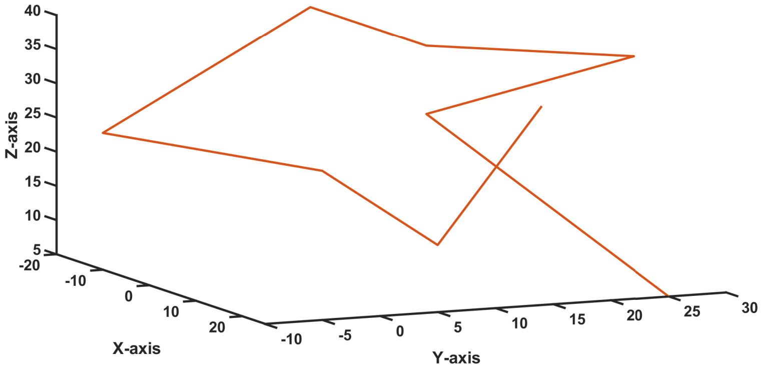


Figure 7.12: Control polygon of the α-Bézier curve

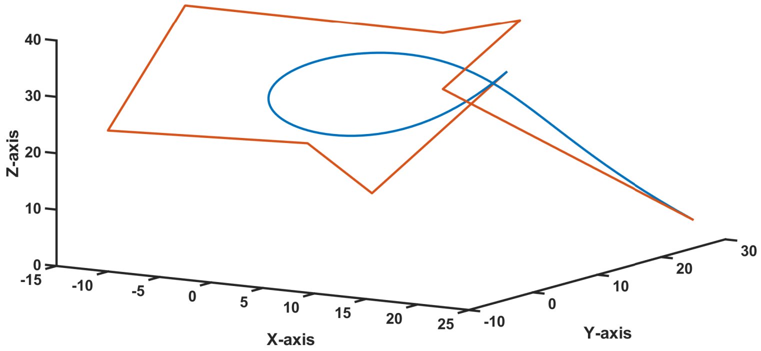


Figure 7.13: Control polygon and α-Bézier curve

7.7 Conclusion

With the help of the numerical examples, we can demonstrate that the parameter α in α-Bézier curves allows us to adjust the shape of a classical Bézier curve. We may modify the shape of the curve to any desired level by changing the value of α. As a result, drawing homogeneous curves with the same attributes become simple. Although we can determine these forms with the help of classical Bézier curves, we must adjust the control points for each curve every time, that would be both time-consuming and expensive. As a result, parametric Bézier curves help us save money and time. We can observe about α-Bézier surfaces in a similar way. We can find numerous surfaces that are identical to each other by varying values of the parameters. The chapter also outlines a systematic procedure for determining control points, allowing precise curve manipulation.

The parametric generalizations of Bézier curves helps us to find the more flexible Bézier curves. We used this idea to introduce an another form of Bézier curves by using the parameter q in the next chapter.

Chapter 8

q-Bézier curves with shifted nodes

8.1 Introduction

In [76], Han *et al.* modified the Bézier curves by incorporating Lupaş q-analogue in order to regulate the curves' shape. When $q = 1$, the generalization transforms into traditional Bézier curves. Similarly, Khan *et al.* [92] generalized Bézier curves based on shifted nodes using two real and non-negative parameters. The authors showed that the curves generated over any sub-interval of $[0, 1]$ with the shifted nodes are similar to classical Bézier curves.

Khatri and Mishra [93] defined the generalized Bézier curves by choosing Bernstein-Stancu Chlodowsky basis polynomials defined by Aral *et al.* [26] by using a positive increasing sequence $(d_n)_{n\geq 1}$ such that $\lim\limits_{n\to\infty} d_n = \infty$, $\lim\limits_{n\to\infty} \dfrac{d_n}{n} = 0$ in the following form:

$$H^{a,b}_{n,k}(t) = \binom{n}{k} \left(\frac{n+b}{n}\right)^{n} \left(\frac{t}{d_n} - \frac{a}{n+b}\right)^{k} \left(\frac{n+a}{n+b} - \frac{t}{d_n}\right)^{n-k}, \tag{8.1}$$

where $\dfrac{a}{n+b} d_n \leq t \leq \dfrac{n+a}{n+b} d_n$ and a, b are the real numbers such that $0 \leq a \leq b$. The authors studied the properties of basis polynomials and the curves induced by these polynomials. They also defined De Casteljau algorithm and degree elevation for these curves. Motivated by the advantages of parametric generalizations, we define the q-Bézier curves of basis polynomials (8.1).

The present chapter is structured as: In the next section 8.2, we present fundamental results of q-calculus, which serves as the foundation for the subsequent sections. Section 8.3 focuses on the properties and characteristics of q-Bernstein Chlodowsky basis polynomials, exploring their applications and associated results. In section 8.4, we study a generalized approach to Bézier curves utilizing q-Bernstein Chlodowsky basis polynomials and their properties to preserve specific shapes. Section 8.5 discusses the degree elevation of these Bézier curves and in section 8.6, we give De Casteljau algorithm tailored for these curves. These algorithms are essential tools for manipulating and optimizing these curves. In the last section, we present some examples to show the flexibility in the shape of the curve with the choice of parameters.

8.2 Preliminaries

Quantum Calculus or q-calculus is an advanced mathematical framework that extends the conventional notions of differentiation and integration by the incorporation of a parameter q. This generalization provides a significant versatility in mathematical modeling, allowing for the examination and resolution of intricate problems for which conventional approaches may not be capable. It is extremely helpful in the field of quantum mechanics, as it offers profound understanding of the dynamics of discrete and continuous systems. Its broad applicability contains several disciplines including Combinatorics, Number Theory, and special functions, making it a versatile instrument for solving many mathematical and scientific problems. The q-analogs of classical polynomials and functions in numerical analysis improves approximation and interpolation techniques, resulting in more precise solutions. q-calculus provides crucial tools for the development of algorithms and comprehension of quantum systems, so enhancing the theoretical and practical foundations of quantum computing and information science.

Let us recall the basic definitions of q-analogs [80]. For a given real number $q > 0$ and any $m \in \mathbb{N}$, we have

$$[m]_q = \begin{cases} \dfrac{1-q^m}{1-q}, & q \neq 1 \\ m, & q = 1 \end{cases}. \tag{8.2}$$

Let $\mathbb{N}_q = \{[m]_q : m \in \mathbb{N}\}$ be the generalization of the set of non-negative integers $\mathbb{N}_0$. For $q = 1$, $\mathbb{N}_q$ reduces to $\mathbb{N}_0$.

For given $m \in \mathbb{N}_0$, $[m]_q!$ is defined as

$$[m]_q! = \begin{cases} [m]_q\,[m-1]_q \cdots [1]_q, & m \geq 1 \\ 1, & m = 0 \end{cases}. \tag{8.3}$$

The q-Binomial coefficient $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q$ is defined as

$$\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q = \frac{[m]_q!}{[j]_q!\,[m-j]_q!}, \tag{8.4}$$

for $m \geq j \geq 1$ and 0 otherwise.

The Pascal type relation between q-Binomial coefficients is given by

$$\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} m-1 \\ j-1 \end{smallmatrix}\right]_q + q^j \left[\begin{smallmatrix} m-1 \\ j \end{smallmatrix}\right]_q, \tag{8.5}$$

$$\text{or} \tag{8.6}$$

$$\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q = q^{m-j} \left[\begin{smallmatrix} m-1 \\ j-1 \end{smallmatrix}\right]_q + \left[\begin{smallmatrix} m-1 \\ j \end{smallmatrix}\right]_q. \tag{8.7}$$

The product in q-analog is defined in the following way:

$$(1+\mathsf{v})_q^m = \prod_{j=0}^{m-1} (1 + q^j \mathsf{v}) = \sum_{j=0}^{m} q^{\frac{j(j-1)}{2}} \left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_q \mathsf{v}^j. \tag{8.8}$$

8.3 Properties of q-Bernstein Chlodowsky Basis Polynomials

Firstly, we define the q-analogue of (8.1) as the basis polynomial for $n \in \mathbb{N}$ in the following form:

$$\begin{aligned} p_{n,k}^{q}(\mathsf{v}) &= \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n}\right)^{n-k} q^{\frac{k(k-1)}{2}}}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{j-1}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \\ &:= \frac{1}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{j-1}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \mu_{n,k}^{q}(\mathsf{v}), \end{aligned} \tag{8.9}$$

where $\mathsf{v} \in \left[\frac{\alpha d_n}{[n]_q+\beta}, \frac{[n]_q+\alpha}{[n]_q+\beta} d_n\right]$.

Theorem 8.3.1. *The basis polynomial* (8.9) *has the following properties:*

1. *Non-negativity:* $p_{n,k}^{q}(\mathsf{v}) \geq 0$ *for* $\mathsf{v} \in \left[\frac{\alpha d_n}{[n]_q+\beta}, \frac{[n]_q+\alpha}{[n]_q+\beta} d_n\right]$.
2. *Partition of unity:* $\sum\limits_{k=0}^{n} p_{n,k}^{q}(\mathsf{v}) = 1$.
3. *End point interpolation property*
$$p_{n,k}^{q}\left(\frac{\alpha d_n}{[n]_q+\beta}\right) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}, \qquad p_{n,k}^{q}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} d_n\right) = \begin{cases} 1, & k=n \\ 0, & k \neq n \end{cases}.$$
4. *Reducibility: when* $q=1, d_n=1$ *and* $\alpha=\beta=0$, *it reduces to classical Bernstein polynomials.*
5. *Symmetry:* $p_{n,n-k}^{q}(t) = p_{n,k}^{\frac{1}{q}}\left(\frac{[n]_q}{[n]_q+\beta} - t\right)$, *where* $t = \frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}$.

Proof. The proofs for the properties (1), (3), and (4) are quite simple. So, we left it for the reader.

For property (2), consider

$$\begin{aligned}\sum_{k=0}^{n}\mu_{n,k}^{q}(\mathsf{v}) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k} q^{\frac{k(k-1)}{2}} \\ &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}}\right)^k \left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n} q^{\frac{k(k-1)}{2}}. \end{aligned} \tag{8.10}$$

Using (8.8), it becomes

$$\begin{aligned}\sum_{k=0}^{n}\mu_{n,k}^{q}(\mathsf{v}) &= \left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^n \left(1+\frac{\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}}\right)\left(1+q\frac{\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}}\right)\cdots \\ &\quad\times\left(1+q^{n-1}\frac{\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}}{\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}}\right) \\ &= \left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)\cdots \\ &\quad\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{n-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right) \\ &= \prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right). \end{aligned} \tag{8.11}$$

Thus, by using (8.9) and (8.11), we get

$$\sum_{k=0}^{n}p_{n,k}^{q}(\mathsf{v}) = \sum_{k=0}^{n}\frac{\mu_{n,k}^{q}(\mathsf{v})}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)} = 1.$$

Now, for property (5), consider $t=\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}$.

$$p_{n,n-k}^{q}(t) = \frac{\mu_{n,n-k}^{q}(\mathsf{v})}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}$$

$$=\frac{\mu^{q}_{n,n-k}(\mathsf{v})}{\prod\limits_{j=1}^{n} q^{j-1}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)+\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}$$

$$=\frac{\begin{bmatrix} n \\ n-k \end{bmatrix}_q \left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^{n-k}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{k} q^{\frac{(n-k)(n-k-1)}{2}}}{q^{\frac{n(n-1)}{2}}\prod\limits_{j=1}^{n}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)+\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}$$

$$=\frac{\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} \left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^{n-k}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{k}}{q^{\frac{k(k-1)}{2}}\prod\limits_{j=1}^{n}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)+\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}$$

$$=\frac{\mu^{\frac{1}{q}}_{n,k}\left(\frac{[n]_q}{[n]_q+\beta}-t\right)}{\prod\limits_{j=1}^{n}\left(\frac{1}{q^{j-1}}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)+\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}=p^{\frac{1}{q}}_{n,k}\left(\frac{[n]_q}{[n]_q+\beta}-t\right).$$

Hence, the proof is completed.

Now, we present the effect of different values of the parameter q on the basis functions by choosing $n=5, \alpha=0.2, \beta=0.8, d_n=\sqrt{n}$. In Fig. 8.1 the value of $q=1$, whereas $q=2$ in Fig. 8.2. □

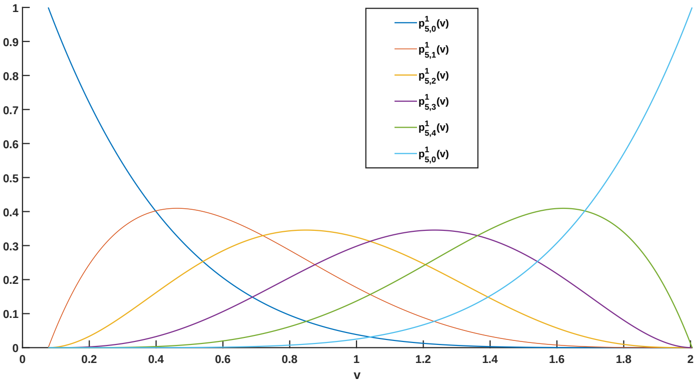


Figure 8.1: Basis polynomials for $n=5, \alpha=0.2, \beta=0.8, d_n=\sqrt{n}$ and $q=1$

Theorem 8.3.2. *For $n \geq 1$, each n^{th} degree basis polynomial can be written as*

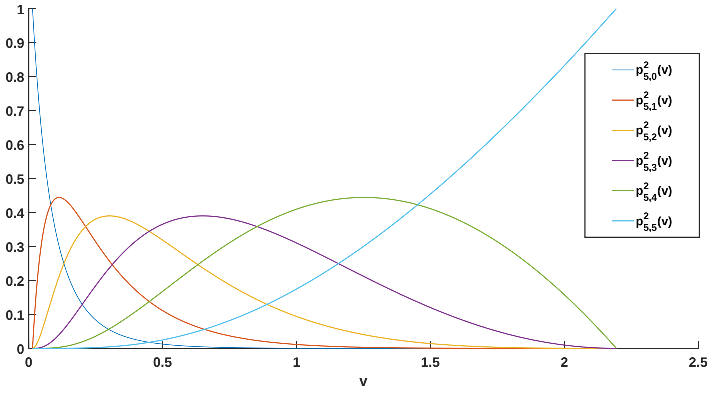


Figure 8.2: Basis polynomials for $n = 4, \alpha = 0.2, \beta = 0.8, d_n = \sqrt{n}$ and $q = 2$

combination of two $(n-1)^{th}$ *degree basis polynomials, i.e.*

$$\begin{aligned} p^q_{n,k}(\mathsf{v}) = & \frac{q^{k-1}}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{n-1}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right) p^q_{n-1,k-1}(\mathsf{v}) \\ & + \frac{q^{k}}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{n-1}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n}\right) p^q_{n-1,k}(\mathsf{v}) . \end{aligned}$$

Proof. Consider

$$\begin{aligned} p^q_{n,k}(\mathsf{v}) = & \frac{q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{j-1}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)^k \\ & \times \left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n}\right)^{n-k} . \end{aligned}$$

Using the recurrence equation (8.5) of q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q ,$$

we get

$$\begin{aligned}
p_{n,k}^{q}(\mathsf{v}) =& \frac{q^{\frac{k(k-1)}{2}}}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\left(\begin{bmatrix}n-1\\k-1\end{bmatrix}_q+q^k\begin{bmatrix}n-1\\k\end{bmatrix}_q\right)\\
&\times\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}\\
=& \frac{q^{\frac{k(k-1)}{2}}}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k\\
&\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}+\frac{q^{\frac{k(k-1)}{2}}q^k}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n-1\\k\end{bmatrix}_q\\
&\times\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}\\
=& \frac{q^{\frac{(k-1)(k-2)}{2}+k-1}}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k\\
&\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}+\frac{q^{\frac{k(k-1)}{2}+k}}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n-1\\k\end{bmatrix}_q\\
&\times\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}.
\end{aligned}$$

Hence, we reach at desired result. ☐

Theorem 8.3.3. *Each n^{th} degree basis polynomial can be expressed as linear combination of two $(n+1)^{th}$ degree basis polynomials in the following way:*

$$p_{n,k}^{q}(\mathsf{v})=\frac{[n-k+1]_q}{[n+1]_q}p_{n+1,k}^{q}(\mathsf{v})+\left(1-\frac{[n-k]_q}{[n+1]_q}\right)p_{n+1,k+1}^{q}(\mathsf{v}).$$

Proof.

$$p_{n,k}^{q}(\mathsf{v}) = p_{n,k}^{q}(\mathsf{v})\left(1-\frac{q^n\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}{\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^n\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}+\frac{q^n\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}{\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^n\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}\right)$$

$$\begin{aligned}
=&\frac{\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)q^{\frac{k(k-1)}{2}}}{\prod\limits_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^{k}\\
&\times\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}+\frac{q^nq^{\frac{k(k-1)}{2}}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)}{\prod\limits_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n\\k\end{bmatrix}_q\\
&\times\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^{k}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}. \qquad (8.12)
\end{aligned}$$

Since, we know $\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n-k+1]_q}{[n+1]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q$, $\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[k+1]_q}{[n+1]_q}\begin{bmatrix}n+1\\k+1\end{bmatrix}_q$, and $q^{n-k}\frac{[k+1]_q}{[n+1]_q}=\left(1-\frac{[n-k]_q}{[n+1]_q}\right)$. Using these equalities and (8.12), we get

$$\begin{aligned}
p_{n,k}^q(\mathsf{v})=&\frac{[n-k+1]_q}{[n+1]_q}\frac{q^{\frac{k(k-1)}{2}}}{\prod\limits_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n+1\\k\end{bmatrix}_q\\
&\times\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^{k}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k+1}\\
&+\frac{[k+1]_q}{[n+1]_q}\frac{q^{\frac{k(k+1)}{2}+n-k}}{\prod\limits_{j=1}^{n+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}\begin{bmatrix}n+1\\k+1\end{bmatrix}_q\\
&\times\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^{k+1}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}.
\end{aligned}$$

Hence, we obtain the required result. □

8.4 Construction of Generalized q-Bézier Curves

We define the generalization of Bézier curves with the help of the q-Bernstein Chlodowsky basis polynomials (8.9) in the following way:

$$C(\mathsf{v},q)=\sum_{k=0}^{n}p_{n,k}^q(\mathsf{v})\,P_k, \qquad (8.13)$$

where P_k, $k=0,1,\cdots,n$ are the control points of the curves.

Theorem 8.4.1. *The Bézier curves defined by* (8.13) *have the following properties:*

1. *Bézier curves lie inside the control polygon determined by the control points.*
2. *End point interpolation property:* $C\left(\frac{\alpha d_n}{[n]_q+\beta},q\right)=P_0$ *and* $C\left(\frac{[n]_q+\alpha}{[n]_q+\beta}d_n,q\right)=P_n$.
3. *Reducibility: for* $q=1$, $\alpha=\beta=0$, *and* $d_n=1$, *the curves* (8.13) *reduce to classical Bézier curves.*

Proof. The above properties are easy to prove by using the results of Theorem 8.3.1. Hence, we omit the details. □

Theorem 8.4.2. *The Bézier curves* (8.13) *have the end-point derivative property as*

$$C'\left(\frac{\alpha d_n}{[n]_q+\beta},q\right)=\frac{[n]_q(P_1-P_0)}{d_n\left(\frac{[n]_q}{[n]_q+\beta}\right)},\quad C'\left(\frac{([n]_q+\alpha)d_n}{[n]_q+\beta},q\right)=\frac{[n]_q(P_n-P_{n-1})}{d_nq^{n-1}\left(\frac{[n]_q}{[n]_q+\beta}\right)}.$$

Proof. Let

$$C(\mathsf{v},q)=\sum_{k=0}^{n}p_{n,k}^{q}(\mathsf{v})\,P_k=\sum_{k=0}^{n}\frac{\mu_{n,k}^{q}(\mathsf{v})}{\prod\limits_{j=1}^{n}\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}+q^{j-1}\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)\right)}P_k:=\frac{C_1(\mathsf{v},q)}{C_2(\mathsf{v},q)}.$$

Thus,

$$C(\mathsf{v},q)\,C_2(\mathsf{v},q)=C_1(\mathsf{v},q).$$

Now, differentiating on both sides w.r.t. v, we get

$$C(\mathsf{v},q)\,C_2'(\mathsf{v},q)+C_2(\mathsf{v},q)\,C'(\mathsf{v},q)=C_1'(\mathsf{v},q). \tag{8.14}$$

$C_1(\mathsf{v},q)=\sum\limits_{k=0}^{n}\mu_{n,k}^{q}(\mathsf{v})\,P_k$, where $\mu_{n,k}^{q}(\mathsf{v})=\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{\mathsf{v}}{d_n}-\frac{\alpha}{[n]_q+\beta}\right)^k\left(\frac{[n]_q+\alpha}{[n]_q+\beta}-\frac{\mathsf{v}}{d_n}\right)^{n-k}q^{\frac{k(k-1)}{2}}$,

and by using (8.11), $C_2(\mathsf{v},q)=\sum\limits_{k=0}^{n}\mu_{n,k}^{q}(\mathsf{v})$.

$$\begin{aligned}
\text{Now, } (\mu_{n,k}^{q})'(\mathsf{v}) = {} & \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{k}{d_n} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q + \beta} \right)^{k-1} \left(\frac{[n]_q + \alpha}{[n]_q + \beta} - \frac{\mathsf{v}}{d_n} \right)^{n-k} q^{\frac{k(k-1)}{2}} \\
& - \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{n-k}{d_n} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q + \beta} \right)^{k} \left(\frac{[n]_q + \alpha}{[n]_q + \beta} - \frac{\mathsf{v}}{d_n} \right)^{n-k-1} q^{\frac{k(k-1)}{2}} \\
= {} & \frac{[n]_q}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{k}{d_n} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q + \beta} \right)^{k-1} \left(\frac{[n]_q + \alpha}{[n]_q + \beta} - \frac{\mathsf{v}}{d_n} \right)^{n-k} q^{\frac{k(k-1)}{2}} \\
& - \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{n-k}{d_n} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q + \beta} \right)^{k} \left(\frac{[n]_q + \alpha}{[n]_q + \beta} - \frac{\mathsf{v}}{d_n} \right)^{n-k-1} q^{\frac{k(k-1)}{2}} \\
= {} & \frac{[n]_q}{[k]_q} \frac{k}{d_n} q^{k-1} \, \mu_{n-1,k-1}^{q}(\mathsf{v}) - \frac{[n]_q}{[n-k]_q} \frac{n-k}{d_n} \, \mu_{n-1,k}^{q}(\mathsf{v}). \qquad (8.15)
\end{aligned}$$

We can easily calculate

$$C\left(\frac{\alpha d_n}{[n]_q + \beta}, q \right) = P_0, \; C_2\left(\frac{\alpha d_n}{[n]_q + \beta}, q \right) = \left(\frac{[n]_q}{[n]_q + \beta} \right)^{n},$$

$$C_1'\left(\frac{\alpha d_n}{[n]_q + \beta}, q \right) = \frac{[n]_q P_1 - n P_0}{d_n} \left(\frac{[n]_q}{[n]_q + \beta} \right)^{n-1}, \; C_2'\left(\frac{\alpha d_n}{[n]_q + \beta}, q \right) = \frac{([n]_q - n)}{d_n} \left(\frac{[n]_q}{[n]_q + \beta} \right)^{n-1}.$$

Using these equalities and (8.14), we get

$$C'\left(\frac{\alpha d_n}{[n]_q + \beta}, q \right) = \frac{[n]_q (P_1 - P_0)}{d_n \left(\frac{[n]_q}{[n]_q + \beta} \right)}.$$

Similarly, Using (8.15)

$$\begin{aligned}
C\left(\frac{[n]_q + \alpha}{[n]_q + \beta} d_n, q \right) = {} & P_n, \; C_2\left(\frac{[n]_q + \alpha}{[n]_q + \beta} d_n, q \right) = \left(\frac{[n]_q}{[n]_q + \beta} \right)^{n} q^{\frac{n(n-1)}{2}}, \\
C_1'\left(\frac{[n]_q + \alpha}{[n]_q + \beta} d_n, q \right) = {} & \frac{(n q^{n-1} P_n - [n]_q P_{n-1}) q^{\frac{(n-1)(n-2)}{2}}}{d_n} \left(\frac{[n]_q}{[n]_q + \beta} \right)^{n-1}, \\
C_2'\left(\frac{[n]_q + \alpha}{[n]_q + \beta} d_n, q \right) = {} & \frac{(n q^{n-1} - [n]_q) q^{\frac{(n-1)(n-2)}{2}}}{d_n} \left(\frac{[n]_q}{[n]_q + \beta} \right)^{n-1}.
\end{aligned}$$

Again, with the help of identity (8.14), we find

$$C'\left(\frac{([n]_q+\alpha)}{[n]_q+\beta}d_n, q\right) = \frac{[n]_q(P_n - P_{n-1})}{d_n q^{n-1}\left(\frac{[n]_q}{[n]_q+\beta}\right)}.$$

□

Remark 8.4.1. *This property represents that tangent at the point* $\mathsf{v} = \dfrac{\alpha d_n}{[n]_q+\beta}$ *is the resultant of vector from* P_0 *to* P_1*. Similarly, the tangent at the end point* $\mathsf{v} = \dfrac{([n]_q+\alpha)d_n}{[n]_q+\beta}$ *is also the resultant of vector from* P_{n-1} *to* P_n*.*

8.5 Degree Elevation

Degree elevation has applications in computer graphics, CAD, and font design, where it improves curve smoothness and complexity control. It is also used for data fitting and interpolation, ensuring accurate curve representations.

Degree elevation of Bézier curves is a mathematical technique used to increase the degree of a Bézier curve and control points while preserving its shape. This process involves introducing new control points to create a higher-degree curve, allowing for comparison with compatible curves. It is a valuable tool in computer graphics and design for achieving smoother and more precise curves.

Let $C(\mathsf{v}, q) = \sum\limits_{k=0}^{n} p_{n,k}^{q}(\mathsf{v})\ P_k$, where $k = 0, 1, \cdots, n$.

This curve can be represented as $C(\mathsf{v}, q) = \sum\limits_{k=0}^{n+1} p_{n+1,k}^{q}(\mathsf{v})\ D_k$, by applying the technique of degree elevation. In this process, we get $(n+2)$ control points D_k to determine new curves in the following way:

$$D_0 = P_0, \quad D_{n+1} = P_n, \quad D_k = \left(1 - \frac{[n-k+1]_q}{[n+1]_q}\right) P_{k-1} + \frac{[n-k+1]_q}{[n+1]_q} P_k,$$

where $k = 1, 2, \cdots, n$. Let $A = [P_0, P_1, \cdots, P_n]^T$ be the vector of (n+1) control points for the given Bézier curves of degree n and $B = [D_0, D_1, \cdots, D_{n+1}]^T$ be the vector of (n+2) control points of the new Bézier curves of degree $n+1$. We can find the relationship between vectors A and B denoted by $B = M_{n+1}A$, where M_{n+1} is the

matrix of order $(n+2)\times(n+1)$ defined as follows:
$M_{n+1} = \dfrac{1}{[n+1]_q}\times$

$$\begin{bmatrix} [n+1]_q & 0 & 0 & \cdots & 0 & 0 & 0 \\ [n+1]_q-[n]_q & [n]_q & 0 & \cdots & 0 & 0 & 0 \\ 0 & [n+1]_q-[n-1]_q & [n-1]_q & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n+1]_q-[2]_q & [2]_q & 0 \\ 0 & 0 & 0 & \cdots & 0 & [n+1]_q-[1]_q & [1]_q \\ 0 & 0 & 0 & \cdots & 0 & 0 & [n+1]_q \end{bmatrix}.$$

Similarly, we can also represent this curve to any higher degree curve without changing its shape.

For $m\in\mathbb{N}$, we can find Bézier curves of degree $n+m$. The vector of control points of degree elevated curves having degree $n+m$ is $R^T = [R_0, R_1, \cdots, R_{n+m}]$, where $R = M_{n+m}\cdots M_{n+2}M_{n+1}A$. For $m\to\infty$, the control polygon converges to Bézier curves.

8.6 De Casteljau Algorithm

The De Casteljau algorithm is a fundamental method for evaluating Bézier curves and surfaces. It works by recursively dividing control points to find a point on the curve. Starting with the original control points, it repeatedly computes intermediate points along the curve or surface until the desired level of precision is achieved.
Consider $P_k^0 = P_k$, where $k = 0, 1, \cdots, n$.

$$\begin{aligned} P_k^r(\mathsf{v};q) =& \frac{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n}\right)}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{n-r}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} P_k^{r-1}(\mathsf{v};q) \\ &+ \frac{q^{n-r}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)}{\left(\frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} + q^{n-r}\left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta}\right)\right)} P_{k+1}^{r-1}(\mathsf{v};q), \end{aligned} \tag{8.16}$$

$$r = 1, 2, \cdots, n, \; k = 0, 1, \cdots, n - r.$$

The matrix to represent these points

$$Q_{n,r} = \left(\frac{[n]_q + \alpha}{[n]_q + \beta} - \frac{\mathsf{v}}{d_n} + q^{n-r} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q + \beta} \right) \right)^{-1}$$
$$\times \begin{bmatrix} \frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} & q^{n-r} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta} \right) & \cdots & 0 & 0 \\ 0 & \frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{[n]_q+\alpha}{[n]_q+\beta} - \frac{\mathsf{v}}{d_n} & q^{n-r} \left(\frac{\mathsf{v}}{d_n} - \frac{\alpha}{[n]_q+\beta} \right) \end{bmatrix}.$$

8.7 Numerical Examples

In the present section, we provide some examples to represent the control of the new introduced parameters on the shape of Bézier curves.

Example 8.7.1. *In Fig. 8.3, we choose the parameters* $\alpha = 0.2, \beta = 0.8, d_n = \sqrt{n}$, *and the control points* $(5, 0), (0, 10), (5, 20), (15, 20), (20, 10)$ *and* $(15, 0)$. *In Fig. 8.4, we choose* $\alpha = 0.5, \beta = 0.9, d_n = \sqrt{n}$, *and the control points* $(2, 5), (5, 0), (15, 10), (15, 15),$ $(-5, 20), (-5, 25), (5, 35)$ *and* $(8, 30)$. *From the figures, it is clear that the parameter* q *is observed to alter the shape of the curves while keeping all other properties constant as well as maintaining the same control polygon. This ability to modify the curves' shape without changing the control polygon enhances to the curves' flexibility.*

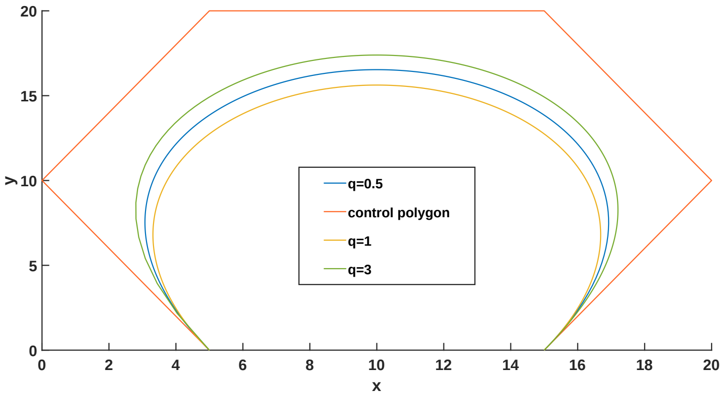


Figure 8.3: Shape modification with different values of $q = 0.5, 1, 3$

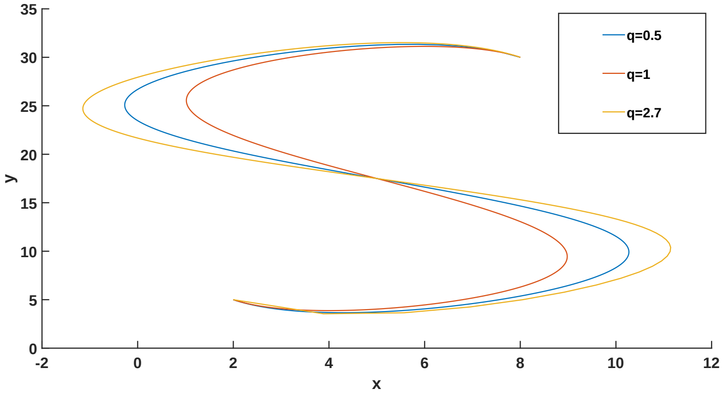


Figure 8.4: Shape modification with different values of $q = 0.5, 1, 2.7$

8.8 Conclusion

The present chapter deals with the Bézier curve's generalization, a technique that empowers us to fine-tune the shape of these curves through the manipulation of diverse parameters. Our exploration primarily focuses on q-analogue of Chlodowsky Bézier curves, a versatile extension of the classical Bézier curves. Additionally, we investigate the process of degree elevation and De Casteljau's algorithm, an essential tool for working with Bézier curves. This novel approach will give the flexibility to control the shape of the curves by choosing the parameters.

Conclusion

This thesis has significantly advanced the understanding and application of positive linear operators across various contexts. One of the primary contributions is the improvement of the convergence rates of classical α-Bernstein Durrmeyer type operators $Q_{n,\rho}^{\alpha}(g;x)$ by introducing three modifications. These modifications have been shown to achieve a better order of approximation, with the convergence of the operators being independent of the parameters involved. While this chapter primarily focused on operators of a single variable, we extended this study to multivariable functions, highlighting the importance of bivariate operators. The subsequent chapter focuses into the approximation characteristics of these operators, providing a comprehensive analysis of convergence and the influence of specific parameters.

In addition, we have expanded the scope of some positive linear operators by incorporating relevant parameters, thereby enhancing their flexibility and accuracy in various applications. We study both univariate and bivariate operators designed to preserve specific polynomials under certain conditions, with a detailed examination of their fundamental approximation properties. Also, we study the Durrmeyer-variant of Lupaş type operators to approximate Lebesgue integrable functions and find its approximation properties for functions of one as well as two variables. Our research also extends to operators defined on infinite intervals, such as Baskakov type operators, demonstrating their superior approximation capabilities compared to classical ones by introducing parameters, that helps to reduce the error of approximation of the operators. All the theoretical results were validated with numerical examples and graphical representations using MAPLE, MATLAB, and Mathematica.

In the realm of computer graphics, we explore the practical applications of positive linear operators particularly in defining generalized Bézier curves. We demonstrate the flexibility and efficiency of α-Bézier curves and surfaces. We introduced an algorithms to determine control points for given α-Bézier curves. Exploring the flexibility of the Bézier curves by introducing a parameter, we led to the introduction of q-Bézier curves, offering greater control over curve shapes through parameter manipulation and providing valuable tools for creating flexible and homogeneous curves. We discuss its various applications such degree elevation and De Casteljau algorithm. In the end, some examples and the graphics of Bézier curves and surfaces were given to show the effect of our introduced parameters.

In conclusion, this thesis tackles many significant deficiencies in the existing knowledge base and paves the road for further investigation and real-world implementations. The theoretical frameworks and practical solutions explored here emphasize the wide-ranging influence and promise of positive linear operators. This study establishes a solid basis for further progress in the discipline by offering fresh perspectives and innovative solutions to emerging challenges. Subsequent research can utilize these discoveries to deepen comprehension and utilization of positive linear operators, providing innovative strategies for intricate issues in Mathematics, Computer graphics, and other domains.

Future Scope

The research undertaken in this Ph.D. thesis sets the stage for numerous future investigations and developments. Building on the completed work, the following avenues present significant potential for further exploration:

1. We can further extend the applications of positive linear operators in solving integral and differential equations by using it to a broader class of these equations. Also, their convergence and efficacy can be studied to solve complex systems of equations encountered in various scientific and engineering problems. We can develop the computational algorithms to implement these solutions efficiently in practical scenarios.

2. As we have studied the operators on positive real line, we can also explore the definition and properties of novel positive linear operators defined on symmetric intervals and conduct a comprehensive study on the theoretical underpinnings and practical applications of these operators.

3. We can find the application of positive linear operators in the field of robotics, particularly in areas such as control systems, path planning, and sensor data fusion by developing the algorithms that leverage these operators to enhance the precision and reliability of robotic operations. In this direction, one can collaborate with robotics researchers and practitioners to test and refine these algorithms in real-world robotic systems.

4. As we discussed the modification of existing positive linear operators to improve their convergence rates on finite domain. Similarly, we can develop new techniques in order to improve the rate of convergence of the positive linear operators that are defined on infinite domain and study the mathematical properties and

convergence behavior of these modified operators to apply these enhanced operators to solve practical problems on infinite domains, such as in Physics and Finance.

5. There are several other fields, in which we can extend the applications of positive linear operators, such as Economics, Biology, and Environmental Science, to solve their problems and identify potential interdisciplinary collaborations that can benefit from the application of these operators.

By pursuing these future research directions, the contributions of this Ph.D. work can be significantly amplified, leading to continued advancements in the field and addressing emerging challenges and opportunities in both theoretical and applied contexts.

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