A STUDY ON FUZZY RISK ANALYSIS BASED ON FUZZY NUMBERS

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DEDICATED

TO

GOD, MY PARENTS AND MEHAR

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"A study on fuzzy risk analysis based on fuzzy numbers"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

Fuzzy set theory 1965 is a powerful tool to deal with real-life situations. Real numbers can be linearly ordered by $\leq$ or $\geq$; however, this type of inequality does not exist in fuzzy numbers. Since fuzzy numbers are represented by possibility distribution, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other. An efficient approach for ordering the fuzzy numbers is by using a ranking function $\Re : F(\mathbb{R}) \rightarrow \mathbb{R}$, where $F(\mathbb{R})$ is a set of fuzzy numbers defined on the real line, where a natural order exists. Thus, specific ranking of fuzzy numbers is an important procedure for decision making in a fuzzy environment and, generally, has become one of the main problems in fuzzy set theory.

It is obvious that we often face the difficulty of lacking precise information to assess the risk of component made by a manufactory in an uncertain environment. In order to overcome this problem, fuzzy number have been used to represent the fuzziness of evaluating value in fuzzy risk analysis problem, where the task of ranking fuzzy number is very important.

The present thesis contains four chapters.

Chapter 1 is introducing in nature in which literature related to the work is presented. Fuzzy risk analysis based on fuzzy numbers with different shapes and different deviations (Lee and Chen, [33]) has been reviewed in Chapter 2. In Chapter 3, Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights [19] has been reviewed. Singh [40] proposed a ranking approach of fuzzy sets with different heights, which is reviewed in Chapter 4.

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Chapter 1

Introduction

Fuzzy set theory, 1965 is a powerful tool to deal with real-life situations. Real numbers can be linearly ordered by ≤ or ≥, however, this type of inequality does not exist in fuzzy numbers. Since fuzzy numbers are represented by possibility distribution, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other. An efficient approach for ordering the fuzzy numbers is by using a ranking function $\Re : F(\mathbb{R}) \to \mathbb{R}$, where $F(\mathbb{R})$ is a set of fuzzy numbers defined on the real line, where a natural order exists. Thus, specific ranking of fuzzy numbers is an important procedure for decision making in a fuzzy environment and, generally, has become one of the main problems in fuzzy set theory.

It is obvious that we often face the difficulty of lacking precise information to assess the risk of component made by a manufactory in an uncertain environment. In order to overcome this problem, fuzzy number have been used to represent the fuzziness of evaluating value in fuzzy risk analysis problem [9, 12, 14, 39, 43], where the task of ranking fuzzy number is very important. In recent years, the topic of fuzzy risk analysis has been studied by some researchers [12, 14, 15, 18, 20, 28, 33, 39, 46]. To deal with fuzzy risk analysis problems, the evaluating values of the risk of each sub-component are usually represented by fuzzy numbers.

Dubois and Prade [26] proposed a complete set of comparison indices in the framework of Zadeh's possibility theory. This approach is related to previous ones, and its possible extension to the ranking of n fuzzy numbers is discussed at length. Buckley [6] investigates the problem of employing expert opinion to rank alternatives across a set of criteria and also employees fuzzy arithmetic to compute an issue's fuzzy ranking. This leads to a partition of the alternatives into sets H_1, H_2,… where H_1 contains the highest ranked issues; H_2 has all the second highest ranked alternatives, etc. Liou and Wang [35] developed a ranking method based on the integral value index. Kwang and Lee [31] considered the overall possibility distributions of fuzzy numbers in their evaluations and proposed a ranking method. Chen and Chen [12] presented a method called the simple centre of gravity method (SCGM) to calculate the centre-of-gravity (COG) points of generalized fuzzy numbers and proposed a new method to measure the degree of similarity between generalized fuzzy numbers.

Chang et al. [7] introduced a new ranking fuzzy numbers approach that can adjust expert's confidence and optimistic index of decision maker using two parameters (α and β) to handle the problems and find the best solutions. Chen and Chen [14] proposed a method for ranking generalized trapezoidal fuzzy numbers and compare the ranking results of the proposed method with the existing centroid-index ranking methods. Lee and Chen [33] presented a new method for ranking trapezoidal fuzzy numbers based on their shapes and deviations and presented a new fuzzy risk analysis algorithm. Chen and Chen [15] presented a new method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. Wang and Liou [45] presented an alternative ranking approach for fuzzy numbers called area ranking based on positive and negative ideal points, which defines two new alternative indices for the purpose of ranking. The two new indices are defined in terms of a decision maker attitude towards risks, and the left and the right areas between fuzzy numbers and the two ideal points.

Wei and Chen [46] presented a new method for fuzzy risk analysis based on similarity measures between interval-valued fuzzy numbers. It combines the concepts of geometric distance, the perimeter, the height and the centre-of-gravity-points of interval-valued fuzzy numbers for calculating the degree of similarity between interval-valued fuzzy numbers. They also proposed a new division operator and an interval-valued fuzzy number adjustment algorithm. Chen and Wang

[21] proposed a new method for ranking fuzzy numbers using the α-cuts, the belief feature and the signal/noise ratios, where $\alpha \in [0,1]$ and use the value of α as the weight of the signal/noise ratio of each α-cut of a fuzzy number to calculate the ranking index of each fuzzy number.

Assady [3] improved Wang's method and presented a revised method for ranking of *LR* deviation degree fuzzy number. Chen et al. [18] presented a new method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights. Chen and Sanguansat [20] presented a new method for ranking generalized fuzzy numbers and with the help of proposed ranking approach, they develop a new method for dealing with fuzzy risk analysis problems. Chen et al [19] presented a new method for fuzzy risk analysis based on the proposed new fuzzy ranking method for ranking generalized fuzzy numbers with different left heights and rights heights. Singh [40] proved that the ranking method proposed by Lee and Chen [33] is not genuine and proposed a new approach for the ranking of fuzzy sets with different heights.

Chapter 2

Fuzzy Risk Analysis Based on Fuzzy Numbers with Different Shapes and Different Deviations

2.1 Introduction

In this chapter, an existing method [33] for fuzzy risk analysis based on fuzzy numbers with different shapes and different deviations is presented. To compare the method with the existing methods [13, 22, 25, 33, 37, 48] some existing method is discussed. Finally, a fuzzy risk analysis algorithm presented to deal with fuzzy risk analysis problems by using the existing fuzzy ranking method [13, 22, 25, 33, 37, 48].

2.2 Preliminaries

In this section, some basic definition and arithmetic operations of fuzzy numbers are presented [33].

2.2.1 Basic Definition

Definition 2.1 Let X be a classical set of objects. Then, the set of Ordered Pair $\tilde{A}$= $\left\{\left(x, f_{\tilde{A}}\left(x\right)\right): x \in X\right\}$, where $f_{\tilde{A}} : X \rightarrow \left[0,1\right]$, is called a fuzzy set in *X*.

Definition 2.2 A fuzzy set $\tilde{A}$ in the universe of discourse X with the membership function, $f_{\tilde{A}}, f_{\tilde{A}} : X \rightarrow [0,1]$. If $\exists\, x_i \in X$ such that $f_{\tilde{A}}(x_i) = 1$, then the fuzzy set $\tilde{A}$ is called a normal fuzzy set.

Definition 2.3 A fuzzy set $\tilde{A}$ is called convex fuzzy set if $\exists\, x_1, x_2 \in X$, such that $f\left(\lambda x_1 + (1-\lambda)x_2\right) \geq$ Min $\left(f_{\tilde{A}}\left(x_1\right),\ f_{\tilde{A}}\left(x_2\right)\right)$, where $\lambda \in [0,1]$.

Definition 2.4 A fuzzy number is a fuzzy set in the universe of discourse *X* which is both normal and convex.

Definition 2.5 A fuzzy number is said to be non normal fuzzy number if $\forall\, x \in X$, such that $0 \leq \mu_{\tilde{A}}(x) < 1$.

Definition 2.6 A fuzzy number $\tilde{A} = (a,b,c,d;1)$ is said to be trapezoidal fuzzy number, if its membership function is given by

$$f_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}^{L} \ , & a \leq x \leq b \\ 1 \ , & b \leq x \leq c \\ f_{\tilde{A}}^{R} \ , & c \leq x \leq d \\ 0 \ , & \text{otherwise} \end{cases}$$

Where, $f_{\tilde{A}}^{L} : [a,b] \to [0,1]$ denotes the left membership function of the fuzzy number $\tilde{A}$, and $f_{\tilde{A}}^{R} : [a,b] \to [0,1]$ denotes the right membership function of the fuzzy number $\tilde{A}$. The membership function $f_{\tilde{A}}$ of the fuzzy number $\tilde{A}$ is shown in Fig. 2.1 and Fig. 2.2 shows a non-normal fuzzy number $\tilde{B} = (a,b,c,d;w)$.

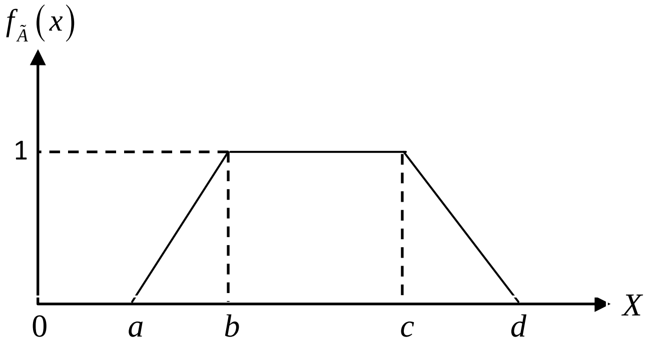


Fig. 2.1 A trapezoidal fuzzy number.

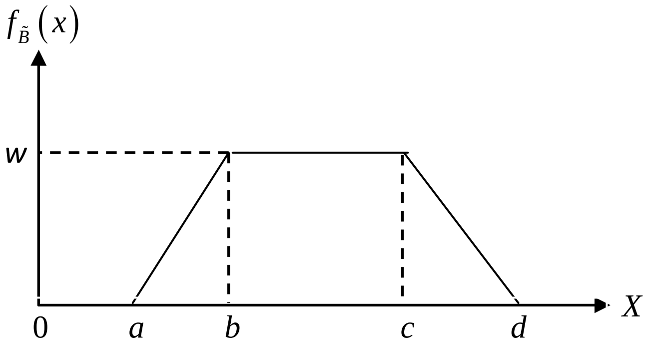


Fig. 2.2 A non- normal fuzzy number.

2.2.2 Arithmetic Operations

In this section, some arithmetic operations between two trapezoidal fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are presented [33].

Let $\tilde{A}_1 = (a_{11}, a_{12}, a_{13}, a_{14}; L_{1H}, R_{1H})$ and $\tilde{A}_2 = (a_{21}, a_{22}, a_{23}, a_{24}; L_{2H}, R_{2H})$ be two trapezoidal fuzzy numbers. Then ,

i. $\tilde{A}_1 \oplus \tilde{A}_2 = (a_{11}, a_{12}, a_{13}, a_{14}; L_{1H}, R_{1H}) \oplus (a_{21}, a_{22}, a_{23}, a_{24}; L_{2H}, R_{2H})$

$$= \left(a_{11}+a_{21},\, a_{12}+a_{22},\, a_{13}+a_{23},\, a_{14}+a_{24}; \min\{L_{1H}, L_{2H}\},\, \min\{R_{1H}, R_{2H}\}\right)$$

ii. $\tilde{A}_1 \Theta \tilde{A}_2 = (a_{11}, a_{12}, a_{13}, a_{14}; L_{1H}, R_{1H}) \Theta (a_{21}, a_{22}, a_{23}, a_{24}; L_{2H}, R_{2H})$

$$= \left(a_{11}-a_{24},\, a_{12}-a_{23},\, a_{13}-a_{22},\, a_{14}-a_{21}; \min\{L_{1H}, L_{2H}\},\, \min\{R_{1H}, R_{2H}\}\right)$$

iii. $\tilde{A}_1 \otimes \tilde{A}_2 = (a_{11}, a_{12}, a_{13}, a_{14}; L_{1H}, R_{1H}) \otimes (a_{21}, a_{22}, a_{23}, a_{24}; L_{2H}, R_{2H})$

$$= \left(a_{11}\times a_{21},\, a_{12}\times a_{22},\, a_{13}\times a_{23},\, a_{14}\times a_{24}; \min\{L_{1H}, L_{2H}\},\, \min\{R_{1H}, R_{2H}\}\right)$$

iv. $\tilde{A}_1 \oslash \tilde{A}_2 = (a_{11}, a_{12}, a_{13}, a_{14}; L_{1H}, R_{1H}) \oslash (a_{21}, a_{22}, a_{23}, a_{24}; L_{2H}, R_{2H})$

$$= \left(a_{11}/a_{24},\, a_{12}/a_{23},\, a_{13}/a_{22},\, a_{14}/a_{21}; \min\{L_{1H}, L_{2H}\},\, \min\{R_{1H}, R_{2H}\}\right)$$

2.3 A review of some existing methods for ranking fuzzy numbers

In this section, some existing methods [7, 17, 25, 22] for ranking fuzzy numbers is briefly discussed.

2.3.1 Cheng's method : Cheng [22] presented a distance method for ranking fuzzy numbers by calculating the centroid points of fuzzy numbers, where the distance is measured from the original point to the centroid point of a fuzzy number. Let us consider a trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;1)$ with the membership function $f_{\tilde{A}}$, shown as follows:

$$f_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ 1 \quad , & b \le x \le c \\ \dfrac{x-d}{c-d}, & c \le x \le d \\ 0 \quad , & \text{otherwise} \end{cases}$$

The centroid point $(x_{\tilde{A}}, y_{\tilde{A}})$ of fuzzy number $\tilde{A}$ is defined as follow:

$$x_{\tilde{A}} = \frac{\int_a^b \left(x\, f_{\tilde{A}}^L\right)dx + \int_b^c (x)\,dx + \int_c^d \left(x\, f_{\tilde{A}}^R\right)dx}{\int_a^b \left(f_{\tilde{A}}^L\right)dx + \int_b^c dx + \int_c^d \left(f_{\tilde{A}}^R\right)dx}$$

$$y_{\tilde{A}} = \frac{\int_0^1 \left(y\, g_{\tilde{A}}^L\right)dx + \int_0^1 \left(y\, g_{\tilde{A}}^R\right)dx}{\int_0^1 \left(g_{\tilde{A}}^L\right)dx + \int_0^1 \left(g_{\tilde{A}}^R\right)dx}$$

Let us consider a non-normal fuzzy number $\tilde{B} = (a,b,c,d;w)$ with the membership function $f_{\tilde{B}}$, shown as follow:

$$f_{\tilde{B}}(x) = \begin{cases} \dfrac{w(x-a)}{b-a}, & a \le x \le b \\ 1 \qquad , & b \le x \le c \\ \dfrac{w(x-d)}{c-d}, & c \le x \le d \\ 0 \qquad , & \text{otherwise} \end{cases}$$

The centroid point $\left(x_{\tilde{B}}, y_{\tilde{B}}\right)$ of the fuzzy number $\tilde{B}$ is defined as follows:

$$x_{\tilde{B}} = \frac{\int_a^b \left(x\, f_{\tilde{B}}^L\right) dx + \int_b^c (x)\, dx + \int_c^d \left(x\, f_{\tilde{B}}^R\right) dx}{\int_a^b \left(f_{\tilde{B}}^L\right) dx + \int_b^c dx + \int_c^d \left(f_{\tilde{B}}^R\right) dx}$$

$$y_{\tilde{B}} = \frac{\int_0^1 \left(y\, g_{\tilde{B}}^L\right) dx + \int_0^1 \left(y\, g_{\tilde{B}}^R\right) dx}{\int_0^1 \left(g_{\tilde{B}}^L\right) dx + \int_0^1 \left(g_{\tilde{B}}^R\right) dx}$$

Then, the ranking values numbers $\tilde{A}$ and $\tilde{B}$ are $\sqrt{\left(x_{\tilde{A}}\right)^2 + \left(y_{\tilde{A}}\right)^2}$ and $\sqrt{\left(x_{\tilde{B}}\right)^2 + \left(y_{\tilde{B}}\right)^2}$ respectively. The larger the ranking value, the better the ranking of the fuzzy number.

2.3.2 Chen-and-Lu's method : Chen and Lu [17] presented an approach for ranking fuzzy numbers based on the left and the right dominance. Their method only requires a few left and right spreads at some α-levels of fuzzy numbers to determine the respective dominance of a fuzzy number over the others. They define the lower and the upper limits of the k^{th} α-cut for a fuzzy number $\tilde{A}_i$ as follows:

$$l_{i,k} = \inf_{x \in \mathbb{R}} \left\{ x \mid \mu_{\tilde{A}} \ge \alpha_k \right\}$$

$$r_{i,k} = \sup_{x \in \mathbb{R}} \left\{ x \mid \mu_{\tilde{A}} \ge \alpha_k \right\}$$

where $l_{i,k}$ and $r_{i,k}$ are the left and right spreads , respectively. The left dominance $D_{i,j}^{L}$ and right dominance $D_{i,j}^{R}$ of a fuzzy number $\tilde{A}_i$ over a fuzzy number $\tilde{A}_j$ is define as follows:

$$D_{i,j}^{L} = \frac{1}{n+1}\sum_{k=0}^{n}\left(l_{i,k} - l_{j,k}\right)$$

$$D_{i,j}^{R} = \frac{1}{n+1}\sum_{k=0}^{n}\left(r_{i,k} - r_{j,k}\right)$$

Where "*n+1*" α-cuts are used to calculate the dominance. Then, the total dominance $D_{i,j}\left(\beta\right)$ of $\tilde{A}_i$ over $\tilde{A}_j$ with respect to the index of optimism $\beta \in \left[0,1\right]$ is defined by the convex combinations of $D_{i,j}^{L}\left(\beta\right)$ and $D_{i,j}^{R}\left(\beta\right)$, shown as follows:

$$\begin{aligned} D_{i,j}\left(\beta\right) &= \beta D_{i,j}^{R} - \left(1-\beta\right) D_{i,j}^{L} \\ &= \beta\left[\frac{1}{n+1}\sum_{k=0}^{n}\left(r_{i,k} - r_{j,k}\right)\right] + \left(1-\beta\right)\left[\frac{1}{n+1}\sum_{k=0}^{n}\left(l_{i,k} - l_{j,k}\right)\right] \\ &= \frac{1}{n+1}\left\{\beta\left[\sum_{k=0}^{n}\left(r_{i,k} - r_{j,k}\right)\right] + \left(1-\beta\right)\left[\sum_{k=0}^{n}\left(l_{i,k} - l_{j,k}\right)\right]\right\} \end{aligned}$$

A decision-maker can rank a pair of fuzzy numbers $\tilde{A}_i$ and $\tilde{A}_j$ based on the value of $D_{i,j}\left(\beta\right)$ using the following rules:

i. If $D_{i,j}\left(\beta\right) > 0,$ then $\tilde{A}_i > \tilde{A}_{j,}$

ii. If $D_{i,j}\left(\beta\right) = 0,$ then $\tilde{A}_i = \tilde{A}_{j,}$

iii. If $D_{i,j}\left(\beta\right) < 0,$ then $\tilde{A}_i < \tilde{A}_{j,}$

2.3.3 **Chang et al.'s method :** Chang et al. [7] presented a conceptual procedure for ranking fuzzy numbers based on the adaptive two-dimensional dominance. *A* trapezoidal fuzzy number $\tilde{A}_i$ can be defined as $\tilde{A}_i = \left(a_1, a_2, a_3, a_4; L_{iH}, R_{iH}\right)$ as shown in Fig. 2.3, where a_i and b_i are called the left elements of the fuzzy number $\tilde{A}_i$; c_i and d_i are called the right elements of the fuzzy number $\tilde{A}_i$. Chang et al.'s [7] method for ranking fuzzy numbers is as follows:

Step 1. Get the values of R_{iH}, L_{iH}, R_{iS} and L_{iS} of each fuzzy number $\tilde{A}_i$,where $\tilde{A}_i = \left(a_1, a_2, a_3, a_4; L_{iH}, R_{iH}\right)$, R_{iH} denotes the right height of the fuzzy number $\tilde{A}_i$, L_{iH} denotes the left height of the fuzzy number $\tilde{A}_i$, R_{iS} denotes the right spread of the fuzzy number A_i i.e., $R_{iS} = c_i + d_i$ and L_{iS} denotes the left spread of the fuzzy number $\tilde{A}_i$ i.e., $L_{iS} = a_i + b_i$

Step 2. If a fuzzy number is normal, then the decision- maker only needs to consider the index of optimism, β i.e., $\alpha = 0$. Otherwise, the decision-maker must further consider the degree of experts' confidence α i.e., he/she must consider the values of α and β, simultaneously, where $\alpha \in [0,1]$ and $\beta \in [0,1]$.

Step 3. Calculate the ranking value D_i of each fuzzy number $\tilde{A}_i$ as follows:

$$D_i = \frac{1}{2}\left\{\alpha\left[\beta R_{iH}\left(\tilde{A}_i\right) + (1-\beta) L_{iH}\left(\tilde{A}_i\right)\right] + (1-\alpha)\left[\beta R_{iS}\left(\tilde{A}_i\right) + (1-\beta) L_{iS}\left(\tilde{A}_i\right)\right]\right\}$$

Where D_i denotes the ranking value of the fuzzy number $\tilde{A}_i$, R_{iH} denotes the right height of the fuzzy number $\tilde{A}_i$, L_{iH} denotes the left height of the fuzzy number $\tilde{A}_i$, R_{iS} denotes the right spread of the fuzzy number $\tilde{A}_i$ i.e., $R_{iS} = c_i + d_i$ and L_{iS} denotes the left spread of the fuzzy number $\tilde{A}_i$ i.e., $L_{iS} = a_i + b_i$ .The larger the value of D_i the better the ranking of the fuzzy number $\tilde{A}_i$.

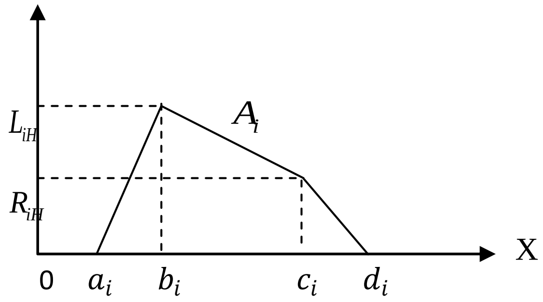


Fig. 2.3 A trapezoid fuzzy number

2.3.4 Deng-and-Liu's method : Deng and Liu [25] presented a new centroid index ranking method based on TOPSIS. The steps of the Deng and Liu's method for ranking fuzzy numbers is as follows:

Step 1. Translate generalized trapezoidal fuzzy numbers into standardized generalized trapezoidal fuzzy numbers. Let $\tilde{A}$ be a generalized trapezoidal fuzzy numbers in the universe of distance $\bigcup$, where $\bigcup = [-m, m]$, $\tilde{A} = (a, b, c, d)$ and $m \geq \max(|a|, |b|, |c|, |d|)$. If $0 \leq m \leq 1$, then the trapezoidal fuzzy number $\tilde{A}$ is called a standardized trapezoidal fuzzy number $\bar{A}$. If $m > 1$ then the trapezoidal fuzzy number $\tilde{A}$ can be translated into the standardized trapezoidal fuzzy number $\bar{A}$, shown as follows:

$$\bar{A} = \left(\frac{a}{m}, \frac{b}{m}, \frac{c}{m}, \frac{d}{m} \right)$$

Step 2. Determine the *COG* point of each standardized trapezoidal fuzzy number.

Step 3. Translate the *COG* point into the index point. Assume that the *COG* point of the standardized trapezoidal fuzzy number $\bar{A}$ is $\left(x_A^*, y_A^*\right)$. Then, the index point $\left(x_{\tilde{A}}^*, y_{\tilde{A}}^*\right)$ can be obtained as follows:

$$x_{\tilde{A}}^{*} = x_{A}^{*} \tag{2.1}$$

$$y_{\tilde{A}}^{*} = \left(w_{A} - y_{A}^{*}\right)^{S_{A}} \times \left(y_{A}^{*} - 0.5\right)^{1-w_{A}} \tag{2.2}$$

Step 4. Determine the positive point and the negative point. The positive alternative is represented as $(1,1,1,1;1)$ and the negative alternative is represented as $(-1,-1,-1,-1;1)$ By applying Equation 2.1 and 2.2, the positive point $\left(x_{P}^{*}, y_{P}^{*}\right)$ and negative point $\left(x_{N}^{*}, y_{N}^{*}\right)$ can be obtained, where

$$\left(x_{P}^{*}, y_{P}^{*}\right) = \left(1, \frac{1}{2}\right).$$

$$\left(x_{N}^{*}, y_{N}^{*}\right) = \left(-1, \frac{1}{2}\right).$$

Step 5. Calculate the distance of the index point of each alternative from the positive $\left(x_{P}^{*}, y_{P}^{*}\right)$ point and the negative $\left(x_{N}^{*}, y_{N}^{*}\right)$ point , respectively, as follow:

$$d^{+} = \sqrt{\left(x_{\tilde{A}}^{*} - x_{P}^{*}\right)^{2} + \left(y_{\tilde{A}}^{*} - y_{P}^{*}\right)^{2}}$$

$$d^{-} = \sqrt{\left(x_{\tilde{A}}^{*} - x_{N}^{*}\right)^{2} + \left(y_{\tilde{A}}^{*} - y_{N}^{*}\right)^{2}}$$

Step 6. Calculate the index co-efficient, where the index co-efficient is defined to determine the ranking order of the fuzzy numbers once the values of d^{+} and d^{-} of each fuzzy number have been calculated. The index co-efficient $IC_{\tilde{A}}$ of fuzzy number $\tilde{A}$ is calculated as follows:

$$IC_{\tilde{A}} = \frac{d^{-}}{d^{+} + d^{-}}$$

The larger the value of $IC_{\tilde{A}}$, the better the ranking of the fuzzy number $\tilde{A}$.

2.4 A method for ranking fuzzy numbers based on the shapes and the deviations of fuzzy numbers

In this section, an existing method [33] is presented.

Let $\tilde{A}_1, \tilde{A}_2, \ldots \text{and} \tilde{A}_n$ be a set of trapezoidal fuzzy numbers, where $\tilde{A}_i = \left(a_{i1}, a_{i2}, a_{i3}, a_{i4}; L_{iH}, R_{iH}\right)$

and $1 \le i \le n$ The method for ranking fuzzy numbers is now presented as follows:

Step 1. Get the values of L_{iH}, R_{iH}, L_{iM}, R_{iM}, L_{iS}, R_{iS} and T_{iS} of each fuzzy number $\tilde{A}_i$ where $\tilde{A}_i = \left(a_{i1}, a_{i2}, a_{i3}, a_{i4}; L_{iH}, R_{iH}\right)$.

L_{iH} denotes the left height of the fuzzy number $\tilde{A}_i$.

R_{iH} denotes the right height of the fuzzy number $\tilde{A}_i$.

R_{iM} denotes the average of the right elements a_{i3} and a_{i4} i.e., $\left(R_{iM} = \dfrac{a_{i3} + a_{i4}}{2}\right)$.

L_{iM} denotes the average of the left elements a_{i1} and a_{i2} i.e., $\left(L_{iM} = \dfrac{a_{i1} + a_{i2}}{2}\right)$.

R_{iS} denotes the standard deviation of the right element a_{i3} and a_{i4}

i.e., $R_{iS} = \sqrt{\dfrac{1}{2}\sum_{j=3}^{4}\left(a_{ij} - \dfrac{1}{2}\sum_{j=3}^{4} a_{ij}\right)^2}$.

L_{iS} denotes the standard deviation of the left element a_{i1} and a_{i2}

i.e., $L_{iS} = \sqrt{\frac{1}{2}\sum_{j=1}^{2}\left(a_{ij} - \frac{1}{2}\sum_{j=1}^{2} a_{ij} \right)^2}$

T_{iS} denotes the standard deviation of the elements a_{i1} , a_{i2} , a_{i3} and a_{i4}

i.e., $T_{iS} = \sqrt{\frac{1}{4}\sum_{j=1}^{4}\left(a_{ij} - \frac{1}{4}\sum_{j=1}^{4} a_{ij} \right)^2}$.

Step 2. Choose two proper values for α and β where α denotes the degree of experts' confidence, $\alpha \in [0,1]$, β denotes the index of optimism of the decision-maker and $\beta \in [0,1]$. In general, let $\alpha = 0.5$ and $\beta = 0.5$ be proper values for ranking fuzzy numbers.

Step 3. Calculate the ranking values $\Re\left(\tilde{A}_i\right)$ of each fuzzy number $\tilde{A}_i$, shown as follows:

$$\Re\left(\tilde{A}_i\right) = \alpha\left[\beta R_{iH}\left(\tilde{A}_i\right) + (1-\beta) L_{iH}\left(\tilde{A}_i\right)\right] + (1-\alpha)\Big[\beta R_{iM}\left(\tilde{A}_i\right) + (1-\beta) L_{iM}\left(\tilde{A}_i\right) - \frac{1}{3}\left(\beta R_{iS}\left(\tilde{A}_i\right) + (1-\beta) L_{iS}\left(\tilde{A}_i\right) + T_{iS}\left(\tilde{A}_i\right)\right)\Big] \tag{2.3}$$

The larger the ranking value $\Re\left(\tilde{A}_i\right)$, the better the ranking of the fuzzy number $\tilde{A}_i$, where $1 \le i \le n$

Example 2.1 [33] Assume that there are two sets of fuzzy numbers (i.e., Set 1 and Set 2) shown in Fig. 2.4, where Set 1 has three fuzzy numbers, i.e., $\tilde{A}_1 = (0.1, 0.3, 0.3, 0.5; 1,1)$, $\tilde{B}_1 = (0.2, 0.3, 0.3, 0.4; 1,1)$, $\tilde{C}_1 = (1,1,1,1;1,1)$; Set 2 has two fuzzy numbers, i.e. $\tilde{A}_2 = (-0.5, -0.3, -0.3, -0.1; 1,1)$, $\tilde{B}_2 = (0.1, 0.3, 0.3, 0.5; 1,1)$

First we calculate rank of Set 1, In Set 1 given $\tilde{A}_1 = (0.1, 0.3, 0.3, 0.5; 1,1)$, Because $\alpha = 0.5$ and $\beta = 0.5$, the ranking values Now Equation 2.3 is

$$\Re\left(\tilde{A}_i\right)=\alpha\left[\beta R_{iH}\left(\tilde{A}_i\right)+(1-\beta)L_{iH}\left(\tilde{A}_i\right)\right]+(1-\alpha)\left[\beta R_{iM}\left(\tilde{A}_i\right)+(1-\beta)L_{iM}\left(\tilde{A}_i\right)-\frac{1}{3}\left(\beta R_{iS}\left(\tilde{A}_i\right)+(1-\beta)L_{iS}\left(\tilde{A}_i\right)+T_{iS}\left(\tilde{A}_i\right)\right)\right]$$

To calculate the value of $\Re\left(\tilde{A}_1\right)$ firstly, Calculate the value of $R_M\left(\tilde{A}_1\right)$ and $L_M\left(\tilde{A}_1\right)$.

$$R_M\left(\tilde{A}_1\right)=\frac{a_{13}+a_{14}}{2}=\frac{0.3+0.5}{2}=0.4$$

$$L_M\left(\tilde{A}_1\right)=\frac{a_{11}+a_{12}}{2}=\frac{0.1+0.2}{2}=0.2$$

Calculate the value of $R_S\left(\tilde{A}_1\right)$ and $L_S\left(\tilde{A}_1\right)$.

$$R_S\left(\tilde{A}_1\right)=\sqrt{\frac{1}{2}\sum_{j=3}^{4}\left(a_{1j}-\frac{1}{2}\sum_{j=3}^{4}a_{1j}\right)^2}$$

$$=\sqrt{\frac{1}{2}\left(\left(a_{13}-\frac{1}{2}(a_{13}+a_{14})\right)^2+\left(a_{14}-\frac{1}{2}(a_{13}+a_{14})\right)^2\right)}$$

$$=\sqrt{\frac{1}{2}\left(\left(0.3-\frac{1}{2}(0.3+0.5)\right)^2+\left(0.5-\frac{1}{2}(0.3+0.5)\right)^2\right)}=0.1$$

$$L_S\left(\tilde{A}_1\right)=\sqrt{\frac{1}{2}\sum_{j=3}^{4}\left(a_{1j}-\frac{1}{2}\sum_{j=3}^{4}a_{1j}\right)^2}$$

$$=\sqrt{\frac{1}{2}\left(\left(a_{11}-\frac{1}{2}(a_{11}+a_{12})\right)^2+\left(a_{12}-\frac{1}{2}(a_{11}+a_{12})\right)^2\right)}$$

$$=\sqrt{\frac{1}{2}\left(\left(0.1-\frac{1}{2}(0.1+0.3)\right)^2+\left(0.3-\frac{1}{2}(0.1+0.3)\right)^2\right)}=0.1$$

$$T_s\left(\tilde{A}_1\right) = \sqrt{\frac{1}{4}\sum_{j=1}^{4}\left(a_{1j} - \frac{1}{4}\sum_{j=1}^{4} a_{1j}\right)^2}$$

$$= \sqrt{\frac{1}{2}\left(\begin{array}{l}\left(a_{11} - \frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2 + \left(a_{12} - \frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2 \\ +\left(a_{13} - \frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2 + \left(a_{14} - \frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2\end{array}\right)}$$

$$= \sqrt{\frac{1}{4}\left(\begin{array}{l}\left(0.1 - \frac{1}{4}\left(0.1+0.3+0.3+0.5\right)\right)^2 + \left(0.3 - \frac{1}{4}\left(0.1+0.3+0.3+0.5\right)\right)^2 \\ +\left(0.3 - \frac{1}{4}\left(0.1+0.3+0.3+0.5\right)\right)^2 + \left(0.5 - \frac{1}{4}\left(0.1+0.3+0.3+0.5\right)\right)^2\end{array}\right)}$$

$$= \sqrt{\frac{1}{4}\left(0.04+0+0+0.04\right)} \ = \ 0.14142$$

$$R_M\left(\tilde{A}_1\right)=1,\ L_M\left(\tilde{A}_1\right)=1$$

Substitute all the above value in Equation 2.3

$$\begin{aligned}\Re\left(\tilde{A}_1\right) = 0.5\left[0.5\times 1+\left(1-0.5\right)\times 0.5\right] + \left(1-0.5\right)\Big[0.5\times 0.4+\left(1-0.5\right)\times 0.2 - \\ \frac{1}{3}\left(0.5\times 0.1+\left(1-0.5\right)\times 0.1+0.14142\right)\Big]=0.6098\end{aligned}$$

Now, calculate rank of $\tilde{B}_1$, where $\tilde{B}_1 = \left(0.2, 0.3, 0.3, 0.4; 1, 1\right)$, Because $\alpha = 0.5$ and $\beta = 0.5$, the ranking values. Now Equation 2.3 is

$$\begin{aligned}\Re\left(\tilde{B}_i\right) = \alpha\left[\beta R_{iH}\left(\tilde{B}_i\right)+\left(1-\beta\right)L_{iH}\left(\tilde{B}_i\right)\right] + \left(1-\alpha\right)\Big[\beta R_{iM}\left(\tilde{B}_i\right)+\left(1-\beta\right)L_{iM}\left(\tilde{B}_i\right) - \\ \frac{1}{3}\left(\beta R_{iS}\left(\tilde{B}_i\right)+\left(1-\beta\right)L_{iS}\left(\tilde{B}_i\right)+T_{iS}\left(\tilde{B}_i\right)\right)\Big]\end{aligned}$$

To calculate the value of $\Re\left(\tilde{B}_1\right)$ firstly, Calculate the value of $R_M\left(\tilde{B}_1\right)$ and $L_M\left(\tilde{B}_1\right)$.

$$R_M\left(\tilde{B}_1\right) = \frac{a_{13}+a_{14}}{2} = \frac{0.3+0.4}{2} = 0.35$$

$$L_M\left(\tilde{B}_1\right) = \frac{a_{11}+a_{12}}{2} = \frac{0.2+0.3}{2} = 0.25$$

Calculate the value of $R_S\left(\tilde{B}_1\right)$ and $L_S\left(\tilde{B}_1\right)$.

$$R_S\left(\tilde{B}_1\right) = \sqrt{\frac{1}{2}\sum_{j=3}^{4}\left(a_{1j} - \frac{1}{2}\sum_{j=3}^{4}a_{1j}\right)^2}$$

$$= \sqrt{\frac{1}{2}\left(\left(a_{13}-\frac{1}{2}\left(a_{13}+a_{14}\right)\right)^2 + \left(a_{14}-\frac{1}{2}\left(a_{13}+a_{14}\right)\right)^2\right)}$$

$$= \sqrt{\frac{1}{2}\left(\left(0.3-\frac{1}{2}\left(0.3+0.4\right)\right)^2 + \left(0.4-\frac{1}{2}\left(0.3+0.4\right)\right)^2\right)} = 0.05$$

$$L_S\left(\tilde{B}_1\right) = \sqrt{\frac{1}{2}\sum_{j=3}^{4}\left(a_{1j} - \frac{1}{2}\sum_{j=3}^{4}a_{1j}\right)^2}$$

$$= \sqrt{\frac{1}{2}\left(\left(a_{11}-\frac{1}{2}\left(a_{11}+a_{12}\right)\right)^2 + \left(a_{12}-\frac{1}{2}\left(a_{11}+a_{12}\right)\right)^2\right)}$$

$$= \sqrt{\frac{1}{2}\left(\left(0.2-\frac{1}{2}\left(0.2+0.3\right)\right)^2 + \left(0.3-\frac{1}{2}\left(0.2+0.3\right)\right)^2\right)} = 0.05$$

$$T_S\left(\tilde{B}_1\right) = \sqrt{\frac{1}{4}\sum_{j=1}^{4}\left(a_{1j} - \frac{1}{4}\sum_{j=1}^{4}a_{1j}\right)^2}$$

$$= \sqrt{\frac{1}{2}\left(\begin{aligned}&\left(a_{11}-\frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2+\left(a_{12}-\frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2\\&+\left(a_{13}-\frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2+\left(a_{14}-\frac{1}{2}\left(a_{11}+a_{12}+a_{13}+a_{14}\right)\right)^2\end{aligned}\right)}$$

$$= \sqrt{\frac{1}{4}\left(\begin{aligned}&\left(0.2-\frac{1}{4}\left(0.2+0.3+0.3+0.4\right)\right)^2+\left(0.3-\frac{1}{4}\left(0.2+0.3+0.3+0.4\right)\right)^2\\&+\left(0.3-\frac{1}{4}\left(0.2+0.3+0.3+0.4\right)\right)^2+\left(0.4-\frac{1}{4}\left(0.2+0.3+0.3+0.4\right)\right)^2\end{aligned}\right)}$$

$$= \sqrt{\frac{1}{4}\left(0.01+0+0+0.01\right)} \; = \; 0.070710$$

$$R_M\left(\tilde{B}_1\right)=1, \; L_M\left(\tilde{B}_1\right)=1$$

Substitute all the above value in Equation 2.3

$$\begin{aligned}\Re\left(\tilde{B}_1\right) &= 0.5\left[0.5\times1+\left(1-0.5\right)\times0.5\right]+\left(1-0.5\right)\left[0.5\times0.35+\left(1-0.5\right)\times0.25-\right.\\&\left.\frac{1}{3}\left(0.5\times0.05+\left(1-0.5\right)\times0.05+0.070710\right)\right]=0.6299\end{aligned}$$

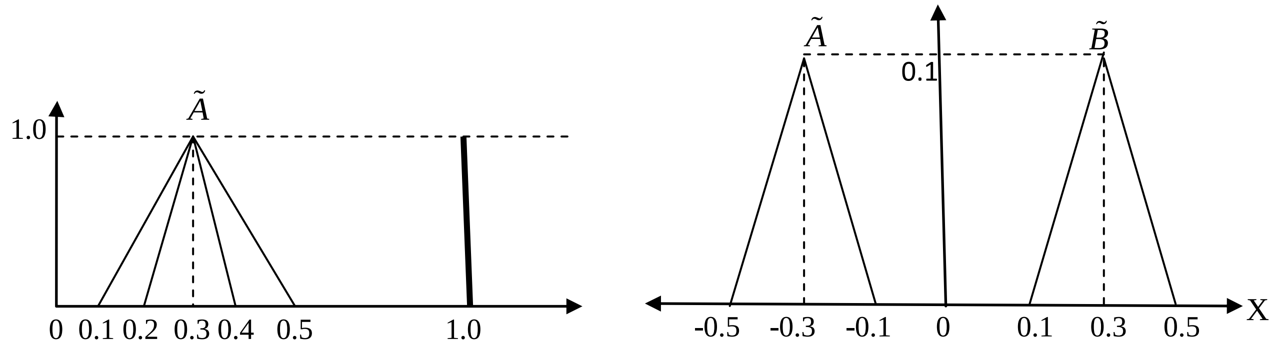
Similarly find the value of $\Re\left(\tilde{C}_1\right)$ in set 1 and $\Re\left(\tilde{A}_2\right)$ and $\Re\left(\tilde{B}_2\right)$ in Set 2.

$$\begin{aligned}\Re\left(\tilde{C}_1\right) &= 0.5\left[0.5\times1+\left(1-0.5\right)\times0.5\right]+\left(1-0.5\right)\left[0.5\times1+\left(1-0.5\right)\times0.2-\right.\\&\left.\frac{1}{3}\left(0.5\times0+\left(1-0.5\right)\times0+0\right)\right]=1\end{aligned}$$

$$\begin{aligned}\Re\left(\tilde{A}_2\right) &= 0.5\left[0.5\times1+\left(1-0.5\right)\times0.5\right]+\left(1-0.5\right)\left[0.5\times-0.2+\left(1-0.5\right)\times-0.4-\right.\\&\left.\frac{1}{3}\left(0.5\times0.1+\left(1-0.5\right)\times0.1+0.141213\right)\right]=0.3098\end{aligned}$$

$$\Re\left(\tilde{B}_2\right)=0.5\left[0.5\times 1+(1-0.5)\times 1\right]+(1-0.5)\Big[0.5\times 0.4+(1-0.5)\times 0.2-\frac{1}{3}\left(0.5\times 0.1+(1-0.5)\times 0.1+0.14142\right)\Big]=0.6098$$

Based on Fig. 2.4, a comparison of the ranking values of the method presented in this chapter with some existing methods [13, 22, 25, 33, 37, 48] are shown in Table 2.1



Set 1 Set 2

Fig. 2.4 Two sets of fuzzy numbers.

Table 2.1 A comparison of ranking values with different methods.

Ranking Methods	Set 1			Set 2	
	A	B	C	A	B
Yager's method [48]	0.3	0.3	-	-0.3	0.3
Murakami et al.'s method [37]	0.3	0.3	-	-0.3	0.3
Cheng's method [22]	0.5831	0.5831	-	0.5831	0.5831
Chen-and-Chen's method [13]	1.2359	1.2674	2	0.6359	1.2359
Deng-and-Liu's method [25]	0.6214	0.6244	1	0.3756	0.6214
The presented method [33]	0.6098	0.6299	1	0.3098	0.6900

From Table 2.1, we can see the drawbacks of the existing ranking methods, which are described as follows:

i. The fuzzy numbers $\tilde{A}$ and $\tilde{B}$ of Set 1 are different, but, by the existing methods [22, 37, 48], same ranking values are obtained..

ii. The existing methods [22, 37, 48] presented by cannot calculate the ranking value of the fuzzy number $\tilde{C}$ due to the fact that they make the denominator to zero.

iii. The fuzzy numbers $\tilde{A}$ and $\tilde{B}$ of Set 2 are different, however, the method presented by Cheng [11] gets the same ranking values.

Table 2.1, also shows that the fuzzy ranking method presented in this chapter and the fuzzy ranking methods presented by Chen and Chen [15] and Deng and Liu [25] get the same ranking results. i.e., they get the same ranking order: "$\tilde{C} > \tilde{B} > \tilde{A}$" in Set 1 and get the same ranking order: "$\tilde{B} > \tilde{A}$" in Set 2.

2.5 Fuzzy risk analysis problems

Assume that $\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n$ are *n* manufactories and assume that the component $\tilde{A}_i$ consists of *p* sub-components $\tilde{A}_{i1}, \tilde{A}_{i2}, ..., \tilde{A}_{ip}$ made by manufactory $\tilde{C}_i$. Assume that two evaluating items $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ are used to evaluate each sub-component $\tilde{A}_{ik}$ to obtain the probability of failure $\tilde{R}_i$ of the manufactory $\tilde{C}_i$, where $\tilde{R}_{ik}$ denotes the probability of failure of the sub-component $\tilde{A}_{ik}$, $\tilde{W}_{ik}$ denotes the severity of loss of the sub-component $\tilde{A}_{ik}$, $1 \leq i \leq n$ and $1 \leq k \leq p$. The structure we used to analyze the probability of failure $\tilde{R}_i$ of component $\tilde{A}_i$ made by manufactory C_i is shown in Fig. 2.5 [39].

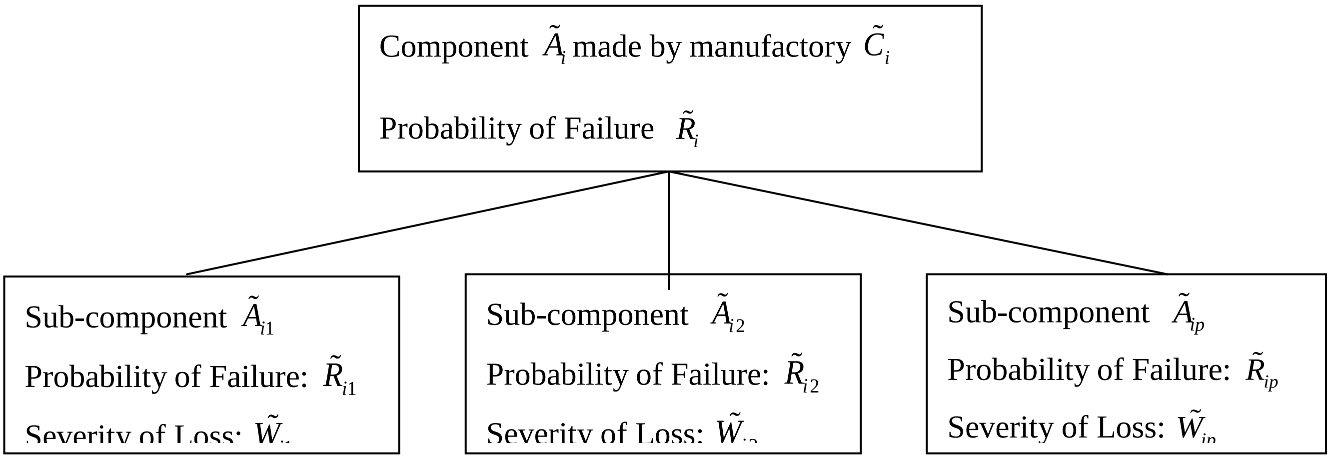


Fig. 2.5 The structure for analyzing the probability of failure R_i of component A_i made by manufactory C_i [39].

2.5.1 Fuzzy risk analysis algorithm

In this section, the existing algorithm [33] for fuzzy risk analysis problems is presented. The step of the existing algorithm [33] is as follow:

Step 1. Aggregate the evaluating items $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ of each sub-component $\tilde{A}_{ik}$ of each component $\tilde{A}_i$ made by manufactory $\tilde{C}_i$ by using Eq. (2.4) to obtain the probability of failure $\tilde{R}_i$ of each component $\tilde{A}_i$ made by manufactory $\tilde{C}_i$, shown as follows:

$$\tilde{R}_i = \frac{\sum_{k=1}^{p}\left(\tilde{W}_{ik} + \tilde{R}_{ik}\right)}{\sum_{k=1}^{p}\tilde{W}_{ik}} \tag{2.4}$$

where, $\tilde{R}_i = \left(r_{i1},\ r_{i2}, r_{i3}, r_{i4}; LR_{iH}, RR_{iH}\right), \tilde{R}_{ik} = \left(r_{ik1},\ r_{ik2}, r_{ik3}, r_{ik4}; LR_{ikH}, RR_{ikH}\right)$

$\tilde{W}_{ik} = \left(w_{ik1},\ w_{ik2}, w_{ik3}, w_{ik4}; LW_{ikH}, RW_{ikH}\right)$ $1 \le i \le n$ and $1 \le k \le p$

Step 2. With the help of Eq. 2.3, choose two proper values of parameters α and β and calculate the ranking values $\Re\left(R_i\right)$ of the probabilities of failure $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, ..., \tilde{R}_n$ of the n manufactories $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, ..., \tilde{C}_n$, respectively. The bigger the ranking value $\Re\left(\tilde{R}_i\right)$ of the probability of failure $\tilde{R}_i$, the higher the risk of the manufactory $\tilde{C}_i$, where $1 \leq i \leq n$.

Example 2.2 [11] Assume that $\tilde{C}_1, \tilde{C}_2$ and $\tilde{C}_3$ are three manufactories and assume that the component $\tilde{A}_i$ consists of three sub-components $\tilde{A}_{i1}, \tilde{A}_{i2}$ and $\tilde{A}_{i3}$ made by manufactory $\tilde{C}_i$, Assume that two evaluating items $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ are used to evaluate each sub-component $\tilde{A}_{ik}$ to obtain the probability of failure $\tilde{R}_i$ of the manufactory $\tilde{C}_i$ where $\tilde{R}_{ik}$ denotes the probability of failure of the sub-component $\tilde{A}_{ik}$, $\tilde{W}_{ik}$ denotes the severity of loss of the sub-component $\tilde{A}_{ik}$.

Table 2.2 The severity of loss W_{ik} and probability of failure R_{ik} of the component A_{ik} .

Sub-components	Severity of loss	Probability of failure
$\tilde{A}_{11}$	$\tilde{W}_{11} = \left(0.04, 0.1, 0.18, 0.23; 1.0, 1.0\right)$	$\tilde{R}_{11} = \left(0.17, 0.22, 0.36, 0.42; 0.9, 0.9\right)$
$\tilde{A}_{12}$	$\tilde{W}_{12} = \left(0.58, 0.63, 0.80, 0.86; 1.0, 1.0\right)$	$\tilde{R}_{12} = \left(0.32, 0.41, 0.58, 0.65; 0.7, 0.7\right)$
$\tilde{A}_{13}$	$\tilde{W}_{13} = \left(0.0, 0.0, 0.02, 0.07; 1.0, 1.0\right)$	$\tilde{R}_{13} = \left(0.58, 0.63, 0.80, 0.86; 0.8, 0.8\right)$
$\tilde{A}_{21}$	$\tilde{W}_{21} = \left(0.04, 0.1, 0.18, 0.23; 1.0, 1.0\right)$	$\tilde{R}_{21} = \left(0.93, 0.98, 1.0, 1.0; 0.85, 0.85\right)$
$\tilde{A}_{22}$	$\tilde{W}_{22} = \left(0.58, 0.63, 0.80, 0.86; 1.0, 1.0\right)$	$\tilde{R}_{22} = \left(0.58, 0.63, 0.80, 0.86; 0.9, 0.9\right)$
$\tilde{A}_{23}$	$\tilde{W}_{23} = \left(0.0, 0.0, 0.02, 0.07; 1.0, 1.0\right)$	$\tilde{R}_{23} = \left(0.32, 0.41, 0.58, 0.65; 0.9, 0.9\right)$
$\tilde{A}_{31}$	$\tilde{W}_{31} = \left(0.04, 0.1, 0.18, 0.23; 1.0, 1.0\right)$	$\tilde{R}_{31} = \left(0.17, 0.22, 0.36, 0.42; 0.95, 0.95\right)$
$\tilde{A}_{32}$	$\tilde{W}_{32} = \left(0.58, 0.63, 0.80, 0.86; 1.0, 1.0\right)$	$\tilde{R}_{32} = \left(0.72, 0.78, 0.92, 0.97; 0.8, 0.8\right)$
$\tilde{A}_{33}$	$\tilde{W}_{33} = \left(0.0, 0.0, 0.02, 0.07; 1.0, 1.0\right)$	$\tilde{R}_{33} = \left(0.58, 0.63, 0.80, 0.86; 1.0, 1.0\right)$

Assume that $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ of the sub-component $\tilde{A}_{ik}$ are as shown in Table 2.2, respectively, where $1 \le i \le 3$ and $1 \le k \le 3$ and take $\alpha = 0.5$ and $\beta = 0.5$.

Step 1. Based on Eq. 2.4, the probability of failure $\tilde{R}_i$ of the component $\tilde{A}_i$ made by manufactory $\tilde{C}_i$ can be calculated, where $1 \le i \le 3$, shown as follows:

$$\tilde{R}_1 = \left(\tilde{W}_{11} \otimes \tilde{R}_{11} \oplus \tilde{W}_{12} \otimes \tilde{R}_{12} \oplus \tilde{W}_{13} \otimes \tilde{R}_{13}\right) \oslash \left(\tilde{W}_{11} \oplus \tilde{W}_{12} \oplus \tilde{W}_{13}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.17, 0.22, 0.36, 0.42; 0.9, 0.9) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1.0) $\otimes$ (0.32, 0.41, 0.58, 0.65; 0.7, 0.7) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0) $\otimes$ (0.58, 0.63, 0.80, 0.86; 0.8, 0.8)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1:0) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)]

= (0.1659, 0.2803, 0.7463, 1.1545; 0.7, 0.7)

$$\tilde{R}_2 = \left(\tilde{W}_{21} \otimes \tilde{R}_{21} \oplus \tilde{W}_{22} \otimes \tilde{R}_{22} \oplus \tilde{W}_{23} \otimes \tilde{R}_{23}\right) \oslash \left(\tilde{W}_{21} \oplus \tilde{W}_{22} \oplus \tilde{W}_{23}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.93, 0.98, 1.0, 1.0; 0.85, 0.85) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1.0) $\otimes$ (0.58, 0.63, 0.80, 0.86; 0.9, 0.9) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0) $\otimes$ (0.32, 0.41, 0.58, 0.65; 0.9, 0.9)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1:0) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)]

= (0.3221, 0.4949, 1.1392, 1.6373; 0.85, 0.85)

$$\tilde{R}_3 = \left(\tilde{W}_{31} \otimes \tilde{R}_{31} \oplus \tilde{W}_{32} \otimes \tilde{R}_{32} \oplus \tilde{W}_{33} \otimes \tilde{R}_{33}\right) \oslash \left(W_{31} \oplus W_{32} \oplus W_{33}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.17, 0.22, 0.36, 0.42; 0.95, 0.95) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1.0) $\otimes$ (0.72, 0.78, 0.92, 0.97; 0.8, 0.8) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0) $\otimes$ (0.58, 0.63, 0.80, 0.86; 1.0, 1.0)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1:0) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)]

= (0.3659, 0.5134, 1.1189, 1.5984; 0.8, 0.8)

Step 2. Now, $\alpha = 0.5$ and $\beta = 0.5$, using equation 2.3, the ranking value $\Re\left(\tilde{R}_1\right)$, $\Re\left(\tilde{R}_2\right)$ and $\Re\left(\tilde{R}_3\right)$ of probability of failures $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$ of the manufactories $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ can be calculated, where

$$\begin{aligned}\Re\left(\tilde{R}_1\right) &= 0.5\left[0.5\times 0.7+(1-0.5)\times 0.7\right]+(1-0.5)\left[0.5\times 0.9504+(1-0.5)\times 0.2231-\right.\\ &\quad \left.\frac{1}{3}\left(0.5\times 0.2041+(1-0.5)\times 0.0572+0.3933\right)\right]=0.5560\\ \Re\left(\tilde{R}_2\right) &= 0.5\left[0.5\times 0.85+(1-0.5)\times 0.85\right]+(1-0.5)\left[0.5\times 1.3883+(1-0.5)\times 0.4085-\right.\\ &\quad \left.\frac{1}{3}\left(0.5\times 0.2491+(1-0.5)\times 0.0864+0.5241\right)\right]=0.7589\\ \Re\left(\tilde{R}_3\right) &= 0.5\left[0.5\times 0.8+(1-0.5)\times 0.8\right]+(1-0.5)\left[0.5\times 1.3587+(1-0.5)\times 0.4397-\right.\\ &\quad \left.\frac{1}{3}\left(0.5\times 0.2398+(1-0.5)\times 0.0738+0.4925\right)\right]=0.0.7414\end{aligned}$$

Therefore, the ranking order of the probabilities of failures $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$ of the three manufactories $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ is" $\tilde{R}_1 < \tilde{R}_3 < \tilde{R}_2$". It indicates that the ranking order of the risk of the three manufactories $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ is ‘‘ $\tilde{C}_1 < \tilde{C}_3 < \tilde{C}_2$”. The result coincides with the existing method [13, 22, 25, 33, 37, 48].

Chapter 3

Fuzzy Risk Analysis Based on Ranking Generalized Fuzzy Numbers with Different Left Heights and Right Heights

3.1 Introduction

In this chapter, an existing method [19] for fuzzy risk analysis based fuzzy ranking method for ranking generalized fuzzy numbers with different left heights and right heights is presented to overcome the drawbacks of the existing fuzzy ranking method [19]. Then, a method for fuzzy risk analysis based on the fuzzy ranking method, in which evaluating values are represented by generalized fuzzy numbers is presented.

3.2 A method for ranking generalized fuzzy numbers with different left heights and right heights

In this section, an existing a method for ranking generalized fuzzy number with different left heights and right heights is reviewed. Also some properties, which are used in existing method [19] are discussed.

3.2.1 Existing method

Let n generalized fuzzy numbers $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, .., \tilde{A}_n$ to be ranked, where $\tilde{A}_i = \left(x_{i1}, x_{i2}, x_{i3}, x_{i4};\ \mu_{iL}, \mu_{iR}\right), -\infty \le x_1 \le x_2 \le x_3 \le x_4 \le \infty\ ;\mu_{iL} \in \left[0,1\right], \mu_{iR} \in \left[0,1\right], \mu_{iL}$ denotes the left height of the fuzzy number $\tilde{A}$, μ_{iR} denotes the right height of the fuzzy number $\tilde{A}$ and $1 \le i \le n$. The steps of method for ranking generalized fuzzy number with different left heights and right heights calculates the area on the positive side, the area of negative side, the centroid of generalized fuzzy numbers to evaluate the ranking scores of generalized fuzzy number are as follows**:**

Step 1. Calculate the left area and the right area of each generalized fuzzy number. The area of generalized fuzzy number $\tilde{A}_i$ is divided into the left negative area LN_i and the right negative area RN_i, the left positive area LP_i, and the right positive area RP_i, where $1 \le i \le n$, describe as follows:

Case 1. The Left Negative area LN_i denotes the area from the generalized fuzzy number $\left(-1,-1,-1,-1;\max\left\{\mu_L,\mu_R\right\},\max\left\{\mu_L,\mu_R\right\}\right)$ shown in Figs. 3.1(a), 3.2(a) and 3.3(a) to the membership function curves of g_1 or g_2 are depending on the values of μ_L and μ_R:

Case (1a). If $\mu_L > \mu_R$ or $\mu_L = \mu_R$. Then

$$LN_i = \int_{-1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} g_1(x)\,dx \tag{3.1}$$

Case (1b). If $\mu_L < \mu_R$. Then

$$LN_i = \int_{-1}^{x_3} f(x)\,dx - \left(\int_{x_1}^{x_2} g_1(x)\,dx + \int_{x_2}^{x_3} g_2(x)\,dx \right) \tag{3.2}$$

Case 2. The Right Negative area RN_i denotes the area from the generalized fuzzy number $\left(-1,-1,-1,-1;\max\left\{\mu_L,\mu_R\right\},\max\left\{\mu_L,\mu_R\right\}\right)$shown in Figs. 3.1(b), 3.2(b) and 3.3(b) to the membership function curves of g_3 are depending on the values of μ_L and μ_R:

Case (2a). If $\mu_L < \mu_R$ or $\mu_L = \mu_R$. Then

$$RN_i = \int_{-1}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} g_3(x)\,dx \tag{3.3}$$

Case (2b). If $\mu_L > \mu_R$ Then

$$RN_i = \int_{-1}^{x_2} f(x)\,dx + \left(\int_{x_2}^{x_3} g_2(x)\,dx + \int_{x_3}^{x_4} g_3(x)\,dx \right) \tag{3.4}$$

Case 3. The left positive area LP_i denotes the area from the generalized fuzzy number $\left(1,1,1,1;\max\{\mu_L,\mu_R\},\max\{\mu_L,\mu_R\}\right)$ shown in Figs. 3.1(c), 3.2(c) and 3.3(c) to the membership function curves of g_1 are depending on the values of μ_L and μ_R:

Case (3a). If $\mu_L > \mu_R$ or $\mu_L = \mu_R$. Then

$$LP_i = \int_{x_2}^{1} f(x)\,dx + \int_{x_1}^{x_2} g_1(x)\,dx \tag{3.5}$$

Case (3b). If $\mu_L < \mu_R$. Then

$$LP_i = \int_{x_3}^{1} f(x)\,dx + \left(\int_{x_1}^{x_2} g_1(x)\,dx + \int_{x_2}^{x_3} g_2(x)\,dx \right) \tag{3.6}$$

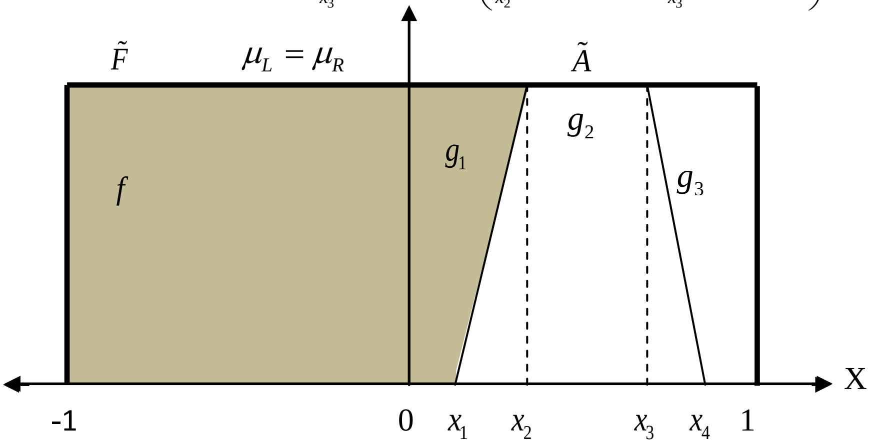
Case 4 The right positive area RP_i denotes the area from the generalized fuzzy number $\left(1,1,1,1;\ \max\{\mu_L,\mu_R\},\max\{\mu_L,\mu_R\}\right)$ shown in Figs. 3.1(d), 3.2(d) and 3.3(d) to the membership function curves of g_2 or g_3 are depending on the values of μ_L and μ_R:

Case (4a). If $\mu_L < \mu_R$ or $\mu_L = \mu_R$. Then

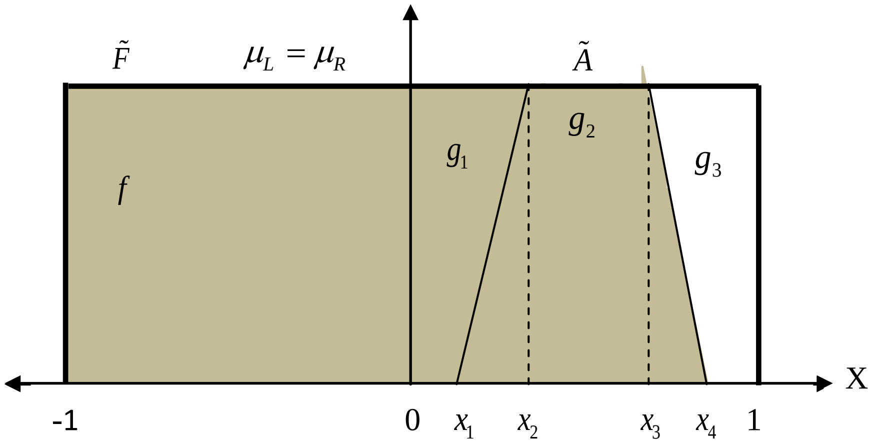
$$RP_i = \int_{x_3}^{1} f(x)\,dx - \int_{x_3}^{x_4} g_3(x)\,dx \tag{3.7}$$

Case 4b. If $\mu_L > \mu_R$. Then

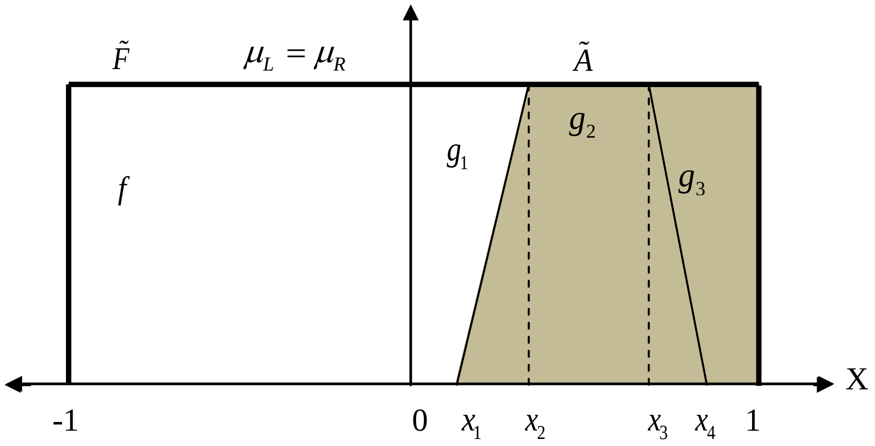
$$RP_i = \int_{x_3}^{1} f(x)\,dx - \left(\int_{x_2}^{x_3} g_2(x)\,dx + \int_{x_3}^{x_4} g_3(x)\,dx \right) \tag{3.8}$$



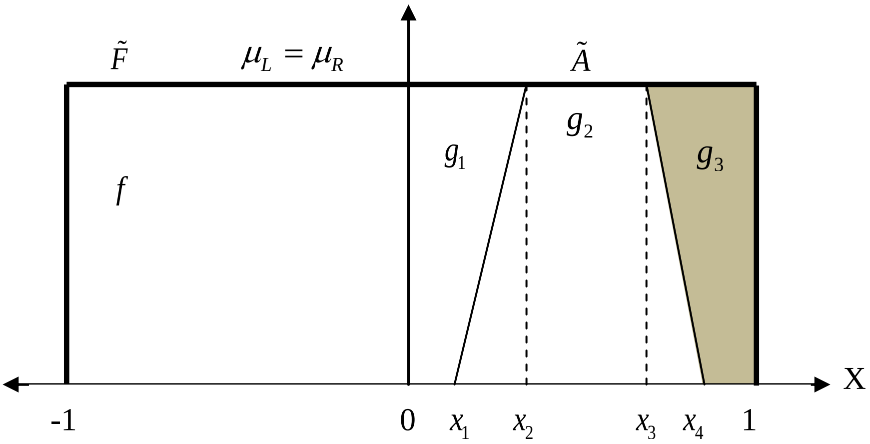
(a)



(b)

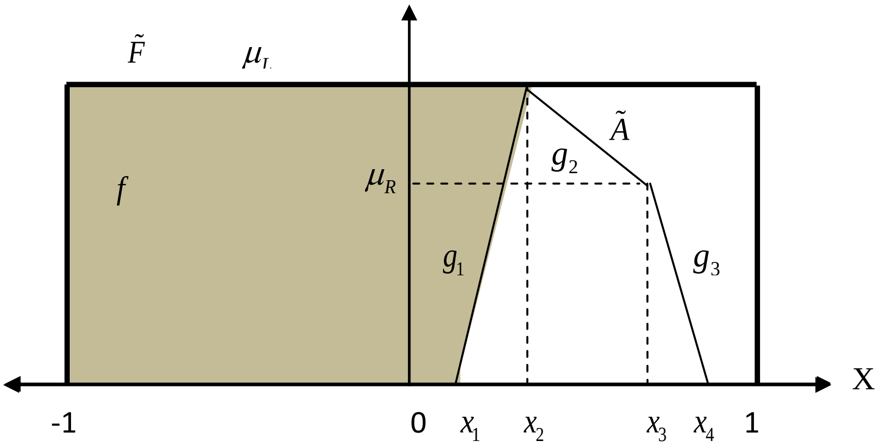


(c)

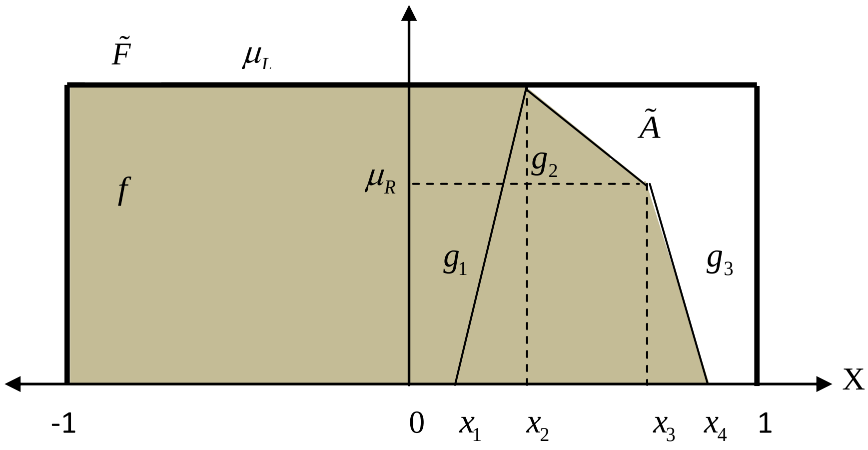


(d)

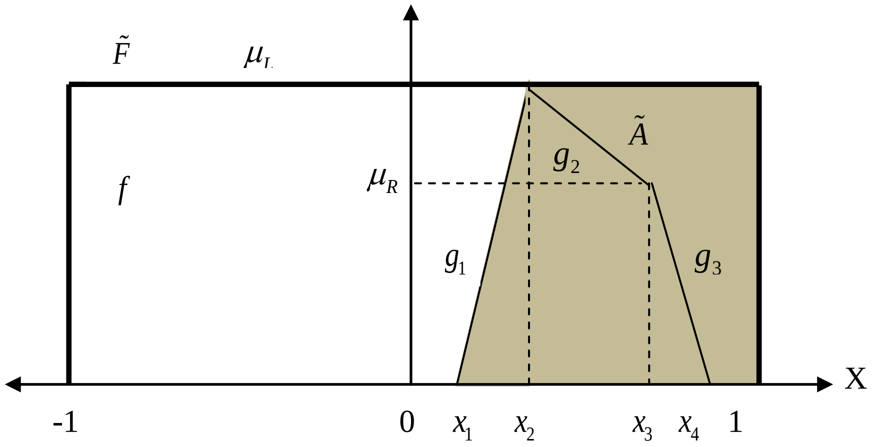
Fig. 3.1 The areas on the negative side and positive side of generalized fuzzy number $\tilde{A}_i$ when $\mu_L = \mu_R$, (a) LN_i, (b) RN_i, (c) LP_i, (d) RP_i.



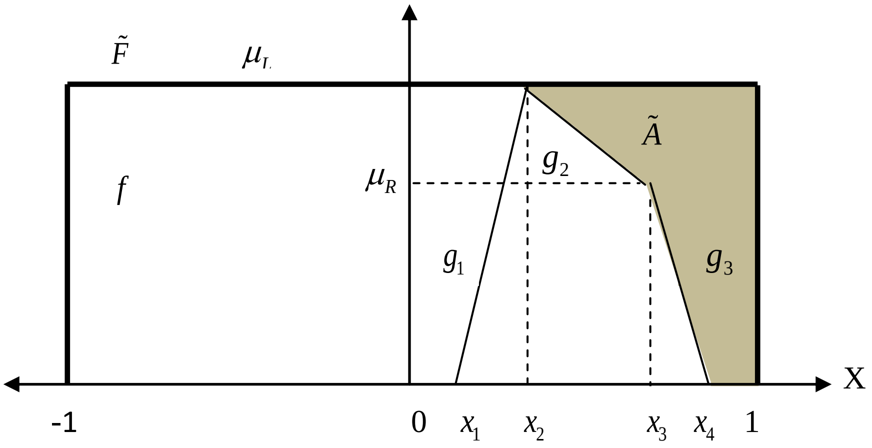
(a)



(b)

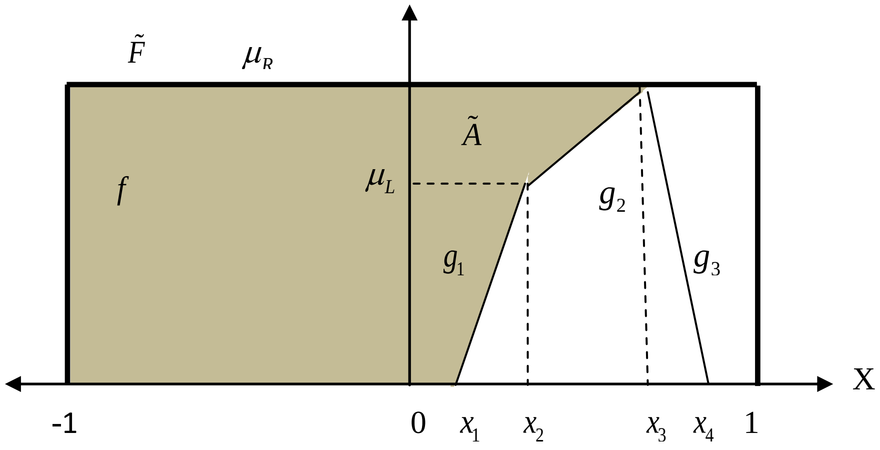


(c)

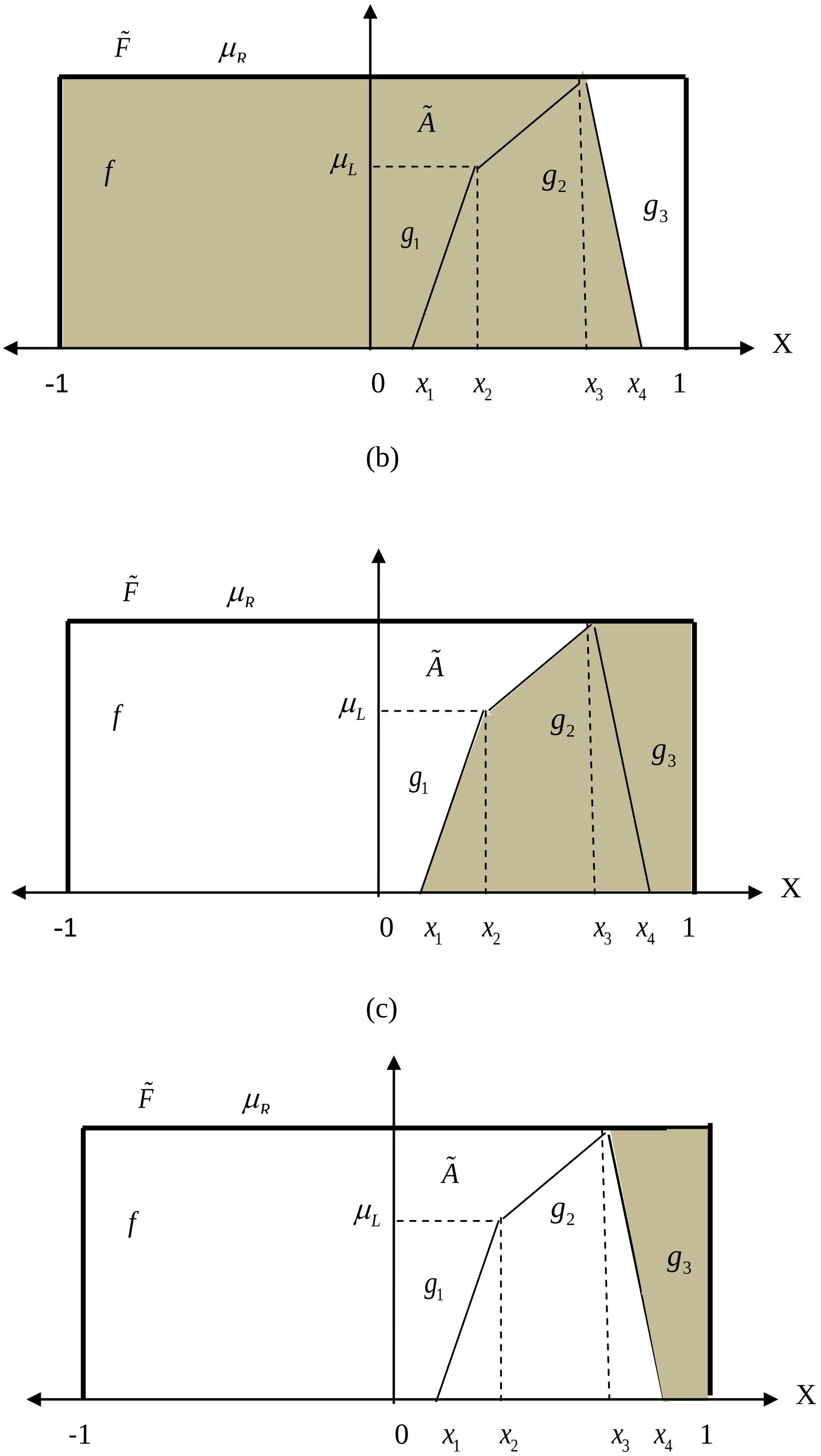


(d)

Fig. 3.2 The areas on the negative side and positive side of generalized fuzzy number $\tilde{A}_i$ when $\mu_L > \mu_R$, (a) LN_i, (b) RN_i, (c) LP_i, (d) RP_i.



(a)



(b)

(c)

(d)

Fig. 3.3 The area on the negative side and positive side of generalized fuzzy number $\tilde{A}_i$ when μ_L < μ_R, (a) LN_i, (b) RN_i, (c) LP_i, (d) RP_i.

i. when $\mu_L = \mu_R$, the area of $\mathrm{LN_i}$, RN_i, LP_i and $\mathrm{RP_i}$ are shown in Fig. 3.1 respectively. .

ii. when $\mu_L > \mu_R$, the area of $\mathrm{LN_i}$, RN_i, LP_i and $\mathrm{RP_i}$ are shown in Fig. 3.2 respectively.

iii. when $\mu_L < \mu_R$, the area of $\mathrm{LN_i}$, RN_i, LP_i and $\mathrm{RP_i}$ are shown in Fig. 3.3 respectively.

Step 2. Calculate the sum M_i of the negative areas and the sum N_i of the positive areas of each generalized fuzzy number $\tilde{A}_i$ where

$$M_i = LN_i + RN_i \quad , \qquad 1 \le i \le n \tag{3.9}$$

$$N_i = LP_i + RP_i \quad , \qquad 1 \le i \le n \tag{3.10}$$

Step 3. Calculate the centroid $c\left(\tilde{A}_i\right)$ of each generalized fuzzy number $\tilde{A}_i$, shown as follows:

$$c\left(\tilde{A}_i\right) = \frac{\sum_{k=1}^{n} \mu_{ik}\, x_{ik}}{\sum_{k=1}^{n} \mu_{ik}} \quad , \qquad 1 \le i \le n \tag{3.11}$$

Step 4: Calculate the ranking $Score\left(\tilde{A}_i\right)$ of each generalized fuzzy number $\tilde{A}_i$, shown as follow

$$Score\left(\tilde{A}_i\right) = \frac{M_i - N_i}{M_i + N_i + \left(1 - |c\left(\tilde{A}_i\right)|\right)} \quad , \qquad 1 \le i \le n \tag{3.12}$$

The larger the value of $Score\left(\tilde{A}_i\right)$, the better the ranking order of $\tilde{A}_i$, where $1 \le i \le n$.

3.2.2 Proporties

Property 1 (Zero property)

Let $\tilde{A}_1 = \left(x_1, x_2, x_3, x_4; \mu_L, \mu_R\right)$ be a generalized fuzzy number. If $-1 \le x_1 \le x_2 \le x_3 \le x_4 \le 1$, $\mu_L = \mu_R$ and $x_1 + x_2 + x_3 + x_4 = 0$, then $Score\left(\tilde{A}_1\right)$=0.

Proof: To satisfy the equation $x_1 + x_2 + x_3 + x_4 = 0$, let $x_3 = -x_2$ and $x_4 = x_1$, such that the equation becomes $x_1 + x_2 + x_3 + x_4 = 0$. If $\tilde{A}_1 = (x_1, x_2, x_3, x_4; \mu_L, \mu_R)$, where $-1 \le x_1 \le x_2 \le x_3 \le x_4 \le 1$ $\mu_L = \mu_R$ and $x_1 + x_2 + x_3 + x_4 = 0$, as shown in Fig. 3.4. From equation 3.1, 3.3, 3.5 and 3.7, we can see that

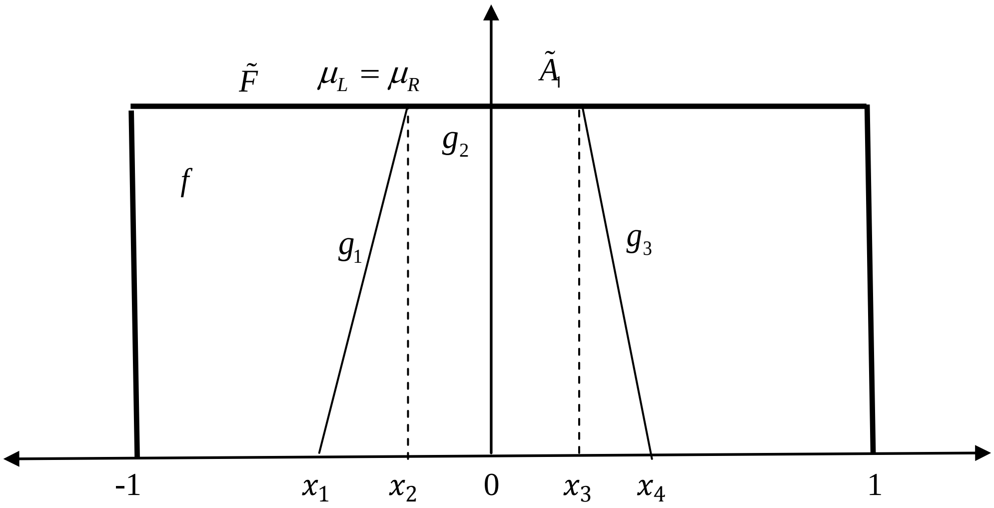


Fig. 3.4 Generalized fuzzy number $\tilde{A}_1$ and $\tilde{F}$.

$$LN_1 = \int_{-1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} g_1(x)\,dx$$

$$LP_1 = \int_{x_2}^{1} f(x)\,dx + \int_{x_1}^{x_2} g_1(x)\,dx$$

$$RN_1 = \int_{-1}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} g_3(x)\,dx$$

$$RP_1 = \int_{x_3}^{1} f(x)\,dx - \int_{x_3}^{x_4} g_3(x)\,dx$$

Then, based on equation 3.9 and 3.10,

$$M_1 = LN_1 + RN_1 = \int_{-1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} g_1(x)\,dx + \int_{-1}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} g_3(x)\,dx$$

$$N_1 = LP_1 + RP_1 = \int_{x_2}^{1} f(x)\,dx + \int_{x_1}^{x_2} g_1(x)\,dx + \int_{x_3}^{1} f(x)\,dx - \int_{x_3}^{x_4} g_3(x)\,dx$$

Therefore,

$$M_1 - N_1 = \left(\int_{-1}^{x_2} f(x)dx - \int_{x_1}^{x_2} g_1(x)dx + \int_{-1}^{x_3} f(x)dx + \int_{x_3}^{x_4} g_3(x)dx \right) - \left(\int_{x_2}^{1} f(x)dx + \int_{x_1}^{x_2} g_1(x)dx + \int_{x_3}^{1} f(x)dx - \int_{x_3}^{x_4} g_3(x)dx \right)$$

$$= \int_{-1}^{x_2} f(x)dx - \int_{x_2}^{1} f(x)dx - 2\int_{x_1}^{x_2} g_1(x)dx + \int_{-1}^{x_3} f(x)dx - \int_{x_3}^{1} f(x)dx + 2\int_{x_3}^{x_4} g_3(x)dx$$

The area of $\int_{x_3}^{x_4} g_3(x)dx$ is equal to the area of $\int_{x_1}^{x_2} g_1(x)dx$, the area of $\int_{-1}^{x_2} f(x)dx$ is equal to area of $\int_{x_3}^{1} f(x)dx$, and the area of $\int_{x_2}^{1} f(x)dx$ is equal to the area of $\int_{-1}^{x_3} f(x)dx$ shown in Fig. 3.4

$$M_1 - N_1 = \int_{x_3}^{1} f(x)dx - \int_{-1}^{x_3} f(x)dx - 2\int_{x_1}^{x_2} g_1(x)dx + \int_{-1}^{x_3} f(x)dx - \int_{x_3}^{1} f(x)dx + 2\int_{x_1}^{x_2} g_1(x)dx = 0$$

$$M_1 + N_1 = \left(\int_{-1}^{x_2} f(x)dx - \int_{x_1}^{x_2} g_1(x)dx + \int_{-1}^{x_3} f(x)dx + \int_{x_3}^{x_4} g_3(x)dx \right) + \left(\int_{x_2}^{1} f(x)dx + \int_{x_1}^{x_2} g_1(x)dx + \int_{x_3}^{1} f(x)dx - \int_{x_3}^{x_4} g_3(x)dx \right)$$

$$= \int_{-1}^{x_2} f(x)dx + \int_{x_2}^{1} f(x)dx + \int_{-1}^{x_3} f(x)dx + \int_{x_3}^{1} f(x)dx$$

As $\int_{-1}^{x_2} f(x)dx + \int_{x_2}^{1} f(x)dx = \int_{-1}^{1} f(x)$ and $\int_{-1}^{x_3} f(x)dx + \int_{x_3}^{1} f(x)dx = \int_{-1}^{1} f(x)\, dx$ Therefore,

$$= \int_{-1}^{1} f(x)dx + \int_{-1}^{1} f(x)dx = 2\int_{-1}^{1} f(x)dx$$

From Fig. 3.4 and based on equation 3.11,

$$c\left(\tilde{A}_1\right) = \frac{\sum_{k=1}^{n} \mu_{1k}\, x_{1k}}{\sum_{k=1}^{n} \mu_{1k}} = \frac{0 \times x_1 + \mu_L \times x_2 + \mu_R \times x_3 + 0 \times x_4}{0 + \mu_L + \mu_R + 0}$$

$$= \frac{\mu_L \times x_2 + \mu_R\, x_3}{\mu_L + \mu_R}$$

Because $\mu_L = \mu_R = \mu$, we get

$$c\left(\tilde{A}_1\right) = \frac{\mu\left(x_2 + x_3\right)}{2\mu} = \frac{\left(x_2 + x_3\right)}{\mu}$$

On using equation 3.12, the ranking is

$$Score\left(\tilde{A}_1\right) = \frac{M_1 - N_1}{M_1 + N_1 + \left(1 - \left|c\left(\tilde{A}_1\right)\right|\right)} = \frac{0}{2\int_{-1}^{1} f(x)dx + \left(1 - \left|\frac{x_2 + x_3}{2}\right|\right)} = 0$$

Property 2 (Symmetric property)

If $\tilde{A}_1 = \left(x_1, x_2, x_3, x_4; \mu_L, \mu_R\right)$ and $\tilde{A}_2 = \left(-x_4, -x_3, -x_2, -x_1; \mu_L, \mu_R\right)$, where $-1 \le x_1 \le x_2 \le x_3 \le x_4 \le 1$ and $\mu_L = \mu_R$, then $Score\left(\tilde{A}_1\right) = -Score\left(\tilde{A}_2\right)$.

Proof: If $\tilde{A}_1 = (x_1, x_2, x_3, x_4; \mu_L, \mu_R)$, where $-1 \le x_1 \le x_2 \le x_3 \le x_4 \le 1$ and $\mu_L = \mu_R$, as shown in Fig. 3.5, then based on Equation 3.1, 3.3, 3.5 and 3.7, we can see that

$$LN_1 = \int_{-1}^{x_2} f(x)\, dx - \int_{x_1}^{x_2} g_{11}(x)\, dx$$

$$LP_1 = \int_{x_2}^{1} f(x)\, dx + \int_{x_1}^{x_2} g_{11}(x)\, dx$$

$$RN_1 = \int_{-1}^{x_3} f(x)\, dx + \int_{x_3}^{x_4} g_{13}(x)\, dx$$

$$RP_1 = \int_{x_3}^{1} f(x)\, dx - \int_{x_3}^{x_4} g_{13}(x)\, dx$$

Based on Equation 3.9 and 3.10,

$$M_1 = LN_1 + RN_1 = \int_{-1}^{x_2} f(x)\, dx - \int_{x_1}^{x_2} g_{11}(x)\, dx + \int_{-1}^{x_3} f(x)\, dx + \int_{x_3}^{x_4} g_{13}(x)\, dx$$

$$N_1 = LP_1 + RP_1 = \int_{x_2}^{1} f(x)\, dx + \int_{x_1}^{x_2} g_{11}(x)\, dx + \int_{x_3}^{1} f(x)\, dx - \int_{x_3}^{x_4} g_{13}(x)\, dx$$

Therefore,

$$\begin{aligned} M_1 - N_1 = & \left(\int_{-1}^{x_2} f(x)dx - \int_{x_1}^{x_2} g_{11}(x)dx + \int_{-1}^{x_3} f(x)dx + \int_{x_3}^{x_4} g_{13}(x)dx \right) - \\ & \left(\int_{x_2}^{1} f(x)dx + \int_{x_1}^{x_2} g_{11}(x)dx + \int_{x_3}^{1} f(x)dx - \int_{x_3}^{x_4} g_{13}(x)dx \right) \end{aligned}$$

$$= \int_{-1}^{x_2} f(x)dx - \int_{x_2}^{1} f(x)dx - 2\int_{x_1}^{x_2} g_{11}(x)dx + \int_{-1}^{x_3} f(x)dx - \int_{x_3}^{1} f(x)dx + 2\int_{x_3}^{x_4} g_{13}(x)dx$$

$$M_1+N_1=\left(\int_{-1}^{x_2} f(x)dx-\int_{x_1}^{x_2} g_{11}(x)dx+\int_{-1}^{x_3} f(x)dx+\int_{x_3}^{x_4} g_{13}(x)dx\right)+$$
$$\left(\int_{x_2}^{1} f(x)dx+\int_{x_1}^{x_2} g_{11}(x)dx+\int_{x_3}^{1} f(x)dx-\int_{x_3}^{x_4} g_{13}(x)dx\right)$$

$$=\int_{-1}^{x_2} f(x)dx+\int_{x_2}^{1} f(x)dx+\int_{-1}^{x_3} f(x)dx+\int_{x_3}^{1} f(x)dx$$

As $\int_{-1}^{x_2} f(x)dx+\int_{x_2}^{1} f(x)dx=\int_{-1}^{1} f(x)$ and $\int_{-1}^{x_3} f(x)dx+\int_{x_3}^{1} f(x)dx=\int_{-1}^{1} f(x)\,dx$ Therefore,

$$M_1+N_1=\int_{-1}^{1} f(x)dx+\int_{-1}^{1} f(x)dx=2\int_{-1}^{1} f(x)dx$$

Using Equation 3.11 and Fig.3.5,

$$c\left(\tilde{A}_1\right)=\frac{\sum_{k=1}^{n}\mu_{1k}\,x_{1k}}{\sum_{k=1}^{n}\mu_{1k}}=\frac{0\times x_1+\mu_L\times x_2+\mu_R\times x_3+0\times x_4}{0+\mu_L+\mu_R+0}$$

$$=\frac{\mu_L\times x_2+\mu_R\,x_3}{\mu_L+\mu_R}$$

Because $\mu_L=\mu_R=\mu$,

$$c\left(\tilde{A}_1\right)=\frac{\mu\left(x_2+x_3\right)}{2\mu}=\frac{x_2+x_3}{\mu}$$

Based on Equation 3.12, the ranking $Score\left(\tilde{A}_1\right)$ is calculated as follow:

$$Score\left(\tilde{A}_1\right)=\frac{M_1-N_1}{M_1+N_1+\left(1-\left|c\left(\tilde{A}_1\right)\right|\right)}$$

$$=\frac{\int_{-1}^{x_2} f\left(x\right)dx-\int_{x_2}^{1} f\left(x\right)dx-2\int_{x_1}^{x_2} g_{11}\left(x\right)dx+\int_{-1}^{x_3} f\left(x\right)dx-\int_{x_3}^{1} f\left(x\right)dx+2\int_{x_3}^{x_4} g_{13}\left(x\right)dx}{2\int_{-1}^{1} f\left(x\right)dx+\left(1-\left|\frac{x_2+x_3}{2}\right|\right)}$$

$$=\frac{\int_{-1}^{x_2} f\left(x\right)dx-\int_{x_2}^{1} f\left(x\right)dx-2\int_{x_1}^{x_2} g_{11}\left(x\right)dx+\int_{-1}^{x_3} f\left(x\right)dx-\int_{x_3}^{1} f\left(x\right)dx+2\int_{x_3}^{x_4} g_{13}\left(x\right)dx}{2\int_{-1}^{1} f\left(x\right)dx+\left(1-\frac{x_2+x_3}{2}\right)}$$

If $\tilde{A}_2=\left(-x_4,-x_3,-x_2,-x_1;\mu_L,\mu_R\right)$, where $-1\le x_1\le x_2\le x_3\le x_4\le 1$ and $\mu_L=\mu_R$, as shown in Fig. 3.5, then based on Equation 3.1, 3.3, 3.5 and 3.7,

$$LN_2=\int_{-1}^{-x_3} f\left(x\right)\,dx-\int_{-x_4}^{-x_3} g_{21}\left(x\right)\,dx$$

$$LP_2=\int_{-x_3}^{1} f\left(x\right)\,dx+\int_{-x_4}^{-x_3} g_{21}\left(x\right)\,dx$$

$$RN_2=\int_{-1}^{-x_2} f\left(x\right)\,dx+\int_{-x_2}^{-x_1} g_{23}\left(x\right)\,dx$$

$$RP_2=\int_{-x_2}^{1} f\left(x\right)\,dx-\int_{-x_2}^{-x_1} g_{23}\left(x\right)\,dx$$

Based on Equation 3.9 and 3.10,

$$M_2 = LN_2 + RN_2 = \int_{-1}^{-x_3} f(x)\,dx - \int_{-x_4}^{-x_3} g_{21}(x)\,dx + \int_{-1}^{-x_2} f(x)\,dx + \int_{-x_2}^{-x_1} g_{23}(x)\,dx$$

$$N_2 = LP_2 + RP_2 = \int_{-x_3}^{1} f(x)\,dx + \int_{-x_4}^{-x_3} g_{21}(x)\,dx + \int_{-x_2}^{1} f(x)\,dx - \int_{-x_2}^{-x_1} g_{23}(x)\,dx$$

Therefore,

$$\begin{aligned} M_2 - N_2 = &\left(\int_{-1}^{-x_3} f(x)dx - \int_{-x_4}^{-x_3} g_{21}(x)dx + \int_{-1}^{-x_2} f(x)dx + \int_{-x_2}^{-x_1} g_{23}(x)dx \right) - \\ &\left(\int_{-x_3}^{1} f(x)dx + \int_{-x_4}^{-x_3} g_{21}(x)dx + \int_{-x_2}^{1} f(x)dx - \int_{-x_2}^{-x_1} g_{23}(x)dx \right) \end{aligned}$$

$$= \int_{-1}^{-x_3} f(x)dx - \int_{-x_3}^{1} f(x)dx - 2\int_{-x_4}^{-x_3} g_{21}(x)dx + \int_{-1}^{-x_2} f(x)dx - \int_{-x_2}^{1} f(x)dx + 2\int_{-x_2}^{-x_1} g_{23}(x)dx$$

The area of $\int_{-x_4}^{-x_3} g_{21}(x)dx$ is equal to the area of $\int_{x_3}^{x_4} g_{13}(x)dx$, the area of $\int_{-x_2}^{-x_1} g_{23}(x)dx$ is equal to area of $\int_{x_1}^{x_2} g_{11}(x)dx$, the area of $\int_{-1}^{-x_3} f(x)dx$ is equal to the area of $\int_{x_3}^{1} f(x)dx$, the area of $\int_{-1}^{-x_2} f(x)dx$ is equal to the area of $\int_{x_2}^{1} f(x)dx$, the area of $\int_{-x_2}^{1} f(x)dx$ is equal to the area of $\int_{-1}^{x_2} f(x)dx$, and the area of $\int_{-x_3}^{1} f(x)dx$ is equal to the area of $\int_{-1}^{x_3} f(x)dx$. Therefore, we can get

$$M_2 - N_2 = \int_{x_3}^{1} f(x)dx - \int_{-1}^{x_3} f(x)dx - 2\int_{x_3}^{x_4} g_{13}(x)dx + \int_{x_2}^{1} f(x)dx - \int_{-1}^{x_2} f(x)dx + 2\int_{x_1}^{x_2} g_{11}(x)dx$$

$$= -\left(\int_{-1}^{x_2} f(x)dx - \int_{x_2}^{1} f(x)dx - 2\int_{x_1}^{x_2} g_{11}(x)dx + \int_{-1}^{x_3} f(x)dx - \int_{x_3}^{1} f(x)dx + 2\int_{x_3}^{x_4} g_{13}(x)dx \right)$$

$$M_2 + N_2 = \left(\int_{-1}^{-x_3} f(x)dx - \int_{-x_4}^{-x_3} g_{21}(x)dx + \int_{-1}^{-x_2} f(x)dx + \int_{-x_2}^{-x_1} g_{23}(x)dx \right) +$$
$$\left(\int_{-x_3}^{1} f(x)dx + \int_{-x_4}^{-x_3} g_{21}(x)dx + \int_{-x_2}^{1} f(x)dx - \int_{-x_2}^{-x_1} g_{23}(x)dx \right)$$

$$= \int_{-1}^{-x_3} f(x)dx - \int_{-x_3}^{1} f(x)dx + \int_{-1}^{-x_2} f(x)dx + \int_{-x_2}^{1} f(x)dx$$

As $\int_{-1}^{-x_3} f(x)dx - \int_{-x_3}^{1} f(x)dx = \int_{-1}^{1} f(x)$ and $\int_{-1}^{-x_2} f(x)dx + \int_{-x_2}^{1} f(x)dx = \int_{-1}^{1} f(x)\, dx$ Therefore,

$$M_2 + N_2 = \int_{-1}^{1} f(x)dx + \int_{-1}^{1} f(x)dx = 2\int_{-1}^{1} f(x)dx$$

Using Equation 3.11 and Fig. 3.5,

$$c\left(\tilde{A}_2\right) = \frac{\sum_{k=1}^{n} \mu_{2k}\, x_{2k}}{\sum_{k=1}^{n} \mu_{2k}} = \frac{0 \times -x_4 + \mu_L \times -x_3 + \mu_R \times -x_2 + 0 \times -x_1}{0 + \mu_L + \mu_R + 0}$$

$$= \frac{-\mu_L \times x_3 - \mu_R\, x_2}{\mu_L + \mu_R}$$

Because $\mu_L = \mu_R = \mu$,

$$c\left(\tilde{A}_2\right) = \frac{-\mu\left(x_3 + x_2\right)}{2\mu} = -\frac{x_3 + x_2}{\mu}$$

On using Equation 3.12, the ranking $Score(\tilde{A}_1)$ is calculated as follow:

$$Score\left(\tilde{A}_2\right)=\frac{M_1-N_1}{M_1+N_1+\left(1-\left|c\left(\tilde{A}_1\right)\right|\right)}$$

$$=\frac{-\left(\int_{-1}^{x_2} f(x)dx-\int_{x_2}^{1} f(x)dx-2\int_{x_1}^{x_2} g_{11}(x)dx+\int_{-1}^{x_3} f(x)dx-\int_{x_3}^{1} f(x)dx+2\int_{x_3}^{x_4} g_{13}(x)dx\right)}{2\int_{-1}^{1} f(x)dx+\left(1-\left|\frac{x_2+x_3}{2}\right|\right)}$$

$$=\frac{-\left(\int_{-1}^{x_2} f(x)dx-\int_{x_2}^{1} f(x)dx-2\int_{x_1}^{x_2} g_{11}(x)dx+\int_{-1}^{x_3} f(x)dx-\int_{x_3}^{1} f(x)dx+2\int_{x_3}^{x_4} g_{13}(x)dx\right)}{2\int_{-1}^{1} f(x)dx+\left(1-\frac{x_2+x_3}{2}\right)}$$

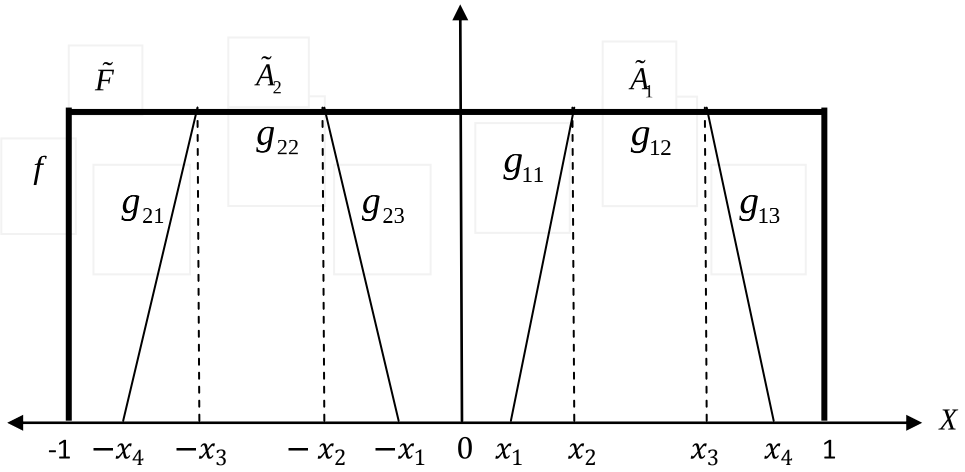


Fig. 3.5 Generalized fuzzy numbers $\tilde{A}_1=\left(x_1,x_2,x_3,x_4;\mu_L,\mu_R\right)$ and $\tilde{A}_2=\left(-x_4,-x_3,-x_2,-x_1;\mu_L,\mu_R\right)$

Therefore, $Score\left(\tilde{A}_1\right)=-Score\left(\tilde{A}_2\right)$

Property 3 If $\tilde{A}_1 = (1,1,1,1;\mu_L,\mu_R)$, where $\mu_L = \mu_R$. Then $Score(\tilde{A}_1) = 1$

Proof: If $\tilde{A}_1 = (1,1,1,1;\mu_L,\mu_R)$, where $\mu_L = \mu_R$, as shown in Fig. 3.6, then based on Equation 3.1, 3.3, 3.5 and 3.7, we can see that

$$LN_1 = \int_{-1}^{1} f(x)\,dx$$

$$LP_1 = \int_{1}^{1} f(x)\,dx = 0$$

$$RN_1 = \int_{-1}^{1} f(x)\,dx$$

$$RP_1 = \int_{1}^{1} f(x)\,dx = 0$$

Based on Equation 3.9 and 3.10,

$$M_1 = LN_1 + RN_1 = \int_{-1}^{1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx = 2\int_{-1}^{1} f(x)\,dx$$

$$N_1 = LP_1 + RP_1 = 0.$$

Therefore,

$$M_1 - N_1 = 2\int_{-1}^{1} f(x)dx - 0 = 2\int_{-1}^{1} f(x)dx$$

$$M_1 + N_1 = 2\int_{-1}^{1} f(x)dx + 0 = 2\int_{-1}^{1} f(x)dx$$

Using Equation 3.11 and Fig.3.6,

$$c\left(\tilde{A}_1\right) = \frac{\sum_{k=1}^{n} \mu_{1k}\ x_{1k}}{\sum_{k=1}^{n} \mu_{1k}} = \frac{0\times 1 + \mu_L \times 1 + \mu_R \times 1 + 0 \times 1}{0 + \mu_L + \mu_R + 0}$$

$$= \frac{\mu_L + \mu_R}{\mu_L + \mu_R} = 1$$

On using Equation 3.12, the ranking score $Score\left(\tilde{A}_1\right)$ is calculated as follow:

$$Score\left(\tilde{A}_1\right) = \frac{M_1 - N_1}{M_1 + N_1 + \left(1 - \left|c\left(\tilde{A}_1\right)\right|\right)} = \frac{\int_{-1}^{1} f\left(x\right) dx}{2\int_{-1}^{1} f\left(x\right) dx + \left(1 - \left|1\right|\right)} = 1$$

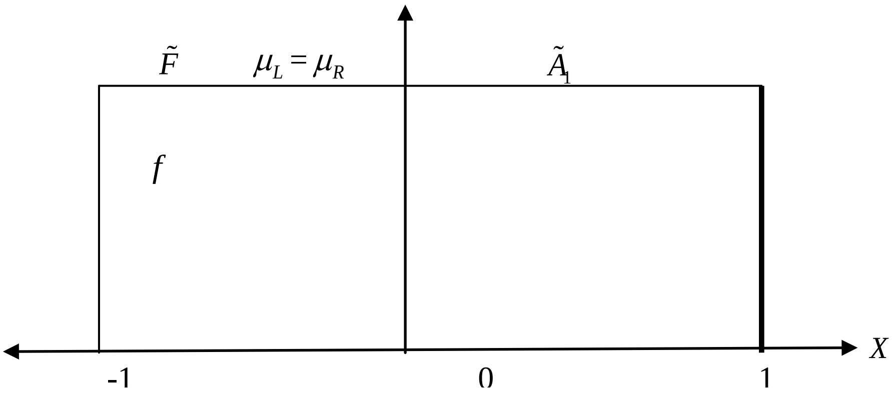


Fig. 3.6 Generalized fuzzy number $\tilde{A}_1 = \left(1,1,1,1; \mu_L, \mu_R\right)$

Property 4 If $\tilde{A}_1 = \left(0,0,0,0; \mu_L, \mu_R\right)$, where $\mu_L = \mu_R$. Then $Score\left(\tilde{A}_1\right) = 0$

Proof: If $\tilde{A}_1 = \left(0,0,0,0; \mu_L, \mu_R\right)$, where $\mu_L = \mu_R$, as shown in Fig. 3.7, then based on Equation 3.1, 3.3, 3.5 and 3.7, we can see that

$$LN_1 = \int_{-1}^{0} f\left(x\right)\ dx$$

$$LP_1 = \int_{0}^{1} f(x)\, dx$$

$$RN_1 = \int_{-1}^{0} f(x)\, dx$$

$$RP_1 = \int_{0}^{1} f(x)\, dx$$

Based on Equation 3.9 and 3.10,

$$M_1 = LN_1 + RN_1 = \int_{-1}^{0} f(x)\, dx + \int_{-1}^{0} f(x)\, dx = 2\int_{-1}^{0} f(x)\, dx$$

$$N_1 = LP_1 + RP_1 = \int_{0}^{1} f(x)\, dx + \int_{0}^{1} f(x)\, dx = 2\int_{0}^{1} f(x)\, dx .$$

Therefore,

$$M_1 - N_1 = 2\int_{-1}^{0} f(x) dx - 2\int_{-1}^{0} f(x) dx = 0$$

The area of $\int_{-1}^{0} f(x) dx$ is equal to the area of $\int_{0}^{1} f(x) dx$.

$$M_1 - N_1 = 2\int_{-1}^{0} f(x) dx - 2\int_{-1}^{0} f(x) dx = 0$$

$$M_1 + N_1 = 2\int_{-1}^{0} f(x) dx + 2\int_{0}^{1} f(x) dx = 2\int_{-1}^{1} f(x) dx$$

Using Equation 3.11 and Fig.3.7,

$$c\left(\tilde{A}_1\right)=\frac{\sum_{k=1}^{n}\mu_{1k}\ x_{1k}}{\sum_{k=1}^{n}\mu_{1k}}=\frac{0\times 0+\mu_L\times 0+\mu_R\times 0+0\times 0}{0+\mu_L+\mu_R+0}$$

$$=\frac{0}{\mu_L+\mu_R}=0$$

On using Equation 3.12, the ranking score $Score\left(\tilde{A}_1\right)$ is calculated as follow:

$$Score\left(\tilde{A}_1\right)=\frac{M_1-N_1}{M_1+N_1+\left(1-\left|c\left(\tilde{A}_1\right)\right|\right)}=\frac{0}{2\int_{-1}^{1}f\left(x\right)dx+\left(1-\left|0\right|\right)}=0$$

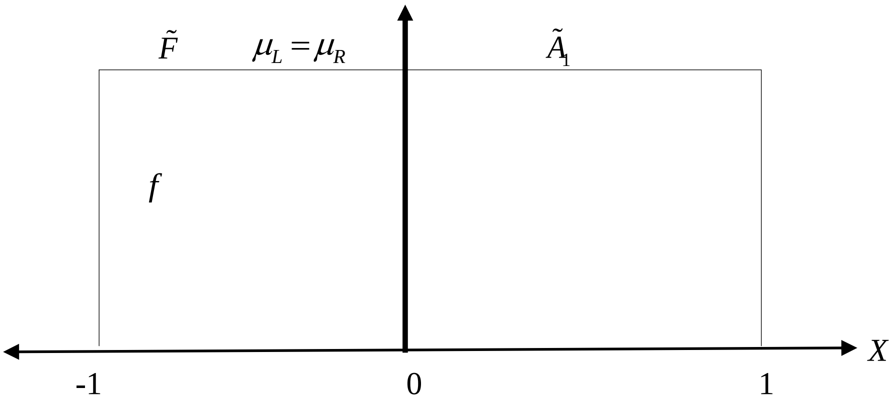


Fig.3.7 Generalized fuzzy numbers $\tilde{A}_1=\left(0,0,0,0;\mu_L,\mu_R\right)$

Property 5 If $\tilde{A}_1=\left(-1,-1,-1,-1;\mu_L,\mu_R\right)$, where $\mu_L=\mu_R$. Then $Score\left(\tilde{A}_1\right)=1$

Proof: If $\tilde{A}_1=\left(-1,-1,-1,-1;\mu_L,\mu_R\right)$, where $\mu_L=\mu_R$, as shown in Fig. 3.8, then based on Equation 3.1, 3.3, 3.5 and 3.7, we can see that

$$LN_1=\int_{-1}^{1}f\left(x\right)\,dx=0$$

$$LP_1 = \int_{1}^{1} f\left(x\right)\, dx$$

$$RN_1 = \int_{-1}^{1} f\left(x\right)\, dx$$

$$RP_1 = \int_{-1}^{1} f\left(x\right)\, dx$$

Then, based on Equation 3.9 and 3.10,

$$M_1 = LN_1 + RN_1 = 0.$$

$$N_1 = LP_1 + RP_1 = \int_{-1}^{1} f\left(x\right)\, dx + \int_{-1}^{1} f\left(x\right)\, dx = 2\int_{-1}^{1} f\left(x\right)\, dx\, .$$

Therefore,

$$M_1 - N_1 = 0 - 2\int_{-1}^{1} f\left(x\right) dx = -2\int_{-1}^{1} f\left(x\right) dx$$

$$M_1 + N_1 = 0 + 2\int_{-1}^{1} f\left(x\right) dx = 2\int_{-1}^{1} f\left(x\right) dx$$

Using Equation 3.11 and Fig.3.6,

$$c\left(\tilde{A}_1\right) = \frac{\sum_{k=1}^{n} \mu_{1k}\, x_{1k}}{\sum_{k=1}^{n} \mu_{1k}} = \frac{0 \times -1 + \mu_L \times -1 + \mu_R \times -1 + 0 \times -1}{0 + \mu_L + \mu_R + 0}$$

$$= \frac{-\mu_L - \mu_R}{\mu_L + \mu_R} = -1$$

On using Equation 3.12, the ranking score $Score\left(\tilde{A}_1\right)$ is calculated as follow:

$$Score\left(\tilde{A}_1\right)=\frac{M_1-N_1}{M_1+N_1+\left(1-\left|c\left(\tilde{A}_1\right)\right|\right)}=\frac{-2\int_{-1}^{1}f(x)dx}{2\int_{-1}^{1}f(x)dx+\left(1-\left|-1\right|\right)}=-1$$

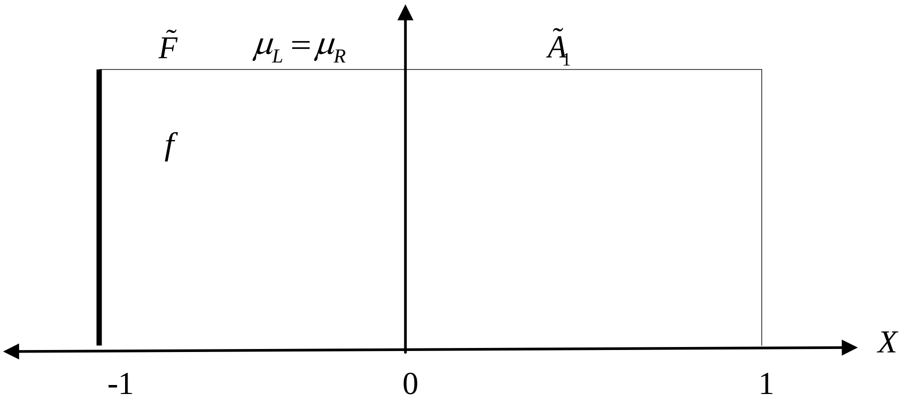


Fig.3.8 Generalized fuzzy numbers $\tilde{A}_1=\left(-1,-1,-1,-1;\mu_L,\mu_R\right)$

3.3 Comparative Study

The results obtained by the ranking method [19] and by the existing methods [14, 15, 20, 22, 23, 37, 47] are compared in Table 3.1.

i. For the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ shown in Set 1 of Fig. 3.9, the ranking order obtained by the existing methods [19, 20, 22, 23, 37, 47] and the presented method [19] is same i.e., $\tilde{B}$ > $\tilde{A}$ which coincides with intuition of human being due to the fact that the center of gravity of $\tilde{B}$ on X-axis is larger than the center of gravity of $\tilde{A}$ on the X-axis.

ii. For the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ shown in Set 2 of Fig. 3.9, the ranking order obtained by the existing methods [20, 22, 23, 47] and the presented method [19] is same i.e., $\tilde{A}$ = $\tilde{B}$ which coincides with intuition of human being due to the fact that the center of gravity of $\tilde{B}$ on *X-axis* is larger than the center of gravity of $\tilde{B}$ on the *X-axis*. But by the existing methods [14, 15, 37], the ranking order is $\tilde{B} > \tilde{A}$ which does not coincide with the intuition of human beings.

iii. For the fuzzy number $\tilde{A}$ and $\tilde{B}$ shown in Set 3 of Fig. 3.9, the ranking order obtained by the existing methods [19, 20, 22, 23, 37, 47] and the presented method [19] is same, i.e., $\tilde{A} = \tilde{B}$, which coincides with intuition of human being due to the fact that the center of gravity of $\tilde{B}$ on *X-axis* is larger than the center of gravity of $\tilde{A}$ on the *X-axis*. But by the existing methods [14, 15, 37], the ranking order is $\tilde{B} > \tilde{A}$, which does not coincide with the intuition of human beings.

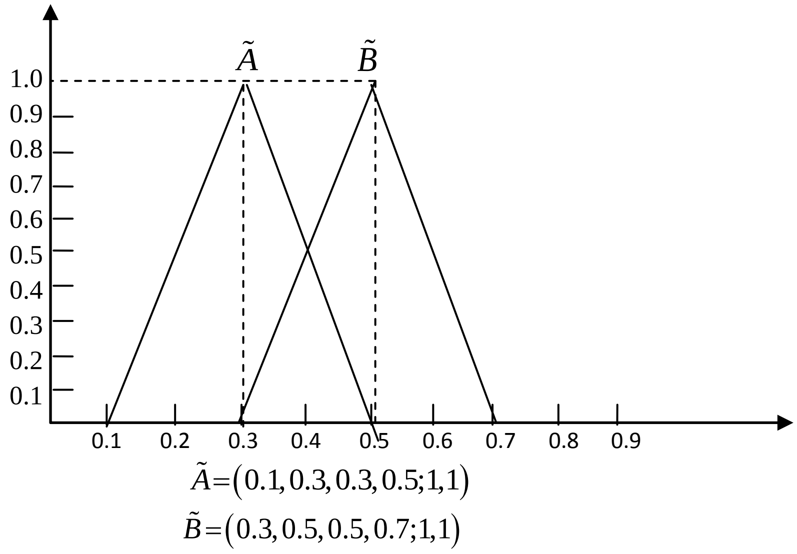
iv. For the fuzzy number $\tilde{A}$, and $\tilde{B}$ shown in Set 4 of Fig. 3.9, the existing method [47] is $\tilde{A} = \tilde{B}$, which does not coincide with the intuition of human beings due to the fact that the height $\tilde{A}$ is less than the height of the $\tilde{B}$, the ranking order obtained by the existing methods [14, 15, 20, 22, 23, 37] and the presented method [19] is same,, $\tilde{B} > \tilde{A}$, which coincide with the intuition of human beings.

v. For the fuzzy number $\tilde{A}$ and $\tilde{B}$ shown Set 5 of Fig. 3.9, the existing methods [22, 23, 37, 47], cannot calculate the ranking score of the crisp valued fuzzy number $\tilde{B}$, whereas the ranking order obtained by the existing methods [14, 15, 20] and the presented method [19] is same,i.e., $\tilde{B} > \tilde{A}$,, which coincides with intuition of human beings.

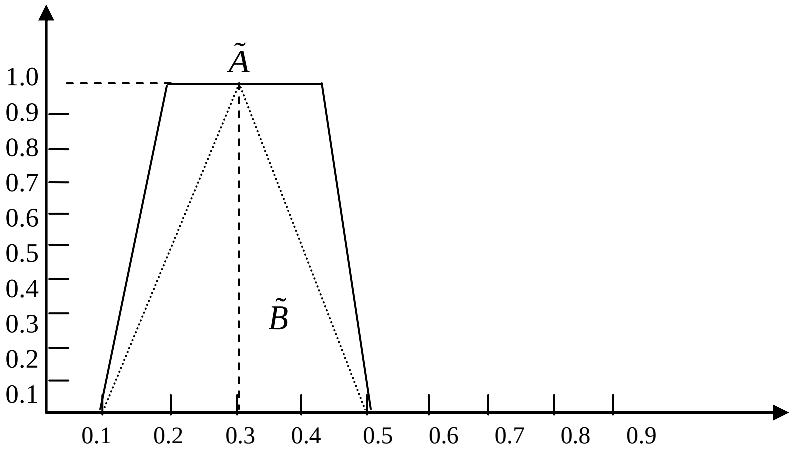
vi. For the fuzzy number $\tilde{A}$ and $\tilde{B}$ shown in Set 6 of Fig. 3.9, the existing method [22] gets the ranking order $\tilde{A} = \tilde{B}$, which does not coincide with the intuition of human beings due to the fact that the center of gravity of $\tilde{B}$ on the *X-axis* is larger than the center of gravity of $\tilde{A}$ on the X-axis. But the ranking order obtained by the existing methods [14, 15, 23, 37, 47] and the presented method [19] is same, $\tilde{B} > \tilde{A}$, which coincide with the intuition of human beings.

vii. For the fuzzy number $\tilde{A}$ and $\tilde{B}$ shown Set 7 of Fig. 3.9 the ranking order obtained by the existing methods [14, 15, 22, 23, 37, 47] and the presented method [19] is same i.e., $\tilde{B} > \tilde{A}$, which coincides with intuition of human beings.

viii. For the fuzzy number $\tilde{A}, \tilde{B}$ and $\tilde{C}$ shown Set 8 of Fig. 3.9, the ranking order obtained by the existing methods [14, 15, 23, 37, 47] and the presented method [19] is same i.e., $\tilde{C} > \tilde{B} > \tilde{A}$, which coincides with intuition of human beings due to the fact that the center of gravity of $\tilde{C}$ on the *X-axis* is larger than the center of gravity of $\tilde{B}$ on the *X-axis*, which is also larger than the center of gravity of $\tilde{A}$ on *X-axis*. But the existing method [14, 37, 47] get the same ranking order $\tilde{B} > \tilde{C} > \tilde{A}$.



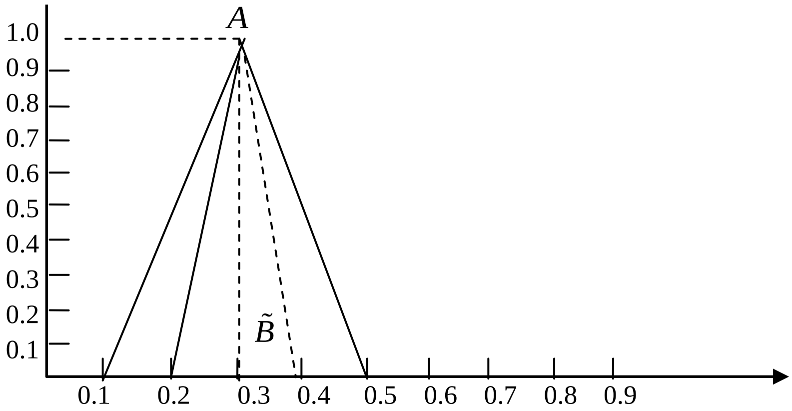
Set 1



$\tilde{A}=(0.1, 0.2, 0.4, 0.5; 1, 1)$

$\tilde{B}=(0.1, 0.3, 0.3, 0.5; 1, 1)$

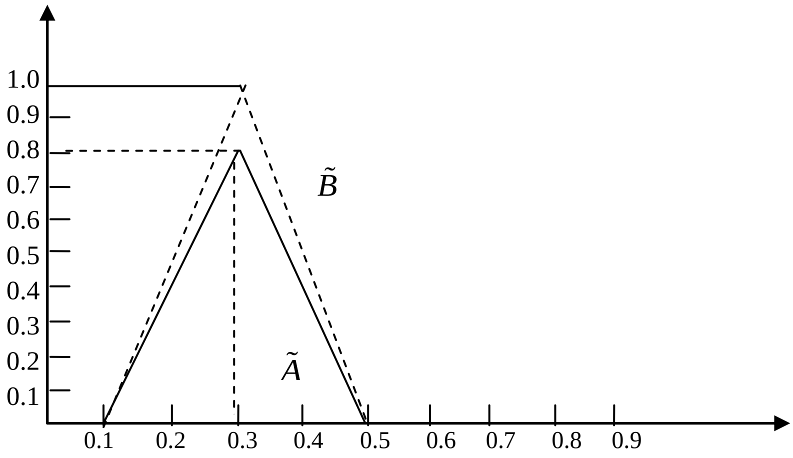
Set 2



$\tilde{A}=(0.1, 0.3, 0.3, 0.5; 1, 1)$

$\tilde{B}=(0.2, 0.3, 0.3, 0.4; 1, 1)$

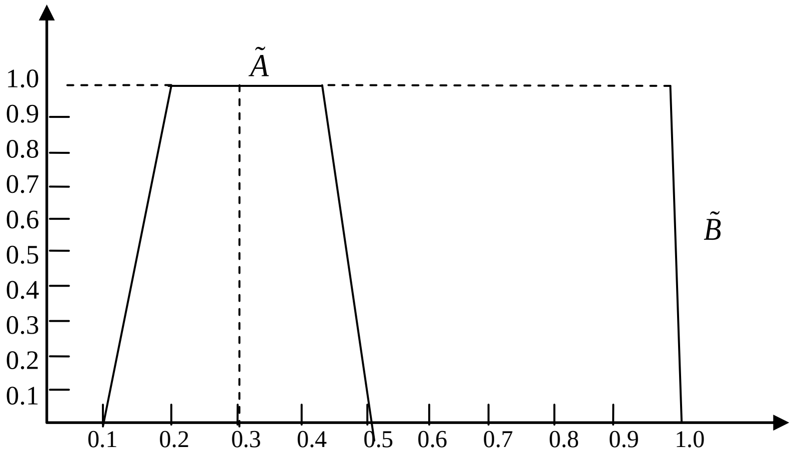
Set 3



$\tilde{A}=(0.1, 0.3, 0.3, 0.5; 0.8, 0.8)$

$\tilde{B}=(0.1, 0.3, 0.3, 0.5; 1, 1)$

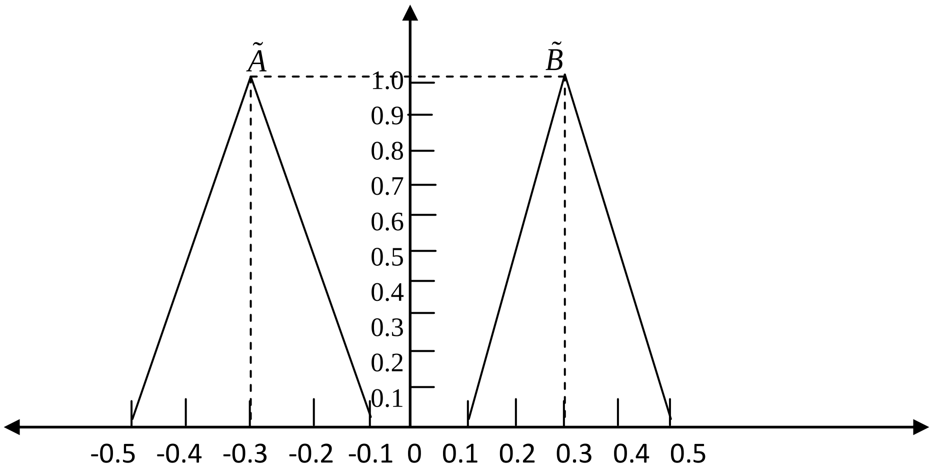
Set 4



$$\tilde{A}=(0.1,0.2,0.4,0.5;1,1)$$

$$\tilde{B}=(1.0,1.0,1.0,1.0;1,1)$$

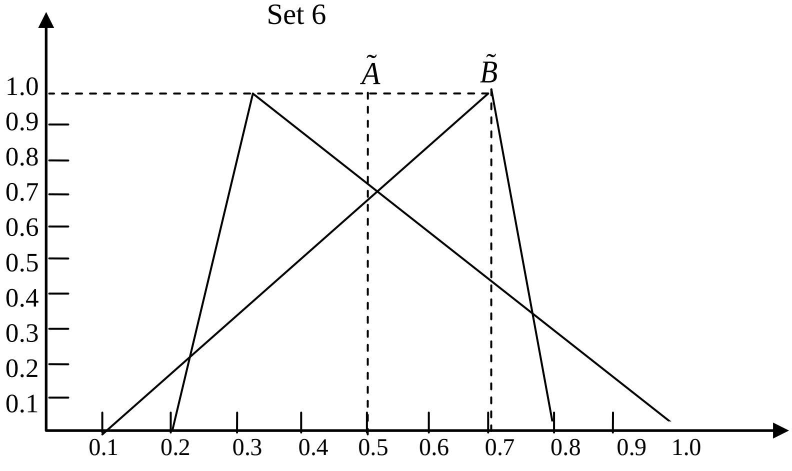
Set 5



$$\tilde{A}=(-0.5,-0.3,-0.3,-0.1;1,1)$$

$$\tilde{B}=(0.1,0.3,0.3,0.5;1,1)$$

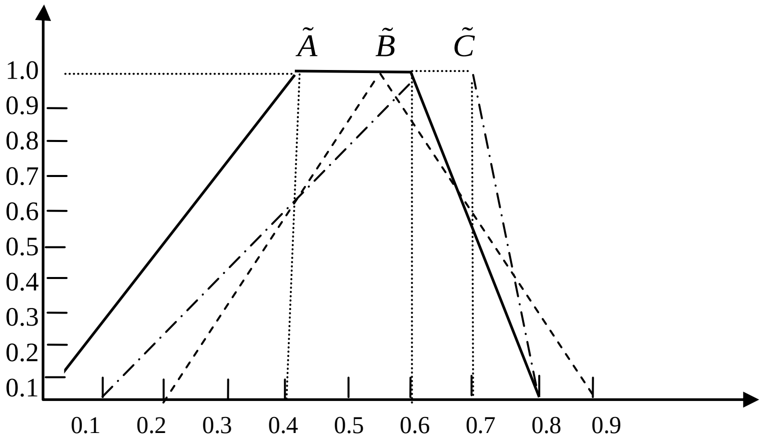
Set 6



$$\tilde{A}=(0.3,0.5,0.5,1.0;1,1)$$

$$\tilde{B}=(0.1,0.6,0.6,0.8;1,1)$$

Set 7



$$\tilde{A}=(0,0.4,0.6,0.8;1,1)$$

$$\tilde{B}=(0.2,0.5,0.5,0.9;1,1)$$

$$\tilde{C}=(0.1,0.6,0.7,0.8;1,1)$$

Set 8

Table 3.1 A comparison of the ranking results of the proposed method with existing method.

Ranking Methods	**Set 1**		**Set 2**		**Set 3**	
	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$
Cheng's method [22]	0.5831	0.7071	0.5831	0.5831	0.5831	0.5831
Chu and Tsao's method [23]	0.1500	0.2500	0.1500	0.1500	0.1500	0.1500
Murakami et al.'s method [37]	0.3000	0.5000	0.3000	0.4167	0.4167	0.3000
Yager's method [47]	0.3000	0.5000	0.3000	0.3000	0.3000	0.3000
Chen-and-Chen's method [14]	0.4456	0.4884	0.4239	0.4456	0.4456	0.4728
Chen-and-Chen's method [15]	0.2579	0.4298	0.2537	0.2579	0.2579	0.2774
Chen and Sanguansat's [20].	0.3000	0.5000	0.3000	0.3000	0.3000	0.3000
The presented method [19]	0.2553	0.4444	0.2553	0.2553	0.2553	0.2553

Ranking Methods	**Set 4**		**Set 5**		**Set 6**	
	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$
Cheng’s method [22]	0.4610	0.5831	0.4243	N/A	-0.5831	0.5831
Chu and Tsao’s method [23]	0.1200	0.1500	0.1500	N/A	-0.1500	0.1500
Murakami et al.’s method [37]	0.2333	0.3000	0.4167	N/A	-0.4167	0.4167
Yager’s method [47]	0.3000	0.3000	0.3000	N/A	-0.3000	0.3000
Chen-and-Chen’s method [14]	0.3565	0.4456	0.424	0.8601	0.4456	0.4456
Chen-and-Chen’s method [15]	0.2063	0.2579	0.2537	1.0000	-0.2579	0.2579
Chen and Sanguansat’s [20]	0.2824	0.3000	0.3000	1.0000	-0.3000	0.3000
The presented method [19]	0.2462	0.2553	0.2553	1.0000	-0.2553	0.2553

Ranking Methods	**Set 7**		**Set 8**		
	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
Cheng’s method [22]	0.7673	0.7241	0.6800	0.7257	0.7462
Chu and Tsao’s method [23]	0.2870	0.2619	0.2281	0.2624	0.2784
Murakami et al.’s method [37]	0.6000	0.5000	0.4400	0.5333	0.5250
Yager’s method [47]	0.6000	0.5000	0.4400	0.5333	0.5250
Chen-and-Chen’s method [14]	0.4128	0.4005	0.3719	0.4155	0.3979
Chen-and-Chen’s method [15]	0.4428	0.4043	0.3354	0.4079	0.4196
Chen and Sanguansat’s [20].	0.5750	0.5250	0.4500	0.5250	0.5500
The proposed method [19]	0.5111	0.4773	0.4000	0.4667	0.6057

In the following we use three sets of generalized fuzzy numbers, shown in Fig. 3.10 [25, 15] to make an experiment to compare the ranking of the presented method [19] with existing method [14, 15, 20,23,25, 37, 47] . The results are shown in Table 3.2.

(1) For the fuzzy numbers $\tilde{A}, \tilde{B}$ and $\tilde{C}$, shown in Set 1 of Fig. 3.10, the presented method 19] and existing method [20, 22, 34, 37] get the ranking order $\tilde{A} = \tilde{B}$, which coincides with intuition of human being due to the fact that the center of gravity of $\tilde{A}$ on X-axis is larger than the center of gravity of $\tilde{B}$ on the X-axis. However, existing methods [22, 37, 47] cannot calculate the ranking score of the crisp- valued fuzzy number $\tilde{C}$, whereas the presented can calculate the ranking score of the crisp-valued fuzzy numbers $\tilde{C}$, where the ranking order is $\tilde{C} > \tilde{A} = \tilde{B}$, which gets the same results with existing method [20]. It can also seen that the existing method [14, 15, 25] calculate the ranking score of crisp-value fuzzy number $\tilde{C}$, but the ranking order is $\tilde{C} > \tilde{B} > \tilde{A}$, which does not coincides with intuition of human beings.

(2) For the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ shown in Set 2 of Fig. 3.10, Cheng's method [22] gets the ranking order $\tilde{A} = \tilde{B}$, which does not coincide with the intuition of human beings due to the fact that the center of gravity of $\tilde{A}$ on X-axis is larger than the center of gravity of $\tilde{B}$ on *X-axis*. It can be easily seen that, the existing methods [47, 37, 14, 25, 20] and presented method [19] get the same ranking order, i.e. $\tilde{A} > \tilde{B}$, which coincides with the intuition of human beings.

(3) For the fuzzy numbers $\tilde{A}, \tilde{B}$ and $\tilde{C}$ shown in Set 3 of Fig. 3.10, the existing methods [22, 37, 47] cannot calculate the ranking score of the crisp- valued fuzzy number $\tilde{A}$ and $\tilde{B}$. But, the methods [15, 20, 25] and presented method [19] get the same ranking order, i.e. $\tilde{A} > \tilde{B}$, which

coincides with the intuition of human beings. Also, It can be seen that Chen and Chen's method [14] gets the ranking order: $\tilde{A} < \tilde{B}$, which does not coincide with the intuition of human beings.

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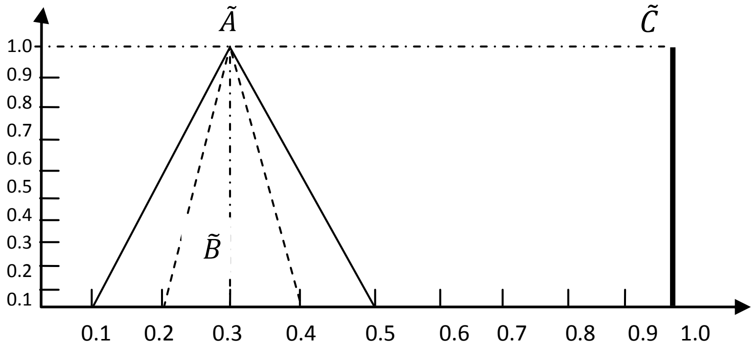
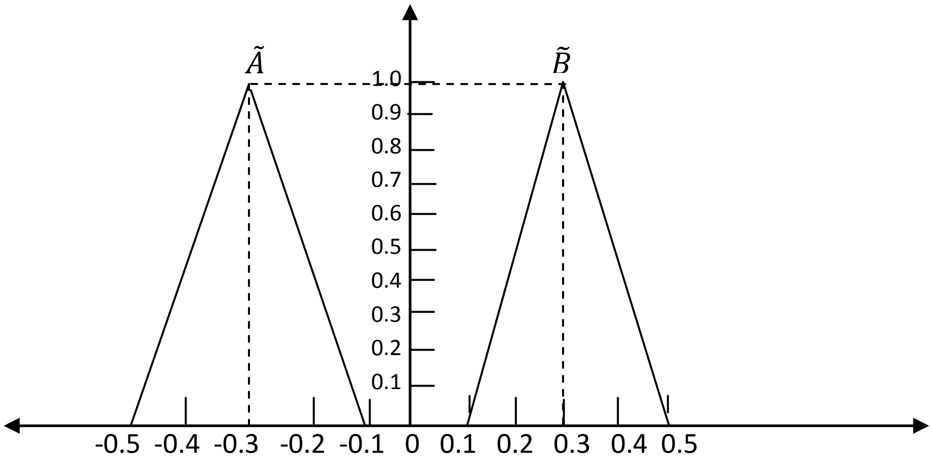


$$\tilde{A} = \ (0.1,0.3, 0.3, 0.5; 1, 1)$$

$$\tilde{B} = \ (0.2, 0.3, 0.3, 0.4; 1, 1)$$

$$\tilde{C} = \ (1.0, 1.0, 1.0, 1.0; 1, 1)$$

Set 1



$$\tilde{A} = \ (0.1, 0.3, 0.3, 0.5; 1, 1; 1, 1)$$

$$\tilde{B} = \ (-0.5, -0.3, -\ 0.3, -\ 0. ; 1, 1)$$

Sets 2

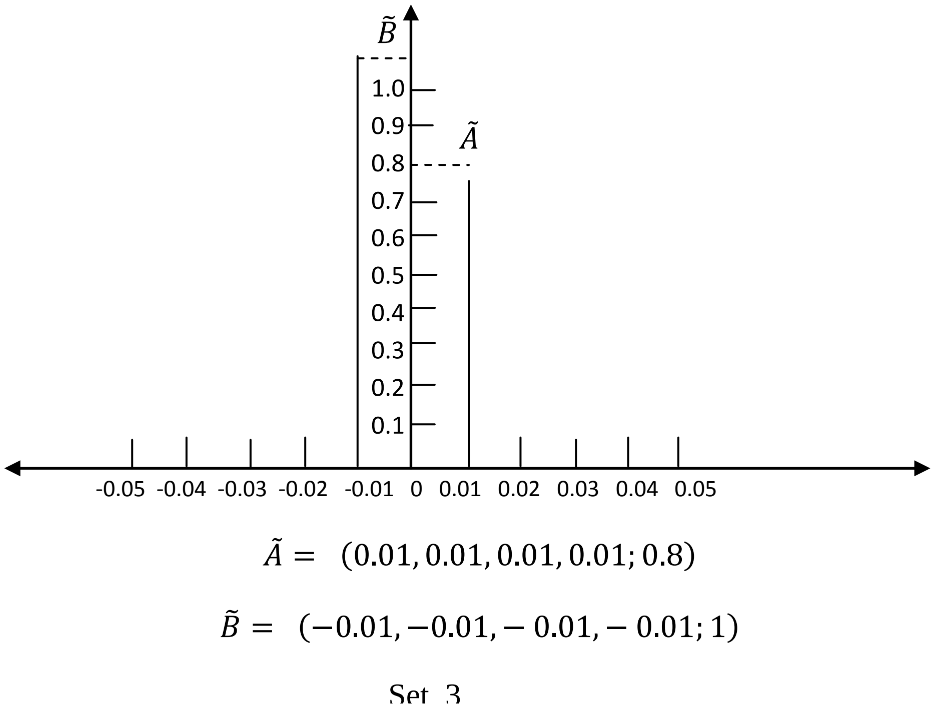


Fig. 12 Three sets of generalized fuzzy numbers (Chen and Chen [18]; Deng and Liu, [25])

Table 3.2 A comparison of the ranking results of the proposed method with existing methods.

Ranking Methods	**Set 1**			**Set 2**		**Set 3**	
	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$\tilde{A}$	$\tilde{B}$	$\tilde{A}$	$\tilde{B}$
Yager's mehod [47]	0.3000	0.3000	N/A	0.3000	-0.3000	N/A	N/A
Murakami et al.'s method [37]	0.3000	0.3000	N/A	0.3000	-0.3000	N/A	N/A
Cheng's method [22]	0.5831	0.5831	N/A	0.5831	0.5831	N/A	N/A
Deng and Liu's method [25]	0.6241	0.6244	1	0.6241	0.6241	0.505	0.495
Chen-and-Chen's method [14]	0.4456	0.4728	0.8602	0.4456	0.4456	0.4	0.5
Chen-and-Chen's method [15]	0.2579	0.2774	1.0000	0.2579	-0.2579	0.0080	-0.0100
Chen and Sanguansat's [23]	0.3000	0.3000	1.0000	0.3000	-0.3000	0.0094	-0.0010
The presented method [19]	0.2553	0.2553	1.0000	0.2553	-0.2553	0.0076	-0.0080

In the summary, from Tables 3.1 and 3.2, the presented method [19] can overcome the drawbacks of the existing methods [14, 15, 22, 25, 37, 47]. Moreover, the presented fuzzy ranking method is better than the Chen and Sanguansat's method [20] due to the fact that the presented fuzzy ranking method can ranking generalized fuzzy numbers with different left heights and rights heights, where as Chen and Sanguansat's method [20] only can deal with generalized fuzzy numbers with the same left heights and rights heighs.

3.4 Fuzzy risk analysis

Assume that there are there are n manufactories $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, ..., \tilde{C}_n$. each manufactories makes the product $\tilde{A}_i$ which is composed of p sub-components $\tilde{A}_{i1}, \tilde{A}_{i2}, \tilde{A}_{i3}, ..., \tilde{A}_{ip}$, where $1 \leq i \leq p$. To evaluate each sub-components $\tilde{A}_{ik}$ and to derive the probability of failure $\tilde{R}_i$ of components $\tilde{A}_i$ made by manufactory $\tilde{C}_i$, we use two evaluating items $\tilde{W}_{ik}$ and $\tilde{R}_{ik}$, where $\tilde{W}_{ik}$ denotes the severity of loss of the sub-component $\tilde{A}_{ik}$, $\tilde{R}_{ik}$ denotes the probability of failure of sub-component $\tilde{A}_{ik}$, $1 \leq i \leq n$ and $1 \leq i \leq p$. Fig. 3.11 shows the structure for fuzzy risk analysis [39].

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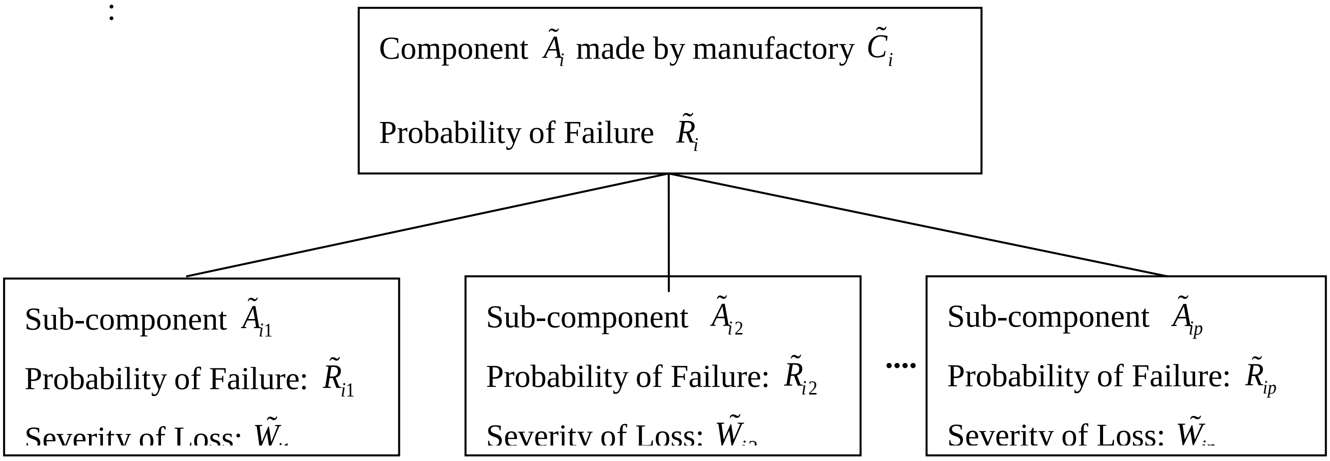


Fig. 3.11 the structure for fuzzy risk analysis (Schmucker, [39])

The steps of method is as follow:

Step 1. Calculate the probability of failure $\tilde{R}_i$ of each components $\tilde{A}_i$ made by manufactory $\tilde{C}_i$, where $1 \le i \le n$. Based on the fuzzy weighted mean method [18] and the generalized fuzzy numbers arithmetic operators given in chapter 2 and using arithmetic operation discussed in section 2.2.2, aggregate the evaluating items $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ of sub-components $\tilde{A}_i$ made by manufactory $\tilde{C}_i$ to get the probability of failure $\tilde{R}_i$ of each components $\tilde{A}_i$ made by manufactory $\tilde{C}_i$,where

$$\tilde{R}_i = \sum_{k=1}^{p}\left(\tilde{R}_{ik} \oplus \tilde{W}_{ik}\right) \ \oslash \ \sum_{k=1}^{p}\tilde{W}_{ik}$$

$\tilde{R}_i$ is a generalized fuzzy number and $1 \le i \le n$.

Step 2. Transform each generalized fuzzy number

$\tilde{R}_i = \left(r_{i1}, r_{i2}, r_{i3}, r_{i4}; \mu_{L\tilde{R}_i}, \mu_{R\tilde{R}_i}\right)$ into a standardized generalized fuzzy number $\tilde{R}_i^*$,

$$\tilde{R}_i^* = \left(\frac{r_{i1}}{k_i}, \frac{r_{i2}}{k_i}, \frac{r_{i3}}{k_i}, \frac{r_{i4}}{k_1}; \mu_{L\tilde{R}_i}, \mu_{R\tilde{R}_i}\right)$$

$$= \left(r_{i1}^*, r_{i2}^*, r_{i3}^*, r_{i4}^*; \mu_{L\tilde{R}_i}, \mu_{R\tilde{R}_i}\right) \qquad (3.13)$$

where $k_i = \max\left\{\left\lceil \left|r_{ij}\right| \right\rceil\right\}$, $\left|r_{ij}\right|$ denotes the absolute value of r_{ij} and $\left\lceil \left|r_{ij}\right| \right\rceil$ denotes the upper bound of $\left|r_{ij}\right||$, $1 \le i \le n$ and $1 \le i \le 4$.

Step 3. On using equation 3.1 to 3.8, calculate the area LN_i, RN_i, LP_i and RP_i of each standardized generalized fuzzy number $\tilde{R}_i^*$, respectively, where $1 \le i \le n.$.

Step 4. On using equation 3.9 and 3.10, we can calculate the area M_i and N_i of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le n.$

Step 5. On using equation 3.11, calculate the area $c\left(\tilde{R}_i^*\right)$ of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le n.$

Step 6. On using equation 3.12, we can calculate the area $Score\left(\tilde{R}_i^*\right)$ of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, where $1 \le i \le n.$ The larger the value of $Score\left(\tilde{R}_i^*\right)$, the higher the probability of failure of components $\tilde{A}_i$ made by manufactory C_i, where $1 \le i \le n.$

Example 3.1 [14, 18, 20, 37] Consider the structure of fuzzy risk analysis shown in Fig. 3.12. Assume that there are three manufactories C_1, C_2 and C_3 and assume that the manufactories C_1, C_2 and C_3 produce the components A_1, A_2 and A_3, respectively, where are the same A_1, A_2 and A_3 product produced by C_1, C_2 and C_3, respectively. Each component A_i is composed of three sub-components A_{i1}, A_{i2} and A_{i3}, where $1 \le i \le 3.$. To derive the probability of failure $\tilde{A}_i$ made by manufactory C_i, we use evaluating items $\tilde{W}_{ik}$ and $\tilde{R}_{ik}$, where $\tilde{W}_{ik}$ denotes the severity of loss of the sub-component A_{ik}, $\tilde{R}_{ik}$ denotes the probability of failure of sub-component A_{ik}, $1 \le i \le 3$ and $1 \le k \le 3.$

Table 3.3 [10, 14, 20, 33,] shows the severity of loss $\tilde{W}_{ik}$ and the probability of failure $\tilde{R}_{ik}$ of the sub-components A_{ik} made by manufactory C_i, where $1 \le i \le 3$ and $1 \le k \le 3.$

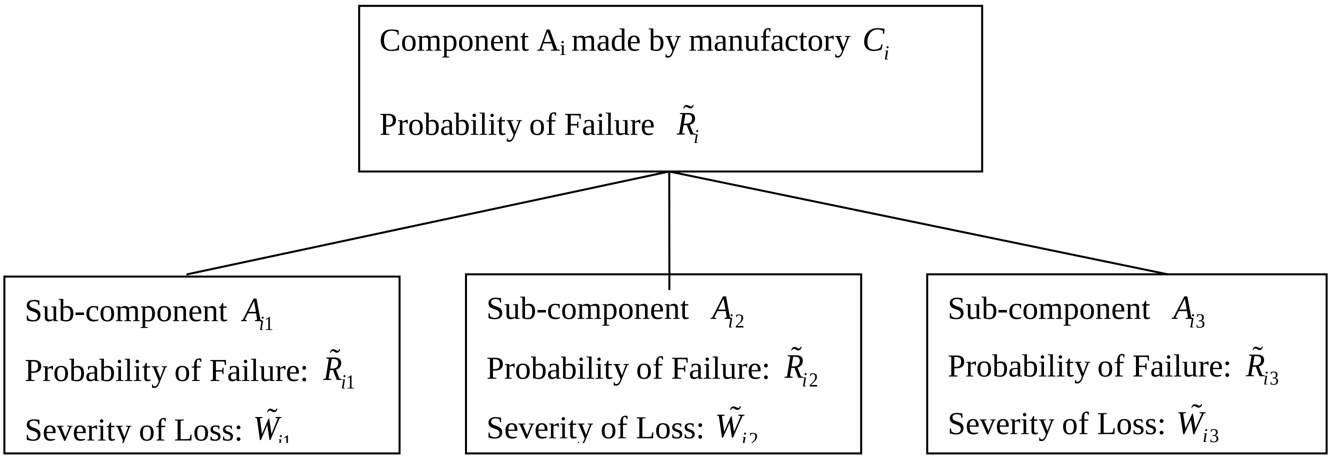


Fig. 3.12. The structure for fuzzy risk analysis [14, 15 20, 33, 39]

Table 3.3 The severity of loss $\tilde{W}_{ik}$ and the probability of failure $\tilde{R}_{ik}$ of sub- component A_{ik} shown [14, 18, 21, 34]

Manufactory	Sub-components	Severity of loss	Probability of failure
C_1	A_{11}	$\tilde{W}_{11}=(0.04,0.1,0.18,0.23;1.0,1.0)$	$\tilde{R}_{11}=(0.17,0.22,0.36,0.42;0.9,0.9)$
	A_{12}	$\tilde{W}_{12}=(0.58,0.63,0.80,0.86;1.0,1.0)$	$\tilde{R}_{12}=(0.32,0.41,0.58,0.65;0.7,0.7)$
	A_{13}	$\tilde{W}_{13}=(0.0,0.0,0.02,0.07;1.0,1.0)$	$\tilde{R}_{13}=(0.58,0.63,0.80,0.86;0.8,0.8)$
C_2	A_{21}	$\tilde{W}_{21}=(0.04,0.1,0.18,0.23;1.0,1.0)$	$\tilde{R}_{21}=(0.93,0.98,1.0,1.0;0.85,0.85)$
	A_{22}	$\tilde{W}_{22}=(0.58,0.63,0.80,0.86;1.0,1.0)$	$\tilde{R}_{22}=(0.58,0.63,0.80,0.86;0.9,0.9)$
	A_{23}	$\tilde{W}_{23}=(0.0,0.0,0.02,0.07;1.0,1.0)$	$\tilde{R}_{23}=(0.32,0.41,0.58,0.65;0.9,0.9)$
C_3	A_{31}	$\tilde{W}_{31}=(0.04,0.1,0.18,0.23;1.0,1.0)$	$\tilde{R}_{31}=(0.17,0.22,0.36,0.42;0.95,0.95)$
	A_{32}	$\tilde{W}_{32}=(0.58,0.63,0.80,0.86;1.0,1.0)$	$\tilde{R}_{32}=(0.72,0.78,0.92,0.97;0.8,0.8)$
	A_{33}	$\tilde{W}_{33}=(0.0,0.0,0.02,0.07;1.0,1.0)$	$\tilde{R}_{33}=(0.58,0.63,0.80,0.86;1.0,1.0)$

The steps of the method is as follows:

Step 1. Using arithmetic operation discussed in section 2.2.2 calculate the probability of failure $\tilde{R}_i$ of each component A_i made by manufactory C_i by aggregating the evaluating items $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ of sub-components A_{ik} made by manufactory C_i, where $1 \le i \le 3$ and $1 \le k \le 3.$, shown as follows:

$$\tilde{R}_1 = \left(\tilde{W}_{11} \otimes \tilde{R}_{11} \oplus \tilde{W}_{12} \otimes \tilde{R}_{12} \oplus \tilde{W}_{13} \otimes \tilde{R}_{13}\right) \oslash \left(\tilde{W}_{11} \oplus \tilde{W}_{12} \oplus \tilde{W}_{13}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.17, 0.22, 0.36, 0.42; 0.9, 0.9) $\oplus$(0.58, 0.63, 0.80, 0.86; 1.0, 1.0) $\otimes$ (0.32, 0.41, 0.58, 0.65; 0.7, 0.7) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0) $\otimes$ (0.58, 0.63, 0.80, 0.86; 0.8, 0.8)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1:0) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)]

= (0.1659, 0.2803, 0.7463, 1.1545; 0.7, 0.7)

$$\tilde{R}_2 = \left(\tilde{W}_{21} \otimes \tilde{R}_{21} \oplus \tilde{W}_{22} \otimes \tilde{R}_{22} \oplus \tilde{W}_{23} \otimes \tilde{R}_{23}\right) \oslash \left(\tilde{W}_{21} \oplus \tilde{W}_{22} \oplus \tilde{W}_{23}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.93, 0.98, 1.0, 1.0; 0.85, 0.85) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1.0) $\otimes$ (0.58, 0.63, 0.80, 0.86; 0.9, 0.9) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0) $\otimes$ (0.32, 0.41, 0.58, 0.65; 0.9, 0.9)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1:0) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)]

= (0.3221, 0.4949, 1.1392, 1.6373,; 0.85, 0.85)

$$\tilde{R}_3 = \left(\tilde{W}_{31} \otimes \tilde{R}_{31} \oplus \tilde{W}_{32} \otimes \tilde{R}_{32} \oplus \tilde{W}_{33} \otimes \tilde{R}_{33}\right) \oslash \left(\tilde{W}_{31} \oplus \tilde{W}_{32} \oplus \tilde{W}_{33}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.17, 0.22, 0.36, 0.42; 0.95, 0.95) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1.0) $\otimes$ (0.72, 0.78, 0.92, 0.97; 0.8, 0.8) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0) $\otimes$(0.58, 0.63, 0.80, 0.86; 1.0, 1.0)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 1:0) $\oplus$ (0.0, 0.0, 0.02, 0.07; 1.0, 1.0)]

= (0.3659, 0.5134, 1.1189, 1.5984; 0.8, 0.8)

Step 2. Using on equation 3.13, transform each generalized fuzz number $\tilde{R}_i$ in to a standardized fuzzy number $\tilde{R}_i^*$, where $1 \le i \le 3$, shown as follows:

$$\tilde{R}_1 = \left(0.1659, 0.2803, 0.7463, 1.154; 0.7, 0.7\right),$$

$$k_1 = \max\left(\lceil |0.1659| \rceil, \lceil |0.2803| \rceil, \lceil |0.7463| \rceil, \lceil |1.154| \rceil, 1\right)$$

= max (1,1,1,2,1) =2

$$\tilde{R}_1^* = \left(\frac{0.1659}{2}, \frac{0.2803}{2}, \frac{0.7463}{2}, \frac{1.154}{2}; 0.7, 0.7\right)$$

= (0.0830, 0.1402, 0.3732, 0.5770;0.7;0.7)

$$\tilde{R}_2 = \left(0.3221, 0.4949, 1.1359, 1.6373; 0.85, 0.85\right)$$

$$k_2 = \max\left(\lceil |0.3221| \rceil, \lceil |0.4949| \rceil, \lceil |1.1392| \rceil, \lceil |1.6373| \rceil, 1\right)$$

= max (1,1,2,2,1)

$$\tilde{R}_2^* = \left(\frac{0.3221}{2}, \frac{0.4949}{2}, \frac{1.1392}{2}, \frac{1.6373}{2}; 0.85, 0.85\right)$$

= (0.1611, 0.2475, 0.5696, 0.8187; 0.85, 0.85)

$$\tilde{R}_3 = \left(0.3659, 0.5134, 1.1189, 1.5984; 0.8, 0.8\right),$$

$$k_3 = \max\left(\lceil |0.3659| \rceil, \lceil |0.5134| \rceil, \lceil |0.1.1189| \rceil, \lceil |1.5984| \rceil, 1\right)$$

= max (1,1,2,2,1)

$$\tilde{R}_3^* = \left(\frac{0.3659}{2}, \frac{0.5134}{2}, \frac{1.1189}{2}, \frac{1.5984}{2}; 0.8, 0.8 \right)$$

= (0.1830, 0.2567, 0.5595, 0.7992; 0.8;0.8)

Step 3. Using on equation 3.1 to 3.8, calculate area LN_i, RN_i, LP_i and RP_i of each standardized generalized fuzzy number $\tilde{R}_i^*$, respectively, where $1 \le i \le 3$.., shown as follows:

Calculate the value of LN_i, RN_i, LP_i and RP_i, when left heights and rights heights are equal

$\tilde{R}_1^*$ = (0.0830, 0.1402, 0.3732, 0.5770;0.7;0.7)

$$LN_1 = \int_{-1}^{x_2} f(x)\, dx - \int_{x_1}^{x_2} g_1(x)\, dx$$

$$= 0.7 \left(\int_{-1}^{0} f(x)\, dx + \int_{0}^{x_2} f(x)\, dx \right) - 0.7 \int_{x_1}^{x_2} g_1(x)\, dx$$

$$= \ 0.7 \left(\int_{-1}^{0} 1 dx + \int_{0}^{0.1402} 1\, dx \right) - 0.7 \int_{0.0830}^{0.1420} \frac{x - 0.0830}{0.1420 - 0.0830}\, dx$$

= 0.7 ×1.1402 – 0.02002 = 0.77812.

$$RN_1 = \int_{-1}^{x_3} f(x)\, dx + \int_{x_3}^{x_4} g_3(x)\, dx$$

$$= 0.7 \left(\int_{-1}^{0} f(x)\, dx + \int_{0}^{x_1} f(x)\, dx + \int_{x_1}^{x_2} f(x)\, dx + \int_{x_2}^{x_3} f(x)\, dx \right) - 0.7 \int_{x_3}^{x_4} g_3(x)\, dx$$

= 0.7 ×1.3732 + 0.07133 = 1.0326.

$$LP_1 = \int_{x_2}^{1} f(x)\, dx + \int_{x_1}^{x_2} g_1(x)\, dx$$

$$=0.7\left(\int_{x_2}^{x_3} 1dx+\int_{x_3}^{x_4} 1\,dx+\int_{x_4}^{1} 1\,dx\right)+0.7\int_{x_1}^{x_2} g_1\,\,dx$$

$$=0.7\left(\int_{0.1402}^{0.3732} 1dx+\int_{0.3732}^{0.5770} 1\,dx+\int_{0.5770}^{1} 1\,dx\right)+0.7\int_{0.0830}^{0.1402}\frac{x-0.0830}{0.1402-0.0830.}\,dx$$

$$=\ 0.7\times 0.60180+0.02002=0.6219.$$

$$RP_1=\int_{x_3}^{1} f\left(x\right)\,dx-\int_{x_3}^{x_4} g_3\left(x\right)\,dx$$

$$=\ 0.7\left(\int_{x_3}^{x_4} f\left(x\right)\,dx+\int_{x_4}^{1} f\left(x\right)\,dx\right)-0.7\int_{x_3}^{x_4} g_3\left(x\right)\,dx$$

$$=\ 0.7\left(\int_{0.3732}^{0.5770} 1dx+\int_{0.5770}^{1} 1\,dx\right)-0.7\int_{0.3732}^{0.5770}\frac{x-0.5770}{0.3732-0.5770}\,dx$$

$$=\ 0.7\times 0.6268-\ 0.07133\ =\ 0.36743$$

$$LN_1=0.7781,\quad RN_1=1.0326,\quad LP_1=0.6219,\quad RP_1=0.3674$$

Similarly, find the area of other two standardized generalized fuzzy numbers $\tilde{R}_2^*$ and $\tilde{R}_3^*$

$$LN_2=1.0236,\quad RN_2=1.4400,\quad LP_2=0.6764,\quad RP_2=0.2600$$

$$LN_3=0.9759,\quad RN_3=1.3435,\quad LP_3=0.6241,\quad RP_3=0.2565$$

Step 4. Using on equation 3.9 and 3.10, we can calculate the area M_i and N_i of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le 3.$, *s*hown as follows:

$$M_1=LN_1+RN_1=1.8107,\qquad N_1=LP_1+RP_1=0.9893,$$

$$M_2 = LN_2 + RN_2 = 2.4636, \quad N_2 = LP_2 + RP_2 = 0.9364,$$

$$M_3 = LN_3 + RN_3 = 2.3193, \quad N_3 = LP_3 + RP_3 = 0.8807,$$

Step 5. Using on equation 3.11, calculate the area $c\left(\tilde{R}_i^*\right)$ of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le 3.$, shown as follows:

$$c\left(\tilde{R}_1^*\right) = \frac{\sum_{k=1}^{n} \mu_{1k}\, x_{1k}}{\sum_{k=1}^{n} \mu_{1k}} = \frac{0 \times 0.0830 + \mu_L \times 0.1402 + \mu_R \times 0.3732 + 0 \times 0.5770}{0 + \mu_L + \mu_R + 0}$$

$$= \frac{0 \times 0.0830 + 0.7 \times 0.1402 + 0.7 \times 0.3732 + 0 \times 0.5770}{0 + 0.7 + 0.7 + 0}$$

$$= 0.2567$$

Similarly we can find the area of other two standardized generalized fuzzy numbers $c\left(\tilde{R}_2^*\right)$ and $c\left(\tilde{R}_3^*\right)$

$$c\left(\tilde{R}_2^*\right) = 0.4085, c\left(\tilde{R}_3^*\right) = 0.4081$$

Step 6. Using on equation 3.12, calculate the area $Score\left(\tilde{R}_i^*\right)$ of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le 3.$, shown as follows:

$$Score\left(\tilde{R}_1^*\right) = \frac{M_1 - N_1}{M_1 + N_1 + \left(1 - \left|c\left(\tilde{R}_1^*\right)\right|\right)} = \frac{1.8107 - 0.9893}{1.8107 + 0.98936 + \left(1 - 0.2567\right)} = 0.2318$$

Similarly we can find the area of other two standardized generalized fuzzy numbers $Score\left(\tilde{R}_2^*\right)$ and $Score\left(\tilde{R}_3^*\right)$

$$Score\left(\tilde{R}_2^*\right) = 0.3826, Score\left(\tilde{R}_3^*\right) = 0.3794$$

Because $Score\left(\tilde{R}_2^*\right) > Score\left(\tilde{R}_3^*\right) > Score\left(\tilde{R}_1^*\right)$, the ranking order of the standardize fuzzy numbers $\tilde{R}_1^*$, $\tilde{R}_2^*$ and $\tilde{R}_3^*$ *is* $\tilde{R}_2^* > \tilde{R}_3^* > \tilde{R}_1^*$ i.e., the order of the risk of the manufactories C_1, C_2 and C_3 is : $C_2 > C_3 > C_1$. It means that the component A_2 made by the manufactory C_2 has the highest probability of failure, followed by C_3 and C_1. This result coincides with existing method [14, 15, 20, 33].

Example 3.2 [19] Consider the structure of fuzzy risk analysis shown in Fig. 3.13. Assume that there are three manufactories C_1, C_2 and C_3 and assume that the manufactories C_1, C_2 and C_3 produce the components A_1, A_2 and A_3, respectively, where are the same A_1, A_2 and A_3 product produced by C_1, C_2 and C_3, respectively. Each component A_i is composed of three sub-components A_{i1}, A_{i2} and A_{i3}, where $1 \le i \le 3$..In table 3.4 To derive the probability of failure $\tilde{A}_i$ made by manufactory C_i, we use evaluating items $\tilde{W}_{ik}$ and $\tilde{R}_{ik}$, where $\tilde{W}_{ik}$ denotes the severity of loss of the sub-component A_{ik}, $\tilde{R}_{ik}$ denotes the probability of failure of sub-component A_{ik}, $1 \le i \le 3$ and $1 \le k \le 3$.

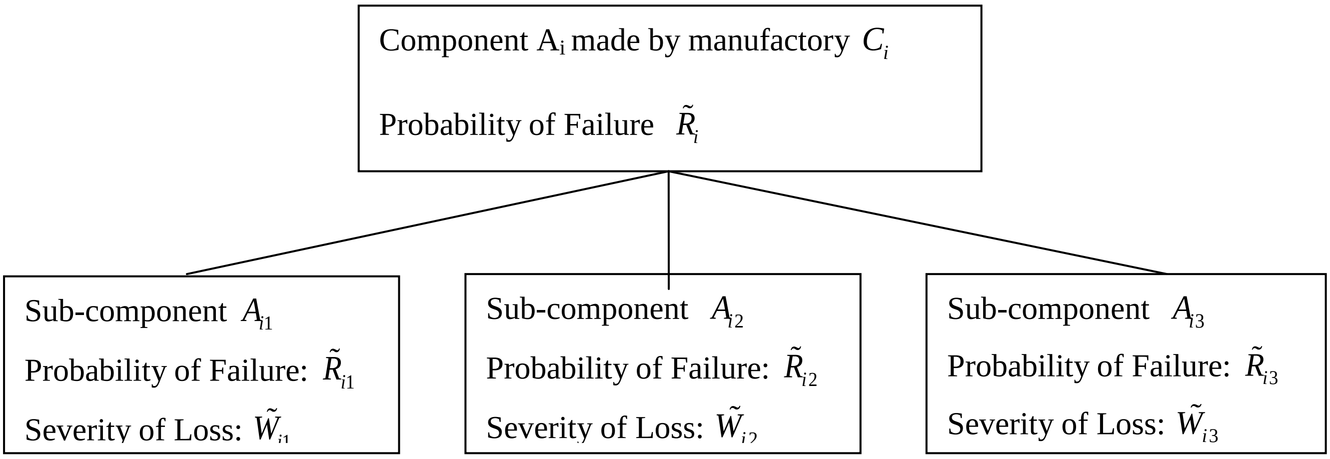


Fig. 3.13 The structure for fuzzy risk analysis [14, 15, 20, 33, 39]

Table 3.4 The severity of loss $\tilde{W}_{ik}$ and the probability of failure $\tilde{R}_{ik}$ of sub- component A_{ik} [14,15, 20, 33]

Manufactory	Sub-components	Severity of loss	Probability of failure
C_1	A_{11}	$\tilde{W}_{11} = (0.04, 0.1, 0.18, 0.23; 0.8, 0.9)$	$\tilde{R}_{11} = (0.17, 0.22, 0.36, 0.42; 0.9, 0.9)$
	A_{12}	$\tilde{W}_{12} = (0.58, 0.63, 0.80, 0.86; 0.65, 0.$	$\tilde{R}_{12} = (0.32, 0.41, 0.58, 0.65; 0.9, 0.7)$
	$\tilde{A}_{13}$	$\tilde{W}_{13} = (0.0, 0.0, 0.0, 0.0; 0.5, 0.5)$	$\tilde{R}_{13} = (0.58, 0.63, 0.80, 0.86; 0.8, 0.9)$
C_2	A_{21}	$\tilde{W}_{21} = (0.04, 0.1, 0.18, 0.23; 0.8, 0.7)$	$\tilde{R}_{21} = (0.93, 0.98, 1.0, 1.0; 0.85, 0.8)$
	A_{22}	$\tilde{W}_{22} = (0.58, 0.63, 0.80, 0.86; 1.0, 0.5$	$\tilde{R}_{22} = (0.58, 0.63, 0.80, 0.86; 0.9, 0.9)$
	A_{23}	$\tilde{W}_{23} = (0.0, 0.0, 0.02, 0.07; 0.4, 0.8)$	$\tilde{R}_{23} = (0.32, 0.41, 0.58, 0.65; 0.7, 0.9)$
C_3	A_{31}	$\tilde{W}_{31} = (0.04, 0.1, 0.18, 0.23; 1.0, 1.0)$	$\tilde{R}_{31} = (0.17, 0.22, 0.36, 0.42; 0.95, 0.$
	A_{32}	$\tilde{W}_{32} = (0.58, 0.63, 0.80, 0.86; 0.8, 0.8$	$\tilde{R}_{32} = (0.72, 0.78, 0.92, 0.97; 0.5, 0.6)$
	A_{33}	$\tilde{W}_{33} = (0.0, 0.0, 0.07, 0.2; 0.9, 0.7)$	$\tilde{R}_{33} = (0.58, 0.63, 0.80, 0.86; 1.0, 1.0)$

The steps of the method is as follow:

Step 1. Using arithmetic operation discussed in section 2.2.2, calculate the probability of failure $\tilde{R}_i$ of each component A_i made by manufactory C_i by aggregating the evaluating items $\tilde{R}_{ik}$ and $\tilde{W}_{ik}$ of sub-components A_{ik} made by manufactory C_i, where $1 \leq i \leq 3$ and $1 \leq k \leq 3$., shown as follows:

$$\tilde{R}_1 = \left(\tilde{W}_{11} \otimes \tilde{R}_{11} \oplus \tilde{W}_{12} \otimes \tilde{R}_{12} \oplus \tilde{W}_{13} \otimes \tilde{R}_{13}\right) \oslash \left(\tilde{W}_{11} \oplus \tilde{W}_{12} \oplus \tilde{W}_{13}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 0.8, 0.9) ⊗ (0.17, 0.22, 0.36, 0.42; 0.9, 0.9) ⊕ (0.58, 0.63, 0.80, 0.86; 0.65, 0.7) ⊗ (0.32, 0.41, 0.58, 0.65; 0.9, 0.7) ⊕ (0.0, 0.0, 0.0, 0.0; 0.5, 0.6) ⊗ (0.58, 0.63, 0.80, 0.86; 0.8, 0.9)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 0.8, 0.9) $\oplus$ (0.58, 0.63, 0.80, 0.86;0.65, 0.7) $\oplus$ (0.0, 0.0, 0.0, 0.0; 0.5, 0.6)]

= (0.1765, 0.2860, 0.7244, 1.0574; 0.5, 0.6)

$$\tilde{R}_2 = \left(\tilde{W}_{21} \otimes \tilde{R}_{21} \oplus \tilde{W}_{22} \otimes \tilde{R}_{22} \oplus \tilde{W}_{23} \otimes \tilde{R}_{23}\right) \oslash \left(\tilde{W}_{21} \oplus \tilde{W}_{22} \oplus \tilde{W}_{23}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 0.8, 0.7) $\otimes$ (0.93, 0.98, 1.0, 1.0; 0.85, 0.8) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 0.5) $\otimes$ (0.58, 0.63, 0.80, 0.86; 0.9, 0.9) $\oplus$ (0.0, 0.0, 0.02, 0.07; 0.4, 0.8) $\otimes$ (0.32, 0.41, 0.58, 0.65; 0.7, 0.9)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 0.8, 0.7) $\oplus$ (0.58, 0.63, 0.80, 0.86; 1.0, 0.5) $\oplus$ (0.0, 0.0, 0.02, 0.07; 0.4, 0.8)]

= (0.3221, 0.4949, 1.1392, 1.6373; 0.85, 0.85)

$$\tilde{R}_3 = \left(\tilde{W}_{31} \otimes \tilde{R}_{31} \oplus \tilde{W}_{32} \otimes \tilde{R}_{32} \oplus \tilde{W}_{33} \otimes \tilde{R}_{33}\right) \oslash \left(\tilde{W}_{31} \oplus \tilde{W}_{32} \oplus \tilde{W}_{33}\right)$$

= [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\otimes$ (0.17, 0.22, 0.36, 0.42; 0.95, 0.95) $\oplus$ (0.58, 0.63, 0.80, 0.86; 0.8, 0.8) $\otimes$ (0.72, 0.78, 0.92, 0.97; 0.5, 0.6) $\oplus$ (0.0, 0.0, 0.02, 0.07; 0.9, 0.7) $\otimes$(0.58, 0.63, 0.80, 0.86; 0.8, 0.8)] $\oslash$ [(0.04, 0.1, 0.18, 0.23; 1.0, 1.0) $\oplus$ (0.58, 0.63, 0.80, 0.86; 0.8, 0.8) $\oplus$ (0.0, 0.0, 0.02, 0.07; 0.9, 0.7)]

= (0.3290, 0.4890, 1.1737, 1.7787; 0.5, 0.6)

Step 2. Based on equation 3.13, transform each generalized fuzz number $\tilde{R}_i$ in to a standardized fuzzy number $\tilde{R}_i^*$, where $1 \leq i \leq 3$, shown as follows:

$$\tilde{R}_1 = (0.1765, 0.2860, 0.7244, 1.057; 0.5, 0.6),$$

$$k_1 = \max\left(\lceil 0.1765 \rceil, \lceil |0.2860| \rceil, \lceil |0.7244| \rceil, \lceil |1.057| \rceil, 1\right)$$

$$= \max (1,1,1,2,1) = 2$$

$$\tilde{R}_1^* = \left(\frac{0.1765}{2}, \frac{0.2860}{2}, \frac{0.7244}{2}, \frac{1.0574}{2}; 0.5, 0.6 \right)$$

= (0.0883, 0.1430, 0.3622, 0.5287;0.5;0.6)

$$\tilde{R}_2 = \left(0.3221, 0.4949, 1.1359, 1.6373; 0.4, 0.5\right),$$

$$k_2 = \max\left(\lceil |0.3221| \rceil, \lceil |0.4949| \rceil, \lceil |1.1392| \rceil, \lceil |1.6373| \rceil, 1\right)$$

= max (1,1,2,2,1) = 2

$$\tilde{R}_2^* = \left(\frac{0.3221}{2}, \frac{0.4949}{2}, \frac{1.1392}{2}, \frac{1.6373}{2}; 0.4, 0.5 \right)$$

= (0.1611, 0.2475, 0.5696, 0.8187;0.4;0.5)

$$\tilde{R}_3 = \left(0.3290, 0.4890, 1.1737, 1.7787; 0.5, 0.6\right),$$

$$k_3 = \max\left(\lceil |0.3290| \rceil, \lceil |0.4890| \rceil, \lceil |0.1737| \rceil, \lceil |1.7787| \rceil, 1\right)$$

= max (1,1,2,2,1) = 2

$$\tilde{R}_3^* = \left(\frac{0.3290}{2}, \frac{0.4890}{2}, \frac{1.1737}{2}, \frac{1.7787}{2}; 0.5, 0.6 \right)$$

= (0.1654, 0.2445, 0.5869, 0.8894;0.5;0.6)

Step 3. Using on Equation 3.1 - 3.8, calculate area LN_i, RN_i, LP_i and RP_i of each standardized generalized fuzzy number $\tilde{R}_i^*$, respectively, where $1 \leq i \leq 3$.., shown as follows:

$$LN_1 = 0.7662, \quad RN_1 = 1.1345, \quad LP_1 = 0.4338, \quad RP_1 = 0.0655$$

$$LN_2 = 0.7451, \quad RN_2 = 1.1941, \quad LP_2 = 0.2549, \quad RP_2 = -0.1941$$

$$LN_3 = 0.8876, \quad RN_3 = 1.4857, \quad LP_3 = 0.3125, \quad RP_3 = -0.2858$$

Step 4. Using on equation 3.9 and 3.10, we can calculate the area M_i and N_i of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le 3.$, *s*hown as follows:

$$M_1 = LN_1 + RN_1 = 1.9007, \quad N_1 = LP_1 + RP_1 = 0.4993,$$

$$M_2 = LN_2 + RN_2 = 1.9293 \quad N_2 = LP_2 + RP_2 = 0.0608,$$

$$M_3 = LN_3 + RN_3 = 2.3733, \quad N_3 = LP_3 + RP_3 = 0.0267,$$

Step 5. Using on equation 3.11, calculate the area $c\left(\tilde{R}_i^*\right)$ of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le 3.$, *s*hown as follows:

$$c\left(\tilde{R}_1^*\right) = 0.5251$$

$$c\left(\tilde{R}_2^*\right) = 0.8528$$

$$c\left(\tilde{R}_3^*\right) = 0.8625$$

Step 6. Based on equation 3.12, calculate the area $Score\left(\tilde{R}_i^*\right)$ of each standardized generalized fuzzy numbers $\tilde{R}_i^*$, respectively, where $1 \le i \le 3.$, *s*hown as follows:

$$Score\left(\tilde{R}_1^*\right) = 0.4875$$

$$Score\left(\tilde{R}_2^*\right) = 0.8748$$

$$Score\left(\tilde{R}_3^*\right) = 0.9248$$

Because $Score\left(\tilde{R}_3^*\right) > Score\left(\tilde{R}_2^*\right) > Score\left(\tilde{R}_1^*\right)$, the ranking order of the standardize fuzzy numbers $\tilde{R}_1^*$, $\tilde{R}_2^*$ and $\tilde{R}_3^*$ is :. $\tilde{R}_3^* > \tilde{R}_2^* > \tilde{R}_1^*$ That is, the order of the risk of the manufactories C_1, C_2 and C_3 is : $C_3 > C_2 > C_1$. It means that the component A_3 made by the manufactory C_3 has the highest probability of failure, followed by C_3 and C_1.

Chapter 4

An Approach for the Ranking of Fuzzy Sets with Different Heights

4.1 Introduction

In this chapter the existing approach [40] for the ranking of fuzzy sets with different heights is presented.

4.2 Preliminaries

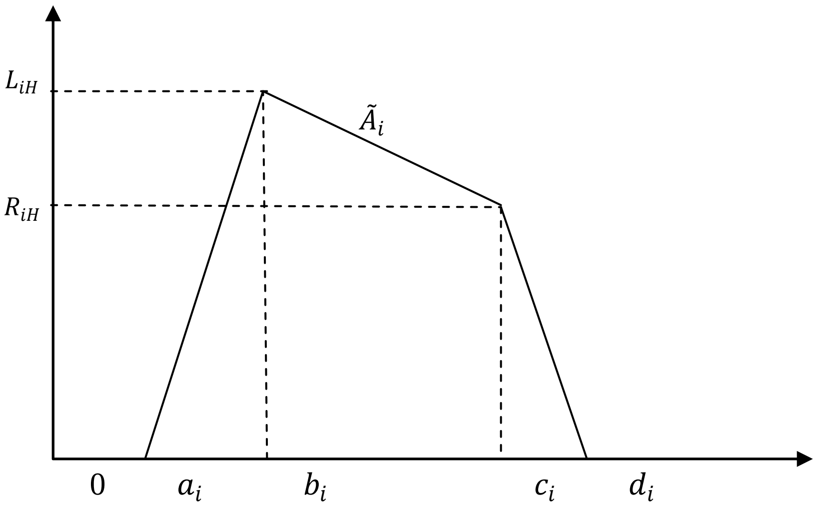
In this section, some basic definitions and arithmetic operations are presented [40].

4.2.1 Basic definitions

Definition 4.1 A fuzzy set $\tilde{A}_i = (a_i, b_i, c_i, d_i; L_{iH}, R_{iH})\ 1 \le i \le n$ is said to be a fuzzy set with different height if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & -\infty \le x \le a_i \\ L_{iH}\left(\dfrac{x-a_i}{b_i-a_i}\right), & a_i \le x \le b_i \\ L_{iH}\left(\dfrac{c_i-x}{c_i-b_i}\right) + R_{iH}\left(\dfrac{x-b_i}{c_i-b_i}\right), & b_i \le x \le c_i \\ R_{iH}\left(\dfrac{x-d_i}{c_i-d_i}\right), & c_i \le x \le d_i \\ 0, & d_i \le x \le \infty \end{cases}$$

where L_{iH} and R_{iH} denotes the left and right height of fuzzy sets with different heights, respectively.



Fuzzy set with different heights.

4.2.2 Arithmetic Operations

Let $\tilde{A}_1=(a_1,b_1,c_1,d_1;L_{1H},R_{1H})$ and $\tilde{A}_2=(a_2,b_2,c_2,d_2;L_{2H},R_{2H})$ be two fuzzy sets with different heights. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2;\ \min(L_{1H},L_{2H}),\min(R_{1H},R_{2H}))$

(ii) $\tilde{A}_1 \Theta \tilde{A}_2=(a_1-d_2,b_1+c_2,c_1-b_2,d_1-a_2;\ \min(L_{1H},L_{2H}),\min(R_{1H},R_{2H}))$

(iii) $\lambda\tilde{A}_1=\begin{cases}(\lambda a_1,\lambda b_1,\lambda c_1,\lambda d_1;L_{1H},R_{1H}); & \lambda\geq 0\\(\lambda d_1,\lambda c_1,\lambda b_1,\lambda a_1;L_{1H},R_{1H}); & \lambda<0\end{cases}$

4.3 Shortcomings of Lee and Chen’s ranking approach

Lee and Chen’s ranking approach explained in Chapter 2. There may exist several fuzzy sets with different heights for which the ranking function given by Lee and Chen [33] do not satisfy the reasonable properties proposed by Wang and Keere [45] for the validation of any ranking function:

If $\tilde{A}$ and $\tilde{B}$ are normal fuzzy sets. Then,

(i) $\tilde{A}\succ\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C})\succ(\tilde{B}\oplus\tilde{C})$

(ii) $\tilde{A}\prec\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C})\prec(\tilde{B}\oplus\tilde{C})$

(iii) $\tilde{A}:\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C}):(\tilde{B}\oplus\tilde{C})$

where, $\tilde{C}$ is a normal fuzzy set.

For the fuzzy sets with different heights, the same property can be written as

If $\tilde{A}=(a_1,b_1,c_1,d_1;L_{1H},R_{1H})$ and $\tilde{B}=(a_2,b_2,c_2,d_2;L_{2H},R_{2H})$ are two fuzzy sets with different heights then

(i) $\tilde{A}\succ\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C})\succ(\tilde{B}\oplus\tilde{C})$

(ii) $\tilde{A}\prec\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C})\prec(\tilde{B}\oplus\tilde{C})$

(iii) $\tilde{A}:\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C}):(\tilde{B}\oplus\tilde{C})$

Where $\tilde{C}=(a_3,b_3,c_3,d_3;L_{3H},R_{3H})$ is a type - II fuzzy set and $(L_{3H},L_{3H})\le\left(\min(L_{1H},L_{2H}),\ \min(R_{1H},R_{2H})\right)$.

$\tilde{A}\succ\tilde{B}\Rightarrow(\tilde{A}\oplus\tilde{C})\succ(\tilde{B}\oplus\tilde{C})$ i.e., according to existing ranking approaches.

$\tilde{A}\succ\tilde{B}\ \not\Rightarrow\ (\tilde{A}\oplus\tilde{C})\succ(\tilde{B}\oplus\tilde{C})$ which is a contradiction to Wang and Keere [45].

Example 1 [33]. Let $\tilde{A}=(1,2,3,4;0.6,0.4)$, $\tilde{B}=(0,3,4,5;0.4,0.2)$ and $\tilde{C}=(1,3,4,5;0.4,0.2)$ be three fuzzy sets with different heights. Then, according to existing ranking approaches Lee and Chen [34], $\tilde{A}\prec\tilde{B}$ but $\tilde{A}\oplus\tilde{C}\succ\tilde{B}\oplus\tilde{C}$ i.e., which is a contradiction.

Example 2 [33]. Let $\tilde{A}=(2,5,6,7;0.6,0.4)$, $\tilde{A}=(3,4,5,6;0.8,0.6)$ and $\tilde{A}=(3,4,7,8;0.6,0.4)$ be three fuzzy sets with different heights. Then, according to existing ranking approaches Lee and Chen [34], $\tilde{A}\prec\tilde{B}$ but $(\tilde{A}\oplus\tilde{C})\succ(\tilde{B}\oplus\tilde{C})$ i.e., which is a contradiction.

4.4. An approach for the ranking of fuzzy sets with different heights

In this section, an approach is reviewed for the ranking of fuzzy sets with different heights. To overcome the shortcomings discussed in Section 4.3, Singh [40] proposed the following definition:

Definition 4.3 [40] For any fuzzy set with different height $\tilde{A}_i = (a_i, b_i, c_i, d_i; L_{iH}, R_{iH})$ $1 \le i \le n$, the expectation value of centroid is defined as follows:

$$M_i = \frac{\int_{a_i}^{d_i} x f_{\tilde{A}_i}(x)}{\int_{a_i}^{d_i} f_{\tilde{A}_i}(x)}$$

Definition 4.4 [40] For any fuzzy set with different heights $\tilde{A}_i = (a_i, b_i, c_i, d_i; L_{iH}, R_{iH})$ $1 \le i \le n$ the transfer coefficient of $\tilde{A}_i$, is given by

$$\lambda_i = \frac{M_i - M_{\min}}{M_{\max} - M_{\min}}$$

where, $M_{\max} = \max(M_1, M_2, ..., M_n)$ and $M_{\min} = \min(M_1, M_2, ..., M_n)$.

Definition 4.5 [40] Let $\tilde{A}_i = (a_i, b_i, c_i, d_i; L_{iH}, R_{iH})$ $1 \le i \le n$ be a fuzzy set with different heights, $a_{\min} = \min(a_1, a_2, ..., a_n)$ and $d_{\max} = \max(d_1, d_2, ..., d_n)$. The areas $S^L(\tilde{A}_i)$ and $S^R(\tilde{A}_i)$ of the left and right side of fuzzy set with different height $\tilde{A}_i$ are defined as follows:

$$S^L(\tilde{A}_i) = \int_0^{L_{iH}} \left(a + \frac{(b-a)y}{L} - a_{\min} \right) dy$$

$$S^R(\tilde{A}_i) = \int_0^{R_{iH}} \left(d_{\max} - d + \frac{(\mathrm{c}-d)y}{R} \right) dy$$

From Definition 4.4 and 4.5, the proposed ranking index of $\tilde{A}_i$ $(i = 1, 2, ..., n)$ is define as follows:

$$S(\tilde{A}) = \frac{S^L(\tilde{A}_i)\lambda_i}{1 + S^R(\tilde{A}_i)(1-\lambda_i)} \tag{4.1}$$

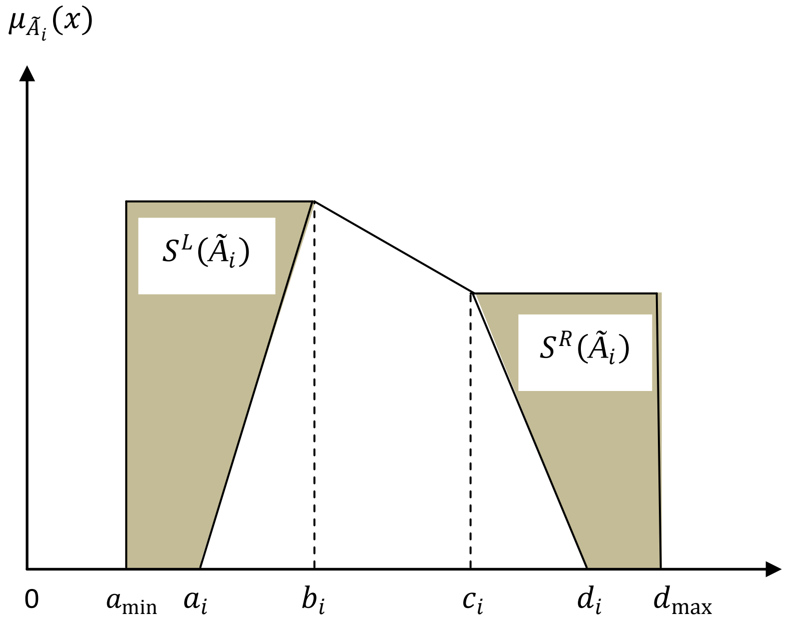


Fig. 4.2 Fuzzy set left and right area.

Definition 4.6 [40] For any two fuzzy sets with different heights, $\tilde{A}_i$ and $\tilde{A}_j$, based on 4.1, their order is defined as follow:

(i) $\tilde{A}_i \succ \tilde{A}_j$ if and only if $S(\tilde{A}_i) \succ S(\tilde{A}_j)$

(ii) $\tilde{A}_i \prec \tilde{A}_j$ if and only if $S(\tilde{A}_i) \prec S(\tilde{A}_j)$

(iii) $\tilde{A}_i : \tilde{A}_j$ if and only if $S(\tilde{A}_i) : S(\tilde{A}_j)$

4.5 Existing method

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1; L_{1H}, R_{1H})$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; L_{2H}, R_{2H})$ be fuzzy sets with different heights in $F(\mathbb{R})$. Use the following step to compare $\tilde{A}_1$ and $\tilde{A}_2$ [40]:

Step 1. Transform $\tilde{A}_1$ and $\tilde{A}_2$ into $\tilde{A}_1^*$ and $\tilde{A}_2^*$ respectively as follows:

$\tilde{A}_1^* = (a_1, b_1, c_1, d_1; L_H, R_H)$ and $\tilde{A}_2^* = (a_2, b_2, c_2, d_2; L_H, R_H)$

where $(L_H, R_H) = (\min(L_{1H}, R_{1H}), \min(L_{2H}, R_{2H}))$.

Step 2. From Definition 4.3, find the expectation values of centroid, M_1 and M_2 of $\tilde{A}_1^*$ and $\tilde{A}_2^*$, respectively.

Step 3. From Definition 4.4, find the transfer coefficients, λ_1 and λ_2 of $\tilde{A}_1^*$ and $\tilde{A}_2^*$, respectively.

Step 4. From Definition 4.5, obtain the value of $S^L(\tilde{A}_1^*)$ and $S^R(\tilde{A}_2^*)$.

Step 5. Using Step 4, find the value of $S(\tilde{A})$.

Step 6. The fuzzy sets with different heights, $\tilde{A}_1^*$ and $\tilde{A}_2^*$ can be compared as follows:

(i) $\tilde{A}_1 \succ \tilde{A}_2$ if and only if $S(\tilde{A}_1^*) \succ S(\tilde{A}_2^*)$

(ii) $\tilde{A}_1 \prec \tilde{A}_2$ if and only if $S(\tilde{A}_1^*) \prec S(\tilde{A}_2^*)$

(iii) $\tilde{A}_1 : \tilde{A}_2$ if and only if $S(\tilde{A}_1^*) : S(\tilde{A}_2^*)$

4.6 Comparative study

In this section, different types of generalized fuzzy sets are taken to compare the results with Lee and Chen's [33] ranking method.

Example 3 [40]. Let $\tilde{A} = (1,2,3,4;0.6,0.4)$ and $\tilde{B} = (0,3,4,5;0.4,0.2)$ be two generalized fuzzy sets. Use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1. Transform $\tilde{A}$ and $\tilde{B}$ into $\tilde{A}^*$ and $\tilde{B}^*$, where

$\tilde{A} = (1,2,3,4;0.4,0.2)$

$\tilde{B} = (0,3,4,5;0.4,0.2)$

Step 2. From Definition 4.3, find the expectation value of centroid, $M_{\tilde{A}^*}$ and $M_{\tilde{B}^*}$ into $\tilde{A}^*$ and $\tilde{B}^*$

.

$$M_{\tilde{A}^*} = \frac{\int_1^4 x f_A(x)dx}{\int_1^4 f_A(x)dx} = \frac{\int_1^2 0.4x\left(\frac{x-1}{2-1}\right)dx + \int_2^3 x\left(0.4\left(\frac{3-x}{3-2}\right)+0.2\left(\frac{x-2}{3-2}\right)\right)dx + \int_3^4 0.2x\left(\frac{x-4}{3-4}\right)dx}{\int_1^2 0.4\left(\frac{x-1}{2-1}\right)dx + \int_2^3\left(0.4\left(\frac{3-x}{3-2}\right)+0.2\left(\frac{x-2}{3-2}\right)\right)dx + \int_3^4 0.2\left(\frac{x-4}{3-4}\right)dx}$$

$$= \frac{0.3333 + 2.0666 + 0.2666 + 0.3333}{0.2 + 0.2 + 0.1 + 0.1} = \frac{2.999}{0.6} = 4.9999$$

$$M_{\tilde{B}^*} = \frac{\int_0^5 x f_A(x)}{\int_0^5 f_A(x)} = \frac{\int_0^3 0.4x\left(\frac{x-0}{3-0}\right)dx + \int_3^4 x\left(0.4\left(\frac{4-x}{4-3}\right) + 0.2\left(\frac{x-3}{4-3}\right)\right)dx + \int_4^5 0.2x\left(\frac{x-5}{4-5}\right)dx}{\int_0^3 0.4\left(\frac{x-0}{3-0}\right)dx + \int_3^4 \left(0.4\left(\frac{4-x}{4-3}\right) + 0.2\left(\frac{x-3}{4-3}\right)\right)dx + \int_4^5 0.2\left(\frac{x-5}{4-5}\right)dx}$$

$$= \frac{1.2 + 0.73333 + 0.36666 - 0.4333}{0.6 + 0.2 + 0.1 - 0.1} = \frac{1.86663}{0.8} = 2.3333$$

Step 3. From Definition 4.4, find the value $\lambda_{\tilde{A}^*}$ and $\lambda_{\tilde{B}^*}$.

$$\lambda_{A^*} = \frac{M_{A^*} - M_{\min}}{M_{\max} - M_{\min}} = \frac{4.9999 - 4.9999}{6.3333 - 4.9999} = 0$$

$$\lambda_{B^*} = \frac{M_{B^*} - M_{\min}}{M_{\max} - M_{\min}} = \frac{2.3333 - 4.9999}{2.3333 - 4.9999} = 1$$

Step 4. From Definition 4.5, find the value of $S^L(\tilde{A}^*), S^R(\tilde{A}^*)$ and $S^L(\tilde{B}^*), S^R(\tilde{B}^*)$.

$$S^L(\tilde{A}^*) = \int_0^{L_{1H}} \left(a_1 + \frac{(b_1 - a_1)y}{L_1} - a_{\min}\right)dy = \int_0^{0.4} \left(1 + \frac{(2-1)y}{0.4} - 1\right)dy$$

$$= \frac{0.4 \times 0.4}{0.4 \times 2} = 0.2$$

$$S^R(\tilde{A}^*) = \int_0^{R_{1H}} \left(d_{\max} - d_1 + \frac{c_1 - d_1}{R_1}\right)dy = \int_0^{0.2} \left(4 - 4 + \frac{(3-4)y}{0.2}\right)dy$$

$$= \frac{0.2 \times 0.2}{0.2 \times 2} = 0.1$$

$$S^L(\tilde{B}^*) = \int_0^{L_{2H}} \left(a_2 + \frac{(b_2 - a_2)y}{L_2} - a_{\min}\right)dy = \int_0^{0.4} \left(0 + \frac{(3-0)y}{0.4} - 0\right)dy$$

$$= \frac{3 \times 0.4 \times 0.4}{0.4 \times 2} = 0.6$$

$$S^R(\tilde{B}^*) = \int_0^{R_{2H}} \left(d_{\max} - d_2 + \frac{c_2 - d_2}{R_2}\right)dy = = \int_0^{0.2} \left(5 - 5 + \frac{(4-5)y}{0.2}\right)dy$$

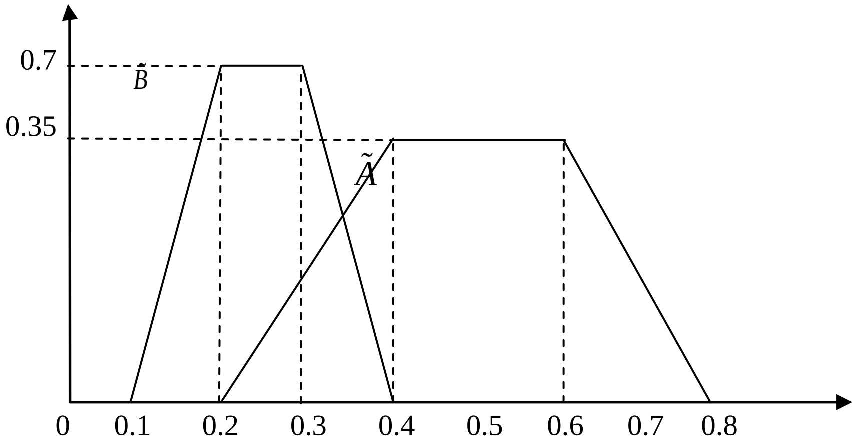
$$= \frac{0.2 \times 0.2}{0.2 \times 2} = 0.1$$

Step 5 . Using Equation 4.1, find the value of $S(\tilde{A}^*)$ and $S(\tilde{B}^*)$.

$$S(\tilde{A}^*) == \frac{S^L(\tilde{A}^*)\lambda}{1+S^R(\tilde{A}^*)(1-\lambda)} = \frac{0.2\times 0}{1+0.1(1-0)} = 0$$

$$S(\tilde{B}^*) == \frac{S^L(\tilde{B}^*)\lambda}{1+S^R(\tilde{B}^*)(1-\lambda)} = \frac{0.6\times 1}{1+0.1(1-0)} = 0.6$$

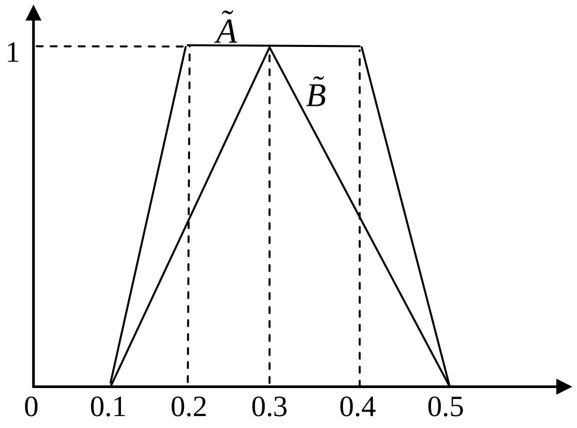
Since, $S(\tilde{A}^*) < S(\tilde{B}^*)$, so $\tilde{A} \prec \tilde{B}$.



$\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$

$\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.35)$

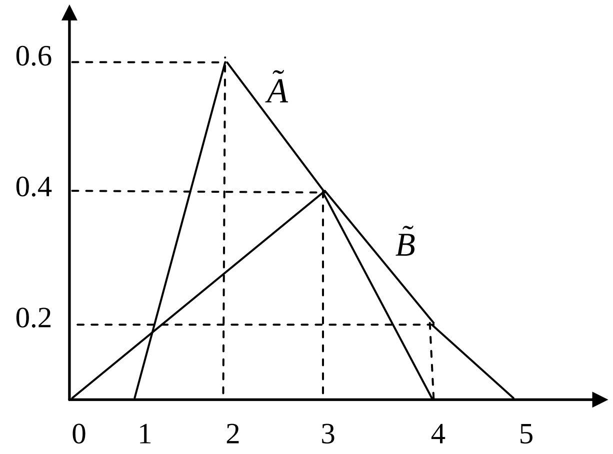
Set 1



$\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$

$\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$

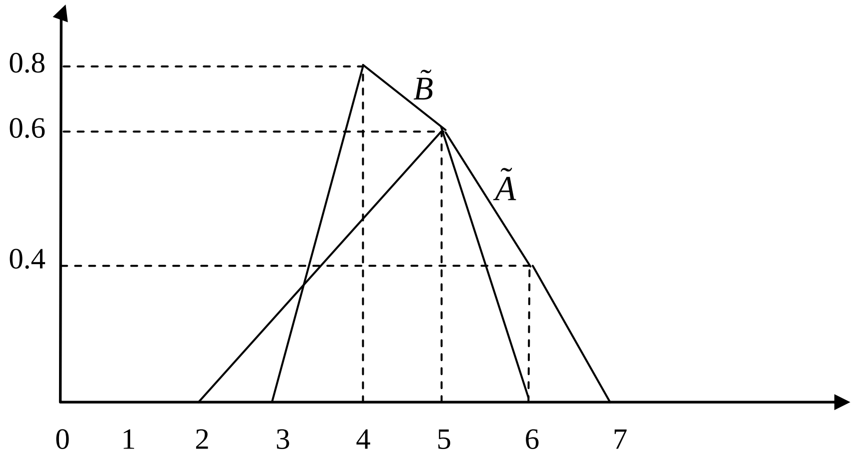
Set 2



$\tilde{A} = (1,2,3,4;0.6;0.4)$

$\tilde{B} = (0,3,4,5;0.4,0.2)$

Set 3



$\tilde{A} = (2,5,6,7;0.6,0.4)$

$\tilde{B} = (3,4,5,6;0.8,0.6)$

Set 4

Fig. 4.3 Generalized fuzzy sets and fuzzy sets with different heights

Similarly, solve all other sets

Example 4 [40]. Let $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized fuzzy sets. Use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1. Transform $\tilde{A}$ and $\tilde{B}$ into $\tilde{A}^*$ and $\tilde{B}^*$, where

$\tilde{A}^* = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B}^* = (0.1, 0.3, 0.3, 0.5; 1)$

Step 2. From Definition 4.3, $M_{\tilde{A}^*} = 0.18771$ and $M_{\tilde{B}^*} = 0.10115$.

Step 3. From Definition 4.4, $\lambda_{\tilde{A}^*} = 1$ and $\lambda_{\tilde{B}^*} = 0$.

Step 4. From Definition 4.5, $S^L(\tilde{A}^*) = 0.5,\ S^R(\tilde{A}^*) = 0.2$ and $S^L(\tilde{B}^*) = 0.1,\ S^R(\tilde{B}^*) = 0.2$.

Step 5. Using Equation 4.1, $S(\tilde{A}^*) = 0.5$ and $S(\tilde{B}^*) = 0$.

Since, $S(\tilde{A}^*) > S(\tilde{B}^*)$, so $\tilde{A}^* \succ \tilde{B}^*$.

Example 5 [40]. Let $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.35)$ be two generalized fuzzy sets. Use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1. Transform $\tilde{A}$ and $\tilde{B}$ into $\tilde{A}^*$ and $\tilde{B}^*$, where

$\tilde{A}^* = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B}^* = (0.1, 0.2, 0.3, 0.4; 0.35)$.

Step 2. From Definition 4.3, $M_{\tilde{A}^*} = -3.766$ and $M_{\tilde{B}^*} = 1.592$.

Step 3. From Definition 4.4, $\lambda_{\tilde{A}^*} = 0$ and $\lambda_{\tilde{B}^*} = 1$.

Step 4. From Definition 4.5, $S^L(\tilde{A}^*) = 0.7,\ S^R(\tilde{A}^*) = 0.035$ and $S^L(\tilde{B}^*) = 0.175,\ S^R(\tilde{B}^*) = 0.0175$

Step 5. Using Equation 4.1, $S(\tilde{A}^*) = 0$ and $S(\tilde{B}^*) = 0.6$.

Since, $S(\tilde{A}^*) < S(\tilde{B}^*)$, so $\tilde{A}^* \prec \tilde{B}^*$.

Example 6 [40]. Let $\tilde{A} = (2, 5, 6, 7; 0.6, 0.4)$ and $\tilde{B} = (3, 4, 5, 6; 0.8, 0.6)$ be two generalized fuzzy sets. Use the following steps to compare $\tilde{A}$ and $\tilde{B}$:

Step 1. Transform $\tilde{A}$ and $\tilde{B}$ into $\tilde{A}^*$ and $\tilde{B}^*$, where

$\tilde{A}^* = (2, 5, 6, 7; 0.6, 0.4)$ and $\tilde{B}^* = (3, 4, 5, 6; 0.6, 0.4)$.

Step 2. From Definition 4.3, $M_{\tilde{A}^*} = 23.83$ and $M_{\tilde{B}^*} = 34.87$.

Step 3. From definition 4.4, $\lambda_{\tilde{A}^*} = 0$ and $\lambda_{\tilde{B}^*} = 1$.

Step 4. Using Equation 4.1, $S(\tilde{A}^*) = 0$ and $S(\tilde{B}^*) = 0.9$.

Since, $S(\tilde{A}^*) < S(\tilde{B}^*)$, so $\tilde{A}^* \prec \tilde{B}^*$.

Table 4.1 Comparison of the presented ranking approach with existing ranking approach.

Method	**Example 3**	**Example 4**	**Example 5**	**Example 6**
Cheng [21]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} : \tilde{B}$	N.A	N.A
Chu and Tsao [23]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} : \tilde{B}$	N.A	N.A
Chen and Chen [14]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	N.A	N.A
Abbasbandy and Hajjari [2]	N.A	$\tilde{A} : \tilde{B}$	N.A	N.A
Chen and Chen [13]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	N.A	N.A
Liou and Wang [35]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} : \tilde{B}$	N.A	N.A
Lee and Chen [32]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$
Rommelfanger [38]	N.A	$\tilde{A} : \tilde{B}$	N.A	N.A
Presented approach	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$

BIBLIOGRAPHY

[1] Abbasbandy, S., and Hajjari, T., A new approach for ranking of trapezoidal fuzzy numbers, Computer and Mathematics with Application, Vol. 57, pp. 413-419, 2009.

[2] Adamo, J. M., Fuzzy decision trees, Fuzzy Sets and System, Vol. 4, pp. 207-219.

[3] Assady, B., The revised method of ranking *LR* fuzzy number based on deviation degree, Expert Systems with Applications, Vol. 37, pp. 5056-5060, 2010.

[4] Baas, S. M., and Kwakeernaak, H., Rating and ranking of multiple aspect alternative using fuzzy sets, Automatica, Vol. 13, pp. 47–58, 1977.

[5] Baldwin, J., and Guild, N. C., Comparision of fuzzy sets on the same decision spaces, Fuzzy Sets and System, Vol. 2, pp. 213-231

[6] Buckley, J., Ranking alternatives using fuzzy numbers, Fuzzy Sets and Systems, Vol. 15, pp. 21–31, 1985.

[7] Chang, J. R., Cheng, C. H., and Kuo, C. Y., Conceptual procedure for ranking fuzzy numbers based on adaptive two-dimensions dominance, Soft Computing, Vol. 10, pp. 94–103, 2006.

[8] Chen, S. H., Ranking fuzzy numbers with maximizing set and minimizing set, Fuzzy Sets and Systems, Vol. 17, pp. 113–129, 1985.

[9] Chen, S. M., New methods for subjective mental workload assessment and fuzzy risk analysis, Cybernetics and Systems, Vol. 27, pp. 449–472, 1996.

[10] Chen, S. H., Ranking generalized fuzzy number with graded mean integration, Proceedings of the Eighth International Fuzzy System Association World Congress, Taiwan, pp. 899-902, 1999.

[11] Chen, S. J., New information fusion and information aggregation methods for handling multicriteria fuzzy decision making problems, PH.D. Dissertation, Department of Computer Science and Information Engineering, National Taiwan University of Science and Technology, Taiwan, 2004

[12] Chen, S. J., and Chen, S. M., Fuzzy risk analysis based on similarity measure of generalized fuzzy numbers, IEEE Transaction on Fuzzy System, Vol. 11, pp. 45-56, 2003.

[13] Chen, S. J., and Chen, S. M., A new method for handling multicriteria fuzzy decision making problems using FN-IOWA operator, Cybernetics and System, Vol. 34, pp. 109-137, 2003.

[14] Chen, S. J. and Chen, S. M., Fuzzy risk analysis based on the ranking of generalized trapezoidal fuzzy numbers, Applied Intelligence, Vol. 26, pp. 1-11, 2007.

[15] Chen, S. M. and Chen, J. H., Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads, Expert Systems with Applications, Vol. 36, pp. 6833-6842, 2009.

[16] Chen, S. J., and Hwang, C. L., Fuzzy multiple attribute decision making: Methods and Applications, Lecture Notes in Economics and Mathematical Systems, New York, Springer, 1992.

[17] Chen, L. H., and Lu, H. W., An approximate approach for ranking fuzzy numbers based on left and right dominance, Computers and Mathematics with Applications, Vol. 41, pp. 1589–1602, 2001.

[18] Chen, S. M., Munif, A., Chen, G. S., Kuo, B. C., and Liu, H. C., A new method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and

right heights, Proceeding of IEEE International Conference on System, Man, and Cybernetics, Alaska, USA, pp. 2307-2312, 2011.

[19] Chen, S. M., Munif, A., Chen, G. S., Liu, H. C. and Kuo, B. C., Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights, Expert Systems with Applications, Vol. 39, pp. 6320-6334, 2012

[20] Chen, S. M., and Sanguansat, K., Analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers, Expert System and Applications, Vol. 38, pp. 2163-2171, 2011.

[21] Chen, S. M. and Wang, C. H., Fuzzy risk analysis based on ranking fuzzy numbers using α-cuts, belief features and signal/noise ratios, Expert Systems with Applications, Vol. 36, pp. 5576-5581, 2009.

[22] Cheng, C. H., A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets and Systems, Vol. 95, pp. 307–317, 1998.

[23] Chu, T. C., and Tsao, C. T., Ranking fuzzy numbers with an area between the centroid point and original point, Computers and Mathematics with Applications, Vol. 43, pp. 111–117, 2002.

[24] Delgado, M., Verdegay, J. L., and Villa, M. A., A procedure for ranking fuzzy numbers using fuzzy relations, Fuzzy Sets and Systems, Vol. 26, pp. 46–62, 1988.

[25] Deng, Y., and Liu, Q., A TOPSIS- based centroid-index ranking method for fuzzy numbers and it’s application in decision making, Cybernetics and Systems, Vol. 36, pp. 581-595, 2005.

[26] Dubois, D., and Prade, H., Ranking of fuzzy numbers in the setting of possibility theory, Information Sciences, Vol. 30, pp. 183–224, 1983.

[27] Efstathiou, J., and Tong, R., Ranking fuzzy sets using linguistic preferrence relations, Proceeding of tenth International Symposium on Multi-Valued Logic, Northwestern University, pp. 137-142, 1980.

[28] Kangari, R., and Riggs, L. S., Construction risk assessment by linguistics. IEEE Transactions on Engineering Management, Vol. 88 36, 126–131, 1989.

[29] Kerre, E. E., The use of fuzzy set theory in electrocarological diagnostics, Approximate Reasoning in Decision analysis, pp. 277-282, 1982.

[30] Kolddziejczyk, W., Orlovsky's Concept of decision-Making with fuzzy preference relation-futher results, Fuzzy Sets and System, Vol. 19, pp. 11-20, 1986.

[31] Kwang, H. C. and Lee, J. H., A method for ranking fuzzy numbers and its application to decision making, IEEE Transactions on Fuzzy Systems, Vol. 7, pp. 677-685, 1999.

[32] Lee, L. W., and Chen, S. M., A new method for ranking fuzzy numbers based on the shapes and the deviations of fuzzy numbers, Proceedings of the second international Conference on natural computation and the third international conference on fuzzy systems and knowledge discovery, China, pp. 269-280, 2006.

[33] Lee, L. W. and Chen, S. M., Fuzzy risk analysis based on fuzzy numbers with different shapes and deviation, Expert Systems with Applications, Vol. 34, pp. 2763-2771, 2008.

[34] Lee, E. G., and Li, R. J., Comparison of fuzzy numbers based on the probability measure of fuzzy events, Computers and Mathematics with Applications, Vol. 10, pp. 887–896, 1988.

[35] Liou, T. S., and Wang, M. J., Ranking fuzzy numbers with integral value, Fuzzy Sets and Systems, Vol. 50, pp. 247–255, 1992.

[36] Mabuchi, G., An approach to the comparison of fuzzy subsets with an a-cut dependent index, IEEE Transactions on Systems, Man and Cybernetics, Vol. 18, pp. 264–272, 1988.

[37] Murakami, S., Maeda, S., and Imamura, S., Fuzzy decision analysis on the development of centralized regional energy control system, Proceedings of the IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis, pp. 363-368, 1983.

[38] Rommelfanger, H., A general concept for solving linear multicriteria programming problem with crisp, fuzzy or stochhhastic values, Fuzzy Set and System, Vol. 158, pp. 1892-1904, 2007.

[39] Schmucker, K. J., Fuzzy Sets: Natural language computations and Risk Analysis, Computer Science Press, 1984.

[40] Singh, P., A new approach for the ranking of fuzzy sets with different heights, Journal of Applied Research and Technology, Vol. 10, pp. 941-949, 2012

[41] Tseng, T. Y., and Klein, C. M., New algorithm for the ranking procedure in fuzzy decision making. IEEE Transactions on Systems, Man and Cybernetics, Vol. 19, pp. 1289–1296, 1989.

[42] Tsukamoto, Y., Nikiforuk, P. N., and Gupta, M. M., On the comparision of fuzzy sets using fuzzy chopping, Control Science and Technology for Process of Society, , NewYork,Pergamon Press, pp. 46-51, 1983.

[43] Wang, Y. M., and Elhag, T., Fuzzy TOPSIS method based on alpha level sets with an application to bridge risk assessment, Expert Systems with Applications, Vol. 31, pp. 309–319, 2006.

[44] Wang, X. and Kerre, E. E., Reasonable properties for the ordering of fuzzy quantities , Fuzzy Sets and Systems, Vol. 118, pp. 375-385, 2001.

[45] Wang, Y. M., and Liou, Y., Area ranking of fuzzy numbers based on positive and negative ideal points, Computers and Mathematics with Applications, Vol. 58, pp. 1769-1779, 2009.

[46] Wei, S. H., and Chen, S. M., Fuzzy risk analysis based on interval-valued fuzzy numbers, Expert Systems with Applications, Vol. 36, pp. 2285-2299, 2009.

[47] Yager, R. R., Ranking fuzzy subset over the unit interval, Proceedings of the seventeenth IEEE International Conference on Decision and Control, San Diego, CA, pp. 1435-1437, 1978.

[48] Yager, R. R., A procedure for ordering fuzzy subsets of the unit interval, Information Sciences, Vol. 24, pp. 143–161, 1981.

[49] Zadeh, L. A., Fuzzy sets, Information and Control, Vol. 8, pp. 338-353, 1965.