ROGERS–RAMANUJAN TYPE IDENTITIES AND COMBINATORICS

A Thesis

submitted in fulfillment of
the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Jasdeep Kaur
(Reg. No. 901211001)

to the

School of Mathematics
THAPAR UNIVERSITY
Patiala, Punjab, India – 147 004

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Declaration of Authorship

I hereby declare that the work which is being presented in this thesis entitled *"Rogers-Ramanujan Type Identities and Combinatorics"* submitted by me, for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar University, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Meenakshi Rana, Assistant Professor, School of Mathematics, Thapar University, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or abroad and that the ideas and references cited herein have been duly acknowledged.

Jasdeep Kaur

Jasdeep Kaur
Reg. No. 901211002

Certificate

This is to certify that the thesis *"Rogers–Ramanujan Type Identities and Combinatorics"* which is submitted by Jasdeep Kaur, in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar University, Patiala, is a record of the candidate's own independent and original research work carried out by her under my supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any university or institute for the award of any degree.

Attestation by supervisor

Dr. Meenakshi Rana
Assistant Professor
School of Mathematics
Thapar University
Patiala, Punjab, India 147 004

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Patiala, April 2016 Jasdeep Kaur

Wings are a constraint that makes
it possible to fly.
— Robert Bringhurst

To the little steps

My son Sukhman

Abstract

In this thesis, we interpret several q–series and q–identities employing combinatorial tools of partitioning of integers, such as $(n+t)$–color partitions introduced by Agarwal and Andrews in 1987 (Agarwal, A. K. and Andrews, G. E. Rogers–Ramanujan identities for partitions with "N copies of N". *Journal of Combinatorial Theory, Series A*, 45:40–49, 1987), lattice paths defined by Agarwal and Bressoud in 1989 (Agarwal, A. K. and Bressoud, D. Lattice paths and multiple basic hypergeometric series. *Pacific Journal of Mathematics*, 136:209–228, 1989) and F–partitions introduced by Andrews in 1984 (Andrews, G. E. Generalized Frobenius partitions. *American Mathematical Society*, 301, 1984).

We have obtained four–way combinatorial indentities. Each four–way combinatorial identity gives us six new combinatorial identities in the usual sense and we get a total of eighteen new combinatorial identities. These new results are contained in Chapter 2 and Chapter 4. The results obtained are accepted for publication as per details given below:

- Sareen, J. K. and Rana, M. Four–way combinatorial interpretations of some Rogers–Ramanujan type identities (Accepted). *Ars Combinatoria*, 2014 (SCI, Impact Factor 0.259).

In Chapter 3 we interpret two tenth order mock theta functions combinatorially using $(n+t)$–color partitions and two mock theta functions generated by Gordon and McIntosh in 2000 (Gordon, B. and McIntosh, R. J. Some eighth order mock theta functions. *Journal of the London Mathematical Society*, 62:321–335, 2000) using signed partitions and ordinary partitions. We have further extended the combinato-

rial interpretation of one of the tenth order mock theta function using F–partitions explicitly given in Chapter 4. The results obtained are accepted/published as per details given below:

- Sareen, J. K. and Rana, M. Combinatorics of tenth order mock theta functions (Accepted). *Proceedings of the Indian Academy of Sciences–Mathematical Sciences*, 2016 (SCI, Impact Factor 0.240).

- Rana, M. and Sareen, J. K. On combinatorial extensions of some mock theta functions using signed partitions. *Advances in Theoretical and Applied Mathematics*, 10(1):15–25, 2015.

Chapter 5 is based on combinatorial interpretations of generalized q–series and split $(n+t)$–color partitions. Each generalized q–series given in this chapter is in conjunction with a Rogers–Ramanujan type identity for a particular value of the parameter. The results obtained in this chapter are accepted for publication as per details given below:

- Rana, M., Sareen, J. K. and Chawla, D. On generalized q–series and split $(n+t)$–color partitions (Accepted). *Utilitas Mathematica*, 2015 (SCI, Impact Factor 0.354).

Further in Chapter 6, the results of Chapter 5 are extended and analogues to the bijections between $(n+t)$–color partitions and F–partitions, new bijections between split $(n+t)$–color partitions and 2–color F–partitions are established for the generalized q–series and hence for Rogers–Ramanujan type identities. Also the similar bijections are established for two Gordon–McIntosh mock theta functions. The results obtained in this chapter are communicated for publication as per details given below:

- Rana, M. and Sareen, J. K. Split $(n+t)$–color partitions and 2–color F–partitions (Communicated).

Contents

Chapter 1

Introduction

The theory of partitions is a significant branch of additive number theory and combinatorics. Liebniz, in 1669, was the first to raise a question to Bernoulli about finding the count in which a given positive integer can be written as sum of more than one positive integers. It leads to defining partitions of a positive integer and a subject connecting theory of numbers with combinatory analysis has been introduced which today we renowned as 'Partition Theory'.

1.1 Preliminaries

Informally, a partition of a non–negative integer ν is the representation of ν as sum of positive integers where the order of the summands is considered irrelevant. Thus the three partitions of 3 are 3, 2+1 and 1 + 1 + 1. The summands are also called the "parts" of a partition, and the order of the parts is considered to be irrelevant. Therefore 2+1, 1+2 are the same partition of 3. Formally, partition of positive integer is defined as below;

Definition 1.1.1. *[23] A partition π of an integer ν into l parts is an l–tuple of positive integers $(\pi_1, \pi_2, \ldots, \pi_l)$ where*

$$\pi_i \geq \pi_{i+1} \quad \textit{for} \quad 1 \leq i \leq l-1 \tag{1.1}$$

such that

$$\sum_{i=1}^{l} \pi_i = \nu. \tag{1.2}$$

The number of parts $l = l(\pi)$, of π is called the length of π. The sum of parts of π is called the weight of π and is denoted by $|\pi|$.

Thus in the above notation, the relevant partitions of 3 are (3), (2,1) and (1,1,1). Also we can represent the partitions of an integer by exponent notation. So the partitions of 3 can be written in the exponent notation as (3), (2 1), (1^3).
About seventy years later, in 1748, Euler showed interest in partitions when Naudé raised a question about finding the number of ways in which 50 can be written as the sum of seven positive integers. It was the time when real developments in partition theory have been started. Euler [40], in his first mathematical paper on partition theory, obtained the generating function for partitions.

Definition 1.1.2. *[40] Let $p(\nu)$ be the partition function of ν and $p(0) = 1$, as the only considered partition of zero is the empty partition. The generating function $f(q)$ for $p(\nu)$ is given by*

$$f(q) = \sum_{\nu=0}^{\infty} p(\nu) q^{\nu} = \frac{1}{(q;q)_\infty}, \qquad \textit{where } |q| < 1. \tag{1.3}$$

$(a;q)_m$ is the q–rising factorial defined as

$$(a;q)_m = \prod_{k=0}^{\infty} \frac{(1-aq^k)}{(1-aq^{k+m})}, \qquad \textit{for any constant } a. \tag{1.4}$$

If m is a positive integer then obviously,

$$(a;q)_m = \prod_{k=0}^{m-1} (1-aq^k) = (1-a)(1-aq)(1-aq^2)\cdot\cdots\cdot(1-aq^{m-1})$$

and for $m \to \infty$,

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots .$$

In order to understand the above generating function, rewrite the right hand side of (1.3) by multiplying and rearranging like terms which results in the relevant partitions of a positive integer, as given below;

$$\begin{aligned}
\frac{1}{(q;q)_\infty} =& \frac{1}{(1-q)(1-q)^2(1-q)^3(1-q)^4\ldots} \\
=& (1+q^{1.1}+q^{1.2}+q^{1.3}+q^{1.4}+q^{1.5}+\cdots)\times \\
& (1+q^{2.1}+q^{2.2}+q^{2.3}+q^{2.4}+q^{2.5}+\cdots)\times \\
& (1+q^{3.1}+q^{3.2}+q^{3.3}+q^{3.4}+q^{3.5}+\cdots)\times \\
& (1+q^{4.1}+q^{4.2}+q^{4.3}+q^{4.4}+q^{4.5}+\cdots). \\
=& 1+q^{1.1}+(q^{1.2}+q^{2.1})+(q^{1.3}+q^{1.1+2.1}+q^{3.1})+ \\
& (q^{1.4}+q^{1.1+3.1}+q^{1.2+2.1}+q^{2.2}+q^{4.1})+\cdots \qquad (1.5) \\
=& 1+q^1+2q^2+3q^3+5q^4+\cdots. \qquad (1.6)
\end{aligned}$$

Expression (1.5) clearly shows all the relevant partitions of a particular value of ν.

Example 1.1.1. *The table for partitions of 4 interpreted by expression* (1.5) *is given below:*

4 as an exponent	*Interpretation*	*Relevant partition*
$q^{1.4}$	1 *is appearing four times*	$1+1+1+1$
$q^{1.1}.q^{3.1}$	*both* 1 *and* 3 *are appearing once*	$1+3$
$q^{1.2}.q^{2.1}$	1 *and* 2 *are appearing twice and once, respectively*	$1+1+2$
$q^{2.2}$	2 *is appearing twice*	$2+2$
$q^{4.1}$	4 *is appearing once*	4

If parts in a partition of ν are taken from a particular set, say S, then the partitions formed are said to be restricted partitions of ν. The following table shows the generating function for some of the restricted partitions of ν.

Partition function	Interpretation	Generating function
$p(S,m,\nu)$	Number of partitions of ν into m parts from the set S	$\prod_{k\in S}\frac{1}{(1-aq^k)}$
$p(m,\nu)$	Number of partitions of ν into m parts or number of partitions of ν with largest part equals to m	$\frac{1}{(q;q)_m}$
$p(O,\nu)$	Number of partitions of ν into odd parts	$\frac{1}{(q;q^2)_\infty}$
$p(D,\nu)$	Number of partitions of ν into distinct parts	$(-q;q)_\infty$
$p(S_{\leq n},m,\nu)$	Number of partitions of ν into at most m parts less than or equal to $n-m$	$\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}$

Euler [40] proved numerous results using the generating functions. One of his famous result is as follows;

Theorem 1.1.1 (Euler Identity). *For a given positive integer ν, the number of partitions of ν into distinct parts is equal to the number of partitions of ν into odd parts. That is,*

$$\prod_{m=1}^{\infty}(1+q^m)=\prod_{m=1}^{\infty}\frac{1}{(1-q^{2m-1})}. \tag{1.7}$$

He gave two separate proofs of (1.7), one using generating functions and another using combinatorial approach.

Analytic proof. For $|q|<1$

$$\begin{aligned}\sum_{\nu=0}^{\infty}p(D,\nu)q^{\nu}&=\prod_{m=1}^{\infty}(1+q^m)\\&=\prod_{m=1}^{\infty}\frac{(1+q^m)(1-q^m)}{(1-q^m)}\\&=\prod_{m=1}^{\infty}\frac{(1-q^{2m})}{(1-q^{2m-1})(1-q^{2m})}\\&=\prod_{m=1}^{\infty}\frac{1}{(1-q^{2m-1})}\\&=\sum_{\nu=0}^{\infty}p(O,\nu)q^{\nu}.\end{aligned}$$

□

Combinatorial proof. For a given partition of ν into distinct parts, split each even part into two halves and repeat splitting till there are only the odd parts in the partition. It becomes the partition into odd parts. For the reverse implication, for a partition of ν into odd parts, keep on adding two equal parts till there are all the distinct parts in the partition. □

Example 1.1.2. *Consider a partition $\pi = 8+6+5+1$ of 20 into distinct parts. Then*

$$\begin{aligned}\pi &= 8+6+5+1\\ &= 4+4+3+3+5+1\\ &= 2+2+2+2+3+3+5+1\\ &= 1+1+1+1+1+1+1+1+3+3+5+1\\ &= 5+3+3+1+1+1+1+1+1+1+1+1.\end{aligned}$$

Conversely, for a partition $\pi = 5+3+3+1+1+1+1+1+1+1+1+1$ of 20 into odd parts, we have

$$\begin{aligned}\pi &= 5+3+3+1+1+1+1+1+1+1+1+1\\ &= 5+6+2+2+2+2+1\\ &= 5+6+4+4+1\\ &= 8+6+5+1.\end{aligned}$$

The more generalized form for the Euler Identity was given by Glashier [44] as follows;

Theorem 1.1.2. *The number of partitions of ν into parts not divisible by d is equal to the number of partitions of ν of the form $\nu = a_1 + a_2 + \ldots + a_s$, where $a_i \geq a_{i+1}$ and $a_i \geq a_{i+d-1} + 1$.*

For $d = 2$, above Theorem 1.1.2 is same as, Euler Identity, given by Theorem 1.1.1. Many authors explored the ordinary partitions to interpret certain q–series/identities and generalized q–series combinatorially, for instance see [10, 21, 29, 31, 39, 44, 51, 66, 67].

1.2 Graphical representation of a partition

Definition 1.2.1. *The Ferrers graph of a partition* $\pi = (\pi_1, \pi_2, ..., \pi_s)$ *of* ν *is a set of rows of equally spaced dots aligned on the left where the* j^{th} *row has* π_j *dots.*

Example 1.2.1. *The Ferrers graph of the partition* $6+4+3+1$ *of* 14 *is given below;*

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & & & & & \end{array} \tag{1.8}$$

Reading this graph horizontally, we see that the number of dots in each row is equal to the value of corresponding part in the given partition.

Definition 1.2.2. *The conjugate of a partition* π *is denoted by* π^c *and is obtained by interchanging rows with its columns in the Ferrers diagram.*

Example 1.2.2. *The conjugate of partition* $\pi = 6+4+3+1$ *is* $\pi^c = 4+3+3+2+1+1$ *which is given by the following Ferrers graph.*

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \\ \bullet & & & \end{array} \tag{1.9}$$

A partition is said to be self conjugate if it is identical with its conjugate. That is, number of dots in i^{th} row are equal to number of dots in i^{th} column.

Example 1.2.3. *$\pi = 4+3+2+1$ is a self conjugate partition of 10 as shown in the Ferrers graph given below;*

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \tag{1.10}$$

Either reading the above Ferrers graph horizontally or vertically we get the same partition $\pi = 4+3+2+1$.

1.3 Frobenius representation of a partition

Frobenius [42] in 1900, while studying the representation theory of groups, introduced the following representation for the ordinary partitions as a two rowed array

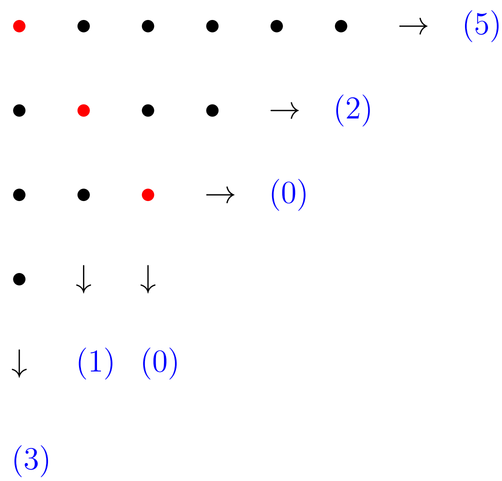
$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

of non–negative integers such that parts in each array are distinct and arranged in decreasing order. The above representation is known as Frobenius representation or symbol of an ordinary partition of ν if

$$\nu = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i. \tag{1.11}$$

Each ordinary partition of ν has a unique such Frobenius symbol associated with it. He established the one to one correspondence between ordinary partitions and their Frobenius symbols with the help of Ferrers Graph.

Example 1.3.1. *Consider a partition $\pi = 6+4+3+1$ of 14, its Ferrers graph is given below;*



Now the diagonal entries (shown in red dots) represent the number of columns, r, in the Frobenius symbol. The number of dots in each horizontal row right to the diagonal entries, form the first array of Frobenius symbol. The number of dots in each vertical column, below the diagonal entries, form the second array of Frobenius symbol.
Thus the Frobenius symbol

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

is associated with the ordinary partition $\pi = 6 + 4 + 3 + 1$ *of* 14.

1.4 Rogers–Ramanujan identities

The contributions of Indian mathematician Srinivasa Ramanujan to the number theory are remarkable. The story begins with the discovery of the Indian genius S. Ramanujan by G. H. Hardy. In his first letter to Hardy [20] in 1913, Ramanujan stated several marvelous theorems on continued fractions. Using generating function for the ordinary partitions, Ramanujan [56] has proved the following congruence relations for the partition function.

$$p(5n + 4) \equiv 0 (mod 5),$$

$$p(7n + 5) \equiv 0 (mod 7),$$

$$p(11n+6) \equiv 0(mod 11).$$

The following two illustrious Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{\substack{k\geq 1 \\ k\equiv \pm 1(mod 5)}} \frac{1}{(1-q^k)}, \tag{1.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{\substack{k\geq 1 \\ k\equiv \pm 2(mod 5)}} \frac{1}{(1-q^k)} \tag{1.13}$$

were found by Rogers [62] in 1894. Somewhat before 1913, S. Ramanujan reproduced the q–series identities (1.12) and (1.13) independently and proved them. Since then these identities are collectively known as Rogers–Ramanujan identities. Many additional q–series and q–identities were found by Ramanujan and were recorded in a book on Ramanujan's collected papers [57] and in his lost notebook [58].
The combinatorial interpretations of (1.12) and (1.13) were given by MacMahon [54] in the following two theorems, respectively.

Theorem 1.4.1. *The number of partitions of ν into parts with minimal difference two equal to the number of partitions of ν into parts which are congruent to* $1, 4(mod 5)$.

Theorem 1.4.2. *The number of partitions of ν into parts with minimal difference two and parts are greater than one equal to the number of partitions of ν into parts which are congruent to* $2, 3(mod 5)$.

The celebrated joint work of Ramanujan with Hardy indeed revolutionized the study of partitions. Because of its numerous applications in different areas like particle physics, statistical mechanics, vertex operator, string theory and Lie algebra, the theory of partitions has become one of the most hot research areas of number theory today.

Analytic generalizations of Rogers–Ramanujan identities

Andrews [22] provided the following analytic generalization of Rogers–Ramanujan identities.

Theorem 1.4.3. *For* $1 \le i \le k,\ \ k \ge 2,\ \ |q| < 1$

$$\sum_{n_1, n_2, \ldots n_{k-1} \ge 0} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+N_i+N_{i+1}+\ldots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i (mod\ \ 2k+1)}}^{\infty} (1-q^n)^{-1}, \qquad (1.14)$$

where $N_j = n_j + n_{j+1} + \ldots + n_{k-1}$.

It can be easily seen that Identities (1.12) and (1.13) are the particular cases of (1.14) for $k = i = 2$ and $k = i + 1 = 2$, respectively.

Combinatorial generalization of Rogers–Ramanujan identities

Combinatorial generalization of Rogers–Ramanujan identities was due to Gordon [47] as given below:

Theorem 1.4.4. *For* $1 \le i \le k, k \ge 2$*, let* $A_{k,i}(\nu)$ *represent the number of partitions of* ν *of the form* $(\pi_1, \pi_2, \cdots, \pi_r)$ *where* $\pi_j - \pi_{j+k-1} \ge 2$*, and at most* $(i-1)$ *of* π_j *equal 1. Let* $B_{k,i}(\nu)$ *represent the number of partitions of* ν *into parts* $\not\equiv 0, \pm i (mod 2k+1)$*. Then*

$$A_{k,i}(\nu) = B_{k,i}(\nu) \qquad \text{for all } \nu. \qquad (1.15)$$

Clearly, Theorems 1.4.1 and 1.4.2 are the particular cases of Theorem 1.4.4 for $k = i = 2$ and $k = i + 1 = 2$, respectively.

1.5 Göllnitz–Gordon identities

The analytic versions of famous Göllnitz–Gordon identities are given below;

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_n (q^4;q^8)_n (q^7;q^8)_n}, \qquad (1.16)$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_n (q^4;q^8)_n (q^5;q^8)_n}, \qquad (1.17)$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^8)_n (q^3;q^8)_n (q^7;q^8)_n}. \qquad (1.18)$$

The famous Göllnitz–Gordon identities (1.16) and (1.17) were given by Slater [64, I(36), I(34)]. However, it has been stated by Sills that the two analytic identities (1.16) and (1.17) were discovered by Ramanujan in his Lost notebook [27] and Ramanujan knew about these identities even before Slater rediscovered them. Identity (1.18) was given in [23, Cor. 2.7, p.21 with $q = q^2$ and $a = -q$] The following theorems are the combinatorial counterparts of identities (1.16)–(1.18).

Theorem 1.5.1. *The number of partitions of ν into parts which differ by at least 2 such that no two consecutive even numbers appear is equal to the number of partitions of ν into parts which are congruent to $\pm 1, 4(mod 8)$.*

Theorem 1.5.2. *The number of partitions of ν into parts which differ by at least 2 such that no two consecutive even numbers appear and parts are greater than 2 is equal to the number of partitions of ν into parts which are congruent to $\pm 3, 4(mod 8)$.*

Theorem 1.5.3. *The number of partitions of ν into parts which differ by at least 2 such that no two consecutive odd numbers appear and with each part being at least 2 is equal to the number of partitions of ν into parts which are congruent to $2, 3, 7(mod 8)$.*

Theorems 1.5.1 and 1.5.2 were first introduced by Göllnitz [45] in 1960 and were included in his unpublished honors baccalaureate thesis. However, the results were popularized in 1965 when Gordon [48] independently rediscovered them. Theorem 1.5.3 was given by Göllnitz [46] in 1967.

Combinatorial generalizations of Göllnitz–Gordon identities

Andrews [21] in 1967 gave the combinatorial generalization of Göllnitz–Gordon identities as follows:

Theorem 1.5.4. *Let i and k be integers such that $0 < i \leq k$. Let $C_{k,i}(\nu)$ be the number of partitions of ν into parts which are $\not\equiv 2(mod 4)$ and also $\not\equiv 0, \pm(2i - 1)(mod 4k)$. Let $D_{k,i}(\nu)$ denote the number of partitions of ν of the form $\nu = \sum_{j=1}^{\infty} a_j.j$ with $a_1 + a_2 \leq i - 1$ and for all $j \geq 1$, $a_{2j-1} \leq 1$ and $a_{2j} + a_{2j+1} + a_{2j+2} \leq k - 1$, Then*

$$C_{k,i}(\nu) = D_{k,i}(\nu). \tag{1.19}$$

The above theorem, for $k = i = 2$ and for $k = i + 1 = 2$ reduces to (1.16) and(1.17), respectively. Later in 1986, Agarwal [1] obtained the interpretation of a generalized basic q–series, given by (1.20) below, which in particular cases gives rise to Göllnitz–Gordon identities.

Theorem 1.5.5. *Given a positive integer k, let $A_k(\nu)$ denote the number of partitions of ν in which each part $\geq k$, minimal difference ≥ 2 between the parts, consecutive odd integers are not allowed if k is even and consecutive even integers are not allowed if k is odd. Then*

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}. \tag{1.20}$$

Theorem 1.5.5 for $k = 1 - -3$, reduces to Theorems 1.5.1–1.5.3 and provide the interpretations of identities (1.16)–(1.18).

1.6 n–color partitions

Agarwal and Andrews [9] in 1987 introduced the concept of colored partitions. These colored partitions together with Rogers–Ramanujan type identities arose naturally in the Baxter's hard hexagonal model [33].

Definition 1.6.1. *[9] A partition with "n copies of n" or n–color partition is a partition in which a part of size n, $(n \geq 0)$, can come in n different colors denoted by the subscripts, $n_1, n_2, n_3, \cdots, n_n$.*

Example 1.6.1. *The six n–color partitions of 3 are,*

$$3_1, \quad 3_2, \quad 3_3, \quad 2_1 1_1, \quad 2_2 1_1, \quad 1_1 1_1 1_1.$$

Definition 1.6.2. *[9] If m_i, n_j, $m \geq n$ are any two parts of an n–color partition, then their weighted difference is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.*

The generating function $F(q)$ for the partition function $P(\nu)$ for n–color partitions is given by,

$$F(q) = \sum_{\nu=0}^{\infty} P(\nu)q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \tag{1.21}$$

which can be easily established once we understand the generating function corresponding to ordinary partitions (1.3).

Using these n–color partitions Agarwal [2] obtained the combinatorial interpretation of a generalized basic q–series

$$\sum_{\nu=0}^{\infty} B_k(\nu)q^{\nu} = \frac{q^{n(1+\frac{(k+3)(n-1)}{2})}}{(q;q)_n(q;q^2)_n} \tag{1.22}$$

in the following Theorem 1.6.1.

Theorem 1.6.1. *For $k \geq -3$, $B_k(\nu)$ represent the number of n color partitions of ν such that each pair of parts $m_i,\ n_j$ satisfies $((m_i - n_j)) > k$.*

For $k = 0, -1, -2$, Theorem 1.6.1, in view of the identities

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q)_n(q;q^2)_n} = \frac{[q^4, q^6, q^{10}; q^{10}]}{(q;q)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n(q;q^2)_n} = \frac{[q^6, q^8, q^{14}; q^{14}]}{(q;q)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n(q;q^2)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}\frac{[q^2, q^5, q^7; q^7]}{[-q, -q^6; q^7]}$$

from [64, I(46), I(61)] and [30, Eq. (3.1)] reduces to the Theorems 1.6.2, 1.6.3 and 1.6.4, respectively, given below:

Theorem 1.6.2. *The number of n–color partitions of ν such that each pair of parts $m_i,\ n_j,\ m_i \geq n_j$ satisfies the weighted difference condition $((m_i - n_j)) > 0$ is equal to the number of ordinary partitions of ν into parts $\not\equiv 0, \pm 4 (mod 10)$.*

Theorem 1.6.3. *The number of n–color partitions of ν such that each pair of parts $m_i,\ n_j,\ m_i \geq n_j$ satisfies the weighted difference condition $((m_i - n_j)) \geq 0$ is equal to the number of ordinary partitions of ν into parts $\not\equiv 0, \pm 6 (mod 14)$.*

Theorem 1.6.4. *The number of n–color partitions of ν such that each pair of parts, $m_i,\ n_j,\ m_i \geq n_j$ satisfies the weighted difference condition $((m_i - n_j)) \geq 0$ is equal to $\sum_{k=0}^{\nu} C(\nu - k)D(k)$, where $C(\nu)$ denote the number of partitions of ν into distinct* *parts* $\equiv \pm 3(mod 7)$ *and* $D(\nu)$ *denote the number of ordinary partitions of* ν *into parts* $\not\equiv 0, \pm 4(mod 14)$.

n–color partition theoretic interpretations of Göllnitz–Gordon identities

Following the approach of [2], Rana and Agarwal [16] used the n–color partitions to obtain the combinatorial interpretations of generalized series (1.20), given by Theorem 1.6.5, which in particular cases, for $1 \leq k \leq 3$, provide the n–color partition theoretic interpretations of identities (1.16)–(1.18).

Theorem 1.6.5. *For a given positive integer* k*, let* $G_k(\nu)$ *represent the* n*–color partitions of* ν *into parts greater than or equals to* k *such that first copy(resp. second copy) of the odd parts(resp. even parts) and second copy(resp. first copy) of the even parts(resp. odd parts) appear if* k *is odd(resp. even). The weighted difference between any two parts is nonnegative and even. Then*

$$\sum_{\nu=0}^{\infty} G_k(\nu) q^{\nu} = \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}. \tag{1.23}$$

Hence, from (1.20) and (1.23), we have

$$A_k(\nu) = G_k(\nu), \quad \text{for } k \geq 1 \tag{1.24}$$

The equation (1.24) give rise to two way infinite combinatorial identities. Also, in some particular cases, it leads to three way combinatorial interpretations of (1.16)–(1.18). These n–color partitions were further explored in [6, 50] to obtain the combinatorial meanings of q–series from Ramanujan's lost notebook [58] and q–identities from Slater's compendium [64]. Also the bijections between n–color partitions and other combinatorial tools like F–partitions, lattice paths, etc. are established in [5, 7, 8, 9, 12, 13, 15, 59, 60, 65]. These bijections are discussed in the forthcoming chapters.

In our present work, we provide the interpretations of several q–identities, q–series, mock theta functions and generalized q–series using some combinatorial tools of partitioning of integers such as ordinary partitions, colored partitions, lattice paths, signed partitions, F–partitions and colored F–partitions.

Chapter 2

Rogers–Ramanujan type identities for *(n+t)*–color partitions and lattice paths

2.1 Introduction

The n–color partitions, introduced by Agarwal and Andrews [9] were discussed in Chapter 1 and further the n–color partitions are extended to $(n+t)$–color partitions where the parts of size n can come in $(n+t)$, $t \geq 0$, different colors.

Note here that zero appears as a part only if $t \geq 1$ and also zeros are not allowed to repeat in any partition.

The parts in an $(n+t)$–color partition can be arranged lexicographically as:

$$1_1 < 1_2 < 1_3 < \cdots < 2_1 < 2_2 < 2_3 < \cdots < 3_1 < 3_2 < 3_3 < \cdots .$$

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Example 2.1.1. *There are twenty $(n+2)$–color partitions of 2 as given below;*

$$\begin{array}{ccccc} 2_1 & 2_1 0_2 & 1_1 1_1 & 1_3 1_2 & 1_1 1_1 0_2 \\ 2_2 & 2_2 0_2 & 1_2 1_1 & 1_3 1_3 & 1_2 1_2 0_2 \\ 2_3 & 2_3 0_2 & 1_3 1_1 & 1_2 1_1 0_2 & 1_3 1_2 0_2 \\ 2_4 & 2_4 0_2 & 1_2 1_2 & 1_3 1_1 0_2 & 1_3 1_3 0_2 \end{array} \tag{2.1}$$

Since the $(n+t)$–color partitions are only the extensions of n–color partitions, so the weighted difference among any two parts, m_i, n_j, in an $(n+t)$–color partition is same as defined for n–color partition, given in Chapter 1, that is, $m-i-n-j$ and denoted by $((m_i - n_j))$.
Using the extended version of n–color partitions, Agarwal [3] in 1988, provided the combinatorial interpretations of seven Rogers–Ramanujan type identities from Slater's compendium [64]. In 1989, Agarwal [4] interpreted two basic analytic q–series using $(n+t)$–color partitions. Later in 2005, Goyal and Agarwal [50] obtained the $(n+t)$–color partition theoretic interpretations of another four q–identities known as Rogers–Selburg identities given in [29, 61, 63, 64].
Colored partitions and lattice paths have a long history in enumerating the q–series. More interestingly in literature we also find the bijections between the two combinatorial tools, for instance see [5, 7, 12, 15]. Next, we recall the following description of lattice paths given by Agarwal and Bressoud [11] in 1989.

Definition 2.1.1. *[11] The length of all paths will be finite and they lie in the first quadrant. They will start from the y–axis and end on the x–axis. The following three moves are allowed at each step;*
northeast : from (a,b) to $(a+1,b+1)$,
southeast : from (a,b) to $(a+1,b-1)$, only allowed if $b>0$,
horizontal: from $(a,0)$ to $(a+1,0)$, allowed along x–axis only.
The terminology used for describing lattice paths is as follows:
Peak *can either be a vertex preceded by a northeast move and followed by a southeast move or a vertex on the y–axis which is followed by a southeast move.*

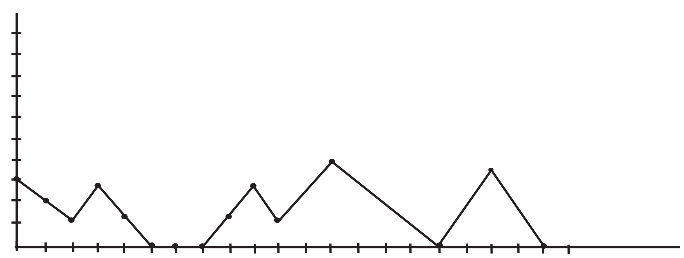
Valley *is a vertex preceded by a southeast move and followed by a northeast move. Note that a southeast move followed by a horizontal move followed by a northeast step does not form a valley.*

Mountain *is a segment of the path which initiate either on x or y–axis, ends on the x–axis, and which does not touch the x–axis anywhere in between the end points. There is at least one peak in a mountain and the number of peaks may exceed one.*

Plain *is a segment of the path comprising of only horizontal steps which starts either on the y–axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.*

The **Height** *of a vertex is its y–coordinate. The* **Weight** *of a vertex is its x–coordinate. The* **Weight** *of a path is the sum of the weights of its peaks.*

Example 2.1.2. *Following is an example of lattice paths defined above:*



In the graph given above, there are three mountains, three valleys, one plain and five peaks.

Following the methods adopted in [3, 4, 5, 7, 12, 15, 50], we, in this chapter, provide the combinatorial interpretations of the following three Rogers–Ramanujan type identities that appear in Slater's compendium [64, I(118), I(117), I(119)] as well as derived in Chu and Zhang's compendium [38, I(192), I(193), I(194)].

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{14},q,q^{13};q^{14}]_\infty[q^{16},q^{12};q^{28}]_\infty, \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4,q^4)_n(q,q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{14},q^3,q^{11};q^{14}]_\infty[q^{20},q^{8};q^{28}]_\infty, \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{14},q^5,q^{9};q^{14}]_\infty[q^{24},q^{4};q^{28}]_\infty. \tag{2.4}$$

In the next section, Identities (2.2)–(2.4) are interpreted using $(n+t)$–color partitions. In the further sections, we then provide the combinatorial interpretations of (2.2)–(2.4) using lattice paths and also establish the bijections between $(n+t)$–color partitions and lattice paths.

2.2 Combinatorial interpretations using *(n+t)*–color partitions

Theorem 2.2.1. *For $\nu \geq 0$, let $A_1(\nu)$ denote the number of partitions of ν with 'n copies of n' into parts greater than or equal to 3 such that if m_i is the least or the only part in the partition then $m - i \equiv 2(mod 4)$ and weighted difference between consecutive parts is non negative and $\equiv 0(mod 4)$ and let*

$$B_1(\nu) = \sum_{k=0}^{\nu} C_1(\nu - k) D_1(k),$$

where $C_1(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 4, \pm 6, \pm 8, \pm 10$ $(mod 28)$(or equivalently $\pm 4, \pm 6$ $(mod 14)$) and $D_1(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm 3, \pm 5, 7(mod 14)$. Then

$$A_1(\nu) = B_1(\nu), \quad \textit{for all } \nu.$$

Example 2.2.1. *We demonstrate Theorem 2.2.1 by showing that*

$$A_1(8) = B_1(8) = 3.$$

The relevant n-color partitions corresponding to $A_1(8)$ are

$$8_2, \qquad 8_6, \qquad 5_1 3_1$$

and $B_1(8) = 3$, since

$$B_1(8) = \sum_{k=0}^{8} C_1(8 - k) D_1(k),$$

where the relevant partitions corresponding to $C_1(\nu)$ and $D_1(\nu)$ are given in the table below:

ν	$C_1(\nu)$	*Partitions enumerated by* $C_1(\nu)$	$D_1(\nu)$	*Partitions enumerated by* $D_1(\nu)$
0	1	*empty partition*	1	*empty partition*
1	0	-	1	1
2	0	-	0	-
3	0	-	1	3
4	1	4	0	-
5	0	-	1	5
6	1	6	0	-
7	0	-	1	7
8	2	$8, 4+4$	1	$5+3$

hence,

$$\begin{aligned} B_1(8) =& C_1(8)D_1(0) + C_1(7)D_1(1) + \cdots + C_1(0)D_1(8) \\ =& 3. \end{aligned}$$

Theorem 2.2.2. *For $\nu \geq 0$, let $A_2(\nu)$ denote the number of partitions of ν with 'n copies of n' into parts such that if m_i is the least or the only part in the partition then $m \equiv i(mod4)$ and weighted difference between consecutive parts is non negative and $\equiv 0(mod4)$ and let*

$$B_2(\nu) = \sum_{k=0}^{\nu} C_2(\nu - k)D_2(k),$$

where $C_2(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 2, \pm 4, \pm 10, \pm 12 (mod 28)$ (or equivalently $\pm 2, \pm 4$ $(mod 14)$) and $D_2(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm 1, \pm 5, 7(mod 14)$. Then

$$A_2(\nu) = B_2(\nu), \quad \textit{for all } \nu.$$

Theorem 2.2.3. *For $\nu \geq 0$, let $A_3(\nu)$ denote the number of partitions of ν with '$n+2$ copies of n' into parts such that for some i, i_{i+2} must be a part and weighted difference between consecutive parts is non negative and $\equiv 0(mod 4)$ and let*

$$B_3(\nu) = \sum_{k=0}^{\nu} C_3(\nu - k) D_3(k),$$

where $C_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 2, \pm 6, \pm 8, \pm 12 (mod 28)$ (or equivalently $\pm 2, \pm 6$ $(mod 14)$) and $D_3(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm 1, \pm 3, 7(mod 14)$. Then

$$A_3(\nu) = B_3(\nu), \quad \textit{for all } \nu.$$

Before constructing the proofs of Theorems 2.2.1–2.2.3, let us consider

$$f_i(z, q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_i(m, \nu) z^m q^{\nu}, \quad \text{for } i = 1, 2, 3. \tag{2.5}$$

where $A_i(m, \nu)$ denote the number of partitions of ν enumerated by $A_i(\nu)$ into m parts.

Proof of Theorem 2.2.1. As considered above, let $A_1(m, \nu)$ denote the number of partitions of ν enumerated by $A_1(\nu)$ into m parts. We split the partitions enumerated by $A_1(m, \nu)$ into three classes:

(i) those that do not contain k_{k-2} as a part,

(ii) those that contain 3_1 as a part,

(iii) those that contain $k_{k-2} (k > 3)$ as a part.

We now transform the partitions into class (i) by subtracting 4 from each part ignoring the subscripts, it will not disturb the inequalities between the parts and transformed partition will be of the type enumerated by $A_1(m, \nu - 4m)$.

Next, transform the partitions in class (ii) by deleting the least part 3_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1((m-1, \nu - 2m - 1)$.

Finally, we transform the partitions in class (iii) by replacing the part k_{k-2} by $(k+1)_{k-3}$ and then subtracting 2 from all the parts. This will produce the partitions of $(\nu - 2m + 1)$ into m parts. Note here that, by this transformation we will get only those partitions of $(\nu - 2m + 1)$ into m parts which contain a part of the form k_{k-2}. Therefore, the actual number of partitions which belong to class (iii) is $A_1(m, \nu - 2m + 1) - A_1(m, \nu - 6m + 1)$ where $A_1(m, \nu - 6m + 1)$ is the number of partitions of $(\nu - 2m + 1)$ into m parts which are free from parts like k_{k-2}.

The above transformations are clearly reversible and so establish a bijection between the partitions enumerated by $A_1(m, \nu)$ and those enumerated by

$$A_1(m, \nu - 4m) + A_1((m-1, \nu - 2m - 1) + A_1(m, \nu - 2m + 1) - A_1(m, \nu - 6m + 1).$$

This leads to the recurrence relation

$$\begin{aligned} A_1(m, \nu) =& A_1(m, \nu - 4m) + A_1((m-1, \nu - 2m - 1) + A_1(m, \nu - \\ & 2m + 1) - A_1(m, \nu - 6m + 1). \end{aligned} \tag{2.6}$$

Now let

$$f_1(z, q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_1(m, \nu) z^m q^\nu, \tag{2.7}$$

substitute $A_1(m, \nu)$ from (2.6) into (2.7), we get

$$f_1(z, q) = f_1(zq^4, q) + zq^3 f_1(zq^2, q) + q^{-1} f_1(zq^2, q) - q^{-1} f_1(zq^6, q). \tag{2.8}$$

Consider

$$f_1(z, q) = \sum_{n=0}^{\infty} a_n(q) z^n \tag{2.9}$$

since $f_1(0,q) = 1$, using (2.9) in (2.8) and then comparing the coefficients of z^n, we get

$$\begin{aligned} a_n(q) &= q^{4n}a_n(q) + q^{2n+1}a_{n-1}(q) + q^{2n-1}a_n(q) - q^{6n-1}a_n(q) \\ (1-q^{4n})(1-q^{2n-1})a_n(q) &= q^{2n+1}a_{n-1}(q) \\ a_n(q) &= \frac{q^{2n+1}}{(1-q^{4n})(1-q^{2n-1})}a_{n-1}(q). \end{aligned}$$

Iterating n–times and noting that $a_0(q) = 1$, we get

$$a_n(q) = \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_n}. \tag{2.10}$$

Put the value of $a_n(q)$ in (2.9), we get

$$\begin{aligned} f_1(z,q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}z^n}{(q^4,q^4)_n(q,q^2)_n}, \\ f_1(1,q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_n}. \end{aligned}$$

Now since

$$\begin{aligned} \sum_{\nu=0}^{\infty} A_1(\nu)q^{\nu} =& \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} A_1(m,\nu)q^{\nu}\right) \\ =& \sum_{\nu,m=0}^{\infty} A_1(m,\nu)q^{\nu} \\ =& f_1(1,q) \\ =& \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_n}. \end{aligned} \tag{2.11}$$

After little manipulation in the right hand of side of Identity (2.2), we have

$$\begin{aligned} \sum_{\nu=0}^{\infty} B_1(\nu)q^{\nu} =& \sum_{k=0}^{\nu} C_1(\nu-k)D_1(k) \\ =& \frac{(q^{-3},q^{-5},q^{-7},q^{-9},q^{-11};q^{14})_\infty}{(q^4,q^6,q^8,q^{10};q^{14})_\infty} \end{aligned}$$

$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{14},q,q^{13};q^{14})_\infty(q^{16},q^{12};q^{28})_\infty. \tag{2.12}$$

Hence, (2.11) and (2.12) implies

$$\sum_{\nu=0}^{\infty} A_1(\nu)q^\nu = \sum_{\nu=0}^{\infty} B_1(\nu)q^\nu. \tag{2.13}$$

which completes the proof of Theorem 2.2.1. □

Proof of Theorem 2.2.2. Split the partitions enumerated by $A_2(m,\nu)$ into three classes as follows:

(i) that do not contain k_k as a part,

(ii) that contain 1_1 as a part,

(iii) that contain $k_k(k \geq 2)$ as a part.

We now transform the partitions into class (i) by subtracting 4 from each part ignoring the subscripts, it will not disturb the inequalities between the parts and transformed partition will be of the type enumerated by $A_2(m,\nu-4m)$.

Next, transform the partitions in class (ii) by deleting the least part 1_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A_2(m-1,\nu-2m+1)$.

Finally, we transform the partitions in class (iii) by replacing the part k_k by $(k+1)_{k-1}$ and then subtracting 2 from all the other parts, ignoring the subscipts. This will produce the partitions of $(\nu-2m+1)$ into m parts. Note here that, by this transformation we will get only those partitions of $(\nu-2m+1)$ into m parts which contain a part of the form k_k. Therefore, the actual number of partitions which belong to class *(iii)* is $A_2(m,\nu-2m+1)-A_2(m,\nu-6m+1)$ where $A_2(m,\nu-6m+1)$ is the number of partitions of $(\nu-2m+1)$ into m parts which are free from parts like k_{k-2}.

The above transformations are reversible and so establish a bijection between the partitions enumerated by $A_2(m,\nu)$ and those enumerated by $A_2(m,\nu-4m)+A_2(m-1,\nu-2m+1)+A_2(m,\nu-2m+1)-A_2(m,\nu-6m+1)$. Hence, we obtain the following recurrence relation

$$A_2(m,\nu) = A_2(m,\nu-4m)+A_2(m-1,\nu-2m+1)+A_2(m,\nu-2m+1)-A_2(m,\nu-6m+1). \tag{2.14}$$

Proceeding same as in proof of Theorem 2.2.1, we get the following q–functional equation

$$f_2(z,q) = f_2(zq^4,q)+zqf_2(zq^2,q)+q^{-1}f_2(zq^2,q)-q^{-1}f_2(zq^6,q). \tag{2.15}$$

Consider

$$f_2(z,q) = \sum_{n=0}^{\infty} b_n(q)z^n \tag{2.16}$$

since $f_2(0,q)=1$, using (2.16) in (2.15) and then comparing the coefficients of z^n, we get

$$\begin{aligned} b_n(q) &= q^{4n}b_n(q)+q^{2n-1}b_{n-1}(q)+q^{2n-1}b_n(q)-q^{6n-1}b_n(q) \\ (1-q^{4n})(1-q^{2n-1})b_n(q) &= q^{2n-1}b_{n-1}(q) \\ b_n(q) &= \frac{q^{2n-1}}{(1-q^{4n})(1-q^{2n-1})}b_{n-1}(q). \end{aligned}$$

Iterating n–times and noting that $b_0(q)=1$, we get

$$b_n(q) = \frac{q^{n^2}}{(q^4,q^4)_n(q,q^2)_n}. \tag{2.17}$$

Put the value of $b_n(q)$ in (2.16), we get

$$\begin{aligned} f_2(z,q) &= \sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q^4,q^4)_n(q,q^2)_n}, \\ f_2(1,q) &= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4,q^4)_n(q,q^2)_n}. \end{aligned}$$

Now since

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_2(\nu)q^{\nu} =& \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} A_2(m,\nu)q^{\nu}\right)\\ =& \sum_{\nu,m=0}^{\infty} A_2(m,\nu)q^{\nu}\\ =& f_2(1,q)\\ =& \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4,q^4)_n(q,q^2)_n}. \end{aligned} \tag{2.18}$$

Again after little manipulation in the right hand side of Identity (2.3), we have

$$\begin{aligned}\sum_{\nu=0}^{\infty} B_2(\nu)q^{\nu} =& \sum_{k=0}^{\nu} C_2(\nu-k)D_2(k)\\ =& \frac{(q^{-1},q^{-5},q^{-7},q^{-9},q^{-13};q^{14})_\infty}{(q^2,q^4,q^{10},q^{12};q^{14})_\infty}\\ =& \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{14},q^3,q^{11};q^{14})_\infty(q^{20},q^8;q^{28})_\infty. \end{aligned} \tag{2.19}$$

Hence, (2.18) and (2.19) implies

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_2(\nu)q^{\nu}. \tag{2.20}$$

which completes the proof of Theorem 2.2.2. □

Proof of Theorem 2.2.3. Combinatorially, we observe that the recurrence relation for $A_3(m,\nu)$ can be easily observed from classes of $A_2(m,\nu)$ as follows:

$$A_3(\nu-1,m-1) = A_2(m,\nu) - A_2(m,\nu-4m)$$

hence, we get the following q–functional equation

$$zqf_3(z,q) = f_2(z,q) - f_2(zq^4,q). \tag{2.21}$$

Analytically we can also derive the q–functional equation $f_3(z,q)$ given above, as follows:

$$
\begin{aligned}
zqf_3(z,q) =& zq \sum_{\nu,m=0}^{\infty} A_3(m,\nu) z^m q^\nu \\
=& zq \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_{n+1} z^n}{(q^2;q^2)_{2n+1}} \\
=& zq \sum_{n=0}^{\infty} \frac{q^{n^2}(zq^2)^n(-q;q^2)_{n+1}}{(q^4;q^4)_n(q^2;q^4)_{n+1}} \\
=& zq \sum_{n=0}^{\infty} \frac{q^{n^2}(zq^2)^n}{(q^4;q^4)_n(q;q^2)_{n+1}} \\
=& zq \sum_{n=1}^{\infty} \frac{q^{(n-1)^2}(zq^2)^{n-1}}{(q^4;q^4)_{n-1}(q;q^2)_n} \\
=& zq \sum_{n=1}^{\infty} \frac{q^{(n-1)^2}(zq^2)^{n-1}(1-q^{4n})}{(q^4;q^4)_n(q;q^2)_n} \\
=& zq \sum_{n=0}^{\infty} \frac{q^{(n-1)^2}(zq^2)^{n-1}}{(q^4;q^4)_n(q;q^2)_n} - zq \sum_{n=0}^{\infty} \frac{q^{(n-1)^2}(zq^2)^{n-1}q^{4n}}{(q^4;q^4)_n(q;q^2)_n} \\
=& \sum_{n=0}^{\infty} \frac{q^{n^2}z^n}{(q^4;q^4)_n(q;q^2)_n} - \sum_{n=0}^{\infty} \frac{q^{n^2}(zq^4)^n}{(q^4;q^4)_n(q;q^2)_n} \\
=& f_2(z,q) - f_2(zq^4,q).
\end{aligned}
$$

Consider

$$
f_3(z,q) = \sum_{n=0}^{\infty} c_n(q) z^n \tag{2.22}
$$

since $f_3(0,q) = 1$, using (2.22) and (2.16) in (2.21) and then comparing the coefficients of z^{n+1}, we get

$$
\begin{aligned}
qc_n(q) &= b_{n+1}(q) - q^{4(n+4)} b_{n+1}(q) \\
qc_n(q) &= \frac{(1-q^{4n+4})q^{(n+1)^2}}{(q^4,q^4)_{n+1}(q,q^2)_{n+1}} \\
c_n(q) &= \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_{n+1}}.
\end{aligned}
$$

Put the value of $c_n(q)$ in (2.22), we get

$$
f_3(z,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)} z^n}{(q^4,q^4)_n(q,q^2)_{n+1}},
$$

$$f_3(1,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_{n+1}}.$$

Now since

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_3(\nu)q^{\nu} &= \sum_{\nu=0}^{\infty}\left(\sum_{\nu=0}^{\infty} A_3(m,\nu)q^{\nu}\right)\\ &= \sum_{\nu,m=0}^{\infty} A_3(m,\nu)q^{\nu}\\ &= f_3(1,q)\\ &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_{n+1}}. \end{aligned} \tag{2.23}$$

After little manipulation in the right hand side of Identity (2.4), we have

$$\begin{aligned}\sum_{\nu=0}^{\infty} B_3(\nu)q^{\nu} &= \sum_{k=0}^{\nu} C_3(\nu-k)D_3(k)\\ &= \frac{(q^{-1},q^{-3},q^{-7},q^{-11},q^{-13};q^{14})_\infty}{(q^2,q^6,q^8,q^{12};q^{14})_\infty}\\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{14},q^5,q^9;q^{14})_\infty(q^{24},q^4;q^{28})_\infty. \end{aligned} \tag{2.24}$$

Hence, (2.23) and (2.24) implies

$$\sum_{\nu=0}^{\infty} A_3(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_3(\nu)q^{\nu}. \tag{2.25}$$

which completes the proof of Theorem 2.2.3. □

2.3 Combinatorial interpretations using lattice paths

In this section, we extend all the results of Section 2.2 using lattice paths. In the following Theorems 2.3.1–2.3.3, $A_i(\nu)$ and $B_i(\nu)$, $1 \le i \le 3$ are same as defined in Theorems 2.2.1–2.2.3, respectively.

Theorem 2.3.1. *For $\nu \ge 0$, let $E_1(\nu)$ denote the number of lattice paths of weight ν which start at $(0,0)$, have no valley above height 0, there is a plain of length $\equiv 2 (mod 4)$*

in the beginning of the path, the length of the other plains, if any, are $\equiv 0(mod 4)$. *Then*

$$E_1(\nu) = B_1(\nu) = A_1(\nu), \text{ for all } \nu.$$

Example 2.3.1. *We demonstrate Theorem 2.3.1 by showing that*

$$E_1(8) = B_1(8) = A_1(8) = 3.$$

The relevant lattice paths corresponding to $E_1(8)$ *are as follows:*

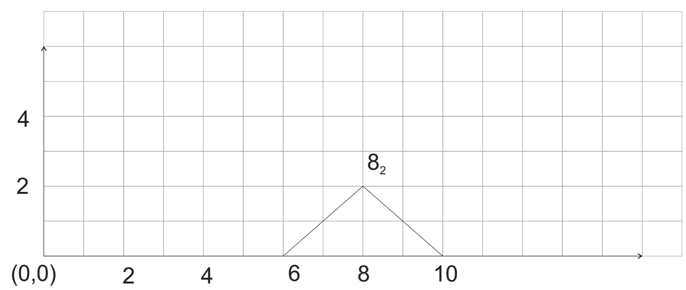


Figure 2-1: Graph 1

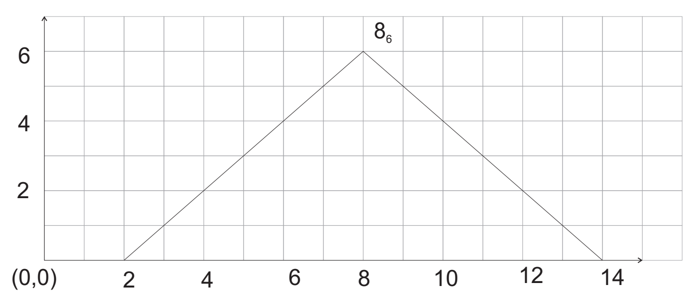


Figure 2-2: Graph 2

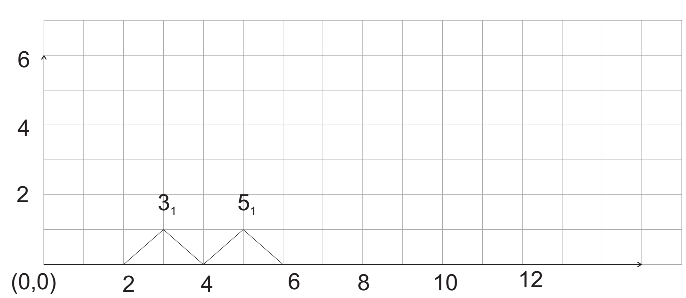


Figure 2-3: Graph 3

and by Example 2.2.1, $A_1(8) = B_1(8) = 3$.

Theorem 2.3.2. *For* $\nu \geq 0$, *Let* $E_2(\nu)$ *denote the number of lattice paths of weight* ν *which start at* $(0, 0)$, *have no valley above height* 0, *the length of the plains, if any, are* $\equiv 0(mod 4)$. *Then*

$$E_2(\nu) = B_2(\nu) = A_1(\nu), \text{ for all } \nu.$$

Theorem 2.3.3. *For* $\nu \geq 0$, *let* $E_3(\nu)$ *denote the number of lattice paths of weight* ν *which start at* $(0, 2)$, *have no valley above height* 0, *the length of the plains, if any, are* $\equiv 0(mod 4)$. *Then*

$$E_3(\nu) = B_3(\nu) = A_3(\nu), \text{ for all } \nu.$$

Proof of Theorem 2.3.1. In $\frac{q^{n(n+2)}}{(q^4;q^4)_n(q;q^2)_n}$ the factor $q^{n(n+2)}$ generates a lattice path from $(0, 0)$ to $(2n + 2, 0)$ having n peaks each of height 1 and plain of length 2 in the beginning of the path.
For $n = 5$, the path begins as

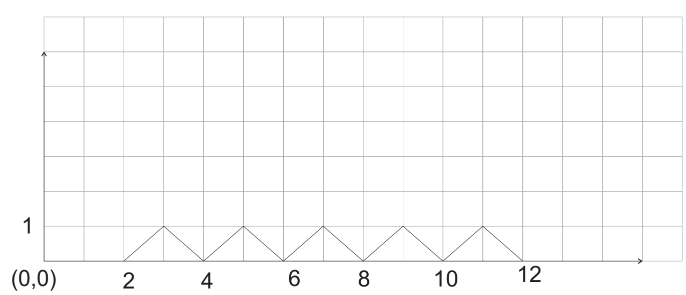


Figure 2-4: Graph A

In the above graph, we consider two successive peaks say, i^{th} and $(i + 1)^{th}$ and denote them by P_1 and P_2, respectively.

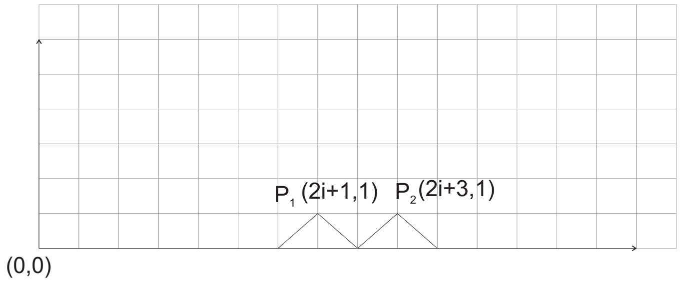


Figure 2-5: Graph B

The factor $1/(q^4;q^4)_n$ generates n nonnegative parts $\equiv 0(mod 4)$, say $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, which are encoded by inserting a_n horizontal steps in front of the first mountain, and $a_i - a_{i+1}$ horizontal steps in front of the $(n-i+1)st$ mountain, $1 \leq i \leq n-1$. Thus the x-coordinate of the i^{th} peak is increased by $a_n + (a_{n-1} - a_n) + (a_{n-2} - a_{n-1}) + \cdots + (a_{n-i+1} - a_{n-i+2}) = a_{n-i+1}$ and the x-coordinate of the $(i+1)^{th}$ peak is increased by a_{n-i}.

Graph B now becomes Graph C.

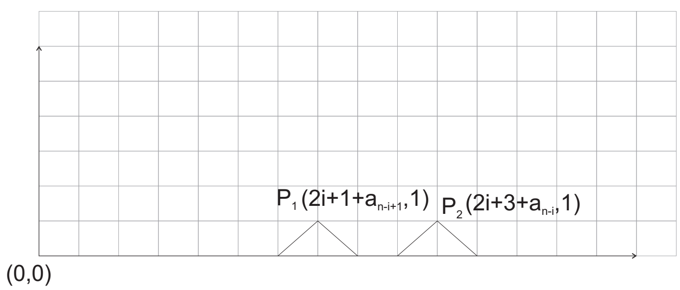


Figure 2-6: Graph C

$P_1 \equiv (1 + 2i + a_{n-i+1}, 1)$,

$P_2 \equiv (3 + 2i + a_{n-i}, 1)$.

The factor $1/(q;q^2)_n$ generates non negative multiples of $(2i-1)$, $1 \leq i \leq n$, say, $b_1 \times 1, b_2 \times 3, \cdots, b_n \times (2n-1)$.

This is encoded by having the i^{th} peak grow to height $b_{n-i+1} + 1$. Each increment by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. Graph C now changes to Graph D or Graph E depending on whether $b_{n-i} > b_{n-i+1}$ or $< b_{n-i+1}$. In the case when $b_{n-i} = b_{n-i+1}$, the new graph looks like Graph D.

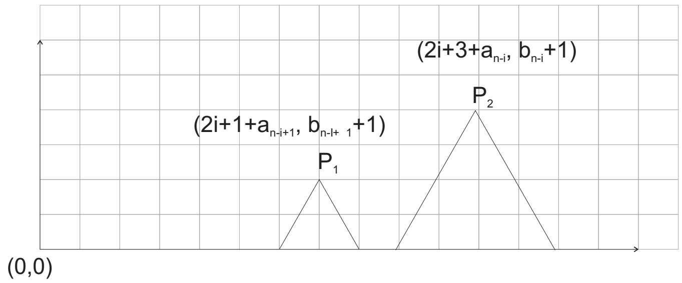


Figure 2-7: Graph D

The Graph E looks like

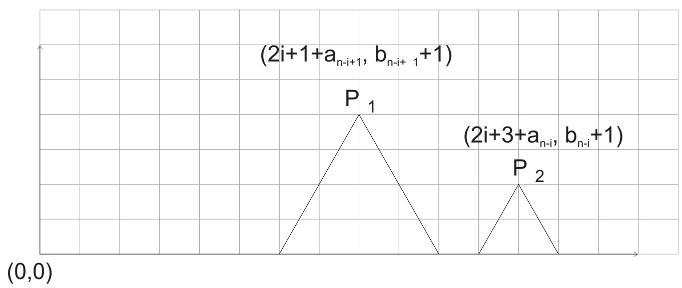


Figure 2-8: Graph E

Every lattice path enumerated by $E_1(\nu)$ is uniquely generated in this manner. This proves Theorem 2.3.1. $\square$

Sketch proofs of Theorems 2.3.2 and 2.3.3. Theorem 2.3.2 is treated in exactly the same manner as the Theorem 2.3.1 except that now the path begins from $(0,0)$.

For Theorem 2.3.3, comparing it with Theorem 2.3.2, we see that in this case there are two extra factors, viz., q^{2n} and $(1-q^{2n+1})^{-1}$. The extra factor q^{2n} puts two south east steps: $(0,2)$ to $(1,1)$ and $(1,1)$ to $(2,0)$. Thus, there are now $n+1$ peaks starting from $(0,2)$ and the extra factor $(1-q^{2n+1})^{-1}$ introduces a non negative multiple of $2n+1$, say $b_{n+1}\times(2n+1)$. This is encoded by having first peak grow to height $b_{n+1}+2$. Clearly, $(b_{n+1})_{b_{n+1}+2}$ which is of the form i_{i+2} will be the colored part corresponding to the first peak. $\square$

2.4 Bijections between *(n+t)*–color partitions and lattice paths

We now establish a 1–1 correspondence between the lattice paths enumerated by $E_1(\nu)$ and the n–color partitions enumerated by $A_1(\nu)$. We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak. Thus if we denote the two peaks in Graph D (or Graph E) by A_x and B_y, $(B \geq A)$ respectively, then

$A = 1 + 2i + a_{n-i+1} + 2(b_n + b_{n-1} + \cdots + b_{n-i+2}) + b_{n-i+1}$

$x = b_{n-i+1} + 1$

$B = 3 + 2i + a_{n-i} + 2(b_n + b_{n-1} + \cdots + b_{n-i+1}) + b_{n-i}$

$y = b_{n-i} + 1.$

Clearly, the weighted difference of these two parts is $((B_y - A_x)) = B - A - x - y = a_{n-i} - a_{n-i+1}$ which is non negative and $\equiv 0(mod 4)$.

To see the reverse implication we consider two n–color parts of a partition enumerated by $A_1(\nu)$, say C_u and D_v with $D \geq C \geq 3$. Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the associated lattice path.

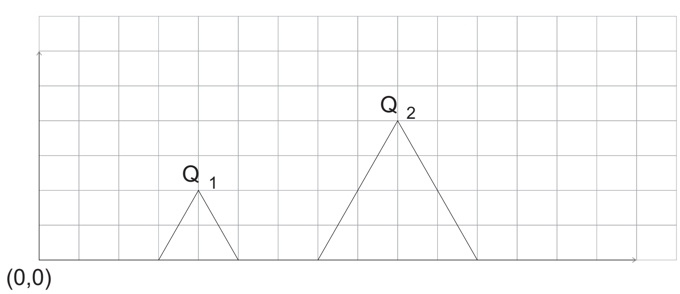


Figure 2-9: Graph F

The length of the plain between the two peaks is $D - C - u - v$ which is the weighted difference between the two parts C_u and D_v and is therefore nonnegative and $\equiv 0(mod 4)$.

If C_u were the smallest part of the partition, the corresponding peak in the associated path would be the first peak preceded by a plain of length $2 + a$, where $a \equiv 0(mod 4)$.

Finally, we show that there can not be a valley above height 0. This can be proved by contradiction.

Suppose there is a valley V of height γ ($\gamma > 0$) between the peaks Q_1 and Q_2.

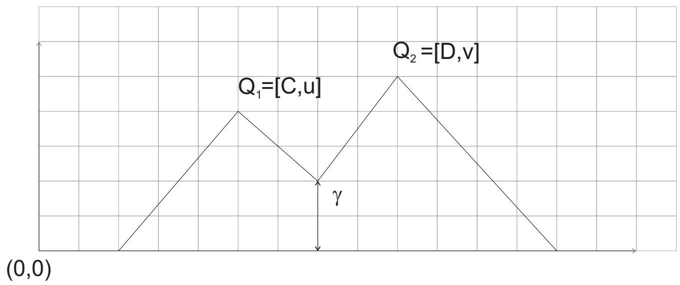


Figure 2-10: Graph G

In this case, there is a descent of $u - -\gamma$ from Q_1 to V and an ascent of $v - -\gamma$ from V to Q_2. This implies that $D = C + (u - -\gamma) + (v - -\gamma)$, or $D - -C - -u - -v = - -2\gamma$. But since the weighted difference is non negative, therefore γ=0. This completes the bijection between the lattice paths enumerated by $E_1(\nu)$ and the n–color partitions enumerated by $A_1(\nu)$.

Proceeding same as above, the similar bijections between $A_i(\nu)$ and $E_i(\nu)$, $i = 2,\ 3$, can be easily established.

2.5 Conclusion

In this chapter, we obtained the combinatorial interpretations to Rogers–Ramanujan type identities (2.2)–(2.4). These interpretations lead to three–way combinatorial identities as follows:

$$A_i(\nu) = B_i(\nu) = E_i(\nu), \quad \text{for } 1 \leq i \leq 3, \tag{2.26}$$

where $A_i(\nu)$ and $B_i(\nu)$, $1 \leq i \leq 3$ are as defined in Theorems 2.2.1–2.2.3 and $E_i(\nu)$, $1 \leq i \leq 3$, are defined in Theorems 2.3.1–2.3.3, respectively.

Each three–way combinatorial identity gives us three new combinatorial identities in the usual sense.

$$A_i(\nu) = B_i(\nu), \tag{2.27}$$

$$A_i(\nu) = E_i(\nu), \tag{2.28}$$

and $$B_i(\nu) = E_i(\nu), \qquad \text{for} \;\; 1 \leq i \leq 3. \tag{2.29}$$

Further, we extended the results of this chapter by providing the combinatorial interpretations of (2.2)–(2.4) using another combinatorial tool, generalized F–partitions, and establishing the bijections between $(n + t)$–color partitions and generalized F–partitions.

Chapter 3

Combinatorics of some mock theta functions

3.1 Introduction

Ramanujan, in his letter dated January 1920 to G. H. Hardy [see for ref., Watson [68]], communicated a list of seventeen functions of which four are of third order, ten are of fifth order and three are of order seven. He called them as mock theta functions, given by (3.1)–(3.17) below:

Mock theta functions of order 3

$$f(q) = \sum_{m=0}^{\infty} q^{m^2}/(-q;q)_m^2, \tag{3.1}$$

$$\phi(q) = \sum_{m=0}^{\infty} q^{m^2}/(-q^2;q^2)_m, \tag{3.2}$$

$$\psi(q) = \sum_{m=1}^{\infty} q^{m^2}/(q;q^2)_m, \tag{3.3}$$

The content of Section 3.2 is accepted for publication in the Journal *'proceedings of the Indian Academy of Sciences–Mathematical Sciences' (SCI, Impact factor: 0.240)* and the contents of Sections 3.3 and 3.4 are published in the journal *'Advances in Theoretical and Applied Mathematics'*, 10(1):15–25, 2015.

$$\chi(q) = \sum_{m=0}^{\infty} q^{m^2}/(1-q+q^2)(1-q^2+q^4)\dots(1-q^m+q^{2m}). \tag{3.4}$$

Mock theta functions of order 5

Group A

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \tag{3.5}$$

$$\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n, \tag{3.6}$$

$$\psi_0(q) = \sum_{n=1}^{\infty} q^{n(n+1)/2}(-q;q)_{n-1}, \tag{3.7}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \tag{3.8}$$

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n}. \tag{3.9}$$

Group B

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n}, \tag{3.10}$$

$$\phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n, \tag{3.11}$$

$$\psi_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}(-q;q)_n, \tag{3.12}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \tag{3.13}$$

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}. \tag{3.14}$$

Mock theta functions of order 7

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}, \tag{3.15}$$

$$\mathcal{F}_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n;q)_n}, \tag{3.16}$$

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}. \tag{3.17}$$

Andrews and Hickerson defined the mock theta functions as below [28] :

Definition 3.1.1. *[28] A mock theta function is a function f of the complex variable q, defined by a q–series of a particular type, which converges for $|q| < 1$ and satisfies the following conditions:*

1. *infinitely many roots of unity are exponential singularities,*
2. *for every root of unity ζ, there is a θ–function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially,*
3. *there is no θ–function that works for all ζ, i.e. f is not the sum of two functions, one of which is a θ–function and the other a function which is bounded in all roots of unity.*

Considering the θ–functions, Ramanujan means sums, products, and quotients of series of the form $\sum_{n\in\mathbb{Z}} \epsilon^n q^{\alpha n^2+\beta n}$ such that $\alpha,\ \beta \in \mathbb{Q}$ and $\epsilon = -1, 1$.
Three more functions were added to this list by G.N. Watson appeared in [68].

$$\omega(q) = \sum_{m=0}^{\infty} \frac{q^{2m(m+1)}}{(q;q^2)_{m+1}^2}, \tag{3.18}$$

$$\nu(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(-q;q^2)_{m+1}}, \tag{3.19}$$

$$\rho(q) = \sum_{m=0}^{\infty} \frac{q^{2m(m+1)}}{(1+q+q^2)(1+q^3+q^6)\ldots(1+q^{2m+1}+q^{4m+2})}. \tag{3.20}$$

In the long list of 17 mock theta functions given by Ramanujan, few have been interpreted combinatorially, for instance, $\psi(q)$ defined by (3.3) has been interpreted by Fine [41] in 1988 as given below;

Theorem 3.1.1. *Let $T_1(\nu)$ represent the number of partitions of ν into odd parts without gap. Then*

$$\sum_{n=0}^{\infty} T_1(\nu) q^\nu = \psi(q).$$

Andrews [25] provided the similar interpretation of $F_0(q)$ in the following theorem.

Theorem 3.1.2. *Let $T_2(\nu)$ represent the number of partitions of ν into odd parts without gap repeating at least twice. Then*

$$\sum_{n=0}^{\infty} T_2(\nu)q^{\nu} = F_0(q).$$

Analogues to above Theorems 3.1.1 and 3.1.2, we obtained the trivial partition theoretic interpretations of $\phi_0(q)$ and $\phi_1(q)$ given below;

Theorem 3.1.3. *Let $T_3(\nu)$ represent the number of partitions of ν into odd parts without gap repeating at most twice. Then*

$$\sum_{n=0}^{\infty} T_3(\nu)q^{\nu} = \phi_0(q).$$

Theorem 3.1.4. *Let $T_4(\nu)$ represent the number of partitions of ν into odd parts without gap repeating at most twice with the largest part appearing once only. Then*

$$\sum_{n=0}^{\infty} T_4(\nu)q^{\nu} = \phi_1(q).$$

Proof of Theorem 3.1.3. The factor $(-q;q^2)_n$ denotes the number of partitions of λ into at most n distinct odd parts. Therefore any partition of λ can be written as

$$\lambda = 1.a_1 + 3.a_2 + 5.a_3 + \cdots + (2n-1).a_n \tag{3.21}$$

such that $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ and $a_i = 0$ or 1.
Now since $q^{n^2} = q^{(1+3+5+\cdots+(2n-1))}$,
therefore any partition enumerated by $q^{n^2}(-q;q^2)_n$ will be of the type given below;

$$\lambda + n^2 = 1.b_1 + 3.b_2 + 5.b_3 + \cdots + (2n-1).b_n \tag{3.22}$$

such that $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_n$ and $b_i = 1$ or 2.
Therefore, $\phi_0(q)$ generates the partitions into odd parts without gap appearing at

most twice. Hence,

$$\sum_{n=0}^{\infty} T_3(\nu)q^{\nu} = \phi_0(q).$$

□

Proof of Theorem 3.1.4. The factor $(-q;q^2)_n$ denotes the number of partitions of λ into at most n distinct odd parts. Therefore any partition of λ can be written as

$$\lambda = 1.a_1 + 3.a_2 + 5.a_3 + \cdots + (2n-1).a_n \tag{3.23}$$

such that $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ and $a_i = 0$ or 1.
Now since $q^{(n+1)^2} = q^{(1+3+5+\cdots+(2n+1))}$,
therefore any partition enumerated by $q^{(n+1)^2}(-q;q^2)_n$ will be of the type given below;

$$\lambda + (n+1)^2 = 1.b_1 + 3.b_2 + 5.b_3 + \cdots + (2n-1).b_n + (2n+1) \tag{3.24}$$

such that $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_n$ and $b_i = 1$ or 2.
Therefore $\phi_1(q)$ generates the partitions of ν into odd parts appearing at most twice with largest part appearing once only. Hence,

$$\sum_{n=0}^{\infty} T_4(\nu)q^{\nu} = \phi_1(q).$$

□

Agarwal [6] in 2004 gave the combinatorial interpretations of four mock theta functions $\psi(q)$, $\phi_0(q)$, $F_0(q)$ and $\phi_1(q)$, given by (3.3), (3.6), (3.8) and (3.11), respectively using n–color partitions.
Agarwal [7] in 2005 translated his results using lattice paths and then Agarwal and Narang [13] in 2011 extended the results using F–partitions. Agarwal and Rana [15] in 2008 gave the combinatorial interpretations of a fifth order mock theta function $F_1(q)$, given by (3.13), using '$(n+2)$ copies of n' and lattice paths. They also extended their results using F–partitions in [60] in 2009.
Together with these seventeen mock theta functions, a list of eight identities involving

following four tenth order mock theta functions are also given in Ramanujan's lost notebook [58].

$$\phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, \tag{3.25}$$

$$\psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}, \tag{3.26}$$

$$X_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \tag{3.27}$$

$$\chi_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}. \tag{3.28}$$

These functions have also been studied analytically by Choi [34, 35, 36, 37].

Later in 2000, Gordon and McIntosh [49] used the modular functions and generated the following eight mock theta functions:

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{3.29}$$

$$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{3.30}$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, \tag{3.31}$$

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, \tag{3.32}$$

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \tag{3.33}$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \tag{3.34}$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}, \tag{3.35}$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{3.36}$$

This chapter makes significant contribution in enumerative combinatorics by giving combinatorial interpretations of two tenth order mock theta functions $\phi_R(q)$ and $\psi_R(q)$ given by (3.25) and (3.26), respectively.

The two Gordon–McIntosh mock theta functions $V_0(q)$ and $V_1(q)$, given by (3.35) and (3.36) are also interpreted combinatorially using 'signed partition', defined in Section 3.3, and using classical partitions with convolution properties.

3.2 Tenth order mock theta functions and *(n+t)*–color partitions

In this section, we provide combinatorial interpretations of $\phi_R(q)$ and $\psi_R(q)$ using $(n+t)$–color partitions.

Theorem 3.2.1. *For $\nu \geq 1$, let $M_1(\nu)$ denote the number of n–color partitions of ν such that for some k, k_k is a part, and the weighted difference of any two consecutive parts is -1. Then,*

$$\sum_{\nu=1}^{\infty} M_1(\nu) q^{\nu} = \psi_R(q). \tag{3.37}$$

Example 3.2.1. *For $\nu = 6$, $M_1(6) = 4$ and the relevant n–color partitions are*

$$6_6, \quad 5_4 1_1, \quad 4_1 2_2, \quad 3_1 2_1 1_1. \tag{3.38}$$

Theorem 3.2.2. *For $\nu \geq 0$, let $M_2(\nu)$ denote the number of $(n+1)$–color partitions of ν such that for some k, k_{k+1} appears as a part, and the weighted difference of any two consecutive parts is -1. Then,*

$$\sum_{\nu=0}^{\infty} M_2(\nu) q^{\nu} = \phi_R(q). \tag{3.39}$$

Example 3.2.2. *For $\nu = 6$, $M_2(6) = 6$. The relevant n–color partitions are*

$$6_7, \quad 6_6 + 0_1, \quad 4_1 + 2_2 + 0_1, \quad 5_3 + 1_2, \quad 3_1 + 2_1 + 1_1 + 0_1, \quad 5_4 + 1_1 + 0_1. \tag{3.40}$$

Note. Theorems 3.2.1 and 3.2.2 are the combinatorial interpretations of (3.26) and (3.25), respectively.

Remark. In the sequel $M_i(m, \nu)$, $(1 \leq i \leq 2)$, will denote the number of partitions of ν enumerated by $M_i(\nu)$ into m parts, and we shall write

$$f_i(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} M_i(m,\nu) z^m q^\nu. \tag{3.41}$$

Proof of Theorem 3.2.1. Splitting the partitions enumerated by $M_1(m,\nu)$ into two classes:

Class (i): having 1_1 as a part, and Class (ii): having $k_k, (k > 1)$ as a part. Now transformed partitions of Class (i), enumerated by $M_1(m-1,\nu-m)$, will be obtained by deleting the part 1_1 and then subtracting 1 from all the remaining parts ignoring the subscripts and the transformed partitions of Class (ii), enumerated by $M_1(m,\nu-2m+1)$, will be obtained by first replacing $(k)_k$ by $(k-1)_{k-1}$ and then subtracting 2 from all of remaining parts ignoring the subscripts.

The above transformations are clearly reversible and leads to the following identity:

$$M_1(m,\nu) = M_1(m-1,\nu-m) + M_1(m,\nu-2m+1). \tag{3.42}$$

From (3.41), we have

$$f_1(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} M_1(m,\nu) z^m q^\nu. \tag{3.43}$$

Substituting for $M_1(m,\nu)$ from (3.42) in (3.43) and then simplifying we get

$$f_1(z,q) = zqf_1(zq,q) + q^{-1}f_1(zq^2,q) \tag{3.44}$$

which is a q–functional equation. Setting

$$f_1(z,q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n \tag{3.45}$$

and then comparing the coefficients of z^n on both sides of (3.44), we see that

$$\alpha_n(q) = \frac{q^n}{1-q^{2n-1}}\alpha_{n-1}(q). \tag{3.46}$$

Iterating (3.46) n times and noting $\alpha_0(q) = 1$, we get,

$$\alpha_n(q) = \frac{q^{n(n+1)/2}}{(q;q^2)_n}. \tag{3.47}$$

Therefore,

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_n} z^n. \tag{3.48}$$

Now,

$$\begin{aligned}\sum_{\nu=0}^{\infty} M_1(\nu)q^\nu =& \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} M_1(m,\nu)\right) q^\nu \\ =& f_1(1,q) \\ =& \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_n} \\ =& 1 + \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}} \\ =& 1 + \psi_R(q)\end{aligned}$$

which completes the proof of Theorem 3.2.1. □

Sketch of proof of Theorem 3.2.2. Proceeding as above, one can easily obtain the following recurrence relation,

$$M_2(m,\nu) = M_1(m-1,\nu) + M_2(m,\nu-2m+1). \tag{3.49}$$

Corresponding q–functional equation is,

$$f_2(z,q) = zf_1(z,q) + q^{-1}f_2(zq^2,q) \tag{3.50}$$

Using above q–functional equation and proceeding as in Theorem 3.2.2, we get

$$\sum_{\nu=0}^{\infty} M_2(\nu)q^\nu = \phi_R(q).$$

□

3.3 Gordon–McIntosh mock theta functions and ‘signed partitions’

Andrews [26] introduced the concept of signed partitions to include the possibility of having negative integers as the parts of a partition. Mclaughlin and Sills [55] used these signed partitions to interpret some q–identities from Slater's Compendium [64] related to moduli 18 and 24. In this section, we are interpreting $V_0(q)$ and $V_1(q)$ using ‘signed partitions’. Before stating the main results, let us recall the definition of signed partitions.

Definition 3.3.1. *[26] A signed partition σ of an integer ν is a partition pair (π^+, π^-) where*

$$\nu = |\pi^+| - |\pi^-|.$$

π^+ (resp. π^-) is the positive (resp. negative) subpartition of σ and $\pi_1^+, \pi_2^+, \ldots, \pi_{l(\pi^+)}^+$ (resp. $\pi_1^-, \pi_2^-, \ldots, \pi_{l(\pi^-)}^-$) are the positive (resp. negative) parts of σ.

Example 3.3.1. $((8\ 2^2\ 1), (3\ 2\ 1^2))$*, which represents* $8+2+2+1-1-1-2-3$*, is a signed partition of* 6*.*

Remark. It is obvious that there are infinitely many unrestricted signed partitions of any integer, but when we place restrictions on how parts may appear, signed partitions arise naturally in the study of certain q–series.

Theorem 3.3.1. *For $\nu \geq 0$, let $S_1(\nu)$ denote the number of signed partitions $\sigma = (\pi^+, \pi^-)$ of ν, where*

(i) π^+ contain the odd parts atleast twice, and

(ii) π^- contain the odd, distinct parts $< 2\lambda$

where λ is the number of distinct parts in π^+. Then,

$$-1 + 2\sum_{\nu=0}^{\infty} S_1(\nu) q^{\nu} = V_0(q). \tag{3.51}$$

Example 3.3.2. *Consider $\nu = 6$, then $B_1(\nu) = 4$, and the relevant signed partitions are;* 1^6, $1^7 - 1$, $1^4 + 3^2 - 1 - 3$, $1^3 + 3^2 - 3$.

Theorem 3.3.2. *For $\nu \geq 0$, let $S_2(\nu)$ denote the number of signed partitions $\sigma = (\pi^+, \pi^-)$ of ν, where*

(i) π^+ contains an odd part r, atleast once and $\frac{(r-1)}{2}$ parts which are odd and appear atleast twice, and

(ii) π^- contain odd, distinct parts $\leq r - 2$.

Then,

$$\sum_{\nu=0}^{\infty} S_2(\nu) q^\nu = V_1(q). \tag{3.52}$$

Example 3.3.3. *Consider $\nu = 6$, then $S_2(\nu) = 3$, and the relevant signed partitions are;* 1^6, $1^7 - 1$, $1^4 + 3 - 1$.

Proof of Theorem 3.3.1. We have,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \tag{3.53}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2} \prod_{k=1}^{n}(1 + q^{(2k-1)})}{(q;q^2)_n} \tag{3.54}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2} \prod_{k=1}^{n}(1 + q^{-(2k-1)})}{(q;q^2)_n}. \tag{3.55}$$

Now, $\frac{q^{2n^2}}{(q;q^2)_n}$ generate the partitions into odd parts without gap appearing atleast twice.

Also, $\prod_{k=1}^{n}(1 + q^{-(2k-1)})$ generate the partitions into distinct odd parts less than 2λ. Taking summation over all n, we obtain that (3.55) generate the signed partitions enumerated by $S_1(\nu)$. Hence,

$$-1 + 2\sum_{\nu=0}^{\infty} S_1(\nu) q^\nu = V_0(q).$$

□

Proof of Theorem 3.3.2. We have,

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{3.56}$$

$$= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} \prod_{k=1}^{n}(1+q^{(2k-1)})}{(q;q^2)_{n+1}} \tag{3.57}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1} \prod_{k=1}^{n}(1+q^{-(2k-1)})}{(q;q^2)_{n+1}} \tag{3.58}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} \frac{q^{2n+1}}{(1-q^{2n+1})} \prod_{k=1}^{n}(1+q^{-(2k-1)}). \tag{3.59}$$

Now, $\frac{q^{2n^2}}{(q;q^2)_n}$ generate the partitions into odd parts without gap appearing atleast twice.

The factor $\frac{q^{2n+1}}{(1-q^{2n+1})}$ generate the partitions consisting of the part $2n+1$ atleast once. Also, $\prod_{k=1}^{n}(1+q^{-(2k-1)})$ generate the partitions into distinct odd parts less than or equals to $2n-1$.

Taking summation over all n, we obtain that (3.59) generate the signed partitions enumerated by $S_2(\nu)$. Hence,

$$\sum_{\nu=0}^{\infty} S_2(\nu)q^{\nu} = V_1(q). \tag{3.60}$$

□

3.4 Gordon–McIntosh mock theta functions and 'classical partitions with convolution properties'

Theorem 3.4.1. *For $m, \nu \geq 0$, let $D_1(m,\nu)$ enumerate the number of partitions of ν into exactly m odd parts without gap and $E_1(m,\nu)$ denote the number of partitions* *of ν into atmost m distinct odd parts less than $2m$. Let*

$$C_1(\nu) = \sum_{m=0}^{\infty} \sum_{k=0}^{\nu} D_1(m,k)E_1(m,\nu-k).$$

Then,

$$V_0(q) = -1 + 2\sum_{\nu=0}^{\infty} C_1(\nu)q^{\nu}.$$

Example 3.4.1. *Consider $\nu = 6$, then $C_1(\nu) = 4$ and the relevant partitions for $D_1(m,k)$ and $E_1(m,k)$, $(0 \le k \le \nu)$ respectively are given in the table below:*

Table 1 : Partitions enumerated by $D_1(m,k)$

m \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0

Table 2 : Partitions enumerated by $E_1(m,k)$

m \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	1	0	1	1	0	0
3	1	1	0	1	1	1	1
4	1	1	0	1	1	1	1

5	1	1	0	1	1	1	1
6	1	1	0	1	1	1	1

$$\begin{aligned}C_1(6) =&[D_1(0,0)E_1(0,6) + D_1(0,1)E_1(0,5) + \cdots + D_1(6,5)E_1(6,1) + D_1(6,6)E_1(6,0)]\\ =&D_1(1,5)E_1(1,1) + D_1(1,6)E_1(1,0) + D_1(2,5)E_1(2,1)\\ &+ D_1(2,6)E_1(2,0)\\ =&1+1+1+1\\ =&4.\end{aligned}$$

Theorem 3.4.2. *For $m,\nu \geq 0$, let $D_2(m,\nu)$ enumerate the number of partitions of ν into exactly $m+1$ odd parts without gap and $E_2(m,\nu)$ denote the number of partitions of ν into distinct odd parts less than $2m$. Let,*

$$C_2(\nu) = \sum_{m=0}^{\infty}\sum_{k=0}^{\nu} D_2(m,k)E_2(m,\nu-k)$$

Then,

$$\sum_{\nu=0}^{\infty} C_2(\nu)q^\nu = V_1(q).$$

Proof of Theorem 3.4.1. We shall show that,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = \sum_{\nu=0}^{\infty} C_1(\nu)q^\nu.$$

The theorem will be proved in the following three steps:

Step 1.

By Theorem 3.1.1, $\frac{q^{m^2}}{(q;q^2)_m}$ generate the partitions into exactly m odd parts without gap such as:

$$a_1.1 + a_2.3 + a_3.5 + \cdots + a_m.(2m-1). \tag{3.61}$$

Therefore,

$$\sum_{\nu=0}^{\infty} D_1(m,\nu)q^\nu = \frac{q^{m^2}}{(q;q^2)_m}. \tag{3.62}$$

Step 2.

Clearly,

$$(-q;q^2)_m = \prod_{k=1}^{m}(1+q^{2k-1}) \tag{3.63}$$

generate the number of partition into distinct odd parts less than $2m$. Therefore,

$$\sum_{\nu=0}^{\infty} E_1(m,\nu)q^{\nu} = (-q;q^2)_m \tag{3.64}$$

Step 3.

Since,

$$\sum_{\nu=0}^{\infty} C_1(\nu)q^{\nu} = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} C_1(m,\nu)q^{\nu} \tag{3.65}$$

We have,

$$\begin{aligned}
\sum_{\nu=0}^{\infty} C_1(\nu)q^{\nu} &= \sum_{\nu=0}^{\infty}\Big(\sum_{m=0}^{\infty}\sum_{k=0}^{\nu} D_1(m,k)E_1(m,\nu-k)\Big)q^{\nu}\\
&= \sum_{\nu=0}^{\infty}\Big(\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} D_1(m,k)E_1(m,\nu)\Big)q^{\nu+k}\\
&= \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{\infty} D_1(m,k)q^{k}\Big)\Big(\sum_{\nu=0}^{\infty} E_1(m,\nu)q^{\nu}\Big)\\
&= \sum_{m=0}^{\infty}\frac{q^{m^2}(-q;q^2)_m}{(q;q^2)_m}.
\end{aligned}$$

Hence the result. □

Proof of Theorem 3.4.2. From Theorem 3.1.1, $\frac{q^{(m+1)^2}}{(q;q^2)_{m+1}}$ generate the partitions into exactly $(m+1)$ odd parts such as:

$$a_1.1 + a_2.3 + a_3.5 + \cdots + a_{m+1}.(2m+1). \tag{3.66}$$

Therefore,

$$\sum_{\nu=0}^{\infty} D_2(m,\nu)q^{\nu} = \frac{q^{(m+1)^2}}{(q;q^2)_{m+1}}. \tag{3.67}$$

Now proceeding same as in Theorem 3.4.1, we get,

$$\sum_{\nu=0}^{\infty} C_2(\nu)q^{\nu} = \sum_{m=0}^{\infty}\frac{q^{(m+1)^2}(-q;q^2)_m}{(q;q^2)_{m+1}}$$

$$=V_1(q).$$

Hence the result. □

3.5 Conclusion

In this chapter, we interpreted two tenth order mock theta functions $\psi_R(q)$ and $\phi_R(q)$ using $(n+t)$–color partitions with weighted difference equals to -1. Also we have provided combinatorial interpretations of two Gordon–McIntosh mock theta functions $V_0(q)$ and $V_1(q)$ by means of signed partitions. The results are further extended giving another combinatorial interpretations of $V_0(q)$ and $V_1(q)$ using ordinary partitions with convolution properties. Hence, we get two–way combinatorial identities of $V_0(q)$ and $V_1(q)$ as given below;

$$S_i(\nu) = C_i(\nu), \quad \text{for } i = 1,\ 2, \tag{3.68}$$

where $S_i(\nu),\ i = 1,\ 2$ are defined in Theorems 3.3.1 and 3.3.2 and $C_i(\nu),\ i = 1,\ 2$ are defined in Theorems 3.4.1 and 3.4.2, respectively.

Chapter 4

F–partitions and q–identities/series

4.1 Introduction

In Chapter 1, we studied the concept of Frobenius representation of an ordinary partition. Andrews in [24] generalizes the concept and introduced generalized Frobenius partitions or simply known as F–partitions. In [9] Agarwal and Andrews obtained the bijections between F–partitions and $(n+t)$–color partitions. In [5, 9, 13, 59, 60, 65] some of Rogers–Ramanujan type identities and q–series have been interpreted using F–partitions. In this chapter, we extended our work done in Chapter 2 and find the new combinatorial interpretations using F–partitions of three Rogers–Ramanujan type identities given by (2.2)–(2.4) in Chapter 2. We also extended the combinatorial interpretation of one of the tenth order mock theta function $\psi_R(q)$, given by 3.26 in Chapter 3, using F–partitions. We accomplish this by establishing the bijections between $(n+t)$–color partitions and F–partitions. First we recall the definition of F–partitions given by Andrews [24] in 1984.

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Definition 4.1.1. *[9] A two rowed array of non-negative integers*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ *and* $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$,
is known as generalized Frobenius partition or more simply an F*–partition of* ν *if* $\nu = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$.

Example 4.1.1. *If* ν*=20=4+(5+3+1+0)+(5+1+1+0) is a partition then the corresponding Frobenius notation is*

$$\begin{pmatrix} 5 & 3 & 1 & 0 \\ 5 & 1 & 1 & 0 \end{pmatrix}$$

Remark. We note here that in Frobenius representation of an ordinary partition, there is strict decrease along the row, while in F–partitions above, entries in the row follow the non–increasing order. So the F–partitions are comparatively higher in number than the ordinary partitions of a positive integer. We associate the F–partitions with the $(n + t)$–color partitions through establishing one to one correspondence between columns of F–partitions and parts of $(n + t)$–color partitions.

4.2 Rogers–Ramanujan type identities and *F*–partition

In this section, we extended the main results of Chapter 2. Throughout the Theorems 4.2.1–4.2.3 given below, the partition function $A_i(\nu)$, $1 \leq i \leq 3$, is same as we defined in Chapter 2 for Theorems 2.2.1–2.2.3. For more clarity, to establish the bijections, the results of Chapter 2 are again written in the form of conditions in Theorems 4.2.1–4.2.3.

Theorem 4.2.1. *For* $\nu \geq 0$*, let* $F_1(\nu)$ *denote the number of* F*–partitions of* ν *such that*

(4.2.1.a) $a_r \equiv 1(mod 2)$,

(4.2.1.b) $a_i \leq b_i$, *and*

(4.2.1.c) $a_i > b_{i+1}$ *and are of opposite parity.*

$A_1(\nu)$ *denote the number of partitions of* ν *with 'n copies of n' such that*

(4.2.1.d) *each part* $\geq$ *3,*

(4.2.1.e) *the weighted difference between consecutive parts is non negative and* $\equiv 0(mod 4)$, *and*

(4.2.1.f) *if* m_i *is the least or only part in the partition then* $m - i \equiv 2(mod 4)$. *Then*

$$F_1(\nu) = A_1(\nu) \quad \textit{for all } \nu.$$

Theorem 4.2.2. *For* $\nu \geq 0$, *let* $F_2(\nu)$ *denote the number of* F*–partitions of* ν *such that*

(4.2.2.a) $a_r \equiv 0(mod 2)$,

(4.2.2.b) $a_i \leq b_i$, *and*

(4.2.2.c) $a_i > b_{i+1}$ *and are of opposite parity.*

$A_2(\nu)$ *denote the number of partitions of* ν *with 'n copies of n' such that*

(4.2.2.d) *the weighted difference between consecutive parts is non negative and* $\equiv 0(mod 4)$, *and*

(4.2.2.e) *if* m_i *is the least or only part in the partition then* $m \equiv i(mod 4)$.

Then

$$F_2(\nu) = A_2(\nu) \quad \textit{for all } \nu.$$

Theorem 4.2.3. *For* $\nu \geq 0$, *let* $F_3(\nu)$ *denote the number of* F*–partitions of* ν *such that*

(4.2.3.a) $a_r = 0$,

(4.2.3.b) $a_i \leq b_i + 2$, *and*

(4.2.3.c) $a_i > b_{i+1} + 2$ *and are of opposite parity.*

$A_3(\nu)$ *denote the number of partitions of* ν *with '*$n+2$ *copies of* n*' such that*

(4.2.3.d) *the weighted difference between consecutive parts is non negative and* $\equiv 0(mod 4)$, *and*

(4.2.3.e) *for some* i, i_{i+2} *must be a part.*

Then

$$F_3(\nu) = A_3(\nu) \quad \textit{for all } \nu.$$

Proof of Theorem 4.2.1. We establish a one to one correspondence between the F–partitions enumerated by $F_1(\nu)$ and the n–color partitions enumerated by $A_1(\nu)$. We do this by mapping each column $\begin{pmatrix} a \\ b \end{pmatrix}$ of the F–partition to a single part m_i of an n–color partition enumerated by $A_1(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a+b+1)_{b-a+1}, \tag{4.1}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i = \begin{pmatrix} (m-i)/2 \\ (m+i-2)/2 \end{pmatrix}. \tag{4.2}$$

Now, suppose we have any two adjacent columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ in an F–partition enumerated by $F_1(\nu)$ with

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} = m_i \text{ and } \phi : \begin{pmatrix} c \\ d \end{pmatrix} = n_j.$$

Then since

$$\begin{pmatrix} a \\ b \end{pmatrix} \to (a+b+1)_{b-a+1} = m_i$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix} \to (c+d+1)_{d-c+1} = n_j,$$

we have

$$\begin{aligned} ((m_i - n_j)) &= m - n - i - j \\ &= (a+b+1) - (c+d+1) - (b-a+1) - (d-c+1) \\ &= 2(a-d-1). \end{aligned} \tag{4.3}$$

Clearly (4.3) and (4.2.1.c) imply (4.2.1.e).

Also (4.2), (4.2.1.a) and (4.2.1.b) imply (4.2.1.d).

Now if $a_r \equiv 1(mod2)$, then

$$m - i = (a_r + b_r + 1) - (b_r - a_r + 1) = 2a_r$$

which imply $(m-i) \equiv 2(mod4)$, hence (4.2.1.f) holds.

To see the reverse implication, we consider the inverse images of two consecutive parts m_i, n_j of an n–color partition enumerated by $A_1(\nu)$

$$\phi^{-1} : m_i = \begin{pmatrix} (m-i)/2 \\ (m+i-2)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\phi^{-1} : n_j = \begin{pmatrix} (n-j)/2 \\ (n+j-2)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{4.4}$$

that is,

$$a = (m-i)/2 \tag{4.5}$$

$$b = (m+i-2)/2 \tag{4.6}$$

$$c = (n - j)/2 \tag{4.7}$$

$$d = (n + j - 2)/2 \tag{4.8}$$

and so

$$b - a = i - 1 \tag{4.9}$$

$$d - c = j - 1 \tag{4.10}$$

$$a - d = \frac{1}{2}((m_i - n_j)) + 1 \tag{4.11}$$

$$\tag{4.12}$$

(4.11) and (4.2.1.e) imply (4.2.1.c).

(4.9) and (4.10) imply (4.2.1.b).

(4.2.1.f) implies that there is a column of the form $\begin{pmatrix} a_r \\ b_r \end{pmatrix}$ such that a_r is odd. Also such a column has to be the last in the F–partition. This completes the proof of the Theorem 4.2.1. □

To illustrate the bijection we have constructed, the example for $\nu = 8$ is shown in the following table:

F–partitions enumerated by $F_1(8)$	Image under ϕ
$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$	8_2
$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$	8_6
$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$	$5_1 + 3_1$

Sketch of proof of Theorem 4.2.2. Proceed in the same manner as in Theorem

4.2.1, here since $a_r \equiv 0(mod 2)$, therefore

$$m - i = (a_r + b_r + 1) - (b_r - a_r + 1) = 2a_r$$

which imply $(m - i) \equiv 0(mod 4)$. □

Sketch of proof of Theorems 4.2.3. The mapping ϕ is

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a + b + 1)_{b-a+3},$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i = \begin{pmatrix} (m - i + 2)/2 \\ (m + i - 4)/2 \end{pmatrix}.$$

Also here the part 0_2 corresponds to a "phantom" column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, which is dropped from the corresponding F–partition. □

4.3 Tenth order mock theta function, $\psi_R(q)$, and F–partitions

In this section, we extended the Theorem 3.2.1 of Chapter 3 using F–partitions and established a two–way combinatorial identity.

Theorem 4.3.1. *For $\nu \geq 1$, Let $N_1(\nu)$ denote the number of F–partitions of ν such that*

(4.3.1.a) *For any two adjacent columns* $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ *and* $\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix}$ *we have*

Case (i): If $a_i \leq b_i$ then $a_{i+1} > b_{i+1}$ and $a_i = a_{i+1}$, for $1 \leq i \leq (r - 2)$,

Case (ii): If $a_i > b_i$ then $a_{i+1} \leq b_{i+1}$ and $b_i = b_{i+1}$, for $1 \leq i \leq (r - 1)$.

(4.3.1.b) $a_r = 0$.

Further, let $M_1(\nu)$ denote the number of n–color partitions of ν such that

(4.3.1.c) *the weighted difference of any two consecutive parts is -1,*

(4.3.1.d) *for some i, i_i is a part.*

Then $N_1(\nu) = M_1(\nu)$, for all ν.

Remark. $M_1(\nu)$ is same as defined in Theorem 3.2.1 of Chapter 3.

Proof of Theorem 4.3.1. We establish a 1–1 correspondence between the F–partitions enumerated by $N_1(\nu)$ and the n–color partitions enumerated by $M_1(\nu)$. We do this by mapping each column $\begin{pmatrix} a \\ b \end{pmatrix}$ of the Frobenius partition to a single part m_i of an n–color partition enumerated by $M_1(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{cases} (a+b+1)_{b-a+1}, & a \le b; \\ (a+b+1)_{a-b}, & a > b. \end{cases} \tag{4.13}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \to \begin{cases} \begin{pmatrix} (m-i)/2 \\ (m+i-2)/2 \end{pmatrix}, & \text{if } m \equiv i(mod2); \\ \begin{pmatrix} (m+i-1)/2 \\ (m-i-1)/2 \end{pmatrix}, & \text{if } m \equiv i+1(mod2). \end{cases} \tag{4.14}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ in an F–partition enumerated by $N_1(\nu)$ with

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} = m_i \text{ and } \phi : \begin{pmatrix} c \\ d \end{pmatrix} = n_j.$$

Then since $a \geq c$ and $b \geq d$ we have

$$((m_i - n_j)) = \begin{cases} 2(b-d)-1, & b < a, d \geq c; \\ 2(a-c)-1, & b \geq a, d < c. \end{cases} \tag{4.15}$$

Therefore (4.15) and (4.3.1.a) imply (4.3.1.c).

Now if $a_r = 0$, then using (4.13) we have

$$\phi : \begin{pmatrix} a_r \\ b_r \end{pmatrix} = (b_r + 1)_{b_r+1}, \text{ for } a \leq b \tag{4.16}$$

which is of the form i_i. Hence, (4.3.1.b) and (4.13) imply (4.3.1.d).

To see the reverse implication, we consider the inverse images of two consecutive parts m_i, n_j of an n–color partition enumerated by $M_1(\nu)$ such that,

$$\phi^{-1} : m_i = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \phi^{-1} : n_j = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Since $((m_i - n_j)) = -1$, we see that if m and i have the same parity then n and j will have the opposite parity and vice–versa. Then

$$a - c = \begin{cases} \frac{((m_i-n_j))+1}{2}, & m \equiv i (mod2), n \not\equiv j(mod2); \\ \frac{((m_i-n_j))-1}{2} + i + j, & m \not\equiv i(mod2), n \equiv j(mod2). \end{cases} \tag{4.17}$$

$$b - d = \begin{cases} \frac{((m_i-n_j))-1}{2} + i + j, & m \equiv i (mod2), n \not\equiv j(mod2); \\ \frac{((m_i-n_j))+1}{2}, & m \not\equiv i(mod2), n \equiv j(mod2). \end{cases} \tag{4.18}$$

Also,

$$a - b = \begin{cases} -i + 1, & m \equiv i(mod2); \\ i, & m \not\equiv i(mod2). \end{cases} \tag{4.19}$$

Similarly,

$$c - d = \begin{cases} -j + 1, & n \equiv j(mod2); \\ j, & n \not\equiv j(mod2). \end{cases} \tag{4.20}$$

Now (4.17), (4.18), (4.19) and (4.20) imply (4.3.1.*a*).

Finally from (4.3.1.*d*), i.e. i_i must appear as a part, and it is the smallest part of the partition, therefore using (4.14), we have

$$\phi^{-1} : i_i = \begin{pmatrix} 0 \\ i-1 \end{pmatrix}$$

hence $a_r = 0$ and (4.3.1.*b*) holds.

This completes the proof of Theorem 4.3.1. □

Example 4.3.1. *To illustrate the bijection we have constructed, the example for* ν*=6 shown in the following table:*

Generalized Frobenius partitions enumerated by $B_1(6)$	*Image under* ϕ
$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	6_6
$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$	$5_4 + 1_1$
$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$	$4_1 + 2_2$
$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$3_1 + 2_1 + 1_1$

4.4 Conclusion

In this chapter, identities (2.2)–(2.4) of Chapter 2 have been interpreted using F–partitions through establishing bijections between parts of $(n+t)$–color partitions and columns of F–partitions. Our results lead to four–way combinatorial interpretations of Rogers–Ramanujan type identities (2.2)–(2.4) as follows;

$$A_i(\nu) = B_i(\nu) = E_i(\nu) = F_i(\nu), \quad \text{for } 1 \le i \le 3, \tag{4.21}$$

where $A_i(\nu)$ and $B_i(\nu)$ are defined in Theorems 2.2.1–2.2.3, $E_i(\nu)$ are defined in Theorems 2.3.1–2.3.3, in Chapter 2 and $F_i(\nu)$ are defined in Theorems 4.2.1–4.2.3 of this chapter. In particular, we get eighteen combinatorial identities of which nine are given in Chapter 2 and rest, given below, are obtained in this chapter.

$$A_i(\nu) = F_i(\nu), \tag{4.22}$$

$$B_i(\nu) = F_i(\nu), \tag{4.23}$$

$$\text{and} \qquad E_i(\nu) = F_i(\nu), \qquad \text{for } 1 \le i \le 3. \tag{4.24}$$

Also the tenth order mock theta function $\psi_R(q)$ is interpreted using F–partitions in the similar manner and we get a two–way combinatorial interpretation of $\psi_R(q)$ as given below;

$$M_1(\nu) = N_1(\nu), \tag{4.25}$$

where $M_1(\nu)$ is defined in Theorem 3.2.1 of Chapter 3 and $N_1(\nu)$ is defined in Theorem 4.3.1 of this chapter.

Chapter 5

Split *(n+t)*–color partitions and generalized *q*–series

5.1 Introduction

In Chapter 2, we gave the $(n+t)$–color partition theoretic interpretations of three Rogers–Ramanujan type identities. Since there are number of q–series/identities available in literature that lack in their combinatorial meaning, so efforts are being made by mathematicians to explore new tools that can be fit for such identities. In 2014, Agarwal and Sood [17] extended the $(n+t)$–color partitions to 'split $(n+t)$–color partitions' as follows:

Definition 5.1.1. *A split $(n+t)$–color partition is defined as a partition in which if m_i be a summand in a $(n+t)$–color partition of a non negative integer ν, split the color 'i' into two parts–'the green part' and 'the red part' and denote them by 'g' and 'r' respectively, such that $1 \leq g \leq i$, $0 \leq r \leq i-1$ and $i = g + r$. Such $(n+t)$–color partition in which subscript of each summand splits in this manner is called a split $(n+t)$–color partitions.*

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Example 5.1.1. *Split* $(n+1)$*–color partitions of 2 are,*

2_1	$2_1 0_1$	$1_1 1_1$	$1_1 1_1 0_1$
2_2	$2_2 0_1$	$1_2 1_1$	$1_2 1_1 0_1$
2_3	$2_3 0_1$	$1_2 1_2$	$1_2 1_2 0_1$
2_{1+1}	$2_{1+1} 0_1$	$1_{1+1} 1_1$	$1_{1+1} 1_1 0_1$
2_{2+1}	$2_{2+1} 0_1$	$1_{1+1} 1_2$	$1_{1+1} 1_2 0_1$
2_{1+2}	$2_{1+2} 0_1$	$1_{1+1} 1_{1+1}$	$1_{1+1} 1_{1+1} 0_1$.

Remark. In the above partitions, note that whenever the red part is 0, then it will not be written separately. That is, 2_{g+0} is written as 2_g only.

These split $(n+t)$–color partitions were used by Agarwal and Sood [17] to provide combinatorial interpretations of following two mock theta functions generated by Gordon and McIntosh [49], also discussed in Chapter 3.

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \tag{5.1}$$

and

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{5.2}$$

In their paper [17], Agarwal and Sood also posed an open problem: whether these split $(n+t)$–color partitions be used to provide combinatorial meaning to Rogers–Ramanujan type identities? We address to this problem in this chapter and use split $(n+t)$–color partitions to interpret four generalized q–series given below;

Let $S = \{-1, 1, 3, 5, \ldots\}$, for $|q| < 1$, $j \in S$ and $1 \le i \le 4$, we define $f^{(i,j)}(q)$ by

$$f^{(1,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q^4;q^4)_n (q;q^2)_n}, \tag{5.3}$$

$$f^{(2,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(j+3)/2}}{(q^4;q^4)_n (q;q^2)_{n+1}}, \tag{5.4}$$

$$f^{(3,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q^4;q^4)_n (q;q^2)_{n+1}}, \tag{5.5}$$

$$f^{(4,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q^4;q^4)_n (q;q^2)_n}. \tag{5.6}$$

These generalized series are in conjunction with the following four q–identities from Chu and Zhang's compendium [38, I(25), I(27), I(29), I(113)].

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n^2}}{(q^4,q^4)_n(-q,q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^2,-q^3;q^5]_\infty, \tag{5.7}$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{2n(n+1)}}{(q^4,q^4)_n(q,q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^3,q^9;q^{12}]_\infty, \tag{5.8}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_{n+1}} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^5,-q^5;q^5]_\infty, \tag{5.9}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q,-q^4;q^5]_\infty. \tag{5.10}$$

In the further sections, our main results and their proofs are given. Also the combinatorial interpretations of Rogers–Ramanujan type identities (appearing in [38]) as particular cases of our main results are obtained.

5.2 Main results

Theorem 5.2.1. *For $j \in S$, let $P_1^j(\nu)$ represent the number of split n–color partitions of ν such that*

(i) parts and their subscripts have the same parity,

(ii) value of red part can be 0 or 1,

(iii) if m_i is the least or only summand of partition, then $m - i \equiv 0 (mod\ 4)$,

(iv) the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1)(mod\ 4)$, then

$$\sum_{\nu=0}^{\infty} P_1^j(\nu)q^\nu = f^{(1,j)}(q). \tag{5.11}$$

Theorem 5.2.2. *For $j \in S$, let $P_2^j(\nu)$ represent the number of split $(n+1)$–color partitions of ν such that*

(i) parts and their subscripts have the opposite parity,

(ii) the value of red part can be 0 or 1,

(iii) the smallest summand is of the form i_{i+1},

(iv) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1)(mod\ 4)$*, then*

$$\sum_{\nu=0}^{\infty} P_2^j(\nu)q^\nu = f^{(2,j)}(q). \tag{5.12}$$

Theorem 5.2.3. *For* $j \in S$*, let* $P_3^j(\nu)$ *represent the number of split* $(n+2)$*–color partitions of* ν *such that*

(i) *parts and their subscripts have the same parity,*

(ii) *the value of red part can be 0 or 1,*

(iii) *the smallest summand is of the form* i_{i+2}*,*

(iv) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1)(mod\ 4)$*, then*

$$\sum_{\nu=0}^{\infty} P_3^j(\nu)q^\nu = f^{(3,j)}(q). \tag{5.13}$$

Theorem 5.2.4. *For* $j \in S$*, let* $P_4^j(\nu)$ *represent the number of split* n*–color partitions of* ν *such that*

(i) *parts and their subscripts have the same parity,*

(ii) *the value of red part can be 0 or 1,*

(iii) *if* m_i *is the least or only summand of partition, then* $m \geq (j+4)$ *and* $m - i \equiv (j+3)(mod\ 4)$*,*

(iv) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1)(mod\ 4)$*, then*

$$\sum_{\nu=0}^{\infty} P_4^j(\nu)q^\nu = f^{(4,j)}(q). \tag{5.14}$$

Remark. The conditions (i), (iii) and (iv) in Theorems 5.2.1–5.2.4 are allowed for the whole subscript i irrespective of green and red parts separately.

5.3 Proofs of Main results

Proof of Theorem 5.2.1. Let $P_1^j(m,\nu)$ denote the number of partitions of ν enumerated by $P_1^j(\nu)$ into m summands. We split the partitions enumerated by $P_1^j(m,\nu)$ into four classes: Class (i): that do not involve k_k or $k_{(k-1)+1}$ as a summand,
Class (ii): that involve 1_1 as a summand,
Class (iii): that involve 2_{1+1} as a summand,
Class (iv): that involve $k_k(k \geq 2)$ and $k_{(k-1)+1}(k \geq 3)$ as a summand.
We transform the partitions which lie in class (i) by subtracting 4 from subsequent summands, ignoring the subscripts. The transformed partitions are enumerated by $P_1^j(m,\nu-4m)$. In the class (ii) delete the summand 1_1 and then subtract $(j+3)$ from the remaining summands ignoring the subscripts, we get partitions enumerated by $P_1^j(m-1,\nu-m(j+3)+j+2)$. Next, in class (iii) delete 2_2 and then subtract $(j+5)$ from the remaining summands ignoring the subscripts. The transformed partitions are enumerated by $P_1^j(m-1,\nu-m(j+5)+j+3)$. In the last class (iv) replace k_k by $(k-1)_{k-1}$ and $k_{(k-1)+1}$ by $(k-1)_{(k-2)+1}$ and subtract 2 from the remaining summands ignoring the subscripts. This will result in partitions enumerated by $P_1^j(m,\nu-2m+1)$. It should be noted here that we are obtaining only those partitions of $\nu-2m+1$ which involve a summand of the type k_k and $k_{(k-1)+1}$. So the number of partitions in class (iv) enumerated is $P_1^j(m,\nu-2m+1) - P_1^j(m,\nu-6m+1)$.
Since the transformations defined above are reversible and they give one to one correspondence between the partitions enumerated by $P_1^j(m,\nu)$ and those by
$P_1^j(m,\nu-4m) + P_1^j(m-1,\nu-m(j+3)+j+2) + P_1^j(m-1,\nu-m(j+5)+j+3) + P_1^j(m,\nu-2m+1) - P_1^j(m,\nu-6m+1)$.
This will give rise to the following recurrence relation:

$$P_1^j(m,\nu) = P_1^j(m,\nu-4m)+P_1^j(m-1,\nu-m(j+3)+j+2)+P_1^j(m-1,\nu-m(j+5)+j+3)+P_1^j(m,\nu-2m+1)-P_1^j(m,\nu-6m+1), \tag{5.15}$$

where $P_1^j(0,0) = 1$ and $P_1^j(m,\nu) = 0$ for $\nu < 0$.

For $|q| < 1$ and $|z| < |q|^{-1}$, let $f_1^j(z,q)$ be defined by

$$f_1^j(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} P_1^j(m,\nu) z^m q^\nu. \tag{5.16}$$

Substitute $P_1^j(m,\nu)$ from (5.15) in (5.16), we get q–functional equation as

$$f_1^j(z,q) = f_1^j(zq^4,q)+zqf_1^j(zq^{j+3},q)+zq^2 f_1^j(zq^{j+5},q)+q^{-1}f_1^j(zq^2,q)- q^{-1}f_1^j(zq^6,q). \tag{5.17}$$

Setting

$$f_1^j(z,q) = \sum_{n=0}^{\infty} \beta(n,q) z^n \tag{5.18}$$

in (5.17) and then examining the coefficients of z^n in the above expression, we get

$$\beta(n,q) = \frac{q^{1+(j+3)(n-1)}(1+q^{2n-1})}{(1-q^{4n})(1-q^{2n-1})}\beta(n-1,q). \tag{5.19}$$

Iterating (5.19) n times and note that $\beta(0,q) = 1$, we find that

$$\beta(n,q) = \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_n (q;q^2)_n}. \tag{5.20}$$

Therefore,

$$f_1^j(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_n (q;q^2)_n} z^n \tag{5.21}$$

$$= f^{1,j}(z,q) \tag{5.22}$$

and

$$\begin{aligned}\sum_{\nu=0}^{\infty} P_1^j(\nu) q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} P_1^j(m,\nu)\right) q^\nu \\ &= f^{1,j}(1,q) \\ &= f^{1,j}(q).\end{aligned}$$

This proves Theorem 5.2.1. □

Proof of Theorem 5.2.2. Let $D^j(\nu)$ represent the number of split n–color partitions of ν enumerated by $P_1^j(\nu)$ with additional constraint that the least summand is of the type k_k and let $D^j(m,\nu)$ represent the number of split n–color partitions of ν enumerated by $D^j(\nu)$ into m summands. Further, let

$$h^j(q) = \sum_{\nu=0}^{\infty} D^j(\nu) q^{\nu}, \tag{5.23}$$

and

$$h^j(z,q) = \sum_{\nu,m=0}^{\infty} D^j(m,\nu) z^m q^{\nu}. \tag{5.24}$$

With the help of (5.15), we have

$$\begin{aligned} D^j(m,\nu) = P_1^j(m-1,\nu-m(j+3)+j+2) + \tfrac{1}{2}[P_1^j(m-1,\nu-m(j+ \\ 5)+j+3) + P_1^j(m,\nu-2m+1) - P_1^j(m,\nu-6m+1), \end{aligned} \tag{5.25}$$

where $P_1^j(0,0) = 1$ and $P_1^j(m,\nu) = 0$ for $\nu < 0$.
Transforming (5.25) into a q–functional equation, we get

$$h^j(z,q) = zqf_1^j(zq^{j+3},q) + \frac{1}{2}zq^2 f_1^j(zq^{j+5},q) + \frac{1}{2}q^{-1}f_1^j(zq^2,q) - \frac{1}{2}q^{-1}f_1^j(zq^6,q). \tag{5.26}$$

Setting

$$h^j(z,q) = \sum_{n=0}^{\infty} \gamma(n,q) z^n, \tag{5.27}$$

and then examining the coefficients of z^n in the above expression (5.26), we get

$$\begin{aligned} 2\gamma(n,q) = 2q^{(j+3)(n-1)+1}\beta(n-1,q) + q^{(j+5)(n-1)+2}\beta(n-1,q) + \\ q^{2n-1}\beta(n,q) - q^{6n-1}\beta(n,q). \end{aligned} \tag{5.28}$$

Replacing $\beta(n,q)$ from (5.19) in (5.28) and then analyzing, we get

$$\gamma(n,q) = \frac{(-q;q^2)_{n-1}q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_{n-1}(q;q^2)_n}. \tag{5.29}$$

Thus

$$h^j(z,q) = \sum_{\nu=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)[1+(j+3)n/2]}}{(q^4;q^4)_n(q;q^2)_{n+1}} z^{n+1} = zqf^{2,j}(zq,q). \tag{5.30}$$

Define $Q^j(m,\nu)$ by

$$g_2^j(z,q) = \sum_{m,\nu=0}^{\infty} Q^j(m,\nu)z^m q^\nu.$$

By examining the coefficients of (5.30), we get

$$D^j(m+1,\nu+m+1) = Q^j(m,\nu).$$

If each summand is subtracted by 1 which is enumerated by $D^j(m+1,\nu+m+1)$ ignoring the subscripts, we have the final partitions enumerated by $P_2^j(m+1,\nu)$. Thus

$$Q^j(m,\nu) = P_2^j(m+1,\nu)$$

and so

$$\sum_{m,\nu=0}^{\infty} P_2^j(m+1,\nu)z^m q^\nu = f^{2,j}(z,q). \tag{5.31}$$

Now

$$\begin{aligned}
\sum_{\nu=0}^{\infty} P_2^j(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} P_2^j(m,\nu)\right)q^\nu \\
&= \sum_{m,\nu=0}^{\infty} P_2^j(m+1,\nu)q^\nu \\
&= f^{2,j}(1,q) \\
&= f^{2,j}(q).
\end{aligned}$$

This proves Theorem 5.2.2. □

Proof of Theorem 5.2.3. Rewrite equation (5.30) as

$$h^j(z,q) = zqf^{3,j}(z,q). \tag{5.32}$$

Define $R^j(m,\nu)$ by

$$g_3^j(z,q) = \sum_{m,\nu=0}^{\infty} R^j(m,\nu) z^m q^\nu. \tag{5.33}$$

By examining the coefficients of (5.32), we get

$$D^j(m+1,\nu+1) = R^j(m,\nu). \tag{5.34}$$

If summand k_k is replaced by $(k-1)_{k+1}$ which is enumerated by $D^j(m+1,\nu+1)$, we have the final partitions enumerated by $P_3^j(m+1,\nu)$. Thus

$$R^j(m,\nu) = P_3^j(m+1,\nu)$$

and

$$\sum_{m,\nu=0}^{\infty} P_3^j(m+1,\nu) z^m q^\nu = f^{3,j}(z,q). \tag{5.35}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} P_3^j(\nu) q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} P_3^j(m,\nu)\right) q^\nu \\ &= \sum_{m,\nu=0}^{\infty} P_3^j(m+1,\nu) q^\nu \\ &= f^{3,j}(1,q) \\ &= f^{3,j}(q).\end{aligned}$$

This proves Theorem 5.2.3. □

Proof of Theorem 5.2.4. Let $P_4^j(m,\nu)$ denote the number of partitions of ν enumerated by $P_4^j(\nu)$ into m summands. We split the partitions enumerated by $P_4^j(m,\nu)$ into four classes:

Class (i): that do not involve $k_{(k-(j+3))+0}$ or $k_{(k-(j+4))+1}$ as a summand, Class (ii): that involve $(j+4)_1$ as a summand,

Class (iii): that involve $(j+5)_{1+1}$ as a summand,

Class (iv): that involve $k_{(k-(j+3))+0}(k \geq (j+5))$ and $k_{(k-(j+4))+1}(k \geq (j+6))$ as a summand.

We transform the partitions which lie in class (i) by subtracting 4 from subsequent summands ignoring the subscripts. The transformed partitions are enumerated by $P_4^j(m, \nu - 4m)$. In the class (ii) delete the summand $(j+4)_1$ and then subtract $(j+3)$ from remaining summands ignoring the subscripts, we get partitions enumerated by $P_4^j(m-1, \nu - m(j+3) - 1)$. Next, in class (iii) delete $(j+5)_{1+1}$ and then subtract $(j+5)$ from the remaining summands ignoring the subscripts. The transformed partitions are enumerated by $P_4^j(m-1, \nu - m(j+5))$. In the last class (iv) replace $k_{(k-(j+3))+0}$ by $(k-1)_{(k-(j+4))+0}$ and $k_{(k-(j+4))+1}$ by $(k-1)_{(k-(j+5))+1}$ and subtract 2 from the remaining summands ignoring the subscripts. This will result in partitions enumerated by $P_4^j(m, \nu - 2m + 1)$. It should be noted here that we are obtaining those partitions of $\nu - 2m + 1$ which involve a summand of the type $k_{(k-(j+3))+0}$ and $k_{(k-(j+4))+1}$. So the number of partitions in class (iv) enumerated is $P_4^j(m, \nu - 2m + 1) - P_4^j(m, \nu - 6m + 1)$.

Since the transformations given above are reversible so we get the following recurrence relation:

$$P_4^j(m, \nu) = P_4^j(m, \nu - 4m) + P_4^j(m-1, \nu - m(j+3) - 1) + P_4^j(m-1, \nu - m(j+5)) + P_4^j(m, \nu - 2m + 1) - P_4^j(m, \nu - 6m + 1), \tag{5.36}$$

where $P_4^j(0,0) = 1$ and $P_4^j(m, \nu) = 0$ for $\nu < 0$.

For $|q| < 1$ and $|z| < |q|^{-1}$, let $f_2^j(z,q)$ be defined by

$$f_2^j(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} P_4^j(m, \nu) z^m q^\nu. \tag{5.37}$$

Using (5.36) and (5.37) we get the following q–functional equation

$$f_2^j(z,q) = f_2^j(zq^4,q)+zq^{j+4}f_2^j(zq^{j+3},q)+zq^{j+5}f_2^j(zq^{j+5},q)+q^{-1}f_2^j(zq^2,q)-q^{-1}f_2^j(zq^6,q). \tag{5.38}$$

Setting

$$f_2^j(z,q) = \sum_{n=0}^{\infty}\alpha(n,q)z^n \tag{5.39}$$

in (5.38) and then examining the coefficients of z^n in the above expression, we get

$$\alpha(n,q) = \frac{q^{1+(j+3)(n)}(1+q^{2n-1})}{(1-q^{4n})(1-q^{2n-1})}\alpha(n-1,q). \tag{5.40}$$

Iterating above n–times and noting $\alpha(0,q)=1$, we find

$$\alpha(n,q) = \frac{(-q;q^2)_n q^{n(1+(j+3)(n+1)/2)}}{(q^4;q^4)_n(q;q^2)_n}. \tag{5.41}$$

Therefore

$$f_2^j(z,q) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(1+(j+3)(n+1)/2)}}{(q^4;q^4)_n(q;q^2)_n}z^n = f^{4,j}(z,q) \tag{5.42}$$

and

$$\begin{aligned}\sum_{\nu=0}^{\infty}P_4^j(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty}P_4^j(m,\nu)\right)q^\nu\\ &= f^{4,j}(1,q)\\ &= f^{4,j}(q).\end{aligned}$$

This proves Theorem 5.2.4. □

5.4 Particular Cases

The Identity (5.7), in conjunction with Theorem 5.2.1, for $j=-1$, leads to the following Theorem 5.4.1:

Theorem 5.4.1. *Let $E_1(\nu)$ denote the number of n–color partitions of ν such that* *parts are distinct and first two copies of parts* $\equiv 5(mod\ 10)$ *are used and only first copy of parts* $\equiv \pm 1(mod\ 10)$ *are used and let* $F_1(\nu)$ *denote the number of* n*–color partitions of* ν *such that first two copies of parts* $\equiv \pm 2(mod\ 10)$ *are used. Further, let*

$$A_1(\nu) = \sum_{k=0}^{\nu} E_1(k)F_1(\nu - k),$$

then

$$A_1(\nu) = P_1^{-1}(\nu), \text{ for all } \nu.$$

Example 5.4.1. *We can verify Theorem 5.4.1 by showing that*

$$P_1^{-1}(10) = A_1(10) = 18.$$

The relevant n*–color partitions corresponding to* $P_1^{-1}(10)$ *are*

10_{10}, 10_{9+1}, 10_6, 10_{5+1}, 10_2, 10_{1+1}, $9_7 1_1$, $9_{6+1} 1_1$, $9_3 1_1$, $9_{2+1} 1_1$, $8_4 2_2$, $8_{3+1} 2_2$, $8_4 2_{1+1}$, $8_{3+1} 2_{1+1}$, $7_1 3_3$, $7_1 3_{2+1}$, $6_2 3_1 1_1$, $6_{1+1} 3_1 1_1$

and $A_1(10) = 18$, *since*

$$A_1(10) = \sum_{k=0}^{10} E_1(k)F_1(10 - k),$$

where the relevant partitions corresponding to $E_1(\nu)$ *and* $F_1(\nu)$ *are given in the table below:*

ν	$E_1(\nu)$	*Partitions enumerated by* $E_1(\nu)$	$F_1(\nu)$	*Partitions enumerated by* $F_1(\nu)$
0	1	*empty partition*	1	*empty partition*
1	1	1_1	0	-
2	0	-	2	2_1, 2_2
3	0	-	0	-

4	0	-	3	$2_12_1,\ 2_22_1,\ 2_22_2$
5	2	$5_1,\ 5_2$	0	-
6	2	$5_11_1,\ 5_21_1$	4	$2_12_12_1,\ 2_22_12_1,\ 2_22_22_1,\ 2_22_22_2$
7	0	-	0	-
8	0	-	7	$8_1,\ 8_2,\ 2_12_12_12_1,\ 2_22_12_12_1,$ $2_22_22_12_1,\ 2_22_22_22_1,\ 2_22_22_22_2$
9	1	9_1	0	-
10	2	$5_25_1,\ 9_11_1$	10	$8_12_1,\ 8_22_1,\ 8_12_2,\ 8_22_2,$ $2_12_12_12_12_1,\ 2_22_12_12_12_1,$ $2_22_22_12_12_1,\ 2_22_22_22_12_1,$ $2_22_22_22_22_1,\ 2_22_22_22_22_2$

Therefore,

$$\begin{aligned} A_1(10) =& E_1(10)F_1(0) + E_1(9)F_1(1) + \cdots + E_1(0)F_1(10) \\ =& 18. \end{aligned}$$

The Identity (5.8), in conjunction with Theorem 5.2.2, for $j = 1$, leads to the following Theorem 5.4.2:

Theorem 5.4.2. *Let $E_2(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm1, \pm5 (mod\ 12)$ and let $F_2(\nu)$ denote the number of partitions of ν into parts $\equiv \pm2, \pm4 (mod\ 12)$. Further, let*

$$A_2(\nu) = \sum_{k=0}^{\nu} E_2(k)F_2(\nu - k),$$

then

$$A_2(\nu) = P_2^1(\nu), \ \textit{for all } \nu.$$

Example 5.4.2. *We can verify Theorem 5.4.2 by showing that*

$$P_2^1(10) = A_2(10) = 8.$$

The relevant $(n+1)$*–color partitions corresponding to* $P_2^1(10)$ *are*

10_{11}, $10_7 0_1$, $10_{6+1} 0_1$, $10_3 0_1$, $10_{2+1} 0_1$, $9_4 1_2$, $9_{3+1} 1_2$, $8_1 2_3$

and $A_2(10) = 8$, *since*

$$A_2(10) = \sum_{k=0}^{10} E_2(k) F_2(10-k),$$

where the relevant partitions corresponding to $E_2(\nu)$ *and* $F_2(\nu)$ *are given in the table below:*

ν	$E_2(\nu)$	*Partitions enumerated by* $E_2(\nu)$	$F_2(\nu)$	*Partitions enumerated by* $F_2(\nu)$
0	1	*empty partition*	1	*empty partition*
1	1	1	0	-
2	0	-	1	2
3	0	-	0	-
4	0	-	2	$4,\ 2+2$
5	1	5	0	-
6	1	$5+1$	2	$4+2,\ 2+2+2$
7	1	7	0	-
8	1	$7+1$	4	$8,\ 4+4,\ 4+2+2,\ 2+2+2+2$
9	0	-	0	-

10	0	-	5	$10,\ 8+2,\ 4+4+2,$ $4+2+2+2,\ 2+2+2+2+2$

Therefore,

$$\begin{aligned} A_2(10) =& E_2(10)F_2(0) + E_2(9)F_2(1) + \cdots + E_2(0)F_2(10) \\ =& 8. \end{aligned}$$

The Identity (5.9), in conjunction with Theorem 5.2.3, for $j = -1$, leads to the following Theorem 5.4.3:

Theorem 5.4.3. *Let $E_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 1$, $\pm 3 (mod\ 10)$ and let $F_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 2, \pm 4 (mod\ 10)$. Further, let*

$$A_3(\nu) = \sum_{k=0}^{\nu} E_3(k) F_3(\nu - k),$$

then

$$A_3(\nu) = P_3^{-1}(\nu),\ \textit{for all}\ \nu.$$

Example 5.4.3. *We can verify Theorem 5.4.3 by showing that*

$$P_3^{-1}(10) = A_3(10) = 12.$$

The relevant $(n+2)$–color partitions corresponding to $P_3^{-1}(10)$ are
10_{12}, $10_8 0_2$, $10_4 0_2$, $10_{7+1} 0_2$, $10_{3+1} 0_2$, $9_5 1_3$, $9_1 1_3$, $9_{4+1} 1_3$, $8_2 2_4$, $8_{1+1} 2_4$, $7_3 3_1 0_2$, $7_{2+1} 3_1 0_2$,
and $A_3(10) = 12$, since

$$A_3(10) = \sum_{k=0}^{10} E_3(k) F_3(10 - k),$$

where the relevant partitions corresponding to $E_3(\nu)$ and $F_3(\nu)$ are given in the table below:

ν	$E_3(\nu)$	*Partitions enumerated by* $E_3(\nu)$	$F_3(\nu)$	*Partitions enumerated by* $F_3(\nu)$
0	1	*empty partition*	1	*empty partition*
1	1	1	0	-
2	0	-	1	2
3	1	3	0	-
4	1	$3+1$	2	$4,\ 2+2$
5	0	-	0	-
6	0	-	3	$6,\ 4+2,\ 2+2+2$
7	1	7	0	-
8	1	$7+1$	5	$8,\ 6+2,\ 4+4,\ 4+2+2,\ 2+2+2+2$
9	1	9	0	-
10	2	$7+3,\ 9+1$	6	$8+2,\ 6+2+2,\ 6+4,\ 4+4+2,\ 4+2+2+2,\ 2+2+2+2+2$

Therefore,

$$\begin{aligned} A_3(10) =& E_3(10)F_3(0) + E_3(9)F_3(1) + \cdots + E_3(0)F_3(10) \\ =& 12. \end{aligned}$$

The Identity (5.10), in conjunction with Theorem 5.2.4, for $j = -1$, leads to the following Theorem 5.4.4:

Theorem 5.4.4. *Let $E_4(\nu)$ denote the number of n–color partitions of ν into distinct* *parts such that first two copies of parts* $\equiv 5 (mod\ 10)$ *are used and only first copy of parts* $\equiv \pm 3 (mod\ 10)$ *are used and let* $F_4(\nu)$ *denote the number of* $n-$*color partitions of* ν *such that first two copies of parts* $\equiv \pm 4 (mod\ 10)$ *are used. Further, let*

$$A_4(\nu) = \sum_{k=0}^{\nu} E_4(k) F_4(\nu - k),$$

then

$$A_4(\nu) = P_4^{-1}(\nu), \ \textit{for all } \nu.$$

Example 5.4.4. *We can verify Theorem 5.4.4 by showing that*

$$P_4^{-1}(10) = A_4(10) = 6.$$

The relevant $n-$*color partitions corresponding to* $P_4^{-1}(10)$ *are*

10_8, 10_{7+1}, 10_4, 10_{3+1}, $7_3 3_1$, $7_{2+1} 3_1$

and $A_4(10) = 6$, *since*

$$A_4(10) = \sum_{k=0}^{10} E_4(k) F_4(10 - k),$$

where the relevant partitions corresponding to $E_4(\nu)$ *and* $F_4(\nu)$ *are given in the table below:*

ν	$E_4(\nu)$	*Partitions enumerated by* $E_4(\nu)$	$F_4(\nu)$	*Partitions enumerated by* $F_4(\nu)$
0	1	*empty partition*	1	*empty partition*
1	0	-	0	-
2	0	-	0	-
3	1	3_1	0	-
4	0	-	2	$4_1, 4_2$
5	2	$5_1, 5_2$	0	-

6	0	-	2	$6_1,\ 6_2$
7	1	7_1	0	-
8	2	$4_1 3_1,\ 5_2 3_1$	3	$4_1 4_1,\ 4_2 4_1,\ 4_2 4_2$
9	0	0	0	-
10	2	$5_1 5_2,\ 7_1 3_1$	4	$6_1 4_1,\ 6_1 4_2,\ 6_2 4_1,\ 6_2 4_2$

Therefore,

$$A_4(10) = E_4(10)F_4(0) + E_4(9)F_4(1) + \cdots + E_4(0)F_4(10)$$
$$= 6.$$

5.5 Conclusion

In this chapter, we provided the combinatorial interpretations of four generalized q–series using split $(n+t)$–color partitions. In particular, we get two–way combinatorial interpretations of four Rogers–Ramanujan type identities as follows;

$$A_i(\nu) = P_i^j(\nu), \qquad \text{for } j = -1 \text{ and } i = 1,\ 3,\ 4 \tag{5.43}$$

and

$$A_2(\nu) = P_2^1(\nu). \tag{5.44}$$

We further extend our results in Chapter 6 to provide the another interpretations of generalized q–series (5.3)–(5.6) using 2–color F–partitions.

Chapter 6

Split *(n+t)*–color partitions and *2*–color *F*–partitions

6.1 Introduction

In Chapter 1, we studied the Frobenius representations of ordinary partitions given by Frobenius [42] and in Chapter 3 we have discussed F–partitions introduced by Andrews [24] in 1984. Andrews in his book [24] further generalized the concept of Frobenius symbol in two classes: The first class contain F–partitions in which parts appear at most k times in any row. Let $\phi_k(\nu)$ denote the number of all such F–partitions of ν and let $\Phi_k(q) = \sum_{\nu=0}^{\infty} \phi_k(\nu) q^{\nu}$ denote the generating function of $\phi_k(\nu)$. Then

$$\Phi_2(q) = 1 + q + 3q^2 + 5q^3 + 9q^4 + \cdots .$$

$$\Phi_3(q) = 1 + q + 3q^2 + 6q^3 + 11q^4 + \cdots .$$

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Example 6.1.1. *The F–partitions enumerated by $\phi_2(2)$ are*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The second class contain colored F–partitions with k copies of the non–negative integers

$$(j_i : 0 \leq j \leq n-1), 1 \leq i \leq k$$

where $j_i \neq j'_{i'}$, unless $j = j'$ and $i = i'$. Also there is a strict decrease among the parts along the rows and the parts follow the order

$$0_1 < 0_2 < \cdots < 1_1 < 1_2 < 1_3 < \cdots < 2_1 < 2_2 < 2_3 < \cdots < 3_1 < 3_2 < 3_3 < \cdots .$$

Consider F–partitions of ν in which the parts in either row appear from k copies and are distinct. Let $c\phi_k(\nu)$ denote the number of all such partitions and $c\Phi_k(q) = \sum_{\nu=0}^{\infty} c\phi_k(\nu)q^\nu$ be the generating function of $c\phi_k(\nu)$. Then

$$c\Phi_2(q) = 1 + 4q + 9q^2 + 20q^3 + 42q^4 + \cdots .$$

Example 6.1.2. *F–partitions enumerated by $c\phi_2(2)$ are*

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix},$$

$$\begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Notation. The partition function $cF_k(\nu)$ denote the restricted k–color F–partitions in which top and bottom row entries of each column appear with same subscripts, where k is used for the color.

Example 6.1.3. *The relevant 2–color F–partitions corresponding to $cF_2(2)$ are*

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Similar to the congruences established by Ramanujan [57], Andrews in his book [24] established the following congruence properties for the two classes of F–partitions.

$$\phi_2(5n+3) \equiv 0(mod 3)$$
$$\text{and} \quad c\phi_k(5n+3) \equiv 0(mod k^2).$$

Many authors have established the congruence relations for the second class of F–partitions for instance see [19, 32, 43, 52]. The relationship between certain generalized colored F–partitions and ordinary partitions are shown by Kolitsch [53].
In this chapter, we explicitly used the second class of F–partitions, colored F–partitions, for providing the bijections between split $(n+t)$–color partitions and second class of F–partitions, analogues to the bijections between F–partitions and $(n+t)$–colored partitions established in Chapter 4 and in [59, 60, 65], for q–series and generalized q–series.

6.2 Gordon–McIntosh mock theta functions and 2–color *F*–partitions

In Chapter 3, we interpreted two Gordon–McIntosh mock theta functions $V_0(q)$ and $V_1(q)$, given by (3.35) and (3.36), using signed partitions. Also these mock theta functions were combinatorial interpreted by Agarwal and Sood [17] using split $(n+t)$–color partitions. Using the colored F–partitions discussed in previous section, we extend the results of Agarwal and Sood [17] and provide the another combinatorial interpretations of $V_0(q)$ and $V_1(q)$ and establish the bijections between the two combinatorial tools. For more clarity, we here reproduce the results using split $(n+t)$–color partitions of Agarwal and Sood [17].

Theorem 6.2.1. *For $\nu \geq 1$, let $A_1(\nu)$ denote the number of 'split n–color partitions' of ν such that*

(6.2.1.a) *the parts and their subscripts have the same parity,*

(6.2.1.b) *the red part of the subscripts cannot exceed 1,*

(6.2.1.c) *the least part is either k_k $(k \geq 1)$ or $k_{(k-1)+1}$ $(k \geq 2)$, and*

(6.2.1.d) *the weighted difference of any two consecutive parts is 0.*

Then

$$V_0(q) = 1 + 2\sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu}. \tag{6.1}$$

Remark. In conditions (6.2.1.a) and (6.2.1.d) the whole subscript i is considered, not its parts g and r, separately.

Theorem 6.2.2. *For $\nu \geq 1$, let $A_2(\nu)$ denote the number of 'split n–color partitions' of ν such that*

(6.2.2.a) *the parts and their subscripts have the same parity,*

(6.2.2.b) *the red part of the subscripts cannot exceed 1,*

(6.2.2.c) *the least part is k_k $(k \geq 1)$, and*

(6.2.2.d) *the weighted difference of any two consecutive parts is 0. Then*

$$V_1(q) = \sum_{\nu=1}^{\infty} A_2(\nu)q^{\nu}. \tag{6.2}$$

Remark. As in Theorem 6.2.1, here also, in conditions (6.2.2.a) and (6.2.2.d) the whole subscript i is considered, not its parts g and r, separately.

The following are our main results providing partition identities of $V_0(q)$ and $V_1(q)$ using 2–color F–partitions.

Theorem 6.2.3. *Let $cF_2^1(\nu)$ denote the number of 2–color F–partitions of ν in which every column has the parts with same subscripts such that*

(6.2.3.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$,

(6.2.3.f) $h = 1$ *or 2,*

(6.2.3.g) *for the last column,* $p_h = 0_h$,

(6.2.3.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p = q' + 1$, *ignoring the subscripts. Then*

$$V_0(q) = \sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} cF_2^1(\nu). \tag{6.3}$$

Theorem 6.2.4. *Let* $cF_2^2(\nu)$ *denote the number of 2–color* F*–partitions of* ν *in which every column has the parts with same subscripts such that*

(6.2.4.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$,

(6.2.4.f) $h = 1$ *or 2,*

(6.2.4.g) *for the last column,* $p_h = 0_1$,

(6.2.4.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p = q' + 1$, *ignoring the subscripts. Then*

$$V_1(q) = \sum_{\nu=1}^{\infty} A_2(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} cF_2^2(\nu). \tag{6.4}$$

Proof of Theorem 6.2.3. We establish a one to one correspondence between the 2–color F–partitions enumerated by $cF_2^1(\nu)$ and the split n–color partitions enumerated by $A_1(\nu)$. We do this by mapping each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ of the 2–color F–partition to a single part m_{g+r} of an split n–color partition enumerated by $A_1(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p + q + 1)_{(q-p-h+2)+(h-1)}, \text{ if } p \leq q - h + 1 \tag{6.5}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \rightarrow \begin{pmatrix} (\frac{m-(g+r)}{2})_{r+1} \\ (\frac{m+(g+r)-2}{2})_{r+1} \end{pmatrix}, \text{ if } m \equiv g + r(mod2). \tag{6.6}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$ in a 2–color F–partition enumerated by $cF_2^1(\nu)$ with

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} = m_{g+r} \text{ and } \phi : \begin{pmatrix} p'_k \\ q'_k \end{pmatrix} = n_{g'+r'}.$$

Then since

$$\begin{pmatrix} p_h \\ q_h \end{pmatrix} \rightarrow (p + q + 1)_{(q-p-h+2)+(h-1)} = m_{g+r}$$

and

$$\begin{pmatrix} p'_k \\ q'_k \end{pmatrix} \rightarrow (p' + q' + 1)_{(q'-p'-k+2)+(k-1)} = n_{g'+r'},$$

we have

$$\begin{aligned} ((m_i - n_j)) =& m - n - i - j \\ =& (p + q + 1) - (p' + q' + 1) - (q - p - h + 2 + (h - 1)) \\ & - (q' - p' - k + 2 + (k - 1)) \\ =& 2(p - q' - 1). \end{aligned} \tag{6.7}$$

Clearly (6.7) and (6.2.3.h) imply (6.2.1.d).

Now using (6.5)

$$m - (g + r) = (p + q + 1) - (q - p - h + 2 + (h - 1)) = 2p \tag{6.8}$$

which imply $(m - (g + r)) \equiv 0(mod2)$, hence (6.2.1.$a$) holds.

(6.2.3.f) and (6.5) imply (6.2.1.b).

(6.2.3.f), (6.2.3.g) and (6.5) imply (6.2.1.c).

To see the reverse implication, we consider the inverse images of two consecutive parts m_{g+r}, $n_{g'+r'}$ of a split n–color partition enumerated by $A_1(\nu)$

$$\phi^{-1} : m_{g+r} = \begin{pmatrix} (\frac{m-(g+r)}{2})_{r+1} \\ (\frac{m+(g+r)-2}{2})_{r+1} \end{pmatrix} \tag{6.9}$$

and

$$\phi^{-1} : n_{g'+r'} = \begin{pmatrix} (\frac{n-(g'+r')}{2})_{r+1} \\ (\frac{n+(g'+r')-2}{2})_{r+1} \end{pmatrix} \tag{6.10}$$

that is,

$$p = \frac{m-(g+r)}{2}, \tag{6.11}$$

$$q = \frac{m+(g+r)-2}{2}, \tag{6.12}$$

$$p' = \frac{n-(g'+r')}{2}, \tag{6.13}$$

$$q' = \frac{n+(g'+r')-2}{2} \tag{6.14}$$

and so

$$m-(g+r) = 2p, \tag{6.15}$$

$$n+(g'+r') = 2q'+2, \tag{6.16}$$

$$2p-2q'-2 = ((m_i-n_j)). \tag{6.17}$$

(6.17) and (6.2.1.d) imply (6.2.3.h).

(6.15) and (6.2.1.c) imply (6.2.3.g).

Also (6.2.3.f) is obvious from (4.2) and (6.2.1.b).

Now from (6.6), (6.11) and (6.12), we have

$$\begin{aligned} q-p-h+2 &= \frac{m+(g+r)-2}{2} - \frac{m-(g+r)}{2} - (r+1) + 2 \\ &= g. \end{aligned} \tag{6.18}$$

Since green part of the subscript, $g \geq 1$, therefore (6.18) implies (6.2.3.e). This completes the proof of Theorem 6.2.3. □

Example 6.2.1. *To illustrate the bijection we have constructed, the example for $\nu = 7$ is shown in the table below:*

Split n–color partitions relevant to $A_1(7)$	*2–color F–partitions relevant to $cF_2^1(7)$*
7_7	$\begin{pmatrix} 0_1 \\ 6_1 \end{pmatrix}$
7_{6+1}	$\begin{pmatrix} 0_2 \\ 6_2 \end{pmatrix}$
$6_4 1_1$	$\begin{pmatrix} 1_1 & 0_1 \\ 4_1 & 0_1 \end{pmatrix}$
$6_{3+1} 1_1$	$\begin{pmatrix} 1_2 & 0_1 \\ 4_2 & 0_1 \end{pmatrix}$
$5_1 2_2$	$\begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 1_1 \end{pmatrix}$
$5_1 2_{1+1}$	$\begin{pmatrix} 2_1 & 0_2 \\ 2_1 & 1_2 \end{pmatrix}$

Hence,

$$A_1(\nu) = cF_2^1(\nu) = 6. \tag{6.19}$$

Sketch of proof of Theorem 6.2.4. In this theorem, the only difference is that in (6.2.2.c), least part is of the type k_k, therefore in the corresponding last column of F–partition, $a_s = 0_1$ and vice versa. □

Example 6.2.2. *To illustrate the bijection we have constructed, the example for $\nu = 7$ is shown in the table below;*

Split n–color partitions relevant to $A_2(7)$	*2–color F–partitions relevant to $cF_2^2(7)$*
7_7	$\begin{pmatrix} 0_1 \\ 6_1 \end{pmatrix}$
$6_4 1_1$	$\begin{pmatrix} 1_1 & 0_1 \\ 4_1 & 0_1 \end{pmatrix}$
$6_{3+1} 1_1$	$\begin{pmatrix} 1_2 & 0_1 \\ 4_2 & 0_1 \end{pmatrix}$
$5_1 2_2$	$\begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 1_1 \end{pmatrix}$

Hence,

$$A_2(\nu) = cF_2^2(\nu) = 4. \tag{6.20}$$

6.3 Bijections between split *(n+t)*–color partitions and 2–color *F*–partitions

In Chapter 5, we interpreted following four generalized q–series using split $(n+t)$–color partitions. Let $S = \{-1, 1, 3, 5, 7, \ldots\}$. For $|q| < 1, j \in S$ and $1 \le i \le 4$, define $f^{(i,j)}(q)$ by

$$f^{(1,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q;q^2)_n (q^4;q^4)_n}, \tag{6.21}$$

$$f^{(2,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(j+3)/2]}}{(q;q^2)_{n+1} (q^4;q^4)_n}, \tag{6.22}$$

$$f^{(3,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q;q^2)_{n+1} (q^4;q^4)_n}, \tag{6.23}$$

$$f^{(4,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q;q^2)_n (q^4;q^4)_n}. \tag{6.24}$$

We here extend our results of Chapter 5 and interpret (6.21)–(6.24) by establishing bijections between split $(n+t)$–color partitions and 2–color F–partitions in the following theorems, respectively.

Theorem 6.3.1. *For $j \in S$, let $P_1^j(\nu)$ represent the number of split n–color partitions of ν such that*

(6.3.1.a) *parts and their subscripts have the same parity,*

(6.3.1.b) *value of red part can be 0 or 1,*

(6.3.1.c) *if m_i is the least or only summand of partition, then $m - i \equiv 0(mod\ 4)$,*

(6.3.1.d) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1)(mod\ 4)$.*

Further, let $cF_2^{(1,j)}(\nu)$ denote the number of 2–color F–partitions of ν in which top and bottom row entries of each column appear with same subscripts such that

(6.3.1.e) *for each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$, ignoring the subscripts,*

(6.3.1.f) *$h = 1$ or 2,*

(6.3.1.g) *for last column $p \equiv 0(mod 2)$, ignoring the subscript,*

(6.3.1.h) *for any two adjacent columns, $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, we have $p \geq q' + (j+3)/2$ and*

$$\begin{cases} p \equiv q'(mod 2), & j \equiv 1(mod 4); \\ p \equiv q' + 1(mod 2), & j \equiv 3(mod 4). \end{cases}$$

ignoring the subscripts. Then

$$\sum_{\nu=0}^{\infty} P_1^j(\nu) q^\nu = \sum_{\nu=o}^{\infty} cF_2^{(1,j)}(\nu) q^\nu.$$

Theorem 6.3.2. *For $j \in S$, let $P_2^j(\nu)$ represent the number of split $(n+1)$–color partitions of ν such that*

(6.3.2.a) *parts and their subscripts have the opposite parity,*

(6.3.2.b) *the value of red part can be 0 or 1,* (6.3.2.c) *the smallest summand is of the form* i_{i+1},

(6.3.2.d) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1)(mod\ 4)$.

Further, let $cF_2^{(2,j)}(\nu)$ *denote the number of* 2*–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(6.3.2.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 2$,

(6.3.2.f) $h = 1$ *or 2,*

(6.3.2.g) *for last column, either* $p_h = 0_1$ *or* $p \geq (j+3)/2$*, ignoring the subscript,*

(6.3.2.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p \geq q' + (j + 5)/2$ *and*

$$\begin{cases} p \equiv q'(mod 2), & j \equiv 3(mod 4); \\ p \equiv q' + 1(mod 2), & j \equiv 1(mod 4). \end{cases}$$

ignoring the subscripts. Then

$$\sum_{\nu=0}^{\infty} P_2^j(\nu)q^\nu = \sum_{\nu=0}^{\infty} cF_2^{(2,j)}(\nu)q^\nu.$$

Theorem 6.3.3. *For* $j \in S$*, let* $P_3^j(\nu)$ *represent the number of split* $(n+2)$*–color partitions of* ν *such that*

(6.3.3.a) *parts and their subscripts have the same parity,*

(6.3.3.b) *the value of red part can be 0 or 1,*

(6.3.3.c) *the smallest summand is of the form* i_{i+2},

(6.3.3.d) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1)(mod\ 4)$.

Further, let $cF_2^{(3,j)}(\nu)$ *denote the number of* 2*–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(6.3.3.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 3$,

(6.3.3.f) $h = 1$ *or 2,*

(6.3.3.g) *for last column, either* $p_h = 0_1$ *or* $p \geq (j+5)/2$, *ignoring the subscript,*

(6.3.3.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p \geq q' + (j + 7)/2$ *and*

$$\begin{cases} p \equiv q'(mod 2), & j \equiv 1(mod 4); \\ p \equiv q' + 1(mod 2), & j \equiv 3(mod 4). \end{cases}$$

ignoring the subscripts. Then

$$\sum_{\nu=0}^{\infty} P_3^j(\nu) q^\nu = \sum_{\nu=0}^{\infty} cF_2^{(3,j)}(\nu) q^\nu.$$

Theorem 6.3.4. *For* $j \in S$ *and* $\nu \geq 0$, *let* $P_4^j(\nu)$ *represent the number of split* n*–color partitions of* ν *such that*

(6.3.4.a) *parts and their subscripts have the same parity,*

(6.3.4.b) *the value of red part can be 0 or 1,*

(6.3.4.c) *if* m_i *is the least or only summand of partition, then* $m \geq (j+4)$ *and* $m - i \equiv (j+3)(mod\ 4)$,

(6.3.4.d) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1)(mod\ 4)$.

Further, let $cF_2^{(4,j)}(\nu)$ *denote the number of* 2*–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(6.3.4.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$,

(6.3.4.f) $h = 1$ *or 2,*

(6.3.4.g) *for last column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p+q-3 \geq j$ *and* $p \equiv \frac{j+3}{2}(mod 2)$, *ignoring the subscripts,*

(6.3.4.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p \geq q' + (j+3)/2$ *and*

$$\begin{cases} p \equiv q'(mod 2), & j \equiv 1(mod 4); \\ p \equiv q'+1(mod 2), & j \equiv 3(mod 4). \end{cases}$$

ignoring the subscripts. Then

$$\sum_{\nu=0}^{\infty} P_4^j(\nu)q^\nu = \sum_{\nu=0}^{\infty} cF_2^{(4,j)}(\nu).$$

Remark. The partition function $P_i^j(\nu)$, $1 \leq i \leq 4$ of Theorems 6.3.1–6.3.4, are same as defined in Theorems 5.2.1–5.2.4 given in Chapter 5.

Agarwal and Sood [18] explored the split $(n+t)$–color partitions to interpret following three basic q–series combinatorially.

Let $S = \{-1, 1, 3, 5, 7, \ldots\}$. For $|q| < 1, j \in S$ and $5 \leq i \leq 7$, define $f^{(i,j)}(q)$ by

$$f^{(5,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q;q)_{2n}}, \tag{6.25}$$

$$f^{(6,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(j+3)/2]}}{(q;q)_{2n+1}}, \tag{6.26}$$

$$f^{(7,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q;q)_{2n+1}}. \tag{6.27}$$

We extended the results of Agarwal and Sood [18] and used the 2–color F–partitions to interpret (6.25)–(6.27) in the following theorems, respectively.

Theorem 6.3.5. *For* $j \in S$ *and* $\nu \geq 0$, *let* $P_5^j(\nu)$ *denote the number of 'split n–color partitions' of* ν *such that*

(6.3.5.a) *the parts and their subscripts have the same parity,*

(6.3.5.b) *the red part of the subscripts cannot exceed 1,* (6.3.5.c) *the weighted difference of any two consecutive parts is greater than* j *and even.*

Further, let $cF_2^{(5,j)}(\nu)$ *denote the number of* 2*–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(6.3.5.d) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$,

(6.3.5.e) $h = 1$ *or 2,*

(6.3.5.f) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p \geq q' + (j + 3)/2$, *ignoring the subscripts. Then*

$$\sum_{\nu=0}^{\infty} P_5^j(\nu)q^\nu = \sum_{\nu=0}^{\infty} cF_2^{(5,j)}(\nu)q^\nu.$$

Remark. In Theorem 6.3.5, in conditions (6.3.5.a) and (6.3.5.c) the whole subscript i is considered, not its parts g and r separately. Similarly, in the Theorems 6.3.6 and 6.3.7, given below, in conditions (6.3.k.a), (6.3.k.c) and (6.3.k.d), $k = 6,\ 7$, the whole subscript i is considered, not its parts g and r separately.

Theorem 6.3.6. *For* $j \in S$ *and* $\nu \geq 0$, *let* $P_6^j(\nu)$ *denote the number of 'split* $(n+1)$*–color partitions' of* ν *such that*

(6.3.6.a) *the parts and their subscripts have the opposite parity,*

(6.3.6.b) *the red part of the subscripts cannot exceed 1,*

(6.3.6.c) *the smallest part is of the form* i_{i+1} *for some* i *and red part of its subscript is 0,*

(6.3.6.d) *the weighted difference between any two consecutive parts is greater than* j *and even.*

Further, let $cF_2^{(6,j)}(\nu)$ *denote the number of* 2*–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(6.3.6.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 2$,

(6.3.6.f) $h = 1$ *or 2,*

(6.3.6.g) *for last column, either* $p_h = 0_1$ *or* $p \geq (j+3)/2$, *ignoring the subscript,*

(6.3.6.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p \geq q' + (j+5)/2$, *ignoring the subscripts. Then*

$$\sum_{\nu=0}^{\infty} P_6^j(\nu) q^\nu = \sum_{\nu=0}^{\infty} cF_2^{(6,j)}(\nu) q^\nu.$$

Theorem 6.3.7. *For* $j \in S$ *and* $\nu \geq 0$, *let* $P_7^j(\nu)$ *denote the number of 'split* $(n+2)$*–color partitions' of* ν *such that*

(6.3.7.a) *the parts and their subscripts have the same parity,*

(6.3.7.b) *the red part of the subscripts cannot exceed 1,*

(6.3.7.c) *the smallest part is of the form* i_{i+2} *for some* i *and red part of its subscript is 0,*

(6.3.7.d) *the weighted difference between any two consecutive parts is greater than* j *and even.*

Further, let $cF_2^{(7,j)}(\nu)$ *denote the number of* 2*–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(6.3.7.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 3$,

(6.3.7.f) $h = 1$ *or 2,*

(6.3.7.g) *for last column, either* $p_h = 0_1$ *or* $p \geq (j+5)/2$, *ignoring the subscript,*

(6.3.7.h) *for any two adjacent columns,* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p \geq q' + (j+7)/2$, *ignoring the subscripts. Then*

$$\sum_{\nu=0}^{\infty} P_7^j(\nu)q^\nu = \sum_{\nu=0}^{\infty} cF_2^{(7,j)}(\nu)q^\nu.$$

Note. The partition functions $P_i^j(\nu),\ 5 \leq i \leq 7$ of Theorems 6.3.5–6.3.7, respectively are due to Agarwal and Sood [18] and the partition functions $cF_2^{(i,j)}(\nu),\ 5 \leq i \leq 7$ are new.

Now we gave the proofs of Theorems 6.3.5–6.3.7.

Proof of Theorem 6.3.5. We establish a one to one correspondence between the 2–color F–partitions enumerated by $cF_2^{(5,j)}(\nu)$ and the split n–color partitions enumerated by $P_5^j(\nu)$. We do this by mapping each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ of the 2–color F–partition enumerated by $cF_2^{(5,j)}(\nu)$ to a single part m_{g+r} of a split n–color partition enumerated by $P_5^j(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+2)+(h-1)},\ \text{if } p \leq q-h+1. \tag{6.28}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_{g+r} \to \begin{pmatrix} (\frac{m-(g+r)}{2})_{r+1} \\ (\frac{m+(g+r)-2}{2})_{r+1} \end{pmatrix},\ \text{if } m \equiv (g+r)(mod2) \tag{6.29}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$ in a 2–color F–partition enumerated by $cF_2^{(5,j)}(\nu)$ with

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} = m_{g+r} \text{ and } \phi : \begin{pmatrix} p'_k \\ q'_k \end{pmatrix} = n_{g'+r'}.$$

Then since

$$\begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+2)+(h-1)} = m_{g+r}$$

and

$$\begin{pmatrix} {p'}_k \\ {q'}_k \end{pmatrix} \to (p'+q'+1)_{(q'-p'-k+2)+(k-1)} = n_{g'+r'},$$

we have

$$\begin{aligned} ((m_i - n_j)) =& m - n - i - j \\ =&(p+q+1) - (p'+q'+1) - (q-p-h+2+(h-1)) \\ & - (q'-p'-k+2+(k-1)) \\ =&2(p-q'-1). \end{aligned} \tag{6.30}$$

Hence, (6.30) and (6.3.5.f) imply (6.3.5.c).

Now using (6.28), we have

$$m-(g+r) = (p+q+1) - (q-p-h+2+(h-1)) = 2p$$

which imply $(m-(g+r)) \equiv 0(mod2)$, hence (6.3.5.a) holds.

Also (6.3.5.b) is obvious from (6.28) and (6.3.5.e).

To see the reverse implication, we consider the inverse images of two consecutive parts m_{g+r}, $n_{g'+r'}$ of a split n–color partition enumerated by $P_5^j(\nu)$

$$\phi^{-1} : m_{g+r} = \begin{pmatrix} (\frac{m-(g+r)}{2})_{r+1} \\ (\frac{m+(g+r)-2}{2})_{r+1} \end{pmatrix}$$

and

$$\phi^{-1} : n_{g'+r'} = \begin{pmatrix} (\frac{n-(g'+r')}{2})_{r'+1} \\ (\frac{n+(g'+r')-2}{2})_{r'+1} \end{pmatrix}$$

that is,

$$p = \frac{m-(g+r)}{2} \tag{6.31}$$

$$q = \frac{m+(g+r)-2}{2} \tag{6.32}$$

$$p' = \frac{n-(g'+r')}{2} \tag{6.33}$$

$$q' = \frac{n+(g'+r')-2}{2} \tag{6.34}$$

and so

$$m-(g+r) = 2p \tag{6.35}$$

$$n+(g'+r') = 2q'+2 \tag{6.36}$$

hence

$$((m_i - n_j)) = 2p - 2q' - 2. \tag{6.37}$$

(6.37) and (6.3.5.c) imply (6.3.5.f).

Also (6.3.5.e) is obvious from (6.29) and (6.3.5.b).

Now from (6.29), (6.31) and (6.32), we have

$$\begin{aligned} q-p-h+2 &= \frac{m+(g+r)-2}{2} - \frac{m-(g+r)}{2} - (r+1)+2 \\ &= g. \end{aligned} \tag{6.38}$$

Hence, (6.38) imply (6.3.5.d) using the fact that the green part, $g \geq 1$.

This completes the proof of Theorem 6.3.5. □

Sketch of proof of Theorem 6.3.6. The map ϕ is given by

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \rightarrow (p+q+1)_{(q-p-h+3)+(h-1)} \text{ if } p \leq q-h+2, \tag{6.39}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \rightarrow \begin{pmatrix} (\frac{m-(g+r)+1}{2})_{r+1} \\ (\frac{m+(g+r)-3}{2})_{r+1} \end{pmatrix}, \text{if } m \equiv (g+r+1)(mod 2). \tag{6.40}$$

Also the part 0_1 in split $(n+1)$–color partition corresponds to the phantom column

$\begin{pmatrix} 0_1 \\ -1_1 \end{pmatrix}$ in the restricted 2–color F–partition. Since the part 0_1 does not have any contribution towards the total value of ν therefore the corresponding column in the 2–color F–partitions is eliminated.

Observe here that if 0_1 appears as a part in the split $(n+1)$–color partition then after ignoring the corresponding phantom column of 2–color F–partition, the next column satisfies the condition $p \geq \frac{j+3}{2}$. □

Sketch of proof of Theorem 6.3.7. The map ϕ is given by

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+4)+(h-1)} \text{ if } p \leq q-h+3, \tag{6.41}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \to \begin{pmatrix} (\frac{m-(g+r)+2}{2})_{r+1} \\ (\frac{m+(g+r)-4}{2})_{r+1} \end{pmatrix}, \text{if } m \equiv (g+r)(mod 2). \tag{6.42}$$

As in case of Theorem 6.3.6 , here also the phantom column $\begin{pmatrix} 0_1 \\ -1_1 \end{pmatrix}$ appears in the colored F–partition corresponding to the part 0_2 in the split $(n+2)$–color partition. Since the part 0_2 does not have any contribution towards the total value of ν therefore the corresponding column in the colored F–partitions is eliminated.

Observe here that if 0_2 appears as a part in the split $(n+2)$–color partition then after ignoring the corresponding phantom column of 2-color F–partition, the next column satisfies the condition $p \geq \frac{j+5}{2}$. □

Since the proofs of Theorems 6.3.1–6.3.3 are similar to those of Theorems 6.3.5–6.3.7 respectively. So, proceeding on the same steps as above one can easily obtain the results.

For the proof of Theorem 6.3.4, proceeding same as the proof of Theorem 6.3.5, one can easily obtain the result.

In the next section, we use the method adopted in [14] and provide the direct proof of Theorems 6.3.1–6.3.7 by classifying the 2–color F–partitions into subclasses.

6.4 Direct proof of generalized q–series using 2–color F–partitions

We here first provide the direct proof of Theorem 6.3.5 in terms of 2–color F–partitions and then establish the proofs of the other theorems.

Let $cF_2^{(5,j)}(\nu)$ enumerate the 2–color F–partitions as described in Theorem 6.3.5. Our goal is to prove that

$$\sum_{\nu=0}^{\infty} cF_2^{(5,j)}(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q;q)_{2n}} \tag{6.43}$$

Proof of Theorem 6.3.5. Let $cF_2^{(5,j)}(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(5,j)}(\nu)$ into m columns. Split the partitions enumerated by $cF_2^{(5,j)}(m,\nu)$ into following four classes:

(i) that contain m^{th} column $= \begin{pmatrix} p_h \\ q_h \end{pmatrix}, p \neq 0$

(ii) that contain m^{th} column $= \begin{pmatrix} 0_1 \\ 0_1 \end{pmatrix}$

(iii) that contain m^{th} column $= \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}$

(iv) that contain m^{th} column $= \begin{pmatrix} 0_1 \\ q_1 \end{pmatrix}, q \geq 1$ or $\begin{pmatrix} 0_2 \\ q_2 \end{pmatrix}, q \geq 2$.

Transforming the partitions of class (i) by subtracting 1 from the top and bottom row entries of each column, we get partitions of $\nu - 2m$ into m parts without disturbing the restrictions on the columns as described in Theorem 6.3.5 Thus the transformed partitions are enumerated by $cF_2^{(5,j)}(m, \nu - 2m)$.

Further, transform the partitions of class (ii) by deleting the column $\begin{pmatrix} 0_1 \\ 0_1 \end{pmatrix}$ and then subtracting $\frac{j+3}{2}$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(5,j)}(m-1, \nu - m(j+3)+j+2)$ as this transformation do not effect the conditions of Theorem 6.3.5

Next, transform the partitions of class (iii) by deleting the column $\begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}$ and then subtracting $\frac{j+5}{2}$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(5,j)}(m-1, \nu-m(j+5)+j+3)$ as this transformation does not effect the conditions of Theorem 6.3.5

Finally, transform the partitions of class (iv) by subtracting 1 from top row entries of each column except the last column and 1 from bottom row entries of each column. We get the partitions enumerated by $cF_2^{(5,j)}(m, \nu-2m+1)$ having the m^{th} column as $\begin{pmatrix} 0_h \\ q_h \end{pmatrix}$. Thus the actual number of partitions in class (iv) are obtained by subtracting the number of those partitions which are enumerated by $cF_2^{(5,j)}(m, \nu-2m+1)$ with the last column as $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \neq 0$ from $cF_2^{(5,j)}(m, \nu-2m+1)$. Thus the transformed partitions are enumerated by $cF_2^{(5,j)}(m, \nu-2m+1) - cF_2^{(5,j)}(m, \nu-4m+1)$.

Hence, we get the following recurrence formula for $cF_2^{(5,j)}(m, \nu)$

$$\begin{aligned} cF_2^{(5,j)}(m,\nu) =& cF_2^{(5,j)}(m, \nu-2m) + cF_2^{(5,j)}(m-1, \nu-m(j+3)+j+2) \\ &+ cF_2^{(5,j)}(m-1, \nu-m(j+5)+j+3) \\ &+ cF_2^{(5,j)}(m, \nu-2m+1) - cF_2^{(5,j)}(m, \nu-4m+1). \end{aligned} \tag{6.44}$$

where $cF_2^{(5,j)}(0,0) = 1$ and $cF_2^{(5,j)}(m,\nu) = 0$ for $\nu < 0$.

let for $|q| < 1$ and $|z| < |q|^{-1}$, let $g^j(z,q)$ be defined by

$$g^j(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} cF_2^{(5,j)}(m,\nu) z^m q^\nu, \quad \text{for all } j \in S \tag{6.45}$$

Substitute $cF_2^{(5,j)}(m,\nu)$ from (6.44) in (6.45), we get q–functional equation

$$\begin{aligned} g^j(z,q) =& g^j(zq^2,q) + zqg^j(zq^{j+3},q) + zq^2 g^j(zq^{j+5},q) \\ &+ q^{-1}g^j(zq^2,q) - q^{-1}g^j(zq^4,q). \end{aligned} \tag{6.46}$$

Setting

$$g^j(z,q) = \sum_{n=0}^{\infty} \alpha(n,q) z^n \tag{6.47}$$

Using (6.46) in (6.47) and then examining the coefficients of z^n we get

$$\alpha(n,q) = \frac{q^{1+(j+3)(n-1)}(1+q^{2n-1})}{(1-q^{2n})(1-q^{2n-1})}\alpha(n-1,q). \tag{6.48}$$

Iterating (6.48) n times and note that $\alpha(0,q) = 1$, we find that

$$\alpha(n,q) = \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^2;q^2)_n(q;q^2)_n}. \tag{6.49}$$

Therefore,

$$g^j(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^2;q^2)_n(q;q^2)_n} z^n \tag{6.50}$$

$$= f^{(5,j)}(z,q) \tag{6.51}$$

and

$$\begin{aligned} \sum_{\nu=0}^{\infty} cF_2^{(5,j)}(\nu) q^\nu &= \sum_{\nu=0}^{\infty} \left(\sum_{m=0}^{\infty} cF_2^{(5,j)}(m,\nu) \right) q^\nu \\ &= f^{(5,j)}(1,q) \\ &= f^{(5,j)}(q). \end{aligned}$$

This proves Theorem 6.3.5. □

Proof of Theorem 6.3.6. Let $\rho^j(\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(6,j)}(\nu)$ with additional condition that the last column is of the type $\begin{pmatrix} 0_h \\ q_h \end{pmatrix}$ and let $\rho^j(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $\rho^j(\nu)$ into m columns. Further, let

$$h^j(q) = \sum_{\nu=0}^{\infty} \rho^j(\nu) q^\nu, \tag{6.52}$$

and

$$h^j(z,q) = \sum_{\nu,m=0}^{\infty} \rho^j(m,\nu) z^m q^\nu. \tag{6.53}$$

With the help of (6.44),we have

$$\rho^j(m,\nu) = cF_2^{(5,j)}(m-1,\nu-m(j+3)+j+2)+\tfrac{1}{2}\{cF_2^{(5,j)}(m-1,\nu-m(j+5)+j+3)+cF_2^{(5,j)}(m,\nu-2m+1)-cF_2^{(5,j)}(m,\nu-4m+1), \tag{6.54}$$

where $cF_2^{(5,j)}(0,0)=1$ and $cF_2^{(5,j)}(m,\nu)=0$ for $\nu<0$.
Transforming (6.54) into a q–functional equation, we get

$$h^j(z,q) = zqg^j(zq^{j+3},q) + \frac{1}{2}zq^2g^j(zq^{j+5},q) + \frac{1}{2}q^{-1}g^j(zq^2,q) - \frac{1}{2}q^{-1}g^j(zq^4,q). \tag{6.55}$$

Setting

$$h^j(z,q) = \sum_{n=0}^{\infty} \beta(n,q) z^n \tag{6.56}$$

and then examining the coefficients of z^n in the above expression (6.55) we get

$$2\beta(n,q) = 2q^{(j+3)(n-1)+1}\alpha(n-1,q)+q^{(j+5)(n-1)+2}\alpha(n-1,q)+q^{2n-1}\alpha(n,q)-q^{4n-1}\alpha(n,q). \tag{6.57}$$

Replacing $\alpha(n,q)$ from (6.49) in (6.57) and then analyzing, we get

$$\beta(n,q) = \frac{(-q;q^2)_{n-1}q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_{n-1}(q;q^2)_n}. \tag{6.58}$$

Thus

$$h^j(z,q) = \sum_{\nu=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)[1+(j+3)n/2]}}{(q^2;q^2)_n(q;q^2)_{n+1}} z^{n+1} = zqf^{(2,j)}(zq,q). \tag{6.59}$$

Define $\psi^j(m,\nu)$ by

$$f^{(2,j)}(z,q) = \sum_{m,\nu=0}^{\infty} \psi^j(m,\nu)z^m q^\nu.$$

By examining the coefficients of (6.59), we get

$$\rho^j(m+1,\nu+m+1) = \psi^j(m,\nu).$$

If bottom row entry of each column is subtracted by 1 which is enumerated by $\rho^j(m+1,\nu+m+1)$ ignoring the subscripts, we have the final partitions enumerated by $cF^{(2,j)}(m+1,\nu)$. Note here that as illustrated in the bijective proof of Theorem 6.3.6, we discard here the phantom column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Thus

$$\psi^j(m,\nu) = cF^{(2,j)}(m+1,\nu)$$

and so

$$\sum_{m,\nu=0}^{\infty} cF^{(2,j)}(m+1,\nu)z^m q^\nu = f^{(2,j)}(z,q). \tag{6.60}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} cF^{(2,j)}(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} cF^{(2,j)}(m,\nu)\right)q^\nu \\ &= \sum_{m,\nu=0}^{\infty} cF^{(2,j)}(m+1,\nu)q^\nu \\ &= f^{(2,j)}(1,q) \\ &= f^{(2,j)}(q).\end{aligned}$$

This proves Theorem 6.3.6. □

Proof of Theorem 6.3.7. Rewrite equation (6.59) as

$$h^j(z,q) = zqf^{(3,j)}(z,q). \tag{6.61}$$

Define $\eta^j(m,\nu)$ by

$$f^{(3,j)}(z,q) = \sum_{m,\nu=0}^{\infty} \eta^j(m,\nu) z^m q^\nu. \tag{6.62}$$

By examining the coefficients of (6.61), we get

$$\rho^j(m+1,\nu+1) = \eta^j(m,\nu). \tag{6.63}$$

If last column $\begin{pmatrix} 0_h \\ q_h \end{pmatrix}$ is replaced by $\begin{pmatrix} 0_h \\ q_h - 1 \end{pmatrix}$ which is enumerated by $\rho^j(m+1,\nu+1)$, we have the final partitions enumerated by $cF_2^{(3,j)}(m+1,\nu)$. Noting that as illustrated in the bijective proof of Theorem 6.3.7 we discard here the phantom column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Thus

$$\eta^j(m,\nu) = cF_2^{(3,j)}(m+1,\nu)$$

and introduce

$$\sum_{m,\nu=0}^{\infty} cF_2^{(3,j)}(m+1,\nu) z^m q^\nu = f^{(3,j)}(z,q). \tag{6.64}$$

Now

$$\begin{aligned}
\sum_{\nu=0}^{\infty} cF_2^{(3,j)}(\nu) q^\nu &= \sum_{\nu=0}^{\infty} \left(\sum_{m=1}^{\infty} cF_2^{(3,j)}(m,\nu) \right) q^\nu \\
&= \sum_{m,\nu=0}^{\infty} cF_2^{(3,j)}(m+1,\nu) q^\nu \\
&= f^{(3,j)}(1,q) \\
&= f^{(3,j)}(q).
\end{aligned}$$

This proves Theorem 6.3.7. □

Since the direct proofs of the Theorems 6.3.1–6.3.3 are analogous to the proofs given above for the Theorems 6.3.5–6.3.7. So proceeding same as above, one can easily obtain the results.

Proof of Theorem 6.3.4. Let $cF_2^{(7,j)}(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(7,j)}(\nu)$ into m columns. Split the partitions enumerated by $cF_2^{(7,j)}(m,\nu)$ into following four classes:

(i) that contain m^{th} column $= \begin{pmatrix} p_h \\ q_h \end{pmatrix}, p \neq (j+3)/2$

(ii) that contain m^{th} column $= \begin{pmatrix} ((j+3)/2)_1 \\ ((j+3)/2)_1 \end{pmatrix}$

(iii) that contain m^{th} column $= \begin{pmatrix} ((j+3)/2)_2 \\ ((j+5)/2)_2 \end{pmatrix}$

(iv) that contain m^{th} column$= \begin{pmatrix} ((j+3)/2)_1 \\ q_1 \end{pmatrix}$, $q \geq (j+5)/2$

or $= \begin{pmatrix} ((j+3)/2)_2 \\ q_2 \end{pmatrix}$, $q \geq (j+7)/2$.

Transforming the partitions of class (i) by subtracting 2 from the top and bottom row entries of each column, we get partitions of $(\nu - 4m)$ into m parts without disturbing the restrictions on the columns as described in Theorem 6.3.4 Thus the transformed partitions are enumerated by $cF_2^{(7,j)}(m,\nu - 4m)$.

Further, transform the partitions of class (ii) by deleting the column $\begin{pmatrix} ((j+3)/2)_1 \\ ((j+3)/2)_1 \end{pmatrix}$ and then subtracting $\frac{j+3}{2}$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(7,j)}(m-1, \nu - m(j+3) - 1)$ as this transformation do not effect the conditions of Theorem 6.3.4.

Next, transform the partitions of class (iii) by deleting the column $\begin{pmatrix} ((j+3)/2)_2 \\ ((j+5)/2)_2 \end{pmatrix}$ and then subtracting $\frac{j+5}{2}$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(7,j)}(m-1,\nu-m(j+5))$ as this transformation does not effect the conditions of Theorem 6.3.4.

Finally, transform the partitions of class (iv) by subtracting 1 from top row entries of each column except the last column and 1 from bottom row entries of each column. We get the partitions enumerated by $cF_2^{(7,j)}(m,\nu-2m+1)$ having the m^{th} column as $\begin{pmatrix} ((j+3)/2)_1 \\ q_1 \end{pmatrix}$, $q \geq (j+5)/2$ or $\begin{pmatrix} ((j+3)/2)_2 \\ q_2 \end{pmatrix}$, $q \geq (j+7)/2$. Thus the actual number of partitions in class (iv) are obtained by subtracting the number of those partitions which are enumerated by $cF_2^{(7,j)}(m,\nu-2m+1)$ with the last column as $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \neq (j+3)/2$ from $cF_2^{(7,j)}(m,\nu-2m+1)$. Thus the transformed partitions are enumerated by $cF_2^{(7,j)}(m,\nu-2m+1) - cF_2^{(7,j)}(m,\nu-6m+1)$.

Hence, we get the following recurrence formula for $cF_2^{(7,j)}(m,\nu)$

$$\begin{aligned} cF_2^{(7,j)}(m,\nu) =& cF_2^{(7,j)}(m,\nu-4m) + cF_2^{(7,j)}(m-1,\nu-m(j+3)-1) + \\ & cF_2^{(7,j)}(m-1,\nu-m(j+5)) + cF_2^{(7,j)}(m,\nu-2m+1) - \\ & cF_2^{(7,j)}(m,\nu-6m+1). \end{aligned} \tag{6.65}$$

where $cF_2^{(7,j)}(0,0)=1$ and $cF_2^{(7,j)}(m,\nu)=0$ for $\nu<0$.

let for $|q|<1$ and $|z|<|q|^{-1}$, $g_1^j(z,q)$ be defined by

$$g_1^j(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} cF_2^{(7,j)}(m,\nu) z^m q^\nu, \quad \text{for all } j \in S \tag{6.66}$$

Substitute $cF_2^{(7,j)}(m,\nu)$ from (6.65) in (6.66), we get q–functional equation

$$\begin{aligned} g_1^j(z,q) =& g_1^j(zq^4,q) + zq^{j+4} g_1^j(zq^{j+3},q) + zq^{j+5} g_1^j(zq^{j+5},q) \\ & + q^{-1} g_1^j(zq^2,q) - q^{-1} g_1^j(zq^4,q). \end{aligned} \tag{6.67}$$

Setting

$$g_1^j(z,q) = \sum_{n=0}^{\infty} \gamma(n,q) z^n \tag{6.68}$$

Using (6.67) in (6.68) and then examining the coefficients of z^n we get

$$\gamma(n,q) = \frac{q^{1+n(j+3)}(1+q^{2n-1})}{(1-q^{4n})(1-q^{2n-1})}\gamma(n-1,q). \tag{6.69}$$

Iterating (6.69) n times and note that $\gamma(0,q)=1$, we find that

$$\gamma(n,q) = \frac{(-q;q^2)_n q^{n(1+(j+3)(n+1)/2)}}{(q^4;q^4)_n(q;q^2)_n}. \tag{6.70}$$

Therefore,

$$g_1^j(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(1+(j+3)(n+1)/2)}}{(q^4;q^4)_n(q;q^2)_n} z^n \tag{6.71}$$

$$= f^{(7,j)}(z,q) \tag{6.72}$$

and

$$\sum_{\nu=0}^{\infty} cF_2^{(7,j)}(\nu)q^\nu = \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} cF_2^{(7,j)}(m,\nu)\right)q^\nu$$
$$= f^{(7,j)}(1,q)$$
$$= f^{(7,j)}(q).$$

This proves Theorem 6.3.4. □

6.5 Particular cases

Particularly, Theorems 6.3.1–6.3.7 translate into following seven Theorems 6.5.1–6.5.7 and provide the combinatorial interpretations to following seven Rogers–Ramanujan type identities, respectively which are listed in Chu and Zhang's Compendium [38] and Slater's compendium [64].

$$\sum_{n=0}^{\infty} \frac{(-1)^n(q,q^2)_n q^{n^2}}{(q^4,q^4)_n(-q,q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^2,-q^3;q^5]_\infty, \qquad ([38,\ \mathrm{I}(25)])$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{2n(n+1)}}{(q^4,q^4)_n(q,q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^3,q^9;q^{12}]_\infty, \quad ([38, I(27)])$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_{n+1}} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^5,-q^5;q^5]_\infty, \quad ([38, I(29)], [64, I(21)])$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q,-q^4;q^5]_\infty, \quad ([38, I(113)])$$

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q,q)_{2n}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^4,q^6;q^6]_\infty, \quad ([38, I(99)])$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{n(n+1)}}{(q,q)_{2n+1}} = \frac{1}{(q;q)_\infty}[q^4,q^8,q^{12};q^{12}]_\infty, \quad ([38, I(104)], [64, I(51)])$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{n(n+2)}}{(q,q)_{2n+1}} = \frac{1}{(q;q)_\infty}[q^2,q^{10},q^{12};q^{12}]_\infty. \quad ([38, I(102)], [64, I(50)])$$

Theorem 6.5.1. *Let $B_1(\nu)$ represent the number of partitions on ν which contain distinct odd parts, even parts $\equiv \pm 2, \pm 4 (mod 12)$ and the parts which are $\equiv \pm 2 (mod 12)$, appear in two copies. Then*

$$B_1(\nu) = P_1^{-1}(\nu) = cF_2^{(1,-1)}(\nu), \text{ for all } \nu,$$

where $P_1^{-1}(\nu)$ and $cF_2^{(1,-1)}$ are defined in Theorem 6.3.1 for $j=-1$.

Example 6.5.1. *The relevant partitions corresponding to $B_1(7) = 13$ are:*

$$7, 5+2_1, 5+2_2, 4+3, 4+2_1+1, 4+2_2+1, 3+2_1+2_1,$$

$$3+2_2+2_2, 3+2_2+2_1, 2_2+2_2+2_2+1, 2_1+2_1+2_1+1,$$

$$2_2+2_2+2_1+1, 2_2+2_1+2_1+1.$$

To illustrate the bijection, example for $\nu = 7$ is shown in the table below:

Split n–color partitions relevant to $P_1^{-1}(7)$	*2–color F–partitions relevant to $cF_2^{(1,-1)}(7)$*
7_1	$\begin{pmatrix} 3_1 \\ 3_1 \end{pmatrix}$

7_3	$\begin{pmatrix} 2_1 \\ 4_1 \end{pmatrix}$
7_5	$\begin{pmatrix} 1_1 \\ 5_1 \end{pmatrix}$
7_7	$\begin{pmatrix} 0_1 \\ 6_1 \end{pmatrix}$
7_{2+1}	$\begin{pmatrix} 2_2 \\ 4_2 \end{pmatrix}$
7_{4+1}	$\begin{pmatrix} 1_2 \\ 5_2 \end{pmatrix}$
7_{6+1}	$\begin{pmatrix} 0_2 \\ 6_2 \end{pmatrix}$
$6_4 1_1$	$\begin{pmatrix} 1_1 & 0_1 \\ 4_1 & 0_1 \end{pmatrix}$
$6_{3+1} 1_1$	$\begin{pmatrix} 1_2 & 0_1 \\ 4_2 & 0_1 \end{pmatrix}$
$6_2 1_1$	$\begin{pmatrix} 2_1 & 0_1 \\ 3_1 & 0_1 \end{pmatrix}$
$6_{1+1} 1_1$	$\begin{pmatrix} 2_2 & 0_1 \\ 3_2 & 0_1 \end{pmatrix}$
$5_1 2_2$	$\begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 1_1 \end{pmatrix}$
$5_1 2_{1+1}$	$\begin{pmatrix} 2_1 & 0_2 \\ 2_1 & 1_2 \end{pmatrix}$

Theorem 6.5.2. *Let $B_2(\nu)$ represent the number of partitions of ν into parts $\equiv \pm 1, \pm 2, \pm 3, \pm 5, 6 (mod 12)$. Then*

$$B_2(\nu) = P_2^{-1}(\nu) = cF_2^{(2,-1)}(\nu), \text{ for all } \nu,$$

where $P_2^{-1}(\nu)$ *and* $cF_2^{(2,-1)}$ *are defined in Theorem 6.3.2 for* $j = -1$.

Theorem 6.5.3. *Let* $B_3(\nu)$ *represent the number of partitions of* ν *into parts* $\equiv \pm1, \pm3, \pm4, \pm5, 6(mod12)$. *Then*

$$B_3(\nu) = P_3^{-1}(\nu) = cF_2^{(3,-1)}(\nu), \text{ for all } \nu,$$

where $P_3^{-1}(\nu)$ *and* $cF_2^{(3,-1)}$ *are defined in Theorem 6.3.3 for* $j = -1$.

Theorem 6.5.4. *Let* $C_4(\nu)$ *represent the number of* n*–color partitions of* ν *which contain the distinct parts such that first two copies of parts* $\equiv 5(mod10)$ *are allowed and only first copy of parts* $\equiv \pm1(mod10)$ *is allowed and let* $D_4(\nu)$ *represent the number of* n*–color partitions of* ν *which contain the first two copies of parts* $\equiv \pm2(mod10)$. *Further let*

$$B_4(\nu) = \sum_{l=0}^{\nu} C_4(l) D_4(\nu - l),$$

then

$$B_4(\nu) = P_4^{-1}(\nu) = cF^{(4,-1)}(\nu), \text{ for all } \nu,$$

where $P_4^{-1}(\nu)$ *and* $cF_2^{(4,-1)}$ *are defined in Theorem 6.3.4 for* $j = -1$.

Theorem 6.5.5. *Let* $C_5(\nu)$ *represent the number of partitions of* ν *which contain distinct parts* $\equiv \pm1, \pm5(mod12)$ *and let* $D_5(\nu)$ *represent the partitions of* ν *which contain parts* $\equiv \pm2, \pm4(mod12)$. *Further, let*

$$B_5(\nu) = \sum_{l=0}^{\nu} C_5(l) D_5(\nu - l),$$

then

$$B_5(\nu) = P_5^{1}(\nu) = cF^{(5,1)}(\nu), \text{ for all } \nu,$$

where $P_5^{1}(\nu)$ *and* $cF_2^{(5,1)}$ *are defined in Theorem 6.3.5 for* $j = 1$.

Theorem 6.5.6. *Let* $C_6(\nu)$ *represent the number partitions of* ν *which contain the parts* $\equiv \pm1, \pm3(mod10)$ *and let* $D_6(\nu)$ *represent the number of partitions of* ν *which* *contain the parts* $\equiv \pm 2, \pm 4 (mod 10)$. *Further, let*

$$B_6(\nu) = \sum_{l=0}^{\nu} C_6(l) D_6(\nu - l),$$

then

$$B_6(\nu) = P_6^{-1}(\nu) = cF^{(6,-1)}(\nu), \text{ for all } \nu,$$

where $P_6^{-1}(\nu)$ *and* $cF_2^{(6,-1)}$ *are defined in Theorem 6.3.6 for* $j = -1$.

Theorem 6.5.7. *Let* $C_7(\nu)$ *represent the number of* n*–color partitions of* ν *which contain the first two copies of distinct parts* $\equiv 5(mod 10)$ *and and only first copy of parts* $\equiv \pm 3(mod 10)$ *and let* $D_7(\nu)$ *represent the number of* n*–color partitions of* ν *which contain the first two copies of parts* $\equiv \pm 4(mod 10)$. *Further, let*

$$B_7(\nu) = \sum_{l=0}^{\nu} C_7(l) D_7(\nu - l),$$

then

$$B_7(\nu) = P_7^{-1}(\nu) = cF^{(7,-1)}(\nu), \text{ for all } \nu,$$

where $P_7^{-1}(\nu)$ *and* $cF_2^{(7,-1)}(\nu)$ *are defined in Theorem 6.3.7 for* $j = -1$.

6.6 Conclusion

In this chapter, we provide the infinite two–way combinatorial identities of the generalized series (6.21)–(6.27) using 2–color F–partitions. Also in some particular cases we obtain the three–way combinatorial interpretations of some Rogers–Ramanujan type identities.

$$P_i^{-1}(\nu) = cF_2^{(i,-1)}(\nu) = B_i(\nu), \qquad \text{for } 1 \leq i \leq 7,\ i \neq 2 \tag{6.73}$$

$$\text{and} \quad P_2^{1}(\nu) = cF_2^{(2,1)}(\nu) = B_2(\nu), \tag{6.74}$$

where $P_i^{j}(\nu)$ and $cF_2^{(i,j)}(\nu)$, $1 \leq i \leq 7$ are defined in Theorems 6.3.1–6.3.7 and $B_i(\nu)$, $1 \leq i \leq 7$ are defined in Theorems 6.5.1–6.5.7, respectively. In total we get twenty–

one new combinatorial identities in the usual sense. Out of which five are obtained in Chapter 5, three are given by Agarwal and Sood [18] and also reproduced in Theorems 6.3.5–6.3.7 and, rest are given as below;

$$P_i^{-1}(\nu) = cF_2^{(i,-1)}(\nu), \qquad 1 \leq i \leq 7,\ i \neq 2 \tag{6.75}$$

$$P_2^{1}(\nu) = cF_2^{(2,1)}(\nu), \tag{6.76}$$

$$cF_2^{(i,-1)}(\nu) = B_i(\nu), \qquad 1 \leq i \leq 7,\ i \neq 2 \tag{6.77}$$

$$cF_2^{(2,1)}(\nu) = B_2(\nu). \tag{6.78}$$

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List of Research Papers

1. Sareen, J. K. and Rana, M. Four–way combinatorial interpretations of some Rogers–Ramanujan type identities(Accepted). *Ars Combinatoria*, 2014 (SCI, Impact Factor 0.259).

2. Rana, M. and Sareen, J. K. On combinatorial extensions of some mock theta functions using signed partitions. *Advances in Theoretical and Applied Mathematics*, 10(1):15-25, 2015.

3. Rana, M., Sareen, J. K. and Chawla, D. On generalized q–series and split $(n+t)$–color partitions(Accepted). *Utilitas Mathematica*, 2015 (SCI, Impact Factor 0.354).

4. Sareen, J. K. and Rana, M. Combinatorics of tenth order mock theta functions(Accepted). *Proceedings of the Indian Academy of Sciences–Mathematical Sciences*, 2106 (SCI, Impact Factor 0.240).

5. Rana, M. and Sareen, J. K. Split $(n+t)$–color partitions and 2–color F–partitions (Communicated).