COMMON FIXED POINT THEOREMS FOR COMPATIBLE AND WEAKLY COMPATIBLE MAPPINGS

A

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SWATI GUPTA

ROLL NO - 301003025

Under the supervision of

Dr. S.S. Bhatia

SMCA,

Thapar University,Patiala.

SCHOOL OF MATHEMATICS AND COMPUTER APPLICATIONS

THAPAR UNIVERSITY

PATIALA- 147001.

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CERTIFICATE

Certified that the dissertation entitled, "**COMMON FIXED POINT THEOREMS FOR COMPATIBLE AND WEAKLY COMPATIBLE MAPPINGS**", which is being submitted by **Miss Swati Gupta** (Roll No. 301003025), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. S.S. Bhatia during the period from January 2012 to june 2012.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

(Swati Gupta)

This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

Dr. S.S. Bhatia
Professor and Head,
SMCA, Thapar University, Patiala.
(Supervisor)

Countersigned by:

Dr. S.S. Bhatia
Professor and Head,
SMCA, Thapar University, Patiala.

Dr. S.K. Mohapatra
Dean of Academic Affairs,
Thapar University, Patiala

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S Gupta

Swati Gupta

(301003025)

ABSTRACT

The intent of this dissertation entitled, "**COMMON FIXED POINT THEOREMS FOR COMPATIBLE AND WEAKLY COMPATIBLE MAPPINGS**", embodies a brief account of investigations carried out by various authors on existence of fixed points of self-mappings in metric spaces under the supervision of **Dr. S.S. Bhatia**, Professor, School of Mathematics and Computer Applications, Thapar University, Patiala.

The aim of this work is to study and obtain some result on existence and uniqueness of common fixed points. Fixed point theory has wide ranging application in many areas of mathematics. For example, in finding the solution of the system of linear equations, in proving the existence of solutions of ordinary and partial differential equation, integral equations, analysis and many other disciplines.

The work presented in this dissertation has been divided into four chapters. The first chapter is introductory. In this chapter we present a brief account of basic definitions and results which will be used in the later chapters. In 2nd chapter, the Banach Contraction principle and some basic fixed point theorems have been studied. Also an application to system of integral equations is given in this chapter.

Whereas, in 3rd chapter, the concept of compatible mappings introduced by Jungck, G. [15] has been studied. Also, some of the properties of this "weakend commutativity" has been given. We study the utility of this concept in the context of metric fixed point theory by replacing the commutativity hypothesis by compatibility. The purpose of the 4th chapter is to study the common fixed point theorem proved by Chugh, R. and Kumar, S. [7], from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity.

Towards the end, references of various publications cited in the present dissertation have been reported.

LIST OF SYMBOLS AND NOTATIONS

- **R** Set of real numbers.
- **N** Set of natural numbers.
- **Z** Set of integers.
- **Q** Set of rational numbers.
- $\mathbf{R}^+$ [0, ∞).
- $\mathbf{R}^n$ Euclidean n-space.
- ∈ Belongs to.
- ⊂ Subset.
- ∀ For all.
- ⇒ Implies.
- ∃ There exists.
- [a, b] Closed interval.
- (a, b) Open interval.

CONTENTS

CHAPTER-1

INTRODUCTION & DEFINITIONS

Introduction

A fixed point of a function is a point that is mapped to itself by the function. Let X be any non-empty set. Given a function $f: X \to X$, a fixed point of f is a point $x \in X$ such that $f(x) = x$, that is, a point which remains invariant under the mapping f. A set of fixed points is sometimes called a fixed set. This is to say, c is a fixed point of the function f(x) if and only if $f(c) = c$. For example, if a function f defined on the real numbers by $f(x) = x^2 - 3x + 4$, then 2 is a fixed point of f. Not all functions have fixed points: for example if f is a function defined on the real numbers as $f(x) = x + 1$, then it has no fixed points. In graphical terms, a fixed point means the point $(x, f(x))$ is on the line $y = x$, or in other words the graph of f has a point in common with the line. The example $f(x) = x + 1$, is a case where the graph and the line are a pair of parallel lines.

The origin of metric contraction principles rest in the method of successive approximations for proving existence and uniqueness of solutions of differential equations. This method is associated with the names of celebrated nineteenth century mathematicians such as Cauchy, Liouville, Lipschitz, Peano and especially Picard. In fact precursors of the fixed point theoretic approach are explicit in the work of Picard. However, it is the Polish Mathematician Stefan Banach who is credited with placing the ideas underlying the method into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations.

Although the basic idea about the fixed point theory was known to others earlier, but the credit of making it useful and popular goes to Polish mathematician Stefan Banach. In 1922, he proved a common fixed point theorem, which ensured the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach [2] is known as Banach fixed point theorem or contraction mapping principle.

Fixed point theory has wide ranging applications in many areas of Mathematics. For example, in proving the existence of solutions of ordinary and partial differential equations, integral equations and many other disciplines. Moreover, fixed point theorems have applications in the theory of matrices, mathematical economics, game theory, optimal control theory, dynamical systems, functional analysis and many other areas.

Basic Definitions

We now present a brief account of basic definitions, which will be used in the subsequent chapters. These definitions have been taken from Rudin [29], Ansari [1], Kreyszig [22] and others.

Metric Space

A set X, whose elements we shall call points, is said to be a metric space if with any two points x and y of X there is associated a real number d(x, y), called the distance from x to y, such that

1) $d(x, y) > 0$ if $x \neq y$; $d(x, x) = 0$;
2) $d(x, y) = d(y, x)$; (Symmetry)
3) $d(x, y) \leq d(x, z) + d(z, y)$, for any $z \in X$. (Triangle inequality)

Any function with these three properties is called a distance function, or a metric.

Examples of Metric Space:-

1) Let X = **R.** For $x, y \in X$, define d: X × X → **R** by

$$d(x, y) = |x - y|$$

   Then (X, d) is a metric space. This is called the metric space **R** with the usual metric.

2) Let X be an arbitrary non-empty set. For $x, y \in X$, define d: X × X → **R** by

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

   Then (X, d) is a metric space. The metric d is called the discrete metric and the space (X, d) is called the discrete metric space.

Limit Point

A point x is a limit point of the set E if every neighborhood of x contains a point $y \neq x$ such that $y \in E$.

Example of Limit Point:-

Let (X, d) be a metric space where X = **R** and $d(x, y) = |x - y|$. Let A = **Q** ⊂ **R**. Then the set of limit points of A is **R.**

Closed Set

A set E is closed if every limit point of E is a point of E.

Example of Closed Set:-

Let (X, d) be a metric space where X = **R** and $d(x, y) = |x - y|$. If we take $A = [0, 1]$ then every limit point of A is contained in A**.** Thus, A is a closed set.

Theorem 1.1: Let (X, d) be a metric space and A be a subset of X. Then $x \in \overline{A}$ if and only if $d(x, A) = 0$, where $d(x, A) = \inf\{d(x, y) : y \in A\}$.

Proof: Let $x \in \overline{A}$. If $x \in A$, then obviously we have $d(x, A) = 0$. Assume $x \notin A$. Then x is a limit point of A. Thus for any $\varepsilon > 0$, there exists a $y \in S_\varepsilon(x) \cap A$, that is, $d(x, y) < \varepsilon$. Therefore $d(x, A) < \varepsilon$ for any $\varepsilon > 0$. Hence $d(x, A) = 0$.

Conversely, suppose that $d(x, A) = 0$. If $x \in A$, then $x \in \overline{A}$. Assume $x \notin A$. Then by the property of the infimum, for any $\varepsilon > 0$, there is a $y \in A$ such that $d(x, y) < \varepsilon$, that is $y \in S_\varepsilon(x) \cap A$. Since $x \notin A$, then $y \neq x$. Therefore, x is a limit point of A, and thus $x \in \overline{A}$.

Compact Set

A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

Examples of Compact Set:-

1) Closed and bounded set is compact in $\mathbf{R^n}$.
2) The discrete metric space (X, d), where X is a finite set, is compact.

Sequence

By a sequence, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots\ldots$ The value of f, that is the element x_n, are called the terms of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a sequence in A, or a sequence of elements of A.

Examples of Sequence:-

1) $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbf{N}$ is a sequence. Here $x_1 = \frac{1}{1}$, $x_2 = \frac{1}{2}$, …so on.

2) $\{x_n\} = (-1)^n, n \in \mathbf{N}$ is a sequence. Here $x_1 = -1, x_2 = 1, x_3 = -1, \dots$ so on.

Convergent Sequence

A sequence $\{x_n\}$ in a metric space X is said to converge if there is a point $x \in X$ with the following property:

For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(x_n, x) < \varepsilon$. (Here 'd' denotes the distance in X.)

In this case we also say that $\{x_n\}$ converges to x, or that x is the limit of $\{x_n\}$, and we write $x_n \to x$, or

$$\lim_{n\to\infty} x_n = x.$$

If $\{x_n\}$ does not converge, it is said to diverge.

Cauchy Sequence

A sequence $\{x_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\varepsilon > 0$, there is an integer N such that

$$d(x_{n,} x_m) < \varepsilon, \forall\, n, m \geq N.$$

Remark

Every convergent sequence is a Cauchy sequence but the converse is not true as shown by the following example.

Example:-

Let $X = (0,1]$ be a metric space with the usual metric 'd'. Let $\{x_n\}$ be a sequence in X, where $\{x_n\} = \frac{1}{n}$. Then, $\{x_n\}$ is a Cauchy sequence since for each $\varepsilon > 0$, we have

$d(x_m,\ x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon, \forall\, m, n > \frac{1}{\varepsilon}$.

On the other hand, $x_n \to 0 \notin X$. Therefore, $\{x_n\}$ is a Cauchy sequence but not a convergent sequence.

Complete Metric Space

A metric space in which every Cauchy sequence converges is said to be complete.

Examples of Complete Metric Space:-

1) Let X = **R** and 'd' be the usual metric on X. Then, (X, d) is a complete metric space.
2) The set of integers **Z** with the usual metric is a complete metric space. Let $\{x_n\}$ be a Cauchy sequence of integers, that is, each term of the sequence belongs to $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Then the sequence must be of the form $\{x_1, x_2, \dots, x_n,\ x,\ x,\ x, \dots\}$. Indeed, if we choose ε = 1/2, then, $x_n, x_m \in \mathbf{Z}$ and $|x_n - x_m| < \frac{1}{2}$ implies $x_n = x_m$. Hence the sequence $\{x_1,\ x_2, \dots, x_n,\ x,\ x,\ x, \dots\}$ converges to x.

Continuous Function

Suppose X and Y are metric spaces, $E \subset X, p \in E$, and f maps E into Y . Then f is said to be continuous at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y\big(f(x), f(p)\big) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$. If f is continuous at every point of E, then f is said to be continuous on E.

Example of Continuous Function:-

1) Let (X, d) be a metric space and $f: X \to X$ be a constant function. Then f is continuous.
2) Let (X, d) be a discrete metric space then every function $f: X \to Y$ is continuous on X.

Lower Semi Continuous Function

Let (X, d) is a metric space. A function $\varphi: X \to \mathbf{R}$ is called lower semi continuous at $x_0 \in X$ if

$$\liminf_{x \to x_0} \varphi(x) \geq \varphi(x_0).$$

Example of Lower Semi Continuous Function:-

Let $\varphi: \mathbf{R} \to \mathbf{R}$ be a function defined by

$$\varphi(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then φ is lower semi continuous at $x_0 = 0$.

Upper Semi Continuous Function

Let (X, d) is a metric space. A function $\varphi: X \to \mathbf{R}$ is called upper semi continuous at $x_0 \in X$ if

$$\lim_{x \to x_0} \sup \varphi(x) \leq \varphi(x_0).$$

Example of Upper Semi Continuous Function:-

Let $\varphi: \mathbf{R} \to \mathbf{R}$ be a function defined by

$$\varphi(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Then φ is upper semi continuous at $x_0 = 0$.

Fixed Point

Let X be a non-empty set and T be a self map on X defined as

$$T : X \to X$$

By a fixed point of T, we mean an element x ∈ X such that T(x) = x.

Examples of Fixed Point:-

1) A translation has no fixed point. Let X be a non empty set. We can define a translation T: X → X as T(x) = x + a where x ∈ X and 'a' is any constant. Clearly, it has no fixed point.
2) The mapping T: **R** → **R** defined as $T(x) = x^2$ has only two fixed points. Here 0 and 1 are only fixed points.
3) The mapping T: $\mathbf{R}^2 \to \mathbf{R}^2$ defined as T(x, y) = (0, y) has infinitely many fixed points.
4) The mapping T: **R** → **R** defined as T(x) = x/2 has unique fixed point. Clearly, 0 is the only fixed point.

Contraction Mapping

Let (X, d) be a metric space. A mapping T: X → X is called contraction of X if there exist a real number α with 0 ≤ α < 1 such that

$$d(Tx, Ty) \leq \alpha d(x, y) \ \forall x, y \in X.$$

Example of Contraction Mapping:-

Let X = [0, 1]. The mapping T: X → X given by $T(x) = \frac{1}{7}(x^3 + x^2 + 1) \ \forall x \in X$. Then T is a contraction mapping.

We have,

$$d(Tx, Ty) = \left|\frac{1}{7}(x^3 + x^2 + 1) - \frac{1}{7}(y^3 + y^2 + 1)\right|$$
$$= \frac{1}{7}|(x^3 - y^3) + (x^2 - y^2)|$$
$$= \frac{1}{7}|x - y||x^2 + xy + y^2 + x + y|$$
$$\leq \frac{1}{7}|x - y||5| \leq \frac{5}{7}d(x, y)$$

Thus $d(Tx, Ty) \leq \alpha d(x, y)$ where $\alpha = \frac{5}{7} < 1$.

Contractive Mapping

Let (X, d) be a metric space. A mapping T: X → X is called contractive if

$$d(Tx, Ty) < d(x, y) \ \ \forall\, x, y \ \in X \text{ and } x \neq y.$$

Commuting Mapping [12]

Two self mappings f and g on a metric space (X, d) are said to be commuting if

$$fog(x) = gof(x) \ \ \forall\, x \in X.$$

Example of Commuting Mapping:-

Let X = $[1, \infty)$ and $d(x, y) = |x - y|$. Let $f, g: X \to X$ by $f(x) = x$ and $g(x) = x^2 + \frac{1}{2}$ for $x \in X$. Then,

$$(fog)x = f\big(g(x)\big) = f\left(x^2 + \frac{1}{2}\right) = x^2 + \frac{1}{2}$$

and

$$(gof)x = g\big(f(x)\big) = g(x) = x^2 + \frac{1}{2}$$

Since (fog)x = (gof)x for all x ∈ X. Thus f and g are commuting mappings.

Weakly Commuting Mapping [30]

Two self mappings f and g on a metric space (X, d) are said to be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx) \ \forall\, x \in X.$$

Example of Weakly Commuting Mapping:-

Let X = [0, 1] with the usual metric d. Define f and g : X $\rightarrow$ X by $f(x) = \frac{x}{2-x}$ and

$g(x) = \frac{x}{2}$ $\forall$ x $\in$ X.

Then $d(fgx, gfx) = \left|\frac{x}{(4-2x)} - \frac{x}{(4-x)}\right| = \left|\frac{x^2}{(4-2x)(4-x)}\right|$

$$\leq \left|\frac{x^2}{(4-2x)}\right| = \left|\frac{x}{(2-x)} - \frac{x}{2}\right|$$

$$= d(fx, gx)$$

Thus, f and g are weakly commuting mappings.

Compatible Mapping [15]

Two self mappings f and g on a metric space (X, d) are said to be compatible if

$$\lim_{n \to \infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \quad \text{for some } t \in X.$$

Example of Compatible Mapping:-

Let X = **R** and d be the metric on X. Let f, g: (X, d) → (X, d) are self maps defined as

$$f(x) = x^3 \text{ and } g(x) = 2 - x$$

Here, $f\big(g(x_n)\big) = f(2 - x_n) = (2 - x_n)^3$ and $g(f(x_n)) = g({x_n}^3) = 2 - {x_n}^3$.

Therefore, $d(fx_n, gx_n) = |{x_n}^3 - 2 + x_n| = |(x_n - 1)({x_n}^2 + x_n + 2)| = |x_n - 1||{x_n}^2 + x_n + 2| \to 0$ if and only if $x_n \to 1$ and $d(fgx_n, gfx_n) = |(8 - {x_n}^3 - 12x_n + 6{x_n}^2) - (2 + {x_n}^3)| = |6{x_n}^2 - 12x_n + 6| = 6|x_n - 1|^2 = 0$ iff $x_n \to 1$. Thus f and g are compatible mappings.

Compatible Mapping of Type (A) [17]

Two self mappings f and g on a metric space (X, d) are said to be compatible of type (A) if

$$\lim_{n \to \infty} d(fgx_n, ggx_n) = 0 \text{ and } \lim_{n \to \infty} d(gfx_n, ffx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \quad \text{for some } t \in X.$$

Example of Compatible Mapping of Type (A):-

Let X = **R**, with the usual metric d. Define f, g: **R** → **R** by

$$f(x) = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 6, & \text{if } 2 < x \leq 5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & \text{if } x = 2 \\ 6, & \text{if } 2 < x < 5 \\ x - 3, & \text{if } x \geq 5 \end{cases}$$

Then, for the sequence $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$, we see that $fx_n = 2$, $gx_n \to 2$, as $n \to \infty$. Also $ffx_n = 2$, $gfx_n = 2$ and $fgx_n = ggx_n = 6$.

Therefore,

$$\lim_{n \to \infty} d(fgx_n, ggx_n) = \lim_{n \to \infty} |6 - 6| = 0,$$

and

$$\lim_{n \to \infty} d(gfx_n, ffx_n) = \lim_{n \to \infty} |2 - 2| = 0,$$

Thus, f and g are Compatible mappings of type (A).

Compatible Mapping of Type (B) [27]

Two self mappings f and g on a metric space (X, d) are said to be compatible of type (B) if

$$\lim_{n \to \infty} d(fgx_n, ggx_n) \leq \frac{1}{2} \left[\lim_{n \to \infty} d(fgx_n, ft) + \lim_{n \to \infty} d(ft, ffx_n)\right],$$

and

$$\lim_{n \to \infty} d(gfx_n, ffx_n) \leq \frac{1}{2} \left[\lim_{n \to \infty} d(gfx_n, gt) + \lim_{n \to \infty} d(gt, ggx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \quad \text{for some } t \in X.$$

Example of Compatible Mapping of Type (B):-

Let X= [0, 6] with usual metric d. Define f, g: X → X by

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 3) \\ 6, & \text{if } x \in [3, 6] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 6 - x, & \text{if } x \in [0, 3) \\ 6, & \text{if } x \in [3, 6] \end{cases}$$

Then f and g are not continuous at x = 3. Here f and g are not compatible but they are compatible of type (A) hence compatible of type (B).

To see this, suppose that $\{x_n\} \subseteq [0, 6]$ and that fx_n, $gx_n \to t$. By definition of f and g, $t \in [3, 6]$.

Since f and g agree on [3, 6], we have only to consider t = 3. So we can suppose that $x_n \to 3$ and $x_n < 3$ for all n. Then $gx_n = 6 - x_n \to 3$ from the right and $fx_n = x_n \to 3$ from the left. Thus, since $x_n < 3$ and $6 - x_n > 3$ for all n,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \lim_{n\to\infty} |6 - (6 - x_n)| \to 3,$$

Further we have,

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = |6 - 6| \to 0,$$

$$\frac{1}{2}[\lim_{n\to\infty} d(fgx_n, f3) + \lim_{n\to\infty} d(f3, ffx_n)] \to \frac{3}{2} \text{ and}$$

$$\lim_{n\to\infty} d(gfx_n, ffx_n) = \lim_{n\to\infty} |(6 - x_n) - x_n| \to 0,$$

$$\frac{1}{2}[\lim_{n\to\infty} d(gfx_n, g3) + \lim_{n\to\infty} d(g3, ggx_n)] \to \frac{3}{2} \text{ as } x_n \to 3$$

Therefore f and g are both compatible mappings of type (A) and compatible mapping of type (B) but they are not compatible.

Compatible Mapping of Type (P) [26]

Two self mappings f and g on a metric space (X, d) are said to be compatible of type (P) if

$$\lim_{n\to\infty} d(ffx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X.$$

Example of Compatible Mapping of Type (P):-

Let X = [0, 1] with the usual metric d(x, y) = |x − y|. Define self maps f and g as

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}) \\ \frac{2}{3}, & \text{if } x = \frac{1}{2} \\ 1 - x, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ \frac{2}{3}, & \text{if } x = \frac{1}{2} \\ x, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Consider a sequence $\{x_n\}$ in X such that $x_n \to \frac{1}{2}$, $x_n > \frac{1}{2}$ $\forall$ n. Then we have

$$fx_n = (1 - x_n) \to \frac{1}{2} = t \text{ and } gx_n = x_n \to \frac{1}{2} = t,$$

Since $1 - x_n \to \frac{1}{2}, \forall$ n. We have $ffx_n = f(1 - x_n) = \frac{1}{2}$ and $ggx_n = gx_n \to \frac{1}{2}$. Also we have f(t) = 2/3 = g(t). Therefore f and g are compatible of type (P).

Weakly Compatible Mapping [18]

Two self mappings f and g on a metric space (X, d) are said to be weakly compatible if they commute at coincidence points.

Example of Weakly Compatible Mapping:-

Let X = [0, 3] with usual metric d(x, y) = |x – y| and define f, g: [0, 3] → [0, 3] as

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1) \\ 3, & \text{if } x \in [1, 3] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3 - x, & \text{if } x \in [0,1) \\ 3, & \text{if } x \in [1,3] \end{cases}$$

Then for any $x \in [1, 3]$, x is a coincident point and fg(x) = gf(x) showing that f, g are weakly compatible maps on [0, 3].

In 2nd chapter, the Banach Contraction principle and some basic fixed point theorems have been studied. Also an application to system of integral equations is given.

Whereas, in 3rd chapter, the concept of compatible mappings introduced by Jungck [15] has been studied. Also, some of the properties of this "weakend commutativity" has been given. We study the utility of this concept in the context of metric fixed point theory by replacing the commutativity hypothesis by compatibility. We study the (ε, δ) contractiblity for four functions introduced by Jungck [15]. Towards the end, we study the existence of unique common fixed point of four mappings.

The purpose of the 4th chapter, is to study the common fixed point theorem proved by Chugh and Kumar [7], from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity. These results of Chugh and Sanjay [7] generalize the results of Jungck [16], Fisher [10], Kang and Kim [19], Jachymski [11] and Rhoades [28].

CHAPTER-2

BANACH FIXED POINT THEOREM

Introduction

First fixed point theorem was proved by Brouwer [4], but the credit of making this concept useful and popular goes to polish mathematician Banach [2] who proved the famous contraction mapping theorem also called the Banach fixed point theorem in 1922. The Banach fixed point theorem concerns certain mappings of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point. The theorem also gives an iterative process by which we can obtain approximations to the fixed point. Banach fixed point theorem has many applications to linear algebraic equations, ordinary differential equations, integral equations and many other fields of mathematics.

In this chapter, we present Banach contraction theorem and some other fixed point theorems. At the end, we present an application of fixed point theorem for system of integral equations.

Fixed Point [1]

Let X be a non-empty set and T be a self map on X defined as

$$T : X \rightarrow X$$

By a fixed point of T, we mean an element $x \in X$ such that T(x) = x.

Examples of Fixed Point:-

1) A translation has no fixed point. Let X be a non empty set. We can define a translation T: $X \rightarrow X$ as T(x) = x + a where $x \in X = \mathbf{R}$. Clearly, it has no fixed point.
2) The mapping T: $\mathbf{R} \rightarrow \mathbf{R}$ defined as $T(x) = x^2$ has only two fixed points. Here 0 and 1 are only fixed points.
3) The mapping T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined as T(x, y) = (0, y) has infinitely many fixed points.
4) The mapping T: $\mathbf{R} \rightarrow \mathbf{R}$ defined as T(x) = x/2 has unique fixed point. Clearly, 0 is the only fixed point.

Contraction Mapping [1]

Let (X, d) be a metric space. A mapping T: X → X is called contraction of X if there exist a real number α with $0 \leq \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \ \forall\, x, y \in X.$$

Example of Contraction Mapping:-

Let X = [0, 1]. The mapping T: X → X given by $T(x) = \frac{1}{7}(x^3 + x^2 + 1)\ \forall\, x \in X$. Then T is a contraction mapping. We have

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{1}{7}(x^3 + x^2 + 1) - \frac{1}{7}(y^3 + y^2 + 1)\right| \\ &= \frac{1}{7}|(x^3 - y^3) + (x^2 - y^2)| \\ &= \frac{1}{7}|x - y||x^2 + xy + y^2 + x + y| \\ &\leq \frac{1}{7}|x - y||5| \\ &\leq \frac{5}{7}d(x, y) \end{aligned}$$

Thus $d(Tx, Ty) \leq \alpha d(x, y)$ where $\alpha = \frac{5}{7} < 1$.

Contractive Mapping [1]

Let (X, d) be a metric space. A mapping T: X → X is called contractive if

$$d(Tx, Ty) < d(x, y) \ \forall\, x, y \in X \text{ and } x \neq y.$$

Complete Metric Space [29]

A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Example of Complete Metric space:-

The set of integers **Z** with the usual metric is a complete metric space. Let $\{x_n\}$ be a Cauchy sequence of integers, that is, each term of the sequence belongs to $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Then the sequence must be of the form $\{x_1, x_2, \ldots, x_n, x, x, x, \ldots\}$. Indeed, if we choose ε = 1/2, then, $x_n, x_m \in \mathbf{Z}$ and $|x_n - x_m| < \frac{1}{2}$ implies $x_n = x_m$. Hence the sequence $\{x_1, x_2, \ldots, x_n, x, x, x, \ldots\}$ converges to x.

Main Theorem

Theorem 2.1(Banach Contraction Theorem) [1]: Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then, T has a unique fixed point in X.

Proof: Since T is a contraction on x so, there exists a number α with $0 \leq \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha\, d(x, y), \quad \forall\, x, y \in X.$$

We prove the theorem in various steps.

Step (i) We construct a sequence $\{x_n\} \subset X$ as follows:

Take any point $x_0 \in X$ and inductively construct the sequence $\{x_n\}$ of points in X as:

$$x_1 = Tx_0$$
$$x_2 = T x_1 = T^2 x_0$$
$$x_3 = T x_2 = T^3 x_0$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$x_n = Tx_{n-1} = T^n x_0$$

Clearly, $\{x_n\}$ is the sequence of images of x_0 under repeated application of T.

Step (ii) $\{x_n\}$ is a Cauchy sequence in X. Let $m < n$. Then

$$\begin{aligned}
d(x_m, x_n) &= d(T^m x_0, T^n x_0) \\
&\leq \alpha\, d(T^{m-1} x_0, T^{n-1} x_0) \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots \\
&\leq \alpha^m\, d(x_0, T^{n-m} x_0) \\
&\leq \alpha^m [d(x_0, Tx_0) + d(Tx_0, T^2 x_0) + \dots + d(T^{n-m-1} x_0, T^{n-m} x_0)] \\
&\qquad\qquad\qquad\qquad \text{(by triangle inequality)} \\
&\leq \alpha^m [d(x_0, Tx_0) + \alpha d(x_0, T x_0) + \dots + \alpha^{n-m-1} d(x_0, Tx_0)] \\
&= \alpha^m [1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}]\, d(x_0, T x_0) \\
&\leq \frac{\alpha^m}{1-\alpha}\, d(x_0, T x_0), \quad (0 \leq \alpha < 1) \\
&\to 0
\end{aligned}$$

as m (and hence n) $\to \infty$. Hence $\{x_n\}$ is a Cauchy sequence.

Step (iii) Since X is complete and $\{x_n\}$ is a Cauchy sequence in X, $\exists$ $x \in X$ such that $x_n \to x$.

Step (iv) x is a fixed point of T. We have

$$d(x,Tx) \le d(x,x_n) + d(x_n,Tx) \qquad (\text{by triangle inequality})$$
$$= d(x,x_n) + d(T(x_{n-1}),Tx) \quad (\because\ x_n = Tx_{n-1})$$
$$\le d(x,x_n) + \alpha\, d(x_{n-1},x)$$
$$\to 0, \text{ as } n \to \infty$$
$$\Rightarrow \quad d(x,Tx) = 0$$
$$x = Tx$$

Hence x is a fixed point of T. Thus the existence of a fixed point is established.

In the last step, we verify the uniqueness of such a fixed point.

Step (v) x is a unique fixed point of T. Let if possible, x and y be two fixed points of T in X. Then $Tx = x$ and $Ty = y$. Now, note that

$$d\,(x,y) = d\,(Tx,Ty) \le d(x,y)$$
$$\Rightarrow \quad d\,(x,y) = 0$$
$$\Rightarrow \quad x = y$$

This completes the proof of the theorem.

Remark: If X is not complete in the Banach Contraction Theorem, then T may not have a fixed point as shown by the following example.

Example: Take $X = (0,1)$ and a mapping $T: X \to X$ defined as $T(x) = \frac{x}{2}$. Here, T is a contraction but X is incomplete. Therefore, T has no fixed point. In fact $T(0) = 0 \notin X$.

Remark: If T is not a contraction in the Banach Contraction Theorem, then T may not have a fixed point as shown by the following example.

Example: Take $X = (0,\infty]$ and a mapping $T: X \to X$ defined as $T(x) = 1/1 + x^2$. Here, X is complete but T is not a contraction. Therefore, T has no fixed point.

Remark: The following example shows that if X is complete metric space and $T: X \to X$ is not a contraction mapping but $T^2 = T \circ T$ is contraction, even then T has a fixed point.

Example: Let $X = \mathbf{R}$ be a metric space with the usual metric and $T: X \to X$ be a mapping defined as

$$T(x) = \begin{cases} 1 & \text{if x is rational} \\ 0 & \text{if x is irrational.} \end{cases}$$

Then T is not continuous and hence not a contraction mapping. But

$$T^2(x) = T\big(T(x)\big) = \begin{cases} T(1) = 1 & \text{if x is rational} \\ T(0) = 1 & \text{if x is irrational.} \end{cases}$$

Then T^2 is a contraction mapping but both T^2 and T have the same fixed point that is 1.

Theorem 2.2 [1]: Let (X, d) be a complete metric space and $T: X \to X$ be a mapping such that for some integer m, $T^m = \underbrace{T \circ T \circ \ldots \circ T}_{\text{m times}}$ is a contraction mapping. Then T has a unique fixed point.

Proof: By the Banach Contraction Theorem, T^m has a unique fixed point $x \in X$, that is, $T^m(x) = x$. Then

$$T(x) = T\big(T^m(x)\big) = T^m(T(x))$$

So T(x) is a fixed point of T^m. Since the fixed point of T^m is unique, $T(x) = x$. To prove the uniqueness, we assume that y is another fixed point of T. Then $T(y) = y$ and so $T^m(y) = y$. Again by the uniqueness of the fixed point of T^m, we have x = y. Hence $x \in X$ is a unique fixed point of T.

The following theorem guarantees the existence of a fixed point for a contractive mapping:

Theorem 2.3 [1]: Let (X, d) be a compact metric space and $T: X \to X$ be a contractive mapping. Then T has a unique fixed point $\bar{x}$. Moreover, for any $x \in X$, $\lim_{n\to\infty} T^n(x) = \bar{x}$, that is, the successive iterates $T(x), T^2(x), \ldots, T^n(x), \ldots$ converges to the unique fixed point $\bar{x}$ of T.

Proof: Define a mapping $\psi: X \to \mathbf{R}^+$ by

$$\psi(x) = d\big(x, T(x)\big) \quad \forall \quad x \in X.$$

Then ψ is continuous, Indeed by contractiveness of T, we have

$$
\begin{aligned}
|\psi(x)-\psi(y)| &= \left|d\big(x, T(x)\big) - d(y, T(y))\right| \\
&\le \left|d\big(x, T(x)\big) - d(T(x), y)\right| + |d(T(x), y) - d(y, T(y))| \\
&\le d(x, y) + d(T(x), T(y)) \\
&< 2d(x, y).
\end{aligned}
$$

Let $\varepsilon > 0$ be given. Then

$$|\psi(x)-\psi(y)| < 2d(x, y) < \varepsilon \textit{ whenever } d(x, y) < \delta = \frac{\varepsilon}{2}.$$

Therefore, ψ is continuous. Clearly, ψ is bounded below. Since X is compact and ψ is continuous, there exists a minimize $\bar{x} \in X$ of ψ, that is, there exists $\bar{x} \in X$ such that $\psi(\bar{x}) \le \psi(y)$ for all $y \in X$. We show that $\bar{x}$ is a fixed point of T.

Suppose on the contrary that $\bar{x}$ is not a fixed point of T. Then $T(\bar{x}) \ne \bar{x}$. By contractiveness of T, we have

$$\psi\big(T(\bar{x})\big) = d(T(\bar{x}), T^2(\bar{x})) < d\big(\bar{x}, T(\bar{x})\big) = \psi(\bar{x})$$

which contracts that $\bar{x}$ is a minimizer of ψ. Hence $\bar{x}$ is a fixed point of T. The uniqueness follows on the lines of the proof of Banach contraction theorem.

Let $x \in X$. If $T^n(x) \ne \bar{x}$, then

$$d(T^{n+1}(x), \bar{x}) = d\big(T^{n+1}(x), T(\bar{x})\big) < d(T^n(x), \bar{x}) \quad \forall \quad n \in \mathbf{N}.$$

Therefore, $\{d(T^{n+1}(x), \bar{x})\}$ is a strictly decreasing sequence of non-negative real numbers and so converges to its infimum. Suppose that $\alpha = \lim_{n\to\infty} d(T^{n+1}(x), \bar{x})$. Since $\{T^n(x)\}$ is a sequence of points of a compact metric space, there exists a subsequence $\{T^{n_k}(x)\}$ which converges to some point, say, $y \in X$. Since $\{T^n(x)\}$ is decreasing,

$$d(y, \bar{x}) = \lim_{k\to\infty} d(T^{n_k}(x), \bar{x}) = \lim_{k\to\infty} d(T^{n_k+1}(x), \bar{x}) = \alpha.$$

If $\alpha \ne 0$, then $y \ne \bar{x}$ and so we have

$$
\begin{aligned}
\alpha = d(y, \bar{x}) &> d\big(T(y), T(\bar{x})\big) = d(T(y), \bar{x}) \\
&= \lim_{k\to\infty} d(T(T^{n_k}(x)), \bar{x}) = d(T^{n_k+1}(x), \bar{x}) \\
&= \alpha
\end{aligned}
$$

a contradiction. Thus $\alpha = 0$, and therefore, $\lim_{n\to\infty} T^n(x) = \bar{x}$.

Application of fixed point theorems for system of integral equations

The most interesting applications of fixed point theorems arise when the underlying metric space is a function space. Here we discuss the existence and uniqueness of the Volterra integral equation by using the Theorem 2.1.

Volterra Integral Equation

Let K be a continuous function on $[a, b] \times [a, b]$ and let φ be a continuous function on $[a, b]$. Consider the equation

$$f(x) = \varphi(x) + \lambda \int_a^x K(x, y) f(y) dy \quad \text{for all } x \in [a, b] \tag{2.1}$$

where λ is a parameter. It is called the Volterra equation.

Theorem 2.4 [1]: For each $\lambda \in \mathbf{R}$, the Volterra equation (2.1) has a unique solution f that is continuous on $[a, b]$.

Proof: Let $X = C[a, b]$ the set of all continuous functions defined on [a, b] with the uniform metric. Since K is continuous, there exists a constant $k > 0$ such that $|K(x, y)| \leq k$ for all $x, y \in [a, b]$. Define the transformation $T: f \to T(f)$ on X by

$$T\big(f(x)\big) = \varphi(x) + \lambda \int_a^x K(x, y) f(y) dy.$$

For all $f, g \in X$, we have

$$\big|T\big(f(x)\big) - T(g(x))\big| = \left|\lambda \int_a^x K(x, y) |f(y) - g(y)| dy\right|$$

$$\leq |\lambda| k (x - a) d(f, g) \quad \text{for all } x \in [a, b].$$

Since $T^2(f) - T^2(g) = T(T(f) - T\big(T(g)\big)$, we have

$$|T^2(f(x)) - T^2(g(x))| = \left|\lambda \int_a^x K(x, y)\, |T(f(y)) - T(g(y))| dy\right|$$

$$\leq |\lambda| \int_a^x |K(x, y)|\, |\lambda|\, k(y - a)\, d(f, g)\, dy$$

$$\leq |\lambda|^2 k^2 \int_a^x (y-a)\, dy\, d(f,g)$$

$$\leq \frac{|\lambda|^2 k^2 (x-a)^2}{2} d(f,g).$$

Continuing this iterative process, we obtain

$$|T^n(f(x)) - T^n(g(x))| \leq \frac{|\lambda|^n k^n (x-a)^n}{n!} d(f,g) \quad \text{for all } x \in [a,b].$$

Hence,

$$|T^n(f(x)) - T^n(g(x))| \leq \frac{[|\lambda| k(b-a)]^n}{n!} d(f,g).$$

Recalling that $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any $r \in$ **R,** we conclude that there exists n such that T^n is a contraction mapping. Taking n sufficiently large to have $\frac{[|\lambda| k(b-a)]^n}{n!} < 1$. By Theorem 2.2, there exists a unique solution $f \in X$ satisfying $T(f) = f$. Obviously, if $T(f) = f$, then f solves Volterra equation.

CHAPTER – 3

COMMON FIXED POINTS FOR COMPATIBLE MAPS

Introduction

In this chapter we will study the results proved by Jungck [15] in 1986. A new concept of compatible mappings was introduced by Jungck [15] which was the generalization of the commuting mappings concept. Also, Jungck [15] derived some of the properties of compatible mappings and obtained related results. The following generalization of Banach contraction Theorem was proved by Jungck [12].

Theorem 3.1: A continuous self map of a complete metric space (X, d) has a fixed point iff there exists $r \in (0, 1)$ and a mapping $g: X \to X$ which commutes with f (gf = fg) and satisfies: $g(X) \subset f(X)$ and $d\big(g(x), g(y)\big) \leq rd\big(f(x), f(y)\big)$ for x, y in X. In fact, f and g have a unique common fixed point.

The purpose of the paper by Jungck [12] was to depict commuting mapping as a tool for generalization. Various extensions, generalizations and applications of Theorem 3.1. related to commuting maps were studied by number of authors like Chang [6], Chang [5], Das and Naik [8], Fisher [9], Jungck [13], Jungck [14], Park [24].

Meir and Keeler [23] proved the following theorem concerning the contraction theorem :

Theorem 3.2: Let (X, d) be a complete metric space and f be a mapping of X into itself if for given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \ implies\ d(f(x), f(y)) < \varepsilon$$

holds then f has a unique fixed point p. Moreover, for any $x \in X$

$$\lim_{n \to \infty} f^n(x) = p.$$

Further Park and Bae [25] generalize this result of Meir and Keeler [23] in the form of following theorem:

Definition: Let f and g be self maps of a complete metric space X and g is an (ε, δ)-f-contraction iff for any $\varepsilon > 0$ there is a $\delta > 0$ such that

(a) $\varepsilon \leq d\big(f(x), f(y)\big) < \varepsilon + \delta$ implies $d\big(g(x), g(y)\big) < \varepsilon$, and

(b) $g(x) = g(y)$ when $f(x) = f(y)$.

Theorem 3.3: Let f and g be is a continuous self map of a complete metric space X and g is an (ε, δ)-f-contraction which commutes with f, then f and g have a unique common fixed point in X.

Sessa [30] generalized commuting maps by calling self maps f and g of a metric space (X, d) a weakly commuting pair iff $d(fgx, gfx) \leq d(fx, gx)$ for x in X. Of course, commuting pair is weakly commuting, but the converse is not true. Baskaran and Subrahmanyam [3] obtained useful fixed point results using this concept. However, since elementary functions as similar as $f(x) = x^3$ and $g(x) = 2x^3$ are not weakly commutative, it was desirable to introduce a less restrictive concept – a concept called as compatibility.

We will study the properties of compatible functions, and examples to illustrate the extent to which the commutativity requirement has been weakened. We will also study the utility of this concept in the context of metric fixed point theory by replacing the commutativity hypothesis in Theorem 3.3. by compatibility. Theorem 3.3. was further extended by Jungck [15] by defining (ε, δ)-contractibility for four functions.

Compatible Mappings

Two self mappings f and g on a metric space (X, d) are said to be compatible if

$$\lim_{n \to \infty} d(fgx_n, gfx_n) = 0,$$

whenever, $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \quad \text{for some } t \in X.$$

Any pair f and g of self maps of a set commute on the set of common fixed points $\{x \in X: f(x) = g(x) = x\}$. But, the above definition of compatible mappings require that f and g commute on a larger set $\{x \in X: f(x) = g(x)\}$.

Remark: Clearly, any weakly commuting pair is compatible but the converse is not true as shown by the following examples.

Example 1: Let X = **R** and d be the metric on X. Let f, g: (X, d) → (X, d) are self maps defined as $f(x) = x^3$ and $g(x) = 2x^3$.

Here

$$f(g(x)) = f(2x^3) = 8x^9 \text{ and } g(f(x)) = g(x^3) = 2x^9$$

Therefore,

$$d(fgx, gfx) = |8x^9 - 2x^9| = 6|x^9| \text{ and } d(fx, gx) = |x^3|$$

Clearly, f and g are not weakly commuting as $6|x^9| > |x^3|$. On the other hand, the functions f and g are compatible since $d(fx, gx) = |x^3| \to 0$ iff $d(fgx, gfx) = 6|x^9| \to 0$. Thus f and g are compatible but are not weakly commuting pair.

Example 2: Let X = **R** and d be the metric on X. Let f, g: (X, d) → (X, d) are self maps defined as $f(x) = x^3$ and $g(x) = 2 - x$.

Here, $f(g(x_n)) = f(2 - x_n) = (2 - x_n)^3$ and $g(f(x_n) = g({x_n}^3) = 2 - {x_n}^3$.

Therefore, $d(fx_n, gx_n) = |{x_n}^3 - 2 + x_n| = |(x_n - 1)({x_n}^2 + x_n + 2)| = |x_n - 1||{x_n}^2 + x_n + 2| \to 0$ iff $x_n \to 1$ and $d(fgx_n, gfx_n) = |(8 - {x_n}^3 - 12x_n + 6{x_n}^2) - (2 + {x_n}^3)| = |6{x_n}^2 - 12x_n + 6| = 6|x_n - 1|^2 = 0$ iff $x_n \to 1$. Thus f and g are compatible pair.

At x = 0, $d(fgx, gfx) = 6|-1|^2 = 6$ and $d(fx, gx) = |-1||2| = 2$. Clearly, f and g are not weakly commuting pair.

Example 3: Let X = **R** and 'd' be the metric on X. Let f, g: (X, d) → (X, d) are self maps defined as f(x) = cosh(x) and g(x) = sinh(x).

Here $|f(x) - g(x)| = e^{-x} \to 0$ iff $x \to \infty$. But $f(x),\ g(x) \to +\infty$ as $x \to +\infty$. Hence f and g do not converge to an element t of X. Thus, condition of compatibility is satisfied vacuously.

Jungck [15] provide the following criterion for identifying compatible mappings:

Proposition 3.4: Let (X, d) be a metric space and $f, g: (X, d) \rightarrow (X, d)$ be continuous self mappings, and let F = $\{x \in X : f(x) = g(x) = x\}$. Then f and g are compatible if any one of the following conditions is satisfied.

(a) If $f(x_n), g(x_n) \rightarrow t\ (\in X)$, then $t \in F$.

(b) $d(f(x_n), g(x_n)) \rightarrow 0$ implies $D(f(x_n), F) \rightarrow 0$.

(c) F compact and $d\big(f(x_n),\ g(x_n)\big) \rightarrow 0$ implies $D(x_n, F) \rightarrow 0$.

where $D(x, F) = \inf\{d(x, y): y \in F\}$.

Proof: Suppose that $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = t$ for some $t \in X$. (3.4.1)

Suppose (a) holds, i.e., $f(x_n),\ g(x_n) \rightarrow t$ then $t \in F$ so, f(t) = g(t) = t. The continuity of f and g on F implies that $fg(x_n) \rightarrow f(t) = t$ and $gf(x_n) \rightarrow g(t) = t$, so that $d\big(fg(x_n), gf(x_n)\big) \rightarrow 0$. Thus f and g are compatible mappings.

Suppose (b) holds, i.e., $d\big(f(x_n), g(x_n)\big) \rightarrow 0$ implies $D(f(x_n), F) \rightarrow 0$. Since f and g are continuous, we obtain that F is closed and by Theorem 1.1 we have

$$D(f(x_n), F) = \inf\{d(f(x_n), y): y \in F\} \rightarrow 0 \ \Rightarrow f(x_n) \in \overline{F} = F. \qquad (3.4.2)$$

From (3.4.1) and (3.4.2) we obtain that $t \in F$. So, f and g are compatible by the condition (a).

Suppose (c) holds, i.e., F is compact and $d\big(f(x_n),\ g(x_n)\big) \rightarrow 0$ implies $D(x_n, F) \rightarrow 0$. Since F is compact there is a subsequence $\{x_{k_n}\}$ of x_n which converges to some element c of F. Then, by the continuity of f, $f\big(x_{k_n}\big) \rightarrow f(c) = c$ as $c \in F$ and by the continuity of f. Consequently, (3.4.1) implies that c = t ∈ F. So, f and g are compatible by the condition of (a).

Example: Let $f(x) = x^4 + ax^2 (a \geq 1)$ and $g(x) = x^2$ with X = **R**. Then $|f(x_n) - gx_n| = {x_n}^2|{x_n}^2 + (a - 1)| \rightarrow 0$ iff $x_n \rightarrow 0 (\in F)$, so that f and g are compatible by proposition 3.4(c). But f and g are not weakly commutative for a = 2 and x = 1.

Corollary 3.5: Suppose that f and g are continuous self maps of **R** such that $f - g$ is strictly increasing. If f and g have a common fixed point, then f and g are compatible.

Example: Let X = **R** and d be the metric on X. Let f, g: (X, d) → (X, d) are self maps defined as $f(x) = x^3 + ax$ and $g(x) = mx$ where $a > m$, then f and g are compatible.

Clearly f, g are continuous self maps of X. Also f-g is increasing as $a > m$ and $f(0) = g(0) = 0$. Therefore, f and g are compatible mappings by Corollary 3.5.

The following result will be useful for proving our main result:

Proposition 3.6: Let (X, d) be a metric space and $f, g: (X, d) \to (X, d)$ are compatible mappings.

(1) If $f(t) = g(t)$, then $fg(t) = gf(t)$.

(2) Suppose that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

(a) If f is continuous at t, then $\lim_{n\to\infty} gf(x_n) = f(t)$.

(b) If f and g are continuous at t, then f(t) = g(t) and fg(t) = gf(t).

Proof: To prove (1) suppose that f(t) = g(t) and let $x_n = t$ for n in **N**. Then $f(x_n)$, $g(x_n) \to f(t)$, so that $d\big(fg(t), gf(t)\big) = d\big(fg(x_n), gf(x_n)\big) \to 0$ by compatibility. Thus,

$$d(fg(t), gf(t) = 0 \Rightarrow fg(t) = gf(t).$$

To prove 2(a) suppose $g(x_n) \to t$, then by the continuity of f, $f(g(x_n)) \to f(t)$. But if $f(x_n) \to t$ also, since $d\big(gf(x_n), f(t)\big) \le d\big(gf(x_n), fg(x_n)\big) + d(fg(x_n), f(t)$. Therefore, by the compatibility of f and g and using $f(g(x_n)) \to f(t)$ so that, $d(gf(x_n), f(t)) \to 0$. Hence $\lim_{n\to\infty} gf(x_n) = f(t)$.

To prove 2(b) using the continuity of f, by 2(a) we have, $g(f(x_n)) \to f(t)$ and by the continuity of g, $g(f(x_n)) \to g(t)$. Thus, by uniqueness of limit f(t) = g(t). Therefore, by part (1), gf(t) = fg(t).

Common Fixed Points for $(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$ Contractions

Definition 3.7: A pair of self maps A and B of a metric space (X, d) are (ε, δ) - S, T - contractions relative to maps S, T: $X \to X$ iff $A(X) \subset T(X), B(X) \subset S(X)$, and there is a function $\delta: \mathbf{R}^+ \to \mathbf{R}^+$ such that $\delta(\varepsilon) > \varepsilon \; for \; all \; \varepsilon \; and \; for \; x, \; y \; \in X$:

(1) $\varepsilon \le d(Sx, Ty) < \delta(\varepsilon)$ implies $d(Ax, By) < \varepsilon, \; and$

(2) Ax = By whenever Sx = Ty.

Clearly, if A and B are (ε, δ)-S, T- contractions, then $d(Ax, By) \leq d(Sx, Ty)$ for all $x, y \in X$ strict inequality holds when $S(x) \neq T(y)$. Consequently, even though A and B are (ε, δ) - S, T- contractions, then it is not necessary that the pair B, A also (ε, δ) - S, T - contractions.

Definition 3.8: Let A, B, S, T be self mappings of a set X such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. For $x_0 \in X$, any sequence $\{y_n\}$ defined by $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \in \mathbf{N}$ will be called an S, T- iteration of x_0 under A and B.

Lemma 3.9: Let S and T be self maps of a metric space (X, d) and let the pair A, B be (ε, δ)-S, T- contractions. If $x_0 \in X$ and $\{y_n\}$ is an S, T- iteration of x_0 under A and B, then

(a) For each $\varepsilon > 0, \varepsilon \leq d(y_p, y_q) < \delta(\varepsilon)$ implies $d(y_{p+1}, y_{q+1}) < \varepsilon$ when p and q are of opposite parity.

(b) $d(y_n, y_{n+1}) \to 0$, and

(c) $\{y_n\}$ is a Cauchy sequence.

Proof: (a) Let $\varepsilon > 0$. Since A and B are (ε, δ)-S, T- contractions,

$$\varepsilon \leq d(Sx, Ty) < \delta(\varepsilon) \text{ implies } d(Ax, By) < \varepsilon. \qquad (3.9.1)$$

Now suppose that $\varepsilon \leq d(y_p, y_q) < \delta(\varepsilon)$ and that p and q are of opposite parity, say $p = 2n$ and $q = 2m - 1$. Then

$$d(y_{p+1}, y_{q+1}) = d(y_{2n+1}, y_{2m}) = d(Ax_{2n}, Bx_{2m-1}). \qquad (3.9.2)$$

and
$$d(y_p, y_q) = d(y_{2n}, y_{2m-1}) = d(Sx_{2n}, Tx_{2m-1}). \qquad (3.9.3)$$

By the above assumption and (3.9.3) we thus have $\varepsilon \leq d(Sx_{2n}, Tx_{2m-1}) < \delta(\varepsilon)$, so that (3.9.2) and (3.9.1) yields $d(Ax_{2n}, Bx_{2m-1}) = d(y_{p+1}, y_{q+1}) < \varepsilon$ as desired.

(b) We know that $d(Ax, By) \leq d(Sx, Ty)\ \forall\, x, y$ by the hypothesis on A, B, S and T. So if m is even, say m = 2n,

$$\begin{aligned} d(y_m, y_{m+1}) &= d(y_{2n}, y_{2n+1}) = d(Bx_{2n-1}, Ax_{2n}) \\ &\leq d(Tx_{2n-1}, Sx_{2n}) = d(y_{2n-1}, y_{2n}) \\ &= d(y_{m-1}, y_m). \end{aligned} \qquad (3.9.4)$$

Similarly, if we take m is odd say $m = 2n - 1$,

$$\begin{aligned} d(y_m, y_{m+1}) = d(y_{2n-1}, y_{2n}) &= d(Ax_{2n-2}, Bx_{2n-1}) \\ &\le d(Sx_{2n-2}, Tx_{2n-1}) = d(y_{2n-2}, y_{2n-1}) \\ &= d(y_{m-1}, y_m). \end{aligned} \quad (3.9.5)$$

Thus by equations (3.9.4) and (3.9.5) the sequence $d\{(y_m, y_{m+1})\}$ is non increasing and converges to the greatest lower bound of its range which we denote by r. Now $r \ge 0$; in fact, r = 0. Otherwise, since m and m + 1 are certainly of opposite parity by part (a) we have $d(y_{m+1,}\, y_{m+2}) < r$ whenever $r \le d(y_m, y_{m+1}) < \delta(r)$. But since $\{d(y_m, y_{m+1})\}$ converges to r, there is a k such that $d(y_k, y_{k+1}) < \delta(r)$ so that $d(y_{k+1}, y_{k+2}) < r$ - which contradicts the designation of r. Hence $r = 0$, and we have $d(y_n, y_{n+1}) \to 0$.

(c) let $\alpha = 2\varepsilon$ be given. With $r = \delta(\varepsilon) - \varepsilon$, part (a) of lemma asserts that

$d(y_{p+1}, y_{q+1}) < \varepsilon, \mathit{whenever}\ \varepsilon \le d(y_p, y_q) < \varepsilon + r\ \mathit{and}\ p\ \mathit{and}\ q$ are of opposite parity.

(3.9.6)

Assume without loss of generality that $r < \varepsilon$. By part (b) of the lemma we can choose $n_0 \in \mathbf{N}$ such that

$$d(y_m, y_{m+1}) < r/6 \text{ for } m \ge n_0. \quad (3.9.7)$$

We now let $q > p \ge n_0$ and show that $d(y_p, y_q) < \alpha$, thereby proving that $\{y_n\}$ is indeed Cauchy. So suppose that

$$d(y_p, y_q) \ge 2\varepsilon = \alpha \quad (3.9.8)$$

In order to show that (3.9.8) yields a contradiction, we first want an $m > p$ such that

$$\varepsilon + r/3 < d(y_p, y_m) < \varepsilon + r \text{ with } p \text{ and } m \text{ of opposite parity.} \quad (3.9.9)$$

To this end let k be the smallest integer greater than p such that $d(y_p, y_k) > \varepsilon + r/2$. Here k exists by (3.9.8) since $r < \varepsilon$. Moreover,

$$d(y_p, y_k) < \varepsilon + (2r)/3. \quad (3.9.10)$$

For otherwise, $\varepsilon + (2r)/3 \le d(y_p, y_{k-1}) + d(y_{k-1}, y_k) < d(y_{p,}\, y_{k-1}) + r/6.$

Since $k - 1 \ge p \ge n_0$, and therefore

$$\varepsilon + r/2 < d(y_p, y_{k-1}). \quad (3.9.11)$$

Since $k - 1 \ge p$, (3.9.11) implies $p < k - 1$. But then (3.9.11) contradicts the choice of k. We thus have

$$\varepsilon + r/2 < d(y_p, y_k) < \varepsilon + (2r)/3. \quad (3.9.12)$$

Thus, if p and k are of opposite parity we can let $k = m$ in (3.9.12) to obtain (3.9.9). If p and k are of like parity, p and k+1 are of opposite parity. Moreover, since $d(y_k, y_{k+1}) < r/6$ by (3.9.7), the triangle inequality and (3.9.12) imply

$$\varepsilon + r/3 < d(y_p, y_{k+1}) < \varepsilon + (5r)/6. \quad (3.9.13)$$

In this instance let m = k+1.

In any event, by (3.9.12) and (3.9.13) we can choose m such that p and m are of opposite parity and (3.9.9) holds. But then $p, m \geq n_0,$ (3.9.7) and (3.9.9) imply

$$\begin{aligned} \varepsilon + r/3 < d(y_p, y_m) &\leq d(y_p, y_{p+1}) + d(y_{p+1}, y_{m+1}) + d(y_{m+1}, y_m) \\ &< r/6 + d(y_{p+1}, y_{m+1}) + r/6 \\ &< r/3 + \varepsilon, \text{by (3.9.6) and (3.9.9).} \end{aligned}$$

This is the contradiction.

The following lemma highlights the role of compatibility in producing common fixed Points:

Lemma 3.10: Let S and T be self maps of a metric space (X, d) and let A and B be (ε, δ)-S, T-contractions such that the pairs A, S and B, T are compatible. If there exists $z \in X$ such that $Az = Sz$ and $Bz = Tz$, then $c = Tz$ is the unique common fixed point of A, B, S and T.

Proof: The definition of (ε, δ)-S, T- contractions implies $d(Ax, By) < d(Sx, Ty)$ if $Sx \neq Ty$. Thus, if we take $Sz \neq Tz$ then the hypothesis yields the contradiction that $d(Az, Bz) < d(Sz, Tz) = d(Az, Bz)$. Thus we can conclude that $Sz = Tz = Az = Bz$. Now let $c = Tz$ and suppose that $c \neq Tc$. Since we have T and B are compatible and $Tz = Bz$ then, by proposition 3.6, we have $TBz = BTz$. Then, $d(c, Tc) = d(Az, TBz) = d(Az, BTz) < d(Sz, TTz) = d(c, Tc)$, which is a contradiction. Thus c must equal to Tc. Similarly, we can prove that $c = Sc$ for this let $c = Sz$ and suppose that $c \neq Sc$. Since S and A are compatible and $Az = Sz$, then $ASz = SAz$ by Proposition 3.6. Then $d(c, Sc) = d(Bz, SAz) = d(Bz, ASz) < d(Tz, SSz) = d(c, Sc)$, which is a contradiction. Thus c must equal to Sc. Since A and S are compatible and $Az = Sz = c$, proposition 3.6. implies

$$Ac = A(Sz) = S(Az) = Sc. \qquad (3.10.1)$$

Similarly if we take B and T are compatible and $Bz = Tz = c$, Proposition 3.6. implies

$$Bc = B(Tz) = T(Bz) = Tc. \qquad (3.10.2)$$

Then (3.10.1) and (3.10.2) yields: $Ac = Sc = Tc = Bc = c$. Thus c is the common fixed point of A, B, S, T.

To prove c is unique. Take t another fixed point of A, B, S, T. Then $d(c,t) = d(Ac, Bt) < d(Sc, Tt) = d(c,t)$ which is a contradiction. Therefore c is unique common fixed point of A, B, S and T.

Main Results

We now study the main results of this chapter:

Theorem 3.11: Let S and T are self maps of a metric space (X, d) and let A and B be (ε, δ)-S, T- contractions such that the pairs A, S and B, T are compatible. Let $x_0 \in X$ and let $\{y_n\}$ be any S, T- iteration of x_0 under A and B. If $\{y_n\}$ has a cluster point z in X, then $\{y_n\}$ converges to z, and Tz is the unique common fixed point of A, B, S and T provided these functions are continuous at z.

Proof: By lemma 3.9, $\{y_n\}$ is a Cauchy sequence and therefore converges to the cluster point z in X. Then $Ax_{2n}, Sx_{2n}, Bx_{2n-1}, Tx_{2n-1} \to z$ where $Ax_{2n}, Sx_{2n}, Bx_{2n-1}, Tx_{2n-1}$ are defined in definition 3.8. Since A and S are compatible and also continuous at z then, by Proposition 3.6.2.b, $Az = Sz$. Similarly, B and T are compatible and also continuous at z then, by Proposition 3.6.2.b, $Bz = Tz$. Therefore, we have $Az = Sz$ and $Bz = Tz$ where $z \in X$. Thus by Lemma 3.10, Tz is the unique common fixed point of A, B, S and T.

Corollary 3.12: Let S and T be continuous self maps of a complete metric space (X, d) and let A and B be (ε, δ)-S, T- contractions such that the pairs A, S and B, T are compatible. If A and B are continuous, then A, B, S and T have unique common fixed point.

Example 3.13: Let $X = [1, \infty)$ and $d(x, y) = |x - y|$. Let $Sx = (x^4 + 1)/2$, $Ax = x^2$, $Bx = x$ and $Tx = (x^2 + 1)/2$ for x in X.

Here (X, d) is a complete metric space and A, B, S and T are continuous self maps of (X, d). Since A1 = S1 =1 and $(S - A)x = (x^2 - 1)^2/2$ is increasing on X. Therefore, by using corollary 3.5 we have A and S are compatible on X. Which is clearly valid on any connected subset of **R.** However, A and S are not even weakly commutative; for this consider x = 2 then $d\big(AS(2), SA(2)\big) > d(A(2), S(2))$. so, we can say that A and S are not weakly commuting. On the other side, B and T are commuting as BT = TB and as we know that every commuting pair is compatible also. Therefore B and T are also compatible on X. Thus we have A, S and B, T as compatible pairs.

Now to see that the hypothesis of Corollary 3.12 is satisfied, we have only to show that A and B are (ε, δ)-S, T-contractions. For this let $\varepsilon > 0$. Consider a function $\delta: \mathbf{R}^+ \to \mathbf{R}^+$ such that $\varepsilon \leq |Sx - Ty| = |(x^4 - y^2)/2| < \delta(\varepsilon)$ implies $|Ax - By| = |x^2 - y| < \varepsilon$. Remembering as $x, y \geq 1$, it is easy to show that $|(x^4 - y^2)/2| \geq \varepsilon$ implies that $(x^2 + y) \geq (1 + \sqrt{2\varepsilon + 1})$. But then $|(x^4 - y^2)/2| < \delta(\varepsilon)$ implies that $|x^2 - y| < 2\delta(\varepsilon)(x^2 + y)^{-1} < 2\delta(\varepsilon)(1 + \sqrt{2\varepsilon + 1})^{-1}$, so that $|x^2 - y| < \varepsilon$ *if* $\delta(\varepsilon) = \varepsilon(1 + \sqrt{2\varepsilon + 1})/2$. Now δ so defined for all $\varepsilon > 0$ is a continuous mapping from $\mathbf{R}^+$ into $\mathbf{R}^+$ such that $\delta(\varepsilon) > \varepsilon$ so that property (1) of definition 3.7 is satisfied. For satisfying property (2) we see that $|Sx - Ty| = |(x^4 - y^2)/2| = |Ax - By||(x^2 + y)/2|$, which requires that $Ax = By$ when $Sx = Ty$ since $x, y \geq 1$. Thus all conditions of (ε, δ)-S, T-contraction are satisfied. Therefore, by Corollary 3.12 A, B, S and T have unique common fixed point.

Remark: The mappings A, B, S and T as defined in example 3.13, the pair A, B are (ε, δ)-S, T- contractions, whereas the pair B, A are not $(\varepsilon, \delta) - S, T -$ contractions since $|Bx - Ay| = |x - y^2| = 3$ and $|Sx - Ty| = |(x^4 - y^2)/2| = 3/2$ with $x = 1$ and $y = 2$.

In Example 3.13. we have used the function δ which is continuous. But, our next result tells us that if we merely require that δ be lower semi continuous then, A and B in Corollary 3.12 need not be continuous.

Theorem 3.14: Let S and T be continuous self maps of a complete metric space (X, d) and let A and B be (ε,δ)-S, T-contractions such that the pairs A, S and B, T are compatible. If δ is lower semi continuous, then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ and let $\{y_n\}$ be an S, T-iteration of x_0 under A and B. Since (X, d) is complete and by lemma 3.9, $\{y_n\}$ is Cauchy. Therefore, $\{y_n\}$ converges to some element z of X. Then

$$Ax_{2n}, Sx_{2n}, Bx_{2n-1}, Tx_{2n-1} \to z. \qquad (3.14.1)$$

where $Ax_{2n}, Sx_{2n}, Bx_{2n-1}, Tx_{2n-1}$ are defined in definition 3.8. Since T is continuous and the pair B, T is compatible, then by using the Proposition 3.6, the continuity of T and by (3.14.1) we have

$$BTx_{2n-1} \to Tz \text{ and } T^2x_{2n-1} = T(Tx_{2n-1}) \to Tz. \qquad (3.14.2)$$

Similarly, since S is continuous and the pair A, S is compatible, then by using the Proposition 3.6, the continuity of S and by (3.14.1) we have

$$ASx_{2n} \to Sz \text{ and } S^2x_{2n} = S(Sx_{2n}) \to Sz. \qquad (3.14.3)$$

Now, we assert that $Sz = Tz$. For proving this suppose that $d(Sz, Tz) = \varepsilon$ for some $\varepsilon > 0$. Since $\delta: \mathbf{R}^+ \to \mathbf{R}^+$ is lower semi continuous and $\delta(\varepsilon) > \varepsilon$. Therefore by using the definition, there is a neighborhood $N(\varepsilon)$ of ε such that $\delta(t) > \varepsilon$ for $t \in N(\varepsilon)$. We can thus choose t_0 such that $0 < t_0 < \varepsilon < \delta(t_0)$. By (3.14.2) and (3.14.3) $d(S^2x_{2n}, T^2x_{2n-1}) \to \varepsilon$, and we can therefore choose n_0 such that $d(S^2x_{2n}, T^2x_{2n-1}) \in (t_0, \delta(t_0))$ for $n \geq n_0$. Then the definition of δ implies that $d(ASx_{2n}, BTx_{2n-1}) < t_0$ for $n \geq n_0$. But then (3.14.2) and (3.14.3) implies that $d(Sz, Tz) \leq t_0 < \varepsilon = d(Sz, Tz)$, the contradiction. Therefore, $Sz = Tz$.

Now $Sz = Tz$ by definition 3.7 implies that $Az = Bz$. Moreover, $Az = Tz$, since $d(Az, Tz) = \lim_{n\to\infty} d(Az, BTx_{2n-1}) \leq \lim_{n\to\infty} d(Sz, T^2x_{2n-1}) = d(Sz, Tz) = 0$. Thus, we have $Az = Bz = Sz = Tz$, by Lemma 3.10, Tz is the unique common fixed point of A, B, S and T.

Corollary 3.15: Let S and T be self maps of a complete metric space (X, d), and let $A, B: X \to S(X) \cap T(X)$. Suppose that S and T are continuous and the pairs A, S and B, T are compatible. If there exists $r \in (0,1)$ such that

$$d(Ax, By) \leq rd(Sx, Ty) \text{ for } x, y \in X. \qquad (3.15.1)$$

Then A, B, S and T have a unique common fixed point.

Proof: Define δ by $\delta(\varepsilon) = \varepsilon/r$. Then δ is a continuous self map of $\mathbf{R}^+$ such that $\delta(\varepsilon) > \varepsilon$. Also, $d(Sx, Ty) < \delta(\varepsilon)$ implies $d(Ax, By) < r(\varepsilon/r) = \varepsilon$. Thus, A and B are (ε, δ)-S, T-contractions and the hypothesis of Theorem 3.14. is satisfied. Therefore, A, B, S and T have a unique common fixed point.

Corollary 3.16: Let S and T be continuous self maps of a complete metric space (X, d). Let $\{A_i : i \in I\}$ be a family of maps $A_i : X \to S(X) \cap T(X)$ compatible with both S and T and let I be any indexing set . If there exists $r \in (0,1)$ such that

$$d\left(A_i x, A_j y\right) \leq rd(Sx, Ty)\ \forall\ x, y \in X \text{ and } i \neq j. \qquad (3.16.1)$$

then there is a unique point $c \in X$ such that $c = Sc = Tc = A_i$ for all $i \in I$.

Proof: Let $i, j \in I\ (i \neq j)$. By corollary 3.15, there is a unique point $c \in X$ such that $c = A_i c = A_j c = Sc = Tc$. Now if $k \in I\ (i \neq k)$, there is a unique point $d \in X$ such that $d = A_i d = A_k d = Sd = Td$. Then (3.16.1) implies:

$$d(c, d) = d(A_i c, A_k d) \leq rd(Sc, Td) = rd(c, d) < d(c, d).$$

Since $r < 1$, c must equal to d and then there is a unique point $c \in X$ such that $c = Sc = Tc = A_i c$ for all $i \in I$.

Remark: The functions in Example 3.13. shows that the concept of (ε, δ)-S, T-contractions does indeed the generalization of the relation (3.15.1) of corollary 3.15, since in example 3.13. if we take x = 1, then we have $|S1 - Ty| = |A1 - By||(1 + y)/2|$ where $(1 + y)/2$ converges to 1 as y approaches 1 from the right; i.e., there exists no $r \in (0,1)$ such that $|Ax - By| \leq r|Sx - Ty|$ for all $x, y \geq 1$.

CHAPTER – 4

COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPS

Introduction

The purpose of this chapter is to study the common fixed point theorem proved by Chugh and Kumar [7], from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity. The results proved by them generalize the results of Jungck [16], Fisher [10], Kang and Kim [19], Jachymski [11] and Rhoades [28].

In 1976, Jungck [12] proved a common fixed point theorem for commuting maps generalizing the Banach fixed point theorem, which states that, 'let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X'. This theorem has many applications but suffers fron one drawback that the definition requires that T be continuous throughout X. There then follows a number of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand, Sessa [30] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further Jungck [15] introduced more generalized commutativity, the so-called compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems, for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors.

It has been known from the paper of Kannan [20] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover , the maps involved in every case were continuous at the fixed point. In 1988, Jungck and Rhoades [18] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but converse need not be true.

Weakly Compatible Mappings

Two self mappings f and g on a metric space (X, d) are said to be weakly compatible if they commute at coincidence points.

Example: Let X = [0, 3] with the usual metric d(x, y) = |x – y| and define f, g: [0, 3] → [0, 3] as

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ 3, & \text{if } x \in [1,3] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3-x, & \text{if } x \in [0,1) \\ 3, & \text{if } x \in [1,3] \end{cases}$$

Then for any x ∈ [1, 3], x is a coincident point and fg(x) = gf(x) showing that f, g are weakly compatible mappings on [0, 3].

Example: Let X = **R** with the usual metric d(x, y) = |x – y| and define f, g: **R** → **R** by

$$f(x) = \frac{x}{3}, x \in \mathbf{R} \text{ and } g(x) = x^2, x \in \mathbf{R}.$$

Here 0 and1/3 are two coincidence points for the maps f and g. Here f and g commute at 0, i.e., $fg(0) = gf(0) = 0$, but $fg\left(\frac{1}{3}\right) = f\left(\frac{1}{9}\right) = \frac{1}{27}$ and $gf\left(\frac{1}{3}\right) = g\left(\frac{1}{9}\right) = \frac{1}{81}$. Therefore f and g not commute at 1/3. Thus f and g are not weakly compatible mappings.

Remark: Clearly, any compatible pair is weakly compatible but the converse is not true as shown by the following example.

Example: Let X = [2, 20] and d be the usual metric on X. Define f, g: X → X as

$$f(x) = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 6, & \text{if } 2 < x \leq 5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x, & \text{if } x = 2 \\ 12, & \text{if } 2 < x \leq 5 \\ x-3, & \text{if } x > 5 \end{cases}$$

The mappings f and g are not compatible as we define a sequence $\{x_n\}$ by $x_n = 5 + \frac{1}{n}, n \geq 1$. Then, $fx_n = 2, gx_n \to 2, fgx_n = 6, gfx_n = 2$. Hence f and g are not compatible mappings. But they are weakly compatible mappings since they commute at coincidence point at x = 2 as $fg(2) = gf(2) = 2$.

Main Results

Let $\mathbf{R}^+$ denote the set of non-negative real numbers and F a family of all mapping

$$\varphi: (\mathbf{R}^+)^5 \to \mathbf{R}^+$$

such that φ is upper semi-continuous, non-decreasing in each coordinate variable and, for any $t > 0$,

$$\varphi(t, t, 0, \alpha t, 0) \le \beta t, \qquad \varphi(t, t, 0, 0, \alpha t) \le \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$,

$$\gamma(t) = \varphi(t, t, \alpha_1 t, \alpha_2 t, \alpha_3 t) < t,$$

where $\gamma: \mathbf{R}^+ \to \mathbf{R}^+$ is a mapping and $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

For the proof of our main result the following lemmas will be used:

Lemma 4.1 [31]: For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n times composition of γ.

Proof: Since φ is upper semi continuous, then γ is upper semi continuous. Assume $\lim_{n\to\infty} \gamma^n(t) = A$, where $A \neq 0$. Then

$$A = \lim_{n\to\infty} \gamma^{n+1}(t) \le \gamma \lim_{n\to\infty} \gamma^n(t) = \gamma(A) < A$$

that is, $A < A$, a contradiction. Therefore $A = 0$ implies

$$\lim_{n\to\infty} \gamma^n(t) = 0.$$

Conversely since φ is non-decreasing, then γ is non-decreasing. Given $\lim_{n\to\infty} \gamma^n(t) = 0$, assume $\gamma(t) > t$ for some $t > 0$, then $\gamma^n(t) > t$ for some $t > 0$ for $n = 1, 2, 3, \ldots$. Thus $\lim_{n\to\infty} \gamma^n(t) \nrightarrow 0$ a contradiction. Also if $\gamma(t) = t$ for some $t > 0$ then $\lim_{n\to\infty} \gamma^n(t) \nrightarrow 0$. Hence, for all $t > 0, \gamma(t) < t$.

Let A, B, C and D be mappings from a metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subset D(X) \text{ and } B(X) \subset C(X), \tag{4.1}$$

$$d(Ax, By) \le \varphi\big(d(Cx, Dy), d(Ax, Cx), d(By, Dy), d(Ax, Dy), d(By, Cx)\big) \tag{4.2}$$

for all $x, y \in X$, where $\varphi \in F$. Then for arbitrary point x_0 in X, by (3.1), we choose a point x_1 such that $Dx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Cx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Dx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Cx_{2n+2}, n = 0, 1, 2, 3, \dots \quad (4.3)$$

Lemma 4.2: $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, where $\{y_n\}$ is the sequence in X defined by (4.3).

Proof: Let $d_n = d(y_n, y_{n+1}), n = 0, 1, 2, \dots$. Now we shall prove that the sequence $\{d_n\}$ is non-increasing in $\mathbf{R}^+$, that is, $d_n \le d_{n-1}$ for $n = 1, 2, 3, \dots$. From (4.2), we have

$$d(Ax_{2n}, Bx_{2n+1}) \le \varphi(d(Cx_{2n}, Dx_{2n+1}), d(Ax_{2n}, Cx_{2n}), d(Bx_{2n+1}, Dx_{2n+1}),$$
$$d(Ax_{2n}, Dx_{2n+1}), d(Bx_{2n+1}, Cx_{2n})).$$
$$d(y_{2n}, y_{2n+1}) \le \varphi(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}),$$
$$d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}))$$
$$= \varphi(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}),$$
$$0, [d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})])$$
$$= \varphi(d_{2n-1}, d_{2n-1}, d_{2n}, 0, d_{2n} + d_{2n-1}). \quad (4.2.1)$$

Suppose that $d_{n-1} < d_n$ for some n. So,

$$d_{n-1} + d_n < d_n + d_n = 2d_n.$$

Then, for some $\alpha < 2, d_{n-1} + d_n = \alpha d_n$. Since φ is non-increasing in each variable and $\beta < 1$ for some $\alpha < 2$. From (4.2.1) we have

$$d_{2n} \le \varphi(d_{2n}, d_{2n}, d_{2n}, 0, \alpha d_{2n}) \le \beta d_{2n} < d_{2n}.$$

Similarly, we have $d_{2n+1} < d_{2n+1}$. Hence, for every $n, d_n \le \beta d_n < d_n$, which is a contradiction. Therefore, $\{d_n\}$ is a non-increasing sequence in $\mathbf{R}^+$. Now, again by (4.2), we have

$$d_1 = d(y_1, y_2) = d(Ax_2, Bx_1)$$
$$\le \varphi(d(Cx_2, Dx_1), d(Ax_2, Cx_2), d(Bx_1, Dx_1),$$
$$d(Ax_2, Dx_1), d(Bx_1, Cx_2))$$
$$= \varphi\big(d(y_1, y_0), d(y_2, y_1), d(y_1, y_0), d(y_2, y_0), d(y_1, y_1)\big)$$
$$\le \varphi(d_0, d_1, d_0, d_0 + d_1, 0)$$
$$\le \varphi(d_0, d_0, d_0, 2d_0, d_0)$$
$$= \gamma(d_0)$$

In general, we have $d_n \le \gamma^n(d_0)$, which implies that, if $d_0 > 0$, by lemma 4.1,

$$\lim_{n\to\infty} d_n \le \lim_{n\to\infty} \gamma^n(d_0) = 0.$$

Therefore, we have $\lim_{n\to\infty} d_n = 0$. For $d_0 = 0$, since $\{d_n\}$ is non-increasing, we have $\lim_{n\to\infty} d_n = 0$. Thus $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. This completes the proof.

Lemma 4.3: The sequence $\{y_n\}$ defined by (4.3) is a Cauchy in X.

Proof: By virtue of lemma 4.2, it is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for each even integer 2k, there exist even integer 2m(k) and 2n(k) with 2m(k) > 2n(k) ≥ 2k such that

$$d(y_{2m(k)}, y_{2n(k)}) > \varepsilon. \tag{4.3.1}$$

For each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying (4.3.1), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \le \varepsilon \text{ and } d(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{4.3.2}$$

Then for each even integer 2k, we have

$$\begin{aligned}\varepsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\le d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

By lemma 4.2 and (4.3.2), it follows that

$$d(y_{2n(k)}, y_{2m(k)}) \to \varepsilon \text{ as } k \to \infty. \tag{4.3.3}$$

By the triangle inequality, we have

$$\left|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \le d(y_{2m(k)-1}, y_{2m(k)})$$

and

$$\left|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \le d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

From lemma 4.2 and eq. (4.3.3), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \text{ and } d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon. \tag{4.3.4}$$

Therefore, by (4.2) and (4.3), we have

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)}) &\le d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)}) \\ &= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}) \\ &\le d(y_{2n(k)}, y_{2n(k)+1}) + \varphi(d(Cx_{2m(k)}, Dx_{2n(k)+1}), \\ &\quad d(Ax_{2m(k)}, Cx_{2m(k)}), d(Bx_{2n(k)+1}, Dx_{2n(k)+1}), \\ &\quad d(Ax_{2m(k)}, Dx_{2n(k)+1}), d(Bx_{2n(k)+1}, Cx_{2m(k)}))\end{aligned}$$

$$= d\big(y_{2n(k)}, y_{2n(k)+1}\big) + \varphi(d\big(y_{2m(k)-1}, y_{2n(k)}\big),$$
$$d\big(y_{2m(k)}, y_{2m(k)-1}\big), d\big(y_{2n(k)+1}, y_{2n(k)}\big),$$
$$d\big(y_{2m(k)}, y_{2n(k)}\big), d(y_{2n(k)+1}, y_{2m(k)-1})). \qquad (4.3.5)$$

Since φ is upper semi continuous, as $k \to \infty$ as in (4.3.4), by lemma 4.2, eqs (4.3.3), (4.3.4) and (4.3.5) we have

$$\varepsilon \le \ \varphi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \gamma(\varepsilon) < \varepsilon,$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X and so is $\{y_n\}$. This completes the proof.

The following is the main result of this chapter:

Theorem 4.4: Let (A, C) and (B, D) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (4.1) and (4.2). Then A, B, C and D have a unique common fixed point in X.

Proof: By Lemma 4.3, $\{y_n\}$ is a Cauchy sequence in X, Since X is complete there exists a point z in X such that

$$\lim_{n\to\infty} y_n = z. \ \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Dx_{2n+1} = z \text{ and } \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z \text{ i.e.},$$
$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Dx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Cx_{2n+2} = z.$$

Since $B(X) \subset C(X)$, there exists a point $u \in X$ such that $z = Cu$. Then, using (4.2),

$$d(Au, z) \le d(Au, Bx_{2n-1}) + d(Bx_{2n-1}, z)$$
$$\le \varphi(d(Cu, Dx_{2n-1}), d(Au, Cu), d(Bx_{2n-1}, Dx_{2n-1}),$$
$$d(Au, Dx_{2n-1}), d(Bx_{2n-1}, Cu)).$$

Taking the limit as $n \to \infty$ yields

$$d(Au, z) \le \varphi(0, d(Au, Cu), 0, d(Au, z), d(z, Su))$$
$$= \varphi(0, d(Au, z), 0, d(Au, z), 0) \le \beta d(Au, z),$$

where $\beta < 1$. Therefore $z = Au = Cu$.

Since $A(X) \subset D(X)$, there exists a point $v \in X$ such that $z = Dv$. Then again using (4.2)

$$d(z, Bv) = d(Au, Bv) \le \varphi(d(Cu, Dv), d(Au, Cu), d(Bv, Dv), d(Au, Tv), d(Bv, Cu))$$
$$= \varphi\big(0, 0, d(Bv, z), 0, d(Bv, z)\big) \le \varphi(t, t, t, 2t, t) < t,$$

where $t = d(z, Bv)$. Therefore $z = Bv = Dv$. Thus $Au = Cu = Bv = Dv = z$. Since pair of maps A and C are weakly compatible, then $ACu = CAu$ i.e., $Az = Cz$. Now we show that z is a fixed point of A. If $Az \neq z$, then by (4.2),

$$d(Az, z) = d(Az, Bv) \leq \varphi(d(Cz, Dv), d(Az, Cz), d(Bv, Dv), d(Az, Dv), d(Bv, Cz))$$

$$= \varphi(d(Az, z), 0, 0, d(Az, z), d(Az, z))$$
$$\leq \varphi(t, t, 2t, t, t) < t, \text{where } t = d(Az, z).$$

Therefore, $Az = z$. Hence $Az = Cz = z$.

Similarly, pair of maps B and D are weakly compatible, we have $Bz = Dz = z$, since

$$d(z, Bz) = d(Az, Bz) \leq \varphi(d(Cz, Dz), d(Az, Cz), d(Bz, Dz), d(Az, Dz), d(Bz, Cz))$$
$$= \varphi(d(z, Dz), 0, 0, d(z, Dz), d(z, Dz)$$
$$\leq \varphi(t, t, 2\,t, t, t) < t, \text{where } t = d(z, Dz) = d(z, Bz).$$

Thus $z = Az = Bz = Cz = Dz$, and z is a common fixed point of A, B, C and D.

Finally, in order to prove the uniqueness of z, suppose that z and w, $z \neq w$, are common fixed points of A, B, C and D. Then by (4.2), we obtain

$$d(z, w) = d(Az, Bw) \leq \varphi(d(Cz, Dw), d(Az, Cz), d(Bw, Dw), d(Az, Dw), d(Bw, Cz))$$
$$= \varphi(d(z, w), 0, 0, d(z, w), d(z, w))$$
$$\leq \varphi(t, t, 2\,t, t, t) < t, \text{where } t = d(z, w).$$

Therefore, $z = w$. Thus A, B, C and D have a unique common fixed point in X. The following corollaries follow immediately from Theorem 4.4.

Corollary 4.5: Let (A, C) and (B, D) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (4.1), (4.3) and (4.5.1)

$$d(Ax, By) \leq hM(x, y), 0 \leq h < 1, x, y \in X, \text{where}$$

$$M(x, y) = \max\{d(Cx, Dy), d(Ax, Cx), d(By, Dy), [d(Ax, Dy) + d(By, Cx)]/2\}. \quad (4.5.1)$$

Then A, B, C and D have a unique common fixed point in X.

Proof: We consider the function $\varphi: [0, \infty)^5 \to [0, \infty)$ defined by

$$\varphi(x_1, x_2, x_3, x_4, x_5) = h \max\left[x_1, x_2, x_3, \frac{1}{2}(x_4 + x_5)\right].$$

Since $\varphi \in F$, we can apply Theorem 4.4 and deduce the corollary.

Corollary 4.6: Let (A, C) and (B, D) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (4.1), (4.3) and (4.6.1).

$$d(Ax, By) \leq h \max\{d(Ax, Cx), d(By, Dy), \frac{1}{2} d(Ax, Dy),$$

$$\frac{1}{2} d(By, Cx), d(Cx, Dy)\} \text{ for all } x, y \text{ in } X, \text{where } 0 \leq h < 1. \qquad (4.6.1)$$

Then A, B, C and D have a unique common fixed point in X.

Proof: We consider the function $\varphi: [0, \infty)^5 \rightarrow [0, \infty)$ defined by

$$\varphi(x_1, x_2, x_3, x_4, x_5) = h \max\{x_1, x_2, x_3, \frac{1}{2} x_4, \frac{1}{2} x_5\}.$$

Since $\varphi \in F$, we can apply Theorem 4.4 to obtain this corollary.

Remark: Theorem 4.4 is a generalization of the result of Jungck [16] by using weakly compatible maps without continuity at S and T. Theorem 4.4 and Corollary 4.6 are also generalization of the result given by Fisher [10] by using compatible maps instead of using the commutativity of maps.

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