ANALOG REALIZATION OF FRACTIONAL ORDER CIRCUITS

Dissertation submitted in the partial fulfillment of requirement for the award of degree of

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in

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DECLARATION

I hereby declare that the work which is being presented in the dissertation entitled, **"Analog Realization of Fractional Order Circuits"** in partial fulfillment of the requirement for the award of degree of Master of Technology in VLSI Design submitted in Electronics and Communication Engineering Department of Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of Mr. Sanjay Kumar, Assistant Professor, ECED and refers other researcher's work which are duly listed in the reference section.

The matter presented in this dissertation has not been submitted in any other University/Institute for the award of degree.

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It is certified that the above statement made by the student is correct to the best of my knowledge and belief.

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ABSTRACT

Calculus of integer orders was once the basic essential mathematical tool for analysis, synthesis response behavior, theorems, and many novel applications for any dynamical system from 1695 until 1960. However, these integer values are a very narrow subset of the real orders, and so during the last five decades, a dramatic shift has taken place and many scientific researchers have been concerned instead with fractional calculus. In particular, these scientists have attempted to broaden the scope of fundamentals and theorems from integer order systems into fractional ones, since many achievements are obtained as a result of employing the extra fractional-order variables, allowing for more flexibility, freedom, best fit, and optimization techniques. Furthermore, many new fundamentals have been investigated only in the fractional order sense.

In recent years it has turned out that many phenomena in engineering, physics, chemistry, and other sciences can be described very successfully by models using mathematical tools from fractional calculus. To obtain better performance, in last few decades, several applications based on fractional order modeling in wide spread fields of science and engineering have been proposed. This includes fluid flow, optics, geology, behavior of viscoelastic material, bioscience, medicine, non-linear control, signal processing, etc.

In this dissertation Studies on analysis and applications of fractance device and fractional order operator is the main objective. Time and frequency domain analysis, different ways of realization of fractance device is presented. Active realization of fractance device of order $s^{0.5}$ using continues fraction expansion is carried out. Later, a general expression for s^{α} is presented using CFE method and time domain response of s^{α} for different input signal and for different values of α is carried out. Further, time and frequency domain analysis of fractance based circuits is considered. The rational approximation of fractional order operator using different methods (Newton, Mastuda, Oustaloup and CFE method) is presented and compared with the ideal response. Later , fractional order filter is studied and performance of the fractional order filter is checked for different input signals (sine wave, trapezoidal wave, sawtooth wave and chirp signal) with random noise and the resulted output is compared with the integer order filter output. All simulation for fractance device , fractional order operator and fractional order filter has done in MATLAB software.

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CHAPTER 1
INTRODUCTION

1.1 Fractional Calculus

Fractional calculus is more than 300 years old topic, which during recent decades became a powerful and widely used tool for better modeling and control of processes in many fields of science and engineering.

All of us are familiar with normal derivatives and integrals, like, $\frac{df}{dt}$, $\frac{d^2f}{dt^2}$, $\int_0^t f(u)du$. We have first-order, second-order derivatives, or first integral, double integral, of a function. Now we wish to have half-order, π^{th}order, or derivative of a function. So, Fractional calculus is equal to derivatives and integrals of arbitrary real or complex order.

Isaac Newton (1642–1727) and Gottfried Wihelm Leibniz (1646–1716) independently discovered calculus in the 17^{th} century. In recognition of this remarkable discovery, John Von Neumann's (1903–1957) thought seems to be worth quoting: '. . . the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance [1]. I think it defines more equivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.' In his discovery of calculus, Leibniz first introduced the idea of a symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the n^{th} derivative, where n is a non-negative integer. L'Hospital asked Leibniz about the possibility that *n* be a fraction. 'What if $n = 1/2$'. Leibniz *(1695)* replied, 'It will lead to a paradox'. But he added prophetically, 'From this apparent paradox, one day useful consequences will be drawn'. Can the meaning of derivatives of integral order Dny be extended to have meaning where n is any number—rational, irrational or complex? In his 700 pages long book on Calculus published in 1819, Lacroix developed the formula for the nth derivative of $y = x^m$, m is a positive integer,

$$d^\alpha x^m = \frac{m!}{(m-\alpha)!} x^m \tag{1.1}$$

The concept of fractional calculus (calculus of integrals and derivatives of any arbitrary real complex order) was raised in 1965 by Marquis L'Hopital to Gottfried Wilhelm Leibniz [2]:

$$\text{for } \frac{d^n y}{dx^n} \quad \text{what if } n = \frac{1}{2}? \tag{1.2}$$

On September 30th 1695, Leibniz replied to L'Hopital

"*...........this is an apparent paradox from which, one day, useful consequences will be drawn..............*"

In the letters to J. Wallis and J. Bernoulli (in 1967) Leibniz mentioned the possible approach to fractional-order differentiation in that sense that for non-integer values of *n* the definition could be the following:

$$\frac{d^n e^{mx}}{dx^n} = mne^{mn} \tag{1.3}$$

Fractional-order calculus is an area of mathematics that deals with derivatives and integrals from non integer orders. In other words, it is a generalization of the traditional calculus that leads to similar concepts and tools, but with a much wider applicability. Fundamental operator ${}_aD^{\alpha}{}_t\ \alpha \in R$, where *a* and *t* are the limits and *α* is the order of the operation [3].

$${}_aD^{\alpha}{}_t = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}, & \alpha > 0 \\ 1, & \alpha = 0 \\ \int_a^t (d\tau)^{-\alpha} & \alpha < 0 \end{cases} \tag{1.4}$$

Some special function used in fractional calculus is Gamma Function, Beta Function, and Mittag-Leffler Function [4].

1.1 Some functions related to fractional order calculus

1.1.1 The Gamma Function

The simplest interpretation of the Gamma function is simply the generalization of the fractional for all real numbers [4]. The definition of Gamma function is given by (1.5)

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{(z-1)} du \qquad \textit{for all } z \in \mathbb{R} \tag{1.5}$$

The ‘beauty’ of the Gamma function can be found in its properties. First, as seen in (1.5), this function is unique in that value for any quantity is, by consequences of the form of the integral, equivalent to that quantity z minus one times the Gamma of the quantity minus one.

$$\Gamma(z-1) = z\Gamma(z), \qquad \text{also, when } z \epsilon \mathbb{N}_{+,} \quad \Gamma(z) = (z-1)! \tag{1.6}$$

1.1.2 Beta function

Beta function is known as the Euler Integral of the First Kind, the Beta Function is in important relationship in fractional calculus. Its solution not is only defined through the use of multiple Gamma Functions, but furthermore shares a form that is characteristically similar to the Fractional Integral/Derivative of many functions, particularly polynomials of the form *t*. Equation (1.7) demonstrates the Beta Integral and its solution in terms of the Gamma function.

$$B(p,q) := \int_0^1 (1-u)^{p-1} u^{q-1} du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(q,p) \qquad p,q \epsilon \mathbb{R}_{+} \tag{1.7}$$

1.1.3 Laplace Transformation and Convolution

The Laplace Transform is a function transformation commonly used in the solution of complicated deferential equations. With the Laplace transform it is frequently possible to avoid working with equations of different deferential order directly by translating the problem into a domain where the solution presents itself algebraically. The formal definition of the Laplace transform is given in (1.8).

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) f(s) \tag{1.8}$$

The Laplace Transform of the function *f (t)* is said to exist if (1.5) is a convergent integral. The requirement for this is that *f(t)* does not grow at a rate higher than the rate at which the exponential term e^{-st} decreases.

Also commonly used is the Laplace convolution, given by (1.9).

$$f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau) d\tau = g(t) * f(t) \tag{1.9}$$

The convolution of two functions in the domain of *t* is sometime complicated to resolve, however, in the Laplace domain (*s*), the convolution results in the simple function multiplication as shown in (1.10).

$$\mathcal{L}\{f(t) * g(t)\} = f\ (s)g(s) \tag{1.10}$$

1.1.4 The Mittag-Leffler Function

The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. In fact, the exponential function itself is a very specific form, one of an infinite set, of this seemingly ubiquitous function. The standard definition of the Mittag-Leffler is given in (1.11)

$$E_\alpha(z) = \frac{\sum_{k=0}^{\infty} z^k}{\Gamma(\alpha k+1)} \tag{1.11}$$

1.2 Definition related to fractional order calculus

1.2.1 L. Euler (1730)

$$\frac{d^n x^m}{dx^n} = m\ (m-1) \dots\dots\dots\dots\dots\dots..(m-n+1)x^m \tag{1.12}$$

$$\Gamma(m+1) = m(m-1) \dots\dots\dots\dots(m-n+1)\Gamma(m-n+1) \tag{1.13}$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^m \tag{1.14}$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$; Euler obtained

$$\frac{d^{0.5}x}{dx^{0.5}} = \sqrt{\frac{4x}{\pi}} \tag{1.15}$$

1.2.2 J. B. J. Fourier (1820{1822})

The first step to generalization of the notion of differentiation for arbitrary functions was done by J. B. J. Fourier

After introducing his famous formula

$$f\ (x) = 1/2\pi \int_{-\infty}^{+\infty} f(z)dz\ \int_{-\infty}^{+\infty} cos(px - pz)\, dp \tag{1.16}$$

Fourier made a remark

$$\frac{d^n f(x)}{dx^n} = 1/2\pi \int_{-\infty}^{+\infty} f(z)dz \int_{-\infty}^{+\infty} cos(px - pz + \frac{n\pi}{2})dp \tag{1.17}$$

and this relationship could serve as a definition of the n^{th} order derivative for non-integer n.

1.2.3 Grunwald-Letnikov, Riemann-Liouville, and Caputo Definitions

There are two main approaches for defining a fractional derivative. The first considers differentiation and integration as limits of finite differences. The Grunwald-Letnikov definition follows this approach. The other approach generalizes a convolution type representation of repeated integration [5]. The Riemann-Liouville and Caputo definitions take this approach. Riemann-Liouville and Caputo fractional derivatives are fundamentally related to fractional integration operators. Consequently, the initial conditions of fractional derivatives are the frequency distributed and infinite dimensional state vector of fractional integrators.

1.3.3.1 Grunwald-Letnikov definition

The Grunwald-Letnikov approach presents limit definitions for higher order derivatives and integrals and shown that

$$_aD^\alpha{}_t f\,(t)\ = lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k\to 0}^{\frac{t}{h}} (-1)^k \binom{\alpha}{k} f(t - \alpha h) \tag{1.18}$$

where h is the time increment.

1.3.3.2 Riemann-Louville definition

$$_aD^\alpha{}_t f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{(dt)^n} \int_a^t (f(\tau)\, d\tau)/(t-\tau)^{\alpha-n+1} \tag{1.19}$$

$$n - 1 \le \alpha < n$$

1.3.3.2.1 Integral according to Riemann-Liouville

According to Riemann-Liouville the notion of fractional integral of Order α $(\alpha > 0)$ for a function $f\ (t)$, is a natural consequence of the well known formula (Cauchy-Dirichlet?), that reduces the calculation of the n-fold primitive of a function $f(t)$ to a single integral of convolution type

$$J^{n}_{a+}f(T) := \frac{1}{n-1}!\int_{a}^{t}(t-\tau)^{n-1} f(\tau)d\tau, \quad (n\in \mathbb{N}) \tag{1.20}$$

vanishes at $t = a$ with its derivatives of order $1, 2. . . n-1$. Require $f(t)$ and $J^{n}_{a}+f(t)$ to be causal functions, that is, vanishing for $t < 0$ .Extend to any positive real value by using the Gamma function,

$$(n-1)! = \Gamma(n) \tag{1.21}$$

Fractional Integral of order α>0 (right-side integral)

$$J^{\alpha}_{a} + f(t) := \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1} f(\tau)d\tau \quad \alpha \in \mathbb{R} \tag{1.22}$$

Define $J^{0}_{a}+ := I, \quad J^{0}_{a}+ f(t) = f(t)$

Alternatively (left-sided integral)

$$J^{\alpha}_{b-}f(t) := \frac{1}{\Gamma(\alpha)}\int_{t}^{b}(\tau-t)^{\alpha-1}f(\tau)d\tau \quad \alpha \in \mathbb{R} \tag{1.23}$$

$(a = 0, b = +\infty)$ Riemann, $(a= -\infty, b = +\infty)$ Liouville

1.3.3.3 Caputo Fractional Derivative

The Caputo definition of fractional differentiation of fractional order α, can be written as

$$D^{\alpha}_{*}f(t)= J^{m-\alpha}D^{m}f(t) \quad \text{with } m-1 <\alpha\leq m, \tag{1.24}$$

$$D^{\alpha}_{*}f(t)= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_{0}^{t} f^{m}(\tau), & m-1 < \alpha < m \\ \frac{d^{m}}{dt^{m}}f(t), & \alpha = m \end{cases} \tag{1.25}$$

where Γ is the Euler's Gamma function

1.3 Application of fractional order calculus

1.3.1 Fractional-order dynamical systems in control theory

New and effective methods for the time-domain analysis of fractional-order dynamical systems are required for solving problems of control theory. As a new generalization of the classical *PID*-controller, the idea of $PI^{\lambda}D^{\mu}$ controller, involving fractional-order integrator and fractional-order differentiator, has been found to be a more efficient control of fractional-order dynamical systems [6].

1.3.2 Electrical circuits with fractance

Classical electrical circuits consist of resistors and capacitors and are described by integer-order models. However, circuits may have the so-called *fractance* which represents an electrical element with fractional-order impedance as suggested by Le Mehaute and Crepy, We consider two kinds of fractances: (i) tree fractance and (ii) chain fractance. Following Nakagawa and Sorimachi, we consider a tree fractance consisting of a finite self-similar circuit with resistors of resistance *R* and capacitors of capacitance *C*. The impedance of the fractance is given by

$$z(i\omega) = \sqrt{\left(\frac{R}{C}\right)}\,\omega^{-0.5}\exp^{-\left(\frac{\Pi i}{4}\right)} \qquad (1.26)$$

For a chain fractance consisting of *N* pairs of resistor-capacitor connected in a chain

1.3.3 Generalized voltage divider

Westerlund observed that both the tree fractance and chain fractance consist not only of resistors and capacitors properties, but also they exhibit electrical properties with noninteger-order impedance. He generalized the classical voltage divider in which the fractional order impedances *F*1 and *F*2 represent impedances not only on Wasteland’s capacitors, classical resistors, and induction coils, but also impedances of tree fractance and chain fractance.

1.3.4 Fractional calculus in electrochemistry and tracer fluid flows

Although the idea of a half-order fractional integral of the current field was known in electrochemistry, Oldham has initiated a mathematical study of some semi-integral electroanalysis with some experimental support. Simultaneously, he and his associates (Oldham and Spanier) have given considerable attention to their new approach to the solution of electrochemical problems that deal with diffusion phenomena. Subsequently, Goto and Ishii developed the idea of semi differential electroanalysis with the fractional-order diffusion equation that may occur in other fields including diffusion, heat conduction, and mass transfer [7].

1.3.5 Ultrasonic wave propagation in human cancellous bone

Fractional calculus is used to describe the viscous interactions between fluid and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellous bone in the elastic frame using Blot's theory. Experimental results are compared with theoretical predictions for slow and fast waves transmitted through human cancellous bone samples

1.3.6 Modelling of speech signals using fractional calculus

In this a novel approach for speech signal modeling using fractional calculus is presented. This approach is contrasted with the celebrated Linear Predictive Coding (LPC) approach which is based on integer order models. It is demonstrated via numerical simulations that by using a few integrals of fractional orders as basic functions, the speech signal can be modeled accurately.

1.3.7 Application of Fractional Calculus to the sound Waves Propagation in Rigid Porous Materials

The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behaviour of sound waves in this kind of materials, including relaxation and frequency dependence.

1.3.8 Application of fractional calculus in the theory of viscoelasticity

The advantage of the method of fractional derivatives in theory of viscoelasticity is that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. Also the fractional derivative method has been used in tudies of the complex moduli and impedances for various models of viscoelastic substances.

1.3.9 Fractional differentiation for edge detection

In image processing, edge detection often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order 2 by the Laplacian. This paper demonstrates how introducing an edge detector based on non-integer (fractional) differentiation can improve the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can interpreted in term of robustness to noise in general [8].

CHAPTER

2

Literature Review

Krishna et. al. [9] presented Studies on analysis, design and applications of analog and digital differentiators and integrators of fractional and. Time and frequency domain analysis, different ways of realization of fractance device is presented. Active and passive realization of fractance device of order ½ using continued fraction expansions is carried out. Later, time and frequency domain analysis of fractance based circuits is considered. The variations of rise time, peak time, settling time, time constant, percent overshoot with respect to fractional order a is presented.

Krishna et. al. [10] proposed active and passive realization of Fractance device of order ½. The crucial point in the realization of fractance device is finding the rational approximation of its impedance function. In this paper, rational approximation is obtained by using continued fraction expansion. The rational approximation thus obtained is synthesized as a ladder network. The results obtained have shown considerable improvement over the previous techniques.

Vinagre et. al. [11] presented the formulation of some possible models of fractional- order systems, several approximations are discussed. For continuous models, some methods for obtaining an approximated rational function using evaluation, interpolation and curve fitting techniques are studied. For discrete models, approximations using Lubich's formula, the trapezoidal rule, and the application of continued fractions expansion technique to integro-differential operators formulated in the Z domain are studied. The methods are compared, in both the time and the frequency domains, using an illustrative example.

Poinot et. al. [12] proposed a method for modeling and simulation of fractional systems and this modeling is based on a new fractional integrator operator, associated to a N dimensional state-space representation and itself approximated by a Ndimensional system composed of an integrator and of a phase-lead filter. Theoretical and numerical comparisons with other techniques commonly used for the simulation of fractional systems. Numerical simulations exhibit the general applicability and flexibility of this new approach to different types of fractional models and to non-conventional non-integer derivation with limited spectral range. A few parameters used for the design of the non-integer action and its spectral range are necessary to characterize this operator's main interest was to propose a general framework for the modelling of fractional systems based on a macro state-space representation, where conventional integration is replaced by fractional one with the help of the integrator operator.

Charef et. al. [13] proposed that the fractional-order differentiator s^{α}, integrator $s^{-\alpha}$ $(0<\alpha< 1)$ and the fractional $PI^{\lambda}D^{\alpha}$ controller are studied and presented a very simple and effective method has for approximating fractional-order integrators and differentiators by rational functions in a given frequency band of practical interest. It has also been shown that, from these rational function approximations, simple analogue circuits can be designed to model fractional-order integrators, differentiators and PID controller.

Djouambi et. al.[14] proposed optimal approximation of the fundamental linear fractional order transfer function using a distribution of the relaxation time function and presented simple method to approximate the irrational transfer function of a class of fractional systems for a given frequency band by a rational function. The optimal parameters of the approximated model are obtained by minimizing simultaneously the gain and the phase error between the irrational transfer function and its rational approximation.

Dorcak et. al.[15] Presented that electronic (analogue) realization of the fractional-order systems controllers or controlled objects whose we earlier used, identified, and analyzed as a mathematical models only – namely a fractional-order differential equation, and solved numerically using a method based on the truncated version of the Grunwald - Letnikov formula for fractional derivative. The electronic realization of the fractional derivative is based on the continued fraction expansion of the rational approximation of the fractional differentiator from which we obtained the values of the resistors and capacitors of the electronic circuit.

Khanra et. al. [16] proposed that Fractional order differentiator (FOD) and fractional order integrator (FOI) have been simulated in Matlab and realized in hardware. The effect of change of order (α) has been observed with different input signals to the differentiator and integrator. The orders are taken as 0.1, 0.5, 0.9, and 1. It has been shown that the value of α has significant effect in system behavior. The simulated results for few typical cases have been compared with the experimentally obtained output of FOD and FOI circuits.

Elwakil et. al. [17] labelled fractional-order continuous-time systems as the "21st century systems". Indeed, this emerging research area is slowly gaining momentum among electrical engineers while its deeply rooted mathematical concepts also slowly migrate to various engineering disciplines. Fractional-order circuits and systems design is definitely an emerging area of interdisciplinary research. Specifically, it is an area where biochemistry, medicine and electrical engineering over-lap giving rise to many new potential application. At the moment, one has to admit that the silicon industry is far from being environmentally friendly.

Das et. al. [18] proposed that Rational approximation of fractional order (FO) elements are now of prime importance for the need of its hardware implementation in control design and signal processing. Among the well known analog realizations methods, the Carlson's approach has been used in this paper due to its simplicity of calculation for designing a certain class of FO differentiators as hybrid filters. Impact of the independent parameters of the hybrid differentiator on the phase response has been depicted along with the achievable accuracies for increasing order of realization for filter design.

Chen et. al. [4] proposed that for fractional-order differentiator s^r where r is a real number, its discretization is a key step in digital implementation. Two discretization methods presented are direct recursive discretization of the Tustin operator and direct discretization method using the Al-Alaoui operator via continued fraction expansion (CFE). The approximate discretization is minimum phase and stable.

Ferdi [6] proposed a novel procedure for the computation of fractional order derivatives and integrals of discrete-time signals. The method was developed by combining the three techniques of *s*-to *z* transform, power series expansion and signal modelling. A mapping function between the *s*-plane and the *z*-plane is first chosen, and then a PSE of this mapping function raised to fractional order is performed to get the desired infinite impulse response of the ideal digital fractional operator. Finally, the desired impulse response is modelled as the impulse response of a linear invariant system whose rational transfer function is determined using deterministic signal modelling techniques.

Sheng et. al. [11] proposed that many complex physical phenomena may be better described using variable-order fractional differential equations. To understand the physical meaning of variable-order fractional calculus, and better know the application potentials of variable-order fractional operators in physical processes, we proposed an experimental study of temperature-dependent variable-order fractional integrator and differentiator.

Li et. al. [14] derived the impulse response of a fractional second order filter of the form $(S^2+aS+b)^{-\lambda}$, where a, b ≥ 0 and $\lambda > 0$ the asymptotic properties of the impulse responses are obtained and based on the derived analytical impulse response, we show how to perform the discretization of the above fractional second order filter.

Radwan et. al. [15] introduced a qualitative revision of the traditional LC tank circuit in the fractional domain. and establish the various conditions under which $L^{\lambda}C^{\alpha}$ impedance may act as a resistor, negative resistor, or a positive or negative pure imaginary inductor or capacitor, in accordance to new frequency definitions; illustrate the process by which the phase response chooses the shortest path from initial to final phase, and ; develop the generalized parameters for the band pass filter response of the $L^{\lambda}C^{\alpha}$ circuit, such as the resonance frequency and quality factor versus $\alpha - \lambda$ lane; discuss sensitivity analyses with respect to the fractional orders, as well as the time domain analyses for the impulse and step responses.

Sierociuk et. al. [16] presented a novel method of analog modelling of fractional order integrators and two special cases are discussed, i.e. integrators of order a = 0.25, and a = 0.5. The method proposed is based on the domino ladder approximation of irrational transfer function. It allows toobtain an analog model using only electric elements like resistors and capacitors of standard produced values.

Bensouici et. al. [19] proposed design method of the digital fractional lead operator z^{α} for $0 < \alpha < 0.5$, in a given frequency band of interest, using digital Infinite Impulse Response (IIR) filters is presented. In this technique, the coefficients of the closed form digital IIR filter derived for the approximation of the fractional lead operator, in a given frequency band, are based on the approximation of fractional order systems. First, analog rational function approximation, for a given frequency band, of the fractional power zero (FPZ) is given. Then the Tustin (bilinear) generating function is used to digitize the FPZ to obtain a closed form IIR digital filter which approximates the digital fractional lead operator z^{α} for ($0 < \alpha < 0.5$).

CHAPTER

3

Fractional Order Differentiator Operator

3.1 Implementation of Fractional Order Differentiator Operator $s^{0.5}$

The output of Fractional order differentiator is such that the order of the differentiation of input signal may be either real or complex i.e. the power of the *s* (in the Laplace domain) is arbitrary instead of integer only as in case of conventional differentiator. The fractional order differentiators or integrators are defined, in the Laplace domain, by the following transfer function

$$H(s) = s^{\alpha} \tag{3.1}$$

where *s* is the Laplace operator.

By making use of well known Regular Newton Process, Carlson and Halijak have obtained rational approximation of $1/\sqrt{s}$ as [19]

$$H(s) = \frac{s^4+36s^3+126s^2+84s+9}{9s^4+84s^3+126s^2+36s+1} \tag{3.2}$$

By approximating an irrational function with rational one, and fitting the original function in a set of logarithmically spaced points, Mastuda has obtained rational approximation of $1/\sqrt{s}$ [20]

$$H(s) = \frac{0.08549s^4+4.877s^3+20.84s^2+12.995s+1}{s^4+13s^3+20.84s^2+4.876s+0.08551} \tag{3.3}$$

Oustaloup has approximated the fractional differentiator operator s^{α} by a rational function and derived the following approximations, [21]

$$\frac{1}{\sqrt{s}} = \frac{s^5+74.97s^4+768.5s^3+1218s^2+298.5s+10}{10s^5+298.5s^4+1218s^3+768.5s^2+74.97s+1} \tag{3.4}$$

$$\sqrt{s}=\frac{10s^5+298.5s^4+1218s^3+768.5s^2+74.97s+1}{s^5+74.97s^4+768.5s^3+1218s^2+298.5s+10} \qquad (3.5)$$

3.2 Continued Fraction Expansion

It is known that the continued fraction expansion for $(1+x)^{\alpha}$ as

$$(1+x)^{\alpha}=\frac{1}{1-}\frac{\alpha x}{1+}\frac{(1+\alpha)x}{2+}\frac{(1-\alpha)x}{3+}\frac{(2+\alpha)x}{2+}\frac{(2-\alpha)x}{5+-----} \qquad (3.6)$$

The above continued fraction expansion converges in the finite complex s-plane, along the negative real axis from $x=-\infty$ to $x=-1$. Substituting $x=s-1$ and taking number of terms of equation, the calculated rational approximations for $\sqrt{s}$ are presented in Table 3.1.

Table 3.1: Rational approximations for $s^{0.5}$

S.No.	No. of terms	Rational approximation
1	2	$\frac{3s+1}{s+3}$
2	4	$\frac{5s^2+10s+1}{s^2+10s+5}$
3	6	$\frac{7s^3+35s^2+21s+1}{s^3+21s^2+35s+7}$
4	8	$\frac{9s^4+84s^3+126s^2+36s+1}{s^4+36s^3+126s^2+84s+9}$
5	10	$\frac{11s^5+165s^4+462s^3+330s^2+55s+1}{s^5+55s^4+330s^3+462s^2+165s+11}$

3.3 Rational approximations for s^{α}

Fractional order systems are systems that are described by fractional differential equations in which the integer order n of the derivative operator $D^n=\frac{d^n}{dt^n}$ is generalized to real or complex order α, such that one can define the operator [10]

$$D^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}} \tag{3.7}$$

Among existing fractional systems, we find the fractance device, $PI^{\lambda}D^{\beta}$ controller, fractional order differentiators or integrators. The fractional order differentiators or integrators are defined, in the Laplace domain, by the following transfer function

$$H(s) = s^{\alpha} \tag{3.8}$$

where s is the Laplace operator. These systems are used to calculate the fractional order time derivative and integral of an input signal. They find applications in many fields of science and engineering particularly in control and signal processing. However, such systems have unlimited memory, thus they cannot be implemented exactly.

Table 3.2: Rational approximations for s^{α} using continued fraction expansion

No. of terms	Rational Approximation
2	$\frac{(1-\alpha)+s(1+\alpha)}{(1+\alpha)+s(1-\alpha)}$
4	$\frac{(\alpha^2+3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2-3\alpha+2)}{(\alpha^2-3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2+3\alpha+2)}$
6	$\frac{(\alpha^3+6\alpha^2+11\alpha+6)s^3+(-3\alpha^3-6\alpha^2+27\alpha+54)s^2+(3\alpha^3-6\alpha^2+27\alpha+54)s+(-\alpha^3+6\alpha^2-11\alpha+6)}{(-\alpha^3+6\alpha^2-11\alpha+6)s^3+(3\alpha^3-6\alpha^2+27\alpha+54)s^2+(-3\alpha^3-6\alpha^2+27\alpha+54)s+(\alpha^3+6\alpha^2+11\alpha+6)}$
8	$\frac{p0s^4+p1s^3+p2s^2+p3s+p4}{q0s^4+q1s^3+q2s^2+q3s+q5}$ where p0= q4= α^4+10 α^3+35 α^2+50 α+24 P1= q3= -4α^4-10 α^3+40 α^2+320 α+384 P2= q2= 6α^4-1 50 α^2+864 P3= q1= -4α^4+20 α^3+40 α^2-320 α+384 P4= q0= α^4-10 α^3+35 α^2-50 α+24
10	$\frac{p0s^5+p1s^4+p2s^3+p3s^2+p4s+p5}{q0s^5+q1s^4+q2s^3+q3s^2+q4s+q5}$ where p0= q5= - $\alpha^5-15\alpha^4$-85 α^3-225 α^2-274 α-120 P1= q4= 5α^5 + 45α^4+5 α^3-1005 α^2-3250 α-3000 P2= q3= -10α^5-30α^4+410 α^3+1230 α^2-4000 α-12000 P3= q2= 10α^5-30α^4-410 α^3+1230 α^2+4000 α-12000 P4= q1= - 5α^5 + 45α^4-5 α^3-1005 α^2+3250 α-3000 P5= q0= $\alpha^5-15\alpha^4$+85 α^3-225 α^2+274

Many algorithms have been developed to best approximate the fractional order operator s^{α} with analogue or digital integer models. Fractional order elements are the building blocks for the fractional order system theory, control and signal processing. The only problem with fractional order elements is its hardware realization due to its infinite dimensional nature. In practice, factional order elements can be approximated as higher order rational transfer functions which have a constant phase curve within a certain frequency band. The fractional order elements can be rationalized as analog filters by various iterative techniques like Carlson's, Oustaloup's, Charef's and CFE (continued fraction expansion) method etc. Rational approximations for s^{α} using continued fraction expansion is shown in Table no.3.2.

3.4 Time-domain response of s^{α} operator

The Time domain response of s^{α} differentiator operator is simulated in MATLAB. The performance of the fractional order operator is checked by giving different input signal (Impulse Step, Sine, and Cosine input) as input of fractional order operator and for different values of α (0.1 to 0.9).

3.4.1 Response with Impulse input

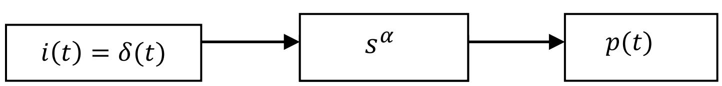


Figure.3.1 Time response of s^{α} operator when input is impulse

Let the input to the fractional order operator s^{α} ,$i(t) = \delta(t)$[10]

In the Laplace domain

$$I(S) = 1,$$

Output in the Laplace domain, $P(S)$and time domain response of above system can be written as

$$p(t) = L^{-1}(I(S) * S^{\alpha})$$

$$p(t) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \tag{3.9}$$

3.4.2 Response with Unit Step input

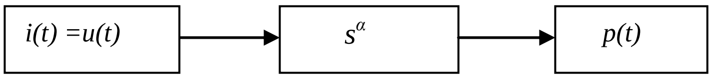


Figure.3.2 Time response of s^{α} operator when input is unit step

Let the input to the fractional order operator s^{α}, $i(t) = u(t)$

In the Laplace domain

$$I(s) = \frac{1}{s},$$

Output in the Laplace domain, $p(s)$ and time domain response of above system can be written as

$$p(t) = L^{-1}(I(s)*s^{\alpha})$$

$$p(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \quad (3.10)$$

3.4.3 Response with Sine input

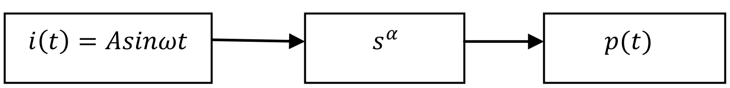


Figure.3.3 Time response of s^{α} operator when input is sine input

Let the input to the fractional order operator s^{α}, $i(t) = Asin\omega t$

In the Laplace domain

$$I(S) = \frac{\omega}{s^2+\omega^2},$$

where $A = 1$, amplitude of sine wave.

Time domain response of above system can be written

$$p(t) = L^{-1}(I(S) * S^{\alpha})$$

$$p(t) = A\omega^{*} t^{1-\alpha} E*_{2,\,2-\alpha}\,(\omega^{2} *t^{2}) \quad (3.11)$$

3.4.4 Response with Cosine input

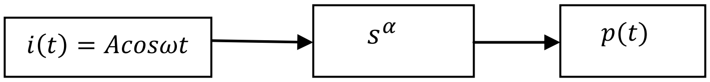


Figure.3.4 Time response of s^{α} operator when input is cosine input

Let the input to the fractional order operator s^{α}, $i(t) = Acos\omega t$

In the Laplace domain

$$I(S) = \frac{s}{s^2+\omega^2},$$

where $A = 1$, amplitude of cosine wave.

Output in the Laplace domain, *p(s)* and time domain response of above system can be written as

$$p(t) = L^{-1}(I(S) * s^{\alpha}$$

$$p(t) = A\omega^* t^{-\alpha} E*_{2,\,1-\,\alpha}(-\omega^2 *t^2) \tag{3.12}$$

CHAPTER

4

FRACTANCE DEVICE & FRACTIONAL ORDER FILTER

4.1 Fractance device

Fractance device is an electrical element which exhibits fractional order impedance properties. The impedance of the fractance device is defined as.

$$z(j\omega) = (j\omega)^{\alpha} \tag{4.1}$$

where ω is the angular frequency and α takes the values as -1, 0, 1 for capacitance, resistance, and the inductance, respectively. Fractance device finds applications in robotics, hard disk drives, signal processing circuits, fractional order control, and so forth [22].

The *Curie law* is as follows. Suppose that the voltage $v(t) = Vu(t)$ is applied to a capacitor possessing no initially stored charged. That is, there is no energy stored in the device before applying the DC voltage V. The current through the device will have the general form [7]

$$i(t) = \frac{V}{ht^{\alpha}} \text{ for t>0 and 0<α<1} \tag{4.2}$$

(h and V are real-valued). This is a power law dependence of terminal current upon the input voltage. The Laplace transform of the input voltage is

$$v(s) = \frac{V}{s} \tag{4.3}$$

However, the Laplace transform of $i(t)$is

$$i(t) = \frac{\Gamma(1-\alpha)V}{hs^{(1-\alpha)}} \tag{4.4}$$

Here Γx is the gamma function. Normally, the impedance of a two-terminal linear time-invariant (LTI) circuit element is defined to be

$$z(s) = \frac{v(s)}{i(s)} \tag{4.5}$$

For the Curie law device of (4.2) from (4.5) it can be seen

$$z(s) = \frac{h}{\Gamma(1-\alpha)} \frac{1}{s^{\alpha}} \tag{4.6}$$

In (4.6) $0 < \alpha < 1$ and so (4.6) is a considered a fractional impedance, or fractance for short.

The following are some of the important points about fractance device:

(i) The phase angle is constant with frequency but depends only on the value of fractional order, α. Hence this device is also called as constant phase angle device or simply *fractor*.

(ii) Moderate characteristics between inductor, resistor, and capacitor can be obtained using fractance device.

(iii) By making use of an operational amplifier, a fractional order differentiation and integration can be accomplished easily.

Design of fractances having given order α can be done easily using any of the rational approximations or a truncated continued fraction expansion (CFE), which also gives a rational approximation. Truncated CFE does not require any further transformation; a rational approximation based on any other methods must be first transformed to the form of a continued fraction; then the values of the electrical elements, which are necessary for building a fractance, are determined from the obtained finite continued fraction.

4.2 Fractional Order Circuits

The invention of the fractional order circuit elements concept has opened the doors to numerous applications with exceptional performance, which were not achievable for integer order standard circuit elements like inductors and capacitors. In contrast to the traditional elements, a fractional element is a function of the parameter value (C or L) and the fractional order a, which brings in immense freedom and versatility towards design and applications. The concept has already found its utility in many applications of electromagnetic, mechanics, signal processing, bioengineering, agriculture and control. We have found many interesting utilities of the fractional concept in RF and microwave domain. Initial investigations predict a paradigm shift in fundamental concepts and techniques traditionally used for these designs. Fractional systems, or more non integer order systems, can be considered as a generalization of integer order systems [Oldham (2006)], [Ross (1975)], [Sabatier (2007)], [Kilbas (2006)] and [Das (2007)].

4.3 Fractance based circuits

The six possible inverted-L type circuits using fractance device as series or as shunt element were shown in Fig. It has been observed that the transfer function H(s) can be expressed in two ways as,

1. For R–F, L–F and C–F circuits,

$$H(s) = \frac{\eta}{\eta + s^{\beta}}$$

2. For F–R, F–L and F–C circuits

$$H(s) = \frac{s^{\beta}}{\eta + s^{\beta}}$$

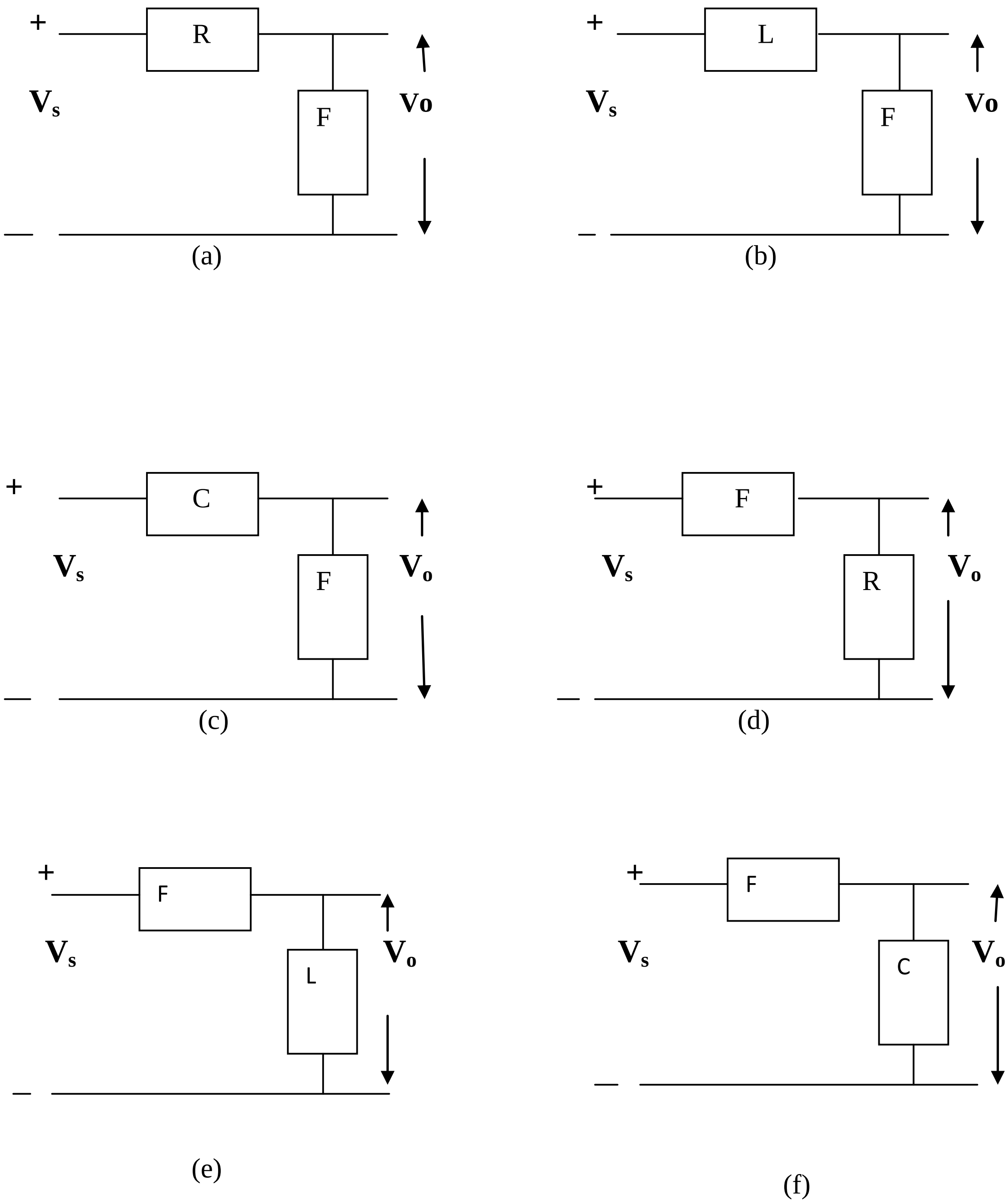
where

$$\beta = \begin{cases} \alpha & \text{For R– F } and \text{ F– R} \\ \alpha + 1 & \text{For L– F } and \text{ F– L} \\ \alpha + 1 & \text{For C– F } and \text{ F– C} \end{cases}$$

and

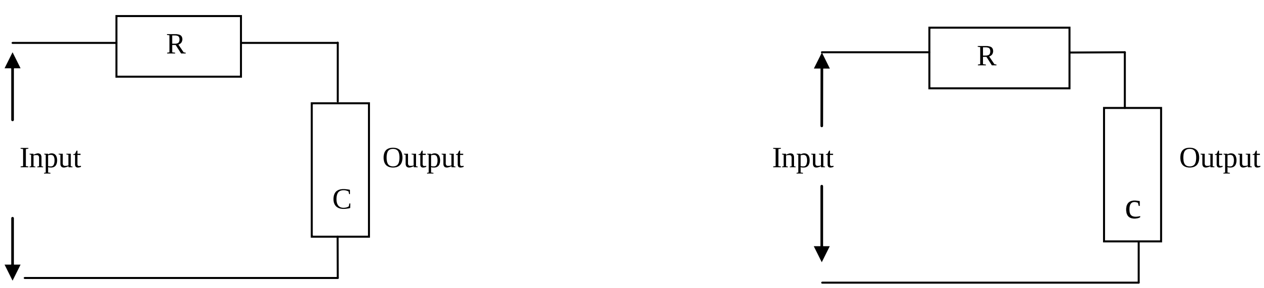
$$\eta = \begin{cases} \frac{k0}{R} & \text{For R– F } and \text{ F– R} \\ \frac{k0}{L} & \text{For L– F } and \text{ F– L} \\ ck0 & \text{For R– F } and \text{ F– R} \end{cases}$$

Figure4.1. Inverted-L type fractance based circuits. (a) R-F circuit, (b) L–F circuit, (c) C–F circuit, (d) F–R circuit, (e) F–L circuit and (f) F–C circuit.



4.4 Fractional Order Filter

Filters are electronic circuits which are traditionally classified as1st, 2nd or nth order circuits with n being an integer number. The describing transfer functions of filters are usually of the form $T(s) = \frac{N(S)}{D(S)}$ where $N(s)$and $D(s)$ are polynomials in the Laplacian operator s raised to an integer exponent, i.e.s, s^2 or s^n. However, a Laplacian of non-integer exponent s^α where $0<\alpha<1$ is mathematically valid and is representative of a fractional order system. Traditional continuous-time filters are of integer order. However, using fractional calculus, filters may also be represented by the more general fractional-order differential equations in which case integer-order filters are only a tight subset of fractional-order filters. In this work, we will show that low-pass filters can be realized with circuits incorporating a single fractance device. For designing passive or active filter, filters necessarily incorporate inductors and capacitors, the total number of inductor or capacitor dictates the filter order. However, an inductor or capacitor is not but a special case of the more general so-called fractance device; which is an electrical element whose impedance in the complex frequency domain is given by $z(j\omega) = (j\omega)^\alpha$ For the special case of $\alpha = 1$ this element represents an inductor while for $\alpha = -1$ it represents a capacitor.



Low pass First Order Filter

Fractional Order Filter were F represents Fractance device

Figure 4.2: (a) low pass filter with integer order, (b) low pass filter with fractional order

4.5 Comparison of Integer order and Fractional order filter performance

The performance of fractional order filter simulated in MATLAB is checked by using Sine, sawtooth wave, trapezoidal wave with random noise as input and resulted output compared with the output of the integer order filter with same input. And for the optimization, the performance of the fractional order filter is checked for different values of α and simulation is done in MATLAB.

CHAPTER

6

SIMULATION RESULTS

The MATLAB software is used for all simulation of fractional order operator and fractional order filter. As we have discussed in chapter 3 the only problem with fractional order elements is its hardware realization due to its infinite dimensional nature and for this factional order elements can be approximated as higher order rational transfer functions which have a constant phase curve within a certain frequency band. The fractional order elements can be rationalized as analog filters by various iterative techniques like Carlson's, Oustaloup's, Charef's and CFE (continued fraction expansion) method etc. The calculated rational approximations for √s using CFE are presented in Table1. Figures 6.1 compare the magnitude and phase responses of the rational approximations of √s with the ideal one. It is observed from Figures 6.1 that fifth-order rational approximation is best fit to ideal response up to certain range of frequencies.

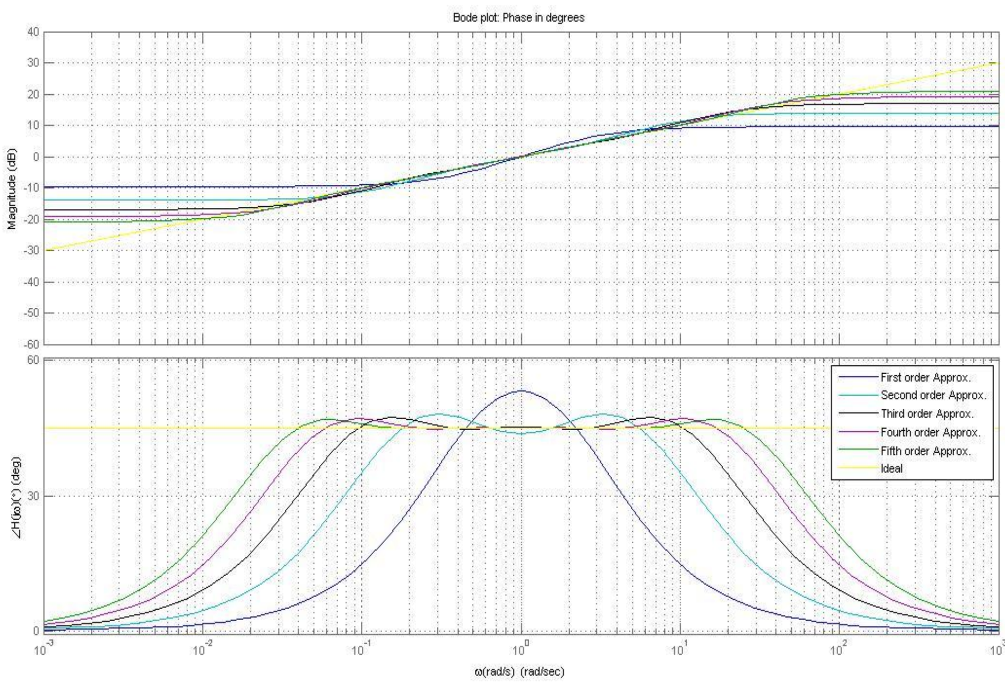


Figure6.1: Comparison of magnitude and phase responses of rational approximation functions with ideal $\sqrt{s}$

The following Figure 6.2 compare the magnitude and phase responses for $\sqrt{s}$ obtained using Oustaloup method and the CFE method with ideal one. It can be observed that the magnitude and phase responses obtained by CFE method have shown considerable improvement than compared to Oustaloup method. So, the CFE method can be used effectively for the realization like fractance device, fractional order differentiators, fractional order integrators, and so forth.

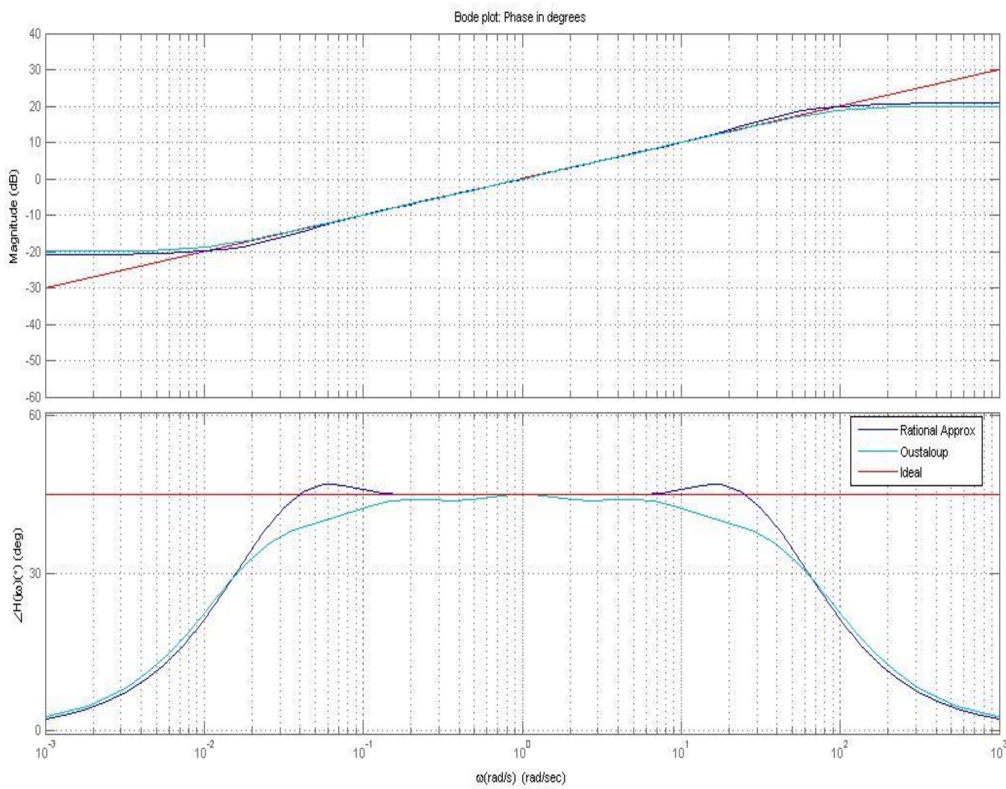


Figure 6.2: Comparison of magnitude and phase responses of rational approximation functions obtained by oustaloup and CFE method with ideal $\sqrt{s}$

The following plots from Figure 6.3 compare the magnitude and phase responses of the rational approximations of s^{α} with the ideal one for different values of α

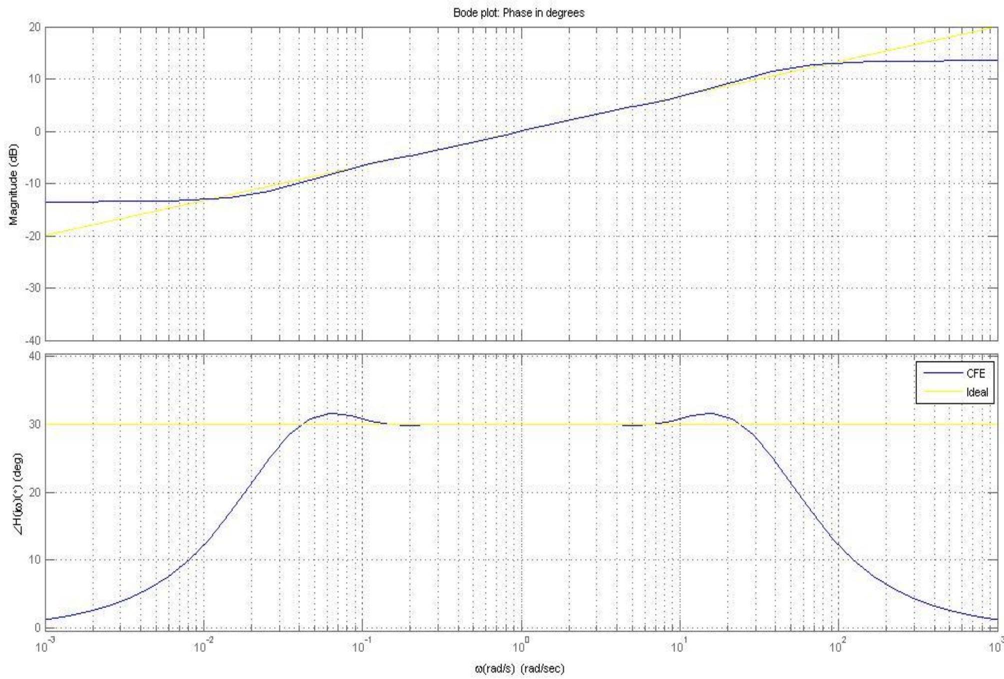


Figure 6.3(a)

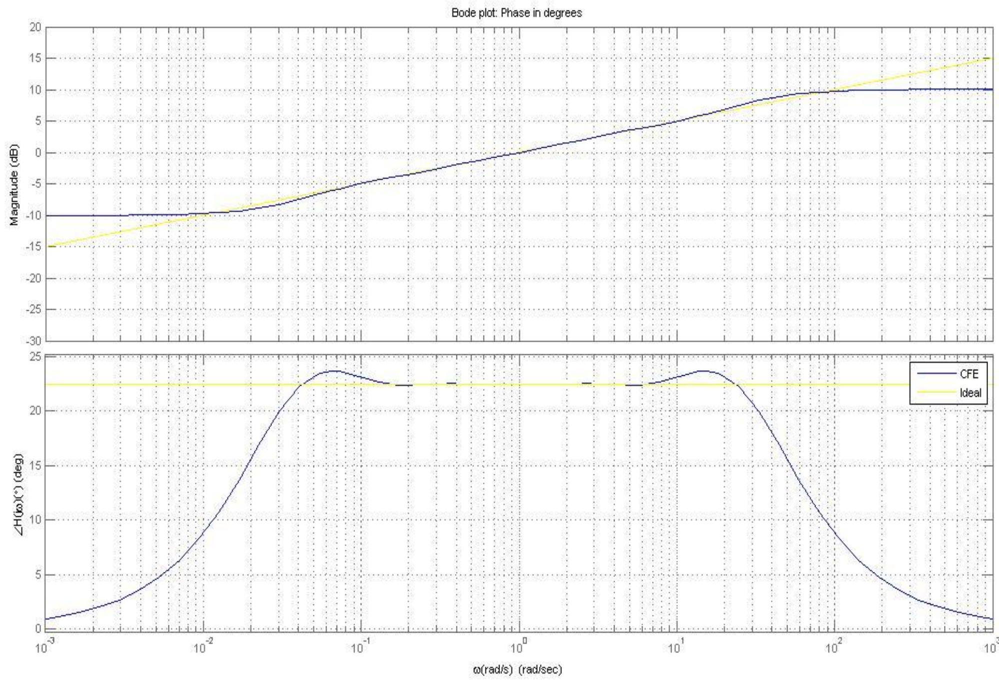


Figure 6.3 (b)

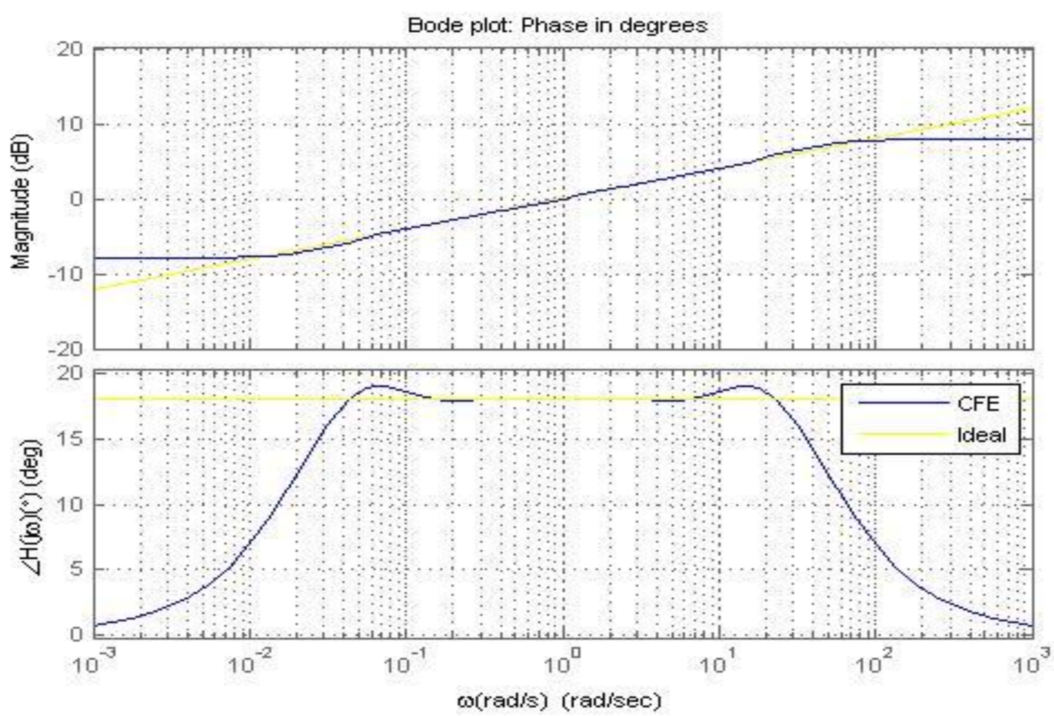


Figure 6.3 (c)

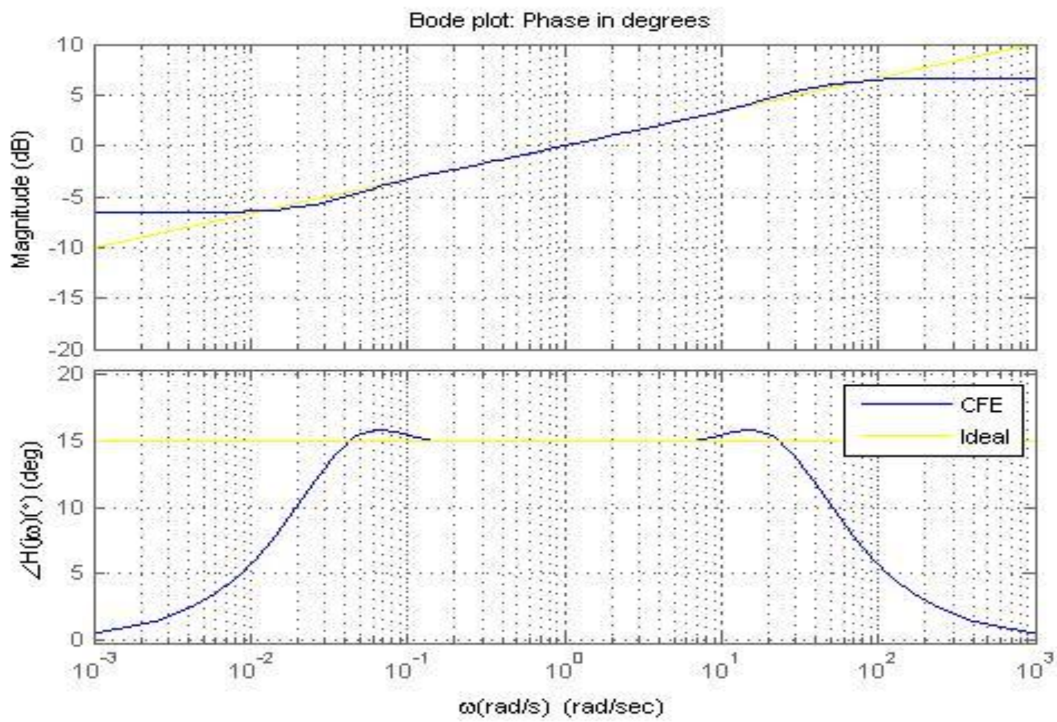


Figure 6.3 (d)

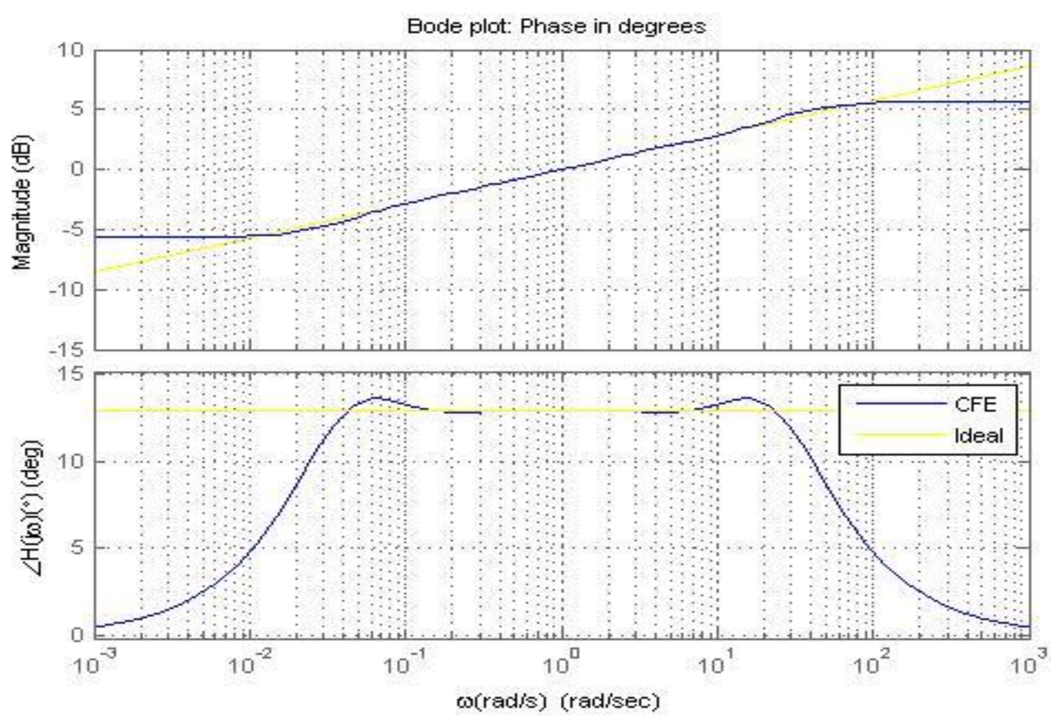


Figure 6.3 (e)

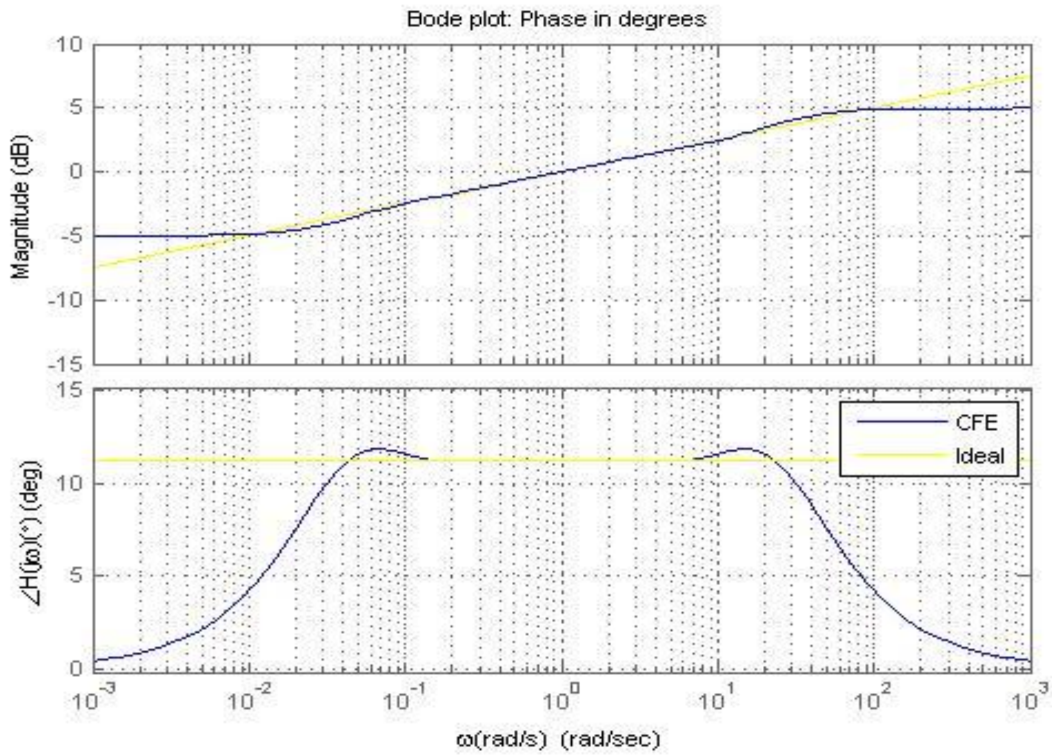


Figure 6.3 (f)

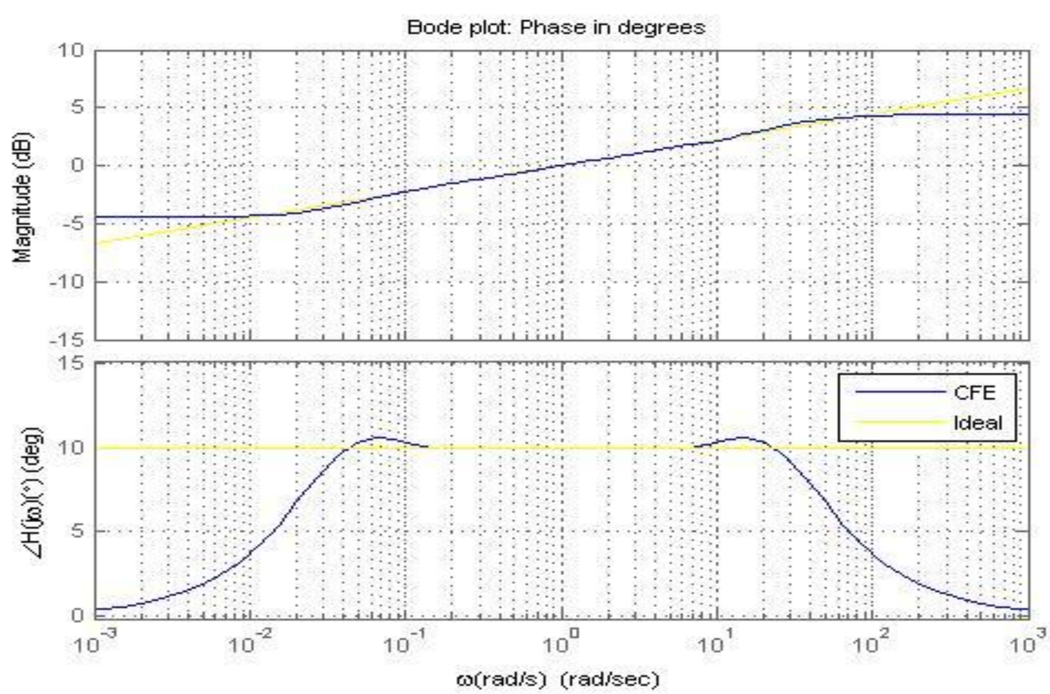


Figure 6.3 (g)

Figure 6.3: Comparison of magnitude and phase responses of rational approximation functions with ideal s^{α}, where α is order of the operator for figure (a) to (g) α = 0.2, 0.3 0.4, 0.6, 0.7, 0.8, and 0.9 respectively.

Figures 6.4 to 6.7 show the time domain responses of the s^{α} for different values of α when the input is impulse signal , unit step signal , sine and cosine wave signal respectively. And the values of α are 0.1 to 0.9 respectively.

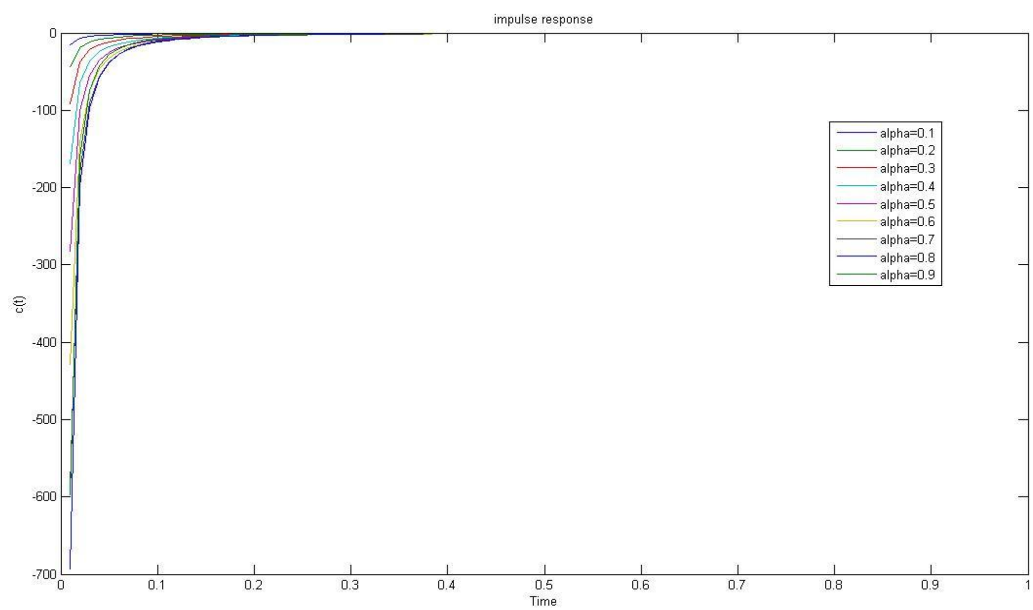


Figure 6.4 Response of the fractional order differentiator operator for impulse input and different values of α (0.1 to 0.9). (α is the order of fractional order differentiator operator)

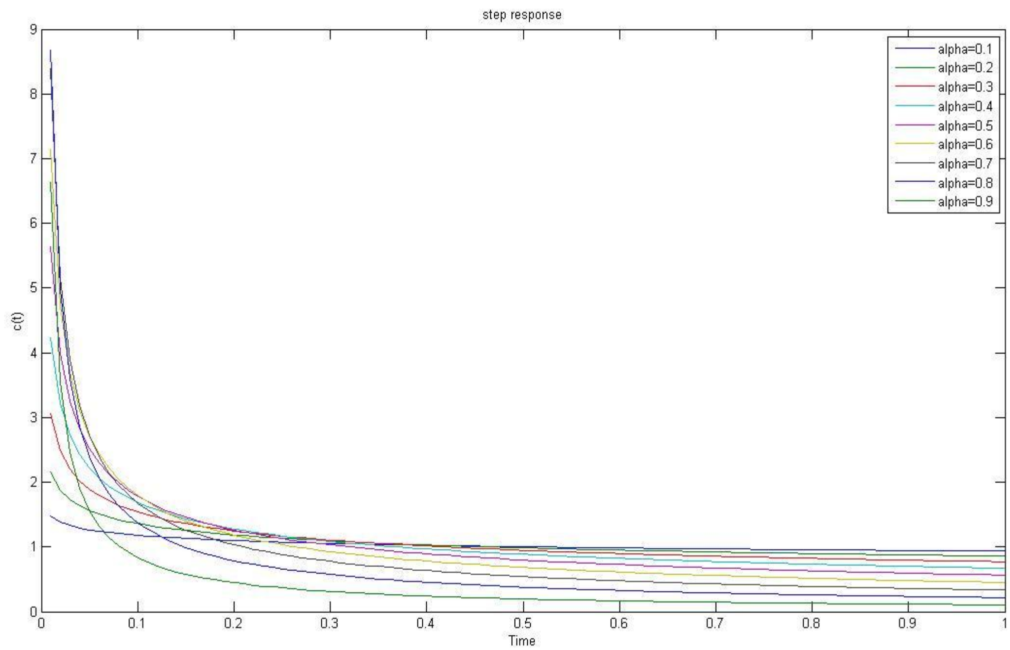


Figure 6.5 Response of the fractional order differentiator operator for step input and different values of α (0.1 to 0.9). (α is the order of Fractional order differentiator operator).

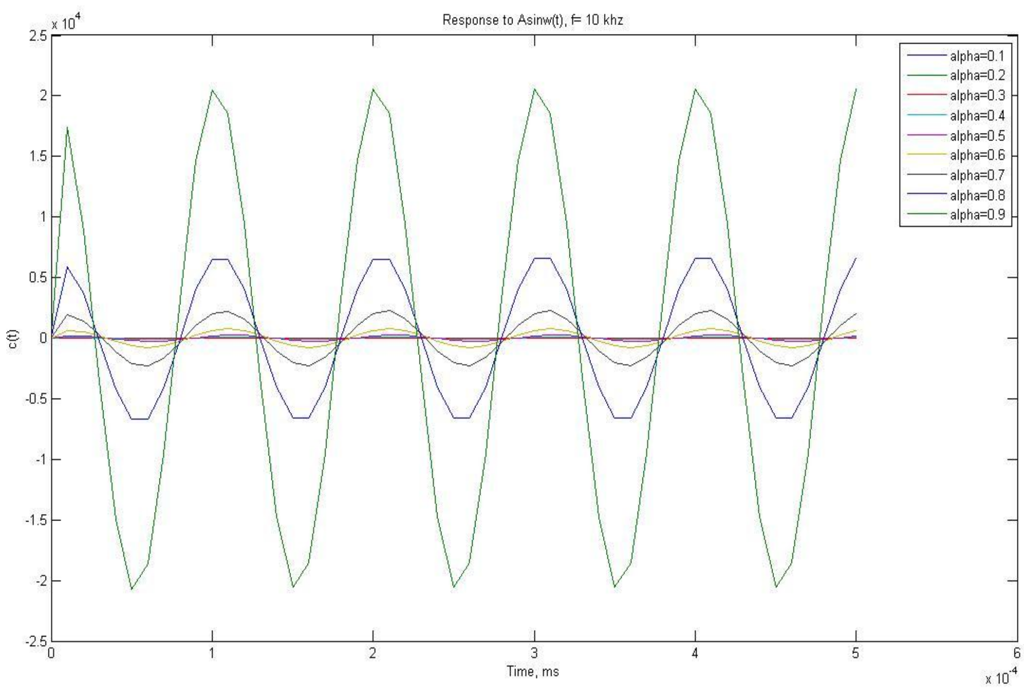


Figure6.6 Response of the fractional order differentiator operator for sine input and different values of α (0.1 to 0.9). (α is the order of fractional order differentiator operator).

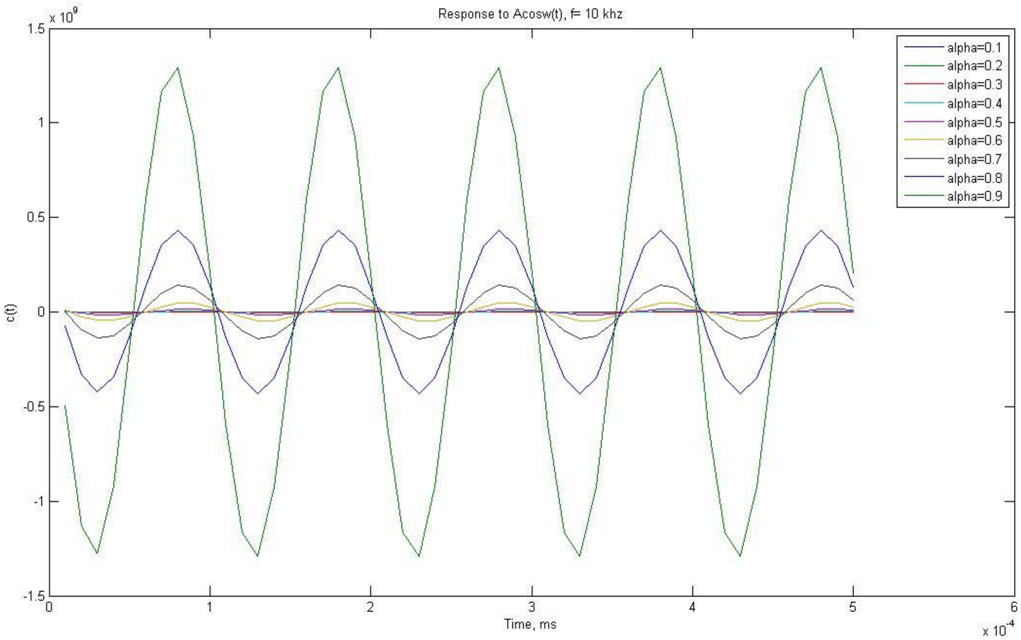


Figure 6.7 Response of the fractional order differentiator operator for cosine input and different values of α (0.1 to 0.9). (α is the order of fractional order differentiator operator).

The performance of fractional order filter simulated in MATLAB software and checked by using Sine wave, sawtooth wave 6.8 to trapezoidal wave with random noise as input and resulted output compared with the output of the integer order filter for the same input. And for the optimization, the performance of the fractional order filter is checked for different values of α and simulation is done in MATLAB. Figures from show the comparison of the response of the fractional order filter with integer order filter. From figures it is observed that the noise is much suppressed in case of fractional order filter.

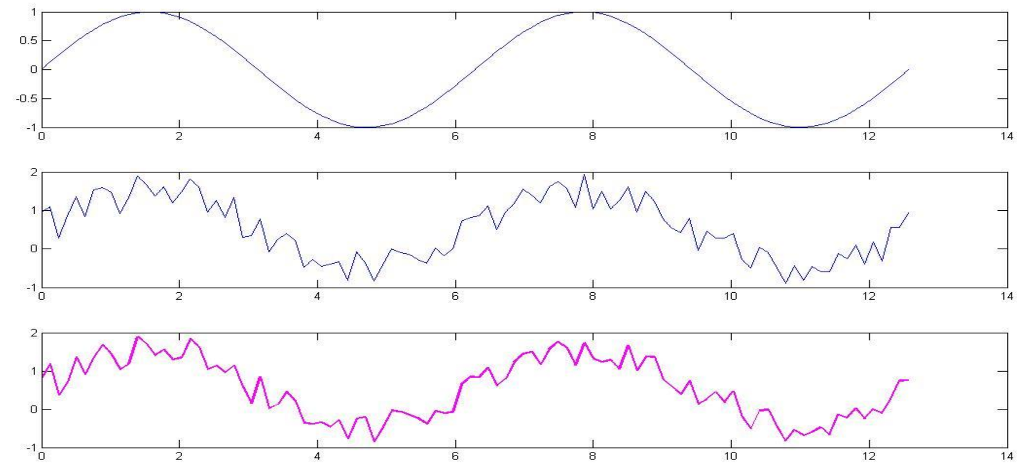


Figure 6.8 (a)

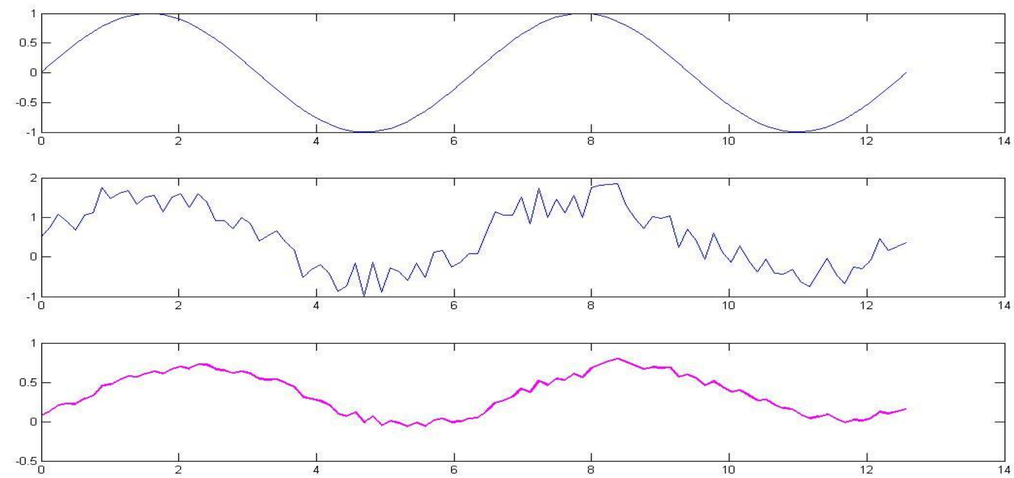


Figure 6.8 (b) α=0.55

Figure 6. 8: (a) performance of integer order filter and (b) performance of fractional order filter for sine wave with random noise as input of filter.

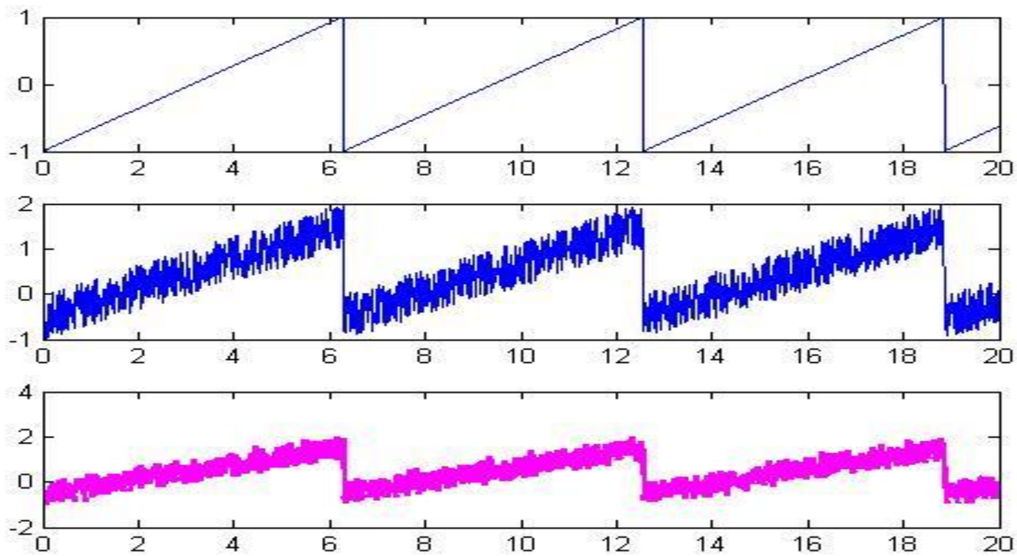


Figure6.9 (a)

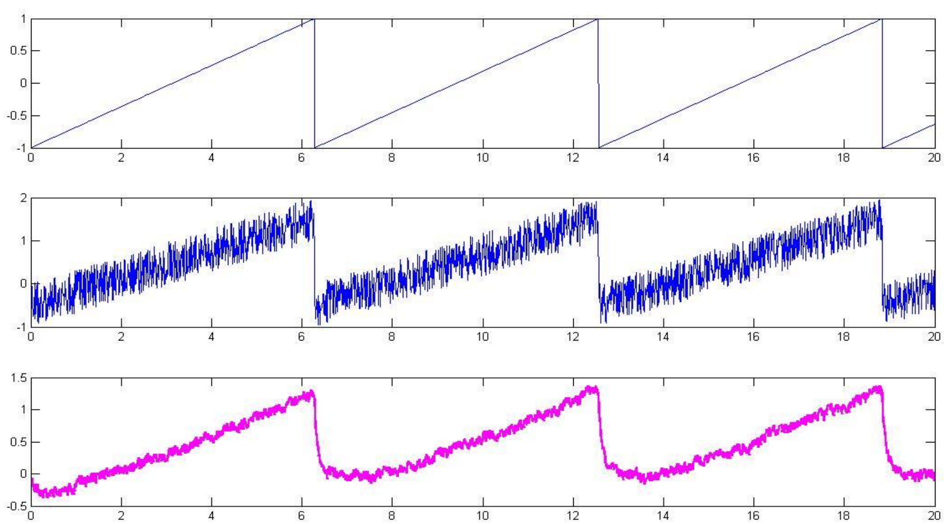


Figure 6.9 (b) α= 0.57

Figure 6.9: (a) performance of integer order filter and (b) performance of fractional order filter for saw tooth wave with random noise as input of filter.

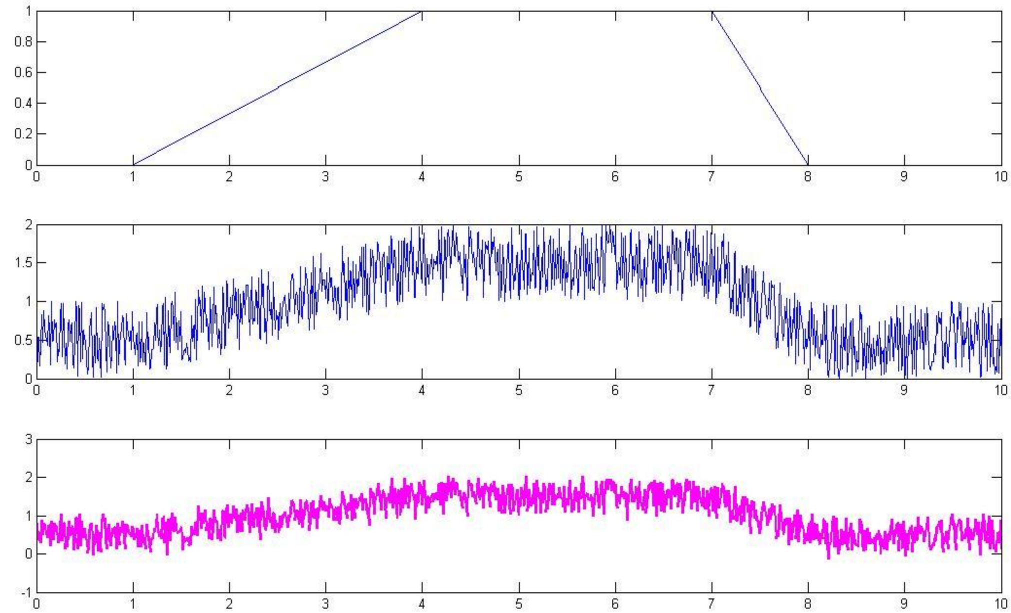


Figure 6.10 (a)

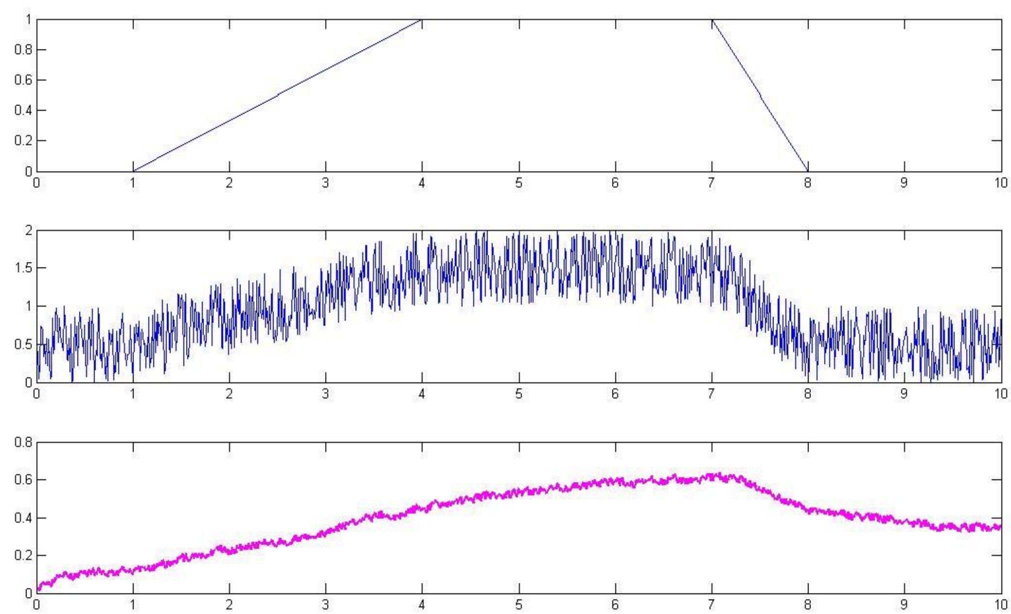


Figure 6.10 (b) α=0.44

Figure 6.10: (a) performance of integer order filter and (b) performance of fractional order filter for trapezoidal wave with random noise as input of filter.

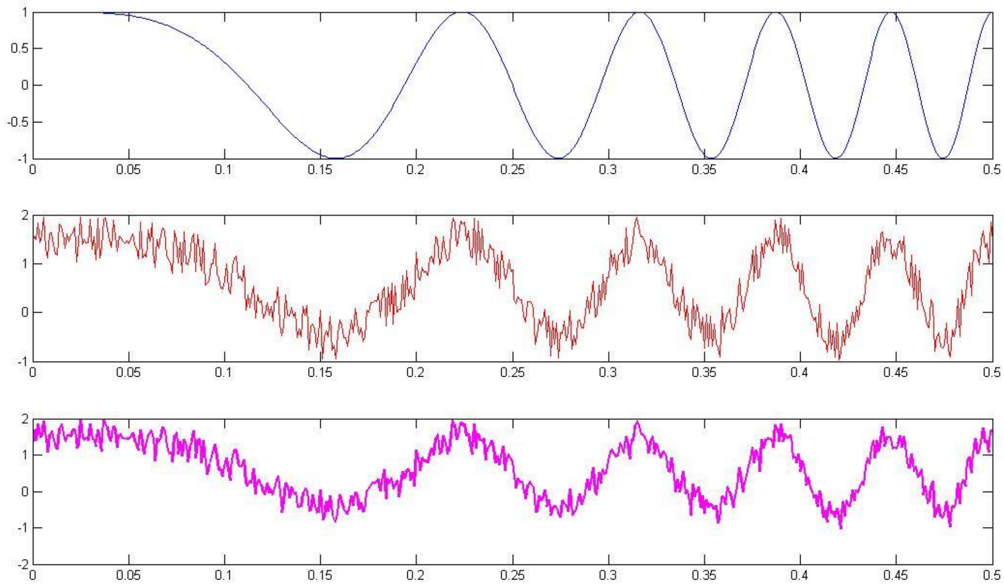


Figure 6.11 (a)

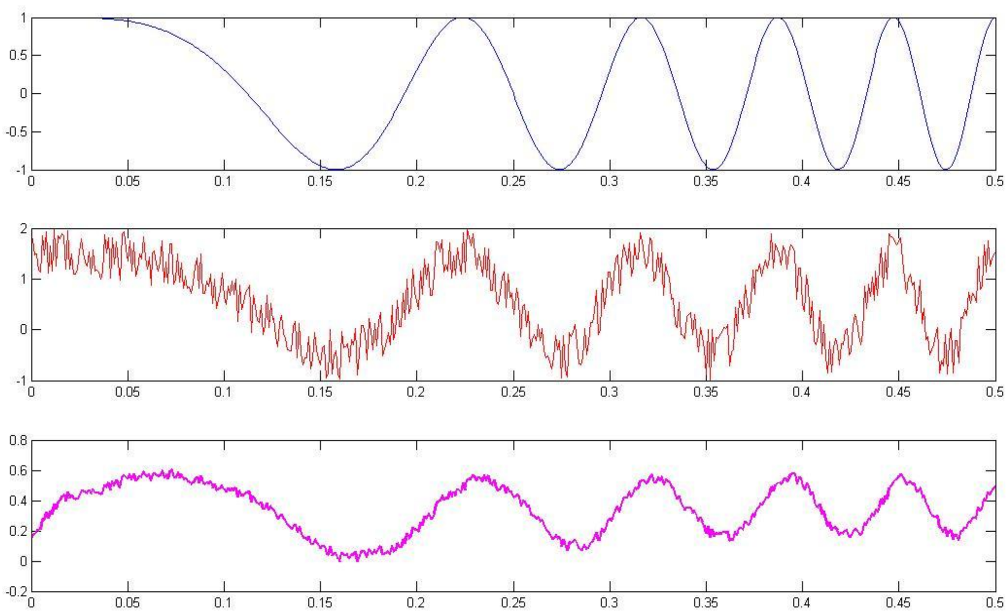


Figure 6.11 (b) α=0.47

Figure 6.11: (a) performance of integer order filter and (b) performance of fractional order filter for Chirp signal with random noise as input of filter

.CHAPTER

7

CONCLUSION AND FUTURE SCOPE

In this thesis fractional order differential operator has been simulated in MATLAB for different Input signals and different value of α (fractional order). The simulated results show that the response of the system is noticeably different for the integer and non-integer values and it is observed that for gradual change of α from 0 to 1, the fractional order system gives the gradual change in the output response. Further fractional order filter is simulated in MATLAB for different input signals (signal with noise) and for different value of α (order of the operation), further compared with the resulted outputs of the integer order filter. So, it can be concluded that fractional order filter gives the better performance in comparison of integer order filter as noise is much suppressed in case of fractional order filter as in integer order filter . For better results optimum value of α (order of operation) is taken. So it can be concluded that output of a fractional order system will give different result if it is approximated by an integer order system. It is expected that the work will help the researchers to understand fractional order system behavior in a better way.

Paper publication

Sanjay Kumar , Ritu Tanwar , “Analysis and Design of Fractional order Filter,” *International Journal of Innovative Research in Electrical , Electronics, Instrumentation and Control Engineering* , vol.1 , issue.3, pp.113-122, 2013.

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