OPTIMAL ITERATIVE FAMILY FOR SOLVING NON-LINEAR EQUATIONS

A Thesis submitted in partial fulfillment of the requirements

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submitted by

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under

the guidance of

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to the

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DEDICATED

TO

GOD, MY PARENTS

AND

SUPERVISOR

CERTIFICATE

I hereby certify that the work which is bring presented in the thesis entitled "**OPTIMAL ITERATIVE FAMILY FOR SOLVING NON-LINEAR EQUATIONS**" *in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Sanjeev Kumar.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

One of the most important and challenging problems in scientific and engineering applications is to find solutions of non-linear equations. In order to find out the approximate solutions of these equations, one has to adopt numerical techniques based on iteration procedures. Newton's method is probably the well-known iterative method for solving these equations. In recent years, many modifications of Newton's method has been proposed in literature, which have either equal or better performance than Newton's method.

The **Chapter 1** is an introductory chapter and gives a brief survey of literature. Several real world problems have been considered for which the numerical solutions are require for solving scalar non-linear equations. The fundamental concepts and classification of iterative methods and their striking features are also stated. The research work on the iterative method carried out in solving non-linear equations.

Chapter 2 presents quartically convergent families of ellipse method for the solution of scalar non-linear equation, permitting $f'(x) = 0$ near the root. Quartically convergent variant of ellipse method (QVEM) have been proposed, convergence analysis and numerical examples are also stated in this chapter.

Chapter 3 presents the basin of attraction of iterative method for solving non-linear equations. Different methods generate different basin of attraction. The different shades of these colors indicate the speed of convergence to the respective roots (in terms of number of iteration required to get very close to the root).

Contents

Glossary of Symbols

$\mathbb{R}$	a set of real numbers
I	an open interval
$\langle x_n \rangle$	the sequence
p	order of convergence
d	number of new pieces of information required per iteration
e_n	an error at n^{th} iteration
x_0	an initial guess
r	a simple root
ρ	computational order of convergence
r_m	a multiple root
m	multiplicity
a_n, b_n	parameters
a, b	arbitrary constants
α, β	real numbers
$f'(x), f''(x)$	first and second – order derivatives of the function $f(x)$ respectively
$g'(x)$	first derivatives of the function $g(x)$

Chapter 1

INTRODUCTION

Computational mathematics is an area of mathematics and computer science that creates, analyzes and implements algorithms for solving numerically, the problems of continuous mathematics. Such problems originate generally from real world applications of algebra, geometry and calculus, and they involve variables which vary continuously. Such problems are encountered throughout the study of natural sciences, social sciences, engineering, medicine and business. A large number of computational problems reduce, naturally to a problem of finding a solution of non-linear equations. Linear equations occur more frequently in numerical calculations, simply because it is difficult and sometimes impossible to solve the corresponding non-linear problem. Most of the equations governing the physical world are non-linear, but very often for simplicity and tractability linear models are considered.

One of the most important and challenging tasks in computational mathematics is that of finding efficiently the approximate solutions of non-linear equation $f(x) = 0$. Therefore, the design of iterative methods for solving the non-linear equation is very interesting and important task in numerical analysis.

When analytical or exact methods are applicable, sometimes formulas for solutions exist. However, these methods are restrictive, often providing insight into the behaviour of only a minor class of real world phenomena. Included in this category are models that can be approximated by linear relationships, simple geometry. Real world phenomena commonly exhibit non-linear relationships, complex geometry, and intricate processes. Consequently, exact methods can be of limited practical value.

1.1 Fundamental concepts

1.1.1 Root

The value of 'x' which satisfies $f(x) = 0$ is called the root of $f(x) = 0$. Sometimes we call root as zero of $f(x)$. In literature, the zero and the root formulation are used interchangeably. Geometrically, a zero of $f(x)$ is that value of x (say $x = r$), where the graph of $y = f(x)$ crosses the x-axis.

1.1.2 Intermediate value property of a continuous function

If $f(x)$ is a continuous function in the interval $[a_0, b_0]$ and $f(a_0)f(b_0) < 0$, then the equation $f(x) = 0$, has at least one root 'r' lying in the interval (a_0, b_0).

1.1.3 Osculating Curves

A curve $y(x)$ is osculating to $f(x)$ at point x_0, if it is tangent at x_0 and has same curvature there. Osculating curves therefore satisfy the following conditions

$$y(x_0) = f(x_0),\ y'(x_0) = f'(x_0) \ and \ y''(x_0) = f''(x_0). \tag{1.1}$$

1.1.4 Order of convergence

If the sequence $\langle x_n \rangle$ tends to a limits 'r' in such a way that

$$\lim_{n\to\infty} \frac{x_{n+1} - r}{(x_n - r)^p} = C, \tag{1.2}$$

for some $C \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be 'p', and 'C' is known as asymptotic error constant.

when $p = 1$, the convergence is linear, while for $p = 2$ and $p = 3$, sequence $\langle x_n \rangle$ is said to converge quadratically and cubically, respectively. The value of 'p' is called the order of convergence of the method which produces the sequence $\langle x_n \rangle$.

Let $e_n = x_n - r$. Then the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}), \tag{1.3}$$

is called an error equation for the method, 'p' being an order of convergence of the sequence.

1.1.5 Efficiency index of iterative methods

There are two types of indices that measure the informational efficiency. They are,

$$EFF = \frac{p}{d} \text{ and } {}^{*}EFF = p^{1/d},$$

where 'p' is the order of method and 'd' is the informational usage which is defined as the number of new pieces of information required per iteration.

1.1.6 Computational order of convergence

Let 'r' be a zero of the function $f(x)$ and suppose that x_{n+1}, x_n and x_{n-1} are three successive iterations closer to the root 'r'. Then the computational order of convergence 'ρ' can be approximated by using the formula

$$\rho \approx \frac{\ln|(x_{n+1} - r)/(x_n - r)|}{\ln|(x_n - r)/(x_{n-1} - r)|}.$$

1.1.7 Kung-Traub conjecture

Kung and Traub, 1947, conjectured that multipoint iterative methods without memory, requiring $n+1$ function evaluations per iteration, have an order of convergence at most 2^n.

Multipoint iterative methods which satisfy the Kung-Traub conjecture are usually called optimal methods and consequently 2^n is the optimal order. Following the Kung-Traub conjecture the optimal computational efficiency would be $2^{n/n+1}$.

1.1.8 Fixed-point iteration

A number 'r' is called fixed point for given function $g(x)$, if $g(r) = r$. This is because 'r' is unchanged when 'g' is applied to it. Root finding problems are equivalent classes in the following sence:

Given a root finding problem $f(x) = 0$, we can define function 'g' with a fixed point at 'r' as $g(x) = x - f(x)$. Conversely, if the function 'g' has a fixed point at 'r', then the function defined by $f(x) = x - g(x)$ has a zero at 'r'. Fixed-point iteration is expressed as $x_{n+1} = g(x_n)$, where 'g' is called an iteration function. The following theorem gives sufficient conditions for existence and uniqueness of a fixed point:

Theorem: Let 'r' be the root of $f(x) = 0$, in an open interval I. Let $g(x)$ and $g'(x)$ are continuous in I, where $g(x)$ is defined by $x = g(x)$, which is equivalent to $f(x) = 0$ and 'K' is positive constant for all 'x' in I.

(I) If $|g'(x)| \leq K < 1$ for all $x \in I$, then the iteration $x = g(x)$ will converge to the unique fixed point $r \in I$. In this case 'r' is said to be an attractive fixed point.

(II) If $|g'(x)| > 1$ for all $x \in I$, then the iteration $x = g(x)$ will not converge to 'r'. In this case 'r' is said to be a repelling fixed point and the iteration exhibits local divergence.

1.2 Types of numerical methods

A root finding algorithm is a numerical method for finding a value 'x' such that $f(x) = 0$, for a given function $f(x)$. These root-finding methods use iteration, producing a sequence of numbers that converge towards a limit which is a root. The first values of this sequence are initial guesses. The numerical methods compute subsequent values based on the old ones, the function values $f(x)$ and its derivatives namely $f'(x)$, $f''(x)$ etc. at these old values. The numerical methods themselves are divided into two categories: bracketing methods(e.g Bisection method, regula -falsi method etc.) and open methods (e.g Newton's method and its variants). The formers require two initial guesses that are on either side of a root. The bracketing methods have guaranteed convergence.

However, a price paid for this property is that the rate of convergence is relatively slow.

Open techniques differ from the bracketing methods in the sence that they use information at single point. Although it leads to quicker convergence, but it also includes a possibility that the solution may diverge. In general, the convergence of open techniques a partially dependent on an quality of the initial guess and the nature of the function. Closer an initial guess to a true root, more likely the methods will converge.

1.2.1 Determining initial guesses

To determining initial guesses, the following different approaches are used:

(i) **Graphical approach:** A simple method for obtaining a rough estimate of the root of equation $f(x) = 0$ is to make a plot of function and determine where it crosses the $x - axis$. This point, which represents the 'x' value for which $f(x) = 0$, provides a rough approximation of the root. Graphical methods can be utilized to obtain rough estimate of roots. Initial rough approximation for the real roots of any equation can be found from the graph of the given equation. Good guesses are usually predicated on knowledge of the physical problem setting or on the devices such a graphs, that provides insight into the behaviour of the solution. Although graphical methods are time-consuming. They provides insight into the behaviour of the function and useful in identifying initial guesses for the required root. Therefore, if time permits a quick sketch or computerized graph can yield valuable information regarding the behaviour of the function.

The problem of finding good initial guesses complicates both the bracketing and open methods, where as the open methods could be susceptible to divergence. Graphical methods could be employed to derive good guesses for the single equation case, no such simple procedure is available for the multiequation version. Although there are some advanced approaches for obtaining acceptable first estimates, often the initial guesses must be obtained on the basis of trail and error and knowledge of the physical system being modeled.

(ii) **Trial and error approach:** This technique consists of guessing a value of 'x' and evaluating whether $f(x)$ is zero. If not, another guess is made and $f(x)$ is again evaluated to determine whether the new value provides a better estimate of the root. The process is repeated untill a guess is obtained that results in an $f(x)$ that is close to zero.

(iii) **Incremental search approach:** This technique consists of starting at one end of a region of interest and then making function evaluations at small increments across the region. When the function changes sign, it is assumed that a root falls within the increment. The 'x' values in the beginning and at the end of increment can then serve as initial guesses for one of the bracketing techniques.

1.3 Classification of iterative methods

Traub has classified iterative methods into following categories:

1.3.1 One-point iterative methods without memory

If estimation of the root is determined only by using new information at one point and no old information is required, then the method is called one point iterative method without memory. Newton's method, Chebyshev's method, Halley's method, super-Halley method etc. are one-point iterative method without memory. Thus if x_{n+1} is determined by new information at x_n and no old information is used, we write

$$x_{n+1} = g(x_n),$$

then 'g' is called one-point iteration function without memory.

1.3.2 One-point iterative methods with memory

If estimation to the root is determined by using new information at one point and by using old information at either one or more than one points, then the method is called one-point iterative method with memory. The most commonly known example is Secant method. Thus if x_{n+1} is determined by new information at x_n and reused information at $x_{n-1}, \ldots\ldots\ldots\ldots, x_{n-s}$, we write

$$x_{n+1} = g(x_n; x_{n-1}, \ldots\ldots\ldots\ldots, x_{n-s}),$$

then 'g' is called one-point iterative function with memory. The semicolon separates the point at which new data are used from the points at which old data are reused.

1.3.3 Multipoint iterative methods without memory

If estimation to the root is determined only by new information at number of points, no old information is used, then method is called multipoint iterative method without memory. Thus if x_{n+1} is determine by new information at $x_n, w_1(x_n), w_2(x_n), \ldots\ldots\ldots w_i(x_n)$, $i \geq 1$, no old information is reused, we write

$$x_{n+1} = g(x_n, w_1(x_n), w_2(x_n), \ldots\ldots\ldots\ldots w_i(x_n)),$$

then 'g' is called multipoint iterative function without memory. Newton-Secant method, Traub-Ostrowski's method, Jarratt method etc. are multipoint iterative methods without memory.

1.3.4 Multipoint iterative methods with memory

If estimation to the root is determine by new information at number of points with reusing the old information at some other points, then method is called multipoint iterative method with memory. There is no well known example of multipoint iterative method with memory.

Let z_j represents $i+1$ quantities $x_j, w_1(x_j), w_2(x_j), \ldots\ldots\ldots\ldots w_i(x_j)$, $i \geq 1$. Let

$$x_{n+1} = g(z_n; z_{n-1}, \ldots\ldots\ldots, z_{n-s}).$$

Then 'g' is called multipoint iterative function with memory. The semicolon separates the point at which new data are used from the points at which old data is reused.

1.4 Survey of some iterative methods for simple roots

Finding solutions of non-linear scalar equation $f(x) = 0$, efficiently and accuratly is an age old problem. A lot of research work [1-33], has been carried out for finding the numerical solutions of these equations for simple roots. Some excellent text books such as Örtega and Rheinboldt [1], Gautschi [5], Traub [6], Ostrowski [7, 8], Conte and Boor [9], Burden and Faires [10], Bradie [11], Chapra and Canale [12], Chapra [13], Gerald and Wheatley [14], Mathews and Fink [15], Atinkson [16], Kelley [17], Ralson and Rabinowitz [18], Denis and Schnable [19] and many others [20-30] are published for a good review of most important iterative methods. In order to derive different iterative methods, many researchers have used variety of techniques namely (1) Geometric approach [6, 33], (2) Functional approach [34-36], (3) Sampling approach [6, 37, 38], (4) Composition approach, (5) Adomain approach, (6) Power-mean approach [39], (7) Weight functional approach [40, 41] and (8) Quadrature approach [32, 42-44] etc. The well-known iterative methods used for solving non-linear equations are Bisection method, Regula falsi method, Secant method, Steffensen's method, Newton's method and its variants. Bisection method and Regula falsi method have guarenteed convergence, but these two methods are linearly convergent, Secant method is superlinearly convergent, where as Newton's method is quadratically convergent. Among all these iterative methods, Newton's method is widely used for the solution of these equations, due to its simplicity, flexibility and geometrical construction and is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.4}$$

This method is quadratically convergent to simple roots and linear to multiple roots. But this method has two well-known drawbacks. The first drawback of the Newton's method is its dependence on a "sufficiently close" initial guess. There are many non-linear equations found in a numerical text book in which the Newton's method fails due to poor initial guess. Secondly, Newton's method will fail misrably if at any stage of computation, the derivative of the function is zero or small in the vicinity of the required root. Moreover, in some cases, Newton's method shows slow or oscillatory convergence. Time to time, Newton's method has been derived in several ways and modified in a variety of ways. One such method derived from Newton's method by approximating the derivative with non-derivative term of difference quotient

$$\frac{f(x+f(x))-f(x)}{f(x)}, \tag{1.5}$$

is Steffensen's method [7]. This method require two fuction evaluations only and avoids evaluation of derivative of $f(x)$. It is quadratically convergent to a simple root. Another such modification is given by Mamta et al. [45] by considering an auxillary function (Taylor's interpolating polynomial)

$$g(x) = \alpha^2(x-x_0)^2\{f(x)\}^2 - \{f(x)\}^2, \tag{1.6}$$

where 'x_0' is an initial guess to the required root 'r'. The advantage of their method is that it can be applied even in cases when $f'(x)$ vanishes at some intermediate steps. Several other modifications of Newton's method are also developed. There also exits several methods that can accelerate the order of convergence from quadratic to cubic and even to higher order. Some of the well-known higher order methods are Chebyshev's method [6, 34, 35, 46], Halley's method, [6, 34, 35] super-Halley method [47, 48], Euler's method [6, 47, 49], Chebyshev-Halley type method [50], Ostrowski's square-root method [6, 7], and many others (for a good review see Traub[6]). These methods are of third-order and successfully applied to find the simple roots of non-linear equations. Lambert [51] used a third-order method for solving stiff system of equations which require quick convergence.

The above mentioned well-known third-order methods which entail the evaluation of second-order derivatives of $f(x)$ can be obtained by admitting geometric derivative from the quadratic curve, e.g parabola and hyperbola. In this category, the Euler's method or irrational Halley's method [6, 47, 49] is constructed by considering the parabola

$$x^2 + ax + by + c = 0, \tag{1.7}$$

and imposing the tangency conditions

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n)\ and\ y''(x_n) = f''(x_n). \tag{1.8}$$

The Chebyshev's method [6, 34, 46] admits the geometric construction from parabola of the form

$$ay^2 + y + bx + c = 0. \tag{1.9}$$

If we consider the hyperbola of the form

$$axy + y + bx + c = 0, \tag{1.10}$$

then using above tangency conditions, we obtain well-known Halley's method or the method of tangent hyperbolas which shares with Secant method the distinction of being most frequently rediscovered methods in literature as pointed out by Traub [6, pp. 91]. This assertion is confirmed by number of publications, as for example, Melman [51]. Convergence analysis of Halley's method can be found in Brown [52], Alefeld [53], Hernández [54] etc.

Amat et al. [47] studied, another type of third-order iterative methods which are obtained by considering the hyperbolas in the following form:

$$a_n y^2 + b_n xy + y + c_n x + d_n = 0, \tag{1.11}$$

and involve a parameter 'b_n' depending on each iterative step 'n'. For particular values of 'b_n', some interesting particular cases of this family are

(i) For $b_n = 0$, the formula obtained from (1.11) corresponds to classical Chebyshev's method [6, 46].

(ii) For $b_n = -\frac{f''(x_n)}{2f'(x_n)}$, the formula obtained from (1.11) corresponds to famous Halley's method [6, 34, 35, 55].

(iii) For $b_n = -\frac{f''(x_n)}{f'(x_n)}$, the formula(1.11) corresponds to super-Halley's method [35, 55].

(iv) Finally, as a limiting case, when $b_n \to \pm\infty$, the formula obtained from corresponds to Newton's method [1,5-22].

Further, Kanwar et al. [36] has presented a new algorithm by considering an osculating circle in the following form:

$$(x - x_n)^2 + \{y - f(x_n)\}^2 + 2a(x - x_n) + 2b\{y - f(x_n)\} = 0, \tag{1.12}$$

where 'a' and 'b' are arbitrary constants.

Similarly, Sharma [35] has proposed a new family of iterative methods by considering a general quadratic equation

$$x^2 + a_n y^2 + b_n x + c_n y + d_n = 0, \tag{1.13}$$

and involve a parameter 'a_n' at every iterative step 'n'.

For particular values of 'a_n', some interesting particular cases of this family are

(i) For $a_n = 0$, the formula obtained from (1.13) corresponds to Euler's method [6, 47, 49].

(ii) For $a_n = \frac{2}{\{f'(x_n)\}^2(2-L(x_n))}$, the formula obtained from (1.13) corresponds to famous Halley's method [6, 56].

(iii) For $a_n = \frac{L(x_n)}{4\{f'(x_n)\}^2(1-L(x_n))}$, the formula obtained from (1.13) corresponds to super-Halley method [34].

Where

$$L(x_n) = \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2}. \tag{1.14}$$

(iv) As a limiting case, when $a_n \to \pm\infty$, the formula obtained from (1.13) corresponds to classical Chebyshev's method [6, 46, 47].

(v) The family of third-order methods studied by Amat et al. [56] is obtained by taking

$$a_n = \frac{1}{b_n f(x_n)f'(x_n)}\left[1-\left(1+b_n\frac{f(x_n)}{f'(x_n)}\right)\left(1+\frac{L(x_n)}{2\left(1+b_n\frac{f(x_n)}{f'(x_n)}\right)}\right)^2\right],$$

where $b_n \in \mathbb{R}$.

(vi) Finally, as an exceptional case, for $a_n = -\frac{1}{\{f'(x_n)\}^2}$, the formula obtained from (1.13) corresponds to the well-known Newton's method [1, 5-22].

All the methods mentioned above are one-point iterative methods. On the other hand, multipoint iterative methods use the information at the number of points. A very restrictive condition of one-point iterative method of order p, is that they depend explicitly on first $p-1$ derivatives of $f(x)$. This implies that their informational efficiency is less than one or equal to unity. Neither these restrictions need to hold for multipoint iterative methods, that is, for iteration which sample $f(x)$ and its derivatives at a variable number of values of the independent variable. There are many multipoint higher-order iterative methods used frequently to solve non-linear equations [6, 41, 42, 57, 58]. Some of these iterative methods are of third-order

[6, 59, 60], while others are of order four [6, 43, 52]. A third-order method which uses only derivatives upto first-order is studied extensively by Traub [6] and Ezquerro et al. [61].

Let us consider iterative methods which require more derivative evaluations than functions. Many iterative methods developed by considering various quadrature rules in Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{1.15}$$

Weerakoon and Fernando [32] obtained the following cubically convergent iterative method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}, \tag{1.16}$$

by using the trapezoidal rule in equation (1.15).

While Frontini and Sormani [59] considered the midpoint quadrature rule in (1.15) and Homeier [19] considered properties of vanishing derivative to obatin the following cubically convergent method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\right)}. \tag{1.17}$$

Furthermore, Homeier [42] derived the following cubically convergent iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}\right), \tag{1.18}$$

by using Newton's theorem (1.15) for the inverse function $x = f(y)$ instead of $y = f(x)$.

In [43] Kou et al. considered Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) - f(x_n)}{f'(x_n)}. \tag{1.19}$$

Recently, Kanwar [44] has generalized the trapezoidal method given in equation (1.16) in order to get a family of trapezoidal-type methods given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)}\right)}. \tag{1.20}$$

Here, the parameter $\alpha \in \mathbb{R}$. In order to obtain cubic convergence of the method, the sign of entity 'α' in the denominator should be chosen so the denominator is largest in magnitude.

All third-order multipoint iterative methods mentioned above require three function evaluations per step. Therefore, they are not optimal in the sence of the Kung-Traub conjecture [58]. The first optimal multipoint iterative method of order four was costructed by Ostrowski [7], several years before Traub's extensive investigation in this area and is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)(y_n - x_n)}{2f(y_n) - f(x_n)}. \end{aligned}\right\}. \tag{1.21}$$

A further generalization of Ostrowski's (also known as Traub-Ostrowski's method) method (1.21) was proposed by King [57] in the following form

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}. \end{aligned}\right\}, \tag{1.22}$$

where 'β' is a disposable parameter. King's family (1.22) is optimal and has fourth order of convergence. In particular, Ostrowski's method [17], Kou's method [62] and Chun's method are the members of this family, when $\beta = 0$, $\beta = 1$ and $\beta = 2$ respectively.

Recently, fourth-order methods proposed by Nedzhibov et al. [63], Petković and Petković (a survey) [64] etc. have optimal order of convergence [45]. The efficiency index [7] for these optimal multipoint iterative methods is $\sqrt[3]{4} \approx 1.587$. These multipoint methods calculate new approximations to a zero of $f(x)$ by sampling $f(x)$ and possibly its derivatives for the number of values of the independent variable, at each step. Nedzhibov et al. [65] has modified Chebyshev-Halley methods [50] to derive several cubic and quartic order convergent multipoint iterative methods free from second-order derivative. Further, Wang and Li [66] has derived a new family of third-order multipoint iterative methods free from first-order derivative for solving non-linear equations numerically. In recent years, many new three-step modifications of Traub-Ostrowski's method [6, 8] and King's family [57] has been reported in the literature. Many of these three-step modifications are of optimal eight order of convergence with efficiency index $\sqrt[4]{8} \approx 1.6818$.

Since all these methods are close variants of classical Newton's method. Therefore will have same defects as Newton's method. **Motivating from this fact in this work author will try to eliminate these defects by constructing alternative algorithms.**

Chapter 2

Optimal quartically convergent families of ellipse methods

In this chapter quartically convergent variants of ellipse method for the solution of scalar non-linear equation, permitting $f'(x) = 0$ near the root has been presented. The methods obtained by combining the quadratically convergent ellipse method with the false position method. Per iteration the new method requires two evaluations of the function and one evaluation of its first order derivative and therefore, has the efficiency index equal to 1.587 i.e optimal in sense of Kung-traub conjecture. Several examples are given to demonstrate the efficiency and performance of the modified ellipse method. Further, numerical experiments demonstrate that the quartically convergent variants of ellipse method outperform the classical Newton's and other variants of Newton's method.

Newton's method [67-69] which converges quadratically to a simple root of $f(x) = 0$, is probably the best known and most widely used algorithm. It is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.1}$$

However, the problem with the Newton's method is that it may fails to converge in some cases if the derivative of the function is very small or zero i.e ($f'(x) \approx 0$) in the vicinity of the required root. There is a class of third-order methods requiring the evaluations of second order derivative such as Halley's irrational method [67-69], Halley's rational method [67-69], Chebyshev's method [67-69] etc and are close relatives of Newton's method. The main practical difficulty associated with these methods is the evaluations of second order derivative. Many researchers developed modifications of Newton's method or Newton-like methods in a number of ways to improve the local order of convergence of Newton's method at the expense of additional evaluations of function and or derivatives mostly at the point iterated by the method. All these modifications are targeted at increasing the local order of convergence with a view of increasing their efficiency index. But as already discussed all these methods are variants of Newton's method and will fail miserably if $f'(x)$ is very small or zero in the vicinity of the root. Therefore, the researches of developing higher-order methods which are suitable in the problem where $f'(x) = 0$ is permitted and free from second order derivative are important for practical applications.

Recently, Gupta et al. [70] have developed a family of ellipse methods given by

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n) + p^2 f^2(x_n)}}, \tag{2.2}$$

where $p \in \Re - \{0\}$ and in which $f'(x) = 0$ is permitted near the root. **The beauty of this method is that it converges quadratically and moreover, has the same error equation as Newton's method. Therefore, this method is an efficient alternative to Newton's method.**

In this chapter, A quartically convergent family of ellipse method based on ellipse method (2.2) and linearly convergent false position method has been presented. Per iteration the method requires two evaluations of the function and one evaluation of its first order derivative. Therefore, this family is very efficient because of the improvement of its local order convergence.

2.1 Development of the method

Assume the equation

$$f(x) = 0, \tag{2.3}$$

has a simple root r which is to be found and let $x_0 = r + e_0$ be our initial approximation to this root. Let $y = f(x)$ represents the graph of the function $f(x)$.

Further, let

$$y_0 = f(x_0), \tag{2.4}$$

and

$$x_1 = x_0 + \frac{f(x_0)}{\sqrt{f'^2(x_0) + p^2 f^2(x_0)}}, \tag{2.5}$$

be the quadratically convergent ellipse method [70].

Let $y_1 = f(x_1)$. If, on the same graph of

$$y = f(x), \tag{2.6}$$

we draw a straight line through the points (x_1, y_1) and $\{(\frac{x_0+x_1}{2}), \frac{f(x_0)}{2}\}$, then equation of this straight is given by

$$y - y_1 = \frac{y_0 - 2y_1}{x_0 - x_1}(x - x_1). \tag{2.7}$$

If initial guess x_0 is sufficiently close to the required root r, then the intersection of a line (2.7) and the $x - axis$ gives a good approximation to the root. Therefore, the first approximation to the root is given by

$$x_2 = x_1 - \frac{(x_0 - x_1)}{y_0 - 2y_1} y_1. \tag{2.8}$$

This is a quartically convergent family of ellipse methods. In general we see that

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n) + p^2 f^2(x_n)}} \left\{ \frac{f(x_n) - f\left(x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right)}{f(x_n) - 2f\left(x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right)} \right\}, \tag{2.9}$$

where the positive sign is taken if $x_0 < r$ and negative sign is taken if $x_0 > r$.

It is interesting to note that by ignoring the term in p, equation (2.9) gives the famous quartically convergent variants of Newton's method namely Traub-Ostrowski's formula [67-69].

For the formula (2.9), we prove the following convergence theorem:

2.1.1 Convergence analysis

Theorem: Suppose $f(x)$ is sufficiently differentiable function in a neighborhood of a simple root r and that x_0 is close to r, then the iteration scheme (2.9) has fourth-order convergence and satisfies the following error equation:

$$e_{n+1} = \left\{ A_2^3 - A_2 A_3 - \frac{1}{2} p^2 A_2 \right\} e_n^4 + O(e_n^5). \tag{2.10}$$

Proof: Since $f(x)$ is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ about $x=r$ by Taylor's expansion, we have

$$f(x_n)=f'(r)[e_n+A_2e_n^2+A_3e_n^3+A_4e_n^4+A_5e_n^5+O(e_n^6)], \tag{2.11}$$

and

$$f'(x_n)=f'(r)[1+2A_2e_n+3A_3e_n^2+4A_4e_n^3+5A_5e_n^4+O(e_n^5)], \tag{2.12}$$

where $e_n=|x_n-r|$ and $A_k=\frac{1}{k!}\frac{f^k(r)}{f'(r)}$, $k=2,3,\ldots$.

Using (2.11) and (2.12), we have

$$\frac{f(x_n)}{f'(x_n)}=e_n-A_2e_n^2-2(A_3-A_2^2)e_n^3-(3A_4-7A_2A_3+4A_2^3)e_n^4+O(e_n^5). \tag{2.13}$$

Therefore,

$$\begin{aligned}\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}&=\frac{f(x_n)}{f'(x_n)\sqrt{1+p^2\left\{\frac{f(x_n)}{f'(x_n)}\right\}^2}},\\ &=e_n-A_2e_n^2-\left\{\frac{1}{2}p^2-2(A_2^2-A_3)\right\}e_n^3\\ &+\left\{\frac{3}{2}p^2A_2+7A_2A_3-4A_2^3-3A_4\right\}e_n^4+O(e_n^5).\end{aligned} \tag{2.14}$$

Furthermore,

$$\begin{aligned}&f\left(x_n\pm\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}\right)\\ &=f'(r)\left[A_2e_n^2+\left\{\frac{1}{2}p^2-2(A_2^2-A_3)\right\}e_n^3-\left\{\frac{3}{2}p^2A_2+7A_2A_3-5A_2^3-3A_4\right\}e_n^4+O(e_n^5)\right].\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}&f(x_n)-2f\left(x_n\pm\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}\right)\\ &=f'(r)\left[e_n-A_2e_n^2+\left\{4A_2^2-3A_3-p^2\right\}e_n^3+\left\{3p^2A_2-10A_2^3-5A_4+14A_2A_3\right\}+O(e_n^5)\right].\end{aligned} \tag{2.16}$$

Using (2.15) and (2.16), we have

$$\begin{aligned}\frac{f(x_n)-f\left(x_n\pm\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}\right)}{f(x_n)-2f\left(x_n\pm\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}\right)}=&1+A_2e_n+\left\{\frac{1}{2}p^2-A_2^2+2A_3\right\}e_n^2\\ &+\left\{3A_4-2A_2A_3\right\}e_n^3+O(e_n^4).\end{aligned} \tag{2.17}$$

Using (2.14) and (2.17) in (2.9), we get the error equation as

$$e_{n+1} = \left\{ A_2^3 - A_2A_3 - \frac{1}{2}p^2A_2 \right\} e_n^4 + O(e_n^5). \tag{2.18}$$

This completes the proof of the theorem.

2.1.2 Numerical examples

We employ the quartically convergent variants of ellipse method (2.9) (QVEM) to solve some non-linear equations and compare this with Newton's method (NM), Halley's method (HM), Chebyshev's method (CM) and Traub-Ostrowski's method (TOM). The formula (2.9) is tested for $p = \frac{1}{2}$ and the results are summarized in Table 2. we use $\in = 10^{-15}$ as tolerance. Computations have been performed using C^{++} in double precision arithmetic.

The following stopping criteria are used for computer programs:

$(1)|x_{n+1} - x_n| < \in$,

$(2)|f(x_{n+1})| < \in$.

Table 1:

Test function

$f(x)$	$Root(r)$
$f_1 = arctan(x)$	0.0000000000000000
$f_2 = e^{x^2+7x-30} - 1$	3.0000000000000000
$f_3 = (x-1)^6 - 1$	2.0000000000000000
$f_4 = x^3 + 4x^2 - 10$	1.365229964256287
$f_5 = cos(x) - x$	0.739085137844086
$f_6 = ln(x)$	1.0000000000000000

Table 2:

Performance of the method(Number of iterations required)

$f(x)$	x_0	NM	HM	CM	TOM	$OVEM$
f_1	-2.0	*Divergent*	4	*Divergent*	5	3
	2.0	*Divergent*	4	*Divergent*	5	3
f_2	2.0	*Divergent*	8	*Divergent*	*Divergent*	2
	2.5	*Divergent*	5	*Divergent*	*Divergent*	6
	2.8	15	4	*Divergent*	5	4
	3.5	11	6	7	5	5
f_3	1.1	58	9	89	25	3
	3.0	8	4	5	4	4
f_4	0.0	*Fails*	*Fails*	*Fails*	*Fails*	3
	0.1	9	5	74	4	2
	2.0	4	3	3	2	2
f_5	-1.0	7	5	*Divergent*	9	3
	2.0	5	3	3	2	3
f_6	3.0	*Divergent*	3	4	*Divergent*	3

2.1.3 Conclusions

The family of fourth-order variants of ellipse method is the main findings of the present work. The presented results in Table 2 indicate that the presented quartically convergent variants of ellipse method improve the computational efficiency of the ellipse method. Moreover, these methods, do not require the evaluation of second order derivative and do not fail even if the derivative of the function is either zero or very small in the vicinity of the root. Moreover, per iteration the new method requires two evaluations of the function and one evaluation of its first order derivative and therefore, has the efficiency index equal to 1.587 i.e optimal in sense of Kung-traub conjecture. Several examples are given to demonstrate the efficiency and performance of the quartically convergent variant of ellipse method (QVEM). Further, numerical experiments demonstrate that the quartically convergent variants of ellipse method outperform the classical Newton's and other variants of Newton's method. Finally, we conclude that the methods presented in this thesis are competitive with other well-known methods, namely, Newton's, Halley's, Chebyshev's, Traub-ostrowski's method etc. and have efficiency index equal to $\sqrt[3]{4} \cong 1.587$ which is better than the one of Newton's method $\sqrt[2]{2} \cong 1414$ and Halley's method $\sqrt[3]{3} \cong 1.442$.

Chapter 3

Basin of attraction

A basin of attraction refers to the set of initial conditions leading to long-time behaviour that approaches the attractor of dynamical system. Interestingly, along the basin boundaries there are nodules that have complex properties. Inside a nodule, the method again becomes regular, i.e. the method converges nicely to a root. Basin boundaries can take on infinitely many shapes. And basin boundaries can be far more complicated than a simple curve, and in most instances are.

A useful insight into the possibilities here can be gained by considering the concept of basin of attraction. Let us start by thinking of bowl containing a ball bearing. This will move around the bowl until eventually it comes to rest at the lowest point. We can say that it can attracted to that point, so each part of bowl can be regarded as leading to that stationary point, and the whole bowl is what we call the basin of attraction of that system. If we place the ball bearing outside the bowl then it will have a tendency to go somewhere else, so we can see that an attractor is only effective within a certain area of space, and we can have many different attractors adjoining each other.

It's pretty easy to write a program that colors Newton basins. Just decide how much of the complex plane to draw, and for each pixel in the image, iterate Newton's method on the corresponding complex number and see that happens. If some iterate gets close enough to a root, then you know which basin it belongs in. If after some number of times it hasn't gotten close to root, then give up and go on to the next pixel. You can tell the computer what the maximum number of iterations it should consider. (If the maximum is too small, there will be a lot of uncolored pixels, and if it's too big it will take the computer too long a time to create the image).

An attractor of a dynamical system is a subset of the state space to which orbits originating from typical initial conditions tend as time increases. It is very common for dynamical systems to have more than one attractor. For each such attractor, its basin of attraction is the set of initial conditions leading to long-time behavior that approaches that attractor. Thus the qualitative behaviour of the long-time motion of a given system can be fundamentally different depending on which basin of attraction the initial condition lies in. Regarding a basin of attraction as a region in the state space, it has been found that the basic topological structure of such regions can vary greatly from system to system.

The basic strategy to generate a picture illustrating the global behaviour of iterative methods for a complex function $f(z)$ is as follows. We choose a rectangular region of the complex plane, frequently a region containing all the roots of $f(z)$. We finely subdivide the region into rectangles, say 300×300 of them, each corresponding to a complex number. We then perform Newton's method, Ellipse method, Traub-Ostrowski's method and QVEM method from each of these complex number and color the corresponding square according to which root the sequence converges. If the sequence does not converge to a root, a real possibility, then the color the rectangle black.

We compared the these four methods namely Newton's method (2.1), Ellipse method (2.2), Traub-Ostrowski's method and QVEM method (2.9).

A lot of research work [71, 72, 73, 74] has been carried out for finding the solution of non-linear equations by using basin of attraction.

Colors of areas of on screen are associated with the root of polynomial equation. The figure shows different basins of attraction with different styles.

For example 1: The complex function has no any root. The black area consists the complex point whose do not approach the root.

$$f(z) = ze^{-z}$$

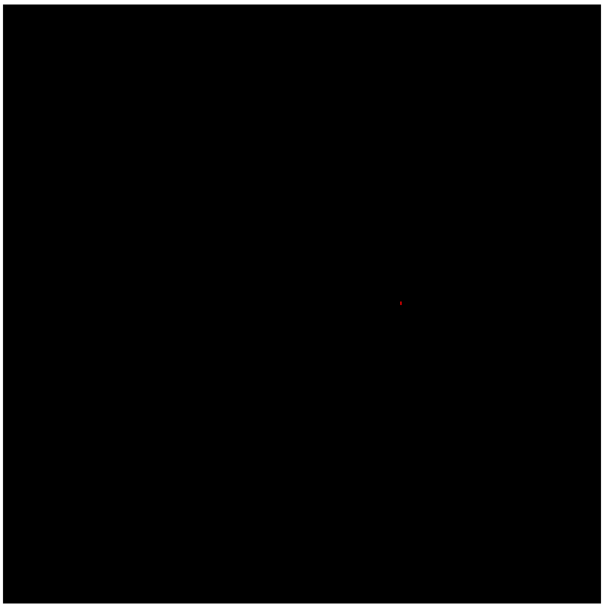

Fig. 1 Newton's Method

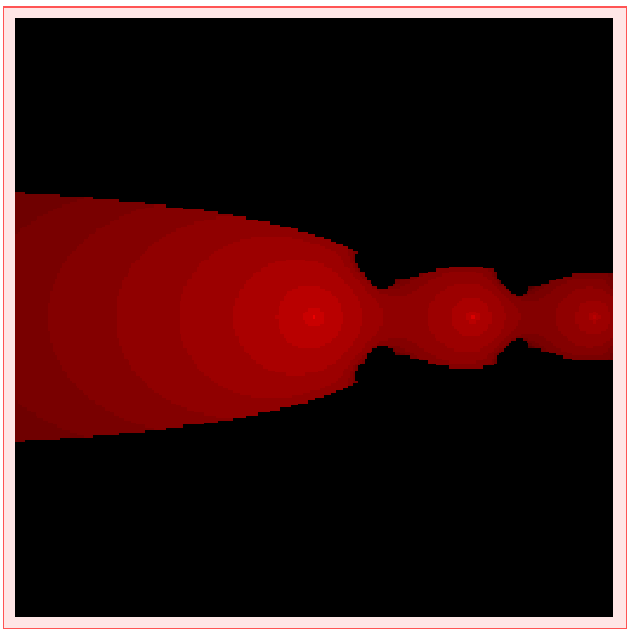

Fig. 2 Ellipse Method

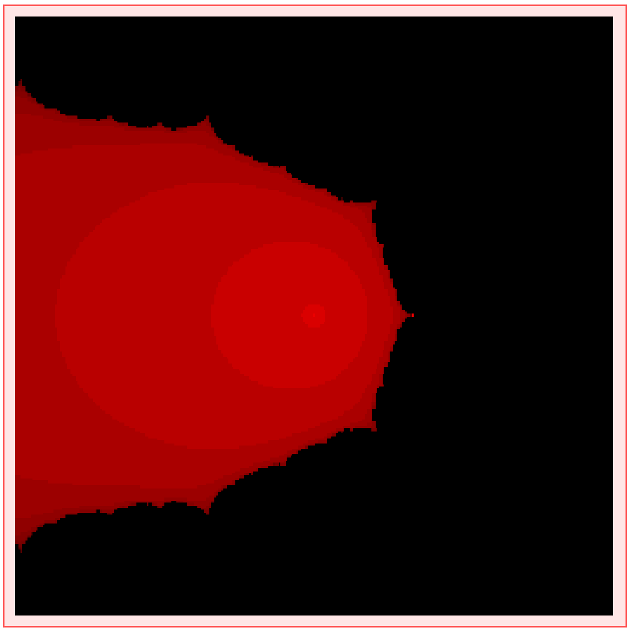

Fig. 3 Traub-Ostrowski's Method

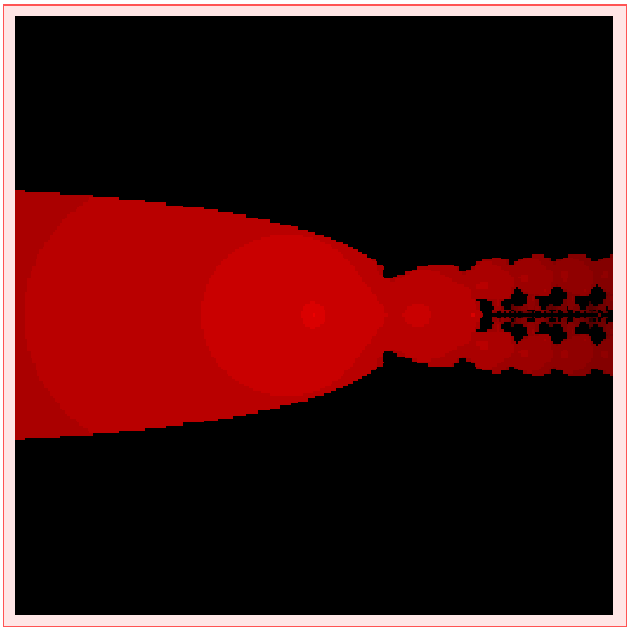

Fig. 4 QVEM Method

The red areas represent the set of convergence points of QVEM method. The black region consists of complex points whose orbit do not approach any root.

For example 2: In this example, the blue and purple areas represents the set of convergence points of Ellipse method and QVEM method respectively.

$$f(z) = z^3 + 4z - 10$$

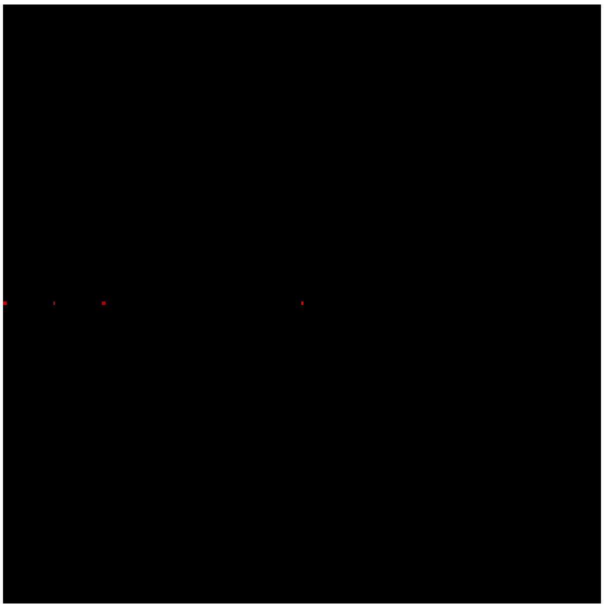

Fig. 1 Newton's Method

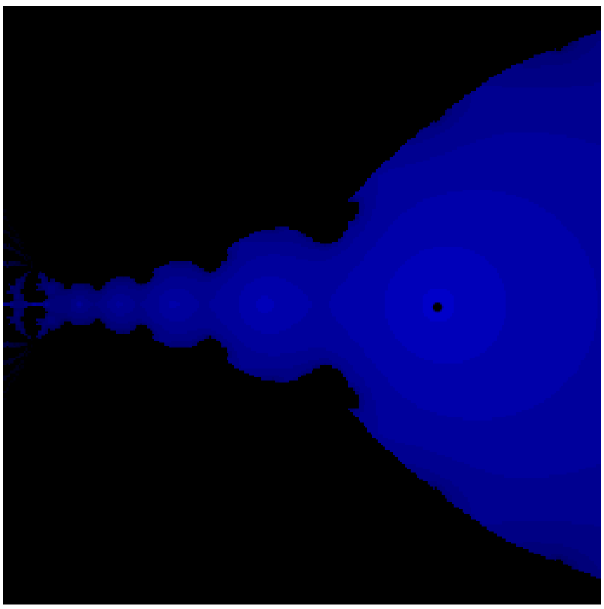

Fig. 2 Ellipse Method

Fig. 3 Traub-Ostrowski's Method

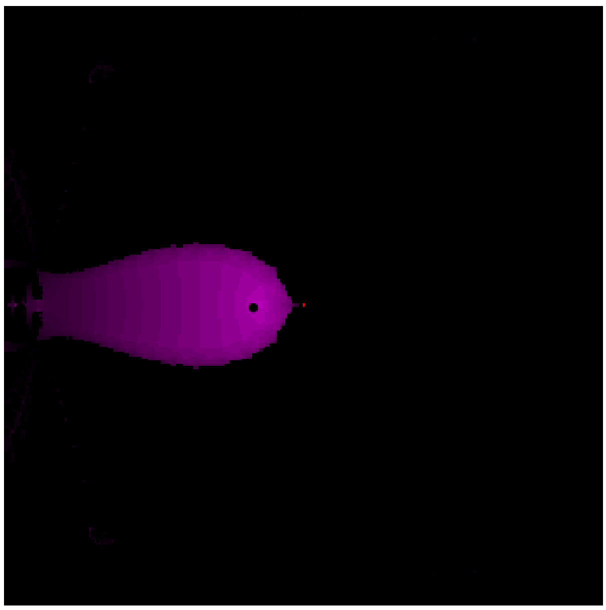

Fig. 4 QVEM Method

In figure third and figure fourth, black area represents the divergence of Traub-Ostrowski's and QVEM method.

For example 3: The complex function has six roots. In fig.1, the six roots have been arbitrarily assigned the colour blue, green, sky-blue, red, yellow, purple.

$$f(z) = z^6 - 1$$

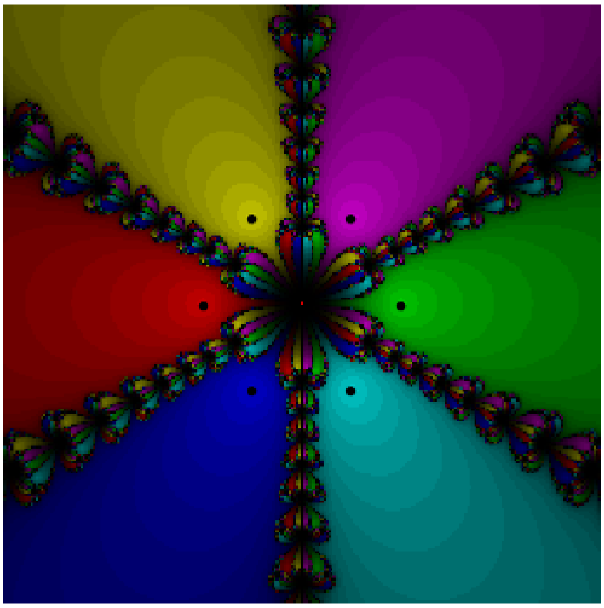

Fig. 1 Newton's Method

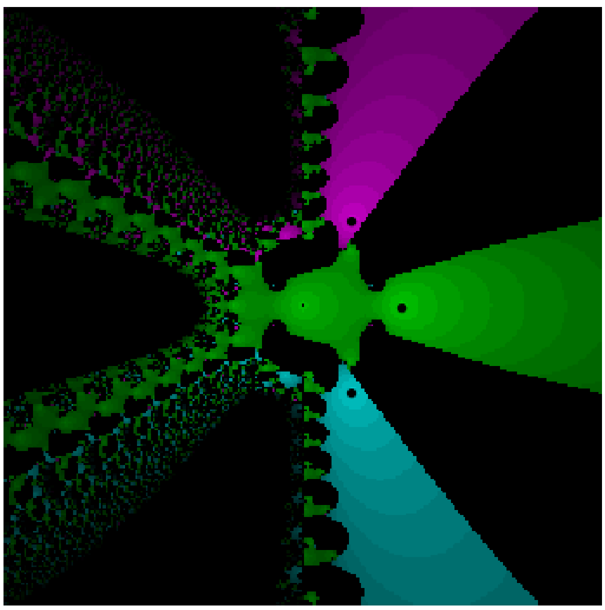

Fig. 2 Ellipse Method

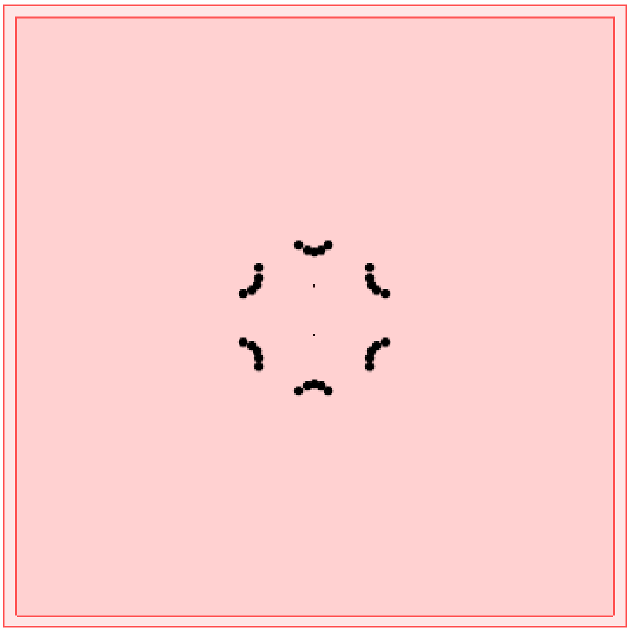

Fig. 3 Traub-Ostrowski's Method

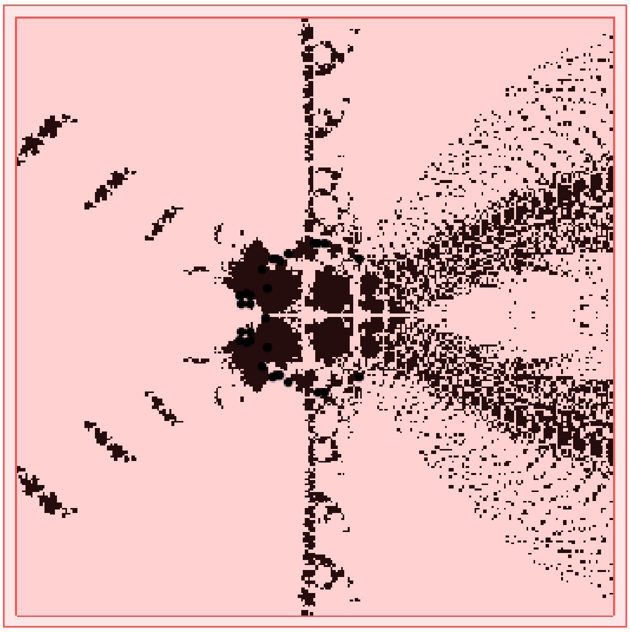

Fig. 4 QVEM Method

For example 4: The complex function has only one root. The black area consists the complex point whose do not approach the root. The red colour represents the set of convergence points.

$$f(z) = Sin(z)$$

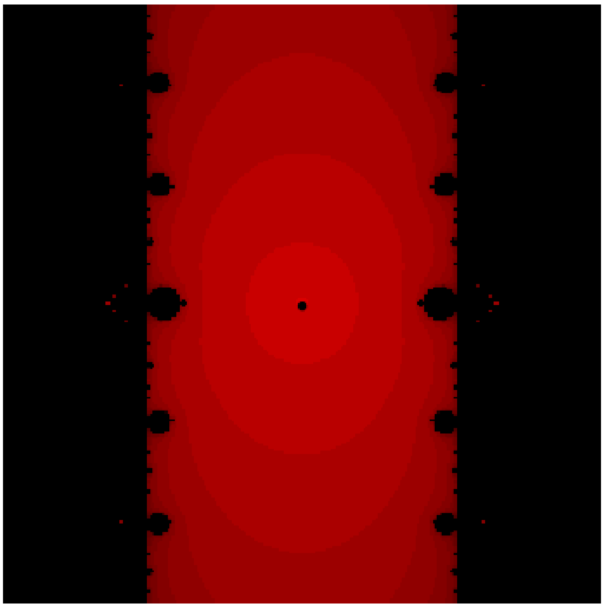

Fig. 1 Newton's Method

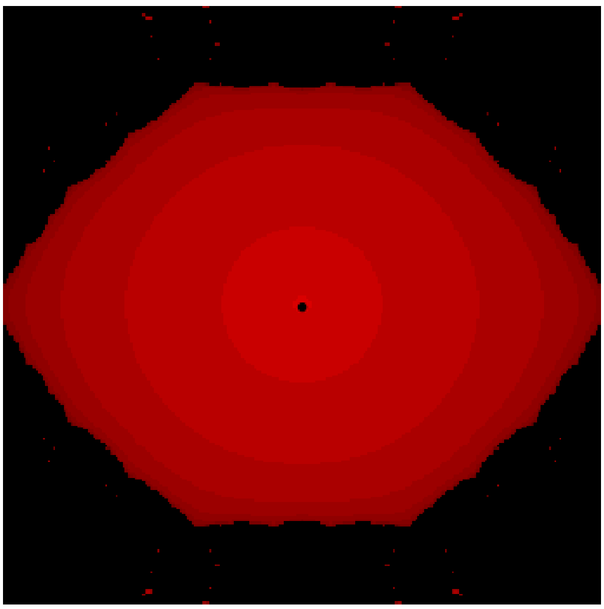

Fig. 2 Ellipse Method

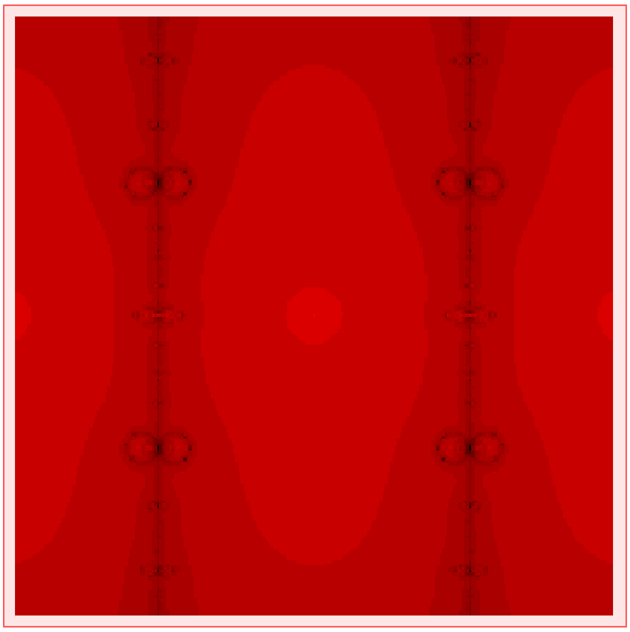

Fig. 3 Traub-Ostrowski's Method

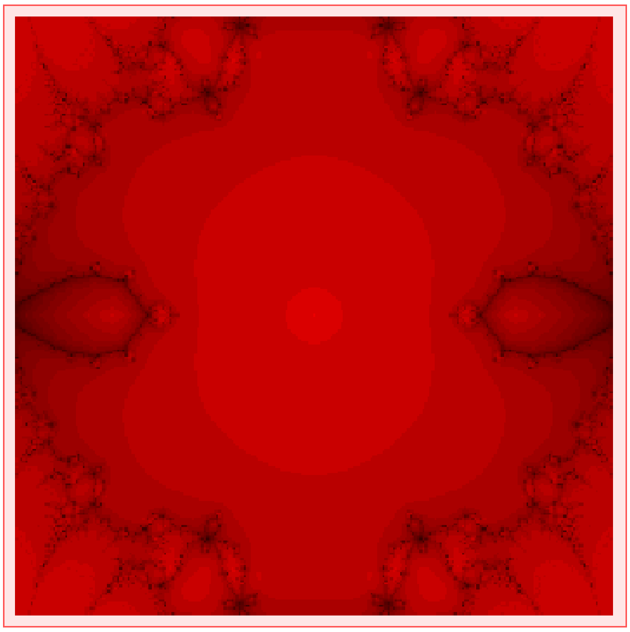

Fig. 4 QVEM Method

For example 5: In figure second, the red colour represents the set of convergence points of ellipse method and black colour represents divergence.

$$f(z) = Cos(z) - z$$

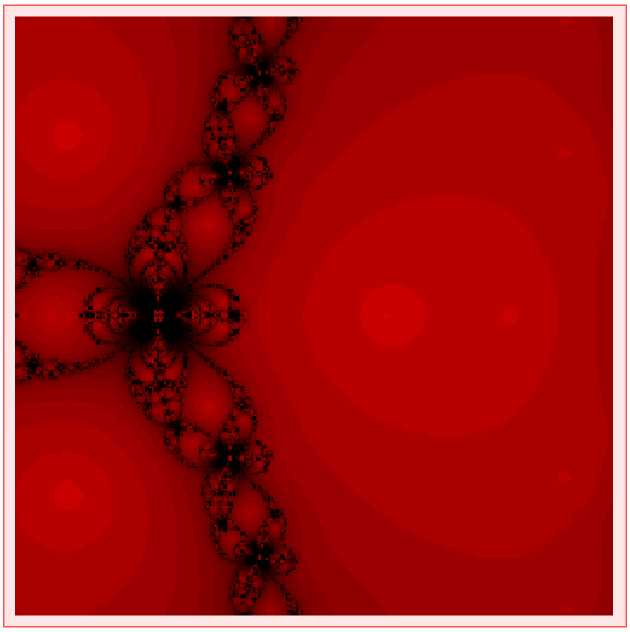

Fig. 1 Newton's Method

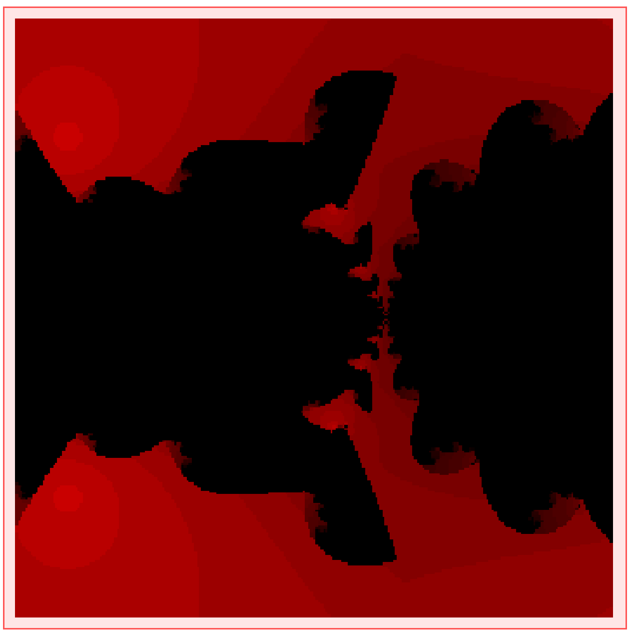

Fig. 2 Ellipse Method

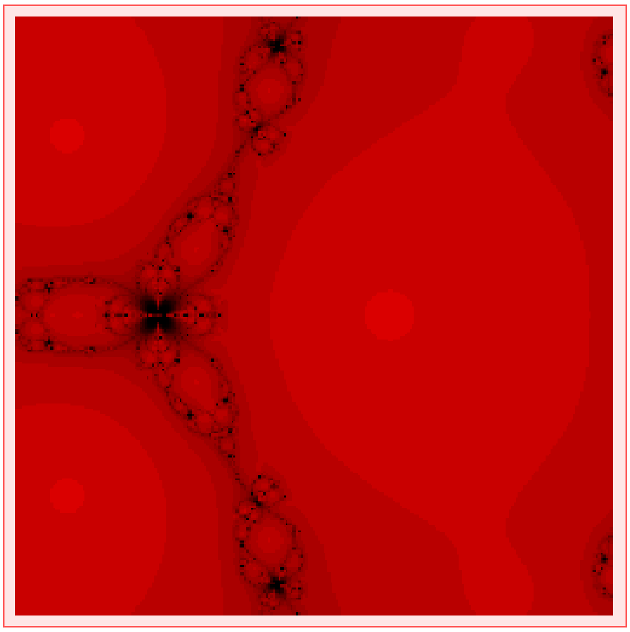

Fig. 3 Traub-Ostrowski's Method

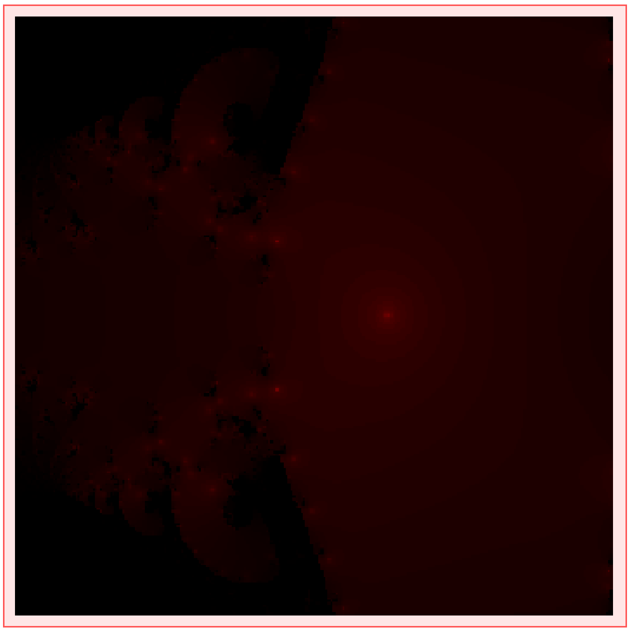

Fig. 4 QVEM Method

The black area represents divergence of QVEM method.

For example 6: The black area consists the complex point whose do not approach the root. The red colour represents the set of convergence points..

$$f(z) = Log(z)$$

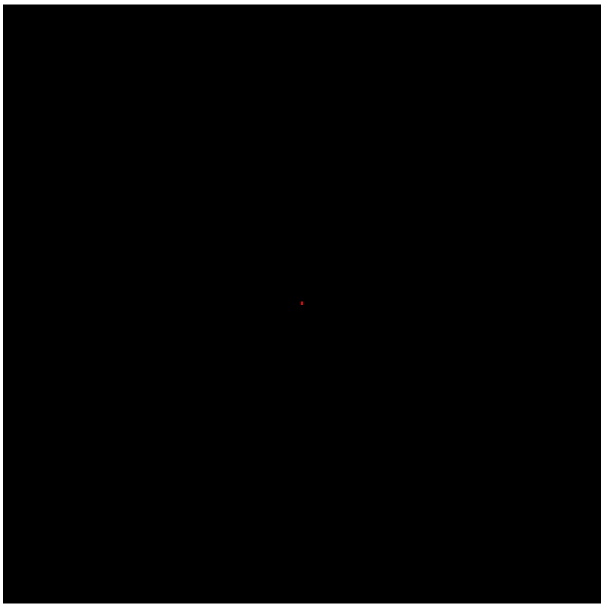

Fig. 1 Newton's Method

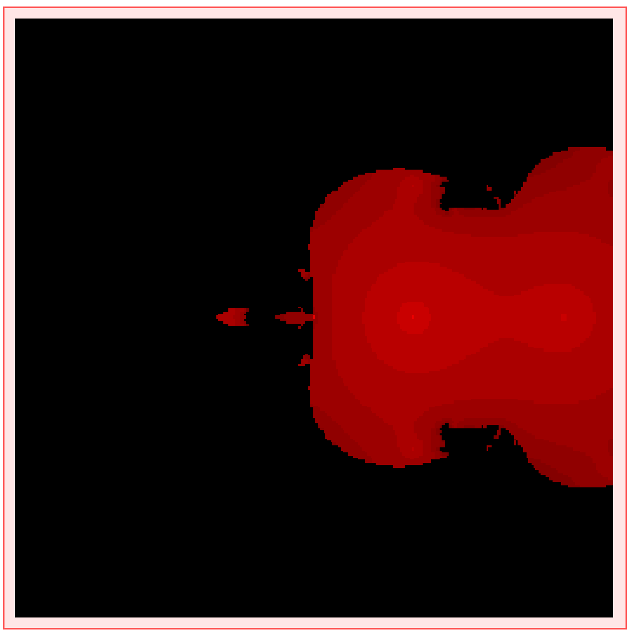

Fig. 2 Ellipse Method

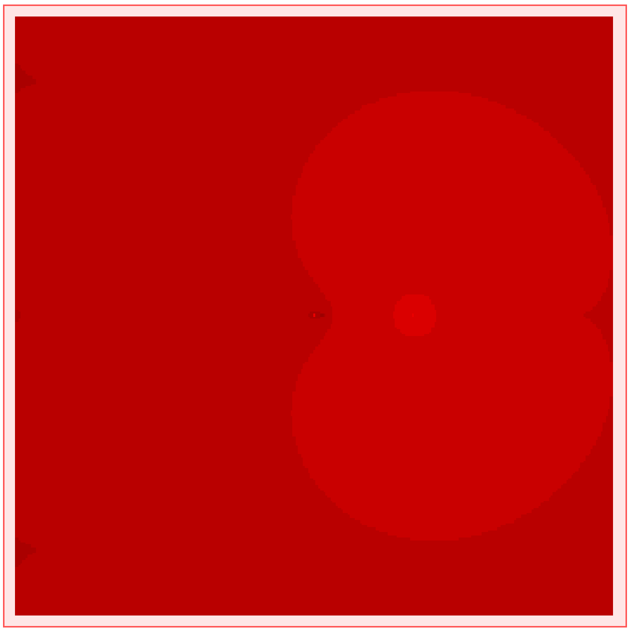

Fig. 3 Traub-Ostrowski's Method

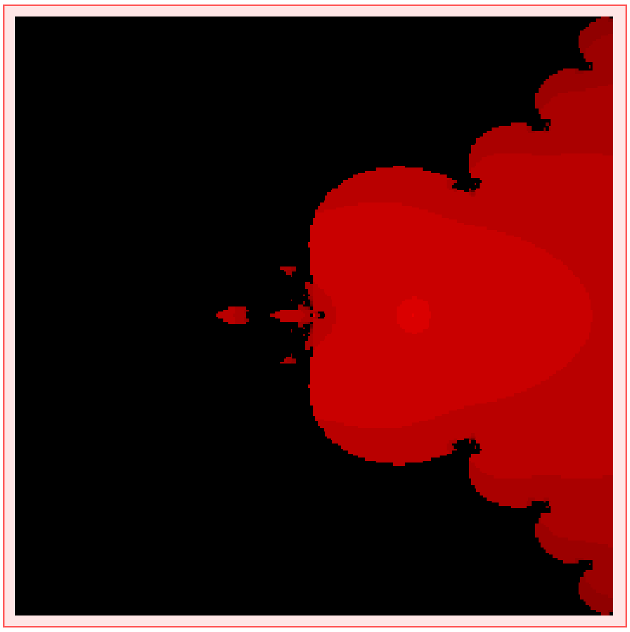

Fig. 4 QVEM Method

For example 7:

$$f(z) = e^{1-z} - 1$$

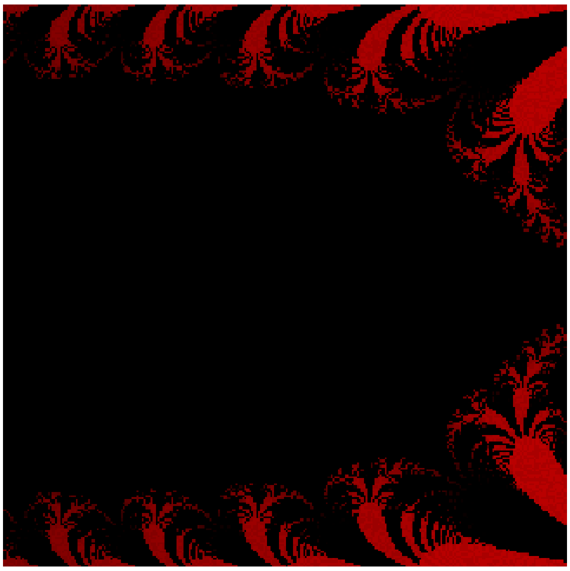

Fig. 1 Newton's Method

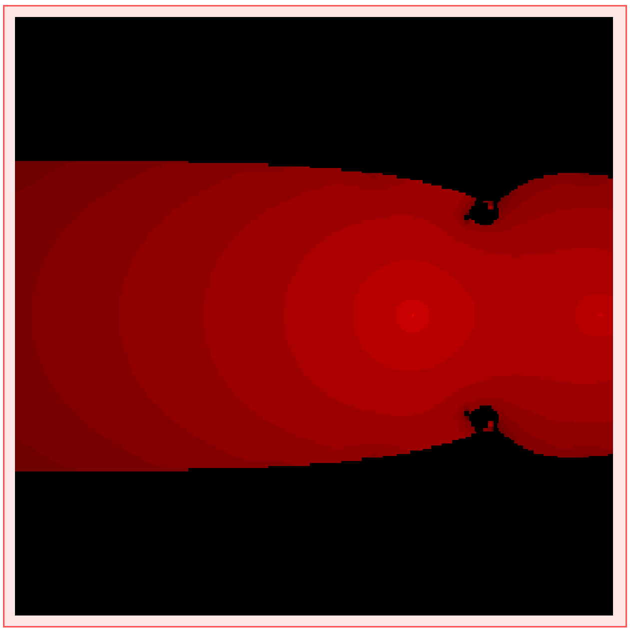

Fig. 2 Ellipse Method

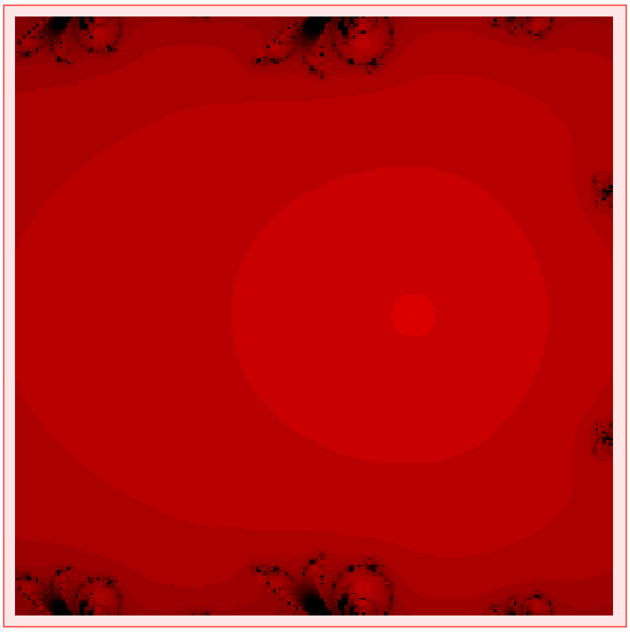

Fig. 3 Traub-Ostrowski's Method

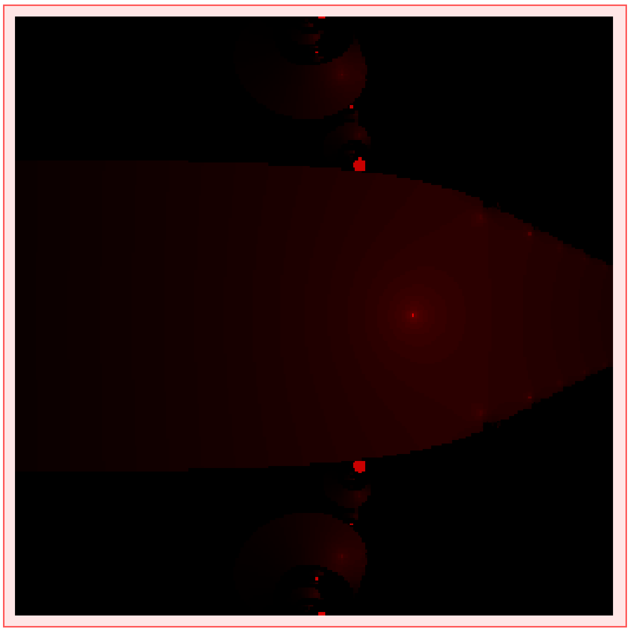

Fig. 4 QVEM Method

For example 8: The red area represents the set of convergence points and black colors shows the divergence of ellipse method.

$$f(z) = e^{-z} - Sinz$$

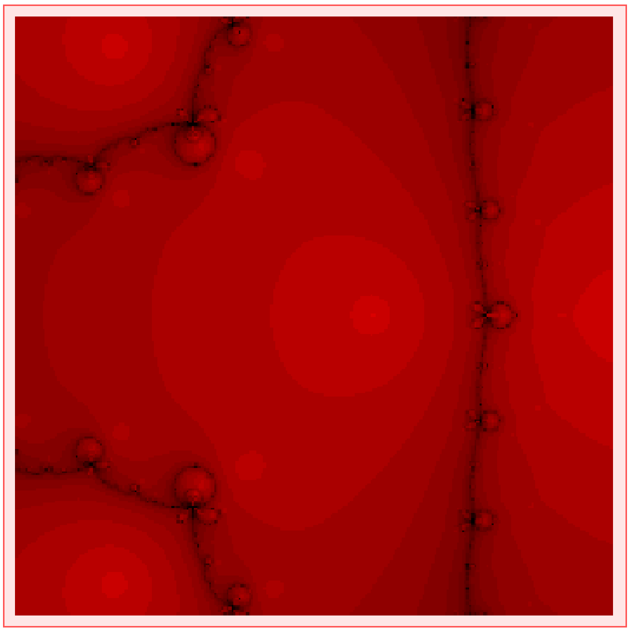

Fig. 1 Newton's Method

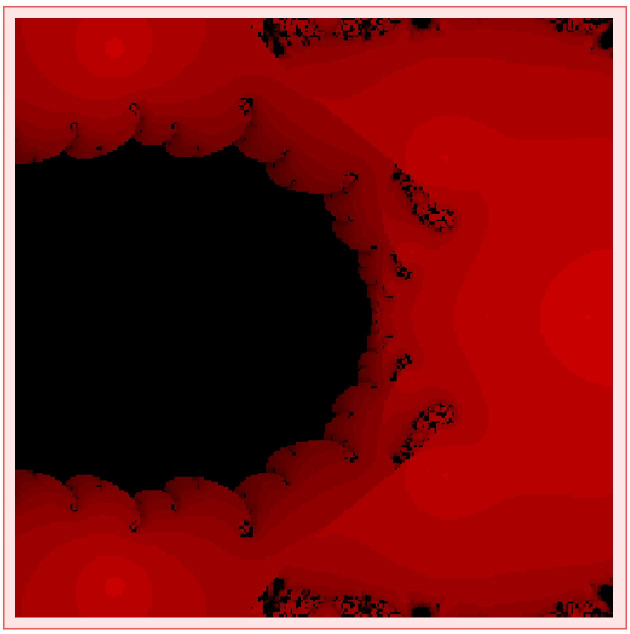

Fig. 2 Ellipse Method

For example 9: In figure first, the complex function has three roots. The different shades of these colors indicate the speed of convergence to the respective roots(in terms of number of iteration required to get very close to the root). The center of the picture is the origin. The three roots have been arbitrarily assigned the colours blue, red, yellow. In this figure, the complex function has two roots. The two roots have been arbitrarily assigned the colours red, yellow.

$$f(z) = z^3 - 0.64310316 - 0.35689687$$

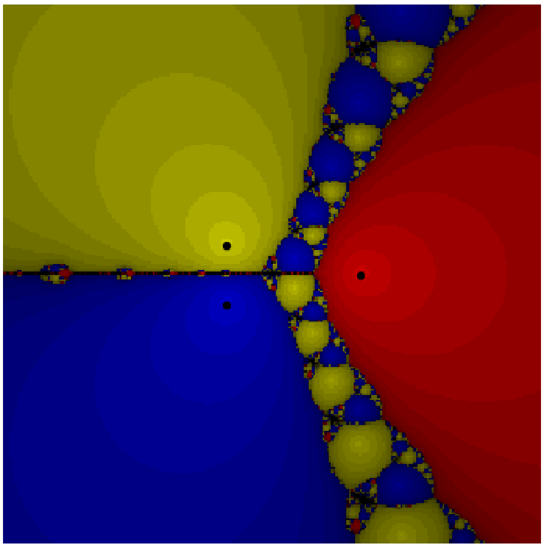

Fig. 1 Newton's Method

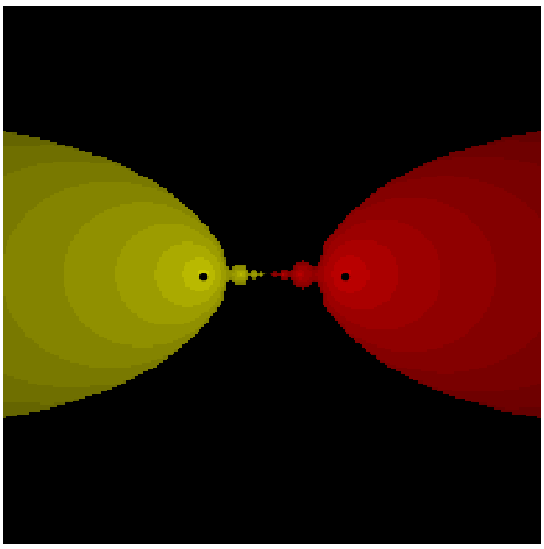

Fig. 2 Ellipse Method

References

[1] Örtega, J.M. and Rheinboldt, W.C.1970.Iterative Solution of Nonlinear Equation in Several Variables. Academic Press, New York,

[2] Wissink, A. M., Lyrintzis, A.S., Chronopoulos, A. T.1996. Efficient iterative methods applied to the solution of transonic flows.Journal of computational Physics,393(2):379-393)

[3] Li, Z. H and Zhang, H. X 2004. Study on gas kinectic unified algorithm for flows from rarefied transition to continuum.Journal of computational Physics,193(2):708-738

[4] Chua, L. O, Desoer, C. A, Kuh, E. S 1987. Linear and Nonlinear Circuits. McGraw Hill, Newyork,

[5] Gautschi, W. 1997. Numerical Analysis:An Introduction. Birkh*ä*user,

[6] Traub, J.F. 1964. Iterative Methods for Solution of Equations. Prentice-Hall, Engle-wood Cliffs, NJ

[7] Ostrowski, A.M. 1973 Solution of Equations in Euclidean and Banach Spaces. Academic Press New York,

[8] Ostrowski, A,M 1966 Solution of Equations and Systems of Equations. Academic Press New York,

[9] Conte, S.D and Boor, C. 1981. Elementary Numerical anaalysis: An Algorithmic Approach. McGraw-Hill, New York,

[10] Burden, R.L and Faires, J.D. 1985. Numerical Analysis. PWS Publishers, Boston, USA, third edition,

[11] Bradie, B. 2005. A Friendly Introduction to Numerical Analysi. Pearson Education, New Delhi,

[12] Chapra, S.C. and Canale, R.P. 2006. Numerical Methods for Engineers. Tata McGraw-Hill, New Delhi,

[13] Chapra, S.C 2007. Applied Numerical Methods with MATLAB for Engineers and Scientists. Tata McGraw-Hill, New Delhi,

[14] Grerald, F.C and Wheatley, P.O 2008. Applied Numerical Analysis. Pearson Education, New Delhi,

[15] Mathews, J.H and Fink, K.D. 2007. Numerical Methods Using MATLAB. Prentice-Hall of India Private Limited, New Delhi,

[16] Atkinson, K.E 1989. An Introduction to Numericall Analysis. John Willey and Sons nc, second edition,

[17] Kelley, C.T. 1995. Iterative Methods for Linear and Nonlinear Equations. SIAM, Philadelphia,

[18] Ralston, A. and Rabinowitz, P.1978. A First Course in Numerical Analysis. McGraw-Hill, New York, pp. 73-122,

[19] Dennis, J.E and Schnable, R.B. 1983. Numerical Methods for Unconstrained Optimization and Nonlinear Equations. Prentice Hall, Englewood Cliffs, NJ,

[20] Householder, A.S. 1953. Principles of Numerical Analysis. McGraw-Hill, New York,

[21] McNamee, J.M. 2007. Numerical Methods for Roots of Polynomials. Elsevier

[22] Wait, R. 1979. The Numerical Solution of Algebraic Equations. John Willey and Sons,

[23] Kantorovich, L.V and Akilov, G.P. 1982. Functional Analysis Pergamon Press. Oxford,

[24] Jarratt, P. 1970. A reviw of methods for solving nonlinear algebraic equations. Gordon and Breach, Science Publishers, London, N ewyork, Paris,

[25] Carnahan, B, Luther, A.H, Wilke, J.D. 1969. Applied Numerical Methods. Wiley New York,

[26] Reklaitis, G.V, Ravindran, A, Raysdel , K.M. 1983. Engineering optimization methods and appications. John Willey and sons, Newyork,

[27] Stoer, J. and Bulirsch, R. 1993. Introduction to Numerical Analysis. second edition, Springer Verlag,

[28] Joshi, M.C and Moudgalya, K.M 2004. Optimization Theory and Practice. Narosa Publication House New Delhi,

[29] Kockler, N. 1994. Numerical Methods and Scientific Computing. Clarendon Press, Oxford,

[30] Lliev, A. and Kyurkchiev, N. 2010. Nontrivial Methods in Numerical Analysis (Selected Topics in Numerical Analysis). Lambert Acad. Publishing,

[31] Wu, X and Wu, H.W. 2000. On a class of quadratic convergence iteration formulae without derivatives. Applied Mathematics and Computation, 10(7) : 77-80

[32] Weerakoon, S. and Fernando, T.G.I. 2000. A variant of Newton's method with accelerated third-order convergence. Applied Mathematics Letters,13(8) : 87-93

[33] Chun, C. 2007. A geometric construction of iterative functions of order three to solve nonlinear equations. Computer and Mathematics with Application, 53(6) :972-976

[34] Chun, C. 2007. A one-parameter family of third-order methods to solve nonlinear equations, Applied Mathematics Computation. 189(1) : 126-130

[35] Sharma, J.R. 2007. A family of third- order methods to solve nonlinear equations by quadratic curves approximation. Applied Mathematics and Computation, 184(2) : 210-215

[36] Kanwar, V, Singh, Sukhjit, Guha, R.K, Mamta. 2006. On method of osculating circle of solving nonlinear equations. Applied Mathematics Computations,176(1) : 379-382

[37] Jarratt, P. 1966. Some fourth-order multipoint iterative methods for solving equations. Mathematics of Computation, 20(95) : 434-437

[38] Jarratt, P. 1969. Some efficient fourth-order methods for nonlinear equations. BIT, 9:119-124

[39] Ralevic, N.M. and Lukic, T. 2009. A New variants of Newton's method based on power means. Intelligent Systems and Informatics, 7th Internation Symposium on, IEEE Conference Publications, pp. 119-122

[40] Babajee, D.K.R. 2012. Several improvement of the 2-point third order midpoint iterative method using weight function. Applied Mathematics and Computation, 218(15) : 7958-7966

[41] Sharifi, M, Babajee, D.K.R, Soleymani, F.2012. Finding the solution of nonliner equations by a class of optimal methods. Computer and Mathematics with Applications, (63)(4) : 764-774

[42] Homeier, H.H.H. 2005. On Newton-type methods with cubic convergence. Journal of Computation and Applied Mathematics, 176(2) : 425-432

[43] Kou, J, Li, Y, Wang, X. 2006. A modification of Newton's method with third-order convergence. Applied Computation and Mathematics, 181 : 1106-1111

[44] Kanwar, V. 2006. A family of third-order multipoint iterative methods for solving nonlinear equation. Applied Mathematics and Computation, 176(2) : 409-413

[45] Mamta, Kanwar, V, Kukreja, V.K, Singh, S. 2005. On class of quadratically convergent iteration formulae. Applied Mathematics and Computation, 166(3) : 633-637

[46] Amat, S, Busquier, S, Gutierrez, J.M, Hernandez, M.A. 2008. On global convergence of Chebyshev's iterative methods. Journal of Computer and Applied Mathematics, 220:17-21

[47] Amat, S, Busquier, S, Gutierrez, J.M. 2003. Geometric construction of iterative functions to solve nonlinear equations. Journal of Computational and Applied Mathematics, 157(1) :197-205

[48] Gutierrez, J.M and Hernandez, M.A 2001. An acceleration of Newton's method: super-Halley method. Applied Mathematics and Computation, 117(2-3) : 223-239

[49] Melman, A. 1997. Geometry and convergence of Euler's and Halley's method. SIAM Rev. 39(4) : 728-735

[50] Gutierrez, J.M and Hernandez, M.A. 1997. A family of Chebyshev-Halley type methods in banach spaces. Bulletin of the Australian Mathematical Society, 55 : 113-130

[51] Melman, A. 1997. Geometry and convergence of Euler's and Halley's method. SIAM Rev. 39(4) : 728-735

[52] Brown, G.H. 1977. On Halley's variation of Newton's method. The american Mathematical Monthly, 84(9) : 726-728

[53] Alefeld, G. 1981. On the convergence of Halley's method. The American Mathematical Monthly, 88(7) : 530-536

[54] Hernandez, M.A 1991. A note on Halley's method. Numerische Mathematik, 59(1) : 273-276

[55] Gutierrez, J.M and Hernandez, M.A 2001. An acceleration of Newton's method: super-Halley method. Applied Mathematics and Computation, 117(2-3) : 223-239

[56] Amat, S, Candela, V.F, Potra, F.A. 2001. Convergence of third-order iterative methods in Banach spaces. Preprint U.P Caratgena, 16

[57] King, R.F. 1973. A family of fourth-order methods for nonlinear equations. SIAM Journal on Numerical Analysis, 10(5) :876-879

[58] Kung, H.T. and Traub, J.F 1974. Optimal order of one-point and multipoint iteration. Journal of Association for Computing Machinery, 21 : 643-551

[59] Frontini, M. and Sornami, F. 2003. Some variants of Newton's method with third-order convergence. Journal of Computer Application and Mathematics, 140 :419-426

[60] Ozban, A. Y. 2004. Some new variants of Newton's method. Applied Mathematics Letters, 17(8) : 677-682

[61] Ezquerro, J.A. and Hernandez, M.A. 2003. A uniparametric Halley-type iteration with free second derivative. International Journal of Pure and Applied Mathematics, 6(1) : 99-110

[62] Kou, J, Li, Y, Wang, X, 2007. A composite fourth-order iterative method. Applied Mathematics and Computation, 184 : 471-475

[63] Cordero, A, Hueso, J.L, Martinez, E, Torregrosa, J.R. 2010. New modifications of Potra-Ptak's method with optimal fourth and eighth orders of convergence. Journal of Computational and Applied Mathematics, 234(10) : 2969-2976

[64] Petkovic, M.S. and Petkovic, L.D. 2010. Families of optimal multipoint method for solving nonlinear equations: a survey. Applicable Analysis and Discrete Mathematics, 4 : 1-22

[65] Nedzhibov, G.H, Hasanov, V.I, Petkov, M.G. 2006. On some families of multipoint iterative methods for solving nonlinear equations. Numerical Algorithm, 42(2) : 127-136

[66] Wang, H. and Li, S. 2010. A family of derivative-free method for nonlinear equations Rev. Mat. Complut. 24(2) : 375-389

[67] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice-hall, Englewood Cliffs, New Jersey, 1964.

[68] A.M. Ostrowski, Solution of Equations in Eucildean and Banach Space, Academic Press, New York, 1973.

[69] J.E. Dennis and R.B Schnable, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1983.

[70] K.C. Gupta, V. Kanwar and Sanjeev Kumar, A family of ellipse methods for solving nonlinear equations, International Journal of Mathematical Education in Science and technology, 40(4)(2009), pp. 571-575.

[71] G. ARDELEAN, Generating the basin of attraction for Newton's method, Creative Math. Inf., 17(2008), 285-292.

[72] A comparison between iterative methods by using the basin of attraction, Appl. Math. Comput., 218(2011), 88-95.

[73] G. ARDELEAN, Graphical views for the basin of attraction of the real roots of a nonlinear equation, Graphical Models (2011), Manuscript number: GMOD-11-65(Under Review).

[74] R. K. ROEDER, Topology For the Basin O Attraction of Newton's Method in two complex variables, A Disertation Presented to the Faculty of the Graduate School of Cornell University (2005). of Computer Application and Math