INTEGRABILITY AND L^1-CONVERGENCE OF CERTAIN COSINE SUMS

A

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CERTIFICATE

Certified that the dissertation entitled, "**INTEGRABILITY AND L^1-CONVERGENCE OF CERTAIN COSINE SUMS**", which is being submitted by **Miss Ritika Bansal** (Roll No. 301003022), in the fulfillment of the requirements for the award of the degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Jatinderdeep Kaur** during the period from January 2012 to June 2012.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University / Institute for the award of any degree.

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ABSTRACT

The present dissertation entitled, **“Integrability and L^1-Convergence of Certain Cosine Sums”,** contains a brief account of investigations carried out by various authors and by me on L^1-convergence of trigonometric cosine series under the supervision of **Dr. Jatinderdeep Kaur,** Professor, School of Mathematices and Computer Applications, Thapar University, Patiala.

The work presented in this dissertation has been divided into four chapters. The first chapter is introductory. In this chapter, apart from setting up the notations and terminology to be used in sequel, we have presented some known results interrelated to our results along with a brief plan of our results presented in the subsequent chapters. The purpose of chapter II is to study the integrability and L^1-convergence of Rees and Stanojevic cosine sum under the Class $\boldsymbol{C}$ of coefficient sequences. In chapter III, I studied the results concerning the L^1-convergence of cosine trigonometric series under the class S^2 of coefficient sequences which is equivalent to class S of Sidon.

In chapter IV, I have studied the generalization of the results of Garrett and Stanojevic by considering the class $(BV)^m$ instead of class BV.

Towards the end, references of various publications cited in the present dissertation have been reported.

CONTENTS

CHAPTER I

INTRODUCTION

1.1 The present thesis embodies certain results studied by the author "**Integrability and $\boldsymbol{L^1}$- Convergence of Certain Cosine Sums".** It is known that if a trigonometric series converges in L^1-metric to a function $f \in L^1(T)$, then it is the Fourier series of the function f. Riesz {[1], Vol II, Ch VIII article 22} gave a counter example to show that converse of above result does not hold good in L^1-metric. This has encouraged various researchers to carry the research on the topic "On L^1-Convergence of Certain Cosine Sums".

Integrability and L^1-convergence of certain cosine sums have been studied by number of authors. The work on this topic was initiated in 1913 by Young W. H. [42] and Kolmogorov A.N. [23] in 1923 by taking the classes of convex sequences $(\Delta^2 a_n \geq 0)$ and quasi – convex sequences $\left(\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty\right)$ respectively.

In 1973 Telijakovskii S.A. [41] studied another class S which was introduced by Sidon [32] in 1939 for L^1-convergence of certain cosine sums. The results obtained by these authors were further generalized and extended by Hardy G.H. and Littelewood J.E. [15], Kano T. [17], Garrett J.W. and Stanojevic C.V. ([11],[12]), Ram B. ([27],[29]), Singh N. and Sharma K.M. ([33],[34],[35]), Bojanic R. and Stanojevic C.V. [4], Chen C.P .[9], Bala, R. and Ram, B. [2], Moricz, F. [25], Bhatia, S.S. and Ram, B. [3], Tomovski, Z. ([36],[37],[38],[39]), Hooda, N. and Ram, B. [16], Kaur, K., Bhatia, S.S. and Ram, B. [18] , Kaur, J. and Bhatia, S.S ([21],[22].[23]) and others by considering various generalizations of classes of sequences mentioned above for one-dimensional trigonometric cosine and sine sums.

During reviewing on this topic, I found that many authors introduced modified trigonometric sums as these sums approximate their limits better than the classical utrigonometric sums, since these sums converge in L^1-metric to the sum of trigonometric series where as the classical series itself may not. In this concern, various authors like Rees, C.S. and Stanojevic C.V. [31], Chen, C.P .[10], Kumari, S. and Ram, B. [24], Ram, B. and Kumari, S. [30], Hooda, N., Ram, B. [16], Kaur, K., Bhatia S.S. and Ram B. [19] Kaur, J. and Bhatia, S.S. ([21],[22],[23]) have introduced various new modified trigonometric cosine and sine sums and have studied their L^1-convergence under different classes of coefficient sequences.

To provide sufficient background for later chapters, a summary of basic concepts, techniques and a brief chapter wise resume of the results contained in the dissertation has been given in this introductory chapter. However, some of the definitions and notations will be repeated occasionally in chapters for the sake of convenience.

1.2 DEFINITIONS AND NOTATIONS :

Let $\{a_n\}$ be a sequence. Then we write

$$\Delta a_n = a_n - a_{n+1}$$
$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

Abel's transformation ([1], VOL.I, p.1). if $a_0, a_1, \ldots a_n, \ldots v_1, v_2, \ldots v_n \ldots$ are any real numbers and we assume that

$$V_n = v_0 + v_1 + \ldots\ldots\ldots\ldots v_n$$

Then, $$\sum_{k=m}^{n} a_k v_k = \sum_{k=m}^{n-1} \Delta a_k V_k + a_n V_n - a_m V_{m-1}$$

is called Abel's transformation, where $\Delta a_k = a_k - a_{k+1}$.

If $m = 0$ and $V_{-1} = 0$, then

$$\sum_{k=0}^{n} a_k v_k = \sum_{k=0}^{n-1} \Delta a_k V_k + a_n V_n$$

Null Sequence. The sequence $\{a_n\}$ is null sequence if $a_n \to 0$ as $n \to \infty$.

Trigonometric series: A series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called trigonometric series, where $a_0, a_n's, b_n's$ are the coefficients. These coefficients may be real or complex.

Fourier Series: A Fourier series may be defined as an expansion of a periodic and intergrable function $f(x)$ over interval $(-\pi, \pi)$ in a series of sines and cosines such as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Convex sequence: A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$.

Quasi-Convex sequence ([1], Vol. II, p.202). A sequence $\{a_n\}$ is said to be quasi-convex if $\sum_{n=1}^{\infty} n\left|\Delta^2 a_n\right| < \infty$.

Semi-Convex sequence[17]. A null sequence $\{a_n\}$ is said to semi – convex if

$$\sum_{n=1}^{\infty} n\left|\Delta^2 a_{n-1} + \Delta^2 a_n\right| < \infty, \quad (a_0 = 0) .$$

O-o Relation: Let u_n and v_n be sequence of real numbers. Then u_n be of order v_n

i.e. $u_n = o(v_n)$ if $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$, and

if $\frac{u_n}{v_n}$ is bounded, then $u_n = O(v_n)$.

Example: We have $u_n = \frac{1}{n}$ and $v_n = 1$

then we see that

$$\frac{u_n}{v_n} \to 0 \quad \text{as} \quad n \to \infty$$

therefore,

$$u_n = o(1)$$

Also,

$$0 \le u_n \le 1 \quad \Rightarrow u_n = O(1)$$

Dirichlet Kernel ([1], Vol. I, p.85) . It can be define as

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + ... + \cos nx$$

Moreover,

$$2\sin\frac{x}{2} D_n(x) = \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + ... + 2\sin\frac{x}{2}\cos nx$$

$$= \sin\left(n + \frac{1}{2}\right)x$$

Hence

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

If $x \ne 0 (\mathrm{mod}\ 2\pi)$, then

$$|D_n(x)| \le \frac{\pi}{2x}, \quad \text{for } 0 < |x| \le \pi$$

and

$$\left|\tilde{D}_n(x)\right| \le \frac{\pi}{x}, \quad \text{for } 0 < |x| \le \pi$$

Conjugate Dirichlet Kernal : Let $\tilde{D}_n(x) = \sin x + \sin 2x + ... + \sin nx$

be called Conjugate Dirichlet Kernel.

Further

$$2\sin\frac{x}{2}\tilde{D}_n(x) = 2\sin\frac{x}{2}\sin x + 2\sin\frac{x}{2}\sin 2x + ... + 2\sin\frac{x}{2}\sin nx$$

$$= \cos\left(\frac{x}{2}\right) - \cos\left(n + \frac{1}{2}\right)x$$

Hence

$$\tilde{D}_n(x) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(n + \frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}$$

Fejer Kernel ([1], [43]): The Fejer Kernel $K_n(x)$ is defined as

$$K_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(x)$$

$$= \frac{1}{n+1}\sum_{j=0}^{n} \frac{\sin\left(j + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

Using $|D_n(x)| \le n+1,$ it follows that $K_n(x) \le n+1.$

It has properties

(i) $K_n(x) \ge 0,$

(ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1.$

The Conjugate Fejer Kernel is defined as

$$\tilde{K}_n(x) = \frac{1}{n+1}\sum_{j=0}^{n}\tilde{D}_j(x)$$

We have $\quad \tilde{K}_n(x) > 0 \quad for \;\; 0 < x < \pi, \quad n = 1,2,3,...$

and $\quad \left|\tilde{K}_n(x)\right| < \frac{1}{2}n.$

Class S of coefficient sequence ([32], [41]): A null sequence $\{a_n\}$ belongs to class S , if there exists a sequence $\{A_n\}$ such that

(i) $A_n \downarrow 0$ as $n \to \infty$,

(ii) $\sum_{n=1}^{\infty} A_n < \infty,$

(iii) $|\Delta a_n| \le A_n \;\; \forall n$.

This was given by S.SIDON [32] in 1939

Class S^2 of coefficient sequence ([14]): A null sequence $\{a_n\}$ belongs to the class S^2 if there exists a null sequence $\{A_n\}$ of nonnegative numbers such that $\sum_{n=1}^{\infty} n\left|\Delta A_n\right| < \infty$ and

$\left|\Delta a_n\right| \le A_n$ for all n .

Remark: The Class S and S^2 are equivalent classes.

Class C ([12]): A null sequence $\{a_n\}$ belongs to the class C if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$, independent of n, and such that for all n ,

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\right| dx < \varepsilon \text{ ,}$$

where D_k is the Dirichlet Kernel.

Class $(BV)^m$ of Bounded Variation ([14]): A null sequence $\{a_n\}$ belongs to the class $(BV)^m$

if for some integer $m>1$, $\sum_{n=1}^{\infty}\left|\Delta^m a_n\right|<\infty$.

where $$\Delta^m a_n = \Delta(\Delta^{m-1}a_n) = \Delta^{m-1}a_n - \Delta^{m-1}a_{n+1}$$

For $m=1$, the class $(BV)^m$ reduces to the class BV.

Remark: (i) $(BV)^{m+1} \subset (BV)^m$; for $m=1,2,3....$

(ii) For $m=1$; $(BV)^m = (BV)$.

1.3 The following results about the behavior of cosine and sine series are known:

Theorem I ([1],[23], [42]). If $\{a_n\}$ is a quasi-convex null sequence, then

(1.3.1) $$f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^1[0,\pi]$$

Theorem II ([1], [40]). If $\{a_n\}$ is a quasi-convex null sequence, then

(1.3.2) $$\sum_{n=1}^{\infty} a_n \sin nx$$

is a Fourier series if and only if $\sum_{k=1}^{\infty}\frac{|a_k|}{k}<\infty$.

Theorem III. If $\{a_n\}$ is a null sequence such that

(1.3.3) $$\sum_{n=1}^{\infty} n^2\left|\Delta^2\left(\frac{a_n}{n}\right)\right|<\infty$$

then (1.3.1) and (1.3.2) are the Fourier series, or equivalently they represent integrable functions.

Concerning the integrability of trigonometric series belonging to the class S ([32], [41] (introduced already in article 1.2)), Teljakovaskii [41] established the following theorems:

Theorem IV. Let the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

belongs to the class S ([32],[41]). Then this is a Fourier series and the following relation holds:

$$\int_0^{\pi}\left|\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\right|dx \le C\sum_{k=0}^{\infty}A_k ,$$

where C is an absolute constant.

Theorem V. Let the sine series

$$\sum_{n=1}^{\infty}a_n\sin nx$$

belongs to the class S ([32],[41]). Then the following relation holds for $p=0,1,2,....$

$$\int_{\pi/(p+1)}^{\pi}\left|\sum_{n=1}^{\infty}a_n\sin nx\right|dx=\sum_{k=1}^{p}\frac{|a_k|}{k}+O\left(\sum_{k=1}^{\infty}A_k\right)$$

We observe that Theorem I and Theorem III provide just only the sufficient conditions for the integrability of cosine series. Rees and Stanojevic [31] showed that $\sum_{k=1}^{\infty}\frac{a_k}{k}<\infty$ is a necessary and sufficient condition for integrability of sine series and for a different type of cosine sums.

Theorem VI. Let $b_k=\frac{a_k}{k}\downarrow 0$. Then

$$g(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{b_k}{2}+\left(\sum_{j=k}^{n}b_j\right)\cos kx\right]$$

exists for $x\in(0,\pi]$ and $g\in L^1[0,\pi]$ if and only if $\sum_{k=1}^{\infty}b_k<\infty$.

Theorem VII. Let $b_k=\frac{a_k}{k}\downarrow 0$. Then

$$\frac{1}{x}\sum_{k=1}^{\infty}\frac{b_k}{2}\sin\left(k+\frac{1}{2}\right)x=\frac{h(x)}{x}$$

converges for $x\neq 0$ and $\frac{h(x)}{x}\in L^1[0,\pi]$ if and only if $\sum_{k=1}^{\infty}b_k<\infty$.

Theorem VIII. Let $(k+1)\left|\Delta^2 a_k\right|\downarrow 0.$ Then

$$h(x)=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{2}(k+1)\left|\Delta^2 a_k\right|+\left(\sum_{j=k}^{n}(j+1)\left|\Delta^2 a_j\right|\right)\cos kx\right]$$

exists for $x \in (0,\pi]$ and $h \in L^1[0,\pi]$ if and only if $\{a_k\}$ is quasi-convex.

Ram [28] showed that the condition S is sufficient for the integrability of Rees-Stanojevic [31] sums

(1.3.4) $$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\cos kx$$

He proved the following theorems:

Theorem IX. Let the sequence $\{a_k\}$ satisfy the condition S. Then $g(x) = \lim_{n\to\infty} g_n(x)$ exists for $x \in (0,\pi]$, and

$$\int_0^{\pi} |g(x)| dx \le C\sum_{k=0}^{\infty} A_k$$

Theorem X. Let $\{a_k\}$ be a sequence satisfying the condition S . Then

$$\frac{1}{x}\sum_{k=1}^{\infty}\Delta a_k \sin\left(k+\frac{1}{2}\right)x = \frac{h(x)}{x},$$

converges for $x \in (0,\pi]$, and $\dfrac{h(x)}{x} \in L^1[0,\pi]$.

The above theorems were further studied by Ram [29], under a condition where the monotonicity of the sequence in the definition of the class S is replaced by quasi-monotonicity.

Consider the cosine series

(1.3.5) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Let the partial sums of (1.3.5) is denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$. Denote the class of sequence of Fourier coefficients $\{a_n\}$ by **F**. There are subclasses of **F** for which

$a_n \log n = o(1),\ n \to \infty$ is a necessary and sufficient condition for $\|S_n - f\|_{L^1} = o(1),\ \ n \to \infty$

A subclass **G** of **F** is called a class of L^1-convergence if $\|S_n - f\|_{L^1} = o(1),\ \ n \to \infty$ if and only if $a_n \log n = o(1),\ n \to \infty$.

Concerning the L^1-convergence of the cosine series, we have the following classical result of Kolmogorov [23].

Theorem XI. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.3.5) in the metric space L , it is necessary and sufficient that

$a_k \log k = o(1),\ k \to \infty$.

The case, in which the sequence $\{a_k\}$ is convex, of this theorem was established by Young [42].

Generalizing the above classical result, Teljakovskii [41] proved the following result:

Theorem XII. If the coefficient sequence $\{a_k\}$ of the cosine series (1.3.5) belongs to the class S, Then a necessary and sufficient condition for L^1 - convergence of (1.3.5) is
$a_k \log k = o(1),\ k \to \infty$

Rees and Stanojevic [31] introduced modified cosine sums (1.3.4) and obtained an analogue of theorem XI for these sums. These modified cosine sums approximate their limits better than the classical cosine series as they converge in L^1 - metric to the sum of the cosine series whereas the classical cosine series itself may not. They proved the following result:

Theorem XIII. Let f be the sum of the cosine series (1.3.5).Then $g_n(x)$ converges to f

in L^1 - metric if and only if $\{a_k\}$ belongs to the class C .

Ram [27] proved the following result on L^1-convergence of Rees-Stanojevic sums (1.3.4).

Theorem XIV. If (1.3.5) belongs to class S. Then

$$\| f - g_n \|_{L^1} = o(1), \quad n \to \infty$$

Theorem XII of Telijakovskii [41] follows as corollary of this theorem.

Singh and Sharma [34] proved the above theorem by replacing the monotonicity of sequence $\{A_n\}$ in the definition of class S by quasi-monotonocity of $\{A_n\}$ .Their result reads as:

Theorem XV. Let $a_n \in S^{'}$, then $f_n(x)$converges to $f(x)$ in L^1 - metric.

Further, Ram and Kumari ([24], [30]) introduced new modified cosine and sine sums as

(1.3.6) $$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\cos kx$$

and

(1.3.7) $$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)k\sin kx$$

and studied their L^1-convergence under the condition that the cosine series and sine series belong to the classes **R** and **S**. They also deduced the results about L^1 -convergence of cosine and sine series. Their results state as below:

Theorem XVI. Let $\{a_n\}$ belongs to the class S . If $\lim_{n\to\infty}|a_{n+1}|\log n = 0$, then

$$\| f - f_n\|_{L^1} = o(1), \quad n \to \infty$$

Theorem XVII. Let $\{a_n\}$ belongs to the class **R**. If $t_n(x)$represents $f_n(x)$ and $g_n(x)$, then

$$\| f - t_n\|_{L^1} = o(1), \quad n \to \infty.$$

Theorem XVII. Let the sequence $\{a_n\}$ belongs to class $S(\delta)$ and $\lim_{n\to\infty}|a_{n+1}|\log n = 0,$

then $\| f - g_n\| = o(1), \quad n \to \infty.$

In chapter II, we have studied the integrability and L^1-convergence of Rees and Stanojevic cosine sums under the class C of coefficient sequences.

In chapter III, We have obtained the results concerning the L^1-convergence of cosine trigonometric series under the class S^2 of coefficient sequences is equivalent to class S of Sidon [32].

The aim of chapter IV is to generate the results of Garrett, Rees and Stanojevic [14] by concerning the class $(BV)^m$ instead of class BV .

CHAPTER II

On L^1 –CONVERGENCE OF CERTAIN COSINE SUMS

2.1 Introduction

Consider the cosine trigonometric series

(2.1.1) $\dfrac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx$

Let the partial sum of (2.1.1) be denoted by $S_n(x)$ and $f(x)=\lim_{n\to\infty} S_n(x)$.

The following well-known theorems provide necessary and sufficient condition for sine and cosine series to be the Fourier series and their convergence in L^1- metric.

Theorem A([1]). Let $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$ where $\Delta b_n \geq 0\left[\Delta b_n = b_n - b_{n+1}\right]$ and $\lim_{n\to\infty} b_n = 0$. Then $f \in L^1[0,\pi]$ or, equivalently $\sum_{n=1}^{\infty} b(n)\sin nx$ is the Fourier series of f if and only if $\sum_{n=1}^{\infty}\left|\Delta b_n\right|\log n < \infty$.

$\sum_{k=1}^{n} b_k \sin kx$

Theorem B([1]). Let $f(x)$ be as in Theorem A. If $f \in L^1[0,\pi]$, Then

converges to f in the L^1- metric.

There are no analogous results of Theorem A for the Cosine series. Further we have the Theorems C and D which give sufficient conditions for the cosine series to be the Fourier series of its sum.

Theorem C([1]). If $\sum_{n=1}^{\infty}\left|\Delta a_n\right|\log n < \infty,$ then $f \in L^1[0,\pi]$ or, equivalently, (2.1.1) is the Fourier series of f.

Theorem D([1]). If $\sum_{n=1}^{\infty}\left|\Delta^2 a_n\right|(n+1) < \infty$, then $f \in L^1[0,\pi]$ or, equivalently, (2.1.1) is the Fourier series of f.

Theorem E([1]). If $\sum_{n=1}^{\infty}\left|\Delta^2 a_n\right|(n+1)<\infty$, then $S_n(x)$ converges to f in the L^1 -metric

if and only if $\lim_{n\to\infty} a_n \log n = 0.$

The aim of this chapter is to study the L^1-convergence of Rees and Stanojevic modified cosine sum under the class ***C*** of coefficient sequences defined as follows:

The Class *C*([12]). A null sequence $\{a_k\}$ belongs to the class ***C*** if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$, independent of n, and such that for all n,

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\right| dx < \varepsilon$$

where $D_k(x)$ is the Dirichlet kernel.

2.2 Lemmas

The following lemmas will be required in the proof of the main results.

Lemma1([12]). Let

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=j}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx$$

then $\lim_{n\to\infty} g_n(x) = f(x)$, for $x \in (0,\pi]$

Proof: Consider

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}[\frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a_j \cos kx]$$

$$= \lim_{n\to\infty}[\frac{1}{2}\sum_{k=0}^{n}(a_k - a_{k+1}) + \sum_{k=1}^{n}(\sum_{j=k}^{n}\Delta a_j)\cos kx]$$

$$= \lim_{n\to\infty}\frac{1}{2}[(a_0 - a_2 + a_2 - a_3 + ... + a_n - a_{n+1})] + \sum_{k=1}^{n}[a_k - a_{k+1} + a_{k+1} - a_{k+2} ..... + a_n - a_{n+1}]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2}[a_0 - a_{n+1}] + \sum_{k=1}^{n} a_k \cos kx - \sum_{k=1}^{n} a_{n+1} \cos kx\right]$$

$$= \lim_{n\to\infty}\left[\frac{a_0}{2} - \frac{a_{n+1}}{2} + \sum_{k=1}^{n} a_k \cos kx + \sum_{k=1}^{n} a_{n+1} \cos kx\right]$$

$$= \lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{2} - \sum_{k=1}^{n} a_{n+1} \cos kx\right]$$

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left[S_n(x) - a_{n+1}D_n(x) - \frac{a_{n+1}}{2}\right]$$

Where , $D_n(x)$ is the Dirichlet kernel .

By given hypothesis, $\{a_n\}$ is a null sequence and $D_n(x)$ is bounded in $(0,\pi)$.

Therefore, $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} S_n(x) + o(1) + o(1) = f(x)$

Lemma 2([31]): Let $b_k = \frac{a_k}{k} \downarrow 0$. Then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sum_{j=k}^{n} b_j\right)\cos kx$$

exists for $x \in (0,\pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Proof: Let

$$S_n(x) = \sum_{k=1}^{n}(\sum_{j=k}^{n} b_j)\cos kx$$

Now applying summation by parts twice gives

$$S_n(x) = \sum_{k=1}^{n-1} b_k\left(D_k(x) - \frac{1}{2}\right) + b_n\left(D_n(x) - \frac{1}{2}\right)$$

$$= \sum_{k=1}^{n-2}(b_k - b_{k+1})(k+1)(F_k(x) - \frac{1}{2}) + b_{n-1}n(F_{n-1} - \frac{1}{2}) + b_n(D_n(x) - \frac{1}{2})$$

(2.2.1) $$= \sum_{k=1}^{n-2}(b_k - b_{k+1})(k+1)F_k(x) - \frac{1}{2}\sum_{k=1}^{n-2}(k+1)(b_k - b_{k+1}) + b_{n-1}nF_{n-1}(x) - b_{n-1}\frac{n}{2} + b_n(D_n(x) - \frac{1}{2})$$

Finally, the equation (2.2.1)

$$S_n(x) = \sum_{k=1}^{n-2}(b_k - b_{k+1})(k+1)F_k(x) - \frac{1}{2}\sum_{k=1}^{n-2} b_k - \frac{b_1}{2} - \frac{b_{n-1}}{2} + b_{n-1}nF_{n-1}(x) + b_n(D_n(x) - \frac{1}{2})$$

(F_k and D_k being the Fejer's and Dirichlet's kernals respectively)

Since $F_k(x) = O\left(\frac{1}{k^2} x^2\right)$ for $x \neq 0$, we have

$$0 \leq \sum_{k=1}^{n-2} (b_k - b_{k+1})(k+1) F_k(x) \leq \frac{c}{x^2}(b_1 - b_{n-1})$$

therfore $\lim_{n\to\infty} \sum_{k=1}^{n-2} (k+1)(b_k - b_{k+1}) F_k(x)$ always exists for $x \neq 0$ and $b_k \downarrow 0$.

Each of the last terms of (2.2.1) tend to zero as $n \to \infty$, hence $\lim_{n\to\infty} S_n(x)$ exists if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

2.3 Main Results .

Theorem 1([12]). Let $g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=j}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx$. Then g_n converges to f in the L^1-metric if and only if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| < \varepsilon \text{ for all } n \geq 0 .$$

Proof**: Necessary part**: let $\varepsilon > 0$ .Then there exists $\delta > 0$ such that

$\int \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| < \frac{\varepsilon}{2}$ for all $n \geq 0$. Then

$$\int_0^{\pi} |f - g_n| = \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx$$

$$= \int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| + \int_{\delta}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx$$

$$< \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_{\delta}^{\pi} |D_k(x)| dx$$

$$\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_{\delta}^{\pi} |D_k(x)| dx$$

$$= \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| \int_{\delta}^{\pi} \csc \frac{1}{2}(x) dx$$

$$= \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a_k| [-2 \log \left| \csc \frac{\delta}{2} - \cot \frac{\delta}{2} \right|] < \varepsilon$$

for sufficiently large n, since $\sum_{k=n+1}^{\infty} |\Delta a_k| < \infty$.

Sufficient part: let $\varepsilon > 0$ .then there exists an integer M such that

$\int_0^{\pi} |f(x) - g_n(x)| < \frac{\varepsilon}{2}$ if $n \geq m$. That is, $\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| < \frac{\varepsilon}{2}$ if $n \geq m$

Now if $\sum_{k=0}^{M} |\Delta a_k| = 0$, for all k.

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| = \int_0^{\pi} \left| \sum_{k=M+1}^{\infty} \Delta a_k D_k(x) \right| < \frac{\varepsilon}{2} < \varepsilon$$

Also,

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| < \frac{\varepsilon}{2} < \varepsilon \text{ and, } 0 \leq n \leq M,$$

If $\sum_{k=0}^{M} |\Delta a_k| \neq 0$ and let $\delta = \frac{\varepsilon}{2M} \sum_{k=0}^{M} |\Delta a_k|$

For $n \geq M$

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| \leq \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| \frac{\varepsilon}{2} < \varepsilon$$

For $0 \leq n < M$

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| \leq \int_0^{\delta} \left| \sum_{k=n}^{M-1} \Delta a_k D_k(x) \right| + \int_0^{\delta} \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right|$$

$$\leq \int_0^{\delta} \sum_{k=n}^{M-1} k |\Delta a_k| + \int_0^{\pi} \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right|$$

$$\leq \delta \sum_{k=0}^{M-1} k |\Delta a_k| + \frac{\varepsilon}{2}$$

$$\leq \delta \,\mathrm{M}\sum_{k=0}^{M-1}\left|\Delta a_{\mathrm{k}}\right| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\right| < \varepsilon$$

So, given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 0$.

If $\lim_{n\to\infty}\int_0^{\pi}\left|f(x) - g_n(x)\right| = 0$ it is clear that $f \in L^1[0,\pi]$.

$\int_0^{\pi}\left|f(x)\right| \leq \int_0^{\pi}\left|f(x) - g_n(x)\right| + \int_0^{\pi}\left|g_n(x)\right| < \infty$. Since $g_n(x)$ is a finite cosine sum.

Hence the results holds.

Theorem 2([31]): Let $b_k = \dfrac{a_k}{k} \downarrow 0$, then

(i) $g(x) = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\dfrac{b_k}{2} + \left(\sum_{j=k}^{n} b_j\right)\cos kx\right]$ exists for $x \in (0,\pi)$.

(ii) $g \in L^1[0,\pi]$ if and only if $\sum_{k=1}^{\infty} b_k < \infty$.

Proof: From (2.2.1) of Lemma1, we have that

$$g(x) = \sum_{k=1}^{\infty}(k+1)(b_k - b_{k+1})F_k(x) - \frac{b_1}{2} \;, \quad x \neq 0$$

Integrating term by term and using summation by parts we get,

$$\int_0^{\pi} g(x)dx = \frac{\pi}{2}\sum_{k=1}^{\infty}(k+1)(b_k - b_{k+1}) - b_1\frac{\pi}{2} = \frac{\pi}{2}\sum_{k=1}^{\infty} b_k$$

This implies $g \in L^1[0,\pi]$ iff $\sum_{k=1}^{\infty} b_k < \infty$.

So, the conclusion follows.

Corollary1([12]). If for $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\int_0^{\delta}\left|\sum_{K=n}^{\infty}\Delta a_k D_k(x)\right| < \varepsilon$ for all $n \geq 0$, then S_n converges to f in the L^1 - metric if and only if $\lim_{n\to\infty} a_n \log n = 0$.

Proof: Consider

$$\int_0^{\pi}|f(x)-S_n(x)| = \int_0^{\pi}|f(x)-g_n(x)+g_n(x)-S_n(x)|$$

$$\leq \int_0^{\pi}|f(x)-g_n(x)| + \int_0^{\pi}|g_n(x)-S_n(x)|$$

$$= \int_0^{\pi}|f(x)-g_n(x)| + \int_0^{\pi}|a_{n+1}D_n(x)|$$

Since $\int_0^{\pi}|a_{n+1}D_n(x)|$ behaves like $a_{n+1}\log n$ for large values of n, and using theorem 1,

We get

$$\lim_{n\to\infty}\int_0^{\pi}|f(x)-S_n(x)| = 0 \ .$$

This proves the corollary.

2.4 Examples :

Example**1**. Let $\sum_{n=1}^{\infty} a_n(n+1) < \infty$. Then g_n converges to f in the L^1-metric space.

Solution: Using summation by parts, we get

$$\int_0^{\pi}|f(x)-g_n(x)| = \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx$$

$$= \int_0^{\pi}\left|\sum (k+1)\Delta^2 a_k F_k(x) - (n+1)\Delta a_n F_n(x)\right| dx$$

$$\leq \sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2 a_k\right|\left|\int_0^{\pi}\right|F_k(x)|\,dx + (n+1)|\Delta a_n|\left|\int_0^{\pi}\right|F_n(x)|\,dx$$

Since $\int_0^{\pi} F_k(x) = \frac{\pi}{2}$ and , by given hypothesis,

$$(n+1)|\Delta a_n| = \sum_{k=n}^{\infty}(n+1)\left[|\Delta a_k| - |\Delta a_{k+1}|\right]$$

$$\leq \sum_{k=n}^{\infty}(n+1)\left|\Delta^2 a_k\right| \leq \sum_{k=n}^{\infty}(k+1)\left|\Delta^2 a_k\right| \quad = O(1)$$

therefore, $\lim_{n\to\infty}\int_0^{\pi}\left|f(x)-g_n(x)\right|=0$.

Example 2. Let $\sum_{k=1}^{\infty}\left|\Delta a_k\right|\log k<\infty$. Then g_n converges to f in the L^1- metric space.

Solution: Consider

$$\int_0^{\pi}\left|f(x)-g_n(x)\right|=\int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right|$$

$$\le\sum_{k=n+1}^{\infty}\left|\Delta a_k\right|\int_0^{\pi}\left|D_k(x)\right|$$

Since $\int_0^{\pi}\left|D_k(x)\right|$ behaves like $\log n$ for large k, and $\sum_{k=1}^{\infty}\left|\Delta a_k\right|\log n<\infty$, we get

$\lim_{n\to\infty}\int_0^{\pi}\left|f(x)-g_n(x)\right|=0$.

CHAPTER III

NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY OF CERTAIN COSINE SUMS

3.1 Introduction

Consider the trigonometric cosine series

(3.1.1) $\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos kx$,

belongs to Class S([32],[41]) of coefficients sequences if there a monotone sequenc$\{A_k\}$

such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \le A_k$, for all k .

Concerning the L^1- convergence of cosine trigonometric series Telijakovskii [41**]** proved the following results.

Theorem A ([41]). Let $a_k \in S$. Then $f \in L^1(0,\pi)$, and $\|S_n - f\| = o(1), \ n \to \infty$

if and only if $a_n \log n = o(1)$ as $n \to \infty$.

In 1976 Garrett and Stanojevic [12] proved the following results.

Theorem B([14]). Let $\{a_k\} \in BV \cap C$. Then $f \in L^1(0,\pi)$, and $\|S_n - f\| = o(1), \ n \to \infty$

if and only if $a_n \log n = o(1)$, $n \to \infty$.

We define more general classes of coefficient sequences as follows:

The Class BV([14]): The coefficient sequences $\{a_n\}$ belongs to class BV if

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty$$

We see that if $\{a_n\} \in S$ **.**

$\Rightarrow \{a_n\} \in BV$ i.e. $S \subset BV$.

The Class *C*([12]). Let $\{a_n\} \in C$ if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$, independent of n , and such that for all n .

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon$$

In 1976 Garrett and Stanojevic [12] prove the following results.

Theorem C([14]). Let $\{a_k\} \in BV$ and let $a_n \log n = o(1)$, $n \to \infty$ .Then

$$\|S_n - f\| = o(1), \quad n \to \infty$$

if and only if $\{a_k\} \in C$.

The aim of this chapter is to study the necessary and sufficient condition for L^1-convergence of cosine trigonometric series under the coefficient class S^2 of sequences

defined as follows.

The Class S^2 ([14]). Let $\{a_n\}$ belongs to class S^2 if there exists a null sequence $\{A_k\}$ of non negative numbers such that $\sum_{k=1}^{\infty} k|\Delta A_k| < \infty$ and $|\Delta a_k| \le A_k$, for all k .

Clearly, class S^2 is an equivalent to class S .

3.3 Main Result:

We prove the following theorem

Theorem 1([14]). Let $\{a_k\} \in S^2$. Then $f \in L^1(0,\pi)$, and $\|S_n - f\| = o(1)$, $n \to \infty$

if and only if $a_n \log n = o(1)$ as $n \to \infty$.

Proof: First we shall show that the point-wise limit f of

(3.2.1) $$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

exists in $(0,\pi)$ and that f is a Fourier series, i.e. that $f \in L^1(0,\pi)$,

As $\{a_n\} \in S^2$, therefore

(3.2.2) $$nA_n \le n\sum_{k=n}^{\infty}|\Delta A_k| \le \sum_{k=n}^{\infty} k|\Delta A_k| = o(1)$$

Thus from

(3.2.3) $$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} k\Delta A_k + nA_n < \infty$$

$$\Rightarrow \ |\Delta a_k| \le A_k \quad \text{for all } k .$$

$$\Rightarrow \ \sum_{k=1}^{n}|\Delta a_k| \le \sum_{k=1}^{n} A_k$$

By (3.2.2), we have

which implies $\sum_{k=1}^{n}|\Delta a_k| < \infty$.

so, $\{a_k\} \in BV$.

Next we show that if $\{a_k\} \in S \Rightarrow \{a_k\} \in C$

To prove this, we first consider

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\right| dx \le \int_0^{\pi}\left|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\right| dx$$

By applying Abel's transformation, we have

$$\le \lim_{n\to\infty}\left[\sum_{k=n+1}^{N-1}(k+1)|\Delta A_k| + (N+1)A_N + (n+1)A_{n+1}\right]$$

By given hypothesis , we have

$\sum_{k=1}^{n} kA_k < \infty$ and using (3.2.2) , we have

$$\int_0^{\delta}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right| dx < \varepsilon$$

therefore, $\{a_k\} \in C$.

Using theorem B, we get the conclusion of main results.

CHAPTER IV

ON L^1-CONVERGENCE OF FOURIER COEFFICIENTS WITH COEFFICIENTS OF BOUNDED VARIATION OF ORDER m

4.1 Introduction

Consider the cosine trigonometric series

(4.1.1) $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$

Let the partial sum of (4.1.1) be denoted as $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Concerning the L^1-convergence of cosine trigonometric series, Garrett and Stanojevic [14] proved the following result:

Theorem A([14]). Let $\{a_k\} \in BV$ and let $a_n \log n = o(1)$, $n \to \infty$. Then

$$\|S_n - f\| = o(1), \; n \to \infty$$

if and only if $\{a_k\} \in C$.

Rees and Stanojevic [31] defined a class as follows:

The Class *C*([12]): Let $\{a_n\} \in C$ if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon)$, independent of n, and such that for all n .

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx$$

In this chapter, we generalized the result of Garrett and Stanojevic [14] considering the generalized class $(BV)^m$ of coefficient sequences defined by as follows:

The Class $(BV)^m$**(**[14]**).** A null sequence $\{a_k\}$ belongs to the class $(BV)^m$ if for some integer $m \geq 1$,

$$\sum_{k=1}^{\infty}\left|\Delta^m a_k\right| < \infty$$

Where,

$$\Delta^m a_k = \Delta(\Delta^{m-1}a_k) = \Delta^{m-1}a_k - \Delta^{m-1}a_{k+1}$$

4.2 Main Results

Theorem 1([14]). If $\{a_k\} \in (BV)^m$, then the point-wise limit g of

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} a_k + \sum_{k=1}^{n}\left(\sum_{j=k}^{n} a_j\right)\cos kx$$

exists in $(0,\pi]$.

Proof: Let $D_k(x)$ and $\tilde{D}_k(x)$ be the Dirichlet and Conjugate Dirichlet kernels,

respectively. Summation by parts (with $n > m$) yields

$$g_n(x) = \sum_{k=0}^{n} a_k D_k(x)$$

Since $\left|D_k(x)\right| < \left|\frac{1}{\left(\frac{2\sin x}{2}\right)}\right|$ for $x \in (0,\pi)$, if $\sum_{k=1}^{\infty}\left|\Delta^0 a_k\right| < \infty (m = 0)$

Now consider $m > 0$, we have

$$g_n(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)}\sum_{k=0}^{n} a_k\left[\frac{\cos x}{2\sin kx} + \frac{\sin x}{2\cos kx}\right]$$

$$= \begin{bmatrix} \frac{1}{2}\cot\frac{x}{2}\left[\sum_{k=0}^{n-1}\Delta a_k \tilde{D}_k(x) + a_n \tilde{D}_n(x)\right] \\ + \frac{1}{2}\left[\frac{a_0}{2} + \sum_{k=0}^{n-1}\Delta a_k D_k(x) + a_n D_n(x)\right] \end{bmatrix}$$

For $m=1$; $\sum_{k=1}^{\infty}\left|\Delta^1 a_k\right|<\infty\,(m=1)$ and $g_n(x)$ is the sum of a constant term, partial sums of two absolutely convergent series, and nth terms of two sequences that converge to zero.

Hence the result holds.

For $m>1$,

$$g_n(x)=\frac{1}{2}\cot\frac{x}{2}\left\{\sum_{k=0}^{n-1}\Delta a_k\left[\frac{\frac{\cos x}{2}-\cos\left(\frac{k+1}{2}\right)x}{\left(2\sin\frac{x}{2}\right)}\right]+a_n\tilde{D}_n(x)\right\}$$
$$+\frac{1}{2}\left[\frac{a_0}{2}+\sum_{k=0}^{n-1}\Delta a_k\frac{\sin\left(\frac{k+1}{2}\right)x}{2\sin\frac{x}{2}}+a_nD_n(x)\right]$$

Appying Abel's transformation m times; we get

$$g_n(x)=B_1(x)+A_n(x)+B_2(x)\sum_{k=0}^{n-m}\Delta_k^m D_k(x)+B_3(x)\sum_{k=0}^{n-m}\Delta^m a_k\tilde{D}_k(x)$$

where $B_1(x), B_2(x)$ *and* $B_3(x)$ do not depend on n and $A_n(x)$ converges to zero.

Thus the limit g exists in $(0,\pi)$.

Theorem 2[14]. Let $\{a_k\}\in(BV)^m$ and $a_n\log n=o(1),\ n\to\infty$. Then $\left\|S_n-f\right\|=o(1)$, $n\to\infty$ if and only if $\{a_n\}\in C$.

Proof: Using Summation by parts we get

$$S_n(x)=\sum_{k=0}^{n-1}\Delta a_k D_k(x)+a_nD_n(x)$$

and $f(x)=\sum_{k=0}^{\infty}\Delta a_k D_k(x)$ since a_n tends to zero and $\left|D_n(x)\right|$ is bounded in $(0,\pi)$.

Necessary part: Let $\varepsilon>0$. Then there exists $\delta>0$ such that

$$\int_0^\delta \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx < \frac{\varepsilon}{3} \quad \text{for all } \ n \ge 0$$

Then
$$\int_0^\pi |f(x) - S_n(x)|\, dx = \int_0^\infty \left|\sum_{k=n}^\infty \Delta a_k D_k(x) - a_n D_n(x)\right| dx$$
$$\le \int_0^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx + |a_n| \int_0^\pi |D_n(x)|\, dx$$
$$= \int_0^\delta \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx + \int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx$$
$$+ |a_n| \int_0^\pi |D_n(x)|\, dx \tag{4.2.1}$$
$$\le \frac{\varepsilon}{3} + \int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx + |a_n| \int_0^\pi |D_n(x)|\, dx$$

for sufficiently large n ,

$$|a_n| \int_0^\pi |D_n(x)|\, dx \approx o(a_n \log n)$$
$$= o(1) \quad \text{as} \quad n \to \infty .$$

Proceeding as in the proof of theorem 1, we get

$$\int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx \le \int_\delta^\pi |A_n(x)|\, dx + \int_\delta^\pi \left| B_1(x) \sum_{k=n}^\infty \Delta^m a_k D_k(x)\right| dx$$
$$+ \int_\delta^\pi \left| B_2(x) \sum_{k=n}^\infty \Delta^m a_k \tilde{D}_k(x)\right| dx$$

Where $\int_\delta^\pi |B_1(x) D_k(x)|\, dx$ and $\int_\delta^\pi |B_2(x) \tilde{D}_k(x)|\, dx$ are both uniformly bounded and $\int_\delta^\pi |A_n(x)|\, dx$ converges to zero.

By given hypothesis for large n , we get

$$\int_\delta^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k(x)\right| dx < \frac{\varepsilon}{3}$$

Therefore, from (4.2.1) $\|S_n - f\| < \varepsilon$; for sufficiently large n .

Sufficient part: Let $\varepsilon > 0$ **.** Then there exists an integer N such that

$$\int_0^{\pi} |f(x) - S_n(x)|\, dx < \frac{\varepsilon}{4} \quad if\ n \geq N$$

That is, $\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) - a_n D_n(x) \right| dx < \frac{\varepsilon}{4}\ if\ n \geq N$. Since $a_n \log n$ tends to zero and $\int_0^{\pi} |D_n(x)|\, dx = o(\log n)$, there exists an integer M such that

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2}\ if\ n \geq M$$

Now if $\sum_{k=0}^{M} |\Delta a_k| = 0$, then for $n < M$,

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx = \int_0^{\pi} \left| \sum_{k=M+1}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon$$

If $\sum_{k=0}^{M} |\Delta a_k| \neq 0$, let $\delta = \dfrac{\varepsilon}{(2M \sum_{k=0}^{M} |\Delta a_k|)}$.

For $n \geq M$, we have

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx \leq \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \frac{\varepsilon}{2} < \varepsilon$$

For $0 \leq n < M$, we get

$$\begin{aligned} \int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx &\leq \int_0^{\delta} \left| \sum_{k=n}^{M-1} \Delta a_k D_k(x) \right| dx + \int_0^{\delta} \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right| dx \\ &\leq \int_0^{\delta} \sum_{k=n}^{M-1} (k+1) |\Delta a_k|\, dx + \int_0^{\pi} \left| \sum_{k=M}^{\infty} \Delta a_k D_k(x) \right| dx \end{aligned}$$

$$< \delta \sum_{k=0}^{M-1} (k+1)\left|\Delta a_k\right| + \frac{\varepsilon}{2}$$

$$\leq \delta M \sum_{k=0}^{M-1} \left|\Delta a_k\right| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus the conclusion holds.

Remark: For $m = 0$; the theorem 2 reduces to Theorem A.

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