MEASURES & AGGREGATION OPERATORS ON FUZZY/INTUITIONISTIC FUZZY SOFT SETS WITH APPLICATIONS TO DECISION-MAKING

A Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Measures & Aggregation Operators On Fuzzy/Intuitionistic Fuzzy Soft Sets with Applications To Decision-Making"** in partial fulfillment of the requirement for the award of degree of Degree of Philosophy and submitted in the School of Mathematics (SoM), Thapar Institute of Engineering & Technology, Patiala is an authentic record of my own carried out during a period from July, 2015 to June, 2020 under the supervision of Dr. Harish Garg, Assistant Professor, SoM, Thapar Institute of Engineering & Technology Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

Multiple-criteria or multiattribute decision-making problems are the imperative part of modern decision theory where a set of alternatives has to be assessed against the multiple influential attributes before the best alternative is selected. In a decision-making(DM) process, an important problem is how to express the preference value. Due to the increasing complexity of the socioeconomic environment and the lack of knowledge or the data about the DM problems, it is difficult for the decision maker to give the exact decision as there is always an imprecise, vague or uncertain information. To deal with this, the theory of the fuzzy sets or its extensions such as intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, soft sets, vague sets, type-2 fuzzy sets, etc., are widely used by the researchers so as to minimize the uncertainty level. During the last decades, the researchers are paying more attention to these theories and have successfully applied it to the various situations in the DM process. Among these, an aggregation operator is an important part of the DM which usually takes the form of mathematical function to aggregate all the input individual data into a single one. However, an information measure such as the distance and similarity measures, complementary to each other, are defined to differentiate between the two or more objects. These two measures can be considered as two diverse perspectives of discrimination. The similarity measure utilized to show the proximity whereas the distance measure is utilized to show the contrast between the objects. In the existing theories and their corresponding work, there occur some limitations such as how to set the membership function in each particular object, insufficiency to consider the parameterizations tool, etc., and hence their corresponding analysis does not give the correct decision to the decision-maker. To overcome these limitations, Molodtsov in 1999, took the soft set (SS) theory where the preferences for each alternative is given on different

parameters. The advantage of this extended theory is that they are capable of facilitating the descriptions of the real-world situation with the help of their parameterized property. Thus in order to handle the information in a more accurate and certain manner, there is a need to plan/adopt suitable methodologies to solve the DM problems.

The objective of this research work is to develop some new methodologies under the intuitionistic fuzzy soft set (IFSS) environment by utilizing available information and uncertain data. For analyzing this, some aggregation operators and information measures are proposed for solving the problems under the intuitionistic and/or interval-valued IFSS information. The various desirable relations between the proposed measures and operators are studied. Later, based on the proposed measures, an efficient method is developed to solve the multicriteria DM (MCDM) problems in which information related to each alternative is assessed under the consideration of the experts as well as parameters features. The presented approaches are the extensions of the existing studies under the intuitionistic fuzzy set environment. Several real-life practical examples are taken to demonstrate the approach and compared their performance with some of the existing studies.

The present thesis is organized into eleven chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of decision making approaches by using several approaches is presented in the first chapter. In **Chapter 2**, the basics and preliminaries related to the intuitionistic fuzzy set, soft sets, aggregation operators, etc., are given.

Chapter 3 presents the concept of the dual hesitant fuzzy soft set to define the several axioms of distance and similarity measures based on the metric of Hamming, Euclidean, and Hausdorff. The desirable inequalities in the form of their relationships between them have also been discussed. Later, a DM approach is presented based on the proposed measures where the features of the objects are taken in the form of dual hesitant fuzzy soft numbers. The effectiveness of the proposed measures is demonstrated through concrete examples of pattern recognition and medical diagnoses and compare the study with the several existing approaches.

Chapter 4 deals with the correlation coefficient and weighted correlation coefficients under the DHFSS environment in which pairs of membership, non-membership are to be considered as vector representation during the formulation and to investigate their properties. Further, under this environment, an MCDM method based on the proposed correlation coefficients are presented. Three numerical examples, one from the selection procedure and other from the medical diagnosis and pattern recognition, are taken to demonstrate the efficiency of the proposed approach and compared their results with the several existing approaches results.

Chapter 5 utilized the idea of dual hesitant fuzzy soft sets (DHFSS) and presents some new aggregation operators, namely, dual hesitant fuzzy soft weighted averaging and geometric operators proposed along with their proofs to aggregate the different preferences of the decision makers. Various desirable properties of its are also investigated in details. Further, these aggregation operators are extended to its generalized operator by incorporating the attitude character of the decision maker towards the data. Later, a MCDM approach is presented based on the proposed operators for solving DM problems. An illustrative example is taken to demonstrate the approach under DHFSS environment and compared their results with some of the existing approaches results.

In **Chapter 6**, we present a nonlinear programming (NLP) model based on the technique for order preference by similarity to ideal solution (TOPSIS), to solve multi-attribute DM problems. In this approach, both ratings of alternatives on attributes and weights of attributes are represented by IVIF sets. Based on the available information, NLP models are constructed on the basis of the concepts of the relative-closeness coefficient (RCC) and the weighted distance. A real example is taken to demonstrate the applicability and validity of the proposed methodology.

In **Chapter 7**, we have defined some new averaging and geometric prioritized aggregation operators in which the preferences related to attributes are taken in the form of IFSSs. Desirable properties of its have also been investigated. Furthermore, based on these operators, an approach to investigate the multi-criteria decision making (MCDM) problem has been presented. The effectiveness of these operators has been demonstrated through a case study.

In **Chapter 8**, we present two new scaled prioritized averaging aggregation operators by considering the interaction between the membership degrees for the IFSSs. The principal advantage of the operators is that they consider the priority relationships between the parameters as well as experts. Furthermore, some properties based on these operators are discussed in detail. Then, we utilized these operators to solve DM problem and validate it with a numerical example.

Chapter 9 presents some new Bonferroni mean(BM) and weighted BM averaging operator for aggregating the different preferences of the decision maker. Some of its desirable properties have also been discussed in details. Further, a DM method based on proposed operators has been presented and then illustrated with a numerical example. A comparative analysis between the proposed and the existing measures under the IFSS environment has been performed in terms of counter-intuitive cases for showing the validity of it.

In **Chapter 10**, we develop some new power aggregation operators based on the Archimedean t-norm operators for IFSSs. The developed AOs are named as intuitionistic fuzzy soft (IFS) power averaging and geometric operators, weighted IFS power averaging and geometric operators, weighted and ordered weighted IFS power averaging and geometric operators. The salient features of these operators are also discussed in details. Further, these operators are extended to generalized IFS power averaging or geometric aggregation operators. Finally, the applicability of these developed operators is explained with a numerical example related to decision making problems. A comparative analysis of existing approaches has been done for showing the validity of the proposed work.

Chapter 11 presents the concept of generalized IFSS (GIFSS), as well as the group-based generalized IFSS (GGIFSS) in which the evaluation of the object is done by the group of experts rather than a single expert. Based on this information, a new weighted averaging and geometric aggregation operator has been proposed by taking the intuitionistic fuzzy parameter. Finally, a DM approach based on the proposed operator is being built to solve the problems under the intuitionistic fuzzy environment. An illustrative example of the selection of the optimal alternative has been given to show the developed method. Comparison analysis between the proposed and the existing operators have been performed in term of counter-intuitive cases for showing the superiority of the approach.

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List of Publications

Refereed Journals

(J1) Harish Garg, Rishu Arora, Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multi criteria decision-making, *International Journal of Intelligent Systems*, 34(2), 215 -246, 2019, doi: 10.1002/int.22048 (**SCI: Impact Factor 7.229**).

(J2) Harish Garg, Rishu Arora, Dual hesitant fuzzy soft aggregation operators and their application in decision making, *Cognitive Computation*, 10(5), 769 - 789, 2018, doi: 10.1007/s12559-018-9569-6 (**SCI: Impact Factor 4.287**)

(J3) Rishu Arora, Harish Garg, A robust correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making, *Engineering Applications of Artificial Intelligence*, 72, 80 - 92, 2018, doi: 10.1016/j.engappai.2018.03.019 (**SCI: Impact Factor 3.526**)

(J4) Harish Garg, Rishu Arora, Novel scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision making, *Engineering Applications of Artificial Intelligence*, 71, 100 - 112, 2018, doi: 10.1016/j.engappai.2018.02.005 (**SCI: Impact Factor 3.526**)

(J5) Harish Garg, Rishu Arora, A nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information, *Applied Intelligence*, 48(8), 2031 - 2046, 2018, doi: 10.1007/s10489-017-1035-8 (**SCI: Impact Factor 2.882**)

(J6) Harish Garg, Rishu Arora, Generalized and Group-based Generalized intuitionistic fuzzy soft sets with applications in decision-making, *Applied Intelligence*, 48(2), 343 - 356, 2018, doi: 10.1007/s10489-017-0981-5 (**SCI: Impact Factor 2.882**).

(J7) Harish Garg, Rishu Arora, Bonferroni mean aggregation operators under intuitionistic fuzzy soft set environment and their applications to decision-making, *Journal of the Operational Research Society*, 69(11), 1711 - 1724, 2018, doi: 10.1080/01605682.2017.1409159 (**SCI: Impact Factor 1.754**)

(J8) Rishu Arora and Harish Garg, Prioritized averaging/geometric aggregation operators under the intuitionistic fuzzy soft set environment, *Scientia Iranica*, 25(1), 466 - 482, 2018, doi: 10.24200/SCI.2017.4410 (**SCI: Impact Factor 0.718**).

(J9) Rishu Arora and Harish Garg, Robust aggregation operators for multi-criteria decision-making with Intuitionistic fuzzy soft set environment, *Scientia Iranica*, 25(2), 931 942, 2018, doi: 10.24200/SCI.2017.4433 (**SCI: Impact Factor 0.718**).

(J10) Harish Garg, Rishu Arora, Distance and similarity measures for dual hesitant fuzzy soft sets and their applications in multi-criteria decision making problem, *International Journal for Uncertainty Quantification*, 7(3), 229 - 248, 2017, doi: 10.1615/Int.J.UncertaintyQuantification.2017019801 (**SCI: Impact Factor 3.259**).

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Chapter 1

Introduction

Decision-making (DM) is a subjective process utilized in both businesses and academic classes for the design of a strategy among the given alternatives scenario by processing the different preferences of the expert(s). To complete the process, the most important problem is to choose the appropriate decision about the task and to find the optimal results. DM is a crucial task for all those professions where experts apply their knowledge in a given zone to take the appropriate decision. However, in many situations, the alternatives among which one must choose or make decisions need to be evaluated in light of multiple criteria. For it, generally, a set of decision makers has taken to evaluate the given alternatives. The focus of such a process is to assign the preferences of decision makers together to solve a specific issue by following some general principle of the decision-making (DM) process. The general framework of the DM problem to select the best option(s) takes the steps shown in Figure 1.1.

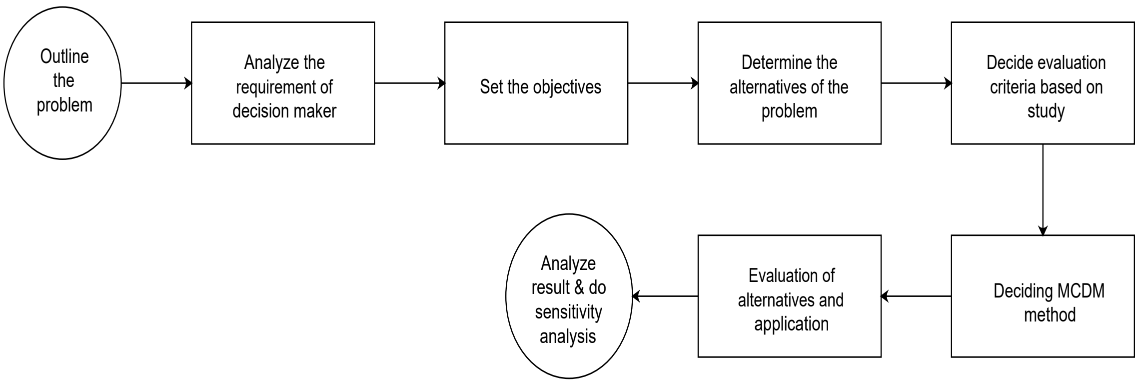

Figure 1.1: Framework of Decision-making process

The development of multi-criteria DM (MCDM) began in 1971, whose main objective is to provide a tool to the decision makers to solve the problem whose several criteria are taken into account simultaneously. In literature, many terms have been used for MCDM such as multi-criteria decision analysis(MCDA), multi-objective DM (MODM), multi-attribute DM (MADM), etc. and have been frequently used by the researchers to solve real-world DM problems. Generally, MCDM issue is explained in the two-stage process: (i) the aggregation of the estimations of criteria for every option (ii) the positioning or ranking between the options.

The general process of MCDM problem consists a set of alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and criteria $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$ such that it is divided into two mutually disjoint sets namely, F_1(the cost criteria) and F_2(the benefit criteria). The importance factor of these criteria is given in the form of weight vector $(w_1, w_2, \ldots, w_m)$ such that $w_j > 0$ and $\sum\limits_{j=1}^{m} w_j = 1$. The decision matrix corresponding to given alternatives is given as:

$$\begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_n \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{G}_1 & \mathcal{G}_2 & \ldots & \mathcal{G}_m \end{array} \\ \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1m} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{n1} & \mathcal{A}_{n2} & \ldots & \mathcal{A}_{nm} \end{pmatrix} \end{array}$$

where $\mathcal{A}_{ij}$ represents the preference of decision maker for alternative $\mathcal{A}_i (i = 1, 2, \ldots, n)$ over the criteria $\mathcal{G}_j (j = 1, 2, \ldots, m)$.

However, due to various constraints in day-to-day life, decision makers may unable to give their judgments in exactly crisp form. Thus, to handle it, they prefer to give their preferences under the uncertain and imprecise in nature by using the theory of fuzzy set(FS)[255]. Using this theory, decision makers use the membership degree(MD) to represent their judgment. But with the growing complexity, there always exists a degree of hesitancy between the preferences of the decision making and hence the analysis conducted under such circumstances is not ideal. To address it, an important extension of FSs such as intuitionistic fuzzy set(IFS)[8], hesitant fuzzy set[194], Neutrosophic set[178], the interval-valued IFS[7] and so on appears simultaneously. All the extensions are characterized by non-membership degree(NMD) along with MD such that their sum is less than one. As these theories have effectively connected, however, at the same time it has been seen that they have their inherent difficulties related to considering the parametrization tool. To deal with them and to provide the decision making more smooth and exact, Molodtsov [139] proposed a new generalized type of fuzzy set called soft set. Later on, this set is hybridized with FS and IFS and Maji et al. [129, 130] came up with the concept of the fuzzy soft set(FSS) and intuitionistic fuzzy soft set(IFSS) respectively. In the present scenario of DM, MCDM is an important topic for a decision maker under the soft set environment to evaluate the considered alternative.

The objective of this work is to develop some novel techniques to access the best alternative(s) for the decision makers under the soft set environment. Brief literature on various issues related to the MCDM process by using existing methods have been reviewed and are given section-wise hereafter.

1.1 Literature Review

In this section, a brief literature review with respect to various MCDM methods under different scenario is given.

1.1.1 Review of distance/similarity measures

Distance and similarity measures, complementary to each other, are defined to differentiate between the two or more objects. These two measures can be considered as two diverse perspectives of discrimination. The similarity measure utilized to show the proximity whereas the distance measure is utilized to show the contrast between the objects. Since the distance and similarity measures play a significant role in real life, numerous analysts have been defined by various distance and similarity measures in different environments. For instance, Szmidt and Kacprzyk [185], firstly defined basic distances such as Hamming and Euclidean to distinguish two IFSs. Later on, Grzegorzewski [69] proposed some distance measures based on the Hausdorff metric. Wang and Xin [208] proposed a generalized definition of distance measure under IFS environment. Dengfeng and

Chuntian [38] introduced the concept of similarity measure to find the closeness degree between the two IFSs. However, Liang and Shi [112] found some drawbacks in [38] and introduced an improved similarity measure to overcome their limitations. Based on a statistical viewpoint, Mitchell [136] developed a modified Dengfeng-Chuntain similarity measure. Afterward, Hung and Yang [86] presented the similarity measures based on Hausdorff distance. Szmidt and Kacprzyk [186] proposed a similarity measure and discussed its importance in medical diagnosis. Li et al. [111] analyzed the existing similarity measures and summarized their limitations with some counter-intuitive examples. Furthermore, Xia and Xu [218] developed the aggregation operator (AO) for IFSs based on the similarity measures. Ye [247] highlighted the errors of existing similarity measure[111] and constructed the cosine similarity, weighted cosine similarity measure to solve pattern recognition and medical diagnosis problems. Besides that, Hwang et al. [88] introduced a new similarity measure induced by Sugeno integral. Based on the medians of intervals and the Hausdorff distance, Chen and Randyanto [29] discussed a similarity measure for IFSs. Later on, Boran and Akay [17] proposed a parametric similarity measure for IFSs and done a comparative analysis with some existing similarity measures[38, 112, 136, 247] in terms of counter-intuitive cases. Song et al. [179] discussed the similarity measure and applied it to the problem of pattern recognition. Chen, Cheng and Lan [27] presented a similarity measure based on the centroid points of transformed right-angled triangular to overcome the limitations of existing measures [17, 38, 86, 112]. Garg [58] characterized the distance and similarity measures for intuitionistic multiplicative preference relation and its application in DM issues. Later on, Song et al. [180] proposed a similarity measure to overcome the restrictions of existing measures [17, 38, 86, 88, 247]. Jiang et al. [90] defined distance and similarity measure based on the transformed triangles and discussed their application in pattern recognition.

Soft set theory is one of the successful extensions of the existing theories by considering the parameterization factor during an analysis. After it, Maji et al. [129, 130] combine the feature of SS with FSs and IFSs and came up with the concept of FSS and IFSS. Under these environments, researchers are paying attention to develop some information measures to solve DM issues. For example, Majumdar and Samanta [131] proposed the distance based similarity measures to compare two SSs. Kharal [100] gave some counterexamples to show the limitations of the measures proposed by Majumdar and Samanta [131]. Majumdar and Samanta [132] introduced three different approaches for defining the concept of a similarity measure for FSSs. Rajarajeswari and Dhanalakshmi [160] and Sut [184] discussed the notion of the similarity measures for handing DM problems under FSS environment. Sulaiman and Mohamad [181] proposed a set-theoretic similarity measure for FSSs and applied it to solve DM problems. Feng and Zheng [53] defined similarity measures based on different distance measures of FSSs. Later on, Liu et al. [126] developed similarity measures associated with fuzzy equivalences for FSSs. Cagman and Deli [21] proposed the similarity measure and applied it to DM problems. Besides that, Rajarajeswari and Dhanalakshmi [161] introduced three distinct approaches such as matching function, distance measure and set-theoretic, of similarity measures to compare two IFSSs. Later on, Muthukumar and Krishnan [142] defined the similarity and weighted similarity measures for IFSSs and illustrated their application in medical diagnosis. The distance-based similarity measure is proposed by Sarala and Suganya [168]. Moreover, Selvachandran, Maji, Faisal and Salleh [171] introduced distance measures based on Hamming and Euclidean distances and applied it to solve DM problems. Apart from them, some other distance/similarity measures and their applications to solve the DM problems are summarized in [96]. Thus, based on the above literature, it can be observed that the distance and similarity measures play an important role for handling the MCDM problems in various fields such as image processing, pattern recognition, medical diagnosis, Machine learning, and market prediction problems.

1.1.2 Review of TOPSIS approaches

The study described in Section 1.1.1 is based on the distance/similarity between the sets to rank the given alternatives. However, apart from them, in our day-to-day life, it is always possible to determine the best alternative(s) by measuring its values from its ideal states. To address it, a TOPSIS ("Technique for order preference by similarity to ideal solution") is one of the pioneer method developed by Hwang and Yoon [87]. The prominent characteristics of TOPSIS method are that they not take into account the distance/similarity between the sets but also examine similar or dissimilar between the sets so as to avoid to draw the conclusion based on the small distance or large similarity. The underlying principle of the TOPSIS method is to select the best alternative whose distance is farthest from its negative ideal. By utilizing these features, several researchers have addressed the issues of TOPSIS method during the process of MCDM methods. For example, Chen [24] initiated the concept of TOPSIS technique under fuzzy environment to solve the MCDM problems. However, Chen et al. [31] extended the concept of TOPSIS method from FSs to IFSs environment by using various existing distance measures [38, 69, 86, 112, 136, 208] and then analyzed their results. Boran et al. [18] developed the TOPSIS technique and discussed their application in handling the DM problem of supplier selection. Li [105] formulated a TOPSIS method based non-linear programming model under the interval-valued IFS (IVIFS) environment, where, as well as the weights, are taken as an interval. Ye [245] presented the TOPSIS method to solve DM problems with incomplete and uncertain information under the IVIFS environment. Tan [191] presented a TOPSIS method for group DM problems using Choquet integral-based Hamming distance for IVIFSs. Park et al. [150] constructed an optimization model based on TOPSIS approach to determine the criterion weights under IVIFS environment.

Zhang and Yu [259] constructed an optimization model based on TOPSIS model by utilizing cross-entropy to determine the weight of attributes under the IVIFS environment. Bai [13] presented an improved score function based TOPSIS method for solving DM problems. Chen, Cheng and Tsai [28] presented a generalized improved score function for IVIFS and hence its based approach to solving the DM problems under IVIFS environment. Joshi and Kumar [93] developed a TOPSIS method based on an entropy and distance measure to solve the problems under the IFS environment. Li et al. [110] connected the quality function deployment framework with the TOPSIS method to handle MCDM problems by considering two different types of criteria including customer and system criteria. Chen [30] defined two optimization models to optimize the TOPSIS approach with the IVIFS information and then inclusion-comparison possibilities to rank the different alternatives. Zhang and Xu [261] presented a soft-computing technique based on maximizing consensus and TOPSIS to solve MCDM problems. Later on, Chen, Cheng and Lan [26] proposed a similarity measure based TOPSIS method under IFS environment to overcome the drawbacks of some existing methods[93, 217]. Later on, Wang and Chen [200] combined the linear programming model with the TOPSIS method to tackle MCDM problems. Vommi [198] developed a TOPSIS method with statistical distance and applied them to solve the DM problem. Gupta et al. [72] presented an algorithm for solving group DM problem based on extended TOPSIS method by adding an indeterminacy degree to the IVIFS values. Shen et al. [174] presented a TOPSIS method based on an improved distance measure to solve MCDM problems under IFS environment and illustrated their advantages over the existing approaches [93, 245]. However, apart from them, some other TOPSIS approaches and their applications to solve the DM problems are summarized in [67, 133].

Researchers are extensively using the above theories and their respective approaches, but parametrization tools are not sufficiently taken into account during the DM process. To overwhelm it, Soft set theory[139] and their extensions[129, 130] are the successful theories which can address the uncertainties more effectively. Based on the SS theory, Eraslan [44] defined a TOPSIS method to solve DM problems. Further, Eraslan and Cagman [45] proposed the technique by combining the TOPSIS and gray method for FSS environment. Liu and Liu [123] presented a method to determine entropy weights and then proposed a technique based on TOPSIS to solve the DM problem. Eraslan and Karaaslan [46] presented a group DM approach based on TOPSIS method for IFSS information. Further, Yang and Peng [244] defined a TOPSIS approach based on IVIFSS models with incomplete weights. Apart from these, some other researchers have investigated the DM problems under different environments, for such reads, we refer to [35, 37, 40, 64, 66, 67, 128, 143, 163, 164, 172].

1.1.3 Review of Correlation coefficients

Correlation coefficient plays a crucial part in statistical analysis to measure the strength of the linear relationship between the two variables. In statistical theory, the observations ought to be depicted by a certain probability distribution. However, in day-to-day life, the observations are sometimes unpredictable due to the lack of information provided by an expert or deficiency of the paper knowledge. Hence, in the light of such information, it is difficult to measure the correlation between the two variables. To describe it, the concept of the correlation coefficient is extended to the IFNs and IVIFNs by Gerstenkorn and Manko [68] and Bustince and Burillo [19] respectively. Saneifard and Hassasi [167] addressed the problem of parametric interval approximation of fuzzy numbers with fuzzy correlation. A fuzzy correlation coefficient method is presented by Wu and Hung [215], which can be utilized for cases with two information sets of fuzzy intervals, the one information set may be a fuzzy interval and second is a real number or both are real numbers. Ruan et al. [166] proposed correlation coefficient method integrated with centroid technique to solve DM problem having data in the form of fuzzy numbers.

After their existence, several researchers put forwards many works on correlation coefficient to solve the DM problems under IFS environment. For example, Hung [83] defined the correlation coefficient based on the statistical view-points for IFSs. Hung and Wu [85] presented a centroid method based correlation coefficient between the two IFSs to show their positively or negatively correlated data. Mitchell [137] improved the existing correlation coefficient [83] by defining new ones for IFSs. Hong [78] defined the correlation coefficient by using an extension principle of a fuzzy number under weakest t-norm operations. Xu [226] defined the correlation coefficients of IFSs and its applications towards medical diagnosis problems. Also, it is mentioned by them that the existing correlation coefficient [19, 83, 85, 137] of IFSs does not satisfy the reflexive property. Besides that, Park, Kwun, Park and Park [147], Park, Lim and Lee [148] proposed the correlation coefficient for IVIFSs and IFSs, respectively and discussed their applications in DM problems. Szmidt and Kacprzyk [187] proposed the correlation coefficient for IFSs which capture the relative strength between the two IFSs. Also, Szmidt and Kacprzyk [188] introduced the Spearman rank correlation coefficient for IFSs.

Szmidt et al. [189] discussed the Pearson correlation coefficient for IFSs which is described by three components i.e. membership, non-membership, and hesitation degrees. Wei et al. [213] defined the weighted correlation coefficient measure in which weights are determined by an optimization model based on the max-min operator. Hanafy et al. [73]

discussed the correlation coefficient based on the centroid method to capture the relationship between two IFSs and also extended this method for IVIFSs. Further, Szmidt et al. [190] developed the concept of the Kendall rank correlation coefficient and illustrated it with the examples. Liu et al. [117] developed the correlation coefficient based on the variance and covariance of IFSs and extended it to describe the correlation coefficient for IVIFSs. Later on, Robinson [162] discussed the correlation coefficient contrasting with distance measure for solving DM problems having interval-valued intuitionistic trapezoidal fuzzy numbers. Qu et al. [158] utilized the Choquet integral for describing the correlation coefficient. However, Huang and Guo [82] highlighted the drawbacks of existing approaches[224, 226] and developed an improved correlation coefficient for IFSs. Recently, Xuan Thao [236] developed the correlation coefficient based on the variance and covariance of IFS and discussed their application for solving the DM problem of pattern recognition. The correlation coefficients between two intuitionistic multiplicative sets have been presented by Garg [60].

In the above theories, the preferences given by the experts are supposed to be either a single number or an interval. But, these theories have their inherent difficulties such as how to establish membership when decision makers hesitate to express their preferences over alternatives. To handle such cases, Torra [194], Torra and Narukawa [195] introduced hesitant fuzzy set(HFS) theory in which preferences are neither a single number nor an interval, but a set of possible values between 0 and 1. Further, Zhu et al. [267] extended this theory to dual hesitant fuzzy set(DHFS) which comprises two sets. Xu and Xia [232] introduced the distance and correlation measure to measure the relationship between two HFSs. Chen et al. [25] proposed the correlation coefficient for HFSs and described their application in clustering analysis. Wang et al. [203] introduced the correlation coefficient under the DHFS environment. Later on, Farhadinia [48] proposed the correlation coefficient for DHFS and discussed its application in DM problem of medical diagnosis. A weighted correlation coefficient is defined by Ye [248]. Under the DHFS environment, Tyagi [196] proposed the correlation coefficient to solve DM problems. Meng and Chen [135] presented the Shapley weighted correlation coefficients to solve the DM problem having HFS data with incomplete weights. Further, Liao et al. [116] pointed out the errors of existing measures[25, 232] and developed an improved correlation coefficient measure for HFSs. Tong and Yu [193] defined some weighted correlation coefficients for HFSSs to solve the DM problem. Liao and Xu [115] defined the entropy and correlation coefficient measures for HFSs. Recently, Sun et al. [182] presented a TOPSIS approach based on correlation coefficient for HFSs. Yang et al. [243] discussed the weighted correlation coefficient and applied this for solving DM problems of supplier selection and medical diagnosis. Guan et al. [70] pointed out the errors of existing correlation coefficients[25, 116, 232] and developed an improved correlation coefficient measure based on the mean, variance and length of each HFS.

The above theories and their respective approaches are used extensively by researchers, but the parameterization tool during DM process is insufficiently considered. To overcome this, Babitha and John [11] introduced the hesitant FSS (HFSS) theory in which preferences are given over different parameters. Zhang and Shu [257] further extended this theory to dual hesitant FSS (DHFSS). Das, Malakar, Kar and Pal [36] introduced the correlation coefficient for measuring the relationship between two HFSSs and illustrated with an example of a DM problem. From the literature, we can conclude that the research is progressing quickly on correlation coefficient under different environments and it has applications in various fields such as image processing, pattern recognition, medical diagnosis, Machine learning, etc.

1.1.4 Review of Aggregation operators

The AO is the most useful information fusion tool that plays an imperative role in the DM process and is important due to the significant applications in different fields. An AO usually takes the form of a mathematical function in which multiple inputs can be combined and fused into one representative value. In the aggregation process, we ought to consider two prospectives: one is the functions and the other is operations. In the functions, the traditional arithmetic mean operator[227] and geometric mean operator[231] can only aggregate all the input individual information into one single value. Several extensions of these AOs have been developed such as power operator[238], prioritized operator[241], Bonferroni mean operator[242], etc. which consider the dependency between the attributes. On the other hand, for operations, there are a lot of t-norms and t-conorms are defined in literature such as algebraic t-norm and t-conorm, einstein t-norm and t-conorm, Hamacher t-norm and t-conorm, frank t-norm and t-conorms. With the combination of these functions and operations, the aggregation process has become an active research area for solving MCDM problems and a lot of research has also been developed in different fields.

For instance, Yager [237] introduced an ordered weighted averaging(OWA) operator to aggregate the data. Chiclana et al. [33] proposed the ordered weighted geometric(OWG) operator and discussed their applications in MCDM problems. Xu [227] defined the weighted averaging(WA) AOs and Xu and Yager [231] developed the weighted geometric(WG) AOs based on algebraic norms. Further, Wei and Wang [214] proposed the OWG AO and intuitionistic hybrid geometric AO to solve the DM issues. Li [104] extended the generalized OWA AO introduced by Yager [240]. Wei [211] suggested and applied induced OWG AOs to the group DM problems. Zhao et al. [263] proposed the generalized intuitionistic WA, generalized intuitionistic OWA and generalized intuitionistic hybrid averaging AOs. Moreover, Beliakov et al. [15] discussed various averaging operators based on triangular norms. Wang and Liu [209] developed some AOs based on Einstein norms. Xia et al. [219] proposed the AOs based on Archimedean norms which are the generalization of other norms such as algebraic, Einstein, Hamacher and frank norms. Zhao and Wei [264] developed some hybrid AOs based on Einstein norms. He et al. [74] introduced some improved operations for IFSs and utilized these to developed averaging and geometric AOs. The main advantage of these operations is that these consider the interaction between the membership and non-membership degrees of the alternative. All the above AOs assume that the aggregated arguments are independent i.e. these operators do not consider any relationship between the input arguments. However, in real life situations, there always exists a dependency between the diverse criteria such as prioritization, support and impact each other which plays a dominant role during the process of aggregation. In order to make an optimum decision, it is essential to take this interdependence into account in the aggregation process.

To overcome this situation, Yager [238] introduced the power average(PA) operator in which the weight vectors depend upon the input contentions and permit values being aggregated to support and strengthen each other. By taking advantage of power operator, Xu and Yager [233] defined the power geometric(PG), power ordered geometric(POG) and power ordered weighted geometric(POWG) and presented an approach for solving DM problem based on uncertain multiplicative preference relations. Xu [223] developed a series of AOs based on PA operator[238] and discussed various properties on these operators. Zhou et al. [265] defined some generalized power AOs and discussed their application in DM problems. Moreover, Yager [241] established the prioritized AO that considers the prioritized relationship between criteria and the associated weights depend on the satisfaction of the higher priority criteria. Yu [249] defined the generalized intuitionistic fuzzy prioritized WG operator based on Archimedean norms and discussed its application in MCDM problems. Yu and Xu [254] proposed different AOs based on the prioritized AO [241]. Another AO, Choquet integral is developed by Choquet [34], in which the operator not only take into account the significance of the elements or their position but can also reflect the correlations between elements or their ordered positions. Further, Yager [239] developed an induced Choquet integral AO and discussed its applications. Xu [229] discussed the averaging and geometric operator based on Choquet integral to aggregate interval-valued intuitionistic fuzzy information. Xu et al. [222] described the AOs based on Choquet integral and Einstein norms. Ngan et al. [146] presented an AO for solving the medical diagnosis problems.

In order to consider the interrelation between two criteria, Bonferroni means (BM) operator is introduced by Bonferroni [16]. The difference between these two operators is that the BM operator can reflect the interrelationship between two criteria but it ignores the interrelationship of criteria with itself. Although the HM operator has the same expression it overcomes the limitation of BM operator. By taking advantage of BM operator, Yager [242] generalized the BM operator by replacing simple average operator with another mean operator such as OWA and Choquet integral to aggregate the crisp values. Xu and Chen [225] established the BM operator to fuse the data given as intuitionistic fuzzy numbers (IFNs). Xu and Chen [225] introduced the BM operator to solve the MCDM problems under the interval-valued intuitionistic fuzzy environment. Later on, Xia et al. [221] described the generalized weighted BM averaging and geometric operators and presented an approach to solving MCDM problems. Liu and Chen [118] proposed the HM AOs based on Archimedean norms for solving the DM problem having intuitionistic fuzzy set information. Besides that, Maclaurin [127] introduced the concept of Maclaurin symmetric mean(MSM) operator which was further developed by DeTemple and Robertson [39]. The conspicuous characteristic of the MSM operator is that the relationship between the multi-input arguments can be captured. The principal difference between MSM and BM is that MSM reflects the interaction between the multi-input arguments, while the BM reflects only the connection between the two arguments. Qin and Liu [155] extended this operator to develop MSM operators for aggregating the intuitionistic fuzzy information. Later on, Ju et al. [94] extended the MSM to the intuitionistic linguistic environment and illustrated it for solving the DM problem. Garg [57] and Garg [54] proposed the intuitionistic fuzzy interactive averaging and geometric AOs based on Einstein norms. Further, Garg [55] defined some intuitionistic fuzzy multiplicative interactive weighted, ordered weighted and hybrid weighted geometric AOs. He, He and Shi [75], He, He, Zhou and Deng [76] developed some scaled prioritized aggregation operations and discussed their application in MCDM problems. Wang, Fu, Meng and He [199] defined scaled prioritized intuitionistic fuzzy geometric interaction averaging and weighted geometric interaction averaging operators. Later on, Garg [59] defined some improved operations based on Einstein norms and utilized these operations to develop the weighted, ordered weighted and hybrid weighted AOs under IFS environment. Garg et al. [63] proposed the scaled prioritized interactive AOs for interval-valued IFNs. Liu and Li [119] established some Muirhead mean operators for IFNs. The main advantage of these operators is that they consider the interconnection between any number of arguments. Liu and Li [120] developed the power Bonferroni mean operator to aggregate the interval-valued intuitionistic numbers. Moreover, Wu et al. [216] developed some Dombi Hamy mean operators based on Dombi operations[43]. Liu, Liu and Chen [121] proposed the Dombi Bonferroni mean AOs and discussed the MCDM process by utilizing these operators. Tao et al. [192] developed weighted AOs by defining the Archimedean copulas and corresponding co-copulas. Besides them, some additional

AOs together with their applications to solve DM problems are summarized in [122, 140]. Recently, Garg [61] highlighted the limitations of existing Hamacher operator and proposed improved Hamacher AOs to overcome the shortcomings. Wang and Mendel [207] investigated that there is a lack of monotonicity property in existing prioritized AOs and resolved this problem by defining a new AO which satisfy the monotonicity property.

Researchers utilize the above theories and their respective approaches extensively, but the parameterization tool is not adequately taken into consideration during the DM process. To handle this difficulty, soft set(SS)[139] and their extensions such as FSS [130] and IFSS [129] etc. are the mathematical tools which can address the uncertainties more effectively. By taking advantage of FSS, Balami and Onyeozili [14] introduced a fuzzy soft AO and discussed their application in MCDM problems. A systematic review of different approaches under the SS theory is done by Khameneh and Kılıçman [99]. Feng et al. [52]developed an algorithm to deal with DM issues with a combined use of the generalized intuitionistic soft set, extended intersectional operation, weighted averaging operator and other related concepts. Recently, Chang [23] defined an intuitionistic fuzzy weighted averaging method to solve the DM problem of supplier selection. Hu et al. [79] developed the weighted intuitionistic fuzzy soft Bonferroni mean operator for the IFSSs, in which weights are determined by the maximizing deviation method. A survey on different algorithms of parameter reduction based on some types of (fuzzy) soft sets is presented in [3, 256]. However, for more studies about the soft sets and its applications, we refer to [2, 5, 9, 10, 20, 22, 42, 51, 125, 144, 145, 165, 169, 170].

1.1.5 Review of dual hesitant fuzzy soft set

The above sections deal with the various methods and algorithms for solving DM problems by using some AOs and information measures with either a crisp or an interval form numbers. However, apart from them, there may be another possible way to express the decision-maker preference in terms of the possible values between 0 and 1 rather than a crisp or in $[0, 1]$. The set which describes such possible information is named as hesitant fuzzy set(HFS)[194, 195]. For example, if decision maker rates certain objects with their satisfaction level as either 0.7 or 0.75 or 0.8 then such values are represented in terms of HFS as $\{0.7, 0.75, 0.8\}$ which is completely differ from interval $[0.7, 0.8]$ information. Hence, due to the much successful representation of the preferences of the decision-maker, the HFS theory is much attention drawn by the scholars and hence solved the various DM problems. In terms of the AOs, Xia and Xu [220] developed some series of operators to aggregate the different hesitant fuzzy numbers. Yu et al. [252] utilized the Choquet integral to define the AOs for HFSs. Zhu et al. [266] defined the Bonferroni mean geometric AOs which reflect the conjunction between two inputs and applied it to MCDM problems. Meng and Chen [134] utilized the Shapley function to define an operator for interval-valued hesitant fuzzy sets and explained them by presenting an algorithm to solve MCDM problems. To capture the interrelation between multi-inputs, Qin et al. [156] developed the Maclaurin symmetric mean operator for HFSs and presented an approach to solve DM problems. Qin et al. [157] defined the AOs based on Frank triangular norms under HFS environment.

Apart from AOs, some information measures under HFSs are also defined by scholars to solve MCDM problems. For example, Farhadinia [47] defined entropy, distance and similarity measures and solve the problem of data analysis and classification based on it. The correlation coefficients of HFSs is defined in [25, 116] and explained with the problem of clustering analysis. Also, Meng and Chen [135] defined the correlation coefficient for solving DM problems by using Shapley fuzzy measures. Li et al. [108] defined the distance and similarity measures for the HFSs and discussed their application in MCDM problems. Hu et al. [81] developed the similarity and entropy measure based on the distance measures for HFSs. Liao and Xu [115] defined the correlation and entropy measures for HFSs. Xu and Zhang [235] developed an approach based on TOPSIS method to solve the MCDM problem with incomplete weight information. However, Zhang and Xu [260] presented a linear programming technique with hesitant fuzzy information to solve DM problems. Sun and Ouyang [183] utilized the entropy to extract the weight and hence developed the TOPSIS method under hesitant fuzzy information. Guan et al. [70] defined the correlation coefficient based on the mean, variance and length of HFSs. Yang et al. [243] pointed out some deficiencies in the existing correlation coefficients[25, 116, 232] and hence defined an improved correlation coefficient measures for them. Sun et al. [182] extended the

TOPSIS method based on the correlation coefficients for HFS information. Farhadinia and Xu [50] presented the distance measures for the HFSs. Apart from them, some other kinds of measures, to handle the uncertainty using HFS information, are developed by the researchers. For more details about them, we refer to read [4, 49, 80, 89, 92, 193].

The above theories are strictly based on the consideration of the membership degrees in HFSs. Besides it, in our day-to-day decision making, it is quite equally important to consider a degree of non-membership into the analysis. To address it properly, Zhu et al. [267] defined the concept of dual HFSs (DHFSs). Under this set, the authors in [32, 203, 248] defined the variate of the correlation coefficient for DHFSs. Also, Farhadinia [48] defined some correlation coefficients for DHFSs and dual interval-valued HFSs and applied them to solve the medical diagnosis problems. However, Wang, Xu, Wang and Ni [205] and Singh [176] defined the distance and similarity measures between the DHFSs to solve the MCDM problems. Besides that, in terms of the AOs for DHFSs, some scholars have also made an investigation on it. For example, in [250, 253], authors have developed some weighted, ordered weighted and the hybrid AOs, while in [95, 262], the authors have presented them by using Einstein and Hamacher t-norm operations respectively. Later on, Wang, Shen and Zhu [204] presented the power AOs based on Archimedean t-norms for DHFSs to solve DM problems. Qu et al. [159] presented the induced Shapley hybrid weighted averaging and geometric operators for DHFSs.

As the HFSs or DHFSs are successfully applied in many domains to solve the problems, but these theories don't consider the uncertainties in a parametric manner. To address it properly, soft set(SS) theory[139] which is a generalization of the fuzzy set theory to deal with the uncertainty in a parametric manner under the notion of hesitant environment, Babitha and John [11] defined the concept of HFSSs by embedding the features of SS with HFS. Wang et al. [202] defined the averaging and geometric operation by using Archimedean t-norms operations for HFSSs. Peng and Yang [153] presented an algorithm for interval-valued HFSS by using AOs. Das, Malakar, Kar and Pal [36] proposed the concept of the correlation coefficient for HFSSs and interval-valued HFSSs. Peng and Dai [151] presented some DM approaches for HFSSs by using Multi-Attributive Border Approximation area Comparison(MABAC), Weighted Aggregated Sum-Product

Assessment(WASPAS) and Complex Proportional Assessment(COPRAS) methods. Garg et al. [62] presented the notion of fuzzy numbers intuitionistic fuzzy soft. Peng and Garg [152] defined an algorithm form solving DM problems for interval-valued FSS by using Weighted Distance Based Approximation (WDBA) and Combinative Distance-based Assessment (CODAS) methods. Furthermore, Peng and Yang [154] initiated the concept of the DHFSS which is the generalization of HFSS. Zhang and Shu [257] investigated the properties of DHFSS. He [77] developed an algorithm for solving the DM problems under the DHFSS environment. However, apart from them, some other researchers have investigated the DM problems under HFSS and/or DHFSS environment, for such reads, we refer to [103, 262].

1.2 Gaps and motivation of the work towards the soft set environment

In the above sections, we have reviewed different techniques which are utilized to deal with DM problems under the environment of the intuitionistic fuzzy set(IFS)[8] theory. As these hypotheses have effectively connected, but at the same time, it has observed that they have their intrinsic flaws which are pointed out by Molodtsov [139]. During their information collection phase, these theories could not model the vagueness of the system because of the inadequacy of its parameterization tool. Therefore, techniques need to be strengthened to resolve these problems in order to obtain a better decision. To deal with them, a concept of soft set[139] and their extensions such as FSS [130] and intuitionistic fuzzy set [129] had been developed. FSS and IFSS theories are the novel research hypothesis with their viable applications. Moreover, the assessment presented by the IFSSs show the positive, negative and reluctant features of experts which can assist experts in making decisions more in keeping with the actual situation. In order to classify the various alternatives having IFSS information, it is necessary to have an AO which aggregate all the preferences of decision-makers into a collective value. However, research on FSSs and IFSSs is confined to fundamental theories and applications. Thus, there is a need to enhance the techniques for fusing the data given in the form of IFSSs.

On the other hand, from the literature of hesitant fuzzy set[194, 195] and dual hesitant fuzzy set [267], it has been observed that researchers are extensively using these theories and approaches. However, in these existing theories, a concept of the parametrization tool is not appropriately taken in the DM process. To address it, a concept of HFSS [11] and DHFSS[257] are utilized. In these theories, the preferences of the experts are given as a set of possible values rather than a single number. Also, from the literature, it has been noticed that the existing methods[77, 258] under DHFSS environment are unable to fuse the different preferences of experts given in terms of DHFSSs. Thus, there is a need to develop some information measures and AOs for the DHFSSs in order to solve the DM problems.

1.3 Objective of the Thesis

The present work is intended to find some new techniques/ methods to solve DM problems under intuitionistic fuzzy soft sets. By motivating from the above literature and gaps, the main objectives of the work are summarized as:

(O1) To develop some aggregation operators for solving decision-making problems.

(O2) To develop and apply some information measures in decision-making problems under soft sets.

(O3) To develop a prioritized weighted aggregation operator for soft set information.

1.4 Structure of the Thesis

The present thesis is organized into eleven chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of decision making approaches by using several approaches is presented in the first chapter. In **Chapter 2**, the basics and preliminaries related to the intuitionistic fuzzy set, soft sets, AOs, etc., are given.

Chapter 3 presents the concept of the dual hesitant fuzzy soft set to define the several axioms of distance and similarity measures based on the metric of Hamming, Euclidean, and Hausdorff. The desirable inequalities in the form of their relationships between them have also been discussed. Later, a DM approach is presented based on the proposed measures where the features of the objects are taken in the form of dual hesitant fuzzy soft numbers. The effectiveness of the proposed measures is demonstrated through concrete examples of pattern recognition and medical diagnoses and compare the study with the several existing approaches.

Chapter 4 deals with the correlation coefficient and weighted correlation coefficients under the DHFSS environment in which pairs of membership, non-membership are to be considered as vector representation during the formulation and to investigate their properties. Further, under this environment, a multicriteria DM method based on the proposed correlation coefficients are presented. Three numerical examples, one from the selection procedure and other from the medical diagnosis and pattern recognition, are taken to demonstrate the efficiency of the proposed approach and compared their results with the several existing approaches results.

Chapter 5 utilized the idea of dual hesitant fuzzy soft sets and presents some new AOs, namely, dual hesitant fuzzy soft weighted averaging and geometric operators proposed along with their proofs to aggregate the different preferences of the decision makers. Various desirable properties of its are also investigated in details. Further, these AOs are extended to its generalized operator by incorporating the attitude character of the decision maker towards the data. Later, an MCDM approach is presented based on the proposed operators for solving the DM problems. An illustrative example present to demonstrate the approach under DHFSS environment and compared their results with some of the existing approaches results.

In **Chapter 6**, we present a nonlinear programming (NLP) model based on the technique for order preference by similarity to ideal solution (TOPSIS), to solve multi-attribute DM problems. In this approach, both ratings of alternatives on attributes and weights of attributes are represented by IVIF sets. Based on the available information, NLP models are constructed on the basis of the concepts of the relative-closeness coefficient (RCC) and the weighted distance. A real example is taken to demonstrate the applicability and validity of the proposed methodology.

Chapter 7, we have defined some new averaging and geometric prioritized AOs in which the preferences related to attributes are taken in the form of IFSSs. Desirable properties of its have also been investigated. Furthermore, based on these operators, an approach to investigate the multi-criteria decision making (MCDM) problem has been presented. The effectiveness of these operators has been demonstrated through a case study.

In **Chapter 8**, we present two new scaled prioritized averaging AOs by considering the interaction between the membership degrees for the intuitionistic fuzzy soft sets. The principal advantage of the operators is that they consider the priority relationships between the parameters as well as experts. Furthermore, some properties based on these operators are discussed in detail. Then, we utilized these operators to solve the DM problem and validate it with a numerical example.

Chapter 9 presents some new Bonferroni mean(BM) and weighted BM averaging operator for aggregating the different preferences of the decision maker. Some of its desirable properties have also been discussed in details. Further, a DM method based on proposed operators has been presented and then illustrated with a numerical example. A comparative analysis between the proposed and the existing measures under the IFSS environment has been performed in terms of counter-intuitive cases for showing the validity of it.

In **Chapter 10**, we develop some new power AOs based on the Archimedean t-norm operators for intuitionistic fuzzy (IF) soft numbers. The developed AOs are named as intuitionistic fuzzy soft power averaging and geometric operators, weighted intuitionistic fuzzy soft power averaging and geometric operators, weighted and ordered weighted intuitionistic fuzzy soft power averaging and geometric operators. The salient features of these operators are also discussed in details. Further, these operators are extended to its generalized version and called generalized intuitionistic fuzzy soft power averaging or geometric AOs. Finally, the applicability of these developed operators is explained with a numerical example related to DM problems. A comparative analysis of existing approaches has been done for showing the validity of the proposed work.

Chapter 11 presents the concept of generalized IFSS (GIFSS), as well as the group-based generalized intuitionistic fuzzy soft set (GGIFSS) in which the evaluation of the object is done by the group of experts rather than a single expert. Based on this information, a new weighted averaging and geometric AO has been proposed by taking the intuitionistic fuzzy parameter. Finally, a DM approach based on the proposed operator is being built to solve the problems under the intuitionistic fuzzy environment. An illustrative example of the selection of the optimal alternative has been given to show the developed method. Comparison analysis between the proposed and the existing operators have been performed in term of counter-intuitive cases for showing the superiority of the approach.

Chapter 2

Preliminaries

In this chapter, we present the basic concepts and the mathematical structure related to IFS, IFSS, DHFS, HFS, aggregation operators, etc., over the universal set $\mathcal{X}$.

2.1 Intuitionistic fuzzy sets and its extensions

Definition 2.1.1. [8] An IFS $\mathcal{A}$ is given as

$$\mathcal{A} = \Big\{(x, u_{\mathcal{A}}(x), v_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\Big\}, \tag{2.1}$$

where, $u_{\mathcal{A}}(x), v_{\mathcal{A}}(x) \in [0,1]$, represents MD and NMD of x with $0 \leq u_{\mathcal{A}}(x) + v_{\mathcal{A}}(x) \leq 1$ holds for $\forall x$. We call, this pair $(u_{\mathcal{A}}, v_{\mathcal{A}})$ as an IFN [231].

Definition 2.1.2. [8] For any two IFNs $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ and $\mathcal{B} = (u_{\mathcal{B}}, v_{\mathcal{B}})$, we have

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $u_{\mathcal{A}} \leq u_{\mathcal{B}}$ and $v_{\mathcal{A}} \geq v_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$.

(iii) $\mathcal{A}^c = (v_{\mathcal{A}}, u_{\mathcal{A}})$.

(iv) $\mathcal{A} \cup \mathcal{B} = (\max\{u_{\mathcal{A}}, u_{\mathcal{B}}\}, \min\{v_{\mathcal{A}}, v_{\mathcal{B}}\})$.

(v) $\mathcal{A} \cap \mathcal{B} = (\min\{u_{\mathcal{A}}, u_{\mathcal{B}}\}, \max\{v_{\mathcal{A}}, v_{\mathcal{B}}\})$.

Definition 2.1.3. [231] The score function for an IFN $\mathcal{A} = (u_{\mathcal{A}}, v_{\mathcal{A}})$ is defined as :

$$\mathcal{S}(\mathcal{A}) = u_{\mathcal{A}} - v_{\mathcal{A}}, \tag{2.2}$$

and accuracy function is defined as:

$$\mathcal{H}(\mathcal{A}) = u_{\mathcal{A}} + v_{\mathcal{A}} \tag{2.3}$$

Definition 2.1.4. [7] An IVIFS $\mathcal{A}$ in $\mathcal{X}$ is defined as

$$\mathcal{A} = \{(x, \widetilde{u}_{\mathcal{A}}(x), \widetilde{v}_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}, \tag{2.4}$$

where $\widetilde{u}_{\mathcal{A}}(x) = [u^L_{\mathcal{A}}(x), u^U_{\mathcal{A}}(x)]$ and $\widetilde{v}_{\mathcal{A}}(x) = [v^L_{\mathcal{A}}(x), v^U_{\mathcal{A}}(x)]$ are all subsets of $[0,1]$, and represents the MDs and NMDs of x to $\mathcal{A}$ such that for any $x \in \mathcal{X}, 0 \leq u^U_{\mathcal{A}}(x) + v^U_{\mathcal{A}}(x) \leq 1$. For convenience, we denote the pair as $\mathcal{A} = ([u^L, u^U], [v^L, v^U])$ and called as IVIFN.

Definition 2.1.5. Let $\mathcal{A} = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ and $\mathcal{B} = ([u^L_{\mathcal{B}}, u^U_{\mathcal{B}}], [v^L_{\mathcal{B}}, v^U_{\mathcal{B}}])$ be any two IVIFNs, then

(i) $\mathcal{A} \subseteq \mathcal{B}$ if $u^L_{\mathcal{A}} \leq u^L_{\mathcal{B}}, u^U_{\mathcal{A}} \leq u^U_{\mathcal{B}}, v^L_{\mathcal{A}} \geq v^L_{\mathcal{B}}$ and $v^U_{\mathcal{A}} \geq v^U_{\mathcal{B}}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$.

(iii) $\mathcal{A}^c = ([v^L_{\mathcal{A}}, v^U_{\mathcal{A}}], [u^L_{\mathcal{A}}, u^U_{\mathcal{A}}])$.

(iv) $\mathcal{A} \cup \mathcal{B} = ([\max\{u^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \max\{u^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}], [\min\{v^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \min\{v^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}])$;

(v) $\mathcal{A} \cap \mathcal{B} = ([\max\{u^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \max\{u^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}], [\min\{v^L_{\mathcal{A}}, u^L_{\mathcal{B}}\}, \min\{v^U_{\mathcal{A}}, u^U_{\mathcal{B}}\}])$.

Definition 2.1.6. [228] The score function for an IVIFN $\mathcal{A} = ([u^L_{\mathcal{A}}, u^U_{\mathcal{A}}], [v^L_{\mathcal{A}}, v^U_{\mathcal{A}}])$ is defined as:

$$\mathcal{S}(\mathcal{A}) = \frac{u^L_{\mathcal{A}} + u^U_{\mathcal{A}} - v^L_{\mathcal{A}} - v^U_{\mathcal{A}}}{2}, \tag{2.5}$$

and accuracy function is

$$\mathcal{H}(\mathcal{A}) = \frac{u^L_{\mathcal{A}} + u^U_{\mathcal{A}} + v^L_{\mathcal{A}} + v^U_{\mathcal{A}}}{2}. \tag{2.6}$$

To order the different IFNs and/or IVIFNs, a comparison law between them is defined as follow.

Definition 2.1.7. [227] Let $\mathcal{A}$ and $\mathcal{B}$ be either two IFNs and/or IVIFNs then based on the score and accuracy functions as defined in Definitions 2.1.3 and 2.1.6, an order relation $A \succeq B$, where "$\succeq$" refer "preferred to", occurs if anyone of the following condition met.

(i) If $\mathcal{S}(\mathcal{A}) \geq \mathcal{S}(\mathcal{B})$.

(ii) $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B})$ and $\mathcal{H}(\mathcal{A}) \geq \mathcal{H}(\mathcal{B})$.

Definition 2.1.8. [194, 195] A hesitant fuzzy set (HFS) $\mathcal{A}$ on $\mathcal{X}$ is defined as a mapping from $\mathcal{X}$ to a finite subset of [0, 1], which can be represented as:

$$\mathcal{A} = \{(u, h_{\mathcal{A}}(u^{(s)})) \mid u \in \mathcal{X}\}, \tag{2.7}$$

where s represent the number of values in $h_{\mathcal{A}}(u)$ and $h_{\mathcal{A}}(u^{(s)})$ is a set of some different values in $[0, 1]$, denoting the possible membership degrees of the element $u \in \mathcal{X}$.

Definition 2.1.9. [267] The dual hesitant fuzzy set (DHFS) $\mathcal{A}$ on $\mathcal{X}$ is defined as:

$$\mathcal{A} = \Big\{(u, h_{\mathcal{A}}(u^{(s)}), g_{\mathcal{A}}(u^{(t)})) \mid u \in \mathcal{X}\Big\} \tag{2.8}$$

where s and t represents the number of values in $h_{\mathcal{A}}(u)$ and $g_{\mathcal{A}}(u)$ and $h_{\mathcal{A}}(u^{(s)}), g_{\mathcal{A}}(u^{(t)})$ are two sets of some values in [0,1], denotes the possible MDs and NMDs of the element $u \in \mathcal{X}$ to the set $\mathcal{A}$, respectively, with the condition that $0 \leq \zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}} \leq 1$ and $0 \leq \zeta_{\mathcal{A}}^{+} + \vartheta_{\mathcal{A}}^{+} \leq 1$ where $\zeta_{\mathcal{A}} \in h_{\mathcal{A}}, \vartheta_{\mathcal{A}} \in g_{\mathcal{A}}, \zeta_{\mathcal{A}}^{+} \in h_{\mathcal{A}}^{+} = \bigcup_{\zeta_{\mathcal{A}} \in h_{\mathcal{A}}} \max\{\zeta_{\mathcal{A}}\}, \vartheta_{\mathcal{A}}^{+} \in g_{\mathcal{A}}^{+} = \bigcup_{\vartheta_{\mathcal{A}} \in g_{\mathcal{A}}} \max\{\vartheta_{\mathcal{A}}\}$.

2.2 Soft sets

In this section, we briefly overview the basic terms associated with soft set over the set $\mathcal{X}$. For it, let $\mathcal{E}$ be a set of parameter.

Definition 2.2.1. [139] A pair $(\mathcal{F}, \mathcal{E})$ is called soft set (SS), if $\mathcal{F}$ is a mapping from $\mathcal{E}$ to power set of $\mathcal{X}$, i.e., $\mathcal{P}^{\mathcal{X}}$.

Definition 2.2.2. [130] A SS is said to be FSS, if $\mathcal{F} : \mathcal{E} \rightarrow \mathcal{P}^{\mathcal{X}}$ defined as:

$$\mathcal{F}_{u_i}(e_j) = \{(u_i, \zeta_j(u_i)) \mid u_i \in \mathcal{X}\} \tag{2.9}$$

where $\mathcal{F}^{\mathcal{X}}$ be the set containing fuzzy subsets of $\mathcal{X}$ and $\zeta_j(u)$ represents the MD of u over the parameter $e_j \in \mathcal{E}$.

Definition 2.2.3. [129] A SS is said to be IFSS, if $\mathcal{F} : \mathcal{E} \to \mathcal{IF}^X$ defined as the set containing intuitionistic fuzzy subsets of $\mathcal{X}$ is defined as:

$$\mathcal{F}_{u_i}(e_j) = \{(u_i, \zeta_j(u_i), \vartheta_j(u_i)) \mid u_i \in \mathcal{X}\} \tag{2.10}$$

where $\mathcal{IF}^{\mathcal{X}}$ be the set containing intuitionistic fuzzy subsets of $\mathcal{X}$ and $\zeta_j(u_i)$ and $\vartheta_j(u_i)$ represents the MDs and NMDs of u over e_j such that $0 \leq \zeta_j, \vartheta_j \leq 1$ and $0 \leq \zeta_j + \vartheta_j \leq 1$ for all $u \in \mathcal{X}$. For simplicity, we denote the pair of $\mathcal{F}_{u_i}(e_j)$ as $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$ and known as IF soft number (IFSN).

Definition 2.2.4. [6] The score function of an IFSN $\mathcal{A} = (\zeta, \vartheta)$ is given as $\mathcal{S}(\mathcal{A}) = \zeta - \vartheta$, while an accuracy function is $\mathcal{H}(\mathcal{A}) = \zeta + \vartheta$. Based on it, an order relation "$\succ$" between two IFSNs $\mathcal{A}$ and $\mathcal{B}$ defined as $\mathcal{A} \succ \mathcal{B}$ if either $\mathcal{S}(\mathcal{A}) > \mathcal{S}(\mathcal{B})$ or $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B})$ and $\mathcal{H}(\mathcal{A}) > \mathcal{H}(\mathcal{B})$ holds.

Definition 2.2.5. [129] Let $(\mathcal{F}, \mathcal{E}_1) = \{(u_i, \zeta_{j_1}(u_i), \vartheta_{j_1}(u_i)) \mid u_i \in \mathcal{X}, e_{j_1} \in \mathcal{E}_1\}$ and $(\mathcal{G}, \mathcal{E}_2) = \{(u_i, \zeta_{j_2}(u_i), \vartheta_{j_2}(u_i)) \mid u_i \in \mathcal{X}, e_{j_2} \in \mathcal{E}_2\}$ are two IFSSs, then the following operations hold:

(i) $(\mathcal{F}, \mathcal{E}_1) \cup (\mathcal{G}, \mathcal{E}_2) = (\mathcal{H}, \mathcal{E})$ where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and

$$\mathcal{H}_{u_i}(e_j) = \begin{cases} \mathcal{F}_{u_i}(e_{j_1}) & \text{if } e_j \in \mathcal{E}_1 - \mathcal{E}_2 \\ \mathcal{G}_{u_i}(e_{j_2}) & \text{if } e_j \in \mathcal{E}_2 - \mathcal{E}_1 \\ \mathcal{F}_{u_i}(e_{j_1}) \cup \mathcal{G}_{u_i}(e_{j_2}) & \text{if } e_j \in \mathcal{E}_1 \cap \mathcal{E}_2 \end{cases}$$

(ii) $(\mathcal{F}, \mathcal{E}_1) \cap (\mathcal{G}, \mathcal{E}_2) = (\mathcal{H}, \mathcal{E})$ where $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$ and

$$\mathcal{H}_{u_i}(e_j) = \begin{cases} \mathcal{F}_{u_i}(e_{j_1}) & \text{if } e_j \in \mathcal{E}_1 - \mathcal{E}_2 \\ \mathcal{G}_{u_i}(e_{j_2}) & \text{if } e_j \in \mathcal{E}_2 - \mathcal{E}_1 \\ \mathcal{F}_{u_i}(e_{j_1}) \cap \mathcal{G}_{u_i}(e_{j_2}) & e_j \in \mathcal{E}_1 \cap \mathcal{E}_2 \end{cases}$$

(iii) $(\mathcal{F}, \mathcal{E}_1)^c = \{(u_i, \vartheta_{j_1}(u_i), \zeta_{j_1}(u_i)) \mid u_i \in \mathcal{X}, e_{j_1} \in \mathcal{E}_1\}$

(iv) $(\mathcal{F}, \mathcal{E}_1)$ is soft subset of $(\mathcal{G}, \mathcal{E}_2)$, if

(a) $\mathcal{E}_1 \subseteq \mathcal{E}_2$.

(b) $\forall e_j \in \mathcal{E}_1, \mathcal{F}_{u_i}(e_j) \subseteq \mathcal{G}_{u_i}(e_j)$.

and is denoted by $(\mathcal{F}, \mathcal{E}_1) \Subset (\mathcal{G}, \mathcal{E}_2)$;

(v) $(\mathcal{F}, \mathcal{E}_1) = (\mathcal{G}, \mathcal{E}_2)$ if $(\mathcal{F}, \mathcal{E}_1) \Subset (\mathcal{G}, \mathcal{E}_2)$ and $(\mathcal{G}, \mathcal{E}_2) \Subset (\mathcal{F}, \mathcal{E}_1)$.

Definition 2.2.6. [91] A pair $(\mathcal{F}, \mathcal{E})$ is called an IVIFSS over $\mathcal{X}$ iff $\mathcal{F} : \mathcal{E} \to K^{IU}$ where K^{IU} denotes the set of all IVIF subsets of $\mathcal{X}$ and for any $e \in \mathcal{E}$, the IVIFSS is defined as

$$\mathcal{F}_u(e_j) = \{(u, [\zeta_j^L(u), \zeta_j^U(u)], [\vartheta_j^L(u), \vartheta_j^U(u)]) \mid u \in \mathcal{X}\} \tag{2.11}$$

where $\zeta_j^L(u)$, $\vartheta_j^L(u)$ are the lower bounds and $\zeta_j^U(u)$, $\vartheta_j^U(u)$ are the upper bounds of degree of the MDs and NMDs respectively, such that $\zeta_j^U(u) + \vartheta_j^U(u) \leq 1$ for all $u \in \mathcal{X}$.

Definition 2.2.7. [11] Let $\mathcal{E}$ be a set of parameters and $\mathcal{H}^{\mathcal{X}}$ denote set of all hesitant fuzzy subsets of $\mathcal{X}$. A pair $(\mathcal{F}, \mathcal{E})$ is called a hesitant fuzzy soft set (HFSS) over $\mathcal{X}$, if and only if $\mathcal{F}$ is a mapping denoted by $\mathcal{F} : \mathcal{E} \to \mathcal{H}^{\mathcal{X}}$. For any parameter $e_j \in \mathcal{E}$, $\mathcal{F}(e_j)$ can be written as:

$$\mathcal{F}_{u_i}(e_j) = \{(u_i, \{h_{\mathcal{F}(u_i)}(e_j^{(s)})\}) \mid u_i \in \mathcal{X}\} \tag{2.12}$$

where $h_{\mathcal{F}(u_i)}(e_j)$ is a set of possible values in $[0, 1]$, denoting the membership degrees of the element $u_i \in \mathcal{X}$ to the parameter e_j.

Definition 2.2.8. [257] A pair $(\mathcal{F}, \mathcal{E})$ is called a dual hesitant soft set (DHFSS) over $\mathcal{X}$ if and only if $\mathcal{F}$ is a mapping given by $\mathcal{F} : \mathcal{E} \to \mathcal{D}^{\mathcal{X}}$, dual hesitant fuzzy subsets in $\mathcal{X}$. For any parameter $e_j \in \mathcal{E}$, $\mathcal{F}(e_j)$ can be written as:

$$\mathcal{F}_{u_i}(e_j) = \{(u_i, \{h_{\mathcal{F}(u_i)}(e_j^{(s)})\}, \{g_{\mathcal{F}(u_i)}(e_j^{(t)})\}) \mid u_i \in \mathcal{X}\} \tag{2.13}$$

where $h_{\mathcal{F}(u_i)}(e_j)$ and $g_{\mathcal{F}(u_i)}(e_j)$ are two sets of some values in $[0, 1]$, denoting the possible MDs and NMDs with the conditions $0 \leq \zeta_{\mathcal{F}(u_i)}, \vartheta_{\mathcal{F}(u_i)} \leq 1$ and $\zeta^+ + \vartheta^+ \leq 1$ where $\zeta^+ \in \bigcup_{\zeta_{\mathcal{F}(u_i)}(e_j) \in h_{\mathcal{F}(u_i)}(e_j)} \max\{\zeta_{\mathcal{F}(u_i)}(e_j)\}$ and $\vartheta^+ \in \bigcup_{\vartheta_{\mathcal{F}(u_i)}(e_j) \in g_{\mathcal{F}(u_i)}(e_j)} \max\{\vartheta_{\mathcal{F}(u_i)}(e_j)\}$. For the sake of simplicity, we denoted the pair $\mathcal{F}_{u_i}(e_j)$ to be $\mathcal{A}_{ij} = \{\{h_{ij}\}, \{g_{ij}\}\}$ or $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$ and called as dual hesitant fuzzy soft numbers (DHFSNs).

2.3 Information measures

Let $\Phi(\mathcal{X})$ be the collections of the soft sets defined over the set $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{E}$ be the set of the parameters.

Definition 2.3.1. [205] Let $\mathcal{A}$ and $\mathcal{B}$ be two sets on $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$ then the distance measure between them is defined as $\mathcal{D}(\mathcal{A}, \mathcal{B})$, if it satisfies the following properties for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Phi(\mathcal{X})$

(P1) $0 \leq \mathcal{D}(\mathcal{A}, \mathcal{B}) \leq 1$.

(P2) $\mathcal{D}(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$.

(P3) $\mathcal{D}(\mathcal{A}, \mathcal{B}) = \mathcal{D}(\mathcal{B}, \mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $\mathcal{D}(\mathcal{A}, \mathcal{B}) \leq \mathcal{D}(\mathcal{A}, \mathcal{C})$ and $\mathcal{D}(\mathcal{B}, \mathcal{C}) \leq \mathcal{D}(\mathcal{A}, \mathcal{C})$.

Under the DHFS environment, the researchers [176, 205] have presented some distance measure which are defined as follows. For it, let $\mathcal{A} = (h_{\mathcal{A}}(x), g_{\mathcal{A}}(x))$ and $\mathcal{B} = (h_{\mathcal{B}}(x), g_{\mathcal{B}}(x))$ be two DHFSs. Then, we have

(i) Wang et al. [205] proposed the weighted distance measure:

$$\mathcal{D}(\mathcal{A}, \mathcal{B}) = \sum_{i=1}^{n} \left[w_i \left(\frac{1}{l_{u_i}} \left(\begin{array}{c} \sum\limits_{j=1}^{\#h_{u_i}} \left| \gamma_{\mathcal{A}}^{\sigma(j)}(u_i) - \gamma_{\mathcal{B}}^{\sigma(j)}(u_i) \right|^{\lambda} \\ + \sum\limits_{k=1}^{\#g_{u_i}} \left| \eta_{\mathcal{A}}^{\sigma(k)}(u_i) - \eta_{\mathcal{B}}^{\sigma(k)}(u_i) \right|^{\lambda} \end{array} \right) \right) \right] \tag{2.14}$$

where $\gamma_{\mathcal{A}}^{\sigma(j)}(u_i)$ be the j^{th} largest value in $h_{\mathcal{A}}(u_i)$ and $\eta_{\mathcal{A}}^{\sigma(j)}(u_i)$ be the j^{th} largest value in $g_{\mathcal{A}}(u_i)$ for DHFS and $\lambda > 0$, $\#h_{u_i}$ and $\#g_{u_i}$ are the number of the elements in $h_{\mathcal{A}}(u_i)$ and $g_{\mathcal{A}}(u_i)$ respectively such that $l_{u_i} = \#h_{u_i} + \#g_{u_i}$.

(ii) Singh [176] proposed a generalized normalized and weighted distance measure denoted by d_{dgn} and d_{dgw} respectively

$$d_{dgn}(\mathcal{A},\mathcal{B}) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left\{\begin{array}{l}\left(\frac{1}{l_{u_i}}\sum_{j=1}^{l_{u_i}}\left|\gamma_{\mathcal{A}}^{\sigma(j)}(u_i)-\gamma_{\mathcal{B}}^{\sigma(j)}(u_i)\right|^{\lambda}\right)\\ +\left(\frac{1}{m_{u_i}}\sum_{j=1}^{m_{u_i}}\left|\eta_{\mathcal{A}}^{\sigma(j)}(u_i)-\eta_{\mathcal{B}}^{\sigma(j)}(u_i)\right|^{\lambda}\right)\end{array}\right\}\right]^{\frac{1}{\lambda}} \tag{2.15}$$

and

$$d_{dgw}(\mathcal{A},\mathcal{B}) = \left[\frac{1}{2}\sum_{i=1}^{n}\left\{\begin{array}{l}w_i\left(\frac{1}{l_{u_i}}\sum_{j=1}^{l_{u_i}}\left|\gamma_{\mathcal{A}}^{\sigma(j)}(u_i)-\gamma_{B}^{\sigma(j)}(u_i)\right|^{\lambda}\right)\\ +z_i\left(\frac{1}{m_{u_i}}\sum_{j=1}^{m_{u_i}}\left|\eta_{\mathcal{A}}^{\sigma(j)}(u_i)-\eta_{B}^{\sigma(j)}(u_i)\right|^{\lambda}\right)\end{array}\right\}\right]^{\frac{1}{\lambda}} \tag{2.16}$$

where $\lambda > 0$ and w_i and z_i be the weights assigned to membership degree and non-membership degree, respectively.

(iii) Wang et al.[205] proposed the distance measure as

$$d(\mathcal{A},\mathcal{B}) = \sum_{i=1}^{n}\left[\frac{1}{2n}\left(\begin{array}{l}\left|\frac{\sum\limits_{\gamma_{\mathcal{A}}\in h_{\mathcal{A}}(u_i)}\gamma_{\mathcal{A}}}{\#h_{\mathcal{A}}(u_i)}-\frac{\sum\limits_{\gamma_{\mathcal{B}}\in h_{\mathcal{B}}(u_i)}\gamma_{\mathcal{B}}}{\#h_{\mathcal{B}}(u_i)}\right|^{\lambda}\\ +\left|\frac{\sum\limits_{\eta_{\mathcal{A}}\in g_{\mathcal{A}}(u_i)}\eta_{\mathcal{A}}}{\#g_{\mathcal{A}}(u_i)}-\frac{\sum\limits_{\eta_{\mathcal{B}}\in g_{\mathcal{B}}(u_i)}\eta_{\mathcal{B}}}{\#g_{\mathcal{B}}(u_i)}\right|^{\lambda}\end{array}\right)\right]^{\frac{1}{\lambda}} \tag{2.17}$$

where $\#h_{\mathcal{A}}(u_i)$ and $\#g_{\mathcal{A}}(u_i)$ are the number of the elements in $h_{\mathcal{A}}$ and $g_{\mathcal{A}}$ respectively.

Definition 2.3.2. [205] Let $\mathcal{A}$ and $\mathcal{B}$ be two DHFSs on $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$ then the similarity measure between them is defined as $s(\mathcal{A},\mathcal{B})$, if it satisfies the following properties

(P1) $0 \leq s(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $s(\mathcal{A},\mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$.

(P3) $s(\mathcal{A},\mathcal{B}) = s(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ then $s(\mathcal{A},\mathcal{B}) \geq s(\mathcal{A},\mathcal{C})$ and $s(\mathcal{B},\mathcal{C}) \geq s(\mathcal{A},\mathcal{C})$.

For two DHFSs $\mathcal{A}$ and $\mathcal{B}$, Wang, Xu, Wang and Ni [205] defined the similarity measures between them as follows.

$$S'(\mathcal{A},\mathcal{B}) = 1 - \sum_{i=1}^{n}\left[w_i\left(\frac{1}{l_{u_i}}\left(\sum_{j=1}^{\#h_{u_i}}\left|\gamma_{\mathcal{A}}^{\sigma(j)} - \gamma_{\mathcal{B}}^{\sigma(j)}\right|^{\lambda} + \sum_{k=1}^{\#g_{u_i}}\left|\eta_{\mathcal{A}}^{\sigma(k)} - \eta_{\mathcal{B}}^{\sigma(k)}\right|^{\lambda}\right)\right)\right] \tag{2.18}$$

where $\gamma_E^{\sigma(j)}$ be the j^{th} largest value in $h_E(x)$ and $\gamma_E^{\sigma(j)}$ be the j^{th} largest value in $g_E(x)$ for DHFS $d_E = (h_E(x), g_E(x))$ and $\lambda > 0$, $l_{u_i} = (\#h_{u_i}) + (\#g_{u_i})$, $\#h$ and $\#g$ are the number of the elements in h and g respectively,

Definition 2.3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be any two sets then the correlation coefficient between them is denoted by $\rho(\mathcal{A},\mathcal{B})$ if it satisfies the following properties:

(P1) $0 \leq \rho(\mathcal{A},\mathcal{B}) \leq 1$.

(P2) $\rho(\mathcal{A},\mathcal{B}) = \rho(\mathcal{B},\mathcal{A})$.

(P3) $\rho(\mathcal{A},\mathcal{B}) = 1$ if $\mathcal{A} = \mathcal{B}$.

For two DHFSs $\mathcal{A}$ and $\mathcal{B}$, some of the existing correlation coefficient measure are defined as below.

(i) Wang et al. [203] defined the correlation coefficient as:

$$\rho(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{i=1}^{n}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i} h_{\mathcal{A}^{\sigma(s)}}(u_i)h_{\mathcal{B}^{\sigma(s)}}(u_i) + \frac{1}{l_i}\sum\limits_{t=1}^{l_i} g_{\mathcal{A}^{\sigma(t)}}(u_i)g_{\mathcal{B}^{\sigma(t)}}(u_i)\right)}{\sqrt{\sum\limits_{i=1}^{n}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i} h^2_{\mathcal{A}^{\sigma(s)}}(u_i) + \frac{1}{l_i}\sum\limits_{t=1}^{l_i} g^2_{\mathcal{A}^{\sigma(t)}}(u_i)\right)}\sqrt{\sum\limits_{i=1}^{n}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i} h^2_{\mathcal{B}^{\sigma(s)}}(u_i) + \frac{1}{l_i}\sum\limits_{t=1}^{l_i} g^2_{\mathcal{B}^{\sigma(t)}}(u_i)\right)}} \tag{2.19}$$

(ii) Ye [248] defined the weighted correlation coefficient as:

$$\rho(\mathcal{A},\mathcal{B}) = \frac{\sum\limits_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i} h_{\mathcal{A}^{\sigma(s)}(u_i)}h_{\mathcal{B}^{\sigma(s)}}(u_i) + \frac{1}{l_i}\sum\limits_{t=1}^{l_i} g_{\mathcal{A}^{\sigma(t)}}(u_i)g_{\mathcal{B}^{\sigma(t)}}(u_i)\right)}{\sqrt{\sum\limits_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i} h^2_{\mathcal{A}^{\sigma(s)}}(u_i) + \frac{1}{l_i}\sum\limits_{t=1}^{l_i} g^2_{\mathcal{A}^{\sigma(t)}}(u_i)\right)}\sqrt{\sum\limits_{i=1}^{n} w_i\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i} h^2_{\mathcal{B}^{\sigma(s)}}(u_i) + \frac{1}{l_i}\sum\limits_{t=1}^{l_i} g^2_{\mathcal{B}^{\sigma(t)}}(u_i)\right)}} \tag{2.20}$$

2.4 Archimedean t-norms and Archimedean t-conorms

A t-norm (fuzzy intersection)[101, 138] '$\mathcal{T}$' is a binary operation on $[0,1]$ i.e.

$$\mathcal{T} : [0,1] \times [0,1] \to [0,1] \tag{2.21}$$

defined by

$$(\mathcal{B}_1 \cap \mathcal{B}_2)(x) = \mathcal{T}\left(\mathcal{B}_1(x), \mathcal{B}_2(x)\right) \qquad \forall x \in [0,1] \tag{2.22}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary fuzzy sets. Further, the mapping $\mathcal{T}$ satisfies the following axioms for all $a, b, d \in [0,1]$

Axiom 1: $\mathcal{T}(a,1) = a$ (Boundary Condition)

Axiom 2: If $b \leq d$ then $\mathcal{T}(a,b) \leq \mathcal{T}(a,d)$ (Monotonicity)

Axiom 3: $\mathcal{T}(a,b) = \mathcal{T}(b,a)$ (Commutativity)

Axiom 4: $\mathcal{T}(a, \mathcal{T}(b,d)) = \mathcal{T}(\mathcal{T}(a,b), d)$ (Associativity)

Alternatively, a T-conorm (fuzzy union)[101, 138] 'S' is also a binary operation on $[0,1]$ given by

$$S : [0,1] \times [0,1] \rightarrow [0,1] \tag{2.23}$$

defined by

$$(\mathcal{B}_1 \cup \mathcal{B}_2)(x) = S\left(\mathcal{B}_1(x), \mathcal{B}_2(x)\right) \qquad \forall x \in [0,1] \tag{2.24}$$

Also, the mapping 'S' further satisfies the boundary, monotonicity, commutativity and associativity conditions.

The relation between 'S' and '$\mathcal{T}$' norms is given as

$$S(a,b) = 1 - \mathcal{T}(1-a, 1-b) \quad \forall a, b \in [0,1] \tag{2.25}$$

A class of fuzzy intersection (t-norm) is obtained if t-norm also satisfies the additional axioms [101, 138], i.e.,

Axiom 5: $\mathcal{T}$ is continuous function (Continuity)

Axiom 6: $\mathcal{T}(a,a) < a$ (Subidempotency)

Axiom 7: If $a_1 < a_2$ and $b_1 < b_2$ implies $\mathcal{T}(a_1, b_1) < \mathcal{T}(a_2, b_2)$ (Strict monotonicity)

Similarly, for t-conorm, Axiom 6 is replaced by $S(a,a) > a$ and is called superidempotency.

A continuous t-norm that satisfy the subidempotency i.e., $\mathcal{T}(a,a) < a$ is called an Archimedean t-norm(AT)[101, 138]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-norm. On the other hand, a continuous t-conorm that satisfies the superidempotency i.e. $S(a,a) > a$ is called an Archimedean t-conorm(AC)[101, 138]. If it also satisfies the strict monotonicity then it is called strict Archimedean t-conorm.

Furthermore, strict AT and AC can be expressed in the form of continuous function $k : [0,1) \to [0,1]$ and $l : (0,1] \to [0,1]$ respectively for $a, b \in [0,1]$ as

$$T(a,b) = l^{-1}(l(a) + l(b)) \quad \text{and} \quad S(a,b) = k^{-1}(k(a) + k(b))$$

where l(or k) is a decreasing(or increasing) function with $l(1) = 0$, $k(0) = 0$ and $l(a) = k(1-a)$. However, some standard union and intersection form for $a, b \in [0,1]$ are defined as [101, 138]:

(i) Standard intersection and union[101, 138]

$$\mathcal{T}(a,b) = \min(a,b) \qquad ; \qquad S(a,b) = \max(a,b)$$

(ii) Algebraic product and algebraic sum

$$\mathcal{T}(a,b) = ab \qquad ; \qquad S(a,b) = a + b - ab$$

(iii) Bounded Difference and Sum

$$\mathcal{T}(a,b) = \max(0, a+b-1) \qquad ; \qquad S(a,b) = \min(1, a+b)$$

(iv) Drastic intersection and union

$$\mathcal{T}(a,b) = \begin{cases} a & ;\text{when } b = 1 \\ b & ;\text{when } a = 1 \\ 0 & ;\text{otherwise} \end{cases} \quad ; \quad S(a,b) = \begin{cases} a & ;\text{when } b = 0 \\ b & ;\text{when } a = 0 \\ 1 & ;\text{otherwise} \end{cases}$$

(v) Yagar class of t-norm and t-conorm

$$\mathcal{T}(a,b) = 1 - \min\Big(1, [(1-a)^p + (1-b)^p]^{1/p}\Big)\,;$$
$$S(a,b) = \min\Big[1, (a^p + b^p)^{1/p}\Big]$$

where $p > 0$.

Apart from them, some other well known ATs and ACs with their generator function are summarized in Table 2.1 [101, 138].

Table 2.1: Some AT and AC with their relative additive generators

Name	T-norm	Additive generator	S-norm	Additive generator
	$\mathcal{T}(a,b)$	$z(t)$	$S(a,b)$	$y(t)$
Algebraic	ab	$-\log(t)$	$a+b-ab$	$-\log(1-t)$
Einstein	$\frac{ab}{1+(1-a)(1-b)}$	$\log\left(\frac{2-t}{t}\right)$	$\frac{a+b}{1+ab}$	$\log\left(\frac{1+t}{1-t}\right)$
Hamacher ($\gamma>0$)	$\frac{ab}{\gamma+(1-\gamma)(a+b-ab)}$	$\log\left(\frac{\gamma+(1-\gamma)t}{t}\right)$	$\frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}$	$\log\left(\frac{\gamma+(1-\gamma)(1-t)}{1-t}\right)$
Frank($\eta>1$)	$\log_\eta\left(1+\frac{(\eta^a-1)(\eta^b-1)}{\eta-1}\right)$	$-\log\left(\frac{\eta-1}{\eta^t-1}\right)$	$\log_\eta\left(1+\frac{(\eta^{1-a}-1)(\eta^{1-b}-1)}{\eta-1}\right)$	$-\log\left(\frac{\eta-1}{\eta^{1-t}-1}\right)$

2.5 Aggregation operator

The operations AT and AC are commonly used to combine several fuzzy sets into a single set by using desirable pattern is called as aggregation operators (AOs).

Definition 2.5.1. An aggregation operator on 'n' fuzzy sets ($n \geq 2$) is defined as

$$\mathcal{F} : [0,1]^n \to [0,1] \tag{2.26}$$

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_n$ be first 'n' fuzzy sets defined on $\mathcal{X}$, then function '$\mathcal{F}$' produce an aggregate fuzzy set $\mathcal{B}$ by operating the membership degrees of these sets for each $x \in \mathcal{X}$, i.e.,

$$\mathcal{B}(x) = \mathcal{F}\left(\mathcal{B}_1(x), \mathcal{B}_2(x), \ldots, \mathcal{B}_n(x)\right) \tag{2.27}$$

Further, an aggregation operator $\mathcal{F}$ must satisfies the three axiomatic conditions.

Axiom 1: $\mathcal{F}(0,0,\ldots,0) = 0$ and $\mathcal{F}(1,1,\ldots,1) = 1$ (Boundary condition)

Axiom 2: If $a_i \leq b_i$ for $i = 1,2,\ldots n$, then $\mathcal{F}(a_1,a_2,\ldots,a_n) \leq \mathcal{F}(b_1,b_2,\ldots,b_n)$ (Monotonicity)

Axiom 3: $\mathcal{F}$ is continuous. (Continuity)

Definition 2.5.2. [6] For a positive real number λ and three IFSNs $\mathcal{A} = (\zeta, \vartheta), \mathcal{A}_{11} = (\zeta_{11}, \vartheta_{11})$, $\mathcal{A}_{12} = (\zeta_{12}, \vartheta_{12})$, we have

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \big(1-(1-\zeta_{11})(1-\zeta_{12}), \vartheta_{11}\vartheta_{12}\big)$

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \big(\zeta_{11}\zeta_{12}, 1-(1-\vartheta_{11})(1-\vartheta_{12})\big)$

(iii) $\lambda\mathcal{A} = \big(1-(1-\zeta)^{\lambda}, \vartheta^{\lambda}\big)$

(iv) $\mathcal{A}^{\lambda} = \big(\zeta^{\lambda}, 1-(1-\vartheta)^{\lambda}\big)$

Based on these operations, Arora and Garg [6] proposed the following weighted aggregation operators for the collection of IFSNs $\mathcal{A}_{ij}(i=1,2,\ldots,n; j=1,2,\ldots,m)$ with weight vectors $\eta=(\eta_1,\eta_2,\ldots,\eta_n)^T$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_m)^T$ for the experts and the parameters respectively such that $\eta_i, \xi_j > 0$, $\sum\limits_{i=1}^{n}\eta_i = 1$ and $\sum\limits_{j=1}^{m}\xi_j = 1$.

(i) IF soft weighted average (IFSWA) operator

$$\text{IFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\eta_i}\right)^{\xi_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\vartheta_{ij}^{\eta_i}\right)^{\xi_j}\right) \tag{2.28}$$

(ii) IF soft weighted geometric (IFSWG) operator

$$\text{IFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\zeta_{ij}^{\eta_i}\right)^{\xi_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\vartheta_{ij})^{\eta_i}\right)^{\xi_j}\right) \tag{2.29}$$

Definition 2.5.3. [242] Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be a collection of alternatives. For any real numbers $p, q > 0$, the Bonferroni mean(BM) of the given values is defined as:

$$B^{p,q}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathcal{A}_i^p\mathcal{A}_j^q\right)^{\frac{1}{p+q}} \tag{2.30}$$

which satisfies the following properties:

(i) $B^{p,q}(0,0,\ldots,0) = 0$.

(ii) $B^{p,q}(\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}) = \mathcal{A}$, if $\mathcal{A}_i = \mathcal{A}$ for all i.

(iii) $B^{p,q}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \geq B^{p,q}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$ i.e., $B^{p,q}$ is monotonic, if $\mathcal{A}_i \geq \mathcal{B}_i$, for all i.

(iv) $\min\{\mathcal{A}_i\} \leq B^{p,q}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \leq \max\{\mathcal{A}_i\}$

Definition 2.5.4. [238] Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are the attributes then the power averaging (PA) operator is given as

$$\mathrm{PA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \sum_{i=1}^{n} \frac{(1+T(\mathcal{A}_i))\mathcal{A}_i}{\sum\limits_{i=1}^{n}(1+T(\mathcal{A}_i))} \tag{2.31}$$

where $T(\mathcal{A}_i) = \sum\limits_{\substack{k=1 \\ i \neq k}}^{n} \mathrm{Sup}(\mathcal{A}_i, \mathcal{A}_k)$ and $\mathrm{Sup}(\mathcal{A}_i, \mathcal{A}_k)$ is the support for $\mathcal{A}_i$ from $\mathcal{A}_k$, defined as

$$\mathrm{Sup}(\mathcal{A}_i, \mathcal{A}_k) = 1 - d(\mathcal{A}_i, \mathcal{A}_k) \tag{2.32}$$

where $d(\mathcal{A}_i, \mathcal{A}_k)$ is the hamming distance between $\mathcal{A}_i$ and $\mathcal{A}_k$. Also, it satisfies the properties as given below:

(i) $\mathrm{Sup}(\mathcal{A}_i, \mathcal{A}_k) \in [0, 1]$;

(ii) $\mathrm{Sup}(\mathcal{A}_i, \mathcal{A}_k) = \mathrm{Sup}(\mathcal{A}_k, \mathcal{A}_i)$;

(iii) if $d(\mathcal{A}_i, \mathcal{A}_k) \leq d(\mathcal{A}_r, \mathcal{A}_s)$ then $\mathrm{Sup}(\mathcal{A}_i, \mathcal{A}_k) \geq \mathrm{Sup}(\mathcal{A}_r, \mathcal{A}_s)$.

Definition 2.5.5. [241] Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are the attributes then the prioritized mean average(PMA) operator is given as

$$\mathrm{PMA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} \mathcal{A}_i \tag{2.33}$$

where $T_1 = 1$ and $T_i = \prod\limits_{k=1}^{i-1} S(\mathcal{A}_k); i = 1, 2, \ldots, n$, $S(\mathcal{A}_k)$ represents the score of $\mathcal{A}_k$.

2.6 Description of the decision making model

Assume that $\mathcal{A} = \{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(t)}\}$ be the set of different alternatives and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ be the set of different parameters whose weight vector is $\xi_j (j = 1, 2, \ldots, m)$ such that $\xi_j > 0$, $\sum\limits_{j=1}^{m} \xi_j = 1$. Consider $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$ as a set of different experts with weight vector $\eta_i (i = 1, 2, \ldots, n)$ such that $\eta_i > 0$ and $\sum\limits_{i=1}^{n} \eta_i = 1$, which give their preferences after evaluating the alternatives $\mathcal{A}^{(b)} (b = 1, 2, \ldots, t)$ over different parameters

$e_k(k = 1, 2, \ldots, m)$ in terms of IFSNs $\mathcal{A}_{ik} = (\zeta_{ik}, \vartheta_{ik})(i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$. Here ζ_{ik} represents the degree of satisfaction and ϑ_{ik} represents the degree of rejection of the alternative over the experts and parameters. Thus, the complete decision matrix of the given alternative is summarized as $\mathcal{M} = (\mathcal{A}_{ij})_{n\times m}$. The general procedure to access the best alternative(s) are summarized in the following steps:

Step 1: Arrange the information related to each alternative are assessed in terms of intuitionistic fuzzy soft decision matrix $\mathcal{M}^{(b)} = (\mathcal{A}_{ij}^{(b)})_{n\times m}$ for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$, as follows

$$\mathcal{M}^{(b)} = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array} \\ \begin{bmatrix} \mathcal{A}_{11}^{(b)} & \mathcal{A}_{12}^{(b)} & \cdots & \mathcal{A}_{1m}^{(b)} \\ \mathcal{A}_{21}^{(b)} & \mathcal{A}_{22}^{(b)} & \cdots & \mathcal{A}_{2m}^{(b)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{n1}^{(b)} & \mathcal{A}_{n2}^{(b)} & \cdots & \mathcal{A}_{nm}^{(b)} \end{bmatrix} \end{array}$$

Step 2: Normalize the rating values of the cost type attributes into the benefit type by using the following normalization equation

$$r_{ij}^{(b)} = \begin{cases} (\zeta_{ij}^{(b)}, \vartheta_{ij}^{(b)}) & ; \text{ if } j \in \text{benefit type parameter} \\ (\vartheta_{ij}^{(b)}, \zeta_{ij}^{(b)}) & ; \text{ if } j \in \text{cost type parameter} \end{cases} \tag{2.34}$$

and hence obtained the normalized decision matrices $\mathcal{R}^{(b)} = (r_{ij}^{(b)})$

Step 3: Aggregate the preference values of the all the alternatives into the overall collective values by using appropriate method/technique.

Step 4: Utilize the appropriate defuzzification method to compute the crisp value of the alternatives.

Step 5: Rank the alternatives based on the defuzzified values and select the best one(s) according to them.

Chapter 3

Distance and similarity measures for dual hesitant fuzzy soft sets and their applications[1]

In this chapter, we utilized the concept of the dual hesitant fuzzy soft set to define the several axioms of distance and similarity measures based on the metric of Hamming, Euclidean, and Hausdorff. The desirable inequalities in the form of their relationships between them have also been discussed. Later, a DM approach is presented based on the proposed measures where the features of the objects are taken in the form of dual hesitant fuzzy soft numbers. The effectiveness of the proposed measures is demonstrated through concrete examples of pattern recognition and medical diagnoses and compare the study with the several existing approaches.

3.1 Introduction

The extensive literature related to distance and similarity measures for solving the DM problems under the different environment are summarized in Section 1.1.1 of Chapter 1. During the phase of expressing the preferences over alternatives by an expert, a hesitant fuzzy set HFS) [194, 195] and dual hesitant fuzzy set (DHFSs) [267] provides a useful reference. Since these theories are successfully applied in various disciplines but failed

[1]The content of this chapter is published as "Distance and similarity measures for dual hesitant fuzzy soft sets and their applications in multi-criteria decision making problem", *International Journal for Uncertainty Quantification*, 7(3), 229 - 248, 2017 (**SCI: Impact Factor 3.259**).

to handle the parameterized tool during the analysis and hence they couldn't be applied effectively in real life problems. To overwhelm this, soft set theory [139] pays great attention and successfully deal with these type of conditions. Maji et al. [129, 130] combined the theory of soft set with the fuzzy and IFS theory and come up with the new concept of fuzzy soft set (FSS) and intuitionistic fuzzy soft set (IFSS). Peng and Yang [154] and Zhang and Shu [257] proposed the concept of dual hesitant fuzzy soft set (DHFSS) by combining the theory of DHFS with the SS. The advantages of these extended theories are that they are capable to facilitate the descriptions of the real-world situation with the help of their parameterized property. Due to this fact as well as the consideration of the hesitant membership and nonmembership degrees are very common in decision making (DM), it is necessary to develop such distance measures.

Therefore, by keeping the advantages of the DHFSS to express the uncertainty and fuzzy DM and the distance measures to compare the discrimination between the two sets, we have defined some new families of distance and similarity measures for two or more DHFSS based on Hamming, Euclidean, Hausdorff metric. Later, these distance measures are extended to their generalized measures. The various desirable properties between the proposed measures have been investigated in detail. Further, we developed a DM approach based on the proposed measures under the dual hesitant fuzzy soft set environment. The applicability of the approach is demonstrated through some numerical examples.

3.2 Proposed distance and similarity measure for dual hesitant fuzzy soft sets

In this section, some distance measures and their corresponding similarity measures for DHFSSs have been proposed on the set of universe $\mathcal{X}$. For it, let $\mathcal{E}$ is the set of parameters and $\Phi(\mathcal{X})$ be the collections of DHFSSs.

Definition 3.2.1. A real-valued function $\mathcal{D} : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to [0, 1]$ is called as distance measure, if $\mathcal{D}$ satisfies the following properties for $(\mathcal{F}, \mathcal{E})$, $(\mathcal{G}, \mathcal{E})$, $(\mathcal{H}, \mathcal{E}) \in \Phi(\mathcal{X})$:

(P1) $0 \leq \mathcal{D}((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) \leq 1$.

(P2) If $(\mathcal{F},\mathcal{E})=(\mathcal{G},\mathcal{E})$ then $\mathcal{D}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))=0$.

(P3) $\mathcal{D}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))=\mathcal{D}((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))$.

(P4) If $(\mathcal{F},\mathcal{E})\subseteq(\mathcal{G},\mathcal{E})\subseteq(\mathcal{H},\mathcal{E})$ then $\mathcal{D}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\leq\mathcal{D}((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E}))$ and $\mathcal{D}((\mathcal{G},\mathcal{E}),(\mathcal{H},\mathcal{E}))\leq\mathcal{D}((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E}))$.

For two DHFSSs $(\mathcal{F},\mathcal{E})=\left(\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i),\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)\right)$ and $(\mathcal{G},\mathcal{E})=\left(\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i),\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right)$ defined over the universal set $\mathcal{X}$, we define some series of the distance measures as follows.

(i) Hamming distance

$$\mathcal{D}_1\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)=\frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\\+\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\end{array}\right)\tag{3.1}$$

(ii) Normalized Hamming distance

$$\mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)=\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\\+\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\end{array}\right)\tag{3.2}$$

(iii) Euclidean distance

$$\mathcal{D}_3\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)=\left\{\frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2\\+\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2\end{array}\right)\right\}^{\frac{1}{2}}\tag{3.3}$$

(iv) Normalized Euclidean distance

$$\mathcal{D}_4\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)=\left\{\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2\\+\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2\end{array}\right)\right\}^{\frac{1}{2}}\tag{3.4}$$

where k_i and l_i represent the number of values in $\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)$and $\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)$ respectively.

In order to justify, whether the measures defined as above are valid distance measures for DHFSSs. For it, we prove the results in the following theorems.

Remark 3.2.1. In particular, if we take $m = 1$, we will get the distance measures for DHFS [176, 205] as defined in Eq. (2.18) of Chapter 2. Thus, distance measures of the DHFSs are the special case of the proposed distance measures under DHFSSs.

Proposition 3.2.1. The measure $\mathcal{D}_2$, between two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ satisfies the properties as mentioned in Definition 3.2.1.

Proof. Let $(\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})$ be two DHFSSs with n alternatives and m parameters; then we have

(P1) As $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| \geq 0$ and $\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right| \geq 0$. Therefore,

$$\sum_{i=1}^{n}\left[\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| + \frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right] \geq 0$$

Hence $\mathcal{D}_2\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) \geq 0$.

Furthermore, $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| \leq 1$ and $\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right| \leq 1$ which implies that

$$\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| + \frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right] \leq 1$$

i.e., $\mathcal{D}_2\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) \leq 1$

Hence $0 \leq \mathcal{D}_2\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) \leq 1$.

(P2) Let $(\mathcal{F}, \mathcal{E}) = (\mathcal{G}, \mathcal{E}) \Leftrightarrow \zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) = \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)$ and $\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) = \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)$, for each i and j, if and only if, $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| = 0$ and $\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right| = 0$. Hence $\mathcal{D}_2\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) = 0$.

(P3) For any two real numbers a and b, we know that $\left|a - b\right|$ is equivalent to $\left|b - a\right|$. Thus by definition of $\mathcal{D}_2$, we have

$$\begin{aligned}
&\mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)\\
=\ &\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|+\frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right]\\
=\ &\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)\right|+\frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)\right|\right]\\
=\ &\mathcal{D}_2\big((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E})\big).
\end{aligned}$$

(P4) Let $(\mathcal{F},\mathcal{E})\subseteq(\mathcal{G},\mathcal{E})\subseteq(\mathcal{H},\mathcal{E})\Rightarrow\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)\le\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\le\zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)$ and $\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)\ge\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\ge\vartheta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)$. Thus, $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\le\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\right|$, $\left|\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\right|\le\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\right|$, $\left|\vartheta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\le\left|\vartheta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\right|$ and $\left|\vartheta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)-\vartheta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\right|\le\left|\vartheta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\right|$. Therefore, by the definition of $\mathcal{D}_2$, we can easily obtain that $\mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)\le\mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E})\big)$ and $\mathcal{D}_2\big((\mathcal{G},\mathcal{E}),(\mathcal{H},\mathcal{E})\big)\le\mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E})\big)$.

Hence $\mathcal{D}_2$ is a valid distance measure. □

Proposition 3.2.2. The measure $\mathcal{D}_4$ as defined in Eq. (3.4) is also a valid measure.

Proof. Follows from Preposition 3.2.1, so we omit it here. □

The working of these proposed measures are explained with a numerical example as below.

Example 3.2.1. Let $\mathcal{X}=\{u_1,u_2,u_3,u_4\}$ represent the set of four houses under the consideration of a person to purchase, $\mathcal{E}$ is a set of parameters such that $\mathcal{E}$={e_1("expensive"), e_2("beautiful"), e_3("in good location")}. An expert gives their preferences in terms of DHFSSs, $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$, for describing the "attractiveness of the houses" and their rating values are summarized in the form of the following decision matrices as follows:

$$(\mathcal{F},\mathcal{E})=\begin{array}{c}\\ u_1\\ u_2\\ u_3\\ u_4\end{array}\begin{array}{c}\begin{array}{ccc}\quad e_1\quad & \quad e_2\quad & \quad e_3\quad\end{array}\\ \begin{bmatrix}(\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}) & (\{0.6,0.5,0.4\},\{0.4,0.3,0.2\}) & (\{0.3,0.2,0.1\},\{0.7,0.6,0.6\})\\ (\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}) & (\{0.5,0.4,0.3\},\{0.5,0.3,0.3\}) & (\{0.7,0.7,0.5\},\{0.3,0.2,0.2\})\\ (\{0.8,0.7,0.7\},\{0.2,0.1,0.1\}) & (\{0.8,0.8,0.7\},\{0.2,0.2,0.1\}) & (\{0.7,0.6,0.5\},\{0.3,0.2,0.1\})\\ (\{0.4,0.4,0.3\},\{0.6,0.5,0.4\}) & (\{0.6,0.6,0.5\},\{0.4,0.3,0.2\}) & (\{0.8,0.7,0.6\},\{0.2,0.1,0.1\})\end{bmatrix}\end{array}$$

$$(\mathcal{G},\mathcal{E}) = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} & \begin{bmatrix} (\{0.7,0.6,0.4\},\{0.3,0.2\}) & (\{0.6,0.4,0.3\},\{0.3,0.1\}) & (\{0.6,0.4,0.3\},\{0.4,0.1\}) \\ (\{0.9,0.7,0.6\},\{0.1\}) & (\{0.7,0.5,0.4\},\{0.3,0.2\}) & (\{0.4,0.3,0.3\},\{0.6,0.5\}) \\ (\{0.8,0.7,0.6\},\{0.2,0.1\}) & (\{0.5,0.4\},\{0.2,0.1\}) & (\{0.9,0.8,0.6\},\{0.1\}) \\ (\{0.6,0.5,0.5\},\{0.4,0.2\}) & (\{0.8,0.7,0.5\},\{0.2,0.1\}) & (\{0.6,0.4,0.3\},\{0.3,0.1\}) \end{bmatrix} \end{array}$$

By applying Eq. (3.2), we have

$$\begin{aligned}
&\mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) \\
= \;& \frac{1}{24}\sum_{j=1}^{3}\sum_{i=1}^{4}\left[\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| + \frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right] \\
= \;& \tfrac{1}{24}\Big[\tfrac{1}{3}\big(|0.7-0.7|+|0.6-0.6|+|0.5-0.4|\big)+\tfrac{1}{3}\big(|0.6-0.6|+|0.5-0.4|+|0.4-0.3|\big) \\
+ \;& \frac{1}{3}\big(|0.3-0.6|+|0.2-0.4|+|0.1-0.3|\big)+\frac{1}{3}\big(|0.6-0.9|+|0.5-0.7|+|0.4-0.6|\big) \\
+ \;& \frac{1}{3}\big(|0.5-0.7|+|0.4-0.5|+|0.3-0.4|\big)+\frac{1}{3}\big(|0.7-0.4|+|0.7-0.3|+|0.5-0.3|\big) \\
+ \;& \frac{1}{3}\big(|0.8-0.8|+|0.7-0.7|+|0.7-0.6|\big)+\frac{1}{3}\big(|0.8-0.5|+|0.8-0.4|+|0.7-0.4|\big) \\
+ \;& \frac{1}{3}\big(|0.7-0.9|+|0.6-0.8|+|0.5-0.6|\big)+\frac{1}{3}\big(|0.4-0.6|+|0.4-0.5|+|0.3-0.5|\big) \\
+ \;& \frac{1}{3}\big(|0.6-0.8|+|0.6-0.7|+|0.5-0.5|\big)+\frac{1}{3}\big(|0.8-0.6|+|0.7-0.4|+|0.6-0.3|\big) \\
+ \;& \frac{1}{3}\big(|0.3-0.3|+|0.2-0.2|+|0.1-0.2|\big)+\frac{1}{3}\big(|0.4-0.3|+|0.3-0.1|+|0.2-0.1|\big) \\
+ \;& \frac{1}{3}\big(|0.7-0.4|+|0.6-0.1|+|0.6-0.1|\big)+\frac{1}{3}\big(|0.3-0.1|+|0.2-0.1|+|0.1-0.1|\big) \\
+ \;& \frac{1}{3}\big(|0.5-0.3|+|0.3-0.2|+|0.3-0.2|\big)+\frac{1}{3}\big(|0.3-0.6|+|0.2-0.5|+|0.2-0.5|\big) \\
+ \;& \frac{1}{3}\big(|0.2-0.2|+|0.1-0.1|+|0.1-0.1|\big)+\frac{1}{3}\big(|0.2-0.2|+|0.2-0.1|+|0.1-0.1|\big) \\
+ \;& \frac{1}{3}\big(|0.3-0.1|+|0.2-0.1|+|0.1-0.1|\big)+\frac{1}{3}\big(|0.6-0.4|+|0.5-0.2|+|0.4-0.2|\big) \\
+ \;& \frac{1}{3}\big(|0.4-0.2|+|0.3-0.1|+|0.2-0.1|\big)+\frac{1}{3}\big(|0.2-0.3|+|0.1-0.1|+|0.1-0.1|\big)\Big] \\
= \;& 0.1569
\end{aligned}$$

and by applying Eq. (3.4), then we get $\mathcal{D}_4\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = 0.2003$.

If we considered the weights $\xi_i > 0$ of the element $u_i \in \mathcal{X}(i = 1, 2, \dots, n)$ with $\sum\limits_{i=1}^{n} \xi_i = 1$ and $\eta_j > 0$ be the weight vector of the parameters $e_j(j = 1, 2, \dots, m)$ such that $\sum\limits_{j=1}^{m} \eta_j = 1$, then we develop some normalized weighted Hamming and Euclidean distances between the two DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ and a real number $\lambda > 0$.

(i) Normalized weighted Hamming distance

$$\mathcal{D}_5\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \tfrac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\\ +\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\end{array}\right)\right] \tag{3.5}$$

(ii) Normalized weighted Euclidean distance

$$\mathcal{D}_6\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \left\{\tfrac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2\\ +\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2\end{array}\right)\right]\right\}^{\frac{1}{2}} \tag{3.6}$$

(iii) Generalized normalized weighted Hamming distance

$$\mathcal{D}_7\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \left\{\tfrac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i\left(\begin{array}{l}\dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^\lambda\\ +\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^\lambda\end{array}\right)\right]\right\}^{\frac{1}{\lambda}} \tag{3.7}$$

Further, it is easy to check that the weighted distances $\mathcal{D}_k (k = 5, 6, 7)$ between DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ also satisfy the properties (P1)-(P4) as defined in Definition 3.2.1.

Proposition 3.2.3. Let $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ be two DHFSSs in $\mathcal{X}$, then $\mathcal{D}_5((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$, $\mathcal{D}_6((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ and $\mathcal{D}_7((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ are the valid distance measures.

Proof. Since $\eta_j \in [0,1]$, $\xi_i \in [0,1]$ and $\sum_{j=1}^{m}\eta_j = 1$, $\sum_{i=1}^{n}\xi_i = 1$ then we can easily obtain $0 \le \mathcal{D}_5((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \le \mathcal{D}_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$. Thus, $\mathcal{D}_5((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ satisfies (P1). The proofs of (P2)-(P4) are similar to those of proposition 3.2.1. Similarly for $\mathcal{D}_6$ and $\mathcal{D}_7$. $\square$

Proposition 3.2.4. The measures $\mathcal{D}_1$ and $\mathcal{D}_3$ satisfies the following properties for DHFSSs,

(i) $0 \le \mathcal{D}_1 \le mn$.

(ii) $0 \leq \mathcal{D}_3 \leq \sqrt{mn}$.

Proof. For any two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$, we can easily obtain that $\mathcal{D}_1((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) = mn\mathcal{D}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E}))$ and thus by Proposition 3.2.1, we get $0 \leq \mathcal{D}_1((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) \leq mn$. Similarly we can obtain $0 \leq \mathcal{D}_3((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) \leq \sqrt{mn}$. $\square$

Below we give an example to illustrate the proposed distance measures:

Example 3.2.2. If we consider the weight vectors $\xi = (0.2, 0.3, 0.4, 0.1)^T$ and $\eta = (0.35, 0.25, 0.4)^T$ for $u_i (i = 1, 2, 3, 4)$ and $e_j (j = 1, 2, 3)$ respectively on two DHFSS $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ defined in Example 3.2.1, then the measurement values between them by using $\mathcal{D}_5$, $\mathcal{D}_6$ and $\mathcal{D}_7$ are obtained as $\mathcal{D}_5\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) = 0.0351$, $\mathcal{D}_6\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) = 0.0938$ and $\mathcal{D}_7\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big)_{\lambda=1} = 0.0351$; $\mathcal{D}_7\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big)_{\lambda=1.5} = 0.0667$; $\mathcal{D}_7\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big)_{\lambda=2} = 0.0938$; $\mathcal{D}_7\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big)_{\lambda=2.5} = 0.1130$; $\mathcal{D}_7\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big)_{\lambda=3} = 0.1381$. From these results, it has been seen that with the increase of the parameter λ, the corresponding measure values defined by $\mathcal{D}_7$ are increasing.

Proposition 3.2.5. The distance measures $\mathcal{D}_2$ and $\mathcal{D}_5$ satisfies the inequality $\mathcal{D}_5 \leq \mathcal{D}_2$ for DHFSSs.

Proof. Since, $\eta_j \in [0, 1]$, $\xi_i \in [0, 1]$ and $\eta_j \leq 1$, $\xi_i \leq 1$, then we have

$$\begin{aligned}
& \mathcal{D}_5\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) \\
= \ & \frac{1}{2mn} \sum_{j=1}^{m} \eta_j \left[\sum_{i=1}^{n} \xi_i \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \left| \zeta_{\mathcal{F}(e_j)}^{\sigma(s)}(u_i) - \zeta_{\mathcal{G}(e_j)}^{\sigma(s)}(u_i) \right| + \frac{1}{l_i} \sum_{t=1}^{l_i} \left| \vartheta_{\mathcal{F}(e_j)}^{\sigma(t)}(u_i) - \vartheta_{\mathcal{G}(e_j)}^{\sigma(t)}(u_i) \right| \right) \right] \\
\leq \ & \frac{1}{2mn} \sum_{j=1}^{m} \sum_{i=1}^{n} \left[\frac{1}{k_i} \sum_{s=1}^{k_i} \left| \zeta_{\mathcal{F}(e_j)}^{\sigma(s)}(u_i) - \zeta_{\mathcal{G}(e_j)}^{\sigma(s)}(u_i) \right| + \frac{1}{l_i} \sum_{t=1}^{l_i} \left| \vartheta_{\mathcal{F}(e_j)}^{\sigma(t)}(u_i) - \vartheta_{\mathcal{G}(e_j)}^{\sigma(t)}(u_i) \right| \right] \\
= \ & \mathcal{D}_2\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big)
\end{aligned}$$

Since $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ are arbitrary DHFSSs and hence we get $\mathcal{D}_5 \leq \mathcal{D}_2$. is true for all DHFSss. $\square$

Proposition 3.2.6. The distance measures $\mathcal{D}_4$ and $\mathcal{D}_6$ satisfy the inequality $\mathcal{D}_6 \leq \mathcal{D}_4$ for DHFSSs.

Proof. Result is easily derived as similar to the proof of the Proposition 3.2.5, so we omit them here. □

Proposition 3.2.7. The distance measures $\mathcal{D}_2$ and $\mathcal{D}_4$ satisfy the inequalities $\mathcal{D}_4 \leq \sqrt{\mathcal{D}_2}$ for DHFSSs.

Proof. For any two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$, we have $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| \leq 1$, $\left|\vartheta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| \leq 1$. Further, we know that for any $a \in [0, 1]$, we have $a^2 \leq a$. Hence, $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2 \leq \left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|$ and $\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2 \leq \left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|$. Therefore,

$$\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2 + \frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2\right)$$
$$\leq \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right| + \frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right)$$

which implies that $(\mathcal{D}_4((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})))^2 \leq \mathcal{D}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E}))$. Since $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ are arbitrary DHFSSs and hence $\mathcal{D}_4 \leq \sqrt{\mathcal{D}_2}$. □

Proposition 3.2.8. The measures $\mathcal{D}_6$ and $\mathcal{D}_5$ satisfy the inequalities $\mathcal{D}_6 \leq \sqrt{\mathcal{D}_5}$ for DHFSSs.

Proof. Results is easily follows from the Proposition 3.2.7. □

Hausdorff distance between two non-empty closed and bounded sets is a measure of resemblance between them. For example, consider $A = [x_1, x_2]$ and $B = [y_1, y_2]$ in the Euclidean domain $\mathbb{R}$, the Hausdroff distance in additive set environment is given by:

$$d^H(A, B) = \max\left\{\left|x_1 - y_1\right|, \left|x_2 - y_2\right|\right\}$$

Now, for any two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ over $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$, we propose the following utmost distance measures:

(i) Normalized Hamming-Hausdorff distance

$$\mathcal{D}^H_1\big((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})\big) = \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\max\left\{\begin{array}{l}\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|,\\ \max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\end{array}\right\} \tag{3.8}$$

(ii) Normalized weighted Hamming-Hausdorff distance

$$\mathcal{D}_2^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \frac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i \max\left\{\begin{array}{l}\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|,\\ \max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\end{array}\right\}\right] \tag{3.9}$$

(iii) Normalized Euclidean-Hausdorff distance

$$\mathcal{D}_3^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \left\{\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\max\left\{\begin{array}{l}\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2,\\ \max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2\end{array}\right\}\right\}^{\frac{1}{2}} \tag{3.10}$$

(iv) Normalized weighted Euclidean-Hausdorff distance

$$\mathcal{D}_4^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \left\{\frac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i \max\left(\begin{array}{l}\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^2,\\ \max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^2\end{array}\right)\right]\right\}^{\frac{1}{2}} \tag{3.11}$$

(v) Generalized Normalized weighted Hausdorff distance

$$\mathcal{D}_5^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \left\{\frac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i \max\left\{\begin{array}{l}\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|^\lambda,\\ \max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|^\lambda\end{array}\right\}\right]\right\}^{\frac{1}{\lambda}} \tag{3.12}$$

Proposition 3.2.9. The distance $\mathcal{D}_1^H$ defined in Eq. (3.8) for two DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ is a valid distance measure.

Proof. Let $(\mathcal{F},\mathcal{E})$, $(\mathcal{G},\mathcal{E})$ be two DHFSSs with n alternatives and m parameters, then measure $\mathcal{D}_1^H$ satisfies the following properties:

(P1) For DHFSSs, we have $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\geq 0$ and $\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\geq 0$ which implies that $\frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\max\big\{\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|,\max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\big\}\geq 0$ and hence we get $\mathcal{D}_1^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)\geq 0$.

Also, $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\leq 1$ and $\left|\vartheta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|\leq 1$ which implies that $\frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\max\big\{\max\limits_{s}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|,\max\limits_{t}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\big\}\leq 1$ and hence $\mathcal{D}_1^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)\leq 1$. Therefore, we have $0\leq\mathcal{D}_1^H\big((\mathcal{F},\mathcal{E}),\,(\mathcal{G},\mathcal{E})\big)\leq 1$

(P2) Let $(\mathcal{F},\mathcal{E})=(\mathcal{G},\mathcal{E})\Leftrightarrow\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)=\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)$ and $\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)=\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)$, which implies, $\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|=0$ and $\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|=0$. Hence, we get $\mathcal{D}_1^H\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)=0$

(P3) It is straightforward.

(P4) Let $(\mathcal{F},\mathcal{E}) \subseteq (\mathcal{G},\mathcal{E}) \subseteq (\mathcal{H},\mathcal{E}) \Rightarrow \zeta_{\mathcal{F}(e_j)} \leq \zeta_{\mathcal{G}(e_j)} \leq \zeta_{\mathcal{H}(e_j)}$ and $\vartheta_{\mathcal{F}(e_j)} \geq \vartheta_{\mathcal{G}(e_j)} \geq \vartheta_{\mathcal{H}(e_j)}$. Thus, $\sum_{j=1}^{m}\sum_{i=1}^{n} \max\big\{\max_s \big|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\big|, \max_t \big|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\big|\big\}$ $\leq \sum_{j=1}^{m}\sum_{i=1}^{n} \max\big\{\max_s \big|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\big|, \max_t \big|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{H}(e_j)}(u_i)\big|\big\}$ and $\sum_{j=1}^{m}\sum_{i=1}^{n} \max\big\{\max_s \big|\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\big|, \max_t \big|\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{H}(e_j)}(u_i)\big|\big\}$ $\leq \sum_{j=1}^{m}\sum_{i=1}^{n} \max\big\{\max_s \big|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{H}(e_j)}(u_i)\big|, \max_t \big|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{H}(e_j)}(u_i)\big|\big\}$. Therefore, by the definition of $\mathcal{D}^H_1$, we can obtain that $\mathcal{D}^H_1\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) \leq \mathcal{D}^H_1\big((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E})\big)$ and $\mathcal{D}^H_1\big((\mathcal{G},\mathcal{E}),(\mathcal{H},\mathcal{E})\big) \leq \mathcal{D}^H_1\big((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E})\big)$.

Hence, $\mathcal{D}^H_1$ is a distance measure. □

Proposition 3.2.10. The measures $\mathcal{D}^H_2$, $\mathcal{D}^H_3$, $\mathcal{D}^H_4$ and $\mathcal{D}^H_5$ for DHFSSs are the valid distance measures.

Proof. It follows from Proposition 3.2.9. □

Example 3.2.3. If we utilize the distance measures $\mathcal{D}^H_1, \mathcal{D}^H_2, \mathcal{D}^H_3, \mathcal{D}^H_4$ and $\mathcal{D}^H_5$, on the data as considered in Example 3.2.2, then we get $\mathcal{D}^H_1 = 0.2667, \mathcal{D}^H_2 = 0.0221, \mathcal{D}^H_3 = 0.2915, \mathcal{D}^H_4 = 0.0846$ and $\mathcal{D}^H_5|_{\lambda=1} = 0.0221$; $\mathcal{D}^H_5|_{\lambda=1.5} = 0.0534$; $\mathcal{D}^H_5|_{\lambda=2} = 0.0846$; $\mathcal{D}^H_5|_{\lambda=2.5} = 0.1130$; $\mathcal{D}^H_5|_{\lambda=3} = 0.1381$.

Proposition 3.2.11. The measures $\mathcal{D}^H_2$ and $\mathcal{D}^H_1$ satisfies the inequality $\mathcal{D}^H_2 \leq \mathcal{D}^H_1$ for DHFSSs.

Proof. Since $\eta_j, \xi_i \in [0,1]$, therefore

$$\begin{aligned}
&\mathcal{D}^H_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big)\\
= {}& \frac{1}{2mn}\sum_{j=1}^{m}\eta_j\left[\sum_{i=1}^{n}\xi_i \max\left\{\max_s \big|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\big|, \max_t \big|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\big|\right\}\right]\\
\leq {}& \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n} \max\left\{\max_s \big|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i) - \zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\big|, \max_t \big|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i) - \vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\big|\right\}\\
= {}& \mathcal{D}^H_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))
\end{aligned}$$

Hence, $\mathcal{D}^H_2 \leq \mathcal{D}^H_1$. □

Proposition 3.2.12. The measures $\mathcal{D}_3^H$ and $\mathcal{D}_4^H$ satisfies the inequality $\mathcal{D}_4^H \leq \mathcal{D}_3^H$ for DHFSSs.

Proof. Follows from proposition 3.2.11, so we omit their proof here. □

Proposition 3.2.13. The measures $\mathcal{D}_3^H$ and $\mathcal{D}_1^H$ satisfies the inequality $\mathcal{D}_3^H \leq \sqrt{\mathcal{D}_1^H}$ for DHFSSs.

Proof. Since for any $a \in [0,1]$, $a^2 \leq a \leq a^{1/2}$, the remaining proof follows from the above Propositions. □

Proposition 3.2.14. The measures $\mathcal{D}_4^H$ and $\mathcal{D}_2^H$ satisfies the inequality $\mathcal{D}_4^H \leq \sqrt{\mathcal{D}_2^H}$ for DHFSSs.

Proof. Follows from Proposition 3.2.13. □

Proposition 3.2.15. The measures $\mathcal{D}_2$ and $\mathcal{D}_1^H$ satisfy the following inequalities $\mathcal{D}_2 \leq \mathcal{D}_1^H$ for DHFSS.

Proof. Since for any k numbers, $\frac{1}{k}\sum\limits_{z=1}^{k} a_z \leq \max\limits_{z}\{a_z\}$. Thus from the definition of $\mathcal{D}_2$, we have

$$\begin{aligned}
& \mathcal{D}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) \\
= \; & \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|+\frac{1}{l_i}\sum_{t=1}^{l_i}\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right) \\
\leq \; & \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\max\left\{\max_s\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|,\max_t\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right\} \\
\leq \; & \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\max\left\{\max_s\left|\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u_i)-\zeta^{\sigma(s)}_{\mathcal{G}(e_j)}(u_i)\right|,\max_t\left|\vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u_i)-\vartheta^{\sigma(t)}_{\mathcal{G}(e_j)}(u_i)\right|\right\} \\
= \; & \mathcal{D}_1^H((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))
\end{aligned}$$

Therefore, the inequality $\mathcal{D}_2 \leq \mathcal{D}_1^H$ holds. □

Proposition 3.2.16. The measures $\mathcal{D}_4$ and $\mathcal{D}_3^H$ satisfy the inequality $\mathcal{D}_4 \leq \mathcal{D}_3^H$ for DHFSSs.

Proof. Results follows from the proposition 3.2.15. □

Proposition 3.2.17. The measures $\mathcal{D}_2$, $\mathcal{D}_5$ and $\mathcal{D}_6$ satisfy the following inequalities

(i) $\mathcal{D}_1^H \geq (\mathcal{D}_5 + \mathcal{D}_6)/2$,

(ii) $\mathcal{D}_1^H \geq \sqrt{\mathcal{D}_5 \cdot \mathcal{D}_6}$.

Proof. Since $\mathcal{D}_5 \leq \mathcal{D}_2 \leq \mathcal{D}_1^H$ and $\mathcal{D}_6 \leq \mathcal{D}_2 \leq \mathcal{D}_1^H$. So by adding these inequalities, we get $\mathcal{D}_1^H \geq (\mathcal{D}_5 + \mathcal{D}_6)/2$. On the other hand, by multiplying these, we get $\mathcal{D}_1^H \geq \sqrt{\mathcal{D}_5 \cdot \mathcal{D}_6}$. $\square$

It is well known that similarity measures can be generated from distance measures. Therefore, based on the relationship between the distance and similarity measures, we can define some similarity measures $\mathcal{S}_k\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big), (k = 1, 2, 3)$ between two DHFSSs as

(i) Hamming similarity

$$\mathcal{S}_1\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\begin{array}{l} \dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta_{\mathcal{F}(e_j)}^{\sigma(s)}(u_i) - \zeta_{\mathcal{G}(e_j)}^{\sigma(s)}(u_i)\right| \\ + \dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta_{\mathcal{F}(e_j)}^{\sigma(t)}(u_i) - \vartheta_{\mathcal{G}(e_j)}^{\sigma(t)}(u_i)\right| \end{array}\right] \tag{3.13}$$

(ii) Euclidean similarity

$$\mathcal{S}_2\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = 1 - \left\{\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{l} \dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta_{\mathcal{F}(e_j)}^{\sigma(s)}(u_i) - \zeta_{\mathcal{G}(e_j)}^{\sigma(s)}(u_i)\right|^2 \\ + \dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta_{\mathcal{F}(e_j)}^{\sigma(t)}(u_i) - \vartheta_{\mathcal{G}(e_j)}^{\sigma(t)}(u_i)\right|^2 \end{array}\right)\right\}^{\frac{1}{2}} \tag{3.14}$$

(iii) Normalized Hamming similarity

$$\mathcal{S}_3\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = 1 - \left[\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\begin{array}{l} \dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\left|\zeta_{\mathcal{F}(e_j)}^{\sigma(s)}(u_i) - \zeta_{\mathcal{G}(e_j)}^{\sigma(s)}(u_i)\right|^\lambda \\ + \dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\left|\vartheta_{\mathcal{F}(e_j)}^{\sigma(t)}(u_i) - \vartheta_{\mathcal{G}(e_j)}^{\sigma(t)}(u_i)\right|^\lambda \end{array}\right)\right]^{\frac{1}{\lambda}} \tag{3.15}$$

which satisfies the following properties:

(P1) $0 \leq \mathcal{S}_k\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) \leq 1$;

(P2) If $(\mathcal{F},\mathcal{E}) = (\mathcal{G},\mathcal{E})$ then $\mathcal{S}_k\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = 1$;

(P3) $\mathcal{S}_k\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big) = \mathcal{S}_k((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))$.

(P4) If $(\mathcal{F},\mathcal{E}) \subseteq (\mathcal{G},\mathcal{E}) \subseteq (\mathcal{H},\mathcal{E})$ then $\mathcal{S}_k((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E})) \leq \mathcal{S}_k((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ and $\mathcal{S}_k((\mathcal{F},\mathcal{E}),(\mathcal{H},\mathcal{E})) \leq \mathcal{S}_k((\mathcal{G},\mathcal{E}),(\mathcal{H},\mathcal{E}))$.

It is clear that the larger the value of $\mathcal{S}_k\big((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\big),(k=1,2,3)$, the more the similarity between DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$.

3.3 Approach based on the proposed distance measures for decision making process

In this section, a DM method by using above defined distance measures for DHFSSs have been presented.

The general description of the DM problem is summarized in Section 2.6 of Chapter 2. The rating values of each alternative is assessed by a decision maker and represent their information in terms of DHFSNs. Their collective information is represented in the form of the dual hesitant fuzzy soft decision matrix $(\mathcal{F},\mathcal{A}_q) = (\alpha_{\mathcal{F}(e_j)}(u_i))_{n\times m}$ where $\alpha_{\mathcal{F}(e_j)}(u_i) = \Big(\zeta^{\sigma(s)}_{\mathcal{F}(e_j)}(u^q_i), \vartheta^{\sigma(t)}_{\mathcal{F}(e_j)}(u^q_i)\Big)$ for $e_j \in \mathcal{X}$ and $q = 1,2,\ldots,k$. Then, in the following, based on the proposed distance measures, we develop an approach to solve the DM problems with dual hesitant fuzzy soft information. The various steps associated with the approach are given as below.

Step 1: Arrange the collective information of the alternatives $\mathcal{A}_q(q = 1,2\ldots,k)$ as a decision matrix $(\mathcal{F},\mathcal{A}_q)$.

Step 2: Normalize the collective information, if required by using Eq. (2.34) of Chapter 2 and obtained the nornalized decision matrix.

Step 3: Utilize the proposed distance measures $\mathcal{D}$ to compute the overall value of each alternative.

Step 4: Rank the alternatives by finding the minimum value of $\arg\min\{\mathcal{D}\}$ and select the best one(s).

3.4 Illustrative Examples

In order to illustrate the performance and validity of the above proposed distance measures, examples from the field of pattern recognition and medical diagnosis have been taken into account.

3.4.1 Example 1: Pattern recognition

Let u_1, u_2, u_3 be three experts which are going to evaluate the three known patterns $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ over four different parameters e_1, e_2, e_3 and e_4. Assume that the rating values of each pattern given by the expert against the parameters $e_j (j = 1, 2, 3, 4)$ are represented in the form of the DHFSSs which are given in Table 3.1.

Table 3.1: Rating values of the patterns in terms of DHFSNs

	For pattern $\mathcal{A}_1$			
	e_1	e_2	e_3	e_4
u_1	$(\{0.7, 0.6, 0.5\}, \{0.3, 0.2, 0.1\})$	$(\{0.6, 0.5, 0.4\}, \{0.4, 0.3, 0.2\})$	$(\{0.3, 0.2, 0.1\}, \{0.7, 0.6, 0.6\})$	$(\{0.8, 0.7, 0.5\}, \{0.2, 0.1\})$
u_2	$(\{0.6, 0.5, 0.4\}, \{0.3, 0.2, 0.1\})$	$(\{0.5, 0.4, 0.3\}, \{0.5, 0.3, 0.3\})$	$(\{0.7, 0.7, 0.5\}, \{0.3, 0.2, 0.2\})$	$(\{0.4, 0.3, 0.2\}, \{0.5\})$
u_3	$(\{0.8, 0.7, 0.7\}, \{0.2, 0.1\})$	$(\{0.8, 0.8, 0.7\}, \{0.2, 0.2, 0.1\})$	$(\{0.7, 0.6, 0.5\}, \{0.3, 0.2, 0.1\})$	$(\{0.7, 0.6, 0.5\}, \{0.3, 0.2, 0.1\})$
	For pattern $\mathcal{A}_2$			
	e_1	e_2	e_3	e_4
u_1	$(\{0.4, 0.4, 0.3\}, \{0.6, 0.5, 0.4\})$	$(\{0.6, 0.6, 0.5\}, \{0.4, 0.3, 0.2\})$	$(\{0.8, 0.7, 0.6\}, \{0.2, 0.1, 0.1\})$	$(\{0.7, 0.5, 0.4\}, \{0.2, 0.2, 0.1\})$
u_2	$(\{0.7, 0.6, 0.4\}, \{0.3, 0.2\})$	$(\{0.6, 0.4, 0.3\}, \{0.3, 0.1\})$	$(\{0.6, 0.4, 0.3\}, \{0.4, 0.1\})$	$(\{0.5, 0.4, 0.2\}, \{0.4, 0.3, 0.1\})$
u_3	$(\{0.9, 0.7, 0.6\}, \{0.1\})$	$(\{0.7, 0.5, 0.4\}, \{0.3, 0.2\})$	$(\{0.4, 0.3, 0.3\}, \{0.6, 0.5\})$	$(\{0.7, 0.6, 0.4\}, \{0.2, 0.1\})$
	For pattern $\mathcal{A}_3$			
	e_1	e_2	e_3	e_4
u_1	$(\{0.8, 0.7, 0.6\}, \{0.2, 0.1\})$	$(\{0.5, 0.4\}, \{0.2, 0.1\})$	$(\{0.9, 0.8, 0.6\}, \{0.1\})$	$(\{0.9, 0.7, 0.5\}, \{0.1\})$
u_2	$(\{0.6, 0.5, 0.5\}, \{0.4, 0.2\})$	$(\{0.8, 0.7, 0.5\}, \{0.2, 0.1\})$	$(\{0.6, 0.4, 0.3\}, \{0.3, 0.1\})$	$(\{0.3, 0.2, 0.1\}, \{0.5, 0.3, 0.2\})$
u_3	$(\{0.7, 0.6, 0.4\}, \{0.2\})$	$(\{0.6, 0.4, 0.3\}, \{0.3\})$	$(\{0.4, 0.3, 0.3\}, \{0.2, 0.1\})$	$(\{0.6, 0.3, 0.2\}, \{0.2, 0.1\})$

Consider an unknown pattern $\mathcal{B} \in$ DHFSS which will be recognized, where the rating value of it is shown in Table 3.2. Then the target of the considered problem is to classify the pattern $\mathcal{B}$ in one of the classes $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$. For it, the various proposed distance

Table 3.2: Rating values of the pattern $\mathcal{B}$ in terms of DHFSNs

	e_1	e_2	e_3	e_4
u_1	$(\{0.3, 0.2, 0.1\}, \{0.2, 0.1\})$	$(\{0.4\}, \{0.3, 0.2\})$	$(\{0.6, 0.4, 0.3\}, \{0.2\})$	$(\{0.4\}, \{0.5, 0.1\})$
u_2	$(\{0.6, 0.5, 0.2\}, \{0.4, 0.3\})$	$(\{0.5, 0.2\}, \{0.3, 0.1\})$	$(\{0.7, 0.3\}, \{0.3, 0.1\})$	$(\{0.2, 0.1\}, \{0.3\})$
u_3	$(\{0.6, 0.4, 0.3\}, \{0.2\})$	$(\{0.9, 0.7, 0.5\}, \{0.1\})$	$(\{0.4, 0.2\}, \{0.5, 0.3\})$	$(\{0.8\}, \{0.1\})$

measures, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_1^H$ and $\mathcal{D}_3^H$ are utilized to compute the measurement values of the alternatives $\mathcal{A}_q(q = 1, 2, 3)$ from $\mathcal{B}$. The results corresponding to each measure are summarized as below.

$$\mathcal{D}_1((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 2.8000, \quad \mathcal{D}_1((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 2.2667, \quad \mathcal{D}_1((\mathcal{A}_3, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 2.5222$$
$$\mathcal{D}_2((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.2333, \quad \mathcal{D}_2((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.1889, \quad \mathcal{D}_2((\mathcal{A}_3, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.2102$$
$$\mathcal{D}_3((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.8705, \quad \mathcal{D}_3((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7087, \quad \mathcal{D}_3((\mathcal{A}_3, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.9037$$
$$\mathcal{D}_4((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.2513, \quad \mathcal{D}_4((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.2046, \quad \mathcal{D}_4((\mathcal{A}_3, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.2609$$
$$\mathcal{D}_1^H((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.4222, \quad \mathcal{D}_1^H((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.3444, \quad \mathcal{D}_1^H((\mathcal{A}_3, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.4444$$
$$\mathcal{D}_3^H((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.3859, \quad \mathcal{D}_3^H((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.3145, \quad \mathcal{D}_3^H((\mathcal{A}_3, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.4269$$

Thus, from these distance measures, we conclude that the pattern $\mathcal{B}$ belongs to the pattern $\mathcal{A}_2$.

On the other hand, if we assume that weight vector of the experts is $\xi_i = (0.3, 0.5, 0.2)^T$ and the parameter weight is $\eta_j = (0.1, 0.3, 0.4, 0.2)^T$, then by utilizing the distance measures $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_2^H$ and $\mathcal{D}_4^H$, their measurement value are computed as

$$\mathcal{D}_5((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.0429, \quad \mathcal{D}_5((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.0313, \quad \mathcal{D}_5(\mathcal{A}_3, \mathcal{B}) = 0.0396$$
$$\mathcal{D}_6((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.1087, \quad \mathcal{D}_6((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.0818, \quad \mathcal{D}_6(\mathcal{A}_3, \mathcal{B}) = 0.1137$$
$$\mathcal{D}_2^H((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.0353, \quad \mathcal{D}_2^H((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.0249, \quad \mathcal{D}_2^H(\mathcal{A}_3, \mathcal{B}) = 0.0336$$
$$\mathcal{D}_4^H((\mathcal{A}_1, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.1125, \quad \mathcal{D}_4^H((\mathcal{A}_2, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.0799, \quad \mathcal{D}_4^H(\mathcal{A}_3, \mathcal{B}) = 0.1147$$

Thus, the ranking order of the three patterns is $\mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_1$ and hence $\mathcal{A}_2$ is the most desirable pattern to be classified with $\mathcal{B}$. Furthermore, it can be easily verified that these results validate the above-proposed propositions on the distance measures.

In order to compare the performance of the proposed measures with some existing measures, an analysis has been conducted under the DHFS environment in which firstly the different parameters of the alternatives are aggregated by using the parameter weights η_j's and then some existing measures [12, 176, 203, 205, 248] have been applied on it. Their corresponding decision matrix is summarized in Table 3.3.

Table 3.3: Comparative analysis of Example 1: Pattern recognition

	When equal weightage has been taken				When weights ξ and η have been taken			
	Measure value of $\mathcal{B}$ from			Ranking	Measure value of $\mathcal{B}$ from			Ranking
Methods	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	order	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	order
Correlation coefficient [248]	0.7807	0.7798	0.7556	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3$	0.7611	0.7583	0.7383	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3$
Cosine similarity measure [12]	0.9403	0.9449	0.8662	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.9338	0.9413	0.8657	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
Correlation coefficient [203]	0.7266	0.7511	0.6878	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.7231	0.7492	0.6860	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
Distance measure [205]								
$\lambda = 1$	0.1631	0.1388	0.2080	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.1677	0.1260	0.2129	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
$\lambda = 2$	0.1687	0.1432	0.2145	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.2991	0.2476	0.3862	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
$\lambda = 5$	0.1841	0.1583	0.2323	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.4650	0.4170	0.5972	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
Distance measure [176]								
$\lambda = 1$	0.1479	0.1167	0.1585	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.1555	0.1164	0.1620	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
$\lambda = 2$	0.1651	0.1443	0.1926	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.1731	0.1492	0.1959	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
$\lambda = 5$	0.1975	0.1908	0.2656	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$	0.2090	0.2050	0.2678	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$

Table 3.4: Rating values of the diseases in terms of DHFSNs

	For diseases $\mathcal{A}_1$				
	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$(\{0.6,0.4,0.3\},\{0.3,0.1\})$	$(\{0.8,0.7,0.5\},\{0.1\})$	$(\{0.4,0.3\},\{0.2\})$	$(\{0.8,0.7,0.5\},\{0.2,0.1\})$	$(\{0.8,0.7\},\{0.2,0.1\})$
d_2	$(\{0.9,0.8,0.7\},\{0.1\})$	$(\{0.7,0.6,0.4\},\{0.3,0.2\})$	$(\{0.5,0.3,0.1\},\{0.3\})$	$(\{0.5,0.4,0.3\},\{0.5,0.3\})$	$(\{0.5,0.3,0.1\},\{0.4,0.3\})$
d_3	$(\{0.5,0.4,0.2\},\{0.5,0.3\})$	$(\{0.5,0.4,0.3\},\{0.4,0.2,0.1\})$	$(\{0.8,0.6,0.5\},\{0.2,0.1\})$	$(\{0.6,0.3,0.2\},\{0.4,0.1\})$	$(\{0.2,0.1\},\{0.8\})$
d_4	$(\{0.3,0.2\},\{0.4,0.2\})$	$(\{0.7,0.6\},\{0.3,0.2,0.1\})$	$(\{0.2,0.1\},\{0.5,0.4\})$	$(\{0.2,0.1\},\{0.7,0.5\})$	$(\{0.3,0.1\},\{0.6,0.4\})$
	For diseases $\mathcal{A}_2$				
	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$(\{0.8,0.7\},\{0.2,0.1\})$	$(\{0.5,0.4,0.3\},\{0.4,0.3\})$	$(\{0.6,0.4\},\{0.4,0.2\})$	$(\{0.3,0.2\},\{0.6,0.5\})$	$(\{0.6,0.5,0.3\},\{0.2\})$
d_2	$(\{0.4,0.3\},\{0.6,0.2\})$	$(\{0.7,0.6\},\{0.3,0.2\})$	$(\{0.7,0.6\},\{0.3,0.2\})$	$(\{0.7,0.6,0.4\},\{0.2,0.1\})$	$(\{0.7,0.3\},\{0.3,0.1\})$
d_3	$(\{0.5,0.4\},\{0.3,0.2\})$	$(\{0.6,0.4\},\{0.3\})$	$(\{0.6,0.5\},\{0.3\})$	$(\{0.6,0.5\},\{0.3,0.2\})$	$(\{0.4,0.2\},\{0.5,0.3\})$
d_4	$(\{0.8,0.7\},\{0.2,0.1\})$	$(\{0.8,0.7\},\{0.2,0.1\})$	$(\{0.7,0.6\},\{0.2\})$	$(\{0.4,0.3\},\{0.2,0.1\})$	$(\{0.2,0.1\},\{0.3\})$
	For diseases $\mathcal{A}_3$				
	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$(\{0.6,0.5\},\{0.2,0.1\})$	$(\{0.6,0.4\},\{0.3,0.2\})$	$(\{0.7,0.4,0.2\},\{0.2,0.1\})$	$(\{0.7,0.6\},\{0.3,0.2\})$	$(\{0.6,0.4,0.3\},\{0.3,0.1\})$
d_2	$(\{0.5,0.4\},\{0.2\})$	$(\{0.7,0.6,0.4\},\{0.3,0.2\})$	$(\{0.7,0.6,0.5\},\{0.3,0.2,0.1\})$	$(\{0.9,0.7,0.6\},\{0.1\})$	$(\{0.7,0.5,0.4\},\{0.3,0.2\})$
d_3	$(\{0.6,0.5\},\{0.1\})$	$(\{0.4,0.3\},\{0.6,0.5\})$	$(\{0.6,0.5,0.4\},\{0.3,0.2,0.1\})$	$(\{0.8,0.7,0.6\},\{0.2,0.1\})$	$(\{0.5,0.4\},\{0.4,0.3\})$
d_4	$(\{0.5,0.4\},\{0.2,0.1\})$	$(\{0.8,0.7\},\{0.2\})$	$(\{0.4,0.4,0.3\},\{0.6\})$	$(\{0.6,0.5\},\{0.4,0.2\})$	$(\{0.6,0.5\},\{0.2,0.1\})$

3.4.2 Example 2: Medical diagnosis

Consider a set of three diseases $\mathcal{A} = \{\mathcal{A}_1(\text{Viral fever}), \mathcal{A}_2(\text{Malaria}), \mathcal{A}_3(\text{Typhoid})\}$ and a set of five symptoms $\mathcal{E} = \{e_1$ (Temperature), e_2 (HeadAche), e_3 (Stomach Pain), e_4 (Cough), e_5 (Chest pain)$\}$. A team of four doctors d_1, d_2, d_3, d_4 are able to examine the patients in terms of diseases. The rating values of each person who is affected by the disease $\mathcal{A}_k (k = 1, 2, 3)$ under the set of symptoms are summarized in terms of DHFSS given in Table 3.4.

A medical team is conducting research in a laboratory to identify that the most infected diseases suffered by the patient $\mathcal{B}$ using the four set of symptoms $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$. The rating value of the patient $\mathcal{B}$ is given in Table 3.5. The target of this problem is to classify

Table 3.5: Rating values of patient $\mathcal{B}$ in terms of DHFSNs

	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$(\{0.9\},\{0.1\})$	$(\{0.6,0.5\},\{0.4\})$	$(\{0.4,0.3\},\{0.5\})$	$(\{0.8,0.7\},\{0.1\})$	$(\{0.4,0.3\},\{0.6\})$
d_2	$(\{0.6,0.4\},\{0.4\})$	$(\{0.8\},\{0.2\})$	$(\{0.7,0.5\},\{0.3\})$	$(\{0.3,0.2,0.1\},\{0.5\})$	$(\{0.7,0.5\},\{0.3,0.1\})$
d_3	$(\{0.7\},\{0.2\})$	$(\{0.4,0.3\},\{0.2\})$	$(\{0.8,0.7\},\{0.2\})$	$(\{0.6,0.3\},\{0.2\})$	$(\{0.6\},\{0.4\})$
d_4	$(\{0.8\},\{0.2\})$	$(\{0.9,0.6\},\{0.1\})$	$(\{0.6,0.4\},\{0.3\})$	$(\{0.5,0.4\},\{0.2,0.1\})$	$(\{0.6,0.4\},\{0.2\})$

the patient $\mathcal{B}$ with one of the diseases $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$. For it, the proposed distance measures, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_1^H$ and $\mathcal{D}_3^H$ have been computed from $\mathcal{B}$ to $\mathcal{A}_k (k = 1, 2, 3)$ and their corresponding measurement values are given as below.

$$\mathcal{D}_1((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 7.2917, \quad \mathcal{D}_1((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 5.1111, \quad \mathcal{D}_1((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 5.2917$$

$$\mathcal{D}_2((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.3646, \quad \mathcal{D}_2((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.2556, \quad \mathcal{D}_2((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.2646$$

$$\mathcal{D}_3((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 1.5754, \quad \mathcal{D}_3((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 1.1738, \quad \mathcal{D}_3((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 1.2230$$

$$\mathcal{D}_4((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.3523, \quad \mathcal{D}_4((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.2625, \quad \mathcal{D}_4((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.2735$$

$$\mathcal{D}_1^H((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.6167, \quad \mathcal{D}_1^H((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.4083, \quad \mathcal{D}_1^H((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.4833$$

$$\mathcal{D}_3^H((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.5083, \quad \mathcal{D}_3^H((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.3500, \quad \mathcal{D}_3^H((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.4143$$

Thus, from these distance measures we conclude that the patient $\mathcal{B}$ suffers from the $\mathcal{A}_2$ disease.

Furthermore, if we assign weights $\xi_i = (0.5, 0.2, 0.2, 0.1)^T$ and $\eta_j = (0.4, 0.1, 0.2, 0.2, 0.1)^T$ corresponding to experts and parameters respectively, then we utilize the distance measures $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_2^H$ and $\mathcal{D}_4^H$ for obtaining the most suitable disease who affected by the patient $\mathcal{B}$, we have

$$\mathcal{D}_5((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0546, \quad \mathcal{D}_5((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0356, \quad \mathcal{D}_5((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0377$$

$$\mathcal{D}_6((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.1448, \quad \mathcal{D}_6((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0980, \quad \mathcal{D}_6((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.1064$$

$$\mathcal{D}_2^H((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0333, \quad \mathcal{D}_2^H((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0218, \quad \mathcal{D}_2^H((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0264$$

$$\mathcal{D}_4^H((\mathcal{A}_1,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.1234, \quad \mathcal{D}_4^H((\mathcal{A}_2,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0822, \quad \mathcal{D}_4^H((\mathcal{A}_3,\mathcal{E}),(\mathcal{B},\mathcal{E})) = 0.0981$$

Thus, we compute that the patient $\mathcal{B}$ suffer from the disease $\mathcal{A}_2$ i.e. Malaria.

In order to verify the feasibility of the proposed DM approach based on a distance measure, we conducted a comparative analysis on the considered data by using some existing measures [12, 176, 203, 205, 248] under dual hesitant fuzzy set environment. The analysis corresponding to it has been summarized in Table 3.6 and concluded that the patient $\mathcal{B}$ suffers from the disease $\mathcal{A}_2$(Malaria) which coincides with the proposed approach results.

Table 3.6: Comparative analysis of Example 2: Medical diagnosis

Method	Measure values of $\mathcal{B}$ from			Ranking
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	
Correlation coefficient [248]	0.9116	0.9556	0.7952	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
Cosine similarity measure [12]	0.6686	0.4609	0.5533	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2$
Correlation coefficient [203]	0.7235	0.8081	0.7511	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Distance measure [205]				
$\lambda = 1$	0.1650	0.0660	0.0970	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
$\lambda = 2$	0.3574	0.1426	0.2145	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
$\lambda = 5$	0.6067	0.2451	0.3608	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Weighted distance measure [205]				
$\lambda = 1$	0.1795	0.1040	0.1201	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
$\lambda = 2$	0.3806	0.1944	0.2445	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
$\lambda = 5$	0.6867	0.3232	0.4180	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
Distance measure [176]				
$\lambda = 1$	0.1382	0.0845	0.1029	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
$\lambda = 2$	0.1680	0.1036	0.1262	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
$\lambda = 5$	0.2311	0.1352	0.1656	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$

3.4.3 Advantages of proposed method

According to the above comparison analysis, the proposed method for addressing the DM problems has the following advantages.

(i) DHFS theory has some limitations and is not able to model the uncertain data because of their insufficiency for parameterization tool. To overcome these, the dual hesitant fuzzy soft set is a generalization of the hesitant fuzzy set, intuitionistic fuzzy set and dual hesitant fuzzy set which contains more information related to an object than the others. In the proposed theory, information related to an object contains both membership hesitant degrees as well as nonmembership hesitant degrees which are measured with the different parameter factors than the hesitant fuzzy set (only membership hesitant degrees), the intuitionistic fuzzy set (both membership and non-membership degrees) and dual hesitant fuzzy set (both membership and non-membership hesitant degrees). Thus, the proposed distance and similarity measures are the further generalization of the distance and similarity measures of dual hesitant intuitionistic fuzzy sets [12, 176, 203, 205].

(ii) The proposed measures consider the decision makers' risk attitude λ, which provide the decision makers more choice as different values of parameters. Since Tables 3.7 and 3.8 gives the different ranking for different values of the risk parameter λ and hence it provides the different option for the decision-makers to choose the best one accordingly.

(iii) From the proposed distance and similarity measures, it has been seen that the measures defined in the hesitant fuzzy set, dual hesitant fuzzy set, intuitionistic fuzzy set are special cases of the proposed ones. Therefore, the proposed measures can easily be utilized to solve the problems under hesitant fuzzy set, dual hesitant fuzzy set, and intuitionistic fuzzy set, while the existing ones are not able to solve the problem under the environment considered in this chapter.

Table 3.7: Results obtained by measure $\mathcal{D}_7$ for different values of decision makers' parameter λ

λ	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	Ranking
1	0.0429	0.0313	0.0396	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
2	0.1087	0.0818	0.1137	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
5	0.2173	0.1711	0.2549	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
10	0.2982	0.2426	0.3648	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
15	0.3417	0.2819	0.4210	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
30	0.4081	0.3348	0.4982	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$

Table 3.8: Results obtained by measure $\mathcal{D}_5^H$ for different values of decision makers' parameter λ

λ	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	Ranking
1	0.0353	0.0249	0.0336	$\mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_1$
2	0.1125	0.0799	0.1147	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
5	0.2452	0.1775	0.2676	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
10	0.3346	0.2516	0.3746	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
15	0.3777	0.2904	0.4276	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$
30	0.4330	0.3402	0.5013	$\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_3$

3.5 Conclusion

Since the existing theories namely intuitionistic fuzzy set, hesitant fuzzy set and dual hesitant fuzzy set cannot model the data accurately due to its inadequacy for parameterization tool. In order to overcome this problem, a dual hesitant fuzzy soft set contains more information about an object than the other existing theories. In this context, a series of distance measures based on Hamming, Euclidean, and Hausdorff metric has been proposed along with their required properties. Based on these distance measures, a DM method has been proposed and demonstrated with numerical examples of pattern recognition as well as the medical diagnosis. A comparative study, as well as the effect of the parameters on the ranking of the alternative, will support the theory and hence demonstrate that the proposed measures place an alternative way for solving the DM problems.

Chapter 4

Correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making[1]

In this chapter, we develop correlation coefficient and weighted correlation coefficients under the DHFSS environment in which pairs of membership, non-membership are to be considered as vector representation during the formulation and to investigate their properties. Further, under this environment, a MCDM method based on the proposed correlation coefficients are presented. Three numerical examples, one from the selection procedure and other from the medical diagnosis and pattern recognition, are taken to demonstrate the efficiency of the proposed approach and compared their results with the several existing approaches results.

4.1 Introduction

In statistical analysis, the correlation coefficients play a vital role to measure the linear relationship between the two variables, whereas in fuzzy set theory the correlation measure determines the degree of dependency between two fuzzy sets. In that direction, Hung and Wu [84] firstly defined the correlation coefficient for fuzzy numbers. Gerstenkorn and

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Manko [68] introduced the correlation coefficients of IFSs which was later analyzed by Szmidt and Kacprzyk [187]. Garg [60] presented the correlation coefficients for the intuitionistic multiplicative sets and applied them to solve the problems of pattern recognition and DM. In these above theories, it is assumed that preferences given by the experts are either a single number or an interval. However, in such a situation, these conditions may be unfulfilled by an expert. To manage such situations, the concept of the hesitant fuzzy sets (HFSs) and dual HFS (DHFS), was introduced by Torra [194], Torra and Narukawa [195] and Zhu et al. [267] respectively. Under this environment, Chen et al. [25] discussed the correlation coefficients of HFSs and their applications to clustering analysis. Xu and Xia [232] discussed the correlation measures for HFSs. Farhadinia [48], Ye [248] presented the correlation coefficients and applied them to solve the multi-attribute DM problems.

Although, the above theories are successfully applied in various situations they have their inherent difficulties such as how to set the membership degrees in each particular case. For instance, some experts may assign the membership value as 0.2, some other assign 0.5, while rest assigns 0.7. Thus, under situations, the above-mentioned techniques are unable to compute the desirable value. To manage such situations where decision makers are hesitant in expressing their preferences over alternatives, Babitha and John [11] have introduced hesitant fuzzy soft set (HFSS) which is the hybridization of HFS and SS. For the above situation, the satisfactory membership degrees are represented by a hesitant fuzzy element $\{0.2, 0.5, 0.7\}$, which is different from a fuzzy number 0.2 (or 0.7), the interval-valued fuzzy number [0.2, 0.7] and the intuitionistic fuzzy number (0.2, 0.5). Das, Malakar, Kar and Pal [36] presented the correlation coefficients for HFSS. Later on, the concept of the dual HFSS (DHFSS)[154, 257] has explored by embedding the features of the DHFS with SS.

Presently, the correlation measure is one of the most important information measures and has drawn more attention by a number of researchers under the uncertain environment. Also, DHFSSs have the great powerful ability to model the imprecise and ambiguous information in the real-world application than the other existing theories such as IFSs, HFSs, DHFSs, and HFSSs. For example, if on a certain object, three experts discuss the membership and non-membership degrees on a certain criterion and give that preferences in terms of acceptance is 0.5, 0.3, 0.4 respectively while the degree of rejection is 0.4, 0.5, 0.2. For such a case, the degrees are represented by a dual hesitant fuzzy element $(\{0.5\},\{0.4\})$, $(\{0.3\},\{0.5\})$ and $(\{0.4\},\{0.2\})$ which is entirely different from the fuzzy numbers 0.5 (or 0.3), the interval-valued intuitionistic fuzzy numbers $([0.3,0.4],[0.4,0.5])$. Hence, the dual hesitant environment reflects all the possible opinions and appears to be a more flexible method to be valued in multifold ways according to the practical demands than the existing sets.

Therefore, keeping the advantages of this set and taking the importance of the correlation measure, this chapter derived the degree of cohesiveness among the DHFSS. In order to achieve it, we first define some operational laws and their corresponding informational energies, as well as the covariance between the two DHFSSs. Then, based on these, we introduce correlation coefficient for DHFSSs and obtain some properties. Also, in order to deal with the situation where the elements in a set are correlative, the weighted correlation coefficient are defined. Furthermore, we propose a DM algorithm based on the DHFSSs and the correlation coefficients. The feasibility, as well as the superiority of the proposed approach, has been illustrated with a numerical example.

4.2 Correlation coefficient of Dual Hesitant Fuzzy Soft Sets

In this section, we propose some correlation coefficients under the DHFSS environment which can be applied to numerous engineering and scientific fields.

Let $\mathcal{X}=\{u_1,u_2,\ldots,u_n\}$ be a discrete universe of discourse, and $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ be two nonempty DHFSSs defined on the $\mathcal{X}$ by $(\mathcal{F},\mathcal{E})=\{u_i,\zeta_{\mathcal{F}(e_j)}(u_i^{(s)}),\ \vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\mid u_i\in\mathcal{X}\}$ and $(\mathcal{G},\mathcal{E})=\{u_i,\zeta_{\mathcal{G}(e_j)}(u_i^{(s)}),\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\mid u_i\in\mathcal{X}\}$ respectively, then the informational intuitionistic energies of two DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ are defined as

$$E_{\text{DHFSS}}(\mathcal{F},\mathcal{E}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right) \tag{4.1}$$

$$E_{\text{DHFSS}}(\mathcal{G},\mathcal{E}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right) \tag{4.2}$$

where k_i and l_i represent the number of values in $\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})$and $\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})$ respectively.

The correlation of the DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ is defined as

$$\begin{aligned} & C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \\ = & \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right) \end{aligned} \tag{4.3}$$

From Eq.(4.3), it is obvious that the following properties holds.

(P1) $C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})) = E_{\text{DHFSS}}(\mathcal{F},\mathcal{E})$

(P2) $C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = C_{\text{DHFSS}}((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))$

Definition 4.2.1. Let $\mathcal{E}$ be a set of parameters defined on the object of the universe $\mathcal{X}=\{u_1,u_2,\ldots,u_n\}$. If $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ be two DHFSSs, then the correlation coefficient between them is denoted by $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ and defined as

$$\begin{aligned} \mathcal{K}_1\left((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\right) &= \frac{C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{\sqrt{E_{\text{DHFSS}}(\mathcal{F},\mathcal{E})\cdot E_{\text{DHFSS}}(\mathcal{G},\mathcal{E})}} \\ &= \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}{\left\{\begin{array}{c}\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right)}\times \\ \times\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}\end{array}\right\}} \end{aligned} \tag{4.4}$$

Theorem 4.2.1. For two DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$, the measure given in Definition (4.2.1) satisfy the following properties:

(P1) $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = \mathcal{K}_1((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))$.

(P2) $0 \le \mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \le 1$.

(P3) $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1$ if $(\mathcal{F},\mathcal{E}) = (\mathcal{G},\mathcal{E})$.

Proof. Let $(\mathcal{F},\mathcal{E}) = \{u_i, \zeta_{\mathcal{F}(e_j)}(u_i^{(s)}),\ \vartheta_{\mathcal{F}(e_j)}(u_i^{(t)}) \mid u_i \in \mathcal{X}\}$ and $(\mathcal{G},\mathcal{E}) = \{u_i, \zeta_{\mathcal{G}(e_j)}(u_i^{(s)}),$ $\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)}) \mid u_i \in \mathcal{X}\}$ be two DHFSSs. Then, we have

(P1) It is straightforward.

(P2) The inequality $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \geq 0$ is obvious. Let us prove $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \leq 1$.

$$
\begin{aligned}
& C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\begin{array}{l} \dfrac{1}{k_i}\displaystyle\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)}) \\ +\dfrac{1}{l_i}\displaystyle\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)}) \end{array}\right) \\
= \quad & \sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_1)}(u_i^{(s)})\zeta_{\mathcal{G}(e_1)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_1)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_1)}(u_i^{(t)})\right) \\
+ \quad & \sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_2)}(u_i^{(s)})\zeta_{\mathcal{G}(e_2)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_2)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_2)}(u_i^{(t)})\right) \\
\vdots \quad & \\
+ \quad & \sum_{i=1}^{n}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_m)}(u_i^{(s)})\zeta_{\mathcal{G}(e_m)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_m)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_m)}(u_i^{(t)})\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left\{\begin{array}{l}
\left(\dfrac{1}{k_1}\displaystyle\sum_{s=1}^{k_1}\zeta_{\mathcal{F}(e_1)}(u_1^{(s)})\zeta_{\mathcal{G}(e_1)}(u_1^{(s)})+\dfrac{1}{l_1}\displaystyle\sum_{t=1}^{l_1}\vartheta_{\mathcal{F}(e_1)}(u_1^{(t)})\vartheta_{\mathcal{G}(e_1)}(u_1^{(t)})\right) \\
+\left(\dfrac{1}{k_2}\displaystyle\sum_{s=1}^{k_2}\zeta_{\mathcal{F}(e_1)}(u_2^{(s)})\zeta_{\mathcal{G}(e_1)}(u_2^{(s)})+\dfrac{1}{l_2}\displaystyle\sum_{t=1}^{l_2}\vartheta_{\mathcal{F}(e_1)}(u_2^{(t)})\vartheta_{\mathcal{G}(e_1)}(u_2^{(t)})\right) \\
+\ldots+\left(\dfrac{1}{k_n}\displaystyle\sum_{s=1}^{k_n}\zeta_{\mathcal{F}(e_1)}(u_n^{(s)})\zeta_{\mathcal{G}(e_1)}(u_n^{(s)})+\dfrac{1}{l_n}\displaystyle\sum_{t=1}^{l_n}\vartheta_{\mathcal{F}(e_1)}(u_n^{(t)})\vartheta_{\mathcal{G}(e_1)}(u_n^{(t)})\right)
\end{array}\right\} \\
+ \quad & \left\{\begin{array}{l}
\left(\dfrac{1}{k_1}\displaystyle\sum_{s=1}^{k_1}\zeta_{\mathcal{F}(e_2)}(u_1^{(s)})\zeta_{\mathcal{G}(e_2)}(u_1^{(s)})+\dfrac{1}{l_1}\displaystyle\sum_{t=1}^{l_1}\vartheta_{\mathcal{F}(e_2)}(u_1^{(t)})\vartheta_{\mathcal{G}(e_2)}(u_1^{(t)})\right) \\
+\left(\dfrac{1}{k_2}\displaystyle\sum_{s=1}^{k_2}\zeta_{\mathcal{F}(e_2)}(u_2^{(s)})\zeta_{\mathcal{G}(e_2)}(u_2^{(s)})+\dfrac{1}{l_2}\displaystyle\sum_{t=1}^{l_2}\vartheta_{\mathcal{F}(e_2)}(u_2^{(t)})\vartheta_{\mathcal{G}(e_2)}(u_2^{(t)})\right) \\
+\ldots+\left(\dfrac{1}{k_n}\displaystyle\sum_{s=1}^{k_n}\zeta_{\mathcal{F}(e_2)}(u_n^{(s)})\zeta_{\mathcal{G}(e_2)}(u_n^{(s)})+\dfrac{1}{l_n}\displaystyle\sum_{t=1}^{l_n}\vartheta_{\mathcal{F}(e_2)}(u_n^{(t)})\vartheta_{\mathcal{G}(e_2)}(u_n^{(t)})\right)
\end{array}\right\} \\
+ \quad & \vdots \\
& \vdots
\end{aligned}
$$

$$+\left\{\begin{array}{l}\left(\dfrac{1}{k_1}\displaystyle\sum_{s=1}^{k_1}\zeta_{\mathcal{F}(e_m)}(u_1^{(s)})\zeta_{\mathcal{G}(e_m)}(u_1^{(s)})+\dfrac{1}{l_1}\displaystyle\sum_{t=1}^{l_1}\vartheta_{\mathcal{F}(e_m)}(u_1^{(t)})\vartheta_{\mathcal{G}(e_m)}(u_1^{(t)})\right)\\ +\left(\dfrac{1}{k_2}\displaystyle\sum_{s=1}^{k_2}\zeta_{\mathcal{F}(e_m)}(u_2^{(s)})\zeta_{\mathcal{G}(e_m)}(u_2^{(s)})+\dfrac{1}{l_2}\displaystyle\sum_{t=1}^{l_2}\vartheta_{\mathcal{F}(e_m)}(u_2^{(t)})\vartheta_{\mathcal{G}(e_m)}(u_2^{(t)})\right)\\ +\ldots+\left(\dfrac{1}{k_n}\displaystyle\sum_{s=1}^{k_n}\zeta_{\mathcal{F}(e_m)}(u_n^{(s)})\zeta_{\mathcal{G}(e_m)}(u_n^{(s)})+\dfrac{1}{l_n}\displaystyle\sum_{t=1}^{l_n}\vartheta_{\mathcal{F}(e_m)}(u_n^{(t)})\vartheta_{\mathcal{G}(e_m)}(u_n^{(t)})\right)\end{array}\right\}$$

$$\begin{aligned}=\;&\sum_{j=1}^{m}\sum_{s=1}^{k_1}\frac{\zeta_{\mathcal{F}(e_j)}(u_1^{(s)})}{\sqrt{k_1}}\frac{\zeta_{\mathcal{G}(e_j)}(u_1^{(s)})}{\sqrt{k_1}}+\sum_{j=1}^{m}\sum_{s=1}^{k_2}\frac{\zeta_{\mathcal{F}(e_j)}(u_2^{(s)})}{\sqrt{k_2}}\frac{\zeta_{\mathcal{G}(e_j)}(u_2^{(s)})}{\sqrt{k_2}}\\ &+\ldots\ldots+\sum_{j=1}^{m}\sum_{s=1}^{k_n}\frac{\zeta_{\mathcal{F}(e_j)}(u_n^{(s)})}{\sqrt{k_n}}\frac{\zeta_{\mathcal{G}(e_j)}(u_n^{(s)})}{\sqrt{k_n}}+\sum_{j=1}^{m}\sum_{t=1}^{l_1}\frac{\vartheta_{\mathcal{F}(e_j)}(u_1^{(t)})}{\sqrt{l_1}}\frac{\vartheta_{\mathcal{G}(e_j)}(u_1^{(t)})}{\sqrt{l_1}}\\ &+\sum_{j=1}^{m}\sum_{t=1}^{l_2}\frac{\vartheta_{\mathcal{F}(e_j)}(u_2^{(t)})}{\sqrt{l_2}}\frac{\zeta_{\mathcal{G}(e_j)}(u_2^{(t)})}{\sqrt{l_2}}+\ldots\ldots+\sum_{j=1}^{m}\sum_{t=1}^{l_n}\frac{\vartheta_{\mathcal{F}(e_j)}(u_n^{(t)})}{\sqrt{l_n}}\frac{\zeta_{\mathcal{G}(e_j)}(u_n^{(t)})}{\sqrt{l_n}}\end{aligned}$$

According to Cauchy-Schwarz inequality:

$$(u_1y_1+u_2y_2+\ldots+u_ny_n)^2\leq(u_1^2+u_2^2+\cdots+u_n^2)(y_1^2+y_2^2+\cdots+y_n^2)$$

where $(u_1,u_2,\ldots,u_n)\in\mathbb{R}^n$ and $(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$, we can obtain:

$$\begin{aligned}&(C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})))^2\\ \leq\;&\left\{\begin{array}{l}\displaystyle\sum_{j=1}^{m}\frac{1}{k_1}\sum_{s=1}^{k_1}\zeta^2_{\mathcal{F}(e_j)}(u_1^{(s)})+\ldots+\sum_{j=1}^{m}\frac{1}{k_n}\sum_{s=1}^{k_n}\zeta^2_{\mathcal{F}(e_j)}(u_n^{(s)})\\ \displaystyle+\sum_{j=1}^{m}\frac{1}{l_1}\sum_{t=1}^{l_1}\vartheta^2_{\mathcal{F}(e_j)}(u_1^{(t)})+\ldots+\sum_{j=1}^{m}\frac{1}{l_n}\sum_{t=1}^{l_n}\vartheta^2_{\mathcal{F}(e_j)}(u_n^{(t)})\end{array}\right\}\\ \times\;&\left\{\begin{array}{l}\displaystyle\sum_{j=1}^{m}\frac{1}{k_1}\sum_{s=1}^{k_1}\zeta^2_{\mathcal{G}(e_j)}(u_1^{(s)})+\ldots+\sum_{j=1}^{m}\frac{1}{k_n}\sum_{s=1}^{k_n}\zeta^2_{\mathcal{G}(e_j)}(u_n^{(s)})\\ \displaystyle+\sum_{j=1}^{m}\frac{1}{l_1}\sum_{t=1}^{l_1}\vartheta^2_{\mathcal{G}(e_j)}(u_1^{(t)})+\ldots+\sum_{j=1}^{m}\frac{1}{l_n}\sum_{t=1}^{l_n}\vartheta^2_{\mathcal{G}(e_j)}(u_n^{(t)})\end{array}\right\}\\ =\;&\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right)\right\}\\ \times\;&\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)\right\}\\ =\;&E_{\text{DHFSS}}(\mathcal{F},\mathcal{E})\cdot E_{\text{DHFSS}}(\mathcal{G},\mathcal{E})\end{aligned}$$

Therefore, $C_{\text{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \leq \sqrt{E_{\text{DHFSS}}(\mathcal{F},\mathcal{E}) \cdot E_{\text{DHFSS}}(\mathcal{G},\mathcal{E})}$. Hence, from Eq. (4.4), it follows that $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \leq 1$.

(P3) If $(\mathcal{F},\mathcal{E}) = (\mathcal{G},\mathcal{E})$ this implies $\zeta_{\mathcal{F}(e_j)}(u_i^{(s)}) = \zeta_{\mathcal{G}(e_j)}(u_i^{(s)})$ and $\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)}) = \vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})$, $u_i \in \mathcal{X}$, where $s = (1,2,\ldots,k_i)$, $t = (1,2,\ldots,l_i)$ and $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$ and thus from Eq. (4.4), it follows that $\mathcal{K}_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1$.

Hence, the theorem holds. □

Example 4.2.1. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be set of parameters and $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ be two DHFSSs in $\mathcal{X} = \{u_1, u_2, u_3, u_4\}$ defined as

$$(\mathcal{F},\mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\begin{array}{c} \begin{array}{ccc} \quad e_1 \quad & \quad e_2 \quad & \quad e_3 \quad \end{array} \\ \begin{bmatrix} \{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\} & \{\{0.6,0.5,0.4\},\{0.4,0.3,0.2\}\} & \{\{0.3,0.2,0.1\},\{0.7,0.6,0.6\}\} \\ \{\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}\} & \{\{0.5,0.4,0.3\},\{0.5,0.3,0.3\}\} & \{\{0.7,0.7,0.5\},\{0.3,0.2,0.2\}\} \\ \{\{0.8,0.7,0.7\},\{0.2,0.1,0.1\}\} & \{\{0.8,0.8,0.7\},\{0.2,0.2,0.1\}\} & \{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\} \\ \{\{0.4,0.4,0.3\},\{0.6,0.5,0.4\}\} & \{\{0.6,0.6,0.5\},\{0.4,0.3,0.2\}\} & \{\{0.8,0.7,0.6\},\{0.2,0.1,0.1\}\} \end{bmatrix} \end{array}$$

and

$$(\mathcal{G},\mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\begin{array}{c} \begin{array}{ccc} \quad e_1 \quad & \quad e_2 \quad & \quad e_3 \quad \end{array} \\ \begin{bmatrix} \{\{0.7,0.6,0.4\},\{0.3,0.2\}\} & \{\{0.6,0.4,0.3\},\{0.3,0.1\}\} & \{\{0.6,0.4,0.3\},\{0.4,0.1\}\} \\ \{\{0.9,0.7,0.6\},\{0.1\}\} & \{\{0.7,0.5,0.4\},\{0.3,0.2\}\} & \{\{0.4,0.3,0.3\},\{0.6,0.5\}\} \\ \{\{0.8,0.7,0.6\},\{0.2,0.1\}\} & \{\{0.5,0.4\},\{0.2,0.1\}\} & \{\{0.9,0.8,0.6\},\{0.1\}\} \\ \{\{0.6,0.5,0.5\},\{0.4,0.2\}\} & \{\{0.8,0.7,0.5\},\{0.2,0.1\}\} & \{\{0.6,0.4,0.3\},\{0.3,0.1\}\} \end{bmatrix} \end{array}$$

By applying Eq. (4.1), the informational energy of $(\mathcal{F},\mathcal{E})$:

$$\begin{aligned} E_{\text{DHFSS}}(\mathcal{F},\mathcal{E}) &= \sum_{i=1}^{4}\sum_{j=1}^{3}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)}) + \frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right) \\ &= \frac{1}{3}(0.7^2+0.6^2+0.5^2) + \frac{1}{3}(0.3^2+0.2^2+0.1^2) + \frac{1}{3}(0.6^2+0.5^2+0.4^2) \\ &+ \frac{1}{3}(0.4^2+0.3^2+0.2^2) + \frac{1}{3}(0.3^2+0.2^2+0.1^2) + \frac{1}{3}(0.7^2+0.6^2+0.6^2) \\ &+ \frac{1}{3}(0.6^2+0.5^2+0.4^2) + \frac{1}{3}(0.3^2+0.2^2+0.1^2) + \frac{1}{3}(0.5^2+0.4^2+0.3^2) \\ &+ \frac{1}{3}(0.5^2+0.3^2+0.3^2) + \frac{1}{3}(0.7^2+0.7^2+0.5^2) + \frac{1}{3}(0.3^2+0.2^2+0.2^2) \\ &+ \frac{1}{3}(0.8^2+0.7^2+0.7^2) + \frac{1}{3}(0.2^2+0.1^2+0.1^2) + \frac{1}{3}(0.8^2+0.8^2+0.7^2) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{3}(0.2^2+0.2^2+0.1^2)+\frac{1}{3}(0.7^2+0.6^2+0.5^2)+\frac{1}{3}(0.3^2+0.2^2+0.1^2)\\
&+ \quad \frac{1}{3}(0.4^2+0.4^2+0.3^2)+\frac{1}{3}(0.6^2+0.5^2+0.4^2)+\frac{1}{3}(0.6^2+0.6^2+0.5^2)\\
&+ \quad \frac{1}{3}(0.4^2+0.3^2+0.2^2)+\frac{1}{3}(0.8^2+0.7^2+0.6^2)+\frac{1}{3}(0.2^2+0.1^2+0.1^2)\\
&= \quad 5.2200
\end{aligned}
$$

Similarly, the informational energy of $(\mathcal{G},\mathcal{E})$ is

$$
\begin{aligned}
& E_{\mathrm{DHFSS}}(\mathcal{G},\mathcal{E})\\
= \quad & \sum_{i=1}^{4}\sum_{j=1}^{3}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)\\
= \quad & \frac{1}{3}(0.7^2+0.6^2+0.4^2)+\frac{1}{2}(0.3^2+0.2^2)+\frac{1}{3}(0.6^2+0.4^2+0.3^2)+\frac{1}{2}(0.3^2+0.1^2)+\ldots\\
& \ldots+\frac{1}{3}(0.8^2+0.7^2+0.5^2)+\frac{1}{2}(0.2^2+0.1^2)+\frac{1}{3}(0.6^2+0.4^2+0.3^2)+\frac{1}{2}(0.3^2+0.1^2)\\
= \quad & 4.7800
\end{aligned}
$$

Thus, by Eq. (4.3), we get:

$$
\begin{aligned}
& C_{\mathrm{DHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\
= \quad & \sum_{i=1}^{4}\sum_{j=1}^{3}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right)\\
= \quad & \frac{1}{3}(0.7\times 0.7+0.6\times 0.6+0.5\times 0.4)+\frac{1}{3}(0.3\times 0.3+0.2\times 0.2+0.1\times 0.2)\\
+ \quad & \frac{1}{3}(0.6\times 0.6+0.5\times 0.4+0.4\times 0.3)+\frac{1}{3}(0.4\times 0.3+0.3\times 0.1+0.2\times 0.1)\\
+ \quad & \frac{1}{3}(0.3\times 0.6+0.2\times 0.4+0.1\times 0.3)+\frac{1}{3}(0.7\times 0.4+0.6\times 0.1+0.6\times 0.1)\\
+ \quad & \frac{1}{3}(0.6\times 0.9+0.5\times 0.7+0.4\times 0.6)+\frac{1}{3}(0.3\times 0.1+0.2\times 0.1+0.1\times 0.1)\\
+ \quad & \frac{1}{3}(0.5\times 0.7+0.4\times 0.5+0.3\times 0.4)+\frac{1}{3}(0.5\times 0.3+0.3\times 0.2+0.3\times 0.2)\\
+ \quad & \frac{1}{3}(0.7\times 0.4+0.7\times 0.3+0.5\times 0.3)+\frac{1}{3}(0.3\times 0.6+0.2\times 0.5+0.2\times 0.5)\\
+ \quad & \frac{1}{3}(0.8\times 0.8+0.7\times 0.7+0.7\times 0.6)+\frac{1}{3}(0.2\times 0.2+0.1\times 0.1+0.1\times 0.1)
\end{aligned}
$$

$$
\begin{aligned}
+ & \frac{1}{3}(0.8 \times 0.5 + 0.8 \times 0.4 + 0.7 \times 0.4) + \frac{1}{3}(0.2 \times 0.2 + 0.2 \times 0.1 + 0.1 \times 0.1) \\
+ & \frac{1}{3}(0.7 \times 0.9 + 0.6 \times 0.8 + 0.5 \times 0.6) + \frac{1}{3}(0.3 \times 0.1 + 0.2 \times 0.1 + 0.1 \times 0.1) \\
+ & \frac{1}{3}(0.4 \times 0.6 + 0.4 \times 0.5 + 0.3 \times 0.5) + \frac{1}{3}(0.6 \times 0.4 + 0.5 \times 0.2 + 0.4 \times 0.2) \\
+ & \frac{1}{3}(0.6 \times 0.8 + 0.6 \times 0.7 + 0.5 \times 0.5) + \frac{1}{3}(0.4 \times 0.2 + 0.3 \times 0.1 + 0.2 \times 0.1) \\
+ & \frac{1}{3}(0.8 \times 0.6 + 0.7 \times 0.4 + 0.6 \times 0.3) + \frac{1}{3}(0.2 \times 0.3 + 0.1 \times 0.1 + 0.1 \times 0.1) \\
= & \ 4.4500
\end{aligned}
$$

Hence, the measure $\mathcal{K}_1$ between $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ is calculated as:

$$
\begin{aligned}
\mathcal{K}_1((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) &= \frac{C_{\text{DHFSS}}((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E}))}{\sqrt{E_{\text{DHFSS}}(\mathcal{F}, \mathcal{E}) \cdot E_{\text{DHFSS}}(\mathcal{G}, \mathcal{E})}} \\
&= \frac{4.4500}{\sqrt{5.2200}\sqrt{4.7800}} = 0.8909
\end{aligned}
$$

Definition 4.2.2. Let $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ be two DHFSSs, then the correlation coefficient is defined as

$$
\begin{aligned}
\mathcal{K}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) &= \frac{C_{\text{DHFSS}}((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E}))}{\max\left\{E_{\text{DHFSS}}(\mathcal{F}, \mathcal{E}), E_{\text{DHFSS}}(\mathcal{G}, \mathcal{E})\right\}} \qquad (4.5) \\
&= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)}) + \frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}{\max\left\{\begin{array}{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)}) + \frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right), \\ \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{1}{k_i}\sum_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)}) + \frac{1}{l_i}\sum_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)\end{array}\right\}}
\end{aligned}
$$

Theorem 4.2.2. The correlation coefficient of two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ defined as $\mathcal{K}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E}))$ in Eq. (4.5) satisfies the following properties:

(P1) $\mathcal{K}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) = \mathcal{K}_2((\mathcal{G}, \mathcal{E}), (\mathcal{F}, \mathcal{E}))$.

(P2) $0 \leq \mathcal{K}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) \leq 1$.

(P3) $\mathcal{K}_2((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) = 1$ if $(\mathcal{F}, \mathcal{E}) = (\mathcal{G}, \mathcal{E})$.

Proof. Since $(\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\in$ DHFSS(U), then

$$0\leq \zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\leq 1, 0\leq \vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\leq 1, \quad \text{for all } u_i\in\mathcal{X}$$
$$0\leq \zeta_{\mathcal{G}(e_j)}(u_i^{(s)})\leq 1, 0\leq \vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\leq 1, \quad \text{for all } u_i\in\mathcal{X}$$

and thus, by Eq. (4.5), we get $\mathcal{K}_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\geq 0$. The inequality $\mathcal{K}_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\leq 1$ can be proven directly by using the well-known Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\left(\sum_{i=1}^{n} a_i^2\right)\cdot\left(\sum_{i=1}^{n} b_i^2\right)} \tag{4.6}$$

with equality if and only if the two vectors $a=(a_1,a_2,\ldots,a_n)$ and $b=(b_1,b_2,\ldots,b_n)$ are linearly dependent.

In fact, by Eq. (4.6), we have

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\left(\sum_{i=1}^{n} a_i^2\right)\cdot\left(\sum_{i=1}^{n} b_i^2\right)} \leq \sqrt{\left(\max\left\{\sum_{i=1}^{n} a_i^2,\sum_{i=1}^{n} b_i^2\right\}\right)^2} = \max\left\{\sum_{i=1}^{n} a_i^2,\sum_{i=1}^{n} b_i^2\right\}$$

and from Eq. (4.5), it follows that $\mathcal{K}_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\leq 1$ which completes the proof of (P2). In addition, by Eq. (4.5), we have

$$\begin{aligned}
\mathcal{K}_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) &= \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}{\max\left\{\begin{array}{c}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right),\\ \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)\end{array}\right\}}\\
&= \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\right)}{\max\left\{\begin{array}{c}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right),\\ \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right)\end{array}\right\}}\\
&= \mathcal{K}_2((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))
\end{aligned}$$

Thus, (P1) also holds. Similarly, we can complete the proof of (P3).

Hence, the theorem holds. □

From the Definition 4.2.1 and Definition 4.2.2, we observe that the correlation coefficient defined by Eq. (4.4) uses the geometric mean of the informational energies of the DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$, and the correlation coefficient defined by Eq. (4.5) applies the maximum between them. For the optimistic decision makers, they tend to use the correlation coefficient defined by Eq. (4.4). Contrary to the optimism decision makers, the pessimistic decision makers tend to apply the correlation coefficient defined by Eq. (4.5).

However, in order to take the importance of the experts as well as parameters into the analysis, we incorporated their weight vectors into the defined correlation coefficients. For it, consider $\xi_i > 0$ be the weight vector of the elements u_i with $\sum\limits_{i=1}^{n}\xi_i = 1$ and $\eta_j > 0$ be the weight vector of the parameters e_j such that $\sum\limits_{j=1}^{m}\eta_j = 1$, then we have the extended the above formulated correlation coefficient $\mathcal{K}_1$ and $\mathcal{K}_2$ to weighted correlation coefficient as follows:

$$\begin{aligned}
\mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) &= \frac{C_{\text{WDHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{\sqrt{E_{\text{WDHFSS}}(\mathcal{F},\mathcal{E})}\sqrt{E_{\text{WDHFSS}}(\mathcal{G},\mathcal{E})}} \qquad (4.7)\\
&= \frac{\sum\limits_{i=1}^{n}\xi_i\sum\limits_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}{\left(\begin{array}{c}\sqrt{\sum\limits_{i=1}^{n}\xi_i\sum\limits_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right)}\\ \times\sqrt{\sum\limits_{i=1}^{n}\xi_i\sum\limits_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}\end{array}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_4((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) &= \frac{C_{\text{WDHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{\max\left\{E_{\text{WDHFSS}}(\mathcal{F},\mathcal{E}),E_{\text{WDHFSS}}(\mathcal{G},\mathcal{E})\right\}} \qquad (4.8)\\
&= \frac{\sum\limits_{i=1}^{n}\xi_i\sum\limits_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta_{\mathcal{F}(e_j)}(u_i^{(s)})\zeta_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta_{\mathcal{F}(e_j)}(u_i^{(t)})\vartheta_{\mathcal{G}(e_j)}(u_i^{(t)})\right)}{\max\left\{\begin{array}{c}\sum\limits_{i=1}^{n}\xi_i\sum\limits_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)})\right),\\ \sum\limits_{i=1}^{n}\xi_i\sum\limits_{j=1}^{m}\eta_j\left(\frac{1}{k_i}\sum\limits_{s=1}^{k_i}\zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)})+\frac{1}{l_i}\sum\limits_{t=1}^{l_i}\vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)})\right)\end{array}\right\}}
\end{aligned}$$

It can be easily verify that, if $\xi = (1/n, 1/n, \ldots, 1/n)^T$ and $\eta = (1/m, 1/m, \ldots, 1/m)^T$, then equation (4.7) and equation (4.8) reduces to the correlation coefficient (4.4) and (4.5) respectively. It is easy to check that the weighted correlation coefficient $\mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ between DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ also satisfies the property of $0 \leq \mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \leq 1$.

Theorem 4.2.3. The weighted correlation coefficient between the DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ defined by Eq. (4.7), satisfies:

(P1) $\mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = \mathcal{K}_3((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))$.

(P2) $0 \leq \mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \leq 1$.

(P3) $\mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1$ if $(\mathcal{F},\mathcal{E}) = (\mathcal{G},\mathcal{E})$.

Proof. For any two DHFSSs $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$, the proofs of the properties (P1) and (P3) are straight forward so we omit here. Also $\mathcal{K}_3((\mathcal{F},\mathcal{E}),\, (\mathcal{G},\mathcal{E})) \geq 0$ is evident so we need to show only $\mathcal{K}_3((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \leq 1$. Since,

$$
\begin{aligned}
& C_{\text{WDHFSS}}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) \\
= & \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \zeta_{\mathcal{F}(e_j)}(u_i^{(s)}) \zeta_{\mathcal{G}(e_j)}(u_i^{(s)}) + \frac{1}{l_i} \sum_{t=1}^{l_i} \vartheta_{\mathcal{F}(e_j)}(u_i^{(t)}) \vartheta_{\mathcal{G}(e_j)}(u_i^{(t)}) \right) \\
= & \sum_{i=1}^{n} \xi_i \Bigg\{ \eta_1 \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \zeta_{\mathcal{F}(e_1)}(u_i^{(s)}) \zeta_{\mathcal{G}(e_1)}(u_i^{(s)}) + \frac{1}{l_i} \sum_{t=1}^{l_i} \vartheta_{\mathcal{F}(e_1)}(u_i^{(t)}) \vartheta_{\mathcal{G}(e_1)}(u_i^{(t)}) \right) \\
& \quad + \eta_2 \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \zeta_{\mathcal{F}(e_2)}(u_i^{(s)}) \zeta_{\mathcal{G}(e_2)}(u_i^{(s)}) + \frac{1}{l_i} \sum_{t=1}^{l_i} \vartheta_{\mathcal{F}(e_2)}(u_i^{(t)}) \vartheta_{\mathcal{G}(e_2)}(u_i^{(t)}) \right) \\
& \quad + \ldots\ldots + \eta_m \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \zeta_{\mathcal{F}(e_m)}(u_i^{(s)}) \zeta_{\mathcal{G}(e_m)}(u_i^{(s)}) + \frac{1}{l_i} \sum_{t=1}^{l_i} \vartheta_{\mathcal{F}(e_m)}(u_i^{(t)}) \vartheta_{\mathcal{G}(e_m)}(u_i^{(t)}) \right) \Bigg\}
\end{aligned}
$$

$$
= \xi_1 \left\{ \begin{array}{l}
\eta_1 \left(\frac{1}{k_1} \sum_{s=1}^{k_1} \zeta_{\mathcal{F}(e_1)}(u_1^{(s)}) \zeta_{\mathcal{G}(e_1)}(u_1^{(s)}) + \frac{1}{l_1} \sum_{t=1}^{l_1} \vartheta_{\mathcal{F}(e_1)}(u_1^{(t)}) \vartheta_{\mathcal{G}(e_1)}(u_1^{(t)}) \right) \\
+ \eta_2 \left(\frac{1}{k_1} \sum_{s=1}^{k_1} \zeta_{\mathcal{F}(e_2)}(u_1^{(s)}) \zeta_{\mathcal{G}(e_2)}(u_1^{(s)}) + \frac{1}{l_1} \sum_{t=1}^{l_1} \vartheta_{\mathcal{F}(e_2)}(u_1^{(t)}) \vartheta_{\mathcal{G}(e_2)}(u_1^{(t)}) \right) \\
+ \ldots + \eta_m \left(\frac{1}{k_1} \sum_{s=1}^{k_1} \zeta_{\mathcal{F}(e_m)}(u_1^{(s)}) \zeta_{\mathcal{G}(e_m)}(u_1^{(s)}) + \frac{1}{l_1} \sum_{t=1}^{l_1} \vartheta_{\mathcal{F}(e_m)}(u_1^{(t)}) \vartheta_{\mathcal{G}(e_m)}(u_1^{(t)}) \right)
\end{array} \right\}
$$

$$+ \quad \xi_2 \left\{ \begin{array}{l} \eta_1 \left(\frac{1}{k_2} \sum_{s=1}^{k_2} \zeta_{\mathcal{F}(e_1)}(u_2^{(s)}) \zeta_{\mathcal{G}(e_1)}(u_2^{(s)}) + \frac{1}{l_2} \sum_{t=1}^{l_2} \vartheta_{\mathcal{F}(e_1)}(u_2^{(t)}) \vartheta_{\mathcal{G}(e_1)}(u_2^{(t)}) \right) \\ + \eta_2 \left(\frac{1}{k_2} \sum_{s=1}^{k_2} \zeta_{\mathcal{F}(e_2)}(u_2^{(s)}) \zeta_{\mathcal{G}(e_2)}(u_2^{(s)}) + \frac{1}{l_2} \sum_{t=1}^{l_2} \vartheta_{\mathcal{F}(e_2)}(u_2^{(t)}) \vartheta_{\mathcal{G}(e_2)}(u_2^{(t)}) \right) \\ + \ldots + \eta_m \left(\frac{1}{k_2} \sum_{s=1}^{k_2} \zeta_{\mathcal{F}(e_m)}(u_2^{(s)}) \zeta_{\mathcal{G}(e_m)}(u_2^{(s)}) + \frac{1}{l_2} \sum_{t=1}^{l_2} \vartheta_{\mathcal{F}(e_m)}(u_2^{(t)}) \vartheta_{\mathcal{G}(e_m)}(u_2^{(t)}) \right) \end{array} \right\}$$

$$\vdots$$

$$+ \quad \xi_n \left\{ \begin{array}{l} \eta_1 \left(\frac{1}{k_n} \sum_{s=1}^{k_n} \zeta_{\mathcal{F}(e_1)}(u_n^{(s)}) \zeta_{\mathcal{G}(e_1)}(u_n^{(s)}) + \frac{1}{l_n} \sum_{t=1}^{l_n} \vartheta_{\mathcal{F}(e_1)}(u_n^{(t)}) \vartheta_{\mathcal{G}(e_1)}(u_n^{(t)}) \right) \\ + \eta_2 \left(\frac{1}{k_n} \sum_{s=1}^{k_n} \zeta_{\mathcal{F}(e_2)}(u_n^{(s)}) \zeta_{\mathcal{G}(e_2)}(u_n^{(s)}) + \frac{1}{l_n} \sum_{t=1}^{l_n} \vartheta_{\mathcal{F}(e_2)}(u_n^{(t)}) \vartheta_{\mathcal{G}(e_2)}(u_n^{(t)}) \right) \\ + \ldots + \eta_m \left(\frac{1}{k_n} \sum_{s=1}^{k_n} \zeta_{\mathcal{F}(e_m)}(u_n^{(s)}) \zeta_{\mathcal{G}(e_m)}(u_n^{(s)}) + \frac{1}{l_n} \sum_{t=1}^{l_n} \vartheta_{\mathcal{F}(e_m)}(u_n^{(t)}) \vartheta_{\mathcal{G}(e_m)}(u_n^{(t)}) \right) \end{array} \right\}$$

$$\begin{aligned} = & \left\{ \sum_{s=1}^{k_1} \xi_1 \eta_1 \frac{\zeta_{\mathcal{F}(e_1)}(u_1^{(s)})}{\sqrt{k_1}} \frac{\zeta_{\mathcal{G}(e_1)}(u_1^{(s)})}{\sqrt{k_1}} + \sum_{s=1}^{k_2} \xi_1 \eta_2 \frac{\zeta_{\mathcal{F}(e_2)}(u_1^{(s)})}{\sqrt{k_2}} \frac{\zeta_{\mathcal{G}(e_2)}(u_1^{(s)})}{\sqrt{k_2}} + \ldots\ldots \right. \\ + & \left. \sum_{s=1}^{k_n} \xi_n \eta_m \frac{\zeta_{\mathcal{F}(e_m)}(u_n^{(s)})}{\sqrt{k_n}} \frac{\zeta_{\mathcal{G}(e_m)}(u_n^{(s)})}{\sqrt{k_n}} \right\} + \left\{ \sum_{t=1}^{l_1} \xi_1 \eta_1 \frac{\vartheta_{\mathcal{F}(e_1)}(u_1^{(t)})}{\sqrt{l_1}} \frac{\vartheta_{\mathcal{G}(e_1)}(u_1^{(t)})}{\sqrt{l_1}} \right. \\ + & \left. \sum_{t=1}^{l_2} \xi_1 \eta_2 \frac{\vartheta_{\mathcal{F}(e_2)}(u_1^{(t)})}{\sqrt{l_2}} \frac{\vartheta_{\mathcal{G}(e_2)}(u_1^{(t)})}{\sqrt{l_2}} + \ldots\ldots + \sum_{t=1}^{l_n} \xi_n \eta_m \frac{\vartheta_{\mathcal{F}(e_m)}(u_n^{(t)})}{\sqrt{l_n}} \frac{\vartheta_{\mathcal{G}(e_m)}(u_n^{(t)})}{\sqrt{l_n}} \right\} \end{aligned}$$

By using Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} & (C_{\text{WDHFSS}}((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})))^2 \\ \leq & \left\{ \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \zeta^2_{\mathcal{F}(e_j)}(u_i^{(s)}) + \frac{1}{l_i} \sum_{t=1}^{l_i} \vartheta^2_{\mathcal{F}(e_j)}(u_i^{(t)}) \right) \right\} \\ \times & \left\{ \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \eta_j \left(\frac{1}{k_i} \sum_{s=1}^{k_i} \zeta^2_{\mathcal{G}(e_j)}(u_i^{(s)}) + \frac{1}{l_i} \sum_{t=1}^{l_i} \vartheta^2_{\mathcal{G}(e_j)}(u_i^{(t)}) \right) \right\} \\ = & E_{\text{WDHFSS}}(\mathcal{F}, \mathcal{E}) \cdot E_{\text{WDHFSS}}(\mathcal{G}, \mathcal{E}) \end{aligned}$$

Therefore, $C_{\text{WDHFSS}}((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) \leq \sqrt{E_{\text{WDHFSS}}(\mathcal{F}, \mathcal{E}) \cdot E_{\text{WDHFSS}}(\mathcal{G}, \mathcal{E})}$ and hence $0 \leq \mathcal{K}_3((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) \leq 1$. $\square$

Theorem 4.2.4. The correlation coefficient of two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ as defined in Eq. (4.8) i.e. $\mathcal{K}_4$ satisfies the same properties as those in Theorem 4.2.3.

Proof. The proof is similar to Theorem 4.2.2, so we omit here. □

Example 4.2.2. Assume that $\xi = (0.2, 0.3, 0.4, 0.1)^T$, $\eta = (0.35, 0.25, 0.40)$ be the weight vectors of $u_i (i = 1, 2, 3, 4)$ and $e_j (j = 1, 2, 3)$ respectively for two DHFSSs $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E})$ defined in Example 4.2.1 then the correlation coefficients between these two DHFSSs by using Eq. (4.7) and Eq. (4.8) are given by $\mathcal{K}_3((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) = 0.8949$ and $\mathcal{K}_4((\mathcal{F}, \mathcal{E}), (\mathcal{G}, \mathcal{E})) = 0.8800$ respectively.

4.3 Proposed decision making approach based on the correlation coefficients

In this section, we shall utilize the proposed correlation coefficients of DHFSSs to MCDM with dual hesitant fuzzy soft set information.

The general description of the DM problems is given in Section 2.6 of Chapter 2 with dual hesitant fuzzy soft information. The rating values corresponding to each alternative $\mathcal{A}^{(q)}$ $(q = 1, 2, \ldots, k)$ are represented in the form of DHFSS $(\mathcal{A}^{(q)}, \mathcal{E})$ over $\mathcal{X}$ which is given as follows.

$$\mathcal{A}^{(q)}(e_j) = \left(u_i, \left\{\zeta_{\mathcal{A}^{(q)}(e_j)}(u_i^{(s)})\right\}, \left\{\vartheta_{\mathcal{A}^{(q)}(e_j)}(u_i^{(t)})\right\} \mid u_i \in \mathcal{X}\right) \tag{4.9}$$

for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; q = 1, 2, \ldots, k$. Here, $\zeta_{\mathcal{A}^{(q)}(e_j)}(u_i^{(s)}), \vartheta_{\mathcal{A}^{(q)}(e_j)}(u_i^{(t)}) \in [0, 1]$ represents the discrete set of the possible MDs and NMDs for the alternative $\mathcal{A}^{(q)}$ given by the expert u_i under the parameter e_j. For convenience, we denote this DHFSS by $(\mathcal{A}^{(q)}, \mathcal{E}) = (\zeta_{ij}^{(q)}, \vartheta_{ij}^{(q)})$ where $\zeta_{ij}^{(q)} \in [0, 1]$ and $\vartheta_{ij}^{(q)} \in [0, 1]$ with the condition that $\zeta_{ij}^{(q)}(u_i^{(s)}) + \vartheta_{ij}^{(q)}(u_i^{(t)}) \leq 1$.

Then, we summarize the steps of the proposed approach based on the developed correlation coefficients as below to find the best alternative(s).

Step 1: Collect the information related to each alternative from a decision makers and represent them as a DHF soft decision matrix

$$(\mathcal{A}^{(q)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \ldots & e_m \end{array} \\ \begin{pmatrix} (\zeta_{11}^{(q)}, \vartheta_{11}^{(q)}) & (\zeta_{12}^{(q)}, \vartheta_{12}^{(q)}) & \ldots & (\zeta_{1m}^{(q)}, \vartheta_{1m}^{(q)}) \\ (\zeta_{21}^{(q)}, \vartheta_{21}^{(q)}) & (\zeta_{22}^{(q)}, \vartheta_{22}^{(q)}) & \ldots & (\zeta_{2m}^{(q)}, \vartheta_{2m}^{(q)}) \\ \vdots & \vdots & \ddots & \vdots \\ (\zeta_{n1}^{(q)}, \vartheta_{n1}^{(q)}) & (\zeta_{n2}^{(q)}, \vartheta_{n2}^{(q)}) & \ldots & (\zeta_{nm}^{(q)}, \vartheta_{nm}^{(q)}) \end{pmatrix} \end{array}$$

Step 2: Compute the correlation coefficient measures between an DHFSSs $(\mathcal{A}^{(q)}, \mathcal{E})$ and the reference set $(\mathcal{B}, \mathcal{E})$ by using either $\mathcal{K}_1$ or $\mathcal{K}_2$ or by using weighted correlations $\mathcal{K}_3$, $\mathcal{K}_4$ measures.

Step 3: Choose the highest index corresponding to the $\arg\max\{\mathcal{K}\}$ and select the best alternative(s) corresponding to them.

4.4 Illustrative Examples

In order to demonstrate the above-mentioned approach based on correlation coefficients, three illustrative examples are taken as below.

4.4.1 Example 1: Selection of the best candidate

A practical example of faculty recruitment for multiple criteria DM problem is discussed. Assume that there is a central Government university who wants to recruit an Assistant Professor in the Department of Mathematics. The panel of three experts u_1, u_2, u_3 will judge the certain candidates and select one candidate on the basis of certain parameters $E = \{e_1, e_2, e_3, e_4\}$ which stands for the Qualification, Teaching experience, number of publications and Teaching Ability respectively. The weight vector corresponding to these parameters are taken as $\eta = (0.1, 0.3, 0.4, 0.2)^T$ while corresponding to three experts are $\xi = (0.3, 0.5, 0.2)^T$. Assume that there are three candidates $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$ selected for an interview in front of the three experts $u_i (i = 1, 2, 3)$ after the screening test. The department head inform to the administrators regarding the requirement of the suitable candidate with some desired features and hence three experts set up a threshold over each parameter e_j according to their levels in the form of DHFSS $(\mathcal{B},\mathcal{E})$. The rating values of $(\mathcal{B},\mathcal{E})$ is given in Table 4.1, which is called as a "reference candidate set" in order to evaluate the given three candidates. The target of this problem is to find the best candidate out of $\mathcal{A}^{(1)},\mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$ which should be classified with the set $\mathcal{B}$.

Table 4.1: Rating values of the reference set with DHFSN information

$(\mathcal{B},\mathcal{E})$	e_1	e_2	e_3	e_4
u_1	$\{\{0.3,0.2,0.1\},\{0.2,0.1\}\}$	$\{\{0.4\},\{0.3,0.2\}\}$	$\{\{0.6,0.4,0.3\},\{0.2\}\}$	$\{\{0.4\},\{0.5,0.1\}\}$
u_2	$\{\{0.6,0.5,0.2\},\{0.4,0.3\}\}$	$\{\{0.5,0.2\},\{0.3,0.1\}\}$	$\{\{0.7,0.3\},\{0.3,0.1\}\}$	$\{\{0.2,0.1\},\{0.3\}\}$
u_3	$\{\{0.6,0.4,0.3\},\{0.2\}\}$	$\{\{0.9,0.7,0.5\},\{0.1\}\}$	$\{\{0.4,0.2\},\{0.5,0.3\}\}$	$\{\{0.8\},\{0.1\}\}$

Then, the steps of the proposed approach are executed as below to find the best candidate(s):

Step 1: During the interview, experts evaluate each candidate one-by-one and give his own preferences with respect to each parameter e_j in terms of DHFSS environment. The rating values of such candidates $\mathcal{A}^{(q)}(q=1,2,3)$ are summarized in Table 4.2.

Table 4.2: Rating values of the candidates with DHFSN information

$(\mathcal{A}^{(1)},\mathcal{E})$	e_1	e_2	e_3	e_4
u_1	$\{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\}$	$\{\{0.6,0.5,0.4\},\{0.4,0.3,0.2\}\}$	$\{\{0.3,0.2,0.1\},\{0.7,0.6,0.6\}\}$	$\{\{0.8,0.7,0.5\},\{0.2,0.1\}\}$
u_2	$\{\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}\}$	$\{\{0.5,0.4,0.3\},\{0.5,0.3,0.3\}\}$	$\{\{0.7,0.7,0.5\},\{0.3,0.2,0.2\}\}$	$\{\{0.4,0.3,0.2\},\{0.5\}\}$
u_3	$\{\{0.8,0.7,0.7\},\{0.2,0.1\}\}$	$\{\{0.8,0.8,0.7\},\{0.2,0.2,0.1\}\}$	$\{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\}$	$\{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\}$
$(\mathcal{A}^{(2)},\mathcal{E})$	e_1	e_2	e_3	e_4
u_1	$\{\{0.4,0.4,0.3\},\{0.6,0.5,0.4\}\}$	$\{\{0.6,0.6,0.5\},\{0.4,0.3,0.2\}\}$	$\{\{0.8,0.7,0.6\},\{0.2,0.1,0.1\}\}$	$\{\{0.7,0.5,0.4\},\{0.2,0.2,0.1\}\}$
u_2	$\{\{0.7,0.6,0.4\},\{0.3,0.2\}\}$	$\{\{0.6,0.4,0.3\},\{0.3,0.1\}\}$	$\{\{0.6,0.4,0.3\},\{0.4,0.1\}\}$	$\{\{0.5,0.4,0.2\},\{0.4,0.3,0.1\}\}$
u_3	$\{\{0.9,0.7,0.6\},\{0.1\}\}$	$\{\{0.7,0.5,0.4\},\{0.3,0.2\}\}$	$\{\{0.4,0.3,0.3\},\{0.6,0.5\}\}$	$\{\{0.7,0.6,0.4\},\{0.2,0.1\}\}$
$(\mathcal{A}^{(3)},\mathcal{E})$	e_1	e_2	e_3	e_4
u_1	$\{\{0.8,0.7,0.6\},\{0.2,0.1\}\}$	$\{\{0.5,0.4\},\{0.2,0.1\}\}$	$\{\{0.9,0.8,0.6\},\{0.1\}\}$	$\{\{0.9,0.7,0.5\},\{0.1\}\}$
u_2	$\{\{0.6,0.5,0.5\},\{0.4,0.2\}\}$	$\{\{0.8,0.7,0.5\},\{0.2,0.1\}\}$	$\{\{0.6,0.4,0.3\},\{0.3,0.1\}\}$	$\{\{0.3,0.2,0.1\},\{0.5,0.3,0.2\}\}$
u_3	$\{\{0.7,0.6,0.4\},\{0.2\}\}$	$\{\{0.6,0.4,0.3\},\{0.3\}\}$	$\{\{0.4,0.3,0.3\},\{0.2,0.1\}\}$	$\{\{0.6,0.3,0.2\},\{0.2,0.1\}\}$

Step 2a: By applying the correlation coefficient $\mathcal{K}_1$ between the alternative $(\mathcal{B},\mathcal{E})$ and $(\mathcal{A}^{(q)},\mathcal{E})$ $(q=1,2,3)$, we can obtain their corresponding indices measurement values as $\mathcal{K}_1((\mathcal{A}^{(1)},\mathcal{E}),(\mathcal{B},\mathcal{E}))=0.8603$, $\mathcal{K}_1((\mathcal{A}^{(2)},\mathcal{E}),(\mathcal{B},\mathcal{E}))=0.8916$ and $\mathcal{K}_1((\mathcal{A}^{(3)},\mathcal{E}),(\mathcal{B},\mathcal{E}))=0.8276$. On the other hand, if we utilize the correlation coefficient $\mathcal{K}_2$ then their corresponding measurement values are $\mathcal{K}_2((\mathcal{A}^{(1)},\mathcal{E}),(\mathcal{B},\mathcal{E}))=0.7111$, $\mathcal{K}_2((\mathcal{A}^{(2)},\mathcal{E}),(\mathcal{B},\mathcal{E}))=0.7834$, $\mathcal{K}_2((\mathcal{A}^{(3)},\mathcal{E}),(\mathcal{B},\mathcal{E}))=0.7580$.

Step 2b: If we assign the weights of the expert as $\xi = (0.3, 0.5, 0.2)^T$ and the parameters as $\eta = (0.1, 0.3, 0.4, 0.2)^T$, then $\mathcal{K}_3$ measure, we get $\mathcal{K}_3((\mathcal{A}^{(1)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.8390$, $\mathcal{K}_3((\mathcal{A}^{(2)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.8926$ and $\mathcal{K}_3((\mathcal{A}^{(3)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.8356$. Similarly, by using the correlation coefficient $\mathcal{K}_4$, we get the measurement value of each candidate is $\mathcal{K}_4((\mathcal{A}^{(1)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.6878$, $\mathcal{K}_4((\mathcal{A}^{(2)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7946$ and $\mathcal{K}_4((\mathcal{A}^{(3)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7509$.

Step 3: From these computed values, we can obtain the ranking order of the alternatives as $\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$ when either $\mathcal{K}_1$ or $\mathcal{K}_3$ is used while $\mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$ when $\mathcal{K}_2$ or $\mathcal{K}_4$ is used. Since, the similarity index of the candidate $\mathcal{A}^{(2)}$ belongs to the reference candidate $\mathcal{B}$ is highest and hence the best alternative for the required post is $\mathcal{A}^{(2)}$.

As the different correlation has different indices so, based on the behavior of the decision makers, they can use the best and worst alternatives. For instance, for an optimistic decision makers behavior, they tend to prefer $\mathcal{A}^{(1)}$ over $\mathcal{A}^{(3)}$ alternative while contrary to the optimism behavior, the pessimistic choice tend to choose $\mathcal{A}^{(3)}$ over $\mathcal{A}^{(1)}$ candidate.

4.4.2 Sensitivity analysis

As it has been observed from the above results that the best candidate is $\mathcal{A}^{(2)}$ for the desired post. However, if we are interested to find which expert gives his highest recommendation to $\mathcal{A}^{(2)}$ candidate w.r.t. the parameters $e_j (j = 1, 2, 3, 4)$, an analysis has been conducted in which various choices corresponding to expert $u_i (i = 1, 2, 3)$ are aggregated by using the parameter weights $\eta_j = (0.1, 0.3, 0.4, 0.2)^T$. Then the correlation index corresponding to each expert has been obtained as $\mathcal{K}_3(\mathcal{A}^{(2)}, u_1), (\mathcal{B}, u_1)) = 0.8228$, $\mathcal{K}_3((\mathcal{A}^{(2)}, u_2), (\mathcal{B}, u_2)) = 0.6942$ and $\mathcal{K}_3((\mathcal{A}^{(2)}, u_3), (\mathcal{B}, u_3)) = 0.8310$. Thus u_3 expert give his highest recommendation to candidate $\mathcal{A}^{(2)}$ for post of the suitable candidate w.r.t. the parameter e_j.

On the other hand, in order to judge which parameter is the strongest of it and which expert give their highest recommendation to it, an analysis has been done by applying proposed approach to $\mathcal{A}^{(2)}$ candidate only. For this, the different preferences of the experts

$u_i(i = 1, 2, 3)$ towards each parameter $e_j(j = 1, 2, 3, 4)$ are aggregated by using weight vector $\xi_i = (0.3, 0.5, 0.2)^T$. Thus based on it, the correlation coefficient measure between them are $0.9323, 0.7273, 0.7390$, and 0.8851 respectively for e_1, e_2, e_3 and e_4 of the best candidate $\mathcal{A}^{(2)}$ with reference candidate $\mathcal{B}$. Therefore, $\mathcal{A}^{(2)}$ candidate has e_1 parameter as the strongest based on the expert views.

Finally, an element wise correlation coefficient has been summarized in Table 4.3. From this analysis, it has been concluded that the expert u_1 gives e_2 attribute to his best

Table 4.3: Correlation coefficient of $(\mathcal{A}^{(2)}, B)$ corresponding to u_i and e_j

	e_1	e_2	e_3	e_4
u_1	0.8736	0.9759	0.9589	0.8297
u_2	0.9628	0.8848	0.8783	0.8082
u_3	0.9543	0.9541	0.9472	0.9659

recommendation while u_2 and u_3 give to e_1 and e_4 respectively parameter.

4.4.3 Comparative studies

In order to compare the proposed approach with some existing metrics, an analysis has been conducted for the aggregated dual hesitant fuzzy set matrix in which different parameters of the candidate are aggregated using weighted averaging formula using weight vector $\xi = (0.3, 0.5, 0.2)^T$. Then, based on these aggregated values, we conduct the following analysis by using existing approaches.

(i) By taking the correlation coefficient (ρ) as defined by Ye [248], we compute the measurement values of the given alternatives from reference set. The obtained values are $\rho(\mathcal{A}^{(1)}, \mathcal{B}) = 0.7633$, $\rho(\mathcal{A}^{(2)}, \mathcal{B}) = 0.7597$ and $\rho(\mathcal{A}^{(3)}, \mathcal{B}) = 0.7335$. Thus, it has been computed from these results that the best candidate is $\mathcal{A}^{(1)}$.

(ii) By taking cosine similarity measures as (C_{DHFS}) as proposed by Bai [12] to find the best candidate(s), then their corresponding measurement values are $C_{\text{DHFS}}(\mathcal{A}^{(1)}, \mathcal{B}) = 0.9316$, $C_{\text{DHFS}}(\mathcal{A}^{(2)}, \mathcal{B}) = 0.9385$ and $C_{\text{DHFS}}(\mathcal{A}^{(3)}, \mathcal{B}) = 0.8641$. Thus, the ranking order is $\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$ and the best candidate is $\mathcal{A}^{(2)}$.

(iii) By applying correlation coefficient measure [203] to the considered data, we get

$\rho(\mathcal{A}^{(1)}, \mathcal{B}) = 0.7231$, $\rho(\mathcal{A}^{(2)}, \mathcal{B}) = 0.7492$, $\rho(\mathcal{A}^{(3)}, \mathcal{B}) = 0.6836$. Therefore, the best candidate for the post is $\mathcal{A}^{(2)}$.

(iv) By taking the distance measure as defined in Eq. (2.17) of Chapter 2 and for different value of the parameter λ, we get the following values and the ranking order.

	Distance values of $\mathcal{B}$ from			Ranking	Best candidate
λ	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	order	
1	0.1518	0.1101	0.1438	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$
2	0.2652	0.2263	0.2870	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$
5	0.3973	0.3794	0.4736	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$

On the other hand, by taking the similarity measures as defined in (2.18) of Chapter 2, we get the values for different values of λ's are as below.

	Similarities values of $\mathcal{B}$ from			Ranking	Best candidate
λ	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	order	
1	0.8329	0.8744	0.7874	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$
2	0.7015	0.7526	0.6140	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$
5	0.5351	0.5830	0.4028	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$

(v) By taking a generalized distance measures as defined in Eq. (2.15) of Chapter 2 to the considered data, then the measurement indices to each alternative are summarized as below.

	Similarities values of $\mathcal{B}$ from			Ranking	Best candidate
λ	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	order	
1	0.1543	0.1156	0.1612	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$
2	0.1721	0.1489	0.1957	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$
5	0.2086	0.2050	0.2678	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(2)}$

From these results, it has been concluded that the proposed index results coincide with the existing methodology results. Thus, it can suitably solve the problem in a real-life situation and can be found as an alternative place than of the existing measures in a dual hesitant fuzzy soft set environment.

4.4.4 Example 2: Medical diagnosis problem

Consider a set of diagnoses $\mathcal{C} = \{\mathcal{C}^{(1)}(\text{Viral fever}), \mathcal{C}^{(2)}(\text{Malaria}), \mathcal{C}^{(3)}(\text{Typhoid})\}$ defined over the discrete set of doctors $\{d_1, d_2, d_3, d_4\}$, and a set of symptoms $\mathcal{E} = \{e_1$ (Temperature), e_2 (HeadAche), e_3 (Stomach Pain), e_4 (Cough), e_5 (Chest pain)$\}$. Suppose a patient $\mathcal{D}$, with respect to all the symptoms $\mathcal{E}$, can be represented by the following DHFSS denoted by $(\mathcal{D}, \mathcal{E})$ in Table 4.4.

Table 4.4: Rating values of the patient with DHFSN information

$(\mathcal{D}, \mathcal{E})$	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$\{\{0.9\}, \{0.1\}\}$	$\{\{0.6, 0.5\}, \{0.4, 0.3, 0.1\}\}$	$\{\{0.4, 0.3, 0.2\}, \{0.5\}\}$	$\{\{0.8, 0.7\}, \{0.1\}\}$	$\{\{0.4, 0.3\}, \{0.6, 0.5\}\}$
d_2	$\{\{0.6, 0.4\}, \{0.4, 0.2\}\}$	$\{\{0.8\}, \{0.2\}\}$	$\{\{0.7, 0.5\}, \{0.3, 0.2\}\}$	$\{\{0.3, 0.2, 0.1\}, \{0.5, 0.3\}\}$	$\{\{0.7, 0.5\}, \{0.3, 0.1\}\}$
d_3	$\{\{0.7\}, \{0.2, 0.1\}\}$	$\{\{0.4, 0.3\}, \{0.2, 0.1\}\}$	$\{\{0.8, 0.7\}, \{0.2, 0.1\}\}$	$\{\{0.6, 0.3\}, \{0.2\}\}$	$\{\{0.6\}, \{0.4, 0.2\}\}$
d_4	$\{\{0.8\}, \{0.2\}\}$	$\{\{0.9, 0.6\}, \{0.1\}\}$	$\{\{0.6, 0.4\}, \{0.3\}\}$	$\{\{0.5, 0.4\}, \{0.2, 0.1\}\}$	$\{\{0.6, 0.4\}, \{0.2\}\}$

Then, the target of this problem is to diagnose the disease of the patient $\mathcal{D}$ among one of the $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}$ and $\mathcal{C}^{(3)}$. For it, we utilized the steps of the proposed approach to obtain the ranking order of the diagnoses which are summarized as follows:

Step 1: The rating values of each diagnosis $\mathcal{C}^{(q)}(q = 1, 2, 3)$ is expressed by doctors $d_i(i = 1, 2, 3, 4)$ under the set of symptoms $e_j(j = 1, 2, \ldots, 5)$ are summarized in Table 4.5.

Step 2a: By applying the correlation coefficient $\mathcal{K}_1$ between the alternatives $(\mathcal{D}, \mathcal{E})$ and $(\mathcal{C}^{(q)}, \mathcal{E})$, we get the similarity index for each alternative as $\mathcal{K}_1((\mathcal{C}^{(1)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8048$, $\mathcal{K}_1((\mathcal{C}^{(2)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.9013$ and $\mathcal{K}_1((\mathcal{C}^{(3)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8876$. On the other hand, by using $\mathcal{K}_2$ measure, we get $\mathcal{K}_2((\mathcal{C}^{(1)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.7321$, $\mathcal{K}_2((\mathcal{C}^{(2)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8292$ and $\mathcal{K}_2((\mathcal{C}^{(3)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8248$.

Step 2b: If we assign the weight vectors $\xi_i = (0.5, 0.2, 0.2, 0.1)^T$ and $\eta_j = (0.4, 0.1, 0.2, 0.2, 0.1)^T$ respectively to the doctors and symptoms, then by using $\mathcal{K}_3$ measure, we get the measurement values are $\mathcal{K}_3((\mathcal{C}^{(1)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8202$, $\mathcal{K}_3((\mathcal{C}^{(2)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8936$ and $\mathcal{K}_3((\mathcal{C}^{(3)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.8976$. However, if we utilize $\mathcal{K}_4$, then we have $\mathcal{K}_4((\mathcal{C}^{(1)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.6836$, $\mathcal{K}_4((\mathcal{C}^{(2)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.7952$ and $\mathcal{K}_4((\mathcal{C}^{(3)}, \mathcal{E}), (\mathcal{D}, \mathcal{E})) = 0.7476$.

Step 3: Thus, from this study, we concluded that the patient $\mathcal{D}$ suffer from the Malaria $(\mathcal{C}^{(2)})$. Thus, we compute that diagnoses $\mathcal{C}^{(3)}$ or $\mathcal{C}^{(2)}$ is the most desirable diagnoses to be classifier with patient $\mathcal{D}$ according to the correlation coefficient $\mathcal{K}_3$ and $\mathcal{K}_4$ respectively.

Furthermore, if we compute the similarity index by using various existing measures under the dual hesitant environment [12, 176, 203, 205, 248] to the considered problem then the detail analysis corresponding to it are summarized in Table 4.6. From these results, we have concluded that the patient can be assign to the diagnosis $\mathcal{C}^{(2)}$(Malaria) and the proposed result coincides with the existing ones.

4.4.5 Example 3: Pattern recognition

Consider three experts u_1, u_2 and u_3 which are going to evaluate the three unknown patterns $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$ over the three different parameters e_1, e_2 and e_3. Each expert is advised to evaluate the given alternatives under the DHFSS environment. Let $\mathcal{B}$ be the formulated desired standard for the recognitions to get selected and its value are given in the form of the DHFSS in Table 4.7.

The target of this problem is to identify the best unknown pattern among $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$ which are well recognized with the desires of the experts given in DHFSS $\mathcal{B}$. In order to achieve it, the steps of the proposed approach are utilized here to obtain the best pattern as follows.

Step 1: Each unknown pattern $\mathcal{A}^{(q)}(q = 1, 2, 3)$ is evaluated by an expert $u_i(i = 1, 2, 3)$ under the different parameters $e_j(j = 1, 2, 3)$ and their rating values are summarized in Table 4.8 with DHFSS information.

Step 2a: By applying the correlation coefficients $\mathcal{K}_1$ and $\mathcal{K}_2$ between the alternatives $(\mathcal{B}, \mathcal{E})$ and $(\mathcal{A}^{(q)}, \mathcal{E})(q = 1, 2, 3)$, we compute their measurement values are $\mathcal{K}_1((\mathcal{A}^{(1)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7524$, $\mathcal{K}_1((\mathcal{A}^{(2)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7901$, $\mathcal{K}_1((\mathcal{A}^{(3)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7577$ while by $\mathcal{K}_2$ we get $\mathcal{K}_2((\mathcal{A}^{(1)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7365$, $\mathcal{K}_2((\mathcal{A}^{(2)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7734$ and $\mathcal{K}_2((\mathcal{A}^{(3)}, \mathcal{E}), (\mathcal{B}, \mathcal{E})) = 0.7481$.

Step 2b: With weight vector $\xi = (0.5, 0.3, 0.2)^T$ and $\eta = (0.4, 0.3, 0.3)^T$ to experts and parameters, we get the measurement values of each pattern $\mathcal{A}^{(q)}(q = 1, 2, 3)$ from the known pattern $\mathcal{B}$ are $\mathcal{K}_3\left((\mathcal{A}^{(1)},\mathcal{E}),(\mathcal{B},\mathcal{E})\right) = 0.7604$, $\mathcal{K}_3\left((\mathcal{A}^{(2)},\mathcal{E}),(\mathcal{B},\mathcal{E})\right) = 0.8051$ and $\mathcal{K}_3\left((\mathcal{A}^{(3)},\mathcal{E}),(\mathcal{B},\mathcal{E})\right) = 0.7328$. On the other hand, if we utilize the weighted correlation coefficient $\mathcal{K}_4$ then these measurement values are $\mathcal{K}_4\left((\mathcal{A}^{(1)},\mathcal{E}),(\mathcal{B},\mathcal{E})\right) = 0.7466$, $\mathcal{K}_4\left((\mathcal{A}^{(2)},\mathcal{E}),(\mathcal{B},\mathcal{E})\right) = 0.8031$ and $\mathcal{K}_4\left((\mathcal{A}^{(3)},\mathcal{E}),(\mathcal{B},\mathcal{E})\right) = 0.7203$.

Step 3: From these computed values, we concluded that $\mathcal{A}^{(2)}$ is the most desirable pattern to be classified with the known pattern $\mathcal{B}$.

4.5 Superiority and advantages of the proposed measures over the existing approaches

In this section, we present some counterexamples which show that the existing approaches fail to rank the given alternatives while the proposed approach can overcome their shortcoming.

4.5.1 Superiority of the proposed measures

Example 4.5.1. Consider the DM problem in which there are two alternatives denoted by $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, which are evaluated over the set of criteria denoted by $\{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\}$ with weight vector $(0.26,0.48,0.26)^T$. The preferences corresponding to each alternative as evaluated are represented in the form of DHFS as follows.

$$\begin{array}{c}
 \\ \mathcal{A}^{(1)} \\ \mathcal{A}^{(2)}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \quad\mathcal{G}_1 \quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3\quad \end{array} \\
\begin{bmatrix}
\{\{0.70,0.60\},\{0.30,0.10\}\} & \{\{0.8,0.7\},\{0.2,0.1\}\} & \{\{0.6,0.4\},\{0.40,0.32\}\} \\
\{\{0.22,0.19\},\{0.62,0.50\}\} & \{\{0.3,0.2\},\{0.5,0.4\}\} & \{\{0.2,0.1\},\{0.41,0.40\}\}
\end{bmatrix}
\end{array}$$

Now, if we utilize cosine similarity measure proposed by Bai [12] to find the similarity degree between the alternatives and the ideal alternative $\mathcal{B} = ((\{0.7,0.6\},\{0.3,0.1\}),(\{0.5,0.3\},\{0.5,0.4\}),(\{0.3,0.1\},\{0.4,0.3\}))$, then the measure values corresponding to alternative $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are 0.8792 and 0.8792 respectively. Thus we get $\mathcal{A}^{(1)} = \mathcal{A}^{(2)}$. But $\mathcal{A}^{(1)} \neq \mathcal{A}^{(2)}$ and hence the existing measure is unable to distinguish between the alternatives. On the other hand, if we apply proposed measures to this problem, then their respective values are $\mathcal{K}_3(\mathcal{A}^{(1)},\mathcal{B}) = 0.8674$, $\mathcal{K}_3(\mathcal{A}^{(2)},\mathcal{B}) = 0.8446$ and $\mathcal{K}_4(\mathcal{A}^{(1)},\mathcal{B}) = 0.7187$, $\mathcal{K}_4(\mathcal{A}^{(2)},\mathcal{B}) = 0.5121$. Thus, $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)}$ and hence $\mathcal{A}^{(1)}$ is preferred over $\mathcal{A}^{(2)}$.

Example 4.5.2. Consider two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are evaluated by considering certain attributes $\{\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3\}$ with weight vector $(0.25, 0.32, 0.43)^T$ and whose preference values are summarized under DHFS environment as follows:

$$\begin{array}{c} \\ \mathcal{A}^{(1)} \\ \mathcal{A}^{(2)} \end{array}\begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 \quad\quad\quad\quad & \quad\quad\mathcal{G}_2\quad\quad\quad\quad & \quad\quad\mathcal{G}_3 \end{array} \\ \begin{bmatrix} \{\{0.8,0.7\},\{0.2,0.1\}\} & \{\{0.7,0.6\},\{0.3,0.1\}\} & \{\{0.60,0.42\},\{0.3,0.1\}\} \\ \{\{0.3,0.2\},\{0.4,0.3\}\} & \{\{0.5,0.4\},\{0.5,0.4\}\} & \{\{0.41,0.30\},\{0.2,0.1\}\} \end{bmatrix} \end{array}$$

If we apply distance measure $d(.)$ as proposed by Wang, Xu, Wang and Ni [205], then their measure values of the alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ from the ideal alternative $\mathcal{B} = ((\{0.7, 0.6\}, \{0.2, 0.1\}), (\{0.4, 0.3\}, \{0.4, 0.3\}), (\{0.3, 0.2\}, \{0.5, 0.4\}))$ are $d(\mathcal{A}^{(1)},\mathcal{B}) = 0.1941$ and $d(\mathcal{A}^{(2)},\mathcal{B}) = 0.1941$. Thus, it has been seen that by their approach it is unable to distinguish between them. On the other hand, if we apply the proposed measure then corresponding to the alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ the measure values are $\mathcal{K}_3(\mathcal{A}^{(1)},\mathcal{B}) = 0.8829$, $\mathcal{K}_3(\mathcal{A}^{(2)},\mathcal{B}) = 0.8251$ and $\mathcal{K}_4(\mathcal{A}^{(1)},\mathcal{B}) = 0.7425$, $\mathcal{K}_4(\mathcal{A}^{(2)},\mathcal{B}) = 0.5199$. Hence we conclude that the alternative $\mathcal{A}^{(1)}$ is better than the alternative $\mathcal{A}^{(2)}$.

Example 4.5.3. Consider the DM problem in which there are two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, which are evaluated by two judges $\{u_1, u_2\}$ having weight vector $(0.87, 0.13)^T$, under the set of parameters $\{e_1, e_2, e_3\}$ with weight vector $(0.39, 0.36, 0.25)^T$. The preference values corresponding to each alternative are summarized under DHFSS environment as follows:

$$(\mathcal{A}^{(1)},\mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array}\begin{array}{c} \begin{array}{ccc} \quad e_1 \quad\quad\quad\quad & \quad\quad e_2\quad\quad\quad\quad & \quad\quad e_3 \end{array} \\ \begin{bmatrix} \{\{0.5,0.4\},\{0.2,0.1\}\} & \{\{0.43,0.38\},\{0.3,0.2\}\} & \{\{0.5,0.4\},\{0.2,0.1\}\} \\ \{\{0.3,0.2\},\{0.4,0.3\}\} & \{\{0.40,0.30\},\{0.5,0.4\}\} & \{\{0.3,0.2\},\{0.2,0.1\}\} \end{bmatrix} \end{array}$$

and

$$(\mathcal{A}^{(2)},\mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array}\begin{array}{c} \begin{array}{ccc} \quad e_1 \quad\quad\quad\quad & \quad\quad e_2\quad\quad\quad\quad & \quad\quad e_3 \end{array} \\ \begin{bmatrix} \{\{0.55,0.50\},\{0.25,0.10\}\} & \{\{0.45,0.33\},\{0.17,0.15\}\} & \{\{0.58,0.43\},\{0.25,0.19\}\} \\ \{\{0.20,0.10\},\{0.25,0.15\}\} & \{\{0.30,0.20\},\{0.15,0.13\}\} & \{\{0.13,0.14\},\{0.25,0.15\}\} \end{bmatrix} \end{array}$$

By utilizing existing measure as proposed by Ye [248] under DHFS environment, the correlation coefficients between the alternatives and the ideal alternative $(\mathcal{B},\mathcal{E})$=(({0.8,0.7}, {0.2,0.1}), ({0.7, 0.6}, {0.3, 0.1}), ({0.6, 0.5}, {0.3,0.2}) are $\rho(\mathcal{A}^{(1)},\mathcal{B}) = 0.9344$ and $\rho(\mathcal{A}^{(2)},\mathcal{B}) = 0.9344$. As both have same values so it is not possible to choose the best. Now, if we utilize the proposed measure on them then the relative values of the alternatives are $\mathcal{K}_3\big((\mathcal{A}^{(1)},\mathcal{E}),(\mathcal{B},\mathcal{E})\big) = 0.9553$ and $\mathcal{K}_3\big((\mathcal{A}^{(2)},\mathcal{E}),(\mathcal{B},\mathcal{E})\big) = 0.9318$. Since $\mathcal{K}_3\big((\mathcal{A}^{(1)},\mathcal{E}),(\mathcal{B},\mathcal{E})\big)$ is greater than $\mathcal{K}_3\big((\mathcal{A}^{(2)},\mathcal{E}),(\mathcal{B},\mathcal{E})\big)$, therefore, $\mathcal{A}^{(1)}$ is preferred over $\mathcal{A}^{(2)}$.

4.5.2 Advantages of the proposed measure

From the above studies and the comparison, we address the following advantages of the proposed method for solving the DM problems under the dual hesitant fuzzy soft environment.

(i) DHFSS is a generalization of HFSS, DHFS, HFS and IFS which contains more information related to an object than the others. Thus, the proposed correlation coefficient under DHFSS environment is a more generalizations than the existing correlation coefficients [25, 36, 203, 232, 246, 248].

(ii) It is observed from the proposed study that the correlation coefficients of HFS [25, 232], IFS [246], DHFS [203, 248], HFSS [36] are the special cases of the proposed correlation coefficients in this chapter. Therefore, from it, we can say that the proposed correlation coefficients can equivalently be utilized to solve the DM problems under these existing HFS, IFS, DHFS, HFSS environment, whereas the method in Chen et al. [25], Xu and Xia [232] or Ye [246] or Wang et al. [203], Ye [248] or Das, Malakar, Kar and Pal [36] is not suitable to solve the problem under the environment considered in this chapter.

(iii) With respect to the DM approach, the main advantage of the proposed approach over the existing DM approaches under the IFS, HFS, DHFS environment is that it contains much more information to handle the uncertainties in the data. In it, information related to the object is expressed more precisely and objectively and hence is a useful tool for handling the imprecise and ambiguous information during the DM process.

4.6 Conclusion

In this chapter, we addressed the concept of novel correlation and weighted correlation coefficients for the DHFSS by embedding the features of the FSS and DHFS into a single form. The various desirable relations and inequalities are proposed to support the proposed measures. Later, it has been shown that the proposed correlation coefficients satisfy the general properties of the correlations and also some of the existing measures under the DHFS, HFS, etc., are the special cases of the proposed one. In order to enhance the study, we present a DM approach based on the proposed correlation coefficients to find the best alternative(s) under the DHFSS environment. Three numerical examples are taken, from the field of the selection procedure, medical diagnoses and pattern recognition, for illustrating the approach and compared their results with some of the existing approaches. From the studies, it has been concluded that the proposed correlation coefficients can suitably handle the real-life DM problem with their targets. Finally, the superiority of the proposed approach has been validated by giving some examples to show that the proposed approach is suitable for working in those cases when the existing approaches [12, 205, 248] fails.

Table 4.5: Rating values of the diagnose with DHFSN information

$(\mathcal{C}^{(1)},\mathcal{E})$	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$\{\{0.6,0.4,0.3\},\{0.3,0.1\}\}$	$\{\{0.8,0.7,0.5\},\{0.1\}\}$	$\{\{0.4,0.3\},\{0.2\}\}$	$\{\{0.8,0.7,0.5\},\{0.2,0.1\}\}$	$\{\{0.8,0.7\},\{0.2,0.1\}\}$
d_2	$\{\{0.9,0.8,0.7\},\{0.1\}\}$	$\{\{0.7,0.6,0.4\},\{0.3,0.2\}\}$	$\{\{0.5,0.3,0.1\},\{0.3\}\}$	$\{\{0.5,0.4,0.3\},\{0.5,0.3\}\}$	$\{\{0.5,0.3,0.1\},\{0.4,0.3\}\}$
d_3	$\{\{0.5,0.4,0.2\},\{0.5,0.3\}\}$	$\{\{0.5,0.4,0.3\},\{0.4,0.2,0.1\}\}$	$\{\{0.8,0.6,0.5\},\{0.2,0.1\}\}$	$\{\{0.6,0.3,0.2\},\{0.4,0.1\}\}$	$\{\{0.2,0.1\},\{0.8\}\}$
d_4	$\{\{0.3,0.2\},\{0.4,0.2\}\}$	$\{\{0.7,0.6\},\{0.3,0.2,0.1\}\}$	$\{\{0.2,0.1\},\{0.5,0.4\}\}$	$\{\{0.2,0.1\},\{0.7,0.5\}\}$	$\{\{0.3,0.1\},\{0.6,0.4\}\}$
$(\mathcal{C}^{(2)},\mathcal{E})$	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$\{\{0.8,0.7\},\{0.2,0.1\}\}$	$\{\{0.5,0.4,0.3\},\{0.4,0.3\}\}$	$\{\{0.6,0.4\},\{0.4,0.2\}\}$	$\{\{0.3,0.2\},\{0.6,0.5\}\}$	$\{\{0.6,0.5,0.3\},\{0.2\}\}$
d_2	$\{\{0.4,0.3\},\{0.6,0.2\}\}$	$\{\{0.7,0.6\},\{0.3,0.2\}\}$	$\{\{0.7,0.6\},\{0.3,0.2\}\}$	$\{\{0.7,0.6,0.4\},\{0.2,0.1\}\}$	$\{\{0.7,0.3\},\{0.3,0.1\}\}$
d_3	$\{\{0.5,0.4\},\{0.3,0.2\}\}$	$\{\{0.6,0.4\},\{0.3\}\}$	$\{\{0.6,0.5\},\{0.3\}\}$	$\{\{0.6,0.5\},\{0.3,0.2\}\}$	$\{\{0.4,0.2\},\{0.5,0.3\}\}$
d_4	$\{\{0.8,0.7\},\{0.2,0.1\}\}$	$\{\{0.8,0.7\},\{0.2,0.1\}\}$	$\{\{0.7,0.6\},\{0.2\}\}$	$\{\{0.4,0.3\},\{0.2,0.1\}\}$	$\{\{0.2,0.1\},\{0.3\}\}$
$(\mathcal{C}^{(2)},\mathcal{E})$	Temperature	HeadAche	Stomach Pain	Cough	Chest Pain
d_1	$\{\{0.6,0.5\},\{0.2,0.1\}\}$	$\{\{0.6,0.4\},\{0.3,0.2\}\}$	$\{\{0.7,0.4,0.2\},\{0.2,0.1\}\}$	$\{\{0.7,0.6\},\{0.3,0.2\}\}$	$\{\{0.6,0.4,0.3\},\{0.3,0.1\}\}$
d_2	$\{\{0.5,0.4\},\{0.2\}\}$	$\{\{0.7,0.6,0.4\},\{0.3,0.2\}\}$	$\{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\}$	$\{\{0.9,0.7,0.6\},\{0.1\}\}$	$\{\{0.7,0.5,0.4\},\{0.3,0.2\}\}$
d_3	$\{\{0.6,0.5\},\{0.1\}\}$	$\{\{0.4,0.3\},\{0.6,0.5\}\}$	$\{\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}\}$	$\{\{0.8,0.7,0.6\},\{0.2,0.1\}\}$	$\{\{0.5,0.4\},\{0.4,0.3\}\}$
d_4	$\{\{0.5,0.4\},\{0.2,0.1\}\}$	$\{\{0.8,0.7\},\{0.2\}\}$	$\{\{0.4,0.4,0.3\},\{0.6\}\}$	$\{\{0.6,0.5\},\{0.4,0.2\}\}$	$\{\{0.6,0.5\},\{0.2,0.1\}\}$

Table 4.6: Comparative studies for Example 2: Medical diagnoses

Ref.	Method	Overall value from $\mathcal{D}$			Ranking
		$\mathcal{C}^{(1)}$	$\mathcal{C}^{(2)}$	$\mathcal{C}^{(3)}$	order
Ye [248]	Correlation coefficient	0.9228	0.9682	0.8057	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(1)} \succ \mathcal{C}^{(3)}$
Bai [12]	Cosine similarity measure	0.6942	0.4706	0.5731	$\mathcal{C}^{(1)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(2)}$
Wang et al. [203]	Correlation measure	0.7439	0.8317	0.7730	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
Wang, Xu, Wang and Ni [205]					
$\lambda = 1$	Distance measure	0.1319	0.0848	0.0884	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
$\lambda = 2$		0.3139	0.1497	0.1961	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
$\lambda = 5$		0.5698	0.2277	0.3354	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
$\lambda = 1$	Distance measure	0.1785	0.1051	0.1159	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
$\lambda = 2$		0.3784	0.1842	0.2381	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
$\lambda = 5$		0.6818	0.2870	0.4112	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
Singh [176]					
$\lambda = 1$	Distance measure	0.1432	0.0959	0.0958	$\mathcal{C}^{(3)} \succ \mathcal{C}^{(2)} \succ \mathcal{C}^{(1)}$
$\lambda = 2$		0.1685	0.1068	0.1202	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$
$\lambda = 5$		0.2303	0.1314	0.1604	$\mathcal{C}^{(2)} \succ \mathcal{C}^{(3)} \succ \mathcal{C}^{(1)}$

Table 4.7: Rating values of known pattern with DHFSN information

$(\mathcal{B}, \mathcal{E})$	e_1	e_2	e_3
u_1	$\{\{0.7, 0.6\}, \{0.2\}\}$	$\{\{0.2\}, \{0.8, 0.7\}\}$	$\{\{0.6, 0.4\}, \{0.2\}\}$
u_2	$\{\{0.6\}, \{0.3, 0.1\}\}$	$\{\{0.3, 0.2\}, \{0.7\}\}$	$\{\{0.7\}, \{0.1\}\}$
u_3	$\{\{0.5, 0.3\}, \{0.4\}\}$	$\{\{0.5\}, \{0.3, 0.2\}\}$	$\{\{0.3, 0.1\}, \{0.6\}\}$

Table 4.8: Rating values of the patterns with DHFSN information

$(\mathcal{A}^{(1)}, \mathcal{E})$	e_1	e_2	e_3
u_1	$\{\{0.7, 0.6\}, \{0.2\}\}$	$\{\{0.8, 0.7\}, \{0.1\}\}$	$\{\{0.2\}, \{0.4, 0.3\}\}$
u_2	$\{\{0.9\}, \{0.1\}\}$	$\{\{0.7\}, \{0.3, 0.2\}\}$	$\{\{0.6\}, \{0.2, 0.1\}\}$
u_3	$\{\{0.6, 0.5\}, \{0.2, 0.1\}\}$	$\{\{0.4\}, \{0.5, 0.3\}\}$	$\{\{0.7, 0.6\}, \{0.1\}\}$
$(\mathcal{A}^{(2)}, \mathcal{E})$	e_1	e_2	e_3
u_1	$\{\{0.7, 0.6\}, \{0.2, 0.1\}\}$	$\{\{0.5\}, \{0.4, 0.3\}\}$	$\{\{0.3, 0.2\}, \{0.6\}\}$
u_2	$\{\{0.4, 0.3\}, \{0.5\}\}$	$\{\{0.8\}, \{0.2, 0.1\}\}$	$\{\{0.8, 0.7\}, \{0.2, 0.1\}\}$
u_3	$\{\{0.5\}, \{0.3, 0.2\}\}$	$\{\{0.7, 0.6\}, \{0.2\}\}$	$\{\{0.6\}, \{0.2\}\}$
$(\mathcal{A}^{(3)}, \mathcal{E})$	e_1	e_2	e_3
u_1	$\{\{0.2, 0.1\}, \{0.7\}\}$	$\{\{0.7, 0.6\}, \{0.3, 0.1\}\}$	$\{\{0.7\}, \{0.1\}\}$
u_2	$\{\{0.7, 0.6\}, \{0.2, 0.1\}\}$	$\{\{0.4\}, \{0.5\}\}$	$\{\{0.6\}, \{0.2, 0.1\}\}$
u_3	$\{\{0.6\}, \{0.3, 0.1\}\}$	$\{\{0.1\}, \{0.7\}\}$	$\{\{0.6, 0.5\}, \{0.2\}\}$

Chapter 5

Dual hesitant fuzzy soft aggregation operators and their application in decision making[1]

In this chapter, we developed some new aggregation operators, namely, dual hesitant fuzzy soft weighted averaging and geometric operators proposed along with their proofs to aggregate the different preferences of the decision makers. Various desirable properties of its are also investigated in details. Further, these aggregation operators are extended to its generalized operator by incorporating the attitude character of the decision maker towards the data. Later, an MCDM approach is presented based on the proposed operators for solving DM problems. An illustrative example present to demonstrate the approach under DHFSS environment and compared their results with some of the existing approaches results.

5.1 Introduction

MCDM is considered a cognitive-based human activity for selecting the best alternative, and it has been applied widely across various domains. Traditionally, MCDM problem often requires decision makers to provide evaluation information about the criteria and the alternatives with an FS, IFS, and other extended sets. In all these above studies, it

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is assumed that the preferences related to each object to a set are taken either as a single number or in terms of interval-valued. But in some other cases, assume that an evaluation was conducted and the results are given by the different five DMs as $\{0.3, 0.4, 0.5, 0.5, 0.6\}$ which is entirely different from the representation of existing theories such as fuzzy numbers 0.3 (or 0.4 or 0.5), the interval-valued fuzzy number [0.3, 0.6]. Then under such circumstances, the above-mentioned techniques are unable to compute the desired value. To manage such situations, Babitha and John [11] introduced the concepts of hesitant fuzzy soft sets (HFSS). Later on, HFSS concepts have been extended to the dual hesitant fuzzy soft set (DHFSS) [154, 257] by embedding the hesitant degree of non-membership into the analysis. The advantages of these extended theories are that they are capable to facilitate the descriptions of the real-world situation with the help of their parameterized property.

Presently, an aggregation operator is one of the thrust areas in the field of the DM process and has drawn great attention from the researchers under an uncertain environment. The extensive literature on the AOs to solve the DM problems are reviewed in Section 1.1.4 of Chapter 1. Further, DHFSSs have the great potential to handle imprecise and vague information than the existing approaches such as IFSs, SSs, FSSs, HFSSs. For instance, if an evaluator gives their rating values by considering the parameterizations factor during an evaluation towards the objects in terms of the acceptance as 0.3, 0.4, 0.45 while the degree of rejection is 0.2, 0.25, 0.3. For such a case, the degrees are represented by a dual hesitant fuzzy element $(\{0.3, 0.4, 0.45\}, \{0.2, 0.25, 0.3\})$ which is completely different from the fuzzy number or the interval-valued intuitionistic fuzzy numbers $([0.3, 0.45], [0.2, 0.3])$. Hence, the dual hesitant fuzzy number can represent the information in a better way than the other numbers.

Therefore, by considering the advantages of DHFSS and the importance of the AOs, this chapter derives some series of the weighted averaging and geometric aggregation operators among the DHFSS. As per our knowledge, the aggregation operators under the DHFSS environment given in the aforementioned studies, however, cannot be utilized to solve the DM problem. Hence, there is a need to propose such aggregation operators for DHFSSs. In order to achieve it, we define some new operational laws for dual hesitant fuzzy soft numbers (DHFSNs) and investigate their properties. Then, based on these operational laws, some weighted averaging and geometric aggregation operators based on the t-norm are defined for DHFSNs. The various desirable properties of these operators are discussed in details. Also, these aggregation operators are extended to its generalized form by adding the risk attitude parameter λ into the analysis and hence developed their corresponding weighted averaging and geometric operators. Further, we propose a DM approach based on the proposed operators and the feasibility of the proposed approach is illustrated with a numerical example.

5.2 Operational laws for dual hesitant fuzzy soft sets

In this section, we discuss some operational laws and the corresponding properties for DHFSNs.

Definition 5.2.1. [257] A pair $(\mathcal{F}, \mathcal{E})$ is called a dual hesitant fuzzy soft set (DHFSS) over $\mathcal{X}$ if and only if $\mathcal{F}$ is a mapping given by $\mathcal{F} : \mathcal{E} \to D^{\mathcal{X}}$ defined as

$$F_{e_j}(u_i) = \left\{ \Big(u_i, h_{\mathcal{F}(e_j)}(u_i), g_{\mathcal{F}(e_j)}(u_i) \Big) \mid u_i \in \mathcal{X} \right\}, \tag{5.1}$$

where $D^{\mathcal{X}}$ represents the dual hesitant fuzzy subsets in $\mathcal{X}$, $h_{\mathcal{F}(e_j)}(u_i)$ and $g_{\mathcal{F}(e_j)}(u_i)$ are two sets of some values in $[0, 1]$, denoting the possible membership and non-membership degrees with the conditions $0 \leq \gamma, \delta \leq 1$ and $\gamma^+ + \delta^+ \leq 1$ where $\gamma \in h_{\mathcal{F}(e_j)}(u_i)$, $\delta \in g_{\mathcal{F}(e_j)}(u_i)$ and for any $x \in \mathcal{X}$, $\gamma^+ \in h^+_{\mathcal{F}(e_j)}(u_i) = \bigcup\limits_{\gamma \in h_{\mathcal{F}(e_j)}(u_i)} \max\{\gamma\}$ and $\delta^+ \in g_{\mathcal{F}(e_j)}(u_i) = \bigcup\limits_{\delta \in g_{\mathcal{F}(e_j)}(u_i)} \max\{\delta\}$.

For the sake of simplicity, we denoted $\mathcal{A}_{\mathcal{F}(e_j)}(u_i) = \Big(h_{\mathcal{F}(e_j)}(u_i),\ g_{\mathcal{F}(e_j)}(u_i) \Big)$ as a DHFSN, denoted by $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$, with the conditions $\gamma_{ij} \in h_{ij}$, $\delta_{ij} \in g_{ij}$, $\gamma^+_{ij} \in h^+_{ij} = \bigcup_{\gamma_{ij} \in h_{ij}} \max\{\gamma_{ij}\}$, $\delta^+_{ij} \in g^+_{ij} = \bigcup_{\delta_{ij} \in g_{ij}} \max\{\delta_{ij}\}$, $0 \leq \gamma_{ij}, \delta_{ij} \leq 1$, and $0 \leq \gamma^+_{ij} + \delta^+_{ij} \leq 1$.

Definition 5.2.2. For a DHFSN $\mathcal{A} = (\{h\}, \{g\})$, a score function $\mathcal{S}$ of $\mathcal{A}$ is defined as

$$\mathcal{S}(\mathcal{A}) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma - \frac{1}{\#g} \sum_{\delta \in g} \delta \tag{5.2}$$

while an accuracy function $\mathcal{H}$ of $\mathcal{A}$ is defined as

$$\mathcal{H}(\mathcal{A}) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma + \frac{1}{\#g}\sum_{\delta\in g}\delta \tag{5.3}$$

where $\#\gamma$ and $\#\delta$ are the number of values in h and g respectively.

Based on these functions, an order relation between two DHFSNs $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \succ \mathcal{B}$, if either of the following condition is satisfied:

(i) If $\mathcal{S}(\mathcal{A}) > \mathcal{S}(\mathcal{B})$;

(ii) If $\mathcal{S}(\mathcal{A}) = \mathcal{S}(\mathcal{B})$ and $\mathcal{H}(\mathcal{A}) > \mathcal{H}(\mathcal{B})$.

Based on the t-norm operations as defined in Section 2.4 of Chapter 2, and considering a generator k and l, we define the general operational laws for DHFSNs $\mathcal{A}_{ij} = (h_{ij}, g_{ij})$ such that neither h_{ij} =0 or 1 nor g_{ij} = 0 or 1.

Definition 5.2.3. Let $\mathcal{A}_{1j} = (\{h_{1j}\}, \{g_{1j}\})\,(j = 1,2)$ and $\mathcal{A} = (\{h\}, \{g\})$ be DHFSNs and $\lambda > 0$ be a real number; then based on norms' generators k and l, we define the generalized operational laws for them as follows:

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \Big(\{l^{-1}(l(\gamma_{11}) + l(\gamma_{12}))\}, \{k^{-1}(k(\delta_{11}) + k(\delta_{12}))\} \Big).$

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \Big(\{k^{-1}(k(\gamma_{11}) + k(\gamma_{12}))\}, \{l^{-1}(l(\delta_{11}) + l(\delta_{12}))\} \Big).$

(iii) $\lambda\mathcal{A} = \bigcup_{\gamma\in h,\delta\in g} \Big(\{l^{-1}(\lambda l(\gamma))\}, \{k^{-1}(\lambda k(\delta))\} \Big).$

(iv) $\mathcal{A}^{\lambda} = \bigcup_{\gamma\in h,\delta\in g} \Big(\{k^{-1}(\lambda k(\gamma))\}, \{l^{-1}(\lambda l(\delta))\} \Big).$

Theorem 5.2.1. All the operational laws for DHFSNs given in Definition 5.2.3, i.e., $\mathcal{A}_{11} \oplus \mathcal{A}_{12}$, $\mathcal{A}_{11} \otimes \mathcal{A}_{12}$, $\lambda\mathcal{A}$ and $\mathcal{A}^{\lambda}$ are also DHFSNs for a real number $\lambda > 0$.

Proof. For two DHFSNs $\mathcal{A}_{11}$ and $\mathcal{A}_{12}$, we have

$$\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \left(\{l^{-1}(l(\gamma_{11}) + l(\gamma_{12}))\}, \{k^{-1}(k(\delta_{11}) + k(\delta_{12}))\} \right)$$

As $k : [0,1] \to [0,\infty)$ is a decreasing function and $l(t) = k(1-t)$, so, we get $0 \leq k^{-1}\big(k(\gamma_{11}) + k(\gamma_{12})\big) \leq 1$ and $0 \leq l^{-1}\big(l(\delta_{11}) + l(\delta_{12})\big) \leq 1$. Therefore,

$$
\begin{aligned}
& k^{-1}\Big(k(\gamma_{11}) + k(\gamma_{12})\Big) + l^{-1}\Big(l(\delta_{11}) + l(\delta_{12})\Big) \\
& \leq k^{-1}\Big(k(1-\delta_{11}) + k(1-\delta_{12})\Big) + l^{-1}\Big(l(\delta_{11}) + l(\delta_{12})\Big) \\
& \leq 1 - l^{-1}\Big(l(\delta_{11}) + l(\delta_{12})\Big) + l^{-1}\Big(l(\delta_{11}) + l(\delta_{12})\Big) \\
& \leq 1
\end{aligned}
$$

Thus, $\mathcal{A}_{11} \otimes \mathcal{A}_{12}$ is a DHFSN. Similarly, we can prove for the other parts. $\square$

Theorem 5.2.2. Let $\mathcal{A}_{1j}(j = 1, 2)$ and $\mathcal{A}$ be three DHFSNs and $\lambda, \lambda_1, \lambda_2 > 0$ be three real numbers, then we have

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \mathcal{A}_{12} \oplus \mathcal{A}_{11}$.

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \mathcal{A}_{12} \otimes \mathcal{A}_{11}$.

(iii) $\lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) = \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12}$.

(iv) $(\mathcal{A}_{11} \otimes \mathcal{A}_{12})^{\lambda} = \mathcal{A}_{11}^{\lambda} \otimes \mathcal{A}_{12}^{\lambda}$.

(v) $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$.

(vi) $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1+\lambda_2}$.

Proof. We shall prove the parts (i), (iii), (v) and others can be proved analogously.

(i) For two DHFSNs $\mathcal{A}_{11}$ and $\mathcal{A}_{12}$, we have

$$
\begin{aligned}
\mathcal{A}_{11} \oplus \mathcal{A}_{12} &= \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \left(\left\{l^{-1}(l(\gamma_{11}) + l(\gamma_{12}))\right\}, \left\{k^{-1}(k(\delta_{11}) + k(\delta_{12}))\right\}\right) \\
&= \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \left(\left\{l^{-1}(l(\gamma_{12}) + l(\gamma_{11}))\right\}, \left\{k^{-1}(k(\delta_{12}) + k(\delta_{11}))\right\}\right) \\
&= \mathcal{A}_{12} \oplus \mathcal{A}_{11}
\end{aligned}
$$

(iii) For a real number $\lambda > 0$,

$$
\lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) = \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \lambda\left(\left\{l^{-1}\Big(l(\gamma_{11}) + l(\gamma_{12})\Big)\right\}, \left\{k^{-1}\Big(k(\delta_{11}) + k(\delta_{12})\Big)\right\}\right)
$$

$$
\begin{aligned}
&= \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \left(\begin{array}{l} \left\{ l^{-1}\Big(\lambda l\Big(l^{-1}(l(\gamma_{11})+l(\gamma_{12}))\Big)\Big)\right\}, \\ \left\{ k^{-1}\Big(\lambda k\Big(k^{-1}(k(\delta_{11})+k(\delta_{12}))\Big)\Big)\right\} \end{array} \right) \\
&= \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \left(\begin{array}{l} \left\{ l^{-1}\Big(\lambda(l(\gamma_{11})+l(\gamma_{12}))\Big)\right\}, \\ \left\{ k^{-1}\Big(\lambda(k(\delta_{11})+k(\delta_{12}))\Big)\right\} \end{array} \right) \\
&= \bigcup_{\substack{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\\ \delta_{11}\in g_{11},\delta_{12}\in g_{12}}} \left(\begin{array}{l} \left\{ l^{-1}\Big(l\Big(l^{-1}(\lambda l(\gamma_{11}))\Big)+l\Big(l^{-1}(\lambda l(\gamma_{12}))\Big)\Big)\right\}, \\ \left\{ k^{-1}\Big(k\Big(k^{-1}(\lambda k(\delta_{11}))\Big)+k\Big(k^{-1}(\lambda k(\delta_{12}))\Big)\Big)\right\} \end{array} \right) \\
&= \bigcup_{\substack{\gamma_{11}\in h_{11},\\ \delta_{11}\in g_{11}}} \left(\left\{l^{-1}\big(\lambda l(\gamma_{11})\big)\right\}, \left\{k^{-1}\big(\lambda k(\delta_{11})\big)\right\}\right) \\
&\oplus \bigcup_{\substack{\gamma_{12}\in h_{12},\\ \delta_{12}\in g_{12}}} \left(\left\{l^{-1}\big(\lambda l(\gamma_{12})\big)\right\}, \left\{k^{-1}\big(\lambda k(\delta_{12})\big)\right\}\right) \\
&= \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12}
\end{aligned}
$$

(v) For real numbers $\lambda_1, \lambda_2 > 0$,

$$
\begin{aligned}
\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} &= \bigcup_{\gamma\in h,\delta\in g} \left(\left\{l^{-1}(\lambda_1 l(\gamma))\right\}, \left\{k^{-1}(\lambda_1 k(\delta))\right\}\right) \\
&\oplus \bigcup_{\gamma\in h,\delta\in g} \left(\left\{l^{-1}(\lambda_2 l(\gamma))\right\}, \left\{k^{-1}(\lambda_2 k(\delta))\right\}\right) \\
&= \bigcup_{\gamma\in h,\delta\in g} \left(\begin{array}{l} \left\{ l^{-1}\Big(l\Big(l^{-1}(\lambda_1 l(\gamma))\Big)+l\Big(l^{-1}(\lambda_2 l(\gamma))\Big)\Big)\right\}, \\ \left\{ k^{-1}\Big(k\Big(k^{-1}(\lambda_1 k(\delta))\Big)+k\Big(k^{-1}(\lambda_2 k(\delta))\Big)\Big)\right\} \end{array} \right) \\
&= \bigcup_{\gamma\in h,\delta\in g} \left(\left\{l^{-1}(\lambda_1 l(\gamma)+\lambda_2 l(\gamma))\right\}, \left\{k^{-1}(\lambda_1 k(\delta)+\lambda_2 k(\delta))\right\}\right) \\
&= (\lambda_1+\lambda_2)\mathcal{A}
\end{aligned}
$$

□

Theorem 5.2.3. For three DHFSNs $\mathcal{A}, \mathcal{A}_{11}$, $\mathcal{A}_{12}$, and a real number $\lambda > 0$ we have

(i) $(\mathcal{A}^c)^\lambda = (\lambda\mathcal{A})^c$.

(ii) $\lambda(\mathcal{A}^c) = (\mathcal{A}^\lambda)^c$.

(iii) $\mathcal{A}_{11}^c \oplus \mathcal{A}_{12}^c = (\mathcal{A}_{11} \otimes \mathcal{A}_{12})^c$.

(iv) $\mathcal{A}_{11}^c \otimes \mathcal{A}_{12}^c = (\mathcal{A}_{11} \oplus \mathcal{A}_{12})^c$.

where $\mathcal{A}^c = (\{g\}, \{h\})$ denotes the complement of DHFSN $\mathcal{A} = (\{h\}, \{g\})$.

Proof. By the operational laws as defined in Definition 5.2.3, we have

(i) $(\mathcal{A}^c)^\lambda = \bigcup\limits_{\gamma \in h, \delta \in g} \left(\left\{ k^{-1}\big(\lambda k(\delta)\big) \right\}, \left\{ l^{-1}\big(\lambda l(\gamma)\big) \right\} \right) = (\lambda \mathcal{A})^c.$

(ii) $\lambda(\mathcal{A}^c) = \bigcup\limits_{\gamma \in h, \delta \in g} \left(\left\{ l^{-1}\big(\lambda l(\delta)\big) \right\}, \left\{ k^{-1}\big(\lambda k(\gamma)\big) \right\} \right) = (\mathcal{A}^\lambda)^c.$

(iii) $\mathcal{A}_{11}^c \oplus \mathcal{A}_{12}^c = \bigcup\limits_{\substack{\gamma_{11} \in h_{11}, \gamma_{12} \in h_{12}, \\ \delta_{11} \in g_{11}, \delta_{12} \in g_{12}}} \begin{pmatrix} \left\{ l^{-1}\big(l(\gamma_{11}) + l(\gamma_{12})\big) \right\}, \\ \left\{ k^{-1}\big(k(\delta_{11}) + k(\delta_{12})\big) \right\} \end{pmatrix} = (\mathcal{A}_{11} \otimes \mathcal{A}_{12})^c.$

(iv) $\mathcal{A}_{11}^c \otimes \mathcal{A}_{12}^c = \bigcup\limits_{\substack{\gamma_{11} \in h_{11}, \gamma_{12} \in h_{12}, \\ \delta_{11} \in g_{11}, \delta_{12} \in g_{12}}} \begin{pmatrix} \left\{ k^{-1}\big(k(\delta_{11}) + k(\delta_{12})\big) \right\}, \\ \left\{ l^{-1}\big(l(\gamma_{11}) + l(\gamma_{12})\big) \right\} \end{pmatrix} = (\mathcal{A}_{11} \oplus \mathcal{A}_{12})^c$

which completes the proof. □

5.3 Aggregation Operators for Dual Hesitant Fuzzy Soft Sets

In this section, we define some series of the weighted averaging and geometric aggregation operators based on the generalized t-norm operators for a collection of DHFSNs.

5.3.1 Dual hesitant fuzzy soft weighted averaging operator

Definition 5.3.1. Let $\mathcal{A}_{ij} (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be the collection of DHFSNs. A DHFSWA operator is a mapping DHFSWA : $\Omega^n \to \Omega$, and is defined as

$$\text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigoplus_{j=1}^{m} \left(\xi_j \bigoplus_{i=1}^{n} (\eta_i \mathcal{A}_{ij}) \right) \tag{5.4}$$

where Ω is the collection of DHFSNs, $\xi_j (j = 1, 2, \ldots, m)$ and $\eta_i (i = 1, 2, \ldots, n)$ are the weight vectors corresponding to the parameters and the experts respectively such that $\xi_j, \eta_i > 0$ and $\sum\limits_{j=1}^{m} \xi_j = 1$, $\sum\limits_{i=1}^{n} \eta_i = 1$. Then DHFSWA is called dual hesitant fuzzy soft weighted averaging operator.

Theorem 5.3.1. For a collection of DHFSNs $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})\,(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ the aggregated value by using DHFSWA operator is again a DHFSN and is given as

$$\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigcup_{\substack{\gamma_{ij} \in h_{ij},\\ \delta_{ij} \in g_{ij}}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\gamma_{ij}) \right) \right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\delta_{ij}) \right) \right) \right\} \end{array} \right) \tag{5.5}$$

Proof. We will prove the result given in Eq. (5.5) by mathematical induction on n and m. The following steps of the mathematical induction are executed as follows

Step 1: For $n = 1$, we have $\eta_1 = 1$, so

$$\begin{aligned} &\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1m}) \\ =\ & \bigoplus_{j=1}^{m} \xi_j \mathcal{A}_{1j} \\ =\ & \bigcup_{\gamma_{1j} \in h_{1j}, \delta_{1j} \in g_{1j}} \left(\left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j l(\gamma_{1j}) \right) \right\}, \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j k(\delta_{1j}) \right) \right\} \right) \\ =\ & \bigcup_{\gamma_{ij} \in h_{ij}, \delta_{ij} \in g_{ij}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{1} \eta_i l(\gamma_{ij}) \right) \right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{1} \eta_i k(\delta_{ij}) \right) \right) \right\} \end{array} \right) \end{aligned}$$

Similarly, for $m = 1$, we have $\xi_1 = 1$, so

$$\begin{aligned} &\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{n1}) \\ =\ & \bigoplus_{i=1}^{n} \eta_i \mathcal{A}_{i1} \\ =\ & \bigcup_{\gamma_{ij} \in h_{ij}, \delta_{ij} \in g_{ij}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{1} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\gamma_{ij}) \right) \right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{1} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\delta_{ij}) \right) \right) \right\} \end{array} \right) \end{aligned}$$

Thus the result holds for $n = 1$ and $m = 1$.

Step 2: Assume that the result is true for $m = k_1, n = k_2 + 1$ and $m = k_1 + 1, n = k_2$ then for $m = k_1 + 1, n = k_2 + 1$, we have

$$
\begin{aligned}
&\bigoplus_{j=1}^{k_1+1}\left(\xi_j \bigoplus_{i=1}^{k_2+1}(\eta_i \mathcal{A}_{ij})\right) \\
=\;& \bigoplus_{j=1}^{k_1}\left(\xi_j \bigoplus_{i=1}^{k_2+1}(\eta_i \mathcal{A}_{ij})\right) \oplus \xi_{k_1+1}\left(\bigoplus_{i=1}^{k_2+1}(\eta_i \mathcal{A}_{i(k_1+1)})\right) \\
=\;& \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{ij})\right)\right)\right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{ij})\right)\right)\right\} \end{array}\right) \\
\oplus\;& \bigcup_{\substack{\gamma_{i(k_1+1)}\in h_{i(k_1+1)}, \\ \delta_{i(k_1+1)}\in g_{i(k_1+1)}}} \left(\begin{array}{c} \left\{ l^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{i(k_1+1)})\right)\right)\right\}, \\ \left\{ k^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{i(k_1+1)})\right)\right)\right\} \end{array}\right) \\
=\;& \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{c} \left\{ l^{-1}\left(\begin{array}{c} l\left(l^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{ij})\right)\right)\right) \\ +l\left(l^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{i(k_1+1)})\right)\right)\right) \end{array}\right)\right\}, \\ \left\{ k^{-1}\left(\begin{array}{c} k\left(k^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{ij})\right)\right)\right) \\ +k\left(k^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{i(k_1+1)})\right)\right)\right) \end{array}\right)\right\} \end{array}\right) \\
=\;& \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{ij})\right) + \xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{i(k_1+1)})\right)\right)\right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{ij})\right) + \xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{i(k_1+1)})\right)\right)\right\} \end{array}\right) \\
=\;& \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\left\{ l^{-1}\left(\sum_{j=1}^{k_1+1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i l(\gamma_{ij})\right)\right)\right\},\ \left\{ k^{-1}\left(\sum_{j=1}^{k_1+1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i k(\delta_{ij})\right)\right)\right\}\right)
\end{aligned}
$$

which is true for $m = k_1 + 1$ and $n = k_2 + 1$.

Hence, the result given in Eq. (5.5) is true for all positive integers m, n. $\square$

Based on the Theorem 5.3.1, it is observed that the DHFSWA operator satisfies some properties which are stated as below.

Property 5.3.1. Suppose $\mathcal{A} = (\{h\}, \{g\}) = \bigcup_{\gamma \in h, \delta \in g} (\{\gamma\}, \{\delta\})$ and $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be a collection of DHFSNs. If for all i, j, $\gamma_{ij} = \gamma$ and $\delta_{ij} = \delta$, where γ_{ij} are elements of h_{ij}, δ_{ij} are elements of g_{ij}, γ is the element of h, and δ is the element of g, then

$$\text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}.$$

This property is called Idempotency.

Proof. Let $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$ and $\mathcal{A} = (\{h\}, \{g\})$ be DHFSNs such that $\mathcal{A}_{ij} = \mathcal{A}$ for all i, j. Then from Eq. (5.5), we get

$$\begin{aligned}
& \text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
= & \bigcup_{\gamma_{ij} \in h_{ij}, \delta_{ij} \in g_{ij}} \left(\begin{array}{c} \left\{ l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\gamma_{ij}) \right) \right) \right\}, \\ \left\{ k^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\delta_{ij}) \right) \right) \right\} \end{array} \right) \\
= & \bigcup_{\gamma \in h, \delta \in g} \left(\left\{ l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\gamma) \right) \right) \right\}, \left\{ k^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\delta) \right) \right) \right\} \right) \\
= & \bigcup_{\gamma \in h, \delta \in g} \left(\left\{ l^{-1} \left(l(\gamma) \right) \right\}, \left\{ k^{-1} \left(k(\delta) \right) \right\} \right) \\
= & (\{h\}, \{g\}) = \mathcal{A}
\end{aligned}$$

$\square$

Property 5.3.2. Suppose $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be a family of DHFSNs. If $\gamma_{\min} = \min_j \min_i \{\gamma_{ij} \mid \gamma_{ij} \in h_{ij}\}$, $\gamma_{\max} = \max_j \max_i \{\gamma_{ij} \mid \gamma_{ij} \in h_{ij}\}$, $\delta_{\min} = \min_j \min_i \{\delta_{ij} \mid \delta_{ij} \in g_{ij}\}$ and $\delta_{\max} = \max_j \max_i \{\delta_{ij} \mid \delta_{ij} \in g_{ij}\}$ for all i, j, then

$$\mathcal{A}^- \leq \text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \mathcal{A}^+$$

where $\mathcal{A}^- = (\{\gamma_{\min}\}, \{\delta_{\max}\}) = \bigcup_{\substack{\gamma^- \in \gamma_{\min}, \\ \delta^+ \in \delta_{\max}}} (\{\gamma^-\}, \{\delta^+\})$ and $\mathcal{A}^+ = (\{\gamma_{\max}\}, \{\delta_{\min}\}) = \bigcup_{\substack{\gamma^+ \in \gamma_{\max}, \\ \delta^- \in \delta_{\min}}} (\{\gamma^+\}, \{\delta^-\})$. This property is called Boundedness.

Proof. Since for all i, j and for any $\gamma_{ij} \in h_{ij}$, $\delta_{ij} \in g_{ij}$, we have $\min_j \min_i \gamma_{ij} \leq \gamma_{ij} \leq \max_j \max_i \gamma_{ij}$, $\min_j \min_i \delta_{ij} \leq \delta_{ij} \leq \max_j \max_i \delta_{ij}$ and the generator k, l are decreasing and increasing functions respectively, then we get

$$\begin{aligned}
& \sum_{i=1}^{n} \eta_i l \left(\min_j \min_i \gamma_{ij} \right) \leq \sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \leq \sum_{i=1}^{n} \eta_i l \left(\max_j \max_i \gamma_{ij} \right) \\
\Rightarrow \quad & \sum_{i=1}^{n} \eta_i \min_j \min_i \gamma_{ij} \leq \sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \leq \sum_{i=1}^{n} \eta_i \max_j \max_i \gamma_{ij} \\
\Rightarrow \quad & \min_j \min_i \gamma_{ij} \leq \sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \leq \max_j \max_i \gamma_{ij} \\
\Rightarrow \quad & l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\min_j \min_i \gamma_{ij} \right) \right) \leq l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \right) \right) \\
& \qquad \leq l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\max_j \max_i \gamma_{ij} \right) \right) \\
\Rightarrow \quad & \min_j \min_i \gamma_{ij} \leq l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \right) \right) \leq \max_j \max_i \gamma_{ij} \\
\Rightarrow \quad & \gamma_{\min} \leq l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \right) \right) \leq \gamma_{\max} \\
\Rightarrow \quad & \gamma_{\min} \leq \frac{1}{l_1} \sum_{\gamma_{ij} \in h_{ij}} l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l \left(\gamma_{ij} \right) \right) \right) \leq \gamma_{\max}
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
& \min_j \min_i \delta_{ij} \leq k^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k \left(\delta_{ij} \right) \right) \right) \leq \max_j \max_i \delta_{ij} \\
\Rightarrow \quad & \delta_{\min} \leq \frac{1}{l_2} \sum_{\delta_{ij} \in g_{ij}} k^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k \left(\delta_{ij} \right) \right) \right) \leq \delta_{\max}
\end{aligned}$$

where l_1 and l_2 are the numbers of the values in the membership degrees and the non-membership degrees of DHFSWA$(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$, respectively. Therefore, according to Definition 5.2.2, we can obtain

$$(\{\gamma_{\min}\}, \{\delta_{\max}\}) \leq \text{DHFSWA}(\mathcal{A}_{11}, \ldots, \mathcal{A}_{nm}) \leq (\{\gamma_{\max}\}, \{\delta_{\min}\})$$

which implies that $\mathcal{A}^- \leq \text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \mathcal{A}^+$. $\square$

Property 5.3.3. Suppose $\mathcal{A}_{ij} = \left(\{h_{\mathcal{A}_{ij}}\}, \{g_{\mathcal{A}_{ij}}\}\right)(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ and $\mathcal{B}_{ij} = \left(\{h_{\mathcal{B}_{ij}}\}, \{g_{\mathcal{B}_{ij}}\}\right)$ be two families of DHFSNs, where γ_{ij} are elements of $h_{\mathcal{A}_{ij}}$, δ_{ij} are elements of $g_{\mathcal{A}_{ij}}$, θ_{ij} are elements of $h_{\mathcal{B}_{ij}}$ and σ_{ij} are elements of $g_{\mathcal{B}_{ij}}$. If for all i, j, $\gamma_{ij} \geq \theta_{ij}$ and $\delta_{ij} \leq \sigma_{ij}$, then

$$\text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \geq \text{DHFSWA}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm})$$

This property is called Monotonicity.

Proof. By Theorem 5.3.1, we get

$$\text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigcup_{\substack{\gamma_{ij} \in h_{\mathcal{A}_{ij}}, \\ \delta_{ij} \in g_{\mathcal{A}_{ij}}}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\gamma_{ij}) \right) \right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\delta_{ij}) \right) \right) \right\} \end{array} \right)$$

and

$$\text{DHFSWA}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm}) = \bigcup_{\substack{\theta_{ij} \in h_{\mathcal{B}_{ij}}, \\ \sigma_{ij} \in g_{\mathcal{B}_{ij}}}} \left(\begin{array}{c} \left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\theta_{ij}) \right) \right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\sigma_{ij}) \right) \right) \right\} \end{array} \right)$$

Furthermore, since $\gamma_{ij} \geq \theta_{ij}$ and $\delta_{ij} \leq \sigma_{ij}$, for all i, j, then according to the definition of the comparison laws of DHFSNs, we know that Property 5.3.3 is true. $\square$

Property 5.3.4. Suppose $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be a collection of DHFSNs. If $\mathcal{B} = (\{h_{\mathcal{B}}\}, \{g_{\mathcal{B}}\}) = \bigcup_{\gamma_{\mathcal{B}} \in h_{\mathcal{B}}, \delta_{\mathcal{B}} \in g_{\mathcal{B}}} (\{\gamma_{\mathcal{B}}\}, \{\delta_{\mathcal{B}}\})$ is DHFSN, where γ_{ij} are the elements of h_{ij} and δ_{ij} are the elements of g_{ij}, then

$$\text{DHFSWA}(\mathcal{A}_{11} \oplus \mathcal{B}, \mathcal{A}_{12} \oplus \mathcal{B}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{B}) = \text{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathcal{B}$$

This property is called Shift Invariance.

Proof. As $\mathcal{A}_{ij}$ and $\mathcal{B}$ are two DHFSNs then for all i, j, we have

$$\mathcal{A}_{ij} \oplus \mathcal{B} = \bigcup_{\substack{\gamma_{ij} \in h_{ij}, \delta_{ij} \in g_{ij}, \\ \gamma_{\mathcal{B}} \in h_{\mathcal{B}}, \delta_{\mathcal{B}} \in g_{\mathcal{B}}}} \left(\left\{ l^{-1}\left(l(\gamma_{ij}) + l(\gamma_{\mathcal{B}}) \right) \right\}, \left\{ k^{-1}\left(k(\gamma_{ij}) + k(\gamma_{\mathcal{B}}) \right) \right\} \right)$$

Therefore, from Eq. (5.5), we get

$$
\begin{aligned}
&\mathrm{DHFSWA}(\mathcal{A}_{11}\oplus\mathcal{B},\mathcal{A}_{12}\oplus\mathcal{B},\ldots,\mathcal{A}_{nm}\oplus\mathcal{B})\\
=\;& \bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij},\\ \gamma_{\mathcal{B}}\in h_{\mathcal{B}},\delta_{\mathcal{B}}\in g_{\mathcal{B}}}} \left(\begin{array}{l} \left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l\left(l^{-1}\left(l(\gamma_{ij})+l(\gamma_{\mathcal{B}})\right)\right)\right)\right)\right\},\\ \left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k\left(k^{-1}(k(\delta_{ij})+k(\delta_{\mathcal{B}}))\right)\right)\right)\right\} \end{array}\right)\\
=\;& \bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij},\\ \gamma_{\mathcal{B}}\in h_{\mathcal{B}},\delta_{\mathcal{B}}\in g_{\mathcal{B}}}} \left(\begin{array}{l} \left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(l(\gamma_{ij})+l(\gamma_{\mathcal{B}})\right)\right)\right)\right\},\\ \left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(k(\delta_{ij})+k(\delta_{\mathcal{B}})\right)\right)\right)\right\} \end{array}\right)\\
=\;& \bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij},\\ \gamma_{\mathcal{B}}\in h_{\mathcal{B}},\delta_{\mathcal{B}}\in g_{\mathcal{B}}}} \left(\begin{array}{l} \left\{l^{-1}\left(l\left(l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(l(\gamma_{ij})\right)\right)\right)\right)+l(\gamma_{\mathcal{B}})\right)\right\},\\ \left\{k^{-1}\left(k\left(k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(k(\delta_{ij})\right)\right)\right)\right)+k(\delta_{\mathcal{B}})\right)\right\} \end{array}\right)\\
=\;& \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{l} \left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(l(\gamma_{ij})\right)\right)\right)\right\},\\ \left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(k(\delta_{ij})\right)\right)\right)\right\} \end{array}\right) \oplus \bigcup_{\substack{\gamma_{\mathcal{B}}\in h_{\mathcal{B}},\\ \delta_{\mathcal{B}}\in g_{\mathcal{B}}}} \left(\{\gamma_{\mathcal{B}}\},\{\delta_{\mathcal{B}}\}\right)\\
=\;& \mathrm{DHFSWA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\oplus\mathcal{B}
\end{aligned}
$$

Thus, $\mathrm{DHFSWA}(\mathcal{A}_{11}\oplus\mathcal{B},\mathcal{A}_{12}\oplus\mathcal{B},\ldots,\mathcal{A}_{nm}\oplus\mathcal{B}) = \mathrm{DHFSWA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\oplus\mathcal{B}$ holds. □

Property 5.3.5. Suppose $\mathcal{A}_{ij} = (\{h_{ij}\},\{g_{ij}\})\,(i = 1,2,\ldots,n; j = 1,2,\ldots,m)$ be a family of DHFSNs. If $\lambda > 0$ be a real number, then

$$\mathrm{DHFSWA}(\lambda\mathcal{A}_{11},\lambda\mathcal{A}_{12},\ldots,\lambda\mathcal{A}_{nm}) = \lambda\mathrm{DHFSWA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})$$

This property is called Homogeneity.

Proof. For a collection of DHFSNs $\mathcal{A}_{ij}$ and according to Definition 5.2.3, we have

$$\lambda\mathcal{A}_{ij} = \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\left\{l^{-1}\left(l\left(\lambda\gamma_{ij}\right)\right)\right\},\left\{k^{-1}\left(k\left(\lambda\delta_{ij}\right)\right)\right\}\right)$$

According to Theorem 5.3.1, we have

$$
\begin{aligned}
&\mathrm{DHFSWA}(\lambda\mathcal{A}_{11}, \lambda\mathcal{A}_{12}, \ldots, \lambda\mathcal{A}_{nm}) \\
= &\bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{l} \left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(l\left(l^{-1}\left(\lambda l(\gamma_{ij})\right)\right)\right)\right)\right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(k\left(k^{-1}\left(\lambda k(\gamma_{ij})\right)\right)\right)\right)\right) \right\} \end{array} \right) \\
= &\bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{l} \left\{ l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(\lambda l(\gamma_{ij})\right)\right)\right) \right\}, \\ \left\{ k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(\lambda k(\gamma_{ij})\right)\right)\right) \right\} \end{array} \right) \\
= &\bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{l} \left\{ l^{-1}\left(\lambda l\left(l^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(\gamma_{ij})\right)\right)\right)\right) \right\}, \\ \left\{ k^{-1}\left(\lambda k\left(k^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(\gamma_{ij})\right)\right)\right)\right) \right\} \end{array} \right) \\
= &\lambda\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})
\end{aligned}
$$

Thus, $\mathrm{DHFSWA}(\lambda\mathcal{A}_{11}, \lambda\mathcal{A}_{12}, \ldots, \lambda\mathcal{A}_{nm}) = \lambda\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$ holds. $\square$

Property 5.3.6. Suppose $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$ and $\mathcal{B}_{ij} = (\{\phi_{ij}\}, \{\psi_{ij}\})$ be two families of DHFSNs for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$, where γ_{ij}, δ_{ij}, θ_{ij} and σ_{ij} are the elements of h_{ij}, g_{ij}, ϕ_{ij} and ψ_{ij} respectively, then

$$
\begin{aligned}
&\mathrm{DHFSWA}(\mathcal{A}_{11} \oplus \mathcal{B}_{11}, \mathcal{A}_{12} \oplus \mathcal{B}_{12}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{B}_{nm}) \\
&= \mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathrm{DHFSWA}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm})
\end{aligned}
$$

Proof. For a collection of two DHFSNs $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$ and $\mathcal{B}_{ij} = (\{\phi_{ij}\}, \{\psi_{ij}\})$ and by using operational laws of Definition 5.2.3, we have

$$
\mathcal{A}_{ij} \oplus \mathcal{B}_{ij} = \bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}, \\ \theta_{ij}\in\phi_{ij},\sigma_{ij}\in\psi_{ij}}} \left(\left\{ l^{-1}\left(l(\gamma_{ij}) + l(\theta_{ij})\right)\right\}, \left\{ k^{-1}\left(k(\delta_{ij}) + k(\sigma_{ij})\right)\right\}\right)
$$

Therefore, according to Theorem 5.3.1, we have

$$
\begin{aligned}
&\mathrm{DHFSWA}(\mathcal{A}_{11}\oplus\mathcal{B}_{11},\mathcal{A}_{12}\oplus\mathcal{B}_{12},\ldots,\mathcal{A}_{nm}\oplus\mathcal{B}_{nm})\\
=&\bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij},\\ \theta_{ij}\in\phi_{ij},\sigma_{ij}\in\psi_{ij}}}\left(\begin{array}{l}\left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l\left(l^{-1}(l(\gamma_{ij})+l(\theta_{ij}))\right)\right)\right)\right\},\\ \left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k\left(k^{-1}(k(\delta_{ij})+k(\sigma_{ij}))\right)\right)\right)\right\}\end{array}\right)\\
=&\bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij},\\ \theta_{ij}\in\phi_{ij},\sigma_{ij}\in\psi_{ij}}}\left(\begin{array}{l}\left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i(l(\gamma_{ij})+l(\theta_{ij}))\right)\right)\right\},\\ \left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(k(\delta_{ij})+k(\sigma_{ij})\right)\right)\right)\right\}\end{array}\right)\\
=&\bigcup_{\substack{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij},\\ \theta_{ij}\in\phi_{ij},\sigma_{ij}\in\psi_{ij}}}\left(\begin{array}{l}\left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l(\gamma_{ij})\right)+\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l(\theta_{ij})\right)\right)\right\},\\ \left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k(\delta_{ij})\right)+\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k(\sigma_{ij})\right)\right)\right\}\end{array}\right)\\
=&\bigcup_{\substack{\gamma_{ij}\in h_{ij},\\ \delta_{ij}\in g_{ij}}}\left(\left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l(\gamma_{ij})\right)\right)\right\},\left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k(\delta_{ij})\right)\right)\right\}\right)\\
\oplus&\bigcup_{\substack{\theta_{ij}\in\phi_{ij},\\ \sigma_{ij}\in\psi_{ij}}}\left(\left\{l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l(\theta_{ij})\right)\right)\right\},\left\{k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k(\sigma_{ij})\right)\right)\right\}\right)\\
=&\ \mathrm{DHFSWA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\oplus\mathrm{DHFSWA}(\mathcal{B}_{11},\mathcal{B}_{12},\ldots,\mathcal{B}_{nm})
\end{aligned}
$$

□

Property 5.3.7. Suppose $\mathcal{A}_{ij}=(\{h_{ij}\},\{g_{ij}\})\,(i=1,2,\ldots,n;j=1,2,\ldots,m)$ be a family of DHFSNs. If $\lambda>0$ be a real number and $\mathcal{B}=(\{h\},\{g\})=\bigcup_{\gamma\in h,\delta\in g}(\{\gamma\},\{\delta\})$ is a DHFSN, then

$$\mathrm{DHFSWA}(\lambda\mathcal{A}_{11}\oplus\mathcal{B},\lambda\mathcal{A}_{12}\oplus\mathcal{B},\ldots,\lambda\mathcal{A}_{nm}\oplus\mathcal{B})=\lambda\mathrm{DHFSWA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\oplus\mathcal{B}.$$

Proof. According to Properties 5.3.4 and 5.3.5, we can get the proof easily, so we omit here. □

Furthermore, some of the special cases of the aggregation operators are obtained from the proposed DHFSWA operator by assigning the different form of the generator k as follows:

(i) If we take $k(t) = -\log(t)$; $t > 0$ then Eq. (5.5) becomes

$$\begin{aligned}&\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})\\ =& \bigcup_{\substack{\gamma_{ij}\in h_{ij},\\ \delta_{ij}\in g_{ij}}} \left(\left\{ 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1-\gamma_{ij})^{\eta_i} \right)^{\xi_j} \right\}, \left\{ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \delta_{ij}^{\eta_i} \right)^{\xi_j} \right\} \right)\end{aligned}$$

and is called as dual hesitant fuzzy soft Archimedean weighted averaging operator.

(ii) If we take $k(t) = \log\left(\dfrac{2-t}{t}\right)$; $t > 0$ then Eq. (5.5) becomes

$$\begin{aligned}&\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})\\ =& \bigcup_{\gamma_{ij}\in h_{ij},\delta_{ij}\in g_{ij}} \left(\begin{array}{c} \left\{ \dfrac{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1+\gamma_{ij})^{\eta_i} \right)^{\xi_j} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1-\gamma_{ij})^{\eta_i} \right)^{\xi_j}}{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1+\gamma_{ij})^{\eta_i} \right)^{\xi_j} + \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1-\gamma_{ij})^{\eta_i} \right)^{\xi_j}} \right\}, \\ \left\{ \dfrac{2\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \delta_{ij}^{\eta_i} \right)^{\xi_j}}{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (2-\delta_{ij})^{\eta_i} \right)^{\xi_j} + \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \delta_{ij}^{\eta_i} \right)^{\xi_j}} \right\} \end{array} \right)\end{aligned}$$

and is called as dual hesitant fuzzy soft Einstein weighted averaging operator.

(iii) If we take $k(t) = \log\left(\dfrac{\rho + (1-\rho)t}{t}\right)$, $\rho \in (0, \infty)$, then Eq. (5.5) becomes

$$\begin{aligned}&\mathrm{DHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})\\ =& \bigcup_{\substack{\gamma_{ij}\in h_{ij},\\ \delta_{ij}\in g_{ij}}} \left(\begin{array}{c} \left\{ \dfrac{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1+(\rho-1)\gamma_{ij})^{\eta_i} \right)^{\xi_j} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1-\gamma_{ij})^{\eta_i} \right)^{\xi_j}}{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1+(\rho-1)\gamma_{ij})^{\eta_i} \right)^{\xi_j} + (\rho-1)\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1-\gamma_{ij})^{\eta_i} \right)^{\xi_j}} \right\}, \\ \left\{ \dfrac{\rho\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \delta_{ij}^{\eta_i} \right)^{\xi_j}}{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1+(\rho-1)(1-\delta_{ij}))^{\eta_i} \right)^{\xi_j} + (\rho-1)\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \delta_{ij}^{\eta_i} \right)^{\xi_j}} \right\} \end{array} \right)\end{aligned}$$

and is called as dual hesitant fuzzy soft Hammer weighted averaging operator.

5.3.2 Dual hesitant fuzzy soft weighted geometric operator

In this section, we present some geometric aggregation operators for a collection of DHFSNs under the DHFSS environment.

Definition 5.3.2. Let $\mathcal{A}_{ij}, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be a family of DHFSNs. A DHFSWG operator is a mapping DHFSWG : $\Omega^n \to \Omega$ defined by

$$\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigotimes_{j=1}^{m} \left(\bigotimes_{i=1}^{n} (\mathcal{A}_{ij})^{\eta_i} \right)^{\xi_j} \tag{5.6}$$

then DHFSWG is called dual hesitant fuzzy soft weighted geometric operator and $\xi_j (j = 1, 2, \ldots, m)$ and $\eta_i (i = 1, 2, \ldots, n)$ are the weight vectors corresponding to the parameters and the experts respectively such that $\xi_j, \eta_i > 0$ and $\sum\limits_{j=1}^{m} \xi_j = 1$, $\sum\limits_{i=1}^{n} \eta_i = 1$.

Theorem 5.3.2. For a collection of DHFSNs $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\}) \, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, the aggregated value by using DHFSWG operator is still a DHFSN and is given by

$$\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigcup_{\substack{\gamma_{ij} \in h_{ij}, \\ \delta_{ij} \in g_{ij}}} \left(\begin{array}{l} \left\{ k^{-1} \left(\sum\limits_{j=1}^{m} \xi_j \left(\sum\limits_{i=1}^{n} \eta_i k(\gamma_{ij}) \right) \right) \right\}, \\ \left\{ l^{-1} \left(\sum\limits_{j=1}^{m} \xi_j \left(\sum\limits_{i=1}^{n} \eta_i l(\delta_{ij}) \right) \right) \right\} \end{array} \right) \tag{5.7}$$

Proof. Proof of this theorem is similar to Theorem 5.3.1, so we omit here. □

Also, DHFSWG operator satisfies the certain properties such as idempotency, boundedness, monotonicity etc., for a collection of DHFSNs which are stated without proof as follows:

(P1) If $\mathcal{A}_{ij} = \mathcal{A}$ for all i, j, then we have $\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}$.

(P2) If $\mathcal{A}^-$ and $\mathcal{A}^+$ are the lower and upper bounds of DHFSNs $\mathcal{A}_{ij}$, then we have $\mathcal{A}^- \leq \text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \mathcal{A}^+$.

(P3) For two DHFSNs $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ such that $\mathcal{A}_{ij} \leq \mathcal{B}_{ij}$ for all i, j, then we have $\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \text{DHFSWG}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm})$.

(P4) For DHFSNs $\mathcal{B}$ and $\mathcal{A}_{ij}$, we have $\text{DHFSWG}(\mathcal{A}_{11} \oplus \mathcal{B}, \mathcal{A}_{12} \oplus \mathcal{B}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{B}) = \text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathcal{B}$.

(P5) For real number $\lambda > 0$, we have, $\text{DHFSWG}(\lambda\mathcal{A}_{11}, \lambda\mathcal{A}_{12}, \ldots, \lambda\mathcal{A}_{nm}) = \lambda\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$.

(P6) For real number $\lambda > 0$ and DHFSN $\mathcal{B}$, we have $\text{DHFSWG}(\lambda\mathcal{A}_{11} \oplus \mathcal{B}, \lambda\mathcal{A}_{12} \oplus \mathcal{B}, \ldots, \lambda\mathcal{A}_{nm} \oplus \mathcal{B}) = \lambda\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathcal{B}$.

(P7) For DHFSNs $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$, we have $\text{DHFSWG}(\mathcal{A}_{11} \oplus \mathcal{B}_{11}, \mathcal{A}_{12} \oplus \mathcal{B}_{12}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{B}_{nm}) = \text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \text{DHFSWG}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm})$

Furthermore, by assigning the different form of the generator k, the following are some special cases of the aggregation operators which have been derived from the proposed DHFSWG operator.

(i) If we take $k(t) = -\log(t); t > 0$, then Eq. (5.7) becomes

$$\begin{aligned} &\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = &\bigcup_{\substack{\gamma_{ij} \in h_{ij}, \\ \delta_{ij} \in g_{ij}}} \left(\left\{ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \gamma_{ij}^{\eta_i} \right)^{\xi_j} \right\}, \left\{ 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \delta_{ij})^{\eta_i} \right)^{\xi_j} \right\} \right) \end{aligned}$$

and is called as dual hesitant fuzzy soft Archimedean weighted geometric operator.

(ii) If we take $k(t) = \log\left(\frac{2-t}{t}\right); t > 0$, then Eq. (5.7) becomes

$$\begin{aligned} &\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = &\bigcup_{\gamma_{ij} \in h_{ij}, \delta_{ij} \in g_{ij}} \left(\begin{array}{c} \left\{ \dfrac{2 \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} \gamma_{ij}^{\eta_i} \right)^{\xi_j}}{\prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (2 - \gamma_{ij})^{\eta_i} \right)^{\xi_j} + \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} \gamma_{ij}^{\eta_i} \right)^{\xi_j}} \right\}, \\ \left\{ \dfrac{\prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 + \delta_{ij})^{\eta_i} \right)^{\xi_j} - \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 - \delta_{ij})^{\eta_i} \right)^{\xi_j}}{\prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 + \delta_{ij})^{\eta_i} \right)^{\xi_j} + \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 - \delta_{ij})^{\eta_i} \right)^{\xi_j}} \right\} \end{array} \right) \end{aligned}$$

and is called as dual hesitant fuzzy soft Einstein weighted geometric operator.

(iii) If we take $k(t) = \log\left(\frac{\rho + (1-\rho)t}{t}\right)$, $\rho \in (0, \infty)$, then Eq. (5.7) becomes

$$\text{DHFSWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$$
$$= \bigcup_{\substack{\gamma_{ij} \in h_{ij},\\ \delta_{ij} \in g_{ij}}} \left(\begin{array}{l} \left\{ \dfrac{\rho \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} \gamma_{ij}^{\eta_i} \right)^{\xi_j}}{\prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 + (\rho - 1)(1 - \gamma_{ij}))^{\eta_i} \right)^{\xi_j} + (\rho - 1) \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} \gamma_{ij}^{\eta_i} \right)^{\xi_j}} \right\}, \\ \left\{ \dfrac{\prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 + (\rho - 1)\delta_{ij})^{\eta_i} \right)^{\xi_j} - \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 - \delta_{ij})^{\eta_i} \right)^{\xi_j}}{\prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 + (\rho - 1)\delta_{ij})^{\eta_i} \right)^{\xi_j} + (\rho - 1) \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} (1 - \delta_{ij})^{\eta_i} \right)^{\xi_j}} \right\} \end{array} \right)$$

and is called as dual hesitant fuzzy soft Hammer weighted geometric operator.

5.3.3 Generalized aggregation operators for DHFSSs

In this section, the above defined aggregation operators are extended to its generalized version by adding the attitudinal character λ of the decision maker towards the aggregation process.

Definition 5.3.3. Let $\mathcal{A}_{ij} (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be a collection of DHFSNs and ξ_j, η_i are weight vectors of the parameters and the experts respectively, satisfying $\xi_j, \eta_i > 0$ and $\sum\limits_{j=1}^{m} \xi_j = 1, \sum\limits_{i=1}^{n} \eta_i = 1$. A generalized dual hesitant fuzzy soft weighted averaging (GDHFSWA) operator is a mapping from Ω^n to Ω and is defined as:

$$\text{GDHFSWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\bigoplus_{j=1}^{m} \left(\xi_j \bigoplus_{i=1}^{n} \left(\eta_i \mathcal{A}_{ij}^{\lambda} \right) \right) \right)^{\frac{1}{\lambda}} \quad ; \quad \lambda > 0 \tag{5.8}$$

In particular, if we take $\lambda = 1$, then the GDHFSWA operator reduces to the DHFSWA operator.

Theorem 5.3.3. For a collection of DHFSNs $\mathcal{A}_{ij} = (\{h_{ij}\}, \{g_{ij}\})$, the aggregated value by using GDHFSWA operator is still a DHFSN and given by:

$$\begin{aligned}
&\text{GDHFSWA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\
&= \bigcup_{\substack{\gamma_{ij}\in h_{ij},\\ \delta_{ij}\in g_{ij}}} \left(\begin{array}{l} \left\{ k^{-1}\left(\frac{1}{\lambda}k\left(l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l\left(k^{-1}\left(\lambda k\left(\gamma_{ij}\right)\right)\right)\right)\right)\right)\right)\right\}, \\ \left\{ l^{-1}\left(\frac{1}{\lambda}l\left(k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k\left(l^{-1}\left(\lambda l\left(\delta_{ij}\right)\right)\right)\right)\right)\right)\right)\right\} \end{array} \right)
\end{aligned} \tag{5.9}$$

Definition 5.3.4. For a collection of DHFSNs $\mathcal{A}_{ij}(i=1,2,\ldots,n; j=1,2,\ldots,m)$, a generalized dual hesitant fuzzy soft weighted geometric(GDHFSWG) operator is a mapping from Ω^n to Ω and is defined as:

$$\text{GDHFSWG}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) = \frac{1}{\lambda}\bigotimes_{j=1}^{m}\left(\bigotimes_{i=1}^{n}(\lambda\mathcal{A}_{ij})^{\eta_i}\right)^{\xi_j} \quad ; \quad \lambda>0 \tag{5.10}$$

In particular, if we take $\lambda=1$, then the GDHFSWG operator reduces to the DHFSWG operator.

Theorem 5.3.4. For a collection of DHFSNs $\mathcal{A}_{ij} = (\{h_{ij}\},\{g_{ij}\})\,(i=1,2,\ldots,n; j=1,2,\ldots,m)$, the aggregated value by using GDHFSWG operator is still a DHFSN and given by

$$\begin{aligned}
&\text{GDHFSWG}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\
&= \bigcup_{\substack{\gamma_{ij}\in h_{ij},\\ \delta_{ij}\in g_{ij}}} \left(\begin{array}{l} \left\{ l^{-1}\left(\frac{1}{\lambda}l\left(k^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i k(l^{-1}(\lambda l(\gamma_{ij})))\right)\right)\right)\right)\right\}, \\ \left\{ k^{-1}\left(\frac{1}{\lambda}k\left(l^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i l(k^{-1}(\lambda k(\delta_{ij})))\right)\right)\right)\right)\right\} \end{array} \right)
\end{aligned} \tag{5.11}$$

Note that both the GDHFSWA and GDHFSWG operators have the properties such as idempotency, boundedness under some conditions. These properties can be proved similar to the properties of DHFSWA operator.

5.4 Proposed decision making approach with dual hesitant fuzzy soft information

In this section, we shall utilize the proposed operators to develop a MCDM method under dual hesitant fuzzy soft environment.

For it, the brief description of the MCDM problem is given in Section 2.6 of Chapter 2. To evaluate the best alternative among them, we construct a dual hesitant fuzzy soft set $(\mathcal{A}^{(b)}, \mathcal{E})$, where $\mathcal{A}^{(b)}(b = 1, 2, \ldots, z)$ is the mapping defined as $\mathcal{A}^{(b)} : \mathcal{E} \to D^{\mathcal{X}}$. The rating values of each alternative $\mathcal{A}^{(b)}$ are provided by an expert $u_i(i = 1, 2, \ldots, n)$ under DHFSS environment in the form of DHFSNs as $\mathcal{A}_{ij}^{(b)} = \left(\{h_{ij}^{(b)}\}, \{g_{ij}^{(b)}\}\right)$ where $h_{ij}, g_{ij} \in [0, 1]$ represents the possible set of hesitant degrees of the membership and non-membership with the condition that $0 \leq \gamma, \delta \leq 1$ and $\gamma^+ + \delta^+ \leq 1$ where $\gamma \in h_{ij}$, $\delta \in g_{ij}$, $\gamma^+ = \bigcup_{\gamma \in h_{ij}} \max\{\gamma\}$ and $\delta^+ = \bigcup_{\delta \in g_{ij}} \max\{\delta\}$. The following are the proposed steps to determine the best alternative(s) with dual hesitant fuzzy soft information.

Step 1: Arrange the collective information in terms of dual hesitant fuzzy soft decision matrices $(\mathcal{A}^{(b)}, \mathcal{E}) = (\mathcal{A}_{ij}^{(b)})_{n\times m}$ corresponding to each alternative as

$$(\mathcal{A}^{(b)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array} \\ \begin{pmatrix} \left(\{h_{11}^{(b)}\}, \{g_{11}^{(b)}\}\right) & \left(\{h_{12}^{(b)}\}, \{g_{12}^{(b)}\}\right) & \cdots & \left(\{h_{1m}^{(b)}\}, \{g_{1m}^{(b)}\}\right) \\ \left(\{h_{21}^{(b)}\}, \{g_{21}^{(b)}\}\right) & \left(\{h_{22}^{(b)}\}, \{g_{22}^{(b)}\}\right) & \cdots & \left(\{h_{2m}^{(b)}\}, \{g_{2m}^{(b)}\}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\{h_{n1}^{(b)}\}, \{g_{n1}^{(b)}\}\right) & \left(\{h_{n2}^{(b)}\}, \{g_{n2}^{(b)}\}\right) & \cdots & \left(\{h_{nm}^{(b)}\}, \{g_{nm}^{(b)}\}\right) \end{pmatrix} \end{array}$$

Step 2: Normalize the information, if required, by using Eq. (2.34) of Chapter 2 and get the decision matrix $\mathcal{R} = \left(r_{ij}^{(b)}\right)_{n\times m}$.

Step 3: Aggregate all the values of the alternative $\mathcal{A}^{(b)}(b = 1, 2, \ldots, z)$ into the overall assessment value $r^{(b)} = (\{h_b\}, \{g_b\})$ based on the GDHFSWA operator as follows

$$
\begin{aligned}
r^{(b)} &= \text{GDHFSWA}(r_{11}^{(b)}, r_{12}^{(b)}, \ldots, r_{nm}^{(b)}) \\
&= \bigcup_{\substack{\gamma_{ij}^{(b)} \in h_{ij}^{(b)}, \\ \delta_{ij}^{(b)} \in g_{ij}^{(b)}}} \left(\begin{array}{l} \left\{ k^{-1} \left(\frac{1}{\lambda} k \left(l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l \left(k^{-1} \left(\lambda k \left(\gamma_{ij}^{(b)} \right) \right) \right) \right) \right) \right) \right) \right\}, \\ \left\{ l^{-1} \left(\frac{1}{\lambda} l \left(k^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k \left(l^{-1} \left(\lambda l \left(\delta_{ij}^{(b)} \right) \right) \right) \right) \right) \right) \right) \right\} \end{array} \right)
\end{aligned}
$$

On the other hand, if we utilize the GDHFSWG operator then the aggregated values are obtained as

$$
\begin{aligned}
r^{(b)} &= \text{GDHFSWG}(r_{11}^{(b)}, r_{12}^{(b)}, \ldots, r_{nm}^{(b)}) \\
&= \bigcup_{\substack{\gamma_{ij}^{(b)} \in h_{ij}^{(b)}, \\ \delta_{ij}^{(b)} \in g_{ij}^{(b)}}} \left(\begin{array}{l} \left\{ l^{-1} \left(\frac{1}{\lambda} l \left(k^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i k(l^{-1}(\lambda l(\gamma_{ij}^{(b)}))) \right) \right) \right) \right) \right\}, \\ \left\{ k^{-1} \left(\frac{1}{\lambda} k \left(l^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i l(k^{-1}(\lambda k(\delta_{ij}^{(b)}))) \right) \right) \right) \right) \right\} \end{array} \right)
\end{aligned}
$$

where k, l are the generator represents the t-norm and t-conorm respectively, $\eta_i (i = 1, 2, \ldots, n)$ be the expert weights and $\xi_j (j = 1, 2, \ldots, m)$ be the parameter weights.

Step 4: Compute the score values of the overall aggregated values $r^{(b)} = (\{h_b\}, \{g_b\})$ $(b = 1, 2, \ldots, z)$ by using equations

$$\mathcal{S}(r^{(b)}) = \frac{1}{\#h_b} \sum_{\gamma_b \in h_b} \gamma_b - \frac{1}{\#g_b} \sum_{\delta_b \in g_b} \delta_b$$

where $\#h_b$ and $\#g_b$ represents the number of elements in h_b and g_b respectively. If there is no difference between two score values $\mathcal{S}(r^{(b_1)})$ and $\mathcal{S}(r^{(b_2)})$ for any two positive b_1, b_2, then we need to calculate the accuracy values of the alternatives as

$$\mathcal{H}(r^{(b)}) = \frac{1}{\#h_b} \sum_{\gamma_b \in h_b} \gamma_b + \frac{1}{\#g_b} \sum_{\delta_b \in g_b} \delta_p.$$

Step 5: Rank all the feasible alternatives $\mathcal{A}^{(b)} (b = 1, 2, \ldots, z)$ according to the Definition 5.2.2 and hence select the most desirable alternative(s).

The above mentioned steps of the proposed approach are summarized in the flow chart given in Fig. 5.1.

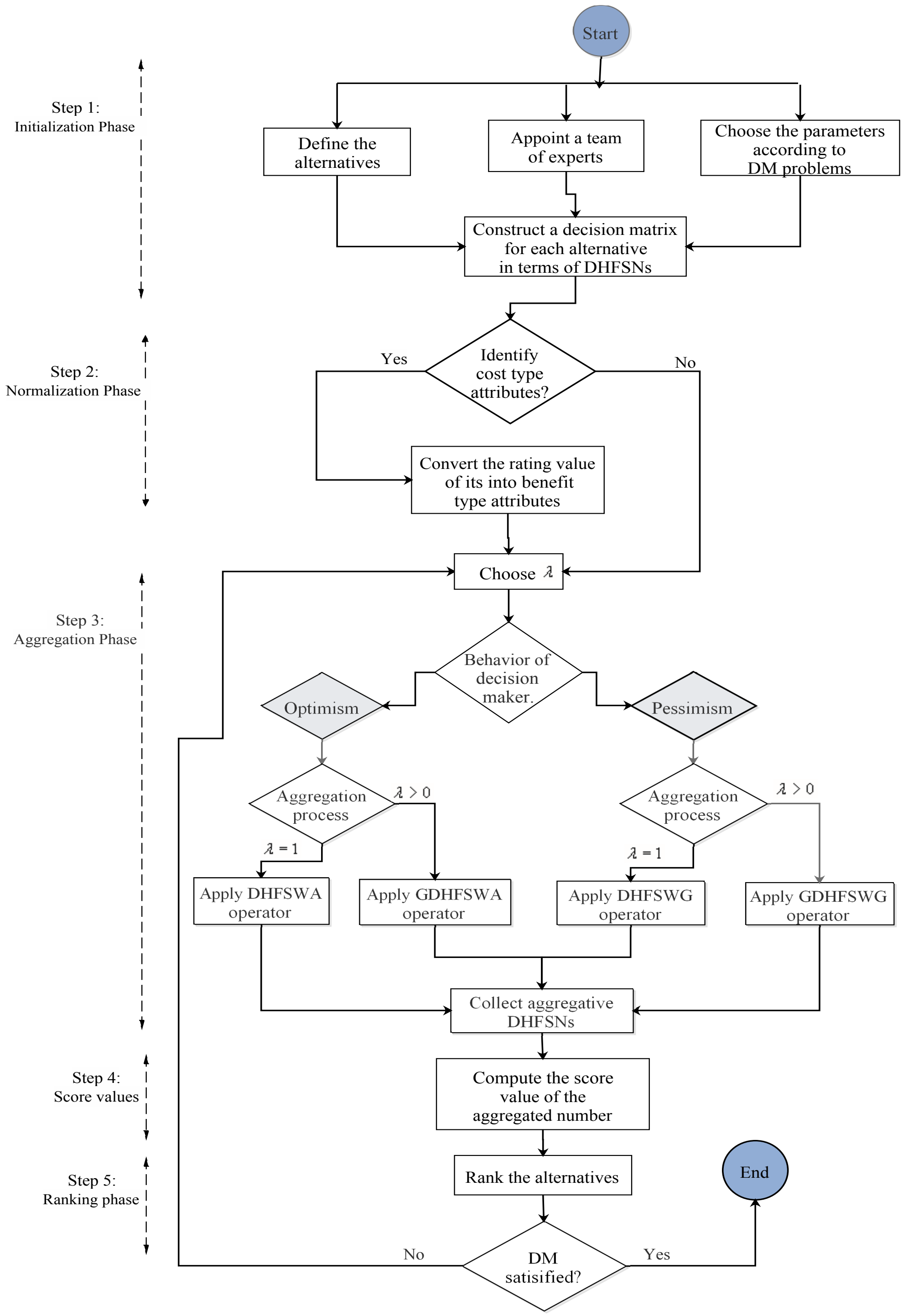

Figure 5.1: Flow chart of the proposed approach

5.5 Illustrative example

The above mentioned approach has been illustrated with a numerical example related to the DM process which can be read as:

5.5.1 A case study

Monsoon rains in river catchments have caused flooding in the northeastern state of Assam, India in July 2016. The Brahmaputra river and its tributaries overflow as a result of monsoon rains which effected over 80,000 people and 4,000 hectares of crop land according to Assam State Disaster Management Agency(ASDMA). The worst affected areas of Assam are treated as an alternative denoted by Lakhimpur ($\mathcal{A}^{(1)}$), Dhemaji ($\mathcal{A}^{(2)}$), Nagaon ($\mathcal{A}^{(3)}$) and Jorhat ($\mathcal{A}^{(4)}$). A team of three experts, denoted by $\{u_1, u_2, u_3\}$ from different departments of ASDMA will inspect the flood affected areas. The weight vector of these experts is $\eta = (0.5, 0.3, 0.2)^T$ and they gave their own report of disaster based on the certain parameters namely, loss of life (e_1), crop loss (e_2), damage of houses (e_3) with weight vector $\xi = (0.4, 0.3, 0.3)^T$. The main task of this problem is to tell us the government that which area is affected more based on the preference values of experts, so as to allocate the funds accordingly to the level of damage. In what follows, we utilize the multicriteria DM method proposed in above section to determine the most desirable alternative(s) under dual hesitant fuzzy soft environment.

By GDHFSWA Operator

First, we utilize the GDHFSWA aggregation operator to aggregate the different preferences. The following steps of it are executed as below.

Step 1: The combined rating values of each alternative based on the expert preferences are summarized in Table 5.1 in the form of DHFSNs $\mathcal{A}_{ij}^{(b)} = \left(\left\{h_{ij}^{(b)}\right\}, \left\{g_{ij}^{(b)}\right\}\right)$; $i = 1, 2, 3; j = 1, 2, 3; b = 1, 2, 3, 4$.

Step 2: Since all the parameters are of the same types, so there is no need of normalizing the data.

Step 3: Without loss of generality, we take $\lambda = 1$ and a generator $k(t) = -\log(t), t > 0$. Now, by utilizing GDHFSWA operator corresponding to the weight vectors $\eta = (0.5, 0.3, 0.2)^T$, $\xi = (0.4, 0.3, 0.3)^T$, we get the aggregated values $r^{(b)}$ corresponding to each alternative $\mathcal{A}^{(b)} (b = 1, 2, 3, 4)$ as follows:

$$r^{(1)} = \bigcup_{\substack{\gamma_{ij}^{(1)} \in h_{ij}^{(1)}, \\ \delta_{ij}^{(1)} \in g_{ij}^{(1)}}} \left(\left\{ 1 - \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(1 - \gamma_{ij}^{(1)}\right)^{\eta_i} \right)^{\xi_j} \right\}, \left\{ \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(\delta_{ij}^{(1)}\right)^{\eta_i} \right)^{\xi_j} \right\} \right)$$

$$= \left(\begin{array}{l} \left\{ \begin{array}{l} 0.6862, 0.6807, 0.6806, 0.6750, 0.6676, 0.6665, 0.6618, 0.6616, \\ 0.6607, 0.6605, 0.6557, 0.6546, 0.6468, 0.6406, 0.6404, 0.6342 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.1935, 0.1876, 0.1865, 0.1853, 0.1830, 0.1818, 0.1809, 0.1797, \\ 0.1786, 0.1775, 0.1765, 0.1763, 0.1753, 0.1752, 0.1741, 0.1732, \\ 0.1719, 0.1711, 0.1700, 0.1699, 0.1690, 0.1688, 0.1678, 0.1668, \\ 0.1658, 0.1647, 0.1639, 0.1628, 0.1608, 0.1597, 0.1588, 0.1540 \end{array} \right\} \end{array} \right)$$

$$r^{(2)} = \bigcup_{\substack{\gamma_{ij}^{(2)} \in h_{ij}^{(2)}, \\ \delta_{ij}^{(2)} \in g_{ij}^{(2)}}} \left(\left\{ 1 - \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(1 - \gamma_{ij}^{(2)}\right)^{\eta_i} \right)^{\xi_j} \right\}, \left\{ \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(\delta_{ij}^{(2)}\right)^{\eta_i} \right)^{\xi_j} \right\} \right)$$

$$= \left(\begin{array}{l} \left\{ \begin{array}{l} 0.6062, 0.5994, 0.5989, 0.5982, 0.5919, 0.5916, 0.5912, 0.5907, \\ 0.5845, 0.5839, 0.5836, 0.5833, 0.5829, 0.5767, 0.5761, 0.5756, \\ 0.5755, 0.5751, 0.5744, 0.5681, 0.5677, 0.5674, 0.5670, 0.5665, \\ 0.5599, 0.5593, 0.5590, 0.5586, 0.5516, 0.5509, 0.5504, 0.5426 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.3017, 0.2921, 0.2890, 0.2835, 0.2797, 0.2744, 0.2715, 0.2663, \\ 0.2628, 0.2627, 0.2578, 0.2551, 0.2543, 0.2516, 0.2469, 0.2468, \\ 0.2435, 0.2389, 0.2363, 0.2318, 0.2288, 0.2244, 0.2221, 0.2150 \end{array} \right\} \end{array} \right)$$

$$r^{(3)} = \bigcup_{\substack{\gamma_{ij}^{(3)} \in h_{ij}^{(3)}, \\ \delta_{ij}^{(3)} \in g_{ij}^{(3)}}} \left(\left\{ 1 - \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(1 - \gamma_{ij}^{(3)}\right)^{\eta_i} \right)^{\xi_j} \right\}, \left\{ \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(\delta_{ij}^{(3)}\right)^{\eta_i} \right)^{\xi_j} \right\} \right)$$

$$= \left(\begin{Bmatrix} 0.5566, 0.5506, 0.5461, 0.5411, 0.5399, 0.5371, 0.5349, 0.5308, \\ 0.5301, 0.5260, 0.5238, 0.5208, 0.5197, 0.5144, 0.5094, 0.5028 \end{Bmatrix}, \begin{Bmatrix} 0.2976, 0.2796, 0.2739, 0.2726, 0.2573, 0.2561, 0.2524, 0.2508, \\ 0.2371, 0.2356, 0.2323, 0.2312, 0.2182, 0.2172, 0.2127, 0.1998 \end{Bmatrix} \right)$$

and

$$r^{(4)} = \bigcup_{\substack{\gamma_{ij}^{(4)} \in h_{ij}^{(4)}, \\ \delta_{ij}^{(4)} \in g_{ij}^{(4)}}} \left(\left\{ 1 - \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(1 - \gamma_{ij}^{(4)} \right)^{\eta_i} \right)^{\xi_j} \right\}, \left\{ \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(\delta_{ij}^{(4)} \right)^{\eta_i} \right)^{\xi_j} \right\} \right)$$

$$= \left(\begin{Bmatrix} 0.5419, 0.5363, 0.5349, 0.5294, 0.5293, 0.5237, 0.5222, 0.5164, \\ 0.5147, 0.5131, 0.5089, 0.5073, 0.5072, 0.5057, 0.5015, 0.5014, \\ 0.4998, 0.4998, 0.4955, 0.4939, 0.4938, 0.4922, 0.4878, 0.4861, \\ 0.4843, 0.4781, 0.4765, 0.4702, 0.4701, 0.4638, 0.4622, 0.4557 \end{Bmatrix}, \begin{Bmatrix} 0.3145, 0.3083, 0.3069, 0.3009, \\ 0.2757, 0.2702, 0.2690, 0.2637 \end{Bmatrix} \right)$$

Step 4: By using score equation, we get $\mathcal{S}(r^{(1)}) = 0.4880$, $\mathcal{S}(r^{(2)}) = 0.3196$, $\mathcal{S}(r^{(3)}) = 0.2850$ and $\mathcal{S}(r^{(4)}) = 0.2115$.

Step 5: Since $\mathcal{S}(r^{(1)}) > \mathcal{S}(r^{(2)}) > \mathcal{S}(r^{(3)}) > \mathcal{S}(r^{(4)})$ and hence based on it, the ranking of all the feasible alternatives $\mathcal{A}^{(b)}(b = 1, 2, 3, 4)$ is shown as follows

$$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)},$$

where the symbol "$\succ$" means "preferred to". Thus, we conclude that the best alternative is $\mathcal{A}^{(1)}$, i.e., $\mathcal{A}^{(1)}$ is the most damaged area than the others.

On the other hand, if we take the generator $k(t) = \log\left(\frac{2-t}{t}\right)$ instead of $k(t) = -\log(t)$ to aggregate the different preference values during the aggregation process then corresponding to $\lambda = 1$ and by using DHFSWA operator, we get the overall score values of each alternative as $\mathcal{S}(r^{(1)}) = 0.4731$, $\mathcal{S}(r^{(2)}) = 0.3020$, $\mathcal{S}(r^{(3)}) = 0.2566$ and $\mathcal{S}(r^{(4)}) = 0.1857$. Based on it, the ranking order of the alternatives is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$ and hence the best alternative remains $\mathcal{A}^{(1)}$.

By GDHFSWG Operator

If in Step 3, instead of using GDHFSWA operator, we utilize GDHFSWG to aggregate the dual hesitant fuzzy soft information, then the following steps of the proposed approach are executed to get the optimal alternative(s):

Step 1: It is similar to above of Step 1.

Step 2: It is similar to above of Step 2.

Step 3: Without loss of generality, we take $\lambda = 1$, $k(t) = -\log(t)$ and their corresponding DHFSWG operator given in Eq. (5.7) to aggregate the dual hesitant fuzzy soft decision matrix $\mathcal{R}$. The aggregated values corresponding to each alternative are obtained as follows:

$$r^{(1)} = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.5745, 0.5693, 0.5662, 0.5631, 0.5610, 0.5580, 0.5571, 0.5550, \\ 0.5520, 0.5499, 0.5490, 0.5461, 0.5440, 0.5410, 0.5381, 0.5332 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.2348, 0.2276, 0.2267, 0.2256, 0.2193, 0.2192, 0.2182, 0.2173, \\ 0.2169, 0.2118, 0.2109, 0.2099, 0.2098, 0.2095, 0.2086, 0.2075, \\ 0.2034, 0.2023, 0.2014, 0.2011, 0.2010, 0.2000, 0.1990, 0.1938, \\ 0.1934, 0.1924, 0.1914, 0.1913, 0.1848, 0.1836, 0.1827, 0.1749 \end{array} \right\} \end{array} \right)$$

$$r^{(2)} = \left(\begin{array}{l} \left\{ \begin{array}{l} 0.5414, 0.5365, 0.5350, 0.5300, 0.5250, 0.5231, 0.5202, 0.5187, \\ 0.5183, 0.5168, 0.5140, 0.5121, 0.5095, 0.5072, 0.5048, 0.5034, \\ 0.5025, 0.5011, 0.4988, 0.4965, 0.4940, 0.4922, 0.4895, 0.4881, \\ 0.4877, 0.4863, 0.4836, 0.4818, 0.4773, 0.4729, 0.4716, 0.4672 \end{array} \right\}, \\ \left\{ \begin{array}{l} 0.3543, 0.3474, 0.3473, 0.3404, 0.3403, 0.3391, 0.3389, 0.3333, \\ 0.3321, 0.3320, 0.3318, 0.3317, 0.3250, 0.3249, 0.3247, 0.3246, \\ 0.3234, 0.3177, 0.3174, 0.3162, 0.3161, 0.3089, 0.3088, 0.3015 \end{array} \right\} \end{array} \right)$$

$$r^{(3)} = \left(\begin{array}{c} \left\{ \begin{array}{c} 0.4449, 0.4401, 0.4368, 0.4348, 0.4320, 0.4300, 0.4268, 0.4222, \\ 0.3873, 0.3831, 0.3802, 0.3785, 0.3761, 0.3744, 0.3716, 0.3675 \end{array} \right\}, \\ \left\{ \begin{array}{c} 0.4135, 0.4072, 0.4051, 0.4015, 0.3988, 0.3952, 0.3930, 0.3909, \\ 0.3866, 0.3844, 0.3823, 0.3786, 0.3757, 0.3719, 0.3697, 0.3630 \end{array} \right\} \end{array} \right)$$

and

$$r^{(4)} = \left(\begin{array}{l} \left\{ \begin{array}{c} 0.4675, 0.4533, 0.4507, 0.4488, 0.4399, 0.4377, 0.4370, 0.4351, \\ 0.4327, 0.4265, 0.4244, 0.4241, 0.4223, 0.4220, 0.4201, 0.4195, \\ 0.4118, 0.4113, 0.4095, 0.4092, 0.4074, 0.4071, 0.4051, 0.3993, \\ 0.3971, 0.3953, 0.3948, 0.3928, 0.3850, 0.3833, 0.3812, 0.3696 \end{array} \right\}, \\ \left\{ \begin{array}{c} 0.4498, 0.4453, 0.4329, 0.4284, \\ 0.4153, 0.4106, 0.3974, 0.3925 \end{array} \right\} \end{array} \right)$$

Step 4: The score values of the alternatives $\mathcal{A}^{(b)}(b = 1, 2, 3, 4)$ are $\mathcal{S}(r^{(1)}) = 0.3483$, $\mathcal{S}(r^{(2)}) = 0.1751$, $\mathcal{S}(r^{(3)}) = 0.0168$, and $\mathcal{S}(r^{(4)}) = -0.0052$.

Step 5: According to the score values $\mathcal{S}(r^{(b)})(b = 1, 2, 3, 4)$, we get the ranking order of the alternatives $\mathcal{A}^{(b)}(b = 1, 2, 3, 4)$ as $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$. Therefore, the most desirable alternative is also $\mathcal{A}^{(1)}$.

Here, if we take the generator $k(t) = \log\left(\frac{2-t}{t}\right)$ instead of $k(t) = -\log(t)$ and $\lambda = 1$ during the aggregation process then the aggregated values of each alternative are obtained by using DHFSWA operator as $\mathcal{S}(r^{(1)}) = 0.4731$, $\mathcal{S}(r^{(2)}) = 0.3020$, $\mathcal{S}(r^{(3)}) = 0.2566$ and $\mathcal{S}(r^{(4)}) = 0.1857$. Thus, based on the ranking order, the most damaged area is $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(4)}$ is the least one.

5.5.2 Validity test

Wang and Triantaphyllou [210] established the following testing criterions to evaluate the validity of MCDM methods.

Test criterion 1: "An effective MCDM method does not change the index of the best alternative by replacing a non optimal alternative with a worse alternative without shifting the corresponding importance of every decision attribute".

Test criterion 2: "To an effective MCDM method must be satisfy transitive property".

Test criterion 3: "If we decomposed a MCDM problem into the sub DM problems and same MCDM method is utilized on sub problems to rank alternatives, collective ranking of alternatives must be identical to ranking of un-decomposed DM problem".

Validity of the proposed approach is tested by using these criteria as follows:

Validity test by criterion 1

For testing the validity of proposed approach under the criterion 1, we replace the non optimal alternative $\mathcal{A}^{(4)}$ with the worse alternative $(\mathcal{A}^{(4)})'$ in original decision matrices with their rating values is summarized in Eq. (5.12).

$$(\mathcal{A}^{(4)})' = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad e_1 \quad & \quad\quad e_2 \quad\quad & \quad e_3 \quad \end{array} \\ \begin{pmatrix} (\{0.5,0.4\},\{0.4\}) & (\{0.1\},\{0.9,0.8\}) & (\{0.4,0.3\},\{0.5\}) \\ (\{0.3\},\{0.4,0.3\}) & (\{0.3,0.2\},\{0.7\}) & (\{0.5\},\{0.3\}) \\ (\{0.3,0.2\},\{0.6\}) & (\{0.4\},\{0.5,0.4\}) & (\{0.2,0.1\},\{0.7\}) \end{pmatrix} \end{array} \tag{5.12}$$

Now by applying the proposed approach by taking $\lambda = 1$ and $k(t) = -\log(t)$ to this modified data, we get the score values of each alternative are $\mathcal{S}(r^{(1)}) = 0.4880$, $\mathcal{S}(r^{(2)}) = 0.3196$, $\mathcal{S}(r^{(3)}) = 0.2850$ and $\mathcal{S}((r^{(4)})') = -0.1747$. Thus $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$ and hence the most desirable alternative remains $\mathcal{A}^{(1)}$ which validates the proposed approach satisfy *test criterion 1.*

Validity test by criteria 2 and 3

For evaluating the proposed MCDM approach under the criteria 2 and 3, we have decomposed original DM problem into three sub DM problems which contains the alternatives $\{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(4)}\}$, $\{\mathcal{A}^{(1)}, \mathcal{A}^{(3)}, \mathcal{A}^{(4)}\}$ and $\{\mathcal{A}^{(2)}, \mathcal{A}^{(3)}, \mathcal{A}^{(4)}\}$. If we apply the proposed MCDM approach on these sub problems then we get ranking order of alternatives as $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)}$, $\mathcal{A}^{(1)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$ and $\mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$ respectively. After combining together the ranking of the alternatives of these smaller problem, we get the final ranking order as $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$ which is same as un-decomposed problem and show transitive property. Hence, the proposed MCDM approach is valid under the criteria 2 and 3.

5.5.3 Sensitivity Analysis

In order to analyze the impact of the parameter λ on the ranking, an investigation has been conducted by taking a different value of λ for the considered problem. For it, Step 3 of the proposed approach has been executed for different values of λ by using aggregation operators GDHFSWA and GDHFSWG corresponding to different generators of $k(t)$. The overall score values and the ranking order of the alternatives are summarized in Table 5.2. From this tabulated value, we observe that as the value of the λ increases, then the score value corresponding to GDHFSWA operator increases while the score values corresponding to GDHFSWG operator decreases. Therefore, our proposed approach is more flexible as the decision-maker(s) can choose the parameters and the operators according to their preferences and practical situations. For instance, if the decision maker's attitude towards the alternative is optimistic in nature and they prefer GDHFSWA operator then he or she can prefer the larger value of λ. On the other hand, if decision maker can take a decision according to the pessimistic nature towards the alternatives, then he or she can choose the lower value of the parameter λ during the aggregation process. Besides this, if a person utilizes GDHFSWG operator and shows an optimistic in nature towards the alternatives, then he or she can assign the lower valve of λ, while for pessimistic nature, they can choose the larger value of λ. Also, it is observed that the best alternative remains the same, which influenced that the results are objective and cannot be changed by decision makes' preference of pessimism and optimism. Thus, the ranking results are reliable. From this analysis, the decision maker can see their goals and hence select the best one according to their attitude.

5.5.4 Comparative Studies

In order to compare the performance of the proposed algorithm with some of the existing metrics under the dual hesitant fuzzy set environment, an analysis has been conducted with some existing approaches [95, 176, 201, 248, 250, 251, 253]. For it, firstly the different parameters of the alternatives are aggregated by using aggregation operator proposed by Yu et al. [253]. The aggregated values of each alternative are summarized in Table 5.3. By using this matrix, the various existing approaches [95, 176, 201, 248, 250, 251, 253] have applied on it to find the most desirable alternative. The results corresponding to these approaches are summarized in Table 5.4. From this analysis, it is noted that the final ranking order of the alternatives is similar to that of the proposed approach. But simultaneously it is noted that the computational procedure of the proposed approach is completely different from these existing approaches. For instance, in [95, 201, 250, 251, 253], weighted averaging and the geometric aggregation operators were proposed by the authors under the dual hesitant fuzzy set environment without considering the parameterizations factor during the analysis. Therefore, there is an information loss during an aggregation process under the existing environment.

On the other hand, in [176, 248] authors have used the correlation coefficient as well as distance and similarity measures to solve the DM problems. In these approaches, the measurement values are computed from the ideal alternatives, which is usually does not exist in our real-life problems and hence results obtained are sometimes inconsistent. Therefore, the proposed technique can suitably utilize to solve the DM problem than the other existing measures.

In the end, it is concluded that the proposed aggregation operators have considered the preferences of the decision makers' risk attitude λ during the aggregation process, which provides more choices to the decision makers to avail their desirable alternatives. Based on these parameters, the decision maker can choose their desire operator to solve their DM problems. Also, the corresponding studies under the dual hesitant fuzzy set environments are considered as a special case of the proposed operators by taking the number of the parameters to be one.

5.5.5 Advantages of the proposed approach

From the existing studies and the proposed operators, we address the following merits of the proposed method to solve the DM problem under the DHFSS environment.

(i) DHFSS is a generalization of the existing studies such as HFSS [11], FSS[130], DHFS [267], HFS [194, 195] and IFS by considering much more information related to an object during the process. For instance, DHFSS contains information (both the membership and non-membership degrees) with different parameter factors under hesitant environment than the HFSS (contains only hesitant membership degree with parameter values), FSS (with a crisp membership degree), DHFS (both membership and non-membership degrees without parameter factor), HFS (with a hesitant membership degree only without parameter factor) and IFSs (with crisp membership and non-membership degrees). Thus, the proposed aggregation operators under DHFSSs environment is more generalizations than the existing ones [54, 59, 65, 95, 113, 114, 192, 201, 250, 251, 253].

(ii) It is revealed from the present study that the aggregation operators under DHFSs [201, 250, 251, 253], HFSs [113, 114], IFSs [54, 59, 192], are the special cases of the proposed aggregation operators. Thus, we conclude that the proposed aggregation operators can equivalently be utilized to solve all the DM problems under DHFS, HFS and IFS environments while the methods under the existing approaches are unable to solve the problems under the environment considered in the present work.

(iii) With respect to the DM problems, the major advantages of the proposed approach are to consider the much more information more precisely and objectively related to the object to reduce the information loss. Further, the risk attitude parameter λ give various choices to the decision makers based on their optimists and pessimists behavior towards the aggregation process.

5.6 Conclusion

This chapter addressed the MCDM problems in which alternatives are assessed under the different parameters and experts in the form of dual hesitant fuzzy soft numbers. To do so, we first present the concept of the operational and comparison laws of DHFSNs. Then, based on the operational laws of DHFSNs, we developed some generalized weighted averaging and geometric aggregation operators, namely, DHFSWA, DHFSWG, GDHFSWA, and GDHFSWG for aggregating the different dual hesitant fuzzy soft information. The various desirable characteristics of these operators are investigated in details. Based on these operators, an approach for solving the DM problems has been presented by taking different values of λ which makes the proposed operators more flexible and offers the various choices to the decision-maker for assessing the decisions. Based on the decision maker's attitude towards the aggregation, he or she may choose a suitable value of λ and change their decision accordingly. Finally, a practical example is given to illustrate the application of the proposed method and to demonstrate its feasibility. A comparative study with some existing approaches shows that the proposed operators provide a more practical nature to the decision-maker during the aggregation process. From the study, it is concluded that existing studies [54, 59, 65, 95, 113, 114, 192, 201, 250, 251, 253] are taken as a special case of the proposed DHFSSs theory. Thus, we conclude that the present study is more generalized one to solve the MCDM problems.

Table 5.1: Rating values for the alternatives in terms of DHFSNs

	For alternative $\mathcal{A}^{(1)}$			For alternative $\mathcal{A}^{(1)}$		
	e_1	e_2	e_3	e_1	e_2	e_3
u_1	$(\{0.7, 0.6\}, \{0.2\})$	$(\{0.8, 0.7\}, \{0.1\})$	$(\{0.2\}, \{0.4, 0.3\})$	$(\{0.7, 0.6\}, \{0.2, 0.1\})$	$(\{0.5\}, \{0.4, 0.3\})$	$(\{0.3, 0.2\}, \{0.6\})$
u_2	$(\{0.9\}, \{0.1\})$	$(\{0.7\}, \{0.3, 0.2\})$	$(\{0.6\}, \{0.2, 0.1\})$	$(\{0.4, 0.3\}, \{0.5\})$	$(\{0.8\}, \{0.2, 0.1\})$	$(\{0.8, 0.7\}, \{0.2, 0.1\})$
u_3	$(\{0.6, 0.5\}, \{0.2, 0.1\})$	$(\{0.4\}, \{0.5, 0.3\})$	$(\{0.7, 0.6\}, \{0.1\})$	$(\{0.5\}, \{0.3, 0.2\})$	$(\{0.7, 0.6\}, \{0.2\})$	$(\{0.6\}, \{0.2\})$
	For alternative $\mathcal{A}^{(3)}$			For alternative $\mathcal{A}^{(4)}$		
	e_1	e_2	e_3	e_1	e_2	e_3
u_1	$(\{0.2, 0.1\}, \{0.7\})$	$(\{0.7, 0.6\}, \{0.3, 0.1\})$	$(\{0.7\}, \{0.1\})$	$(\{0.7, 0.6\}, \{0.2\})$	$(\{0.2\}, \{0.8, 0.7\})$	$(\{0.6, 0.4\}, \{0.2\})$
u_2	$(\{0.7, 0.6\}, \{0.2, 0.1\})$	$(\{0.4\}, \{0.5\})$	$(\{0.6\}, \{0.2, 0.1\})$	$(\{0.6\}, \{0.3, 0.1\})$	$(\{0.3, 0.2\}, \{0.7\})$	$(\{0.7\}, \{0.1\})$
u_3	$(\{0.6\}, \{0.3, 0.1\})$	$(\{0.1\}, \{0.7\})$	$(\{0.6, 0.5\}, \{0.2\})$	$(\{0.5, 0.3\}, \{0.4\})$	$(\{0.5\}, \{0.3, 0.2\})$	$(\{0.3, 0.1\}, \{0.6\})$

Table 5.2: Ranking of alternatives at different values of λ

		When $k(t) = -\log(t)$					When $k(t) = \log(\frac{2-t}{t})$				
		$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	Ranking order	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	Ranking order
$\lambda \to 0$	GDHFSWA	0.4649	0.2924	0.2316	0.1658	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.4724	0.3005	0.2435	0.1763	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
	GDHFSWG	0.4028	0.2281	0.1078	0.0662	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.3977	0.2227	0.0941	0.0551	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
$\lambda = 0.5$	GDHFSWA	0.4766	0.3060	0.2592	0.1889	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.4683	0.2960	0.2380	0.1709	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
	GDHFSWG	0.3770	0.2029	0.0637	0.0317	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.3984	0.2241	0.0993	0.0600	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
$\lambda = 1$	GDHFSWA	0.4880	0.3196	0.2850	0.2115	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.4731	0.3020	0.2566	0.1857	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
	GDHFSWG	0.3483	0.1751	0.0168	-0.0052	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.3720	0.1993	0.0624	0.0320	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
$\lambda = 1.5$	GDHFSWA	0.4988	0.3332	0.3080	0.2328	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.4838	0.3163	0.2866	0.2132	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
	GDHFSWG	0.3179	0.1461	-0.0295	-0.0424	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.3306	0.1585	0.0020	-0.0167	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
$\lambda = 2$	GDHFSWA	0.5089	0.3463	0.3282	0.2525	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.4969	0.3351	0.3171	0.2438	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
	GDHFSWG	0.2871	0.1173	-0.0728	-0.0781	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$	0.2849	0.1134	-0.0629	-0.0717	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$

Table 5.3: Aggregated dual hesitant fuzzy decision matrix

	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$
u_1	$(\{0.6435, 0.5974, 0.6000, 0.5483\}, \{0.2000, 0.1835\})$	$(\{0.5491, 0.5307, 0.4941, 0.4734\}, \{0.3424, 0.3140, 0.2595, 0.2380\})$
u_2	$(\{0.7893\}, \{0.1712, 0.1516, 0.1390, 0.1231\})$	$(\{0.6896, 0.6495, 0.6699, 0.6272\}, \{0.2885, 0.2344, 0.2344, 0.1904\})$
u_3	$(\{0.5856, 0.5483, 0.5469, 0.5061\}, \{0.2138, 0.1835, 0.1621, 0.1390\})$	$(\{0.5988, 0.5627\}, \{0.3104, 0.2521\})$
	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$
u_1	$(\{0.5559, 0.5158, 0.5344, 0.4925\}, \{0.3028, 0.2178\})$	$(\{0.5611, 0.5043, 0.5075, 0.4438\}, \{0.3031, 0.2912\})$
u_2	$(\{0.5974, 0.5483\}, \{0.2633, 0.2138, 0.1995, 0.1621\})$	$(\{0.5660, 0.5483\}, \{0.2782, 0.1793\})$
u_3	$(\{0.4898, 0.4545\}, \{0.3425, 0.2207\})$	$(\{0.4469, 0.4036, 0.3672, 0.3177\}, \{0.4144, 0.3669\})$

Table 5.4: Comparison with existing approaches under DHFS environment

Author	Overall Values of alternative				Ranking
	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
Ju et al. [95]	0.4663	0.2955	0.2777	0.1959	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
Yu [250]	0.4626	0.2935	0.2764	0.1928	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
Wang, Zhao and Wei [201]	0.4880	0.3063	0.2850	0.2115	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
Ye [248]	0.9538	0.8952	0.8985	0.8478	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)}$
Singh [176]	0.0883	0.1170	0.1201	0.1332	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
Yu [251]	0.4846	0.3044	0.2839	0.2090	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$
Yu et al. [253]	0.4880	0.3063	0.2850	0.2115	$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)}$

Chapter 6

TOPSIS-based nonlinear - programming methodology for decision-making problems with interval-valued intuitionistic fuzzy soft sets[1]

In this chapter, an attempt has been made to present a nonlinear-programming (NLP) model based on the technique for order preference by similarity to ideal solution (TOPSIS), to solve decision-making problems. In this approach, both ratings of alternatives on attributes and weights of attributes are represented by IVIF sets. Based on the available information, NLP models are constructed on the basis of the concepts of the relative-closeness coefficient (RCC) and the weighted distance. A real example is taken to demonstrate the applicability and validity of the proposed methodology.

6.1 Introduction

The extensive literature related to the DM approach under the different environment by using TOPSIS approaches are summarized in Section 1.1.2 of Chapter 1. However, there is less literature available on how to solve the MADM problems in which the rating of

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alternatives on attributes are expressed using IF or IVIF sets and preference information on attributes is incomplete. In that direction, Li [105] constructed an NP model based on TOPSIS method to solve the DM problem under IVIFS environment. Li [106] also presented a closeness coefficient based NP method to solve the DM problems with incomplete preference information. Park et al. [150] established an optimization model to determine the weights of attributes and hence the relative closeness coefficient of the TOPSIS method based on the distance measures has been utilized to rank the alternatives. Zhang and Yu [259] presented an optimization model based on the cross-entropy to determine attribute weights for MADM problems with incomplete weight information of criteria under IVIFS environment. Li and Wan [107] presented the possibility linear programming method to solve MADM problems with triangular fuzzy numbers having multiple types of attributes and incomplete weight information. Chen [30] proposed the two optimization models to determine the criterion weights for addressing the situations in which the preference information is completely unknown or incompletely known under the IVIFS environment.

From these studies, it has been concluded that the work conducted under IFS, IFSS and IVIFSS theories by the researchers are based on the assumptions that both the ratings of alternatives on attributes and weights of an attribute are represented by the known crisp numbers. However, the information about the attribute weights provided by the decision-maker is usually incomplete. So, an interesting and important issue is how to utilize the collective IVIF soft decision matrix and the unknown preferences information to find the most desirable alternative(s) during the DM process.

In light of this, in this chapter, we have presented a concept of new nonlinear programming (NLP) method for solving the DM problems with the concept of the closeness coefficient. In it, the ratings of alternatives on attributes are expressed using IVIF sets and the preference information on attributes is incomplete under IVIFSS environment. Also, in the approach, the information extracted by the alternatives on attributes and the preference for attributes is expressed in the form of IVIF soft numbers. A weighted distance between an alternative and the IVIF soft positive ideal solution (IVIFSPIS) as well as the IVIF soft negative ideal solution (IVIFSNIS) is used to define the concept of closeness coefficients, which are continuous and monotonic functions with respect to the MDs and NMDs of the alternatives on attributes and preference on attributes. A pair of auxiliary nonlinear fractional programming model is constructed to calculate these coefficients' intervals of alternatives with respect to the IVIFSPIS which can be used to generate the ranking order of the alternatives.

6.2 Non-Linear Programming method under IVIFSS environment

In this section, an approach has been presented for solving MADM problem by using the nonlinear programming model under IVIFSS environment.

6.2.1 Description of MADM problem under IVIFSSs

The description of the MADM problem is given in Section 2.6 of Chapter 2 under the IVIFSS environment, i.e., the overall representation of these rating values can be framed into the interval-valued DM matrix as $\mathcal{M}= \left(\left[(\zeta_{ij}^{(k)})^L, (\zeta_{ij}^{(k)})^U\right],\ \left[(\vartheta_{ij}^{(k)})^L, (\vartheta_{ij}^{(k)})^U\right]\right)$, $(i = 1,2,\ldots,n; j = 1,2,\ldots,m; k = 1,2,\ldots,l)$. Also, it is quite noticed that different criteria always plays an important role at different priority levels during the evaluations of the alternatives in our real-life problems. To address it, the weight vector for each expert $u_i (i = 1,2,\ldots,n)$ is expressed as an interval set $W_i = \{(u_i, [\omega_i^L, \omega_i^U],\ [\rho_i^L, \rho_i^U],\ |\ u_i \in \mathcal{X}\}$, which is usually denoted by $W_i = ([\omega_i^L, \omega_i^U], [\rho_i^L, \rho_i^U])$ for short, where $0 \leq \omega_i^L \leq \omega_i^U \leq 1$ and $0 \leq \rho_i^L \leq \rho_i^U \leq 1$ and $\omega_i^U + \rho_i^U \leq 1$. Similarly, the weight vector for each parameter e_j is denoted by $\chi_j = ([\xi_j^L, \xi_j^U], [\eta_j^L, \eta_j^U])$ where $0 \leq \xi_j^L \leq \xi_j^U \leq 1$, $0 \leq \eta_j^L \leq \eta_j^U \leq 1$ and $\xi_j^U + \eta_j^U \leq 1$. Therefore, the weight vector of all the experts and the parameters can be concisely expressed as $W = (([\omega_i^L, \omega_i^U], [\rho_i^L, \rho_i^U]))_{n\times 1}$ and $\chi = (([\xi_j^L, \xi_j^U], [\eta_j^L, \eta_j^U]))_{m\times 1}$ respectively.

6.2.2 Determination of ideal solutions

As the matrix $\mathcal{M}$ is an IVIFSNs where $0 \leq (\zeta_{ij}^{(k)})^L \leq (\zeta_{ij}^{(k)})^U \leq 1$ and $0 \leq (\vartheta_{ij}^{(k)})^L \leq (\vartheta_{ij}^{(k)})^U \leq 1$, so accordingly the two ideals namely, IVIF soft positive and negative ideal solutions, denoted by IVIFSPIS ($\mathcal{A}^+$) and IVIFSNIS ($\mathcal{A}^-$) respectively may be taken as $\mathcal{A}^+ = \{(\mathcal{A}_{ij}^+, [1,1], [0,0])\}$ and $\mathcal{A}^- = \{(\mathcal{A}_{ij}^+, [0,0], [1,1])\}$ which are written in short as $\mathcal{A}^+ = ([1,1], [0,0])$ and $\mathcal{A}^- = ([0,0], [1,1])$ respectively. It has been seen from them that $\mathcal{A}^+$ and $\mathcal{A}^-$ are complement to each other.

However, apart from fixed values, these reference points $\mathcal{A}^+$ and $\mathcal{A}^-$ can also be defined as $\mathcal{A}^+ = (([g_{ij}^+, g_{ij}^+],\ [h_{ij}^+,\ h_{ij}^+]))_{n\times m}$ and $\mathcal{A}^- = (([g_{ij}^-, g_{ij}^-], [h_{ij}^-, h_{ij}^-]))_{n\times m}$, where $g_{ij}^+ = \max\{\zeta_{ij}^U \mid i = 1,2,\dots,n; j = 1,2,\dots,m\}$, $h_{ij}^+ = \min\{\vartheta_{ij}^L \mid i = 1,2,\dots,n; j = 1,2,\dots,m\}$, $g_{ij}^- = \min\{\zeta_{ij}^L \mid i = 1,2,\dots,n; j = 1,2,\dots,m\}$ and $h_{ij}^- = \max\{\vartheta_{ij}^U \mid i = 1,2,\dots,n; j = 1,2,\dots,m\}$. Since $g_{ij}^- \le \zeta_{ij}^L \le \zeta_{ij}^U \le g_{ij}^+$ and $h_{ij}^- \le \vartheta_{ij}^L \le \vartheta_{ij}^U \le h_{ij}^+$, therefore, for all i, j, $([g_{ij}^-, g_{ij}^-], [h_{ij}^-, h_{ij}^-]) \subseteq ([\zeta_{ij}^L, \zeta_{ij}^U], [\vartheta_{ij}^L, \vartheta_{ij}^U] \subseteq ([g_{ij}^+, g_{ij}^+], [h_{ij}^+, h_{ij}^+])$.

6.2.3 Separation measurement values

The weighted Hamming distance measure for IVIFSS are taken to compute measurement values between the alternatives $\mathcal{A}^{(k)}(k = 1,2,\dots,l)$ from its reference values $\mathcal{A}^+ = ([1,1],[0,0])$ and $\mathcal{A}^- = ([0,0],[1,1])$. For it, assume that for any $\zeta_{ij}^{(k)} \in [(\zeta_{ij}^{(k)})^L, (\zeta_{ij}^{(k)})^U]$, $\vartheta_{ij}^{(k)} \in [(\vartheta_{ij}^{(k)})^L, (\vartheta_{ij}^{(k)})^U]$, $\omega_i \in [\omega_i^L, \omega_i^U]$, $\rho_i \in [\rho_i^L, \rho_i^U]$, $\xi_j \in [\xi_j^L, \xi_j^U]$ and $\eta_j \in [\eta_j^L, \eta_j^U]$, the separation measure of $\mathcal{A}^{(k)}$ from $\mathcal{A}^+$ and $\mathcal{A}^-$ are defined as

$$\mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^+) = \sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \omega_i (1 - \zeta_{ij}^{(k)}) \right) + \sum_{j=1}^{m} \eta_j \left(\sum_{i=1}^{n} \rho_i \vartheta_{ij}^{(k)} \right). \tag{6.1}$$

and

$$\mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^-) = \sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \omega_i \zeta_{ij}^{(k)} \right) + \sum_{j=1}^{m} \eta_j \left(\sum_{i=1}^{n} \rho_i (1 - \vartheta_{ij}^{(k)}) \right). \tag{6.2}$$

However, if an expert utilize $\mathcal{A}^+ = ([g_{ij}^+, g_{ij}^+],\ [h_{ij}^+, h_{ij}^+])_{1\times n}$, and $\mathcal{A}^- = ([g_{ij}^-, g_{ij}^-], [h_{ij}^-, h_{ij}^-])_{1\times n}$ then the separation measures between them are defined as

$$\begin{aligned} \mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^+) &= \sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \omega_i (g_{ij}^+ - \zeta_{ij}^{(k)}) \right) + \sum_{j=1}^{m} \eta_j \left(\sum_{i=1}^{n} \rho_i (\vartheta_{ij}^{(k)} - h_{ij}^+) \right) & (6.3) \\ &\text{and} \\ \mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^-) &= \sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \omega_i (\zeta_{ij}^{(k)} - g_{ij}^-) \right) + \sum_{j=1}^{m} \eta_j \left(\sum_{i=1}^{n} \rho_i (h_{ij}^- - \vartheta_{ij}^{(k)}) \right) & (6.4) \end{aligned}$$

6.2.4 Relative-closeness coefficient and its monotonicity

To measure the relative strength of the alternatives $\mathcal{A}^{(k)}, (k = 1, 2, \ldots, l)$ with respect to $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$, we define the RCC of the alternatives by Eq. (6.5) as

$$\mathfrak{R}^{(k)}\begin{pmatrix}(\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, \\ (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1}\end{pmatrix} = \frac{\mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{-})}{\mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{-}) + \mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{+})}; \tag{6.5}$$

provided $\mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{+}) \neq 0$ for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$. Here, $(\zeta_{ij}^{(k)})_{n\times m}$ and $(\vartheta_{ij}^{(k)})_{n\times m}$ are $n \times m$ matrices and $(\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}$ be the column weight vector of $n-$ dimension corresponding to the membership and non-membership degrees of an expert u_i while $(\xi_j)_{m\times 1}, (\eta_j)_{m\times 1}$ are the column vectors of $m-$ dimension corresponding to membership and non-membership degrees of a parameter e_j. Further, it has been seen that $0 \leq \mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{-}) \leq \mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{-}) + \mathcal{D}(\mathcal{A}^{(k)}, \mathcal{A}^{+})$ and hence $0 \leq \mathfrak{R}^{(k)}\big((\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1}\big) \leq 1$. Also from the Eq. (6.5), it is clearly seen that $\mathfrak{R}^{(k)}$ is a continuous function of $2[(1+m)(1+n)-1]$ variables, which rises sharply even for a small value of m and n. For instance, if $m = 2$ and $n = 3$ then the unknown variables are 22 whereas if $m = 5$ and $n = 6$ then the unknown variables are 82. Therefore, it is a tedious task to invest effort leading to large computational time and excess cost. So, there is a need for construction of a time-efficient algorithm to resolve this problem so that the number of unknowns gets reduced.

To do so, firstly we check the monotonicity and boundedness of the function $\mathfrak{R}^{(k)}$ with respect to the unknown variables $\zeta_{ij}^{(k)} \in [(\zeta_{ij}^{(k)})^{L}, (\zeta_{ij}^{(k)})^{U}]$ and $\vartheta_{ij}^{(k)} \in [(\vartheta_{ij}^{(k)})^{L}, (\vartheta_{ij}^{(k)})^{U}]$, respectively. For it, by taking the measures given in Eqs. (6.1) and (6.2), we explicit the expression of $\mathfrak{R}^{(k)}$ given in Eq. (6.5) as

$$\begin{aligned}\mathfrak{R}^{(k)} &= \frac{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i\zeta_{ij}^{(k)}\right) + \sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i(1-\vartheta_{ij}^{(k)})\right)}{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i\zeta_{ij}^{(k)}\right) + \sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i(1-\vartheta_{ij}^{(k)})\right) + \sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i(1-\zeta_{ij}^{(k)})\right) + \sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i\vartheta_{ij}^{(k)}\right)} \\ &= \frac{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i\zeta_{ij}^{(k)}\right) + \sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i(1-\vartheta_{ij}^{(k)})\right)}{\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\left(\xi_j\omega_i + \eta_j\rho_i\right)}\end{aligned} \tag{6.6}$$

To address the monotonic behavior of $\mathfrak{R}^{(k)} (k = 1, 2, \ldots, l)$, differentiate it partially with respect variable $\zeta_{ij}^{(k)}$ and get

$$\frac{\partial \mathfrak{R}^{(k)}}{\partial \zeta_{ij}^{(k)}} = \frac{\sum\limits_{j=1}^{m} \xi_j \sum\limits_{i=1}^{n} \omega_i}{\sum\limits_{j=1}^{m} \sum\limits_{i=1}^{n} \left(\xi_j \omega_i + \eta_j \rho_i\right)}$$

Since $\omega_i, \rho_i, \xi_j, \eta_j \in [0,1]$, it follows that $\frac{\partial \mathfrak{R}^{(k)}}{\partial \zeta_{ij}^{(k)}} \geq 0$. For $\xi_j, \omega_i \neq 0$, $\frac{\partial \mathfrak{R}^{(k)}}{\partial \zeta_{ij}^{(k)}} > 0$. Therefore, $\mathfrak{R}^{(k)}(k = 1, 2, \ldots, l)$ are monotonic and non-decreasing functions of the variables $\zeta_{ij}^{(k)}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; k = 1, 2 \ldots, l)$.

Similarly, partial derivatives of $\mathfrak{R}^{(k)}$ with respect to the variables $\vartheta_{ij}^{(k)}$ are computed as follows:

$$\frac{\partial \mathfrak{R}^{(k)}}{\partial \vartheta_{ij}^{(k)}} = -\frac{\sum\limits_{j=1}^{m} \eta_j \sum\limits_{i=1}^{n} \rho_i}{\sum\limits_{j=1}^{m} \sum\limits_{i=1}^{n} \left(\xi_j \omega_i + \eta_j \rho_i\right)} \leq 0$$

For $\rho_i, \eta_j \neq 0$, $\frac{\partial \mathfrak{R}^{(k)}}{\partial \vartheta_{ij}^{(k)}} < 0$, therefore, $\mathfrak{R}^{(k)}$ are monotonic and non-increasing functions of the variables $\vartheta_{ij}^{(k)}(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$.

Since, $\zeta_{ij}, \vartheta_{ij}, \omega_i, \rho_i, \xi_j, \eta_j$ are all closed and bounded elements of the subintervals of the interval $[0,1]$ then the continuous functions $\mathfrak{R}^{(k)}$ are bounded, that is, the value of $\mathfrak{R}^{(k)}$ lies in the range $[0,1]$ which is denoted by $[(\mathfrak{R}^{(k)})^L, (\mathfrak{R}^{(k)})^U]$ respectively. Therefore, by the basic definition as well as the continuous nature of the function $\mathfrak{R}^{(k)}$, we have $0 \leq (\mathfrak{R}^{(k)})^L \leq (\mathfrak{R}^{(k)})^U \leq 1$ for $\zeta_{ij}^{(k)} \in [(\zeta_{ij}^{(k)})^L, (\zeta_{ij}^{(k)})^U]$ and so on. Therefore, the RCC of an alternative $\mathcal{A}^{(k)}$ is an interval-valued set $[(\mathfrak{R}^{(k)})^L, (\mathfrak{R}^{(k)})^U]$ and lies in the interval $[0,1]$. Further, it can be seen that $(\mathfrak{R}^{(k)})^L + (1 - (\mathfrak{R}^{(k)})^U) = 1 + ((\mathfrak{R}^{(k)})^L - (\mathfrak{R}^{(k)})^U) \leq 1$. Hence, $(\mathfrak{R}^{(k)})^L$ and $(\mathfrak{R}^{(k)})^U$ satisfy the following conditions; $0 \leq (\mathfrak{R}^{(k)})^L \leq 1$; $0 \leq 1 - (\mathfrak{R}^{(k)})^U \leq 1$ and $(\mathfrak{R}^{(k)})^L + (1 - (\mathfrak{R}^{(k)})^U \leq 1$. Thus, the set $[(\mathfrak{R}^{(k)})^L, (\mathfrak{R}^{(k)})^U]$ may equivalently be expressed as intuitionistic fuzzy set $\mathfrak{R}^{(k)} = ((\mathfrak{R}^{(k)})^L, 1 - (\mathfrak{R}^{(k)})^U)$ which means that the degree of the closeness and non-closeness of the alternatives $\mathcal{A}^{(k)} \in A$ to the IVIFSPIS $\mathcal{A}^+$ are $(\mathfrak{R}^{(k)})^L$ and $1 - (\mathfrak{R}^{(k)})^U$, respectively. Hence, the set $\mathfrak{R}^{(k)} = [(\mathfrak{R}^{(k)})^L, (\mathfrak{R}^{(k)})^U]$ can be used to determine the ranking of the alternatives $\mathcal{A}^{(k)}(k = 1, 2, \ldots, l)$.

6.2.5 Auxiliary NLP models

Since $(\mathfrak{R}^{(k)})^L$ and $(\mathfrak{R}^{(k)})^U$ are the lower and upper bounds of $\mathfrak{R}^{(k)}$, respectively, which are bounded and continuous functions of the variables $\zeta_{ij}, \vartheta_{ij}, \omega_i, \rho_i, \xi_j$ and η_j $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; k = 1, 2, \ldots, l)$. Now, by combining with Eq. (6.6), $(\mathfrak{R}^{(k)})^L$ and $(\mathfrak{R}^{(k)})^U$ can be captured by solving the NLP models, constructed as follows:

$$(\mathfrak{R}^{(k)})^L = \min \Big\{ \mathfrak{R}^{(k)} \big((\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1} \big) \Big\} \quad (6.7)$$

and

$$(\mathfrak{R}^{(k)})^U = \max \Big\{ \mathfrak{R}^{(k)} \big((\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1} \big) \Big\} \quad (6.8)$$

subject to

$$\Big((\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1} \Big) \\ \in \Omega_\mu \times \Omega_\nu \times \Omega_\omega \times \Omega_\rho \times \Omega_\xi \times \Omega_\eta$$

Here, $\Omega_\mu = \{(\zeta_{ij}^{(k)})_{n\times m} \mid (\zeta_{ij}^{(k)})^L \leq \zeta_{ij}^{(k)} \leq (\zeta_{ij}^{(k)})^U\}$, $\Omega_\nu = \{(\vartheta_{ij}^{(k)})_{n\times m} \mid (\vartheta_{ij}^{(k)})^L \leq \vartheta_{ij}^{(k)} \leq (\vartheta_{ij}^{(k)})^U\}$ represents all the possible values of the membership and non-membership of k alternatives on n experts and m parameters, respectively. $\Omega_\omega = \{(\omega_i)_{n\times 1} \mid \omega_i^L \leq \omega_i \leq \omega_i^U (i = 1, 2, \ldots, n)\}$, $\Omega_\rho = \{(\rho_i)_{n\times 1} \mid \rho_i^L \leq \rho_i \leq \rho_i^U (i = 1, 2, \ldots, n)\}$, respectively, represent all the possible values of the importance or unimportance of the n experts while $\Omega_\xi = \{(\xi_j)_{m\times 1} \mid \xi_j^L \leq \xi_j \leq \xi_j^U (j = 1, 2, \ldots, m)\}$ and $\Omega_\eta = \{(\eta_j)_{m\times 1} \mid \eta_j^L \leq \eta_j \leq \eta_j^U (j = 1, 2, \ldots, m)\}$ respectively represents all the possible values of the importance or unimportance of the m parameters. $\Omega_\mu \times \Omega_\nu \times \Omega_\omega \times \Omega_\rho \times \Omega_\xi \times \Omega_\eta$ is the cartesian product of all the sets $\Omega_\mu, \Omega_\nu, \Omega_\omega, \Omega_\rho, \Omega_\xi, \Omega_\eta$.

As stated above, $\mathfrak{R}^{(k)}$ is a monotonic and continuous functions so it reaches its minimum at the lower bounds of the interval $[(\zeta_{ij}^{(k)})^L, (\zeta_{ij}^{(k)})^U]$ and the upper bounds of the interval $[(\vartheta_{ij}^{(k)})^L, (\vartheta_{ij}^{(k)})^U]$ respectively. Therefore, model (6.7) can be further reduce to

$$(\mathfrak{R}^{(k)})^L = \min \left\{ \frac{\sum_{j=1}^{m} \xi_j \big(\sum_{i=1}^{n} \omega_i (\zeta_{ij}^{(k)})^L \big) + \sum_{j=1}^{m} \eta_j \big(\sum_{i=1}^{n} \rho_i (1 - (\vartheta_{ij}^{(k)})^U) \big)}{\sum_{j=1}^{m} \sum_{i=1}^{n} \big(\xi_j \omega_i + \eta_j \rho_i \big)} \right\} \quad (6.9)$$

$$\text{s.t.} \quad \begin{cases} \omega_i^L \le \omega_i \le \omega_i^U; \quad i = 1, 2, \ldots, n \\ \rho_i^L \le \rho_i \le \rho_i^U; \quad i = 1, 2, \ldots, n \\ \xi_j^L \le \xi_j \le \xi_j^U; \quad j = 1, 2, \ldots, m \\ \eta_j^L \le \eta_j \le \eta_j^U; \quad j = 1, 2, \ldots, m \end{cases} \tag{6.10}$$

Obviously, the model (6.9) is a nonlinear fractional programming model and it contains $2(n+m)$ unknown variables, which is less than that of unknown variables in the model (6.7), and hence it can be solved easily as compared to the model (6.7). Similarly, $(\mathfrak{R}^{(k)})^U$ given in the model (6.8) attains its maximum at the upper bounds $(\zeta_{ij}^{(k)})^U$ of the interval $[(\zeta_{ij}^{(k)})^L, (\zeta_{ij}^{(k)})^U]$ and the lower bounds $(\vartheta_{ij}^{(k)})^L$ of the interval $[(\vartheta_{ij}^{(k)})^L, (\vartheta_{ij}^{(k)})^U]$. Thus, model (6.8) can be further reduced to

$$(\mathfrak{R}^{(k)})^U = \max \left\{ \frac{\sum\limits_{j=1}^{m} \xi_j \left(\sum\limits_{i=1}^{n} \omega_i (\zeta_{ij}^{(k)})^U \right) + \sum\limits_{j=1}^{m} \eta_j \left(\sum\limits_{i=1}^{n} \rho_i (1 - (\vartheta_{ij}^{(k)})^L) \right)}{\sum\limits_{j=1}^{m} \sum\limits_{i=1}^{n} \left(\xi_j \omega_i + \eta_j \rho_i \right)} \right\} \tag{6.11}$$

which consists of $2(n+m)$ unknown variables subject to constraints given in Eq. (6.10). Also, this model has lesser number of unknown variables than the model (6.8).

Now, these models (6.9) and (6.11) can be simplified as linear fractional programming models based on Charnes and Cooper's transformations [71]. For it, let $z = \frac{1}{\sum\limits_{j=1}^{m} \sum\limits_{i=1}^{n} \left(\xi_j \omega_i + \eta_j \rho_i \right)}$, $t_{ij} = z \xi_j \omega_i$ and $y_{ij} = z \eta_j \rho_i$ and hence the above nonlinear fractional programming model is transformed into the equivalent linear programming models as follows.

$$(\mathfrak{R}^{(k)})^L = \min \left\{ \sum_{j=1}^{m} \sum_{i=1}^{n} t_{ij} (\zeta_{ij}^{(k)})^L + y_{ij} (1 - (\vartheta_{ij}^{(k)})^U) \right\}$$

$$s.t. \quad \begin{cases} z \xi_j^L \omega_i^L \le t_{ij} \le z \xi_j^U \omega_i^U; \\ z \eta_j^L \rho_i^L \le y_{ij} \le z \eta_j^U \rho_i^U; \\ \sum\limits_{j=1}^{m} \sum\limits_{i=1}^{n} (t_{ij} + y_{ij}) = 1 \\ z \ge 0 \end{cases} \tag{6.12}$$

and

$$(\mathfrak{R}^{(k)})^U = \max\left\{\sum_{j=1}^{m}\sum_{i=1}^{n} t_{ij}(\zeta_{ij}^{(k)})^U + y_{ij}(1-(\vartheta_{ij}^{(k)})^L)\right\}$$

$$s.t. \quad \begin{cases} z\xi_j^L\omega_i^L \leq t_{ij} \leq z\xi_j^U\omega_i^U; \\ z\eta_j^L\rho_i^L \leq y_{ij} \leq z\eta_j^U\rho_i^U; \\ \sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}(t_{ij}+y_{ij}) = 1 \\ z \geq 0 \end{cases} \tag{6.13}$$

which can be solved easily by using any existing method and thus RCC of each alternative $\mathcal{A}^{(k)}(k=1,2,\ldots,l)$ can be obtained in the form of the interval set $\mathfrak{R}^{(k)} = [(\mathfrak{R}^{(k)})^L, (\mathfrak{R}^{(k)})^U]$.

In order to rank the different alternative, the inclusion-comparison probability of intuitionistic fuzzy soft event $\mathfrak{R}^{(k)} \supseteq \mathfrak{R}^{(v)}; k, v \in \{1,2,\ldots,l\}$ is denoted by $p(\mathfrak{R}^{(k)} \supseteq \mathfrak{R}^{(v)})$ and hence the likelihood of the alternatives $p(\mathcal{A}^{(k)} \succeq \mathcal{A}^{(v)}) = p(\mathfrak{R}^{(k)} \supseteq \mathfrak{R}^{(v)})$ is defined as

$$p(\mathcal{A}^{(k)} \succeq \mathcal{A}^{(v)}) = p(\mathfrak{R}^{(k)} \supseteq \mathfrak{R}^{(v)}) = \max\left\{1 - \max\left\{\frac{(\mathfrak{R}^{(v)})^U - (\mathfrak{R}^{(k)})^L}{L(\mathfrak{R}^{(k)}) + L(\mathfrak{R}^{(v)})}, 0\right\}, 0\right\} \tag{6.14}$$

where $L(\mathfrak{R}^{(k)}) = (\mathfrak{R}^{(k)})^U - (\mathfrak{R}^{(k)})^L$ and $L(\mathfrak{R}^{(v)}) = (\mathfrak{R}^{(v)})^U - (\mathfrak{R}^{(v)})^L$ are the hesitation degrees of $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(v)}$ respectively and their corresponding likelihood matrix is denoted by $P = (p^{(kv)})_{l\times l}$ where $p^{(kv)} = p(\mathcal{A}^{(k)} \succeq \mathcal{A}^{(v)}); (k, v = 1,2,\ldots,l)$. Further, it has been clearly observed that $0 \leq p^{(kv)} \leq 1$ and $p^{(kv)} + p^{(vk)} = 1; (k, v = 1,2,\ldots,l)$ which implies that P is the fuzzy-complementary-judgement matrix. Now, based on the values of $p^{(kv)}$, the optimal degrees of the alternative $\mathcal{A}^{(k)}(k=1,2,\ldots,l)$ are computed as:

$$\theta^{(k)} = \frac{1}{l(l-1)}\left(\sum_{v=1}^{l} p^{(kv)} + \frac{l}{2} - 1\right) \tag{6.15}$$

and select the best alternative(s), according to the decreasing order of values of $\theta^{(k)}$'s.

Based on the above analysis, we summarize the algorithm for solving MADM problem with IVIF soft sets as follows:

Step 1: Arrange the information about alternatives $\mathcal{A}^{(k)}(k=1,2,...,l)$ given by decision makers in the form of IVIF soft matrix $\mathcal{M} = ([(\zeta_{ij}^{(k)})^L, (\zeta_{ij}^{(k)})^U], [(\vartheta_{ij}^{(k)})^L, (\vartheta_{ij}^{(k)})^U])$.

Step 2: Summarize the subjective weight vector of each expert u_i and parameter e_j as $W_i, (i = 1, 2, \ldots, n)$ and $\chi_j, (j = 1, 2, \ldots, m)$, respectively in the form of IVIF numbers.

Step 3: Construct the models (6.12) and (6.13) for each alternative $\mathcal{A}^{(k)}$ and hence obtain their RCCs $\mathfrak{R}^{(k)} = [(\mathfrak{R}^{(k)})^L, (\mathfrak{R}^{(k)})^U]$ for $k = 1, 2, \ldots, l$.

Step 4: Construct the likelihood matrix and obtain the values of $\theta^{(k)}$'s by using Eq. (6.15).

Step 5: Based on the descending values of $\theta^{(k)}$'s, rank the given alternatives.

6.3 Additional features of the proposed NLP models

The proposed designed models (6.9) and (6.11) also has some additional features as compared to the other approaches which are enlisted below:

(i) The above computed RCCs are based on the weighted distance from its reference points which are taken as a fixed with respect to the decision makers, which results that RCCs could not show any variation even if the alternatives are changed. However, instead of fixing these reference points as a *priori*, we can take them as given in Section 6.2.2, and hence by using Eqs. (6.3) and (6.4), we explicitly defined the RCC of the alternatives $\mathcal{A}^{(k)}(k = 1, 2, \ldots, l)$ as follows:

$$\begin{aligned}
\mathfrak{R}^{(k)} &= \frac{\sum\limits_{j=1}^{m} \xi_j\Big(\sum\limits_{i=1}^{n} \omega_i(\zeta_{ij}^{(k)} - g_{ij}^-)\Big) + \sum\limits_{j=1}^{m} \eta_j\Big(\sum\limits_{i=1}^{n} \rho_i(h_{ij}^- - \vartheta_{ij}^{(k)})\Big)}{\sum\limits_{j=1}^{m} \xi_j\Big(\sum\limits_{i=1}^{n} \omega_i(\zeta_{ij}^{(k)} - g_{ij}^-)\Big) + \sum\limits_{j=1}^{m} \eta_j\Big(\sum\limits_{i=1}^{n} \rho_i(h_{ij}^- - \vartheta_{ij}^{(k)})\Big) + \sum\limits_{j=1}^{m} \xi_j\Big(\sum\limits_{i=1}^{n} \omega_i(g_{ij}^+ - \zeta_{ij}^{(k)})\Big) + \sum\limits_{j=1}^{m} \eta_j\Big(\sum\limits_{i=1}^{n} \rho_i(\vartheta_{ij}^{(k)} - h_{ij}^+)\Big)} \\
&= \frac{\sum\limits_{j=1}^{m} \xi_j\Big(\sum\limits_{i=1}^{n} \omega_i(\zeta_{ij}^{(k)} - g_{ij}^-)\Big) + \sum\limits_{j=1}^{m} \eta_j\Big(\sum\limits_{i=1}^{n} \rho_i(h_{ij}^- - \vartheta_{ij}^{(k)})\Big)}{\sum\limits_{j=1}^{m} \xi_j\left(\sum\limits_{i=1}^{n} \omega_i(g_{ij}^+ - g_{ij}^-)\right) + \sum\limits_{j=1}^{m} \eta_j\left(\sum\limits_{i=1}^{n} \rho_i(h_{ij}^- - h_{ij}^+)\right)}
\end{aligned}$$

Thus, in view of Eqs. (6.7) and (6.8), $(\mathfrak{R}^{(k)})^L$ and $(\mathfrak{R}^{(k)})^U$ can similarly be captured by solving the NLP models as follows:

$$(\mathfrak{R}^{(k)})^L = \min\Big\{\mathfrak{R}^{(k)}\big((\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1}\big)\Big\} \tag{6.16}$$

and

$$(\mathfrak{R}^{(k)})^U = \max\Big\{\mathfrak{R}^{(k)}\big((\zeta_{ij}^{(k)})_{n\times m}, (\vartheta_{ij}^{(k)})_{n\times m}, (\omega_i)_{n\times 1}, (\rho_i)_{n\times 1}, (\xi_j)_{m\times 1}, (\eta_j)_{m\times 1}\big)\Big\} \tag{6.17}$$

subject to

$$\big((\zeta_{ij}^{(k)})_{n\times m},(\vartheta_{ij}^{(k)})_{n\times m},(\omega_i)_{n\times 1},(\rho_i)_{n\times 1},(\xi_j)_{m\times 1},(\eta_j)_{m\times 1}\big)$$
$$\in \Omega_\mu\times\Omega_\nu\times\Omega_\omega\times\Omega_\rho\times\Omega_\xi\times\Omega_\eta \tag{6.18}$$

respectively. Thus, RCCs and ranking of the alternatives can be found for the given set of alternatives.

(ii) If the weights of the criterion as well the parameters are known *a priori*, then we can still utilize the Eqs. (6.9) and (6.11) provided all the constraints given in Eq. (6.10) as known constants. Then, $(\mathfrak{R}^{(k)})^L$ and $(\mathfrak{R}^{(k)})^U$ of alternative $\mathcal{A}^{(k)}$ to $\mathcal{A}^+$ are obtained directly as

$$(\mathfrak{R}^{(k)})^L = \frac{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i(\zeta_{ij}^{(k)})^L\right)+\sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i(1-(\vartheta_{ij}^{(k)})^U)\right)}{\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\left(\xi_j\omega_i+\eta_j\rho_i\right)} \tag{6.19}$$

$$(\mathfrak{R}^{(k)})^U = \frac{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i(\zeta_{ij}^{(k)})^U\right)+\sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i(1-(\vartheta_{ij}^{(k)})^L)\right)}{\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\left(\xi_j\omega_i+\eta_j\rho_i\right)} \tag{6.20}$$

respectively.

(iii) If in the proposed approach, we utilize the weighted Euclidean distances

$$\mathcal{D}(\mathcal{A}^{(k)},\mathcal{A}^+) = \sqrt{\sum_{j=1}^{m}\xi_j\Big(\sum_{i=1}^{n}\omega_i(1-\zeta_{ij}^{(k)})^2\Big)+\sum_{j=1}^{m}\eta_j\Big(\sum_{i=1}^{n}\rho_i(\vartheta_{ij}^{(k)})^2\Big)} \tag{6.21}$$

and

$$\mathcal{D}(\mathcal{A}^{(k)},\mathcal{A}^-) = \sqrt{\sum_{j=1}^{m}\xi_j\Big(\sum_{i=1}^{n}\omega_i(\zeta_{ij}^{(k)})^2\Big)+\sum_{j=1}^{m}\eta_j\Big(\sum_{i=1}^{n}\rho_i(1-\vartheta_{ij}^{(k)})^2\Big)} \tag{6.22}$$

in place of the weighted Hamming distances between the alternatives $\mathcal{A}^{(k)}$ and $\mathcal{A}^+$, as well as between $\mathcal{A}^{(k)}$ and $\mathcal{A}^-$, then the closeness-coefficients of the alternatives

$\mathcal{A}^{(k)}$ to $\mathcal{A}^{+}$ is written as follows:

$$\mathfrak{R}^{(k)} = \frac{\sqrt{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i\left(\zeta_{ij}^{(k)}\right)^2\right)+\sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i\left(1-\vartheta_{ij}^{(k)}\right)^2\right)}}{\left\{\begin{array}{l}\sqrt{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i\left(\zeta_{ij}^{(k)}\right)^2\right)+\sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i\left(1-\vartheta_{ij}^{(k)}\right)^2\right)}\\ +\sqrt{\sum\limits_{j=1}^{m}\xi_j\left(\sum\limits_{i=1}^{n}\omega_i\left(1-\zeta_{ij}^{(k)}\right)^2\right)+\sum\limits_{j=1}^{m}\eta_j\left(\sum\limits_{i=1}^{n}\rho_i\left(\vartheta_{ij}^{(k)}\right)^2\right)}\end{array}\right\}} \tag{6.23}$$

Now, according to the monotonicity of $\mathfrak{R}^{(k)}$ as stated earlier, the bounds of RCC i.e., $(\mathfrak{R}^{(k)})^L$ and $(\mathfrak{R}^{(k)})^U$ are obtained by solving the models

$$(\mathfrak{R}^{(k)})^L = \min\left\{\mathfrak{R}^{(k)}\left((\zeta_{ij}^{(k)})^L_{n\times m},(\vartheta_{ij}^{(k)})^U_{n\times m},(\omega_i)_{n\times 1},(\rho_i)_{n\times 1},(\xi_j)_{m\times 1},(\eta_j)_{m\times 1}\right)\right\}$$

and

$$(\mathfrak{R}^{(k)})^U = \max\left\{\mathfrak{R}^{(k)}\left((\zeta_{ij}^{(k)})^U_{n\times m},(\vartheta_{ij}^{(k)})^L_{n\times m},(\omega_i)_{n\times 1},(\rho_i)_{n\times 1},(\xi_j)_{m\times 1},(\eta_j)_{m\times 1}\right)\right\}$$

respectively, subject to $\left((\omega_i)_{n\times 1},(\rho_i)_{n\times 1},(\xi_j)_{m\times 1},(\eta_j)_{m\times 1}\right)\in\Omega_\omega\times\Omega_\rho\times\Omega_\xi\times\Omega_\eta$.

6.4 Illustrative example

In this section, a real example related to the DM problem is used to demonstrate the proposed method under IVIFSS environment.

6.4.1 A case study

The Kedarnath valley, along with other parts of the state of Uttarakhand in the north of India, was hit with unprecedented flash floods in July 2013. A large number of roads, which connect to the Kedarnath valley with other parts of Uttarakhand had been destroyed in this flood. In this context, the Uttrakhand government had taken a considerable number of road building projects either to preserve the roads already built or to undertake the new roads. These projects have been carried out by a limited number of well-established contractors, and the selection process has been on the basis of bid price alone. In recent years, increased project complexity, technical capability, higher performance, and safety and financial requirements have been demanding the use of multi-attribute DM methods. For this, Uttarakhand government had issued the notice in the newspapers, and considered three parameters required for contractor selection, namely, tender price (e_1), completion time (e_2) and technical capability (e_3). The three contractors taken as in the form of the alternatives, namely, Jai Hind Road Builders Private Ltd. ($\mathcal{A}^{(1)}$), J.K. Construction ($\mathcal{A}^{(2)}$) and Build quick Infrastructure Private Ltd. ($\mathcal{A}^{(3)}$) bid for these projects. An expert committee of four members $\mathcal{X} = \{u_1, u_2, u_3, u_4\}$ with weight vector is being set up to choose the best one on the basis of the above parameters $\mathcal{E} = \{e_1, e_2, e_3\}$. Then the objective of the Government is to choose the best contractor among them for the task. In order to fulfill it, the computational procedure to solve this problem is summarized as below:

Step 1: The rating values of the alternatives given by the experts are represented in the form of IVIF soft decision matrices in Table 6.1.

Table 6.1: Input preference values of each alternative in terms of IVIFSNs

	For alternative $\mathcal{A}^1$		
	e_1	e_2	e_3
u_1	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.2,0.4])	([0.1,0.3],[0.5,0.6])
u_2	([0.6,0.7],[0.2,0.3])	([0.6,0.7],[0.2,0.3])	([0.4,0.7],[0.1,0.2])
u_3	([0.3,0.6],[0.3,0.4])	([0.5,0.6],[0.3,0.4])	([0.5,0.6],[0.1,0.3])
u_4	([0.7,0.8],[0.1,0.2])	([0.6,0.7],[0.1,0.3])	([0.3,0.4],[0.1,0.2])
	For alternative $\mathcal{A}^2$		
	e_1	e_2	e_3
u_1	([0.7,0.8],[0.1,0.2])	([0.6,0.8],[0.1,0.2])	([0.2,0.3],[0.3,0.5])
u_2	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.4,0.5],[0.3,0.4])
u_3	([0.4,0.6],[0.2,0.3])	([0.7,0.8],[0.1,0.2])	([0.6,0.7],[0.1,0.2])
u_4	([0.6,0.7],[0.2,0.3])	([0.3,0.4],[0.5,0.6])	([0.5,0.7],[0.1,0.2])
	For alternative $\mathcal{A}^3$		
	e_1	e_2	e_3
u_1	([0.2,0.3],[0.4,0.5])	([0.5,0.6],[0.1,0.3])	([0.4,0.5],[0.2,0.4])
u_2	([0.6,0.7],[0.2,0.3])	([0.5,0.6],[0.1,0.3])	([0.6,0.7],[0.2,0.3])
u_3	([0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.1,0.2])	([0.5,0.6],[0.3,0.4])
u_4	([0.5,0.6],[0.3,0.4])	([0.5,0.7],[0.1,0.3])	([0.7,0.8],[0.1,0.2])

Step 2: Assume that the subjective weights of the experts and the parameters are given in the form of intervals as $W_i = (([0.15, 0.45], [0.2, 0.4]), ([0.3, 0.4], [0.4, 0.5]), ([0.6, 0.7], [0.1, 0.2]), ([0.5, 0.7], [0.1, 0.3]))^T$ and $\chi_j = (([0.1, 0.4], [0.2, 0.55]), ([0.2, 0.5], [0.15, 0.45]), ([0.25, 0.6], [0.15, 0.38]))^T$, respectively.

Step 3: The optimization models (6.12) and (6.13) are constructed based on the given information. For example, for an alternative $\mathcal{A}^{(1)}$, these models becomes:

$$(\mathfrak{R}^{(1)})^L = \min\{0.4t_{11} + 0.6y_{11} + 0.4t_{12} + 0.6y_{12} + 0.1t_{13} + 0.4y_{13} + 0.6t_{21} + 0.7y_{21} + 0.6t_{22} + 0.7y_{22} + 0.4t_{23} + 0.8y_{23} + 0.3t_{31} + 0.6y_{31} + 0.5t_{32} + 0.6y_{32} + 0.5t_{33} + 0.7y_{33} + 0.7t_{41} + 0.8y_{41} + 0.6t_{42} + 0.7y_{42} + 0.3t_{43} + 0.8y_{43}\}$$

$$\text{s.t.}\begin{cases} 0.015z \leq t_{11} \leq 0.180z \quad ; \quad 0.040z \leq y_{11} \leq 0.220z \quad ; \quad 0.030z \leq t_{12} \leq 0.225z \\ 0.030z \leq y_{12} \leq 0.180z \quad ; \quad 0.037z \leq t_{13} \leq 0.270z \quad ; \quad 0.030z \leq y_{13} \leq 0.152z \\ 0.030z \leq t_{21} \leq 0.160z \quad ; \quad 0.080z \leq y_{21} \leq 0.275z \quad ; \quad 0.060z \leq t_{22} \leq 0.200z \\ 0.060z \leq y_{22} \leq 0.225z \quad ; \quad 0.075z \leq t_{23} \leq 0.240z \quad ; \quad 0.060z \leq y_{23} \leq 0.190z \\ 0.060z \leq t_{31} \leq 0.280z \quad ; \quad 0.020z \leq y_{31} \leq 0.110z \quad ; \quad 0.120z \leq t_{32} \leq 0.350z \\ 0.015z \leq y_{32} \leq 0.090z \quad ; \quad 0.150z \leq t_{33} \leq 0.420z \quad ; \quad 0.015z \leq y_{33} \leq 0.076z \\ 0.050z \leq t_{41} \leq 0.280z \quad ; \quad 0.020z \leq y_{41} \leq 0.165z \quad ; \quad 0.100z \leq t_{42} \leq 0.350z \\ 0.015z \leq y_{42} \leq 0.135z \quad ; \quad 0.125z \leq t_{43} \leq 0.420z \quad ; \quad 0.015z \leq y_{43} \leq 0.114z \\ \sum_{j=1}^{3}\sum_{i=1}^{4}(t_{ij} + y_{ij}) = 1 \quad ; \quad z \geq 0 \end{cases} \tag{6.24}$$

and

$$(\mathfrak{R}^{(1)})^U = \max\{0.5t_{11} + 0.7y_{11} + 0.6t_{12} + 0.8y_{12} + 0.3t_{13} + 0.5y_{13} + 0.7t_{21} + 0.8y_{21} + 0.7t_{22} + 0.8y_{22} + 0.7t_{23} + 0.9y_{23} + 0.6t_{31} + 0.7y_{31} + 0.6t_{32} + 0.7y_{32} + 0.6t_{33} + 0.9y_{33} + 0.8t_{41} + 0.9y_{41} + 0.7t_{42} + 0.9y_{42} + 0.4t_{43} + 0.9y_{43}\}$$

subject to the constraints given in Eq. (6.24).

The optimal values of these models are obtained as $(\mathfrak{R}^{(1)})^L = 0.41473$ and $(\mathfrak{R}^{(1)})^U = 0.75279$, i.e., the interval RCC of $\mathcal{A}^{(1)}$ is $\mathfrak{R}^{(1)} = [0.41473, 0.75279]$. In the same way, using models (6.12) and (6.13), the RCCs of the alternatives $\mathcal{A}^{(k)}(k = 2, 3)$

are obtained as $\mathfrak{R}^{(2)} = [0.46473, 0.78963]$ and $\mathfrak{R}^{(3)} = [0.50540, 0.78798]$ respectively.

Step 4: By Eq. (6.14), the likelihood probability matrix is given as

$$P = \begin{bmatrix} 0.5 & 0.4345 & 0.3986 \\ 0.5655 & 0.5 & 0.4679 \\ 0.6014 & 0.5321 & 0.5 \end{bmatrix}$$

and hence by Eq. (6.15) we get $\theta^{(1)} = 0.3055, \theta^{(2)} = 0.3389$ and $\theta^{(3)} = 0.3556$ respectively.

Step 5: Based on values of $\theta^{(k)}$'s, we get the ranking order of alternatives as $\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$ and hence $\mathcal{A}^{(3)}$ is the best company for the above said project, i.e., the Build quick Infrastructure Private Ltd.

6.4.2 Comparison analysis with the existing approaches

In order to conduct a comparative analysis of the proposed method with some existing approaches under IVIFSS environment by taking the weights of each expert as well as the parameters are in the form of constants rather than the interval sets. For it, assume that $\chi = (0.1, 0.4, 0.5)^T$ and $W = (0.1, 0.2, 0.4, 0.3)^T$ and hence the results corresponding to the existing approaches are summarized in Table 6.2. It is clearly seen that the best alternative remains the same with these different approaches and the proposed approach which shows the stability and feasibility of the approach.

Table 6.2: Comparative analysis with IVIFSS studies

		Overall values of			Ranking	Best
	Method	$\mathcal{A}^1$	$\mathcal{A}^2$	$\mathcal{A}^3$	order	alternative
Khalid and Abbas [98]	Distance measure	0.3603	0.3365	0.3103	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Mukherjee and Sarkar [141]	Similarity measure	0.3125	0.3875	0.3292	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$
Liu and Qin [124]	Entropy measure	0.5979	0.5205	0.5144	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Shanthi et al. [173]	Similarity measure	0.5916	0.6047	0.6180	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Cagman and Deli [21]	Similarity measure	0.6397	0.6635	0.6897	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Rajarajeswari and Dhanalakshmi [161]	Similarity measure	0.5235	0.5725	0.6085	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Sarala and Suganya [168]	Similarity measure	0.2795	0.3270	0.3795	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Muthukumar and Krishnan [142]	Similarity measure	0.0400	0.0442	0.0475	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$

However, we also conduct a comparative analysis of the proposed method with some of the existing approaches [19, 147, 149, 175, 214, 230] under the IFS environment. For it, we first aggregate the rating values of each alternatives with respect to the parameters by taking the parameters weight $\chi = (0.1, 0.4, 0.5)^T$ and then their corresponding approaches are applied. The results obtained from them are summarized in Table 6.3. Hence, from the above comparative analysis, it has been concluded that the decision results obtained by our proposed method coincide with the existing approaches results.

Table 6.3: Comparative analysis with IVIFSS studies

		Overall values of			Ranking	Best
	Method	$\mathcal{A}^1$	$\mathcal{A}^2$	$\mathcal{A}^3$	order	alternative
Wei and Wang [214]	Aggregation operator	0.2490	0.2892	0.3645	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Xu and Jian [230]	Similarity measure	0.3157	0.3809	0.4108	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Park et al. [149]	Distance measure	0.3728	0.3349	0.3140	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Bustince and Burillo [19]	Correlation coefficient	0.8569	0.9044	0.9169	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Park, Kwun, Park and Park [147]	Correlation measure	0.7774	0.8458	0.8691	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$
Singh [175]	Cosine similarity measure	0.8678	0.9147	0.9278	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$	$\mathcal{A}^{(3)}$

6.4.3 Superiority of the proposed approach

In this section, we have presented some counterexamples which show that the existing method under the IVIF soft set environment fails to rank the given alternatives while the proposed approach can overcome their shortcoming.

Example 6.4.1. Consider a DM problem by considering the two alternatives, namely $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which have been evaluated by the experts $\{u_1, u_2\}$ over the set of three parameters, say $\{e_1, e_2, e_3\}$. The objective of the problem is to find the best alternative under the given set by paying equal weight to each expert as well as to the parameters, i.e., $\xi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ and $\omega = (\frac{1}{2}, \frac{1}{2})^T$. In order to do so, an expert evaluated these alternatives and gave the preferences in terms of IVIF numbers which are summarized as follows:

$$\mathcal{A}^{(1)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} ([0.6, 0.7], [0.2, 0.3]) & ([0.4, 0.6], [0.1, 0.2]) & ([0.1, 0.3], [0.4, 0.5]) \\ ([0.4, 0.5], [0.3, 0.4]) & ([0.7, 0.8], [0.1, 0.2]) & ([0.6, 0.7], [0.2, 0.3]) \end{bmatrix} \end{array}$$

and

$$\mathcal{A}^{(2)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} ([0.5, 0.6], [0.1, 0.2]) & ([0.5, 0.7], [0.1, 0.2]) & ([0.1, 0.3], [0.6, 0.7]) \\ ([0.5, 0.6], [0.3, 0.4]) & ([0.7, 0.8], [0.1, 0.2]) & ([0.5, 0.6], [0.3, 0.4]) \end{bmatrix} \end{array}$$

Then, the target of this problem is to choose the best alternative among them. For it, if we utilize the existing distance measure, $\mathcal{D}(\cdot)$, as proposed by [98], then their corresponding distance measures for the alternative $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ from its positive ideal measure $\mathcal{A}^{+} = (([1, 1], [0, 0]))_{2\times 3}$ are $\mathcal{D}(\mathcal{A}^{(1)}, \mathcal{A}^{+}) = 0.9333$ and $\mathcal{D}(\mathcal{A}^{(2)}, \mathcal{A}^{+}) = 0.9333$ respectively. Thus, from it, we have analyzed that their measure is unable to distinguish between these two alternatives.

Since all the weights of the criterion as well as the parameters are known *a priori*, then we can apply Eqs. (6.19) and (6.20) of the proposed approach to the considered problem. Based on these equations, we get the optimal values of the alternative $\mathcal{A}^{(1)}$ as $(\mathfrak{R}^{(1)})^L = 0.5750$ and $(\mathfrak{R}^{(1)})^U = 0.6917$, i.e., the interval RCC is $\mathfrak{R}^{(1)} = [0.5750, 0.6917]$. Similarly, we can obtain the interval RCC for alternative $\mathcal{A}_2$ is $\mathfrak{R}^{(2)} = [0.5583, 0.6750]$. The likelihood probabilities of the pairwise components of $\mathfrak{R}^{(k)}(k = 1, 2)$ can be obtained by using Eq. (6.14) and are expressed as the matrix format as follows:

$$P = \begin{bmatrix} 0.5 & 0.5714 \\ 0.4286 & 0.5 \end{bmatrix}$$

The ranking values which represent the optimal degrees of the alternative $\mathcal{A}^{(k)}(k = 1, 2)$ is obtained by Eq. (6.15) as $\theta^{(1)} = 0.5357$ and $\theta^{(2)} = 0.4643$ and thus, $\mathcal{A}^{(1)}$ is the best alternative. Therefore, the proposed approach is suitably working in those cases when the existing approach fails.

Example 6.4.2. Consider an another DM problem with two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are evaluated under the set of the different parameters e_1, e_2 and e_3 with weight vector as $\xi = (0.52, 0.34, 0.14)^T$. Two experts u_1 and u_2, whose weight vector is $\omega = (0.37, 0.63)^T$, have evaluated these alternatives under the set of the three parameters e_1,

e_2 and e_3 and gave their preferences in terms of IVIF numbers, represented here in the form of decision-matrices as follows:

$$\mathcal{A}^{(1)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} ([0.4,0.5],[0.2,0.3]) & ([0.7,0.8],[0.1,0.2]) & ([0.3,0.4],[0.4,0.5]) \\ ([0.5,0.6],[0.3,0.4]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) \end{bmatrix} \end{array}$$

and

$$\mathcal{A}^{(2)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} ([0.5,0.6],[0.3,0.4]) & ([0.6,0.7],[0.1,0.2]) & ([0.3,0.5],[0.3,0.4]) \\ ([0.3,0.5],[0.2,0.3]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) \end{bmatrix} \end{array}$$

Based on this information, if we utilize Cagman and Deli [21] approach to compute the distance measures of the given alternatives from its ideal alternative $\mathcal{A}^+ = (([1,1],[0,0]))_{2\times 3}$ then we get their respective measures values as $\mathcal{D}(\mathcal{A}^{(1)}, \mathcal{A}^+) = 0.3505$ and $\mathcal{D}(\mathcal{A}^{(2)}, \mathcal{A}^+) = 0.3505$. Hence, it is unable to choose the best one.

On the other hand, if we can apply Eqs. (6.19) and (6.20) of the proposed approach to the considered problem, then we get the optimal values of the closeness-coefficient for alternative $\mathcal{A}^{(1)}$ as $(\mathfrak{R}^{(1)})^L = 0.5995$ and $(\mathfrak{R}^{(1)})^U = 0.6995$ and for $\mathcal{A}^{(2)}$ is $(\mathfrak{R}^{(2)})^L = 0.5902$ and $(\mathfrak{R}^{(2)})^U = 0.7091$. Hence, the RCCs of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are $\mathfrak{R}^{(1)} = [0.5995, 0.6995]$ and $\mathfrak{R}^{(2)} = [0.5902, 0.7091]$. Thus, the likelihood probabilities of the pairwise components of $\mathfrak{R}^{(k)}(k = 1, 2)$ are obtained by using Eq. (6.14) and are summarized as follows:

$$P = \begin{bmatrix} 0.5 & 0.4993 \\ 0.5007 & 0.5 \end{bmatrix}$$

The optimal degrees of the alternative based on the likelihood probabilities values are computed by using Eq. (6.15) as $\theta^{(1)} = 0.4997$ and $\theta^{(2)} = 0.5003$ respectively. Thus, the ranking order is $\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$ where $\succ$ refers to "preferred to" and hence $\mathcal{A}^{(2)}$ is the best alternative.

6.5 Conclusion

IVIF soft set is a useful tool to deal with fuzziness and uncertainties in DM problems. In IVIFSS, the hesitancy of the decision makers' judgment over the different parameters has been taken into consideration as compared to the existing IFSs and IVIFSs. In this chapter, we have developed a methodology for solving the DM problems under the IVIFSS environment in which the ratings of all alternatives on attributes and preference information on attributes are incomplete and expressed with the IVIF sets. In this methodology, the fractional programming models based on the TOPSIS method are established to obtain an RCC interval where preference information is independently determined for each alternative. Further, these derived models are transformed into two simpler auxiliary linear programming models using the Charnes and Cooper's transformation to compute the closeness coefficients of alternatives with respect to IVIFSPIS, which are used to generate the ranking order of the alternatives based on the concept of likelihood of interval numbers. Furthermore, the derived auxiliary models are extremely flexible since some extensions are easily implemented from them. Therefore, if a DM problem includes imprecise and uncertain properties of IVIFSSs, the proposed methodology is useful due to their simplicity and flexibility with regard to incomplete and inconsistent preference information. Also, the existing studies such as IFSs and IVIFSs are the special cases of the IVIFSSs and hence the proposed approach can be applied to DM problems with IFSs and IVIFSs. Finally, a numerical study has been presented to verify the developed approach and to demonstrate the effectiveness by comparing their results with some of the existing approaches' results. From this study, it has been concluded that the proposed results coincide with the ones shown in the existing approaches' results.

Chapter 7

Prioritized averaging and geometric aggregation operators for the intuitionistic fuzzy soft set environment[1]

In this chapter, we have defined some new averaging and geometric prioritized aggregation operators in which the preferences related to attributes are taken in the form of IFSSs. Desirable properties of its have also been investigated. Furthermore, based on these operators, an approach to investigate the MCDM problem has been presented. The effectiveness of these operators has been demonstrated through a case study.

7.1 Introduction

The literature related to different AOs are summarized in Section 1.1.4 of Chapter 1. From these studies, it has been concluded that most of the existing approaches are worked under the restriction that the parameters and the decision-makers are at the same priority level. But this assumption needs to be relaxed for better decision analysis. So to investigate this issue, the present work proposed some prioritized aggregation operations under IFSSs environment by taking advantage of the prioritized aggregation operator [241]. Thus, by considering the fact that the IFSS has a powerful tool to deal with the ambiguity and

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vagueness of the data, the work introduced a series of aggregation operators namely, the intuitionistic fuzzy soft prioritized weighted and ordered weighted averaging, intuitionistic fuzzy soft prioritized weighted and ordered weighted geometric. Furthermore, these operators have been tested on the problem of MCDM where the most desirable alternative has been computed under the set of different criteria. Finally, the computed results are compared with the existing operators result for showing their effectiveness.

7.2 Operational law for IFSNs

In this section, we introduce some operations on IFSNs and solved some desirable properties on it.

Definition 7.2.1. Let $\mathcal{A} = (\zeta, \vartheta)$, $\mathcal{A}_{11} = (\zeta_{11}, \vartheta_{11})$ and $\mathcal{A}_{12} = (\zeta_{12}, \vartheta_{12})$ be three IFSNs and for any real number $\lambda > 0$, then by algebraic norms we have

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = (\zeta_{11} + \zeta_{12} - \zeta_{11}\zeta_{12}, \vartheta_{11}\vartheta_{12})$.

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = (\zeta_{11}\zeta_{12}, \vartheta_{11} + \vartheta_{12} - \vartheta_{11}\vartheta_{12})$.

(iii) $\lambda\ \mathcal{A} = (1 - (1 - \zeta)^{\lambda}, \vartheta^{\lambda})$.

(iv) $\mathcal{A}^{\lambda} = (\zeta^{\lambda}, 1 - (1 - \vartheta)^{\lambda})$.

Theorem 7.2.1. All the operations laws for IFSNs as given in Definition 7.2.1 i.e., $\mathcal{A}_{11} \oplus \mathcal{A}_{12}$, $\mathcal{A}_{11} \otimes \mathcal{A}_{12}$, $\lambda\mathcal{A}$ and $\mathcal{A}^{\lambda}$ are also IFSNs.

Proof. Since $\mathcal{A}_{1j} (j = 1, 2)$ are IFSNs this implies $0 \leq \zeta_{1j} \leq 1$, $0 \leq \vartheta_{1j} \leq 1$, $0 \leq \zeta_{1j} + \vartheta_{1j} \leq 1$ and hence $0 \leq (1 - \zeta_{11})(1 - \zeta_{12}) \leq 1 \Leftrightarrow 0 \leq 1 - (1 - \zeta_{11})(1 - \zeta_{12}) \leq 1$ and $0 \leq \vartheta_{11}\vartheta_{12} \leq 1$. Further, $1 - (1 - \zeta_{11})(1 - \zeta_{12}) + \vartheta_{11}\vartheta_{12} \leq 1 - \vartheta_{11}\vartheta_{12} + \vartheta_{11}\vartheta_{12} \leq 1$. Thus, $\mathcal{A}_{11} \oplus \mathcal{A}_{12}$ is IFSN.

Also, $1 - (1 - \zeta)^{\lambda} \geq 0$, $\vartheta^{\lambda} \geq 0$, $1 - (1 - \zeta)^{\lambda} + \vartheta^{\lambda} \leq 1 - (1 - \zeta)^{\lambda} + (1 - \zeta)^{\lambda} \leq 1$. Thus, $\lambda\mathcal{A}$ is also an IFSN. Similarly, for others. □

Theorem 7.2.2. (Commutative law) Let $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})(j = 1, 2)$ be two IFSNs, then

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \mathcal{A}_{12} \oplus \mathcal{A}_{11}$

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \mathcal{A}_{12} \otimes \mathcal{A}_{11}$

Theorem 7.2.3. (Associative law) Let $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})(j = 1, 2, 3)$ be three IFSNs, then

(i) $(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) \oplus \mathcal{A}_{13} = \mathcal{A}_{11} \oplus (\mathcal{A}_{12} \oplus \mathcal{A}_{13})$

(ii) $(\mathcal{A}_{11} \otimes \mathcal{A}_{12}) \otimes \mathcal{A}_{13} = \mathcal{A}_{11} \otimes (\mathcal{A}_{12} \otimes \mathcal{A}_{13})$

Theorems 7.2.2 and 7.2.3 are straightforward and we omit their proofs.

Theorem 7.2.4. Let $\mathcal{A} = (\zeta, \vartheta)$ and $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})(j = 1, 2)$ be three IFSNs then for a real numbers λ's,

(i) $\lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) = \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{11}$.

(ii) $(\mathcal{A}_{11} \otimes \mathcal{A}_{12})^{\lambda} = \mathcal{A}_{11}^{\lambda} \otimes \mathcal{A}_{12}^{\lambda}$.

(iii) $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$.

(iv) $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1+\lambda_2}$.

Proof. We prove the part (i) and (iii) and hence similar for other.

(i) For real number $\lambda > 0$, $\lambda\mathcal{A}_{11} = (1 - (1 - \zeta_{11})^{\lambda}, \vartheta_{11}^{\lambda})$ and $\lambda\mathcal{A}_{12} = (1 - (1 - \zeta_{12})^{\lambda}, \vartheta_{12}^{\lambda})$. Thus,

$$\begin{aligned}
& \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12} \\
= & \left(1 - \left(1 - \left(1 - (1 - \zeta_{11})^{\lambda}\right)\right)\left(1 - \left(1 - (1 - \zeta_{12})^{\lambda}\right)\right), \vartheta_{11}^{\lambda}\vartheta_{12}^{\lambda}\right) \\
= & \left(1 - (1 - \zeta_{11})^{\lambda}(1 - \zeta_{12})^{\lambda}, \vartheta_{11}^{\lambda}\vartheta_{12}^{\lambda}\right) \\
= & \left(1 - ((1 - \zeta_{11})(1 - \zeta_{12}))^{\lambda}, (\vartheta_{11}\vartheta_{12})^{\lambda}\right) \\
= & \lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12})
\end{aligned}$$

Hence the result.

(iii) For $\lambda_1, \lambda_2 > 0$ and IFSN $\mathcal{A} = (\zeta, \vartheta)$, we have

$$\lambda_1\mathcal{A} = \left(1 - (1 - \zeta)^{\lambda_1}, \vartheta^{\lambda_1}\right) \quad ; \quad \lambda_2\mathcal{A} = \left(1 - (1 - \zeta)^{\lambda_2}, \vartheta^{\lambda_2}\right)$$

Thus,

$$
\begin{aligned}
& \lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} \\
= & \Big(1-\Big(1-\big(1-(1-\zeta)^{\lambda_1}\big)\Big)\Big(1-\big(1-(1-\zeta)^{\lambda_2}\big)\Big), \vartheta^{\lambda_1}\vartheta^{\lambda_2}\Big) \\
= & \Big(1-(1-\zeta)^{\lambda_1}(1-\zeta)^{\lambda_2}, \vartheta^{\lambda_1}\zeta^{\lambda_2}\Big) \\
= & \Big(1-(1-\zeta)^{\lambda_1+\lambda_2}, \vartheta^{\lambda_1+\lambda_2}\Big) \\
= & (\lambda_1+\lambda_2)\mathcal{A}
\end{aligned}
$$

Hence the result.

□

7.3 Intuitionistic fuzzy soft Prioritized aggregation operators

In this section, we will investigate the prioritized averaging and geometric AOs for the IFSSs. The formal definition of the Prioritized weighted average operator is already given in Definition 2.5.5 of Chapter 2. We restated here as follows.

Definition 7.3.1. [241] For a collection of attributes $a_1, a_2, \ldots, a_n$ whose rating are lies between [0,1]. Let prioritization linear relation between them are expressed with ordering $a_1 \succ a_2 \succ \ldots \succ a_n$, indicating that the attribute a_j has a higher priority than a_k if $j < k$. Then, the prioritized weighted averaging (PWA) operator is defined as

$$\text{PWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \frac{T_j}{\sum\limits_{r=1}^{n} T_r} a_j,$$

where $T_j = \prod\limits_{k=1}^{j-1} a_k$, $j = 2, 3, \ldots, n$ and $T_1 = 1$.

Based on this definition, we have presented averaging and geometric AOs as follows.

7.3.1 Prioritized weighted average operator

Definition 7.3.2. Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, $(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$ be collection of IFSNs. Then we have

$$\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = \bigoplus_{i=1}^{m} \frac{T_i}{\sum\limits_{i=1}^{m} T_i} \left(\bigoplus_{j=1}^{n} \frac{R_j}{\sum\limits_{r=1}^{n} R_r} \mathcal{A}_{ij} \right) \tag{7.1}$$

where $R_1 = 1$, $T_1 = 1$ and $R_j = \prod\limits_{l=1}^{j-1} S(\mathcal{A}_{il}), (j = 2,3,\ldots,n)$, $T_i = \prod\limits_{k=1}^{i-1} S(\mathcal{A}_k) (i = 2,3,\ldots,m)$ where $S(\mathcal{A}_{ij})$ represents the score function of IFSN $\mathcal{A}_{ij}$.

By using Definition 7.3.2, we can get the following result.

Theorem 7.3.1. The aggregated value of all IFSNs by IFSPWA operator is still an IFSN and is defined as:

$$\begin{aligned} & \text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \\ = & \left(1 - \prod_{i=1}^{m} \left(\prod_{j=1}^{n} (1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{m} \left(\prod_{j=1}^{n} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \end{aligned} \tag{7.2}$$

Proof. We will proof the results by mathematical induction on n. For $n = 1$, we have

$$\begin{aligned} & \text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{m1}) \\ = & \bigoplus_{i=1}^{m} \frac{T_i}{\sum\limits_{s=1}^{m} T_s} \mathcal{A}_{i1} \\ = & \left(1 - \prod_{i=1}^{m} (1-\zeta_{i1})^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{m} \vartheta_{i1}^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \\ = & \left(1 - \prod_{i=1}^{m} \left(\prod_{j=1}^{1} (1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{m} \left(\prod_{j=1}^{1} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \end{aligned}$$

and for $m = 1$,

$$\begin{aligned}
&\mathrm{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1n}) \\
= &\bigoplus_{j=1}^{n} \frac{R_j}{\sum\limits_{r=1}^{n} R_r} \mathcal{A}_{1j} \\
= &\left(1 - \prod_{j=1}^{n} (1 - \zeta_{1j})^{\frac{R_j}{\sum_{r=1}^{n} R_r}}, \prod_{j=1}^{n} \vartheta_{1j}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right) \\
= &\left(1 - \prod_{i=1}^{1} \left(\prod_{j=1}^{n} (1 - \zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{i=1}^{1} T_i}}, \prod_{i=1}^{1} \left(\prod_{j=1}^{n} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{i=1}^{1} T_i}} \right)
\end{aligned}$$

Thus, Eq. (7.2) holds for $n = 1; m = 1$. Assume Eq. (7.2) holds for $m = k_1 + 1, n = k_2$ and $m = k_1, n = k_2 + 1$, i.e.

$$\begin{aligned}
&\bigoplus_{i=1}^{k_1+1} \frac{T_i}{\sum\limits_{s=1}^{m} T_s} \left(\bigoplus_{j=1}^{k_2} \frac{R_j}{\sum\limits_{r=1}^{n} R_r} \mathcal{A}_{ij} \right) \\
= &\left(1 - \prod_{i=1}^{k_1+1} \left(\prod_{j=1}^{k_2} (1 - \zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{k_1+1} \left(\prod_{j=1}^{k_2} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right)
\end{aligned}$$

and

$$\begin{aligned}
&\bigoplus_{i=1}^{k_1} \frac{T_i}{\sum\limits_{s=1}^{m} T_s} \left(\bigoplus_{j=1}^{k_2+1} \frac{R_j}{\sum\limits_{r=1}^{n} R_r} \mathcal{A}_{ij} \right) \\
= &\left(1 - \prod_{i=1}^{k_1} \left(\prod_{j=1}^{k_2+1} (1 - \zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{k_1} \left(\prod_{j=1}^{k_2+1} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right)
\end{aligned}$$

Now,

$$
\begin{aligned}
&\bigoplus_{i=1}^{k_1+1} \frac{T_i}{\sum\limits_{s=1}^{k_1+1} T_s} \left(\bigoplus_{j=1}^{k_2+1} \frac{R_j}{\sum\limits_{r=1}^{k_2+1} R_r} \mathcal{A}_{ij} \right) \\
= \; & \bigoplus_{i=1}^{k_1+1} \frac{T_i}{\sum\limits_{s=1}^{k_1+1} T_s} \left(\bigoplus_{j=1}^{k_2} \frac{R_j}{\sum\limits_{r=1}^{k_2+1} R_r} \mathcal{A}_{ij} \oplus \frac{R_{k_2+1}}{\sum\limits_{r=1}^{k_2+1} R_r} \mathcal{A}_{(k_2+1)j} \right) \\
= \; & \left(\bigoplus_{i=1}^{k_1+1} \frac{T_i}{\sum\limits_{s=1}^{k_1+1} T_s} \bigoplus_{j=1}^{k_2} \left(\frac{R_j}{\sum\limits_{r=1}^{k_2+1} R_r} \mathcal{A}_{ij} \right) \right) \bigoplus_{i=1}^{k_1+1} \frac{T_i}{\sum\limits_{s=1}^{k_1+1} T_s} \left(\frac{R_{k_2+1}}{\sum\limits_{r=1}^{k_2+1} R_r} \mathcal{A}_{(k_2+1)j} \right) \\
= \; & \left(1 - \prod_{i=1}^{k_1+1} \left(\prod_{j=1}^{k_2+1} (1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{k_2+1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1+1} T_s}}, \prod_{i=1}^{k_1+1} \left(\prod_{j=1}^{k_2+1} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{k_2+1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1+1} T_s}} \right)
\end{aligned}
$$

Therefore, Eq. (7.2) holds for $m = k_1 + 1$ and $n = k_2 + 1$ and hence by mathematical induction the result holds for all positive integers $m, n \geq 1$. $\square$

Example 7.3.1. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a set of parameters and $\mathcal{X} = \{u_1, u_2, u_3, u_4\}$ be set of experts which give their preferences $\mathcal{A}_{ij}$ for describing the "attractiveness of a house" in terms of IFSNs and are summarized as below.

$$
(\mathcal{F}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.8, 0.1) & (0.5, 0.3) & (0.4, 0.5) \\ (0.4, 0.3) & (0.3, 0.5) & (0.7, 0.2) \\ (0.7, 0.1) & (0.8, 0.2) & (0.5, 0.1) \\ (0.3, 0.5) & (0.5, 0.2) & (0.6, 0.1) \end{bmatrix} \end{array}
$$

By utilizing these information and $R_1 = 1$, $R_j = \prod\limits_{l=1}^{j-1} S(\mathcal{A}_{il}), (j = 2, 3)$, $T_1 = 1$, $T_i =$

$\prod_{k=1}^{i-1} S(\mathcal{A}_k)(i=2,3,4)$, we get

$$R = \begin{bmatrix} 1 & 0.85 & 0.60 \\ 1 & 0.55 & 0.40 \\ 1 & 0.80 & 0.80 \\ 1 & 0.40 & 0.65 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 \\ 0.71 \\ 0.57 \\ 0.78 \end{bmatrix}$$

Thus, the aggregated IFSN by using Eq. (7.2) becomes

$$\begin{aligned} & \text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{43}) \\ = & \left(1 - \prod_{i=1}^{4} \left(\prod_{j=1}^{3} (1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}}, \prod_{i=1}^{4} \left(\prod_{j=1}^{3} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}} \right) \end{aligned}$$

Now,

$$\begin{aligned} & \prod_{i=1}^{4} \left(\prod_{j=1}^{3} (1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}} = \left((1-0.8)^{0.41} \times (1-0.5)^{0.35} \times (1-0.4)^{0.24} \right)^{0.33} \\ & \times \left((1-0.4)^{0.51} \times (1-0.3)^{0.28} \times (1-0.7)^{0.20} \right)^{0.23} \times \left((1-0.7)^{0.38} \times (1-0.8)^{0.31} \times (1-0.5)^{0.31} \right)^{0.19} \\ & \times \left((1-0.3)^{0.49} \times (1-0.5)^{0.20} \times (1-0.6)^{0.32} \right)^{0.25} \\ = & \ 0.4291 \end{aligned}$$

and

$$\begin{aligned} & \prod_{i=1}^{4} \left(\prod_{j=1}^{3} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}} \\ = & \left((0.1)^{0.41} \times (0.3)^{0.35} \times (0.5)^{0.24} \right)^{0.33} \times \left((0.3)^{0.51} \times (0.5)^{0.28} \times (0.2)^{0.20} \right)^{0.23} \\ & \times \left((0.1)^{0.38} \times (0.2)^{0.31} \times (0.1)^{0.31} \right)^{0.19} \times \left((0.5)^{0.49} \times (0.2)^{0.20} \times (0.1)^{0.32} \right)^{0.25} \\ = & \ 0.2219 \end{aligned}$$

Therefore, by IFSPWA operator, we get

$$\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{43}) = (0.5709, 0.2219)$$

Based on Theorem 7.3.1, we have the following properties for IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$.

Property 7.3.1. (Idempotency) If $\mathcal{A}_{ij} = \mathcal{A}_e = (\zeta, \vartheta)$,(say) for all i, j, then

$$\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = \mathcal{A}_e$$

Proof. Since for all i, j, $\mathcal{A}_{ij}$ are equal i.e. $\mathcal{A}_{ij} = \mathcal{A}_e$, then we have

$$\begin{aligned}
&\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \\
= &\left(1 - \prod_{i=1}^{m}\left(\prod_{j=1}^{n}(1-\zeta)^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{m}\left(\prod_{j=1}^{n}\vartheta^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}\right) \\
= &\left(1 - \left((1-\zeta)^{\frac{\sum_{j=1}^{n} R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{\sum_{i=1}^{m} T_i}{\sum_{s=1}^{m} T_s}}, \left(\vartheta^{\frac{\sum_{j=1}^{n} R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{\sum_{i=1}^{m} T_i}{\sum_{s=1}^{m} T_s}}\right) \\
= &(1-(1-\zeta), \vartheta) \\
= &(\zeta, \vartheta) \\
= &\mathcal{A}_e
\end{aligned}$$

□

Property 7.3.2. (Boundedness) Let $\mathcal{A}_{ij}^{-} = \left(\min_i \min_j \{\zeta_{ij}\}, \max_i \max_j \{\vartheta_{ij}\}\right)$ and $\mathcal{A}_{ij}^{+} = \left(\max_i \max_j \{\zeta_{ij}\}, \min_i \min_j \{\vartheta_{ij}\}\right)$ then

$$\mathcal{A}_{ij}^{-} \leq \text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \leq \mathcal{A}_{ij}^{+}$$

Proof. As $\mathcal{A}_{ij}$ represents an IFSNs so for all i, j, $\min_i \min_j \{\zeta_{ij}\} \leq \zeta_{ij} \leq \max_i \max_j \{\zeta_{ij}\}$ this implies $1 - \max_i \max_j \{\zeta_{ij}\} \leq 1 - \zeta_{ij} \leq 1 - \min_i \min_j \{\zeta_{ij}\} \Leftrightarrow (1 - \max_i \max_j \{\zeta_{ij}\})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \leq$

$(1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \leq (1-\min_i \min_j\{\zeta_{ij}\})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \Leftrightarrow 1-\max_i \max_j\{\zeta_{ij}\} \leq \prod_{j=1}^{n}(1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \leq$ $1-\max_i \min_j\{\zeta_{ij}\} \Leftrightarrow 1-\max_i \max_j\{\zeta_{ij}\} \leq \prod_{i=1}^{m}\left(\prod_{j=1}^{n}(1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \leq 1-\min_i \min_j\{\zeta_{ij}\}$ and hence

$$\min_i \min_j\{\zeta_{ij}\} \leq 1-\prod_{i=1}^{m}\left(\prod_{j=1}^{n}(1-\zeta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \leq \max_i \max_j\{\zeta_{ij}\} \tag{7.3}$$

Furthermore, $\min_i \min_j\{\vartheta_{ij}\} \leq \vartheta_{ij} \leq \max_i \max_j\{\vartheta_{ij}\} \Leftrightarrow (\min_i \min_j\{\vartheta_{ij}\})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \leq \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \leq$ $(\max_i \max_j\{\vartheta_{ij}\})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \Leftrightarrow \min_i \min_j\{\vartheta_{ij}\} \leq \prod_{j=1}^{n} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \leq \max_i \max_j\{\vartheta_{ij}\} \Leftrightarrow (\min_i \min_j\{\vartheta_{ij}\})^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \leq$ $\left(\prod_{j=1}^{n} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \leq (\max_i \max_j\{\vartheta_{ij}\})^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \Leftrightarrow (\min_i \min_j\{\vartheta_{ij}\})^{\frac{\sum_{i=1}^{m} T_i}{\sum_{s=1}^{m} T_s}} \leq \prod_{i=1}^{m}\left(\prod_{j=1}^{n} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \leq$ $(\max_i \max_j\{\vartheta_{ij}\})^{\frac{\sum_{i=1}^{m} T_i}{\sum_{s=1}^{m} T_s}}$ and hence

$$\min_i \min_j\{\vartheta_{ij}\} \leq \prod_{i=1}^{m}\left(\prod_{j=1}^{n} \vartheta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \leq \max_i \max_j\{\vartheta_{ij}\} \tag{7.4}$$

Let $\mathcal{A} \equiv \text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = (\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ then we have from Eq. (7.3) and (7.4), $\min_i \min_j\{\zeta_{ij}\} \leq \zeta_{\mathcal{A}} \leq \max_i \max_j\{\zeta_{ij}\}$ and $\min_i \min_j\{\vartheta_{ij}\} \leq \vartheta_{\mathcal{A}} \leq \max_i \max_j\{\vartheta_{ij}\}$.

Now, by definition of score function, we have

$$\begin{aligned} S(\mathcal{A}) &= \frac{1+\zeta_{\mathcal{A}}-\vartheta_{\mathcal{A}}}{2} \leq \frac{1+\max_i \max_j\{\zeta_{ij}\}-\min_i \min_j\{\vartheta_{ij}\}}{2} = S(\mathcal{A}_{ij}^{+}) \\ S(\mathcal{A}) &= \frac{1+\zeta_{\mathcal{A}}-\vartheta_{\mathcal{A}}}{2} \geq \frac{1+\min_j \min_i\{\zeta_{ij}\}-\max_j \max_i\{\vartheta_{ij}\}}{2} = S(\mathcal{A}_{ij}^{-}) \end{aligned}$$

In that direction, three cases are considered.

Case 1: If $S(\mathcal{A}_{ij}) < S(\mathcal{A}_{ij}^{+})$ and $S(\mathcal{A}_{ij}) > S(\mathcal{A}_{ij}^{-})$ then by comparison laws between two

IFSNs we have

$$\mathcal{A}_{ij}^{-} < \text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) < \mathcal{A}_{ij}^{+}$$

Case 2: If $S(\mathcal{A}_{ij}) = S(\mathcal{A}_{ij}^{+})$ i.e., $\zeta_{\mathcal{A}} - \vartheta_{\mathcal{A}} = \max_j \max_i \{\zeta_{ij}\} - \min_j \min_i \{\vartheta_{ij}\}$ then by above inequalities, we have $\zeta_{\mathcal{A}} = \max_j \max_i \{\zeta_{ij}\}$ and $\vartheta_{\mathcal{A}} = \min_j \min_i \{\vartheta_{ij}\}$. Thus

$$H(\mathcal{A}) = \zeta_{\mathcal{A}} + \vartheta_{\mathcal{A}} = \max_j \max_i \{\zeta_{ij}\} + \min_j \min_i \{\vartheta_{ij}\} = H(\mathcal{A}_{ij}^{+})$$

then by comparison laws between two IFSNs, we have

$$\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}_{ij}^{+}$$

Case 3: If $S(\mathcal{A}_{ij}) = S(\mathcal{A}_{ij}^{-})$ i.e., $\zeta_{\mathcal{A}} - \vartheta_{\mathcal{A}} = \min_j \min_i \{\zeta_{ij}\} - \max_j \max_i \{\vartheta_{ij}\}$ then by above inequalities, we have $\zeta_{\mathcal{A}} = \min_j \min_i \{\zeta_{ij}\}$ and $\vartheta_{\mathcal{A}} = \max_j \max_i \{\vartheta_{ij}\}$. Thus

$$H(\mathcal{A}) = \zeta_{\mathcal{A}} + \vartheta_{\mathcal{A}} = \min_j \min_i \{\zeta_{ij}\} + \max_j \max_i \{\vartheta_{ij}\} = H(\mathcal{A}_{ij}^{-})$$

then it follows that

$$\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}_{ij}^{-}$$

Hence, property holds.

□

Property 7.3.3. (Monotonicity) Let $\mathcal{A}'_{ij}$ be another IFSNs such that $\mathcal{A}_{ij} \leq \mathcal{A}'_{ij}$ for all i, j, then

$$\text{IFSPWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \leq \text{IFSPWA}(\mathcal{A}'_{11}, \mathcal{A}'_{12}, \ldots, \mathcal{A}'_{mn})$$

Proof. It is similar to Property 7.3.2, so we omit here. □

7.3.2 Prioritized ordered weighted averaging operator

Definition 7.3.3. Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij}), (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be IFSNs. Then an intuitionistic fuzzy soft prioritized ordered weighted average (IFSPOWA) operator is defined as follow:

$$\text{IFSPOWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = \bigoplus_{i=1}^{m} \frac{T_i}{\sum_{s=1}^{m} T_s} \left(\bigoplus_{j=1}^{n} \frac{R_j}{\sum_{r=1}^{n} R_r} \mathcal{A}_{\delta(i)\gamma(j)} \right) \tag{7.5}$$

where $R_j = \prod_{l=1}^{j-1} S(\mathcal{A}_{il})$ and $T_i = \prod_{k=1}^{i-1} S(\mathcal{A}_{\delta(k)})$. Let $R_1 = 1$, $T_1 = 1$ and $S(\mathcal{A}_e)$ represent score function of IFSN $\mathcal{A}_e$. and δ, γ are permutations of $(1, 2, \ldots, m)$ and $(1, 2, \ldots, n)$ such that $e_{\delta(i)j} \geq e_{\delta(i-1)j}$ and $e_{i\gamma(j)} \geq e_{i\gamma(j-1)}$ for any i, j.

Theorem 7.3.2. The aggregated value of all IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij}), (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ by using IFSPOWA operator is still an IFSN and is defined as:

$$\begin{aligned} &\mathrm{IFSPOWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \\ = &\left(1 - \prod_{i=1}^{m}\left(\prod_{j=1}^{n}\left(1 - \zeta_{\delta(i)\gamma(j)}\right)^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, \prod_{i=1}^{m}\left(\prod_{j=1}^{n} \vartheta_{\delta(i)\gamma(j)}^{\frac{R_j}{\sum_{r=1}^{n} R_r}}\right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}\right) \end{aligned} \tag{7.6}$$

where $R_1 = 1$, $T_1 = 1$, $R_j = \prod_{l=1}^{j-1} S(\mathcal{A}_{i\gamma(l)}), (j = 2, 3, \ldots, n)$ and $T_i = \prod_{k=1}^{i-1} S(\mathcal{A}_{\delta(k)}), (i = 2, 3, \ldots, m)$ where $S(\mathcal{A}_e)$ represent score function of IFSN $\mathcal{A}_e$.

Proof. Similar of this theorem is same as Theorem 7.3.1. □

Example 7.3.2. As given in Example 7.3.1, $(\mathcal{F}, \mathcal{E})$ is an IFSN. Now, by using score function, we have $S(e_{11}) = 0.85$, $S(e_{12}) = 0.60$, $S(e_{13}) = 0.45$. Thus, $S(e_{11}) \succ S(e_{12}) \succ S(e_{13})$ and therefore, $\mathcal{A}_{1\gamma(1)} = (0.8, 0.1), \mathcal{A}_{1\gamma(2)} = (0.5, 0.3), \mathcal{A}_{1\gamma(3)} = (0.4, 0.5)$. Similarly, we can find the other $\mathcal{A}_{i\gamma(j)}$. Hence, the ordered matrix of $e_{i\gamma(j)}$'s is given as

$$(\mathcal{F}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{ccc} \gamma(e_1) & \gamma(e_2) & \gamma(e_3) \end{array} \\ \begin{bmatrix} (0.8, 0.1) & (0.8, 0.2) & (0.5, 0.1) \\ (0.7, 0.2) & (0.4, 0.3) & (0.3, 0.5) \\ (0.7, 0.1) & (0.8, 0.2) & (0.5, 0.1) \\ (0.6, 0.1) & (0.5, 0.2) & (0.3, 0.5) \end{bmatrix} \end{array}$$

Furthermore, $S(\mathcal{A}_{1\gamma(1)}) = 0.85, S(\mathcal{A}_{2\gamma(1)}) = 0.75, S(\mathcal{A}_{3\gamma(1)}) = 0.8, S(\mathcal{A}_{4\gamma(1)}) = 0.75$, therefore

$$S(\mathcal{A}_{1\gamma(1)}) \succ S(\mathcal{A}_{3\gamma(1)}) \succ S(\mathcal{A}_{2\gamma(1)}) \succ S(\mathcal{A}_{4\gamma(1)}).$$

Thus, $\mathcal{A}_{\delta(1)\gamma(1)} = (0.8, 0.1), \mathcal{A}_{\delta(2)\gamma(1)} = (0.7, 0.1), \mathcal{A}_{\delta(3)\gamma(1)} = (0.7, 0.2), \mathcal{A}_{\delta(4)\gamma(1)} = (0.6, 0.1)$. Then the ordered matrix for IFSS $(\mathcal{F}, \mathcal{E})$ is:

$$(\mathcal{F}, \mathcal{E}) = \begin{array}{c} \\ \delta(u_1) \\ \delta(u_2) \\ \delta(u_3) \\ \delta(u_4) \end{array} \begin{array}{c} \begin{array}{ccc} \gamma(e_1) & \gamma(e_2) & \gamma(e_3) \end{array} \\ \begin{bmatrix} (0.8, 0.1) & (0.8, 0.2) & (0.5, 0.1) \\ (0.7, 0.1) & (0.5, 0.2) & (0.4, 0.5) \\ (0.7, 0.2) & (0.5, 0.3) & (0.3, 0.5) \\ (0.6, 0.1) & (0.4, 0.3) & (0.3, 0.5) \end{bmatrix} \end{array}$$

Now, by using, $R_1 = 1$, $T_1 = 1$, $R_j = \prod\limits_{l=1}^{j-1} S(\mathcal{A}_{i\gamma(l)}), (j = 2, 3)$ and $T_i = \prod\limits_{k=1}^{i-1} S(\mathcal{A}_{\delta(k)}), (i = 2, 3, 4)$, we get

$$R = \begin{bmatrix} 1 & 0.85 & 0.80 \\ 1 & 0.80 & 0.65 \\ 1 & 0.75 & 0.60 \\ 1 & 0.75 & 0.55 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 \\ 0.7868 \\ 0.6376 \\ 0.6583 \end{bmatrix}$$

Hence,

$$\begin{aligned} & \text{IFSPOWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{43}) \\ = & \left(1 - \prod_{i=1}^{4} \left(\prod_{j=1}^{3} \left(1 - \zeta_{\delta(i)\gamma(j)}\right)^{\frac{R_j}{\sum_{r=1}^{3} R_j}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}}, \prod_{i=1}^{4} \left(\prod_{j=1}^{3} \vartheta_{\delta(i)\gamma(j)}^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}} \right) \\ = & (0.5964, 0.2127) \end{aligned}$$

As similar to IFSPWA operator, IFSPOWA operator also satisfies some properties:

Property 7.3.4. Let $\mathcal{A}_{ij}$ and $\mathcal{A}'_{ij}, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be two collections of IFSNs, then

(i) (Idempotancy) If all $\mathcal{A}_{ij} = \mathcal{A}_e$, then $\text{IFSPOWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = \mathcal{A}_e$.

(ii) (Boundedness) Let $\mathcal{A}^-_{ij} = (\min\limits_i \min\limits_j \{\zeta_{ij}\}, \max\limits_i \max\limits_j \{\vartheta_{ij}\})$ and $\mathcal{A}^+_{ij} = (\max\limits_i \max\limits_j \{\zeta_{ij}\}, \min\limits_i \min\limits_j \{\vartheta_{ij}\})$, then $\mathcal{A}^-_{ij} \leq \text{IFSPOWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \leq \mathcal{A}^+_{ij}$.

(iii) (Monotonicity) If $\mathcal{A}_{ij} \leq \mathcal{A}'_{ij}$, for all i, j, then $\text{IFSPOWA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \leq \text{IFSPOWA}(\mathcal{A}'_{11}, \mathcal{A}'_{12}, \ldots, \mathcal{A}'_{mn})$

7.3.3 Prioritized weighted geometric operator

Definition 7.3.4. Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij}), (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be collection of IFSNs. Then an IFSPWG operator is defined as follow:

$$\text{IFSPWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = \bigotimes_{i=1}^{m} \left(\bigotimes_{j=1}^{n} \mathcal{A}_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \tag{7.7}$$

where $R_1 = 1,\ T_1 = 1,\ R_j = \prod_{l=1}^{j-1} S(\mathcal{A}_{il}), (j = 2, 3, \ldots, n)$ and $T_i = \prod_{k=1}^{i-1} S(\mathcal{A}_k)(i = 2, 3, \ldots, m)$ where $S(\mathcal{A}_e)$ represents the score function of IFSN $\mathcal{A}_e$.

Theorem 7.3.3. The aggregated value of all IFSNs $\mathcal{A}_{ij}$ by using IFSPWG operator is still an IFSN and is defined as:

$$\begin{aligned} &\text{IFSPWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \\ &= \left(\prod_{i=1}^{m} \left(\prod_{j=1}^{n} \zeta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, 1 - \prod_{i=1}^{m} \left(\prod_{j=1}^{n} (1 - \vartheta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \end{aligned} \tag{7.8}$$

Proof. We prove this result by mathematical induction on n and m. For $n = 1$,

$$\begin{aligned} &\text{IFSPWG}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{m1}) \\ &= \bigotimes_{i=1}^{m} \mathcal{A}_{i1}^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \\ &= \left(\prod_{i=1}^{m} \zeta_{i1}^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, 1 - \prod_{i=1}^{m} (1 - \vartheta_{i1})^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \\ &= \left(\prod_{i=1}^{m} \left(\prod_{j=1}^{1} \zeta_{ij}^{\frac{R_j}{\sum_{r=1}^{1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, 1 - \prod_{i=1}^{m} \left(\prod_{j=1}^{1} (1 - \vartheta_{ij})^{\frac{R_j}{\sum_{r=1}^{1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \end{aligned}$$

and for $m = 1$,

$$
\begin{aligned}
&\mathrm{IFSPWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1n}) \\
= &\bigotimes_{j=1}^{n} \mathcal{A}_{1j}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \\
= &\left(\prod_{j=1}^{n} \zeta_{1j}^{\frac{R_j}{\sum_{r=1}^{n} R_r}}, 1 - \prod_{j=1}^{n} (1 - \vartheta_{1j})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right) \\
= &\left(\prod_{i=1}^{1} \left(\prod_{j=1}^{n} \zeta_{ij}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{1} T_s}}, 1 - \prod_{i=1}^{1} \left(\prod_{j=1}^{n} (1 - \vartheta_{ij})^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{1} T_s}} \right)
\end{aligned}
$$

Assume the result is true for $m = k_1 + 1, n = k_2$ and $m = k_1, n = k_2 + 1$, i.e.

$$
\begin{aligned}
&\bigotimes_{i=1}^{k_1+1} \left(\bigotimes_{j=1}^{k_2} \mathcal{A}_{ij}^{\frac{R_j}{\sum_{r=1}^{k_2} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1+1} T_s}} \\
= &\left(\prod_{i=1}^{k_1+1} \left(\prod_{j=1}^{k_2} \zeta_{ij}^{\frac{R_j}{\sum_{r=1}^{k_2} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1+1} T_s}}, 1 - \prod_{i=1}^{k_1+1} \left(\prod_{j=1}^{k_2} (1 - \vartheta_{ij})^{\frac{R_j}{\sum_{r=1}^{k_2} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1+1} T_s}} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigotimes_{i=1}^{k_1} \left(\bigotimes_{j=1}^{k_2+1} \mathcal{A}_{ij}^{\frac{R_j}{\sum_{r=1}^{k_2+1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1} T_s}} \\
= &\left(\prod_{i=1}^{k_1} \left(\prod_{j=1}^{k_2+1} \zeta_{ij}^{\frac{R_j}{\sum_{r=1}^{k_2+1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1} T_s}}, 1 - \prod_{i=1}^{k_1} \left(\prod_{j=1}^{k_2+1} (1 - \vartheta_{ij})^{\frac{R_j}{\sum_{r=1}^{k_2+1} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{k_1} T_s}} \right)
\end{aligned}
$$

Now to prove the result is true for $m = k_1 + 1, n = k_2 + 1$

$$\begin{aligned}
&\bigotimes_{i=1}^{k_1+1}\left(\bigotimes_{j=1}^{k_2+1}\mathcal{A}_{ij}^{\frac{R_j}{\sum\limits_{r=1}^{k_2+1}R_r}}\right)^{\frac{T_i}{\sum\limits_{s=1}^{k_1+1}T_s}}\\
&=\bigotimes_{i=1}^{k_1+1}\left(\bigotimes_{j=1}^{k_2}\mathcal{A}_{ij}^{\frac{R_j}{\sum\limits_{r=1}^{k_2+1}R_r}}\otimes\mathcal{A}_{(k_2+1)j}^{\frac{R_{k_2+1}}{\sum\limits_{r=1}^{k_2+1}R_r}}\right)^{\frac{T_i}{\sum\limits_{s=1}^{k_1+1}T_s}}\\
&=\left(\bigotimes_{i=1}^{k_1+1}\left(\bigotimes_{j=1}^{k_2}\mathcal{A}_{ij}^{\frac{R_j}{\sum\limits_{r=1}^{k_2+1}R_r}}\right)^{\frac{T_i}{\sum\limits_{s=1}^{k_1+1}T_s}}\right)\bigotimes_{i=1}^{k_1+1}\left(\mathcal{A}_{(k_2+1)j}^{\frac{R_{k_2+1}}{\sum\limits_{r=1}^{k_2+1}R_r}}\right)^{\frac{T_i}{\sum\limits_{s=1}^{k_1+1}T_s}}\\
&=\left(\prod_{i=1}^{k_1+1}\left(\prod_{j=1}^{k_2+1}\zeta_{ij}^{\frac{R_j}{\sum\limits_{r=1}^{k_2+1}R_r}}\right)^{\frac{T_i}{\sum\limits_{s=1}^{k_1+1}T_s}}, 1-\prod_{i=1}^{k_1+1}\left(\prod_{j=1}^{k_2+1}(1-\vartheta_{ij})^{\frac{R_j}{\sum\limits_{r=1}^{k_2+1}R_r}}\right)^{\frac{T_i}{\sum\limits_{s=1}^{k_1+1}T_s}}\right)
\end{aligned}$$

It shows the result holds for $m = k_1 + 1$ and $n = k_2 + 1$, thus by mathematical induction the result holds for all $m, n \geq 1$. □

Example 7.3.3. Let $\mathcal{X} = \{u_1, u_2, u_3, u_4\}$ be set of experts which give their preferences for describing their "usage of a mobile" on certain parameters $\mathcal{E} = \{e_1, e_2, e_3\}$, then IFSS $(\mathcal{F}, \mathcal{E})$ is defined as

$$(\mathcal{F},\mathcal{E}) = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} & \begin{bmatrix} (0.7,0.2) & (0.4,0.1) & (0.6,0.1) \\ (0.7,0.1) & (0.5,0.3) & (0.3,0.6) \\ (0.6,0.1) & (0.5,0.2) & (0.8,0.1) \\ (0.6,0.4) & (0.8,0.1) & (0.6,0.1) \end{bmatrix} \end{array}$$

Based on these data, we get

$$R = \begin{bmatrix} 1 & 0.75 & 0.65 \\ 1 & 0.80 & 0.60 \\ 1 & 0.75 & 0.65 \\ 1 & 0.60 & 0.85 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 \\ 0.35 \\ 0.37 \\ 0.37 \end{bmatrix}$$

and hence by using Eq. (7.8), we get:

$$\begin{aligned} & \text{IFSPWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{43}) \\ = & \left(\prod_{i=1}^{4} \left(\prod_{j=1}^{3} \zeta_{ij}^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}}, 1 - \prod_{i=1}^{4} \left(\prod_{j=1}^{3} (1-\vartheta_{ij})^{\frac{R_j}{\sum_{r=1}^{3} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}} \right) \\ = & \ (0.84, 0.42) \end{aligned}$$

7.3.4 Prioritized ordered weighted geometric operator

Definition 7.3.5. Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij}), (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be the collection of IFSNs. Then an IFSPOWG operator is defined as follow:

$$\text{IFSPOWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) = \bigotimes_{i=1}^{m} \left(\bigotimes_{j=1}^{n} \mathcal{A}_{\delta(i)\gamma(j)}^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \tag{7.9}$$

where $R_1 = 1$, $T_1 = 1$ $R_j = \prod_{l=1}^{j-1} S(\mathcal{A}_{il})$, $T_i = \prod_{k=1}^{i-1} S(\mathcal{A}_{\delta(k)})$and $S(\mathcal{A}_e)$ represents the score function of IFSN $\mathcal{A}_e$ and δ, γ are permutations of $(1, 2, \ldots, m)$ and $(1, 2, \ldots, n)$ such that $e_{\delta(i)j} \geq e_{\delta(i-1)j}$ and $e_{i\gamma(j)} \geq e_{i\gamma(j-1)}$ for any i, j.

Theorem 7.3.4. The aggregated value of all IFSNs $\mathcal{A}_{ij}$ by using IFSPOWG operator is still an IFSN and is defined as:

$$\begin{aligned} & \text{IFSPOWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{mn}) \\ = & \left(\prod_{i=1}^{m} \left(\prod_{j=1}^{n} \zeta_{\delta(i)\gamma(j)}^{\frac{R_j}{\sum_{r=1}^{m} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}}, 1 - \prod_{i=1}^{m} \left(\prod_{j=1}^{n} \left(1-\vartheta_{\delta(i)\gamma(j)}\right)^{\frac{R_j}{\sum_{r=1}^{n} R_r}} \right)^{\frac{T_i}{\sum_{s=1}^{m} T_s}} \right) \end{aligned} \tag{7.10}$$

where $R_1 = 1$, $T_1 = 1$, $R_j = \prod_{l=1}^{j-1} S(\mathcal{A}_{i\gamma(l)}), j = 2, 3, \ldots, n$, $T_i = \prod_{k=1}^{i-1} S(\mathcal{A}_{\delta(k)}) (i = 2, 3, \ldots, m)$.

Proof. It is similar to the Theorem 7.3.3, so we omit here. $\square$

Example 7.3.4. Considered the Example 7.3.3 and by using score function, we get the ordered matrix as:

$$(\mathcal{F}, \mathcal{E}) = \begin{array}{c} \\ \delta(u_1) \\ \delta(u_2) \\ \delta(u_3) \\ \delta(u_4) \end{array} \begin{array}{c} \begin{array}{ccc} \gamma(e_1) & \gamma(e_2) & \gamma(e_3) \end{array} \\ \begin{bmatrix} (0.8, 0.1) & (0.6, 0.1) & (0.4, 0.1) \\ (0.8, 0.1) & (0.6, 0.1) & (0.5, 0.2) \\ (0.7, 0.1) & (0.6, 0.1) & (0.6, 0.4) \\ (0.7, 0.2) & (0.5, 0.3) & (0.3, 0.6) \end{bmatrix} \end{array}$$

Thus, based on it, values of $R_j = \prod_{l=1}^{j-1} S(\mathcal{A}_{il}), j = 2, 3$ and $T_i = \prod_{k=1}^{i-1} S(\mathcal{A}_k), i = 2, 3, 4$, as:

$$R = \begin{bmatrix} 1 & 0.85 & 0.75 \\ 1 & 0.85 & 0.75 \\ 1 & 0.80 & 0.75 \\ 1 & 0.75 & 0.60 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 \\ 0.35 \\ 0.38 \\ 0.39 \end{bmatrix}$$

Hence, by Eq. (7.10), we get

$$\begin{aligned} & \text{IFSPOWG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{e_{43}}) \\ = & \left(\prod_{i=1}^{4} \left(\prod_{j=1}^{3} \zeta_{\delta(i)\gamma(j)}^{\frac{R_j}{\sum_{r=1}^{3} R_j}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_i}}, 1 - \prod_{i=1}^{4} \left(\prod_{j=1}^{3} \left(1 - \vartheta_{\delta(i)\gamma(j)}\right)^{\frac{R_j}{\sum_{r=1}^{3} R_j}} \right)^{\frac{T_i}{\sum_{s=1}^{4} T_s}} \right) \\ = & (0.85, 0.41) \end{aligned}$$

As similar to IFSPWA and IFSPOWA operators, IFSPWG and IFSPOWG operators also satisfies same properties for the collection of IFSNs $\mathcal{A}_{ij}$.

7.4 Proposed approach

The description of the DM problem is given in Section 2.6 of Chapter 2, where the each alternative $\mathcal{A}^{(b)}$ is assessed by an expert and rates them in the form of IFSNs $\mathcal{A}_{ij}^{(b)} = (\zeta_{ij}^{(b)}, \vartheta_{ij}^{(b)})$. Then, the following steps of the proposed approach are summarized as below to get the best alternative(s).

Step 1: Collect and arrange the information of the alternatives in terms of intuitionistic fuzzy soft matrices $\mathcal{M}^{(b)} = (\mathcal{A}_{ij}^{(b)}) = (\zeta_{ij}^{(b)}, \vartheta_{ij}^{(b)})_{m\times n}$.

Step 2: Normalize the values, if required, by using Eq. (2.34) of Chapter 2 and get the normalized decision matrix $\mathcal{D}^{(b)} = (r_{ij}^{(b)})$.

Step 3: Calculate the $R_j^{(b)}(j = 1, 2, \ldots, n)$, $T_i^{(b)}(i = 1, 2, \ldots, m)$ as follows:

$$T_1^{(b)} = 1, R_1^{(b)} = 1 \tag{7.11}$$

$$R_j^{(b)} = \prod_{l=1}^{j-1} S(\mathcal{A}_{il}^{(b)}),\ j = 1, 2, \ldots, n \tag{7.12}$$

$$\text{and} \qquad T_i^{(b)} = \prod_{k=1}^{i-1} S(\mathcal{A}_{k}^{(b)}),\ i = 1, 2, \ldots, m \tag{7.13}$$

Step 4: Aggregate the IFSNs $r_{ij}^{(b)}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ for each alternative $A^{(b)}(b = 1, 2, \ldots, t)$ into collective preference value $r^{(b)} = (\zeta^{(b)}, \vartheta^{(b)})$ by the proposed IFSPWA (or IFSPWG, IFSPOWA, IFSPOWG) operator.

Step 5: Compute the score function of $r^{(b)}$ by using equation

$$S(r^{(b)}) = \zeta^{(b)} - \vartheta^{(b)} \tag{7.14}$$

Step 6: Rank the alternatives $A^{(b)}, (b = 1, 2, \ldots, t)$ and select the best one(s).

7.5 Numerical Example

The above DM procedure has been illustrated with a practical example of the recruitment of Professor in the Mathematics department for a central Government University.

7.5.1 Case study

For it, consider a panel of five experts u_1, u_2, u_3, u_4, u_5 will judge four candidates $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, $A^{(4)}$ and select one candidate on the basis of certain parameters $\mathcal{E}$={"Qualification (e_1)", "Teaching experience(e_2)", "research experience (e_3)", "number of publications (e_4)" and "Teaching Ability (e_5)"}. Then, we implemented the proposed approach steps to get the most desirable alternative(s).

Step 1: The rating values of each alternative is provided by the five experts under the parameters and summarized in Table 7.1.

Table 7.1: Rating values of the candidates in terms of IFSNs

	For alternative $A^{(1)}$					For alternative $A^{(2)}$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.3, 0.4)$	$(0.5, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.6, 0.2)$	$(0.4, 0.3)$	$(0.5, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.7, 0.2)$
u_2	$(0.6, 0.1)$	$(0.6, 0.2)$	$(0.2, 0.4)$	$(0.5, 0.1)$	$(0.7, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.3)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0.4, 0.1)$
u_3	$(0.5, 0.1)$	$(0.7, 0.2)$	$(0.5, 0.4)$	$(0.2, 0.2)$	$(0.4, 0.2)$	$(0.5, 0.3)$	$(0.5, 0.1)$	$(0.5, 0.3)$	$(0.3, 0.2)$	$(0.6, 0.2)$
u_4	$(0.2, 0.4)$	$(0.5, 0.1)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.7, 0.3)$	$(0.4, 0.2)$	$(0.5, 0.1)$	$(0.5, 0.2)$
u_5	$(0.6, 0.1)$	$(0.3, 0.4)$	$(0.4, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.4, 0.2)$	$(0.5, 0.2)$	$(0.3, 0.3)$	$(0.6, 0.1)$	$(0.4, 0.2)$
	For alternative $A^{(3)}$					For alternative $A^{(4)}$				
	e_1	e_2	e_3	e_4	e_5					
u_1	$(0.4, 0.3)$	$(0.5, 0.4)$	$(0.5, 0.2)$	$(0.6, 0.1)$	$(0.4, 0.2)$	$(0.3, 0.4)$	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.4, 0.3)$	$(0.2, 0.3)$
u_2	$(0.5, 0.1)$	$(0.3, 0.2)$	$(0.3, 0.2)$	$(0.4, 0.2)$	$(0.3, 0.2)$	$(0.5, 0.1)$	$(0.4, 0.2)$	$(0.4, 0.2)$	$(0.6, 0.1)$	$(0.2, 0.6)$
u_3	$(0.5, 0.3)$	$(0.5, 0.1)$	$(0.4, 0.2)$	$(0.2, 0.2)$	$(0.5, 0.4)$	$(0.2, 0.1)$	$(0.4, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.5, 0.2)$
u_4	$(0.5, 0.1)$	$(0.4, 0.5)$	$(0.3, 0.2)$	$(0.7, 0.2)$	$(0.3, 0.2)$	$(0.7, 0.2)$	$(0.5, 0.1)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.7, 0.1)$
u_5	$(0.7, 0.1)$	$(0.4, 0.6)$	$(0.4, 0.2)$	$(0.3, 0.1)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.3, 0.2)$	$(0.7, 0.1)$

Step 2: As all the parameters are of same types, so there is no need of normalization.

Step 3: By Eqs. (7.11)-(7.13), we get the values of $R_j^{(b)}, T_i^{(b)} (b = 1, 2, 3, 4)$ as

$$R_j^{(1)} = \begin{bmatrix} 1 & 0.45 & 0.70 & 0.70 & 0.80 \\ 1 & 0.75 & 0.70 & 0.40 & 0.70 \\ 1 & 0.70 & 0.75 & 0.55 & 0.50 \\ 1 & 0.40 & 0.70 & 0.75 & 0.65 \\ 1 & 0.75 & 0.45 & 0.55 & 0.75 \end{bmatrix}, R_j^{(2)} = \begin{bmatrix} 1 & 0.55 & 0.70 & 0.70 & 0.80 \\ 1 & 0.75 & 0.60 & 0.55 & 0.55 \\ 1 & 0.60 & 0.70 & 0.60 & 0.55 \\ 1 & 0.60 & 0.70 & 0.60 & 0.70 \\ 1 & 0.60 & 0.65 & 0.50 & 0.75 \end{bmatrix}$$

$$R_j^{(3)} = \begin{bmatrix} 1 & 0.55 & 0.55 & 0.65 & 0.75 \\ 1 & 0.70 & 0.55 & 0.55 & 0.60 \\ 1 & 0.60 & 0.70 & 0.60 & 0.50 \\ 1 & 0.70 & 0.45 & 0.55 & 0.75 \\ 1 & 0.80 & 0.40 & 0.60 & 0.60 \end{bmatrix}, R_j^{(4)} = \begin{bmatrix} 1 & 0.45 & 0.85 & 0.80 & 0.55 \\ 1 & 0.70 & 0.60 & 0.60 & 0.75 \\ 1 & 0.55 & 0.60 & 0.55 & 0.60 \\ 1 & 0.75 & 0.70 & 0.75 & 0.65 \\ 1 & 0.65 & 0.55 & 0.60 & 0.55 \end{bmatrix}$$

$$T_i^{(1)} = \begin{bmatrix} 1 \\ 0.6760 \\ 0.6834 \\ 0.6554 \\ 0.6427 \end{bmatrix}, T_i^{(2)} = \begin{bmatrix} 1 \\ 0.7063 \\ 0.6521 \\ 0.6360 \\ 0.6554 \end{bmatrix}, T_i^{(3)} = \begin{bmatrix} 1 \\ 0.6272 \\ 0.6092 \\ 0.6076 \\ 0.6327 \end{bmatrix}, T_i^{(4)} = \begin{bmatrix} 1 \\ 0.6474 \\ 0.6243 \\ 0.6017 \\ 0.7403 \end{bmatrix}$$

Step 4: Without loss of generality, we take the IFSPWA operator to aggregate the different preferences by Eq. (7.2) and hence get the collective values $r^{(b)}$ as $r^{(1)} = (0.5166, 0.1853)$, $r^{(2)} = (0.5157, 0.1942)$, $r^{(3)} = (0.4614, 0.1943)$, and $r^{(4)} = (0.4932, 0.1839)$.

Step 5: By Eq. (7.14), we get $S(r^{(1)}) = 0.6657$, $S(r^{(2)}) = 0.6608$, $S(r^{(3)}) = 0.6335$, and $S(r^{(4)}) = 0.6547$.

Step 6: Since $S(r^{(1)}) \succ S(r^{(2)}) \succ S(r^{(4)}) \succ S(r^{(3)})$ and hence $A^{(1)}$ is best candidate for the required post.

On the other hand, if we utilize IFSPWG operator to aggregate the information, then the steps of the proposed approach are executed as follows:

Step 1: Same as that of above.

Step 2: All parameters are of same types, so no need of normalization.

Step 3: By Eq. (7.11)-(7.13), we get

$$T_i^{(1)} = \begin{bmatrix} 1 \\ 0.6335 \\ 0.629 \\ 0.6132 \\ 0.5889 \end{bmatrix}, T_i^{(2)} = \begin{bmatrix} 1 \\ 0.6803 \\ 0.6280 \\ 0.6178 \\ 0.6376 \end{bmatrix}, T_i^{(3)} = \begin{bmatrix} 1 \\ 0.6075 \\ 0.5967 \\ 0.5795 \\ 0.5873 \end{bmatrix}, T_i^{(4)} = \begin{bmatrix} 1 \\ 0.5736 \\ 0.5624 \\ 0.5692 \\ 0.7223 \end{bmatrix}$$

Step 4: By taking Eq. (7.2), we get the aggregated values of each alternative as $r^{(1)} = (0.4632, 0.2251)$, $r^{(2)} = (0.4897, 0.2152)$, $r^{(3)} = (0.4295, 0.2378)$, and $r^{(4)} = (0.4319, 0.2285)$.

Step 5: The score values of $r^{(b)}$ are computed as $S(r^{(1)}) = 0.6191$, $S(r^{(2)}) = 0.6372$, $S(r^{(3)}) = 0.5958$, and $S(r^{(4)}) = 0.6017$.

Step 6: Therefore, the raking is $S(r^{(2)}) \succ S(r^{(1)}) \succ S(r^{(4)}) \succ S(r^{(3)})$ and hence $A^{(2)}$ is best candidate for the required post.

7.5.2 Comparative studies

To demonstrate the effectiveness of the proposed approach as compared to the existing ones approaches, an analysis has been conducted by using different operators as proposed by various researchers [197, 206, 209, 227, 231]. For this, grades corresponding to different parameters to each candidate are aggregated by geometric operator corresponding to weight vector $(0.2, 0.2, 0.2, 0.2, 0.2)^T$ and their aggregated results are summarized in Table 7.2. By using these aggregated values, the different approaches as proposed by various

Table 7.2: Aggregated intuitionistic fuzzy soft matrix for candidates

	$A^{(1)}$	$A^{(2)}$	$A^{(3)}$	$A^{(4)}$
u_1	$(0.4807, 0.2150)$	$(0.5368, 0.1841)$	$(0.4554, 0.2921)$	$(0.4103, 0.2552)$
u_2	$(0.5796, 0.2052)$	$(0.4786, 0.2063)$	$(0.3508, 0.1761)$	$(0.3704, 0.2990)$
u_3	$(0.4854, 0.1879)$	$(0.4973, 0.1991)$	$(0.4512, 0.2497)$	$(0.3596, 0.1879)$
u_4	$(0.4107, 0.2104)$	$(0.5562, 0.2528)$	$(0.4103, 0.3011)$	$(0.5839, 0.1261)$
u_5	$(0.4407, 0.2512)$	$(0.4440, 0.1959)$	$(0.4947, 0.3264)$	$(0.5111, 0.2550)$

authors [197, 206, 209, 227, 231] have been applied on it and their score values as well as ranking for each candidate are computed and tabulated in Table 7.3. From these results, we have concluded that $\mathcal{A}^{(2)}$ is still the best candidate for the required post.

Table 7.3: Comparative Studies with the existing studies

Method	Score value of				Ranking
	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
Xu and Yager [231]	0.1255	0.1411	0.0920	0.1086	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Xu [227]	0.1271	0.1443	0.0951	0.1175	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Wang and Liu [209]	0.1257	0.1416	0.0925	0.1098	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Wang and Liu [206]	0.1269	0.1438	0.0946	0.1162	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Verma and Sharma [197]	0.6275	0.6435	0.5957	0.6144	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$

7.6 Conclusion

The aim of this chapter is to present the intuitionistic fuzzy soft prioritized aggregation operator to solve the DM problem in which the priority level for each parameter and expert is different. For this, a series of prioritized averaging and geometric aggregation operators such as IFSPWA, IFSPOWA, IFSPWG, and IFSPOWG have been presented. The important characteristic of these operators is that they take the parameters and decision makers according to their priority level. Finally, an approach for solving MCDM problems under the intuitionistic fuzzy soft set environment has been given. To demonstrate the proposed work, a practical example of the recruitment of the candidate is given and the results obtained from the proposed approach are compared with the existing methods.

Chapter 8

Scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision making[1]

In this chapter, we present two new scaled prioritized averaging aggregation operators by considering the interaction between the membership degrees for the intuitionistic fuzzy soft sets. The principal advantage of the operators is that they consider the priority relationships between the parameters as well as experts. Furthermore, some properties based on these operators are discussed in detail. Then, we utilized these operators to solve DM problem and validate it with a numerical example.

8.1 Introduction

Decision-making (DM) is one of the widely interesting topics in these days to choose a suitable alternative from a certain goal. Initially, it is assumed that the information about the alternatives is taken as a crisp number, but in a real-life situation, the collective data always contain imprecise and vague information. From the literature, it is observed that

[1]The content of this chapter is published as "Novel scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision making", *Engineering Applications of Artificial Intelligence*, 71, 100 - 112, 2018 (**SCI: Impact Factor 3.526**)

the aggregation operators are based on the algebraic sum and product to carry out the aggregation process. However, all these aggregation processes do not take into account the interdependency relationship between the attributes and the parameters. Considering the importance of the relationship between the attributes, prioritized aggregation (PA) operators are one of the great significance of the decision problem. For it, Yager [241] introduced the prioritized averaging operator. Xu and Yager [233] investigated the MCDM based on prioritized aggregation operators. Wei [212] investigated the PA operator under hesitant fuzzy environment. In [75, 76], authors proposed the scaled prioritized intuitionistic fuzzy interaction aggregation operator with consideration of the priority relationship between the different criterion.

Thus, in this chapter, by keeping the advantages of PA operators and by taking the specific priority degrees of different parameters and the experts, we develop the scaled prioritized intuitionistic fuzzy soft interaction averaging (SPIFSIA) operator and its generalized form GSPIFSIA. In these proposed operators, the weighting vectors depend upon the input arguments and aggregated the values based on the priority degrees, and study their desirable properties. Then, we developed a new approach to MCDM problems under the IFSS environment based on the proposed aggregation operators. The feasibility, as well as the superiority of the proposed approach, has been illustrated with a numerical example. The main research contents of this chapter are divided into three parts:

(i) to propose some new interaction operations laws for IFSNs.

(ii) to introduce the scaled PA operator under IFSS environment and its generalized form,

(iii) to establish an MCDM method based on these proposed operators.

8.2 Proposed scaled prioritized aggregation operators

In this section, we develop some scaled prioritized aggregation operators under the IFSS environment.

8.2.1 Improved operational laws for IFSNs

Definition 8.2.1. Let $\mathcal{A} = (\zeta, \vartheta), \mathcal{A}_{11} = (\zeta_{11}, \vartheta_{11})$ and $\mathcal{A}_{12} = (\zeta_{12}, \vartheta_{12})$ are IFSNs and a real number $\lambda > 0$. Then the proposed operational laws defined for it are summarized as follows:

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \Big(1 - (1-\zeta_{11})(1-\zeta_{12}), (1-\zeta_{11})(1-\zeta_{12}) - (1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12})\Big)$.

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \Big((1-\vartheta_{11})(1-\vartheta_{12}) - (1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}), 1-(1-\vartheta_{11})(1-\vartheta_{12})\Big)$.

(iii) $\lambda\mathcal{A} = \Big(1-(1-\zeta)^{\lambda}, (1-\zeta)^{\lambda} - (1-\zeta-\vartheta)^{\lambda}\Big)$.

(iv) $\mathcal{A}^{\lambda} = \Big((1-\vartheta)^{\lambda} - (1-\zeta-\vartheta)^{\lambda}, 1-(1-\vartheta)^{\lambda}\Big)$.

Theorem 8.2.1. The operations defined in Definition 8.2.1 for IFSNs are also IFSNs.

Proof. Since $\mathcal{A}_{1j}(j = 1, 2)$ are IFSNs, therefore $0 \leq \zeta_{1j}, \vartheta_{1j} \leq 1$ and $\zeta_{1j} + \vartheta_{1j} \leq 1$ which implies $0 \leq (1-\zeta_{11})(1-\zeta_{12}) \leq 1 \Leftrightarrow 0 \leq 1-(1-\zeta_{11})(1-\zeta_{12}) \leq 1$ and $1-\zeta_{1j}-\vartheta_{1j} \leq 1-\zeta_{1j}$ which implies $(1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}) \leq (1-\zeta_{11})(1-\zeta_{12}) \leq 1 \Leftrightarrow 0 \leq (1-\zeta_{11})(1-\zeta_{12}) - (1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}) \leq 1-(1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}) \leq 1 \Leftrightarrow 0 \leq (1-\zeta_{11})(1-\zeta_{12}) - (1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}) \leq 1$. Further, $1-(1-\zeta_{11})(1-\zeta_{12}) + (1-\zeta_{11})(1-\zeta_{12}) - (1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}) = 1-(1-\zeta_{11}-\vartheta_{11})(1-\zeta_{12}-\vartheta_{12}) \leq 1$. Thus, $\mathcal{A}_{11} \oplus \mathcal{A}_{12}$ is an IFSN.

Also, $0 \leq (1-\zeta)^{\lambda} \leq 1$ which implies $0 \leq 1-(1-\zeta)^{\lambda} \leq 1$ and $(1-\zeta-\vartheta)^{\lambda} \leq (1-\zeta)^{\lambda} \leq 1$ which implies $0 \leq (1-\zeta)^{\lambda} - (1-\zeta-\vartheta)^{\lambda} \leq 1-(1-\zeta-\vartheta)^{\lambda} \leq 1$. Further, $1-(1-\zeta)^{\lambda} + (1-\zeta)^{\lambda} - (1-\zeta-\vartheta)^{\lambda} = 1-(1-\zeta-\vartheta)^{\lambda} \leq 1$. Thus, $\lambda\mathcal{A}$ is also an IFSN. Similarly, we can prove for the other parts. □

The proposed operation laws also satisfy certain properties which are stated as below.

Theorem 8.2.2. (Commutative law) If $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})(j = 1, 2)$ be two IFSNs, then

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \mathcal{A}_{12} \oplus \mathcal{A}_{11}$

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \mathcal{A}_{12} \otimes \mathcal{A}_{11}$

Theorem 8.2.3. (Associative law) If $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})(j = 1, 2, 3)$ be three IFSNs, then

(i) $(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) \oplus \mathcal{A}_{13} = \mathcal{A}_{11} \oplus (\mathcal{A}_{12} \oplus \mathcal{A}_{13})$

(ii) $(\mathcal{A}_{11} \otimes \mathcal{A}_{12}) \otimes \mathcal{A}_{13} = \mathcal{A}_{11} \otimes (\mathcal{A}_{12} \otimes \mathcal{A}_{13})$

Proof of these Theorems are easily obtained from their definitions, so we omit their proofs here.

Theorem 8.2.4. Let $\mathcal{A} = (\zeta, \vartheta)$ and $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})(j = 1, 2)$ be three IFSNs and real numbers $\lambda, \lambda_1, \lambda_2 > 0$, we have

(i) $\lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) = \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12}$.

(ii) $(\mathcal{A}_{11} \otimes \mathcal{A}_{12})^\lambda = \mathcal{A}_{11}^\lambda \otimes \mathcal{A}_{12}^\lambda$.

(iii) $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$.

(iv) $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1+\lambda_2}$.

Proof. Here, we prove the parts (i) & (iii) only and the proof of others are similar.

(i) Since for IFSNs $\mathcal{A}_{11}$ and $\mathcal{A}_{12}$,

$$\lambda\mathcal{A}_{11} = \Big(1 - (1 - \zeta_{11})^\lambda, (1 - \zeta_{11})^\lambda - (1 - \zeta_{11} - \vartheta_{11})^\lambda\Big)$$

and

$$\lambda\mathcal{A}_{12} = \Big(1 - (1 - \zeta_{12})^\lambda, (1 - \zeta_{12})^\lambda - (1 - \zeta_{12} - \vartheta_{12})^\lambda\Big)$$

Thus, we have

$$\begin{aligned}
&\lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) \\
=\ & \Bigg(1 - (1 - \zeta_{11})^\lambda (1 - \zeta_{12})^\lambda, (1 - \zeta_{11})^\lambda (1 - \zeta_{12})^\lambda \\
& - \left\{ \begin{pmatrix} 1 - 1 + (1 - \zeta_{11})^\lambda - (1 - \zeta_{11})^\lambda \\ + (1 - \zeta_{11} - \vartheta_{11})^\lambda \end{pmatrix} \times \begin{pmatrix} 1 - 1 + (1 - \zeta_{12})^\lambda - (1 - \zeta_{12})^\lambda \\ + (1 - \zeta_{12} - \vartheta_{12})^\lambda \end{pmatrix} \right\} \Bigg) \\
=\ & \Big(1 - (1 - \zeta_{11})^\lambda (1 - \zeta_{12})^\lambda, \\
& \qquad (1 - \zeta_{11})^\lambda (1 - \zeta_{12})^\lambda - (1 - \zeta_{11} - \vartheta_{11})^\lambda (1 - \zeta_{12} - \vartheta_{12})^\lambda\Big) \\
=\ & \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12}
\end{aligned}$$

(iii) For real numbers $\lambda_1, \lambda_2 > 0$ and the IFSN $\mathcal{A} = (\zeta, \vartheta)$, we have

$$\lambda_1\mathcal{A} = \big(1 - (1-\zeta)^{\lambda_1}, (1-\zeta)^{\lambda_1} - (1-\zeta-\vartheta)^{\lambda_1}\big)$$

and

$$\lambda_2\mathcal{A} = \big(1 - (1-\zeta)^{\lambda_2}, (1-\zeta)^{\lambda_2} - (1-\zeta-\vartheta)^{\lambda_2}\big).$$

Therefore,

$$\begin{aligned}
\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} &= \Big(1 - (1-\zeta)^{\lambda_1}(1-\zeta)^{\lambda_2}, \\
&\qquad (1-\zeta)^{\lambda_1}(1-\zeta)^{\lambda_2} - (1-\zeta-\vartheta)^{\lambda_1}(1-\zeta-\vartheta)^{\lambda_2}\Big) \\
&= \Big(1 - (1-\zeta)^{\lambda_1+\lambda_2}, (1-\zeta)^{\lambda_1+\lambda_2} - (1-\zeta-\vartheta)^{\lambda_1+\lambda_2}\Big) \\
&= (\lambda_1+\lambda_2)\mathcal{A}
\end{aligned}$$

□

Next, in the below section, we present the scaled prioritized averaging (PA) AOs by considering the prioritized relationship between the experts as well as the parameters.

8.2.2 Scaled prioritized intuitionistic fuzzy soft interaction aggregation operator

Consider a set of 'm' parameters $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ and a set of 'n' experts $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$, such that $e_j \preceq e_l$ if $j \leq l$ and $u_i \preceq u_k$ if $i \leq k$. Note that "$\preceq$" denotes the "higher priority". Assume that these parameters and experts have their own priority coefficients denoted by $b_{j,j+1}$ and $d_{i,i+1}$ respectively, and their degrees are defined as

(i) For "extreme priority", $b_{j,j+1} = 0.1$ and $d_{i,i+1} = 0.1$;

(ii) For "strong priority", $b_{j,j+1} = 0.4$ and $d_{i,i+1} = 0.4$;

(iii) For "slight priority", $b_{j,j+1} = 0.6$ and $d_{i,i+1} = 0.6$;

(iv) For "weak priority", $b_{j,j+1} = 0.8$ and $d_{i,i+1} = 0.8$;

(v) For "equal priority", $b_{j,j+1} = 1$ and $d_{i,i+1} = 1$

Based on these priority degrees, we define the weights of the parameters and experts, respectively are as follows

$$\xi_j = \begin{cases} b_{0,1} = 1 & ; \quad j = 1 \\ \xi_{j-1}.b_{j-1,j} = \prod_{l=1}^{j} b_{l-1,l} & ; \quad j = 2, 3, \ldots, m \end{cases} \tag{8.1}$$

and

$$\eta_i = \begin{cases} d_{0,1} = 1 & ; \quad i = 1 \\ \eta_{i-1}.d_{i-1,i} = \prod_{k=1}^{i} d_{k-1,k} & ; \quad i = 2, 3, \ldots, n \end{cases} \tag{8.2}$$

Definition 8.2.2. A SPIFSIA operator is a mapping SPIFSIA : $\Omega^n \to \Omega$ defined as

$$\text{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigoplus_{j=1}^{m} \frac{\prod_{l=1}^{j} b_{l-1,l}}{\sum_{j=1}^{m} \prod_{l=1}^{j} b_{l-1,l}} \left(\bigoplus_{i=1}^{n} \frac{\prod_{k=1}^{i} d_{k-1,k}}{\sum_{i=1}^{n} \prod_{k=1}^{i} d_{k-1,k}} \mathcal{A}_{ij} \right) \tag{8.3}$$

where Ω be the collection of all IFSNs $\mathcal{A}_{ij}$, $d_{0,1} = b_{0,1} = 1$, $d_{i,i+1}$ and $b_{j,j+1}$ indicates the priority coefficient between the experts u_i and u_{i+1}, as well as the parameters e_j and e_{j+1} respectively. Then SPIFSIA is called as scaled prioritized intuitionistic fuzzy soft interaction aggregation operator.

Theorem 8.2.5. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, the aggregated value by SPIFSIA operator is still an IFSN and given by:

$$\begin{aligned} &\text{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = &\left(1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \zeta_{ij})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \zeta_{ij})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \zeta_{ij} - \vartheta_{ij})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right) \end{aligned} \tag{8.4}$$

where $\widetilde{\xi}_j = \frac{\prod_{l=1}^{j} b_{l-1,l}}{\sum_{j=1}^{m} \prod_{l=1}^{j} b_{l-1,l}}$ and $\widetilde{\eta}_i = \frac{\prod_{k=1}^{i} d_{k-1,k}}{\sum_{i=1}^{n} \prod_{k=1}^{i} d_{k-1,k}}$.

Proof. We prove Eq. (8.4) by principle of mathematical induction (PMI) on n and m. Since for all $i, j; \mathcal{A}_{ij}$ are IFSNs which implies that $\zeta_{ij}, \vartheta_{ij} \in [0, 1]$ and $\zeta_{ij} + \vartheta_{ij} \leq 1$.

Step 1: For $n = 1$, we have $\widetilde{\eta}_i = 1$ and hence from Eq. (8.3), we get

$$
\begin{aligned}
&\mathrm{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1m}) \\
&= \bigoplus_{j=1}^{m} \left(\frac{\prod_{l=1}^{j} b_{l-1,l}}{\sum_{j=1}^{m} \prod_{l=1}^{j} b_{l-1,l}} \mathcal{A}_{1j} \right) \\
&= \left(1 - \prod_{j=1}^{m} (1-\zeta_{1j})^{\widetilde{\xi}_j}, \prod_{j=1}^{m} (1-\zeta_{1j})^{\widetilde{\xi}_j} - \prod_{j=1}^{m} (1-\zeta_{1j}-\vartheta_{1j})^{\widetilde{\xi}_j} \right) \\
&= \left(1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{1} (1-\zeta_{1j})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \prod_{j=1}^{m} \left(\prod_{i=1}^{1} (1-\zeta_{1j})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right. \\
&\qquad \left. - \prod_{j=1}^{m} \left(\prod_{i=1}^{1} (1-\zeta_{1j}-\vartheta_{1j})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right)
\end{aligned}
$$

Similarly, for $m = 1$, we have $\widetilde{\xi}_j = 1$ and

$$
\begin{aligned}
&\mathrm{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{n1}) \\
&= \bigoplus_{i=1}^{n} \left(\frac{\prod_{k=1}^{i} d_{k-1,k}}{\sum_{i=1}^{n} \prod_{k=1}^{i} d_{k-1,k}} \mathcal{A}_{i1} \right) \\
&= \left(1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} (1-\zeta_{i1})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \prod_{j=1}^{1} \left(\prod_{i=1}^{n} (1-\zeta_{i1})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right. \\
&\qquad \left. - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} (1-\zeta_{i1}-\vartheta_{i1})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right)
\end{aligned}
$$

Thus, Eq. (8.4) true for $n = 1$ and $m = 1$.

Step 2: Assume the Eq. (8.4) holds for $m = k_1, n = k_2 + 1$ and $m = k_1 + 1, n = k_2$, then for $m = k_1 + 1, n = k_2 + 1$, we have

$$
\mathrm{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{(k_2+1)(k_1+1)}) = \bigoplus_{j=1}^{k_1+1} \widetilde{\xi}_j \left(\bigoplus_{i=1}^{k_2+1} \widetilde{\eta}_i \mathcal{A}_{ij} \right)
$$

$$
\begin{aligned}
&= \bigoplus_{j=1}^{k_1+1} \widetilde{\xi}_j \left(\left(\bigoplus_{i=1}^{k_2} \widetilde{\eta}_i \mathcal{A}_{ij} \right) \oplus \left(\widetilde{\eta}_{k_2+1} \mathcal{A}_{(k_2+1)j} \right) \right) \\
&= \underbrace{\left(\bigoplus_{j=1}^{k_1+1} \widetilde{\xi}_j \left(\bigoplus_{i=1}^{k_2} \widetilde{\eta}_i \mathcal{A}_{ij} \right) \right)}_{\text{by induction } m=k_1+1, n=k_2} \oplus \underbrace{\left(\bigoplus_{j=1}^{k_1+1} \widetilde{\xi}_j \left(\widetilde{\eta}_{k_2+1} \mathcal{A}_{(k_2+1)j} \right) \right)}_{\text{by induction } m=k_1+1} \\
&= \left(1 - \prod_{j=1}^{k_1+1} \left(\prod_{i=1}^{k_2} \left(1-\zeta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \prod_{j=1}^{k_1+1} \left(\prod_{i=1}^{k_2} \left(1-\zeta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right. \\
&\qquad \left. - \prod_{j=1}^{k_1+1} \left(\prod_{i=1}^{k_2} \left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right) \\
&\oplus \left(1 - \prod_{j=1}^{k_1+1} \left(\left(1-\zeta_{(k_2+1)j}\right)^{\widetilde{\eta}_{k_2+1}} \right)^{\widetilde{\xi}_j}, \prod_{j=1}^{k_1+1} \left(\left(1-\zeta_{(k_2+1)j}\right)^{\widetilde{\eta}_{k_2+1}} \right)^{\widetilde{\xi}_j} \right. \\
&\qquad \left. - \prod_{j=1}^{k_1+1} \left(\left(1-\zeta_{(k_2+1)j}-\vartheta_{(k_2+1)j}\right)^{\widetilde{\eta}_{k_2+1}} \right)^{\widetilde{\xi}_j} \right) \\
&= \left(1 - \prod_{j=1}^{k_1+1} \left(\prod_{i=1}^{k_2+1} \left(1-\zeta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \prod_{j=1}^{k_1+1} \left(\prod_{i=1}^{k_2+1} \left(1-\zeta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right. \\
&\qquad \left. - \prod_{j=1}^{k_1+1} \left(\prod_{i=1}^{k_2+1} \left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right)
\end{aligned}
$$

Hence the result holds for all positive integers m, n. □

Based on Theorem 8.2.5, we have observed that proposed SPIFSIA operator satisfies the idempotency, boundedness and monotonicity properties.

Property 8.2.1. (Idempotency:) If $\mathcal{A}_{ij} = \mathcal{A} = (\zeta, \vartheta)$, for all i, j then

$$\text{SPIFSIA}(\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A}) = \mathcal{A}$$

Proof. If $\mathcal{A}_{ij} = \mathcal{A}$, for all i, j, and $\widetilde{\xi}_j = \frac{\prod_{l=1}^{j} b_{l-1,l}}{\sum_{j=1}^{m} \prod_{l=1}^{j} b_{l-1,l}}$, $\widetilde{\eta}_i = \frac{\prod_{k=1}^{i} d_{k-1,k}}{\sum_{i=1}^{n} \prod_{k=1}^{i} d_{k-1,k}}$ are the weight vectors such that $\sum_{j=1}^{m} \widetilde{\xi}_j = 1$ and $\sum_{i=1}^{n} \widetilde{\eta}_i = 1$, then by Eq. (8.4), we have

$$
\begin{aligned}
&\mathrm{SPIFSIA}(\mathcal{A},\mathcal{A},\ldots,\mathcal{A})\\
= &\left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j},\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta-\vartheta)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)\\
= &\left(1-\left((1-\zeta)^{\sum\limits_{i=1}^{n}\widetilde{\eta}_i}\right)^{\sum\limits_{j=1}^{m}\widetilde{\xi}_j},\left((1-\zeta)^{\sum\limits_{i=1}^{n}\widetilde{\eta}_i}\right)^{\sum\limits_{j=1}^{m}\widetilde{\xi}_j}-\left((1-\zeta-\vartheta)^{\sum\limits_{i=1}^{n}\widetilde{\eta}_i}\right)^{\sum\limits_{j=1}^{m}\widetilde{\xi}_j}\right)\\
= &\big(\zeta,\vartheta\big)
\end{aligned}
$$

□

Property 8.2.2. (Boundedness:) Let $\mathcal{A}^- = \big(\min\limits_j \min\limits_i\{\zeta_{ij}\}, \max\limits_j \max\limits_i\{\zeta_{ij}+\vartheta_{ij}\}-\min\limits_j \min\limits_i\{\zeta_{ij}\}\big)$ and $\mathcal{A}^+ = \big(\max\limits_j \max\limits_i\{\zeta_{ij}\}, \max\left(0, \min\limits_j \min\limits_i\{\zeta_{ij}+\vartheta_{ij}\}-\max\limits_j \max\limits_i\{\zeta_{ij}\}\right)\big)$ then

$$\mathcal{A}^- \leq \mathrm{SPIFSIA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) \leq \mathcal{A}^+.$$

Proof. For IFSNs $\mathcal{A}_{ij}$, we have

$$
\begin{aligned}
&\min_j \min_i\{\zeta_{ij}\} \leq \zeta_{ij} \leq \max_j \max_i\{\zeta_{ij}\}\\
\Leftrightarrow\ & 1-\max_j \max_i\{\zeta_{ij}\} \leq 1-\zeta_{ij} \leq 1-\min_j \min_i\{\zeta_{ij}\}\\
\Leftrightarrow\ & \left(1-\max_j \max_i\{\zeta_{ij}\}\right)^{\widetilde{\eta}_i} \leq (1-\zeta_{ij})^{\widetilde{\eta}_i} \leq \left(1-\min_j \min_i\{\zeta_{ij}\}\right)^{\widetilde{\eta}_i}\\
\Leftrightarrow\ & 1-\max_j \max_i\{\zeta_{ij}\} \leq \prod_{i=1}^{n}(1-\zeta_{ij})^{\widetilde{\eta}_i} \leq 1-\min_j \min_i\{\zeta_{ij}\}\\
\Leftrightarrow\ & 1-\max_j \max_i\{\zeta_{ij}\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \leq 1-\min_j \min_i\{\zeta_{ij}\}
\end{aligned}
$$

and hence

$$\min_j \min_i\{\zeta_{ij}\} \leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \leq \max_j \max_i\{\zeta_{ij}\} \tag{8.5}$$

Furthermore, we have

$$
\begin{aligned}
&\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\} \leq \zeta_{ij}+\vartheta_{ij} \leq \max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}\\
\Leftrightarrow\ & 1-\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\} \leq 1-\zeta_{ij}-\vartheta_{ij} \leq 1-\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}
\end{aligned}
$$

$$\Leftrightarrow \quad \left(1-\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}\right)^{\widetilde{\eta}_i} \leq (1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i} \leq \left(1-\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}\right)^{\widetilde{\eta}_i}$$

$$\Leftrightarrow \quad 1-\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\} \leq \prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i} \leq 1-\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}$$

$$\Leftrightarrow \quad \left(1-\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}\right)^{\widetilde{\xi}_j} \leq \left(\prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \leq \left(1-\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}\right)^{\widetilde{\xi}_j}$$

and hence, we get

$$1-\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \leq 1-\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\} \tag{8.6}$$

From Eqs. (8.5) and (8.6), we have

$$\left\{\begin{array}{l}\min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}\\ -\max_j \max_i\{\zeta_{ij}\}\end{array}\right\} \leq \left\{\begin{array}{l}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\\ -\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\end{array}\right\} \leq \left\{\begin{array}{l}\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}\\ -\min_j \min_i\{\zeta_{ij}\}\end{array}\right\}$$

Also, we have

$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \geq 0$$

Thus,

$$\max\left\{\begin{array}{l}0, \min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}\\ -\max_j \max_i\{\zeta_{ij}\}\end{array}\right\} \leq \left\{\begin{array}{l}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\\ -\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\end{array}\right\} \leq \left\{\begin{array}{l}\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}\\ -\min_j \min_i\{\zeta_{ij}\}\end{array}\right\} \tag{8.7}$$

Take $\gamma = \text{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = (\zeta_\gamma, \vartheta_\gamma)$, $\mathcal{A}^- = \big(\min_j \min_i\{\zeta_{ij}\}, \max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}-\min_j \min_i\{\zeta_{ij}\}\big)$ and $\mathcal{A}^+ = \big(\max_j \max_i\{\zeta_{ij}\}, \max\big(0, \min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}-\max_j \max_i\{\zeta_{ij}\}\big)\big)$ then from Eqs. (8.5) and (8.7), we have

$$\min_j \min_i\{\zeta_{ij}\} \leq \zeta_\gamma \leq \max_j \max_i\{\zeta_{ij}\}$$

and

$$\max\left\{\begin{array}{l}0, \min_j \min_i\{\zeta_{ij}+\vartheta_{ij}\}\\ -\max_j \max_i\{\zeta_{ij}\}\end{array}\right\} \leq \vartheta_\gamma \leq \left\{\begin{array}{l}\max_j \max_i\{\zeta_{ij}+\vartheta_{ij}\}\\ -\min_j \min_i\{\zeta_{ij}\}\end{array}\right\} \tag{8.8}$$

So by definition of score function, we get

$$\begin{aligned} S(\gamma) &= \zeta_\gamma - \vartheta_\gamma \\ &\leq \max_j \max_i \{\zeta_{ij}\} - \max\left\{0, \min_j \min_i \{\zeta_{ij} + \vartheta_{ij}\} - \max_j \max_i \{\zeta_{ij}\}\right\} \\ &= S(\mathcal{A}^+) \\ \text{and } S(\gamma) &= \zeta_\gamma - \vartheta_\gamma \\ &\geq \min_j \min_i \{\zeta_{ij}\} - (\max_j \max_i \{\zeta_{ij} + \vartheta_{ij}\} - \min_j \min_i \{\zeta_{ij}\}) \\ &= S(\mathcal{A}^-) \end{aligned}$$

Now based on it, the following three cases arises:

Case 1. If $S(\gamma) < S(\mathcal{A}^+)$ and $S(\gamma) > S(\mathcal{A}^-)$, then, we have $\mathcal{A}^- < \gamma < \mathcal{A}^+$.

Case 2. If $S(\gamma) = S(\mathcal{A}^+)$, i.e., $\zeta_\gamma - \vartheta_\gamma = \max_j \max_i \{\zeta_{ij}\} - \max\{0, \min_j \min_i \{\zeta_{ij} + \vartheta_{ij}\} - \max_j \max_i \{\zeta_{ij}\}\}$, then by above inequalities, we have $\zeta_\gamma = \max_j \max_i \{\zeta_{ij}\}$ and $\vartheta_\gamma = \max\{0, \min_j \min_i \{\zeta_{ij} + \vartheta_{ij}\} - \max_j \max_i \{\zeta_{ij}\}\}$. Therefore, $H_\gamma = \zeta_\gamma + \vartheta_\gamma = \max_j \max_i \{\zeta_{ij}\} + \max\{0, \min_j \min_i \{\zeta_{ij} + \vartheta_{ij}\} - \max_j \max_i \{\zeta_{ij}\}\}$ and $\gamma = \mathcal{A}^+$.

Case 3. If $S(\gamma) = S(\mathcal{A}^-)$, i.e., $\zeta_\gamma - \vartheta_\gamma = \min_j \min_i \{\zeta_{ij}\} - (\max_j \max_i \{\zeta_{ij} + \vartheta_{ij}\} - \min_j \min_i \{\zeta_{ij}\})$, then by above inequalities, we have $\zeta_\gamma = \min_j \min_i \{\zeta_{ij}\}$ and $\vartheta_\gamma = \max_j \max_i \{\zeta_{ij} + \vartheta_{ij}\} - \min_j \min_i \{\zeta_{ij}\}$. Therefore, we have $H_\gamma = \zeta_\gamma + \vartheta_\gamma = \min_j \min_i \{\zeta_{ij}\} + (\max_j \max_i \{\zeta_{ij} + \vartheta_{ij}\} - \min_j \min_i \{\zeta_{ij}\})$ and $\gamma = \mathcal{A}^-$.

Hence, by combining all these cases, we have

$$\mathcal{A}^- \leq \text{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \mathcal{A}^+$$

□

Property 8.2.3. (Monotonicity:) If $\mathcal{A}_{ij} \leq \mathcal{A}'_{ij}$ for all $i, j,$, where $\mathcal{A}'_{ij}$ is IFSN, then

$$\text{SPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \text{SPIFSIA}(\mathcal{A}'_{11}, \mathcal{A}'_{12}, \ldots, \mathcal{A}'_{nm})$$

Proof. Follows from Property 8.2.2, so we omit here. □

8.2.3 Generalized SPIFSIA operator

Definition 8.2.3. A GSPIFSIA operator for a collection of IFSNs $\mathcal{A}_{ij}$ is a mapping GSPIFSIA : $\Omega^n \to \Omega$ and is defined as

$$\text{GSPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\bigoplus_{j=1}^{m} \frac{\prod\limits_{l=1}^{j} b_{l-1,l}}{\sum\limits_{j=1}^{m} \prod\limits_{l=1}^{j} b_{l-1,l}} \left(\bigoplus_{i=1}^{n} \frac{\prod\limits_{k=1}^{i} d_{k-1,k}}{\sum\limits_{i=1}^{n} \prod\limits_{k=1}^{i} d_{k-1,k}} \mathcal{A}_{ij}^{\lambda} \right) \right)^{\frac{1}{\lambda}} \quad ; \quad \lambda > 0$$

Theorem 8.2.6. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the aggregated value by GSPIFSIA is still an IFSN and given by:

$$\begin{aligned}
&\text{GSPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
&= \left(\left(\begin{array}{c} 1 - \prod\limits_{j=1}^{m} \left[\prod\limits_{i=1}^{n} \begin{pmatrix} 1 - \left(1 - \vartheta_{ij}\right)^{\lambda} \\ + \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \end{pmatrix}^{\widetilde{\eta}_i} \right]^{\widetilde{\xi}_j} \\ + \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} \left(\left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \end{array} \right)^{\frac{1}{\lambda}} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \zeta_{ij} - \vartheta_{ij})^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \right. \\
&\qquad \left. 1 - \left(\begin{array}{c} 1 - \prod\limits_{j=1}^{m} \left[\prod\limits_{i=1}^{n} \begin{pmatrix} 1 - \left(1 - \vartheta_{ij}\right)^{\lambda} \\ + \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \end{pmatrix}^{\widetilde{\eta}_i} \right]^{\widetilde{\xi}_j} \\ + \prod\limits_{j=1}^{m} \left(\prod\limits_{i=1}^{n} \left(\left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \end{array} \right)^{\frac{1}{\lambda}} \right)
\end{aligned} \tag{8.9}$$

where $\widetilde{\eta}_i = \frac{\prod_{k=1}^{i} d_{k-1,k}}{\sum_{i=1}^{n} \prod_{k=1}^{i} d_{k-1,k}}$, $\widetilde{\xi}_j = \frac{\prod_{l=1}^{j} b_{l-1,l}}{\sum_{j=1}^{m} \prod_{l=1}^{j} b_{l-1,l}}$ be the weight vector of the experts and the parameters respectively.

Proof. It can be easily obtained as parallel to the Theorem 8.2.5. □

Also, GSPIFSIA operator satisfies the properties such as idempotency, boundedness, and monotonicity. Further, we assume $\ln 0 = 0$ and $\ln\left(\frac{0}{0}\right) = 0$, if necessary.

Lemma 8.2.1. If $y = \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(1-\vartheta_{ij}\right)^{\lambda}+\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}$, then

$$\lim_{\lambda\to 0}(y)' = \sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\ln\left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}}\right)\right); \tag{8.10}$$

where y' denotes $\frac{dy}{d\lambda}$.

Proof. Since, for a given y, we have

$$\lim_{\lambda\to 0}(y)' = \lim_{\lambda\to 0}(e^{\ln(y)})' = \lim_{\lambda\to 0} y(\ln(y))' \tag{8.11}$$

But, $\lim_{\lambda\to 0} y = \lim_{\lambda\to 0}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\begin{pmatrix}1-\left(1-\vartheta_{ij}\right)^{\lambda}\\ +\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\end{pmatrix}^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} = 1$. Therefore, Eq. (8.11) becomes

$$\begin{aligned}
\lim_{\lambda\to 0}(y)' &= \lim_{\lambda\to 0}(\ln(y))' \\
&= \lim_{\lambda\to 0}\left(\ln\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(1-\vartheta_{ij}\right)^{\lambda}+\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)\right)' \\
&= \lim_{\lambda\to 0}\left(\sum_{j=1}^{m}\ln\left(\prod_{i=1}^{n}\left(1-\left(1-\vartheta_{ij}\right)^{\lambda}+\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)' \\
&= \lim_{\lambda\to 0}\left(\sum_{j=1}^{m}\widetilde{\xi}_j\ln\left(\prod_{i=1}^{n}\left(1-\left(1-\vartheta_{ij}\right)^{\lambda}+\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)\right)' \\
&= \lim_{\lambda\to 0}\left(\sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\ln\left(1-\left(1-\vartheta_{ij}\right)^{\lambda}+\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\right)\right)\right)' \\
&= \lim_{\lambda\to 0}\left(\sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\left(\frac{-(1-\vartheta_{ij})^{\lambda}\ln(1-\vartheta_{ij})+(1-\zeta_{ij}-\vartheta_{ij})^{\lambda}\ln(1-\zeta_{ij}-\vartheta_{ij})}{1-\left(1-\vartheta_{ij}\right)^{\lambda}+\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}}\right)\right)\right) \\
&= \sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\Big(-\ln(1-\vartheta_{ij})+\ln(1-\zeta_{ij}-\vartheta_{ij})\Big)\right) \\
&= \sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\ln\left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}}\right)\right)
\end{aligned}$$

Hence the result. ∎

Lemma 8.2.2. If $z = \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}$, then

$$\lim_{\lambda \to 0} (z)' = \ln \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right) \tag{8.12}$$

where z' denotes $\frac{dz}{d\lambda}$.

Proof. For a given z, we have $\lim_{\lambda \to 0} z = 1$ and thus

$$\begin{aligned}
\lim_{\lambda \to 0} (z)' &= \lim_{\lambda \to 0} (e^{\ln(z)})' = \lim_{\lambda \to 0} z \; (\ln(z))' \\
&= \lim_{\lambda \to 0} \left(\ln \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right) \right)' \\
&= \lim_{\lambda \to 0} \left(\sum_{j=1}^{m} \widetilde{\xi}_j \left(\sum_{i=1}^{n} \widetilde{\eta}_i \ln \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\lambda} \right) \right)' \\
&= \lim_{\lambda \to 0} \left(\sum_{j=1}^{m} \widetilde{\xi}_j \left(\sum_{i=1}^{n} \widetilde{\eta}_i \big(\lambda \ln \left(1 - \zeta_{ij} - \vartheta_{ij}\right) \big) \right) \right)' \\
&= \lim_{\lambda \to 0} \left(\sum_{j=1}^{m} \widetilde{\xi}_j \left(\sum_{i=1}^{n} \widetilde{\eta}_i \ln \left(1 - \zeta_{ij} - \vartheta_{ij}\right) \right) \right) \\
&= \sum_{j=1}^{m} \widetilde{\xi}_j \left(\sum_{i=1}^{n} \ln \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\widetilde{\eta}_i} \right) \\
&= \ln \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right)
\end{aligned}$$

Hence, the result. $\square$

Theorem 8.2.7. For IFSNs $\mathcal{A}_{ij}$, if $\lambda \to 0$, then GSPIFSIA operator becomes

$$\begin{aligned}
&\lim_{\lambda \to 0} \text{GSPIFSIA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
&= \left(\begin{array}{l} e^{-\sum_{j=1}^{m} \widetilde{\xi}_j \left(\sum_{i=1}^{n} \widetilde{\eta}_i \ln \left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}} \right) \right) + \ln \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right)} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \zeta_{ij} - \vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j}, \\ \\ 1 - e^{-\sum_{j=1}^{m} \widetilde{\xi}_j \left(\sum_{i=1}^{n} \widetilde{\eta}_i \ln \left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}} \right) \right) + \ln \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i} \right)^{\widetilde{\xi}_j} \right)} \end{array} \right)
\end{aligned}$$

where $\widetilde{\xi}_j = \frac{\prod_{l=1}^{j} b_{l-1,l}}{\sum_{j=1}^{m}\prod_{l=1}^{j} b_{l-1,l}}$ and $\widetilde{\eta}_i = \frac{\prod_{k=1}^{i} d_{k-1,k}}{\sum_{i=1}^{n}\prod_{k=1}^{i} d_{k-1,k}}$.

Proof. Firstly, we shall prove that

$$\begin{aligned}
&\lim_{\lambda\to 0}\left(1-\prod_{j=1}^{m}\left[\prod_{i=1}^{n}\begin{pmatrix}1-\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\\ +\left(1-\vartheta_{ij}\right)^{\lambda}\end{pmatrix}^{\widetilde{\eta}_i}\right]^{\widetilde{\xi}_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left((1-\zeta_{ij}-\vartheta_{ij})^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)^{\frac{1}{\lambda}}\\
= \;& e^{\left(-\sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\ln\left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}}\right)\right)+\ln\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)\right)}
\end{aligned}\tag{8.13}$$

By L'Hopital's rule, and by using Lemma 8.2.1 and 8.2.2, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\left(1-\prod_{j=1}^{m}\left[\prod_{i=1}^{n}\begin{pmatrix}1-\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\\ +\left(1-\vartheta_{ij}\right)^{\lambda}\end{pmatrix}^{\widetilde{\eta}_i}\right]^{\widetilde{\xi}_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left((1-\zeta_{ij}-\vartheta_{ij})^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)^{\frac{1}{\lambda}}\\
= \;& \lim_{\lambda\to 0}\left(1-y+z\right)^{\frac{1}{\lambda}} \qquad\qquad (1^{\infty}\text{form})\\
= \;& \lim_{\lambda\to 0} e^{\ln(1-y+z)^{\frac{1}{\lambda}}}\\
= \;& e^{\lim_{\lambda\to 0}\left\{\frac{\ln(1-y+z)}{\lambda}\right\}} \qquad\qquad \left(\frac{0}{0}\text{form}\right)\\
= \;& e^{\lim_{\lambda\to 0}\left\{\ln(1-y+z)\right\}'}\\
= \;& e^{\lim_{\lambda\to 0}\frac{-y'+z'}{1-y+z}}\\
= \;& e^{-\lim_{\lambda\to 0}y'+\lim_{\lambda\to 0}z'} \qquad\qquad |\because 1-y+z=1 \;\text{ as }\; \lambda\to 0\\
= \;& e^{-\sum_{j=1}^{m}\widetilde{\xi}_j\left(\sum_{i=1}^{n}\widetilde{\eta}_i\ln\left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}}\right)\right)+\ln\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)}
\end{aligned}$$

Hence, Eq. (8.13) holds.

Therefore,

$$
\begin{aligned}
&\lim_{\lambda\to 0} \mathrm{GSPIFSIA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\
&= \left(\left(\lim_{\lambda\to 0} \begin{pmatrix} 1-\prod\limits_{j=1}^{m}\left[\prod\limits_{i=1}^{n}\begin{pmatrix} 1-\left(1-\vartheta_{ij}\right)^{\lambda} \\ +\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\end{pmatrix}^{\widetilde{\eta}_i}\right]^{\widetilde{\xi}_j} \\ +\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\left((1-\zeta_{ij}-\vartheta_{ij})^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \end{pmatrix}^{\frac{1}{\lambda}} - \lim_{\lambda\to 0}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij}-\vartheta_{ij})^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \right), \right.\\
&\qquad\qquad \left. 1-\lim_{\lambda\to 0}\begin{pmatrix} 1-\prod\limits_{j=1}^{m}\left[\prod\limits_{i=1}^{n}\begin{pmatrix} 1-\left(1-\vartheta_{ij}\right)^{\lambda} \\ +\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\end{pmatrix}^{\widetilde{\eta}_i}\right]^{\widetilde{\xi}_j} \\ +\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j} \end{pmatrix}^{\frac{1}{\lambda}} \right)\\
&= \left(e^{-\sum\limits_{j=1}^{m}\widetilde{\xi}_j\left(\sum\limits_{i=1}^{n}\widetilde{\eta}_i\ln\left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}}\right)\right)+\ln\left(\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}, \right.\\
&\qquad \left. 1-e^{-\sum\limits_{j=1}^{m}\widetilde{\xi}_j\left(\sum\limits_{i=1}^{n}\widetilde{\eta}_i\ln\left(\frac{1-\zeta_{ij}-\vartheta_{ij}}{1-\vartheta_{ij}}\right)\right)+\ln\left(\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\left(1-\zeta_{ij}-\vartheta_{ij}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)} \right)
\end{aligned}
$$

Hence the result. $\square$

8.3 Proposed operators based decision-making approach

The brief description of the MCDM problem is given in Section 2.6 of Chapter 2. The rating values of each alternative is represented in the form of intuitionistic fuzzy soft decision matrix $\mathcal{M} = (\zeta_{ij}^{(b)}, \vartheta_{ij}^{(b)})$. The priority relationship between the parameters e_j and e_{j+1} as well as between the experts u_i and u_{i+1} are denoted by priority coefficients $b_{j,j+1}(j = 1, 2, \ldots, m-1)$ and $d_{i,i+1}(i = 1, 2, \ldots, n-1)$, respectively. Then, the following steps based on the proposed operators are stated to solve the DM problems, whose flowchart is given in Fig. 8.1.

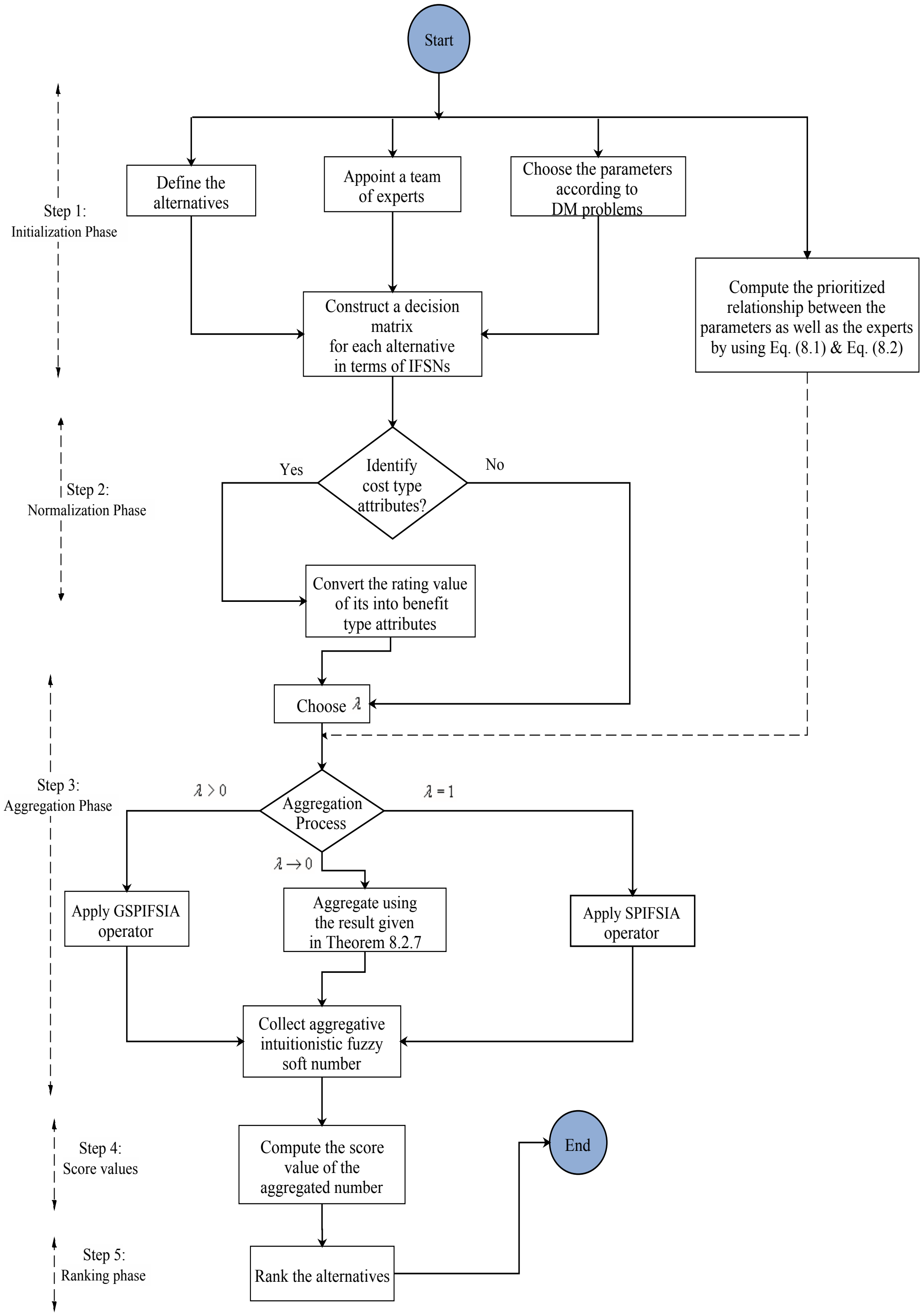

Figure 8.1: Flow chart of the proposed approach

Step 1: Arrange the information of each alternative $\mathcal{A}^{(b)}(b=1,2,\ldots,t)$ in terms of decision matrices $\mathcal{M}^{(b)}$.

Step 2: Normalize the data information by using Eq. (2.34) of Chapter 2 and obtained the decision matrices $\mathcal{R}^{(b)}=(r_{ij}^{(b)})_{n\times m}$.

Step 3: Aggregate the preference values $r_{ij}^{(b)}$ of the alternatives $\mathcal{A}^{(b)}(b=1,2,\ldots,t)$ into the collective one $r^{(b)}=(\zeta^{(b)},\vartheta^{(b)})$, for a real positive number λ, either by using SPIFSIA aggregation operator as

$$\begin{aligned}
r^{(b)} &= \mathrm{SPIFSIA}(r_{11}^{(b)},r_{12}^{(b)},\ldots,r_{nm}^{(b)})\\
&= \left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}^{(b)}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j},\right. \qquad (8.14)\\
&\qquad\left.\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}^{(b)}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}^{(b)}-\vartheta_{ij}^{(b)}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\right)
\end{aligned}$$

or by using GSPIFSIA operator as

$$\begin{aligned}
r^{(b)} &= \mathrm{GSPIFSIA}(r_{11}^{(b)},r_{12}^{(b)},\ldots,r_{nm}^{(b)})\\
&= \left(\left(\begin{array}{c}1-\prod_{j=1}^{m}\left[\prod_{i=1}^{n}\left(\begin{array}{c}1-\left(1-\vartheta_{ij}^{(b)}\right)^{\lambda}\\ +\left(1-\zeta_{ij}^{(b)}-\vartheta_{ij}^{(b)}\right)^{\lambda}\end{array}\right)^{\widetilde{\eta}_i}\right]^{\widetilde{\xi}_j}\\ +\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1-\zeta_{ij}^{(b)}-\vartheta_{ij}^{(b)}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\end{array}\right)^{\frac{1}{\lambda}}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\zeta_{ij}^{(b)}-\vartheta_{ij}^{(b)}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j},\right.\\
&\qquad\left.1-\left(\begin{array}{c}1-\prod_{j=1}^{m}\left[\prod_{i=1}^{n}\left(\begin{array}{c}1-\left(1-\vartheta_{ij}^{(b)}\right)^{\lambda}\\ +\left(1-\zeta_{ij}^{(b)}-\vartheta_{ij}^{(b)}\right)^{\lambda}\end{array}\right)^{\widetilde{\eta}_i}\right]^{\widetilde{\xi}_j}\\ +\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1-\zeta_{ij}^{(b)}-\vartheta_{ij}^{(b)}\right)^{\lambda}\right)^{\widetilde{\eta}_i}\right)^{\widetilde{\xi}_j}\end{array}\right)^{\frac{1}{\lambda}}\right) \qquad (8.15)
\end{aligned}$$

where $\widetilde{\eta}_i=\frac{\prod_{k=1}^{i}d_{k-1,k}}{\sum_{i=1}^{n}\prod_{k=1}^{i}d_{k-1,k}}$, $\widetilde{\xi}_j=\frac{\prod_{l=1}^{j}b_{l-1,l}}{\sum_{j=1}^{m}\prod_{l=1}^{j}b_{l-1,l}}$.

Step 4: Compute the score values $S(r^{(b)})=\zeta^{(b)}-\vartheta^{(b)}$ for each aggregated number $r^{(b)}(b=1,2,\ldots,t)$.

Step 5: Rank the alternatives $\mathcal{A}^{(b)}(b = 1, 2, \ldots, t)$ and choose the desirable one(s).

8.4 An illustrative example

The above algorithm has been illustrated with a numerical example which is stated as follows.

8.4.1 A case study

Monsoon rains in river catchments have caused flooding in the northeastern state of Assam, India in July 2016. The Brahmaputra river and its tributaries overflow as a result of monsoon rains which effected over 80,000 people and 4,000 hectares of crop land according to Assam State Disaster Management Agency(ASDMA). The worst affected areas of Assam are {Lakhimpur($\mathcal{A}_1$), Dhemaji($\mathcal{A}^{(2)}$), Nagaon($\mathcal{A}^{(3)}$), Jorhat($\mathcal{A}^{(4)}$)}. A team of five experts, denoted by $\{u_1, u_2, u_3, u_4, u_5\}$ from different departments of ASDMA will inspect the flood affected areas on the different parameters {loss of life (e_1), loss of livestock (e_2), crop loss (e_3), damage of houses (e_4), ecological damage (e_5)}. Consider the prioritized relationship between the parameters and between the experts which are indicated by the priority coefficients $b_{j,j+1}$ and $d_{i,i+1}$ respectively, given in Tables 8.1 and 8.2. Based on

Table 8.1: Priority relationship between the parameters

	PR(e_1, e_2)	PR(e_2, e_3)	PR(e_3, e_4)	PR(e_4, e_5)
Priority relationship	Strong priority	Slight priority	Weak priority	Extreme priority
Priority coefficient	0.4	0.6	0.8	0.1

Table 8.2: Priority relationship between the experts

	PR(u_1, u_2)	PR(u_2, u_3)	PR(u_3, u_4)	PR(u_4, u_5)
Priority relationship	Strong priority	Weak priority	Slight priority	Extreme priority
Priority coefficient	0.4	0.8	0.6	0.1

their priority level, the government has to allocate funds according to the level of damage. In order to achieve it, given experts have evaluated the alternatives in terms of IFSNs. Then, the following steps have been followed to find the best alternative(s):

Step 1: The collective information of each alternatives $\mathcal{A}^{(b)}(b = 1, 2, 3, 4)$ is summarized in Table 8.3.

Table 8.3: Intuitionistic fuzzy soft matrices for alternatives

	For alternative $(\mathcal{A}^{(1)}, \mathcal{E})$					For alternative $(\mathcal{A}^{(2)}, \mathcal{E})$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.3, 0.4)$	$(0.5, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.6, 0.2)$	$(0.4, 0.3)$	$(0.5, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.7, 0.2)$
u_2	$(0.6, 0.1)$	$(0.6, 0.2)$	$(0.2, 0.4)$	$(0.5, 0.1)$	$(0.7, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.3)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0.4, 0.1)$
u_3	$(0.5, 0.1)$	$(0.7, 0.2)$	$(0.5, 0.4)$	$(0.2, 0.2)$	$(0.4, 0.2)$	$(0.5, 0.3)$	$(0.5, 0.1)$	$(0.5, 0.3)$	$(0.3, 0.2)$	$(0.6, 0.2)$
u_4	$(0.2, 0.4)$	$(0.5, 0.1)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.7, 0.3)$	$(0.4, 0.2)$	$(0.5, 0.1)$	$(0.5, 0.2)$
u_5	$(0.6, 0.1)$	$(0.3, 0.4)$	$(0.4, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.4, 0.2)$	$(0.5, 0.2)$	$(0.3, 0.3)$	$(0.6, 0.1)$	$(0.4, 0.2)$
	For alternative $(\mathcal{A}^{(3)}, \mathcal{E})$					For alternative $(\mathcal{A}^{(4)}, \mathcal{E})$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.4, 0.3)$	$(0.5, 0.4)$	$(0.5, 0.2)$	$(0.6, 0.1)$	$(0.4, 0.2)$	$(0.3, 0.4)$	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.4, 0.3)$	$(0.2, 0.3)$
u_2	$(0.5, 0.1)$	$(0.3, 0.2)$	$(0.3, 0.2)$	$(0.4, 0.2)$	$(0.3, 0.2)$	$(0.5, 0.1)$	$(0.4, 0.2)$	$(0.4, 0.2)$	$(0.6, 0.1)$	$(0.2, 0.6)$
u_3	$(0.5, 0.3)$	$(0.5, 0.1)$	$(0.4, 0.2)$	$(0.2, 0.2)$	$(0.5, 0.4)$	$(0.2, 0.1)$	$(0.4, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.5, 0.2)$
u_4	$(0.5, 0.1)$	$(0.4, 0.5)$	$(0.3, 0.2)$	$(0.7, 0.2)$	$(0.3, 0.2)$	$(0.7, 0.2)$	$(0.5, 0.1)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.7, 0.1)$
u_5	$(0.7, 0.1)$	$(0.4, 0.6)$	$(0.4, 0.2)$	$(0.3, 0.1)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.3, 0.2)$	$(0.7, 0.1)$

Step 2: Since all the parameters are of same type, so there is no need of normalizing the data.

Step 3: Without loss of generality, we take $\lambda = 1$ and SPIFSIA operator given in Eq. (8.14) to aggregate the different preferences and get $r^{(1)} = (0.4792, 0.2492)$, $r^{(2)} = (0.5059, 0.3668)$, $r^{(3)} = (0.4571, 0.5429)$ and $r^{(4)} = (0.4989, 0.2235)$.

Step 4: The score values are $S(r^{(1)}) = 0.2300, S(r^{(2)}) = 0.1391, S(r^{(3)}) = -0.0857$ and $S(r^{(4)}) = 0.2754$.

Step 5: Since $S(r^{(4)}) > S(r^{(1)}) > S(r^{(2)}) > S(r^{(3)})$ and thus ranking order of the alternatives is $\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$. Note that "$\succ$" means "preferred to". From it, it is concluded that the most damaged area is $\mathcal{A}^{(4)}$.

However, in order to analyze the impact of λ on the ranking, the value of λ has been varied from 0 to 20 in Step 3 of the approach and the overall score values of each alternative are presented in Table 8.4. From it, the following observations have been noted.

(i) When $\lambda \in (0, 0.0887]$, then the ranking order of alternatives is $\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$.

(ii) When $\lambda \in (0.0887, 0.0978]$, then $\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$.

(iii) When $\lambda \in (0.0978, 1.3891]$, then $\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$.

(iv) When $\lambda \in (1.3891, 1.4803]$, then $\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$.

(v) When $\lambda \in (1.4803, 1.5373]$, then $\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$.

(vi) When $\lambda \in (1.5373, 1.5708]$, then $\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$.

(vii) When $\lambda \in (1.5708, 1.6793]$, then $\mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)}$.

(viii) When $\lambda \in (1.6793, 1.8670]$, then $\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)}$.

(ix) When $\lambda \in (1.8670, 20]$, then $\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$ which gives $\mathcal{A}^{(3)}$ is the most damaged area.

As we see, based on the preference of the decision makers' parameter λ, the ordering of the alternative is different, which is useful for selecting the most desirable one.

8.4.2 Validity test

Wang and Triantaphyllou [210] proposed the following testing criteria to validate any MCDM problem:

Test criterion 1: "The best alternative should never be changed on replacing a non-optimal alternative by other worse alternative without changing the relative importance of each decision criteria."

Test criterion 2: "The transitivity property should be followed by an efficient DM method."

Test criterion 3: "The ranking of DM problem should never be changed by applying same DM method to the smaller problems which are decomposed from the original problem."

The applicability of test criteria to proposed DM method is discussed below:

Table 8.4: Effect of λ on ranking order of the alternatives based on GSPIFSIA operator

	Score value of the alternatives			
	$\lambda \to 0$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1.0$
$\mathcal{A}^{(1)}$	0.2365	0.2179	0.1911	0.2300
$\mathcal{A}^{(2)}$	0.4157	0.2099	-0.0633	0.1391
$\mathcal{A}^{(3)}$	-1.000	-1.000	-0.7800	-0.0857
$\mathcal{A}^{(4)}$	0.2536	0.2581	0.2702	0.2754
Ranking	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
	$\lambda = 1.5$	$\lambda = 2.0$	$\lambda = 2.5$	$\lambda = 3.0$
$\mathcal{A}^{(1)}$	0.2664	0.2972	0.3213	0.3396
$\mathcal{A}^{(2)}$	0.2853	0.3723	0.4224	0.4516
$\mathcal{A}^{(3)}$	0.2529	0.4087	0.4879	0.5318
$\mathcal{A}^{(4)}$	0.2811	0.2934	0.3089	0.3243
Ranking	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$
	$\lambda = 5.0$	$\lambda = 7.5$	$\lambda = 12.5$	$\lambda = 15.5$
$\mathcal{A}^{(1)}$	0.3797	0.4047	0.4355	0.4480
$\mathcal{A}^{(2)}$	0.4952	0.5162	0.5480	0.5635
$\mathcal{A}^{(3)}$	0.5955	0.6227	0.6579	0.6744
$\mathcal{A}^{(4)}$	0.3684	0.3988	0.4317	0.4441
Ranking	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$	$\mathcal{A}^{(3)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)}$

Validity test with criterion 1

For testing the validity of our approach under test criterion 1, the alternative $\mathcal{A}^{(3)}$ (non-optimal alternative) is to be substituted with any arbitrary alternative $(\mathcal{A}^{(3)})'$ (worse alternative) with their rating values is summarized in Eq. (8.16).

$$(\mathcal{A}^{(3)})' = \begin{array}{c c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} & \begin{pmatrix} (0.3, 0.5) & (0.3, 0.6) & (0.4, 0.5) & (0.4, 0.3) & (0.2, 0.5) \\ (0.4, 0.3) & (0.2, 0.4) & (0.2, 0.4) & (0.2, 0.5) & (0.2, 0.4) \\ (0.3, 0.6) & (0.3, 0.3) & (0.3, 0.5) & (0.1, 0.4) & (0.3, 0.6) \\ (0.4, 0.4) & (0.2, 0.7) & (0.1, 0.4) & (0.5, 0.3) & (0.2, 0.5) \\ (0.5, 0.3) & (0.3, 0.7) & (0.2, 0.5) & (0.2, 0.3) & (0.4, 0.3) \end{pmatrix} \end{array} \tag{8.16}$$

Now, by applying the proposed SPIFSIA operator to this modified data we get the aggregated numbers of each alternative are $r^{(1)} = (0.4792, 0.2492)$, $r^{(2)} = (0.5059, 0.3668)$, $(r^{(3)})' = (0.3174, 0.6826)$, and $r^{(4)} = (0.4989, 0.2235)$. The score values of these numbers are $S(r^{(1)}) = 0.2300$, $S(r^{(2)}) = 0.1391$, $S((r^{(3)})') = -0.3652$, and $S(r^{(4)}) = 0.2754$ and thus $\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ (\mathcal{A}^{(3)})'$. Therefore, the desirable alternative is remain $\mathcal{A}^{(4)}$ which validates the proposed approach satisfy *test criterion 1*.

Validity test with criteria 2 and 3

Under this test, if we decomposed the given MCDM problem into the smaller subproblems such as "$\{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)}\}$", "$\{\mathcal{A}^{(2)}, \mathcal{A}^{(3)}, \mathcal{A}^{(4)}\}$" and "$\{\mathcal{A}^{(3)}, \mathcal{A}^{(4)}, \mathcal{A}^{(1)}\}$" and then by applying the proposed approach, we get the ranking order of these problems are "$\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$", "$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$" and "$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$" respectively. Thus, by combining these ordering we get $\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$ which is identical to the original MCDM problem and hence follows transitivity. Therefore, the proposed method validates the *test criteria 2 and 3*.

8.4.3 Superiority of the proposed operator over the existing approaches

Here, we have presented some examples which show that under some certain cases, the existing approaches are unable to classify the alternatives while the proposed one does.

Example 8.4.1. Assume there are two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, which are assessed under three parameters e_1, e_2, e_3 with their weight vector $0.34, 0.33$ and 0.33 respectively. Two different experts u_1 and u_2, whose weight vector is 0.54 and 0.46, have evaluated these alternatives and represent their information in the form of IFSNs as follows.

$$\mathcal{A}^{(1)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad e_1 \quad & \quad e_2 \quad & \quad e_3 \quad \end{array} \\ \begin{bmatrix} (0.6, 0.4) & (0.5, 0.3) & (0.7, 0.2) \\ (0.4, 0.5) & (0.7, 0.2) & (0.5, 0.4) \end{bmatrix} \end{array}$$

and

$$\mathcal{A}^{(2)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad e_1 \quad & \quad e_2 \quad & \quad e_3 \quad \end{array} \\ \begin{bmatrix} (0.5, 0.2) & (0.6, 0.2) & (0.5, 0.4) \\ (0.3, 0.2) & (0.6, 0.2) & (0.5, 0.3) \end{bmatrix} \end{array}$$

If we followed an aggregation operator as mentioned in Liu and Li [119] then we get the score value of the alternatives as $S(\mathcal{A}^{(1)}) = 0.2697$ and $S(\mathcal{A}^{(2)}) = 0.2697$ and hence conclude that $\mathcal{A}^{(1)} \sim \mathcal{A}^{(2)}$. But, clearly seen from the input argument that $\mathcal{A}^{(1)} \nsim \mathcal{A}^{(2)}$ and therefore, Liu and Li [119] approach is unable to classify these alternatives. On the other hand, by applying the proposed SPIFSIA operator corresponding to equal priority levels between the parameters and the expert, we get their respective score values are $S(\mathcal{A}^{(1)}) = 0.1623$ and $S(\mathcal{A}^{(2)}) = 0.2402$. Since $S(\mathcal{A}^{(2)}) > S(\mathcal{A}^{(1)})$ and thus $\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$.

Example 8.4.2. Consider two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are evaluated by considering set of parameters $\{e_1, e_2, e_3\}$ and two judges $\{u_1, u_2\}$ whose preference values are summarized under IFSS environment as follows:

$$\mathcal{A}^{(1)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.5, 0.4) & (0.7, 0.1) & (0.6, 0.2) \\ (0.3, 0.6) & (0.8, 0.1) & (0.6, 0.3) \end{bmatrix} \end{array}$$

and

$$\mathcal{A}^{(2)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.7, 0.1) & (0.6, 0.3) & (0.3, 0.4) \\ (0.5, 0.4) & (0.6, 0.1) & (0.8, 0.2) \end{bmatrix} \end{array}$$

By applying Khalid and Abbas [98] approach to compute the measurement values 'd' of the given alternatives from its ideal values $B = ((1, 0))_{2\times 3}$, we get $d(\mathcal{A}^{(1)}, B) = 0.4167$ and $d(\mathcal{A}^{(2)}, B) = 0.4167$. From this, it is unable to classify the alternatives. However, if we SPIFSIA operator then the score values are $S(\mathcal{A}^{(1)}) = 0.3520$ and $S(\mathcal{A}^{(2)}) = 0.2260$. Thus, we conclude that $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)}$.

Example 8.4.3. Consider MCDM problem with two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are evaluated under three parameters $\{e_1, e_2, e_3\}$ by the two judges $\{u_1, u_2\}$. The rating values corresponding to its are summarized as follows:

$$\mathcal{A}^{(1)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.7, 0.1) & (0.8, 0.2) & (0.6, 0.2) \\ (0.3, 0.5) & (0.5, 0.4) & (0.7, 0.3) \end{bmatrix} \end{array}$$

and

$$\mathcal{A}^{(2)} = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.8, 0.2) & (0.7, 0.3) & (0.5, 0.3) \\ (0.6, 0.2) & (0.3, 0.4) & (0.6, 0.2) \end{bmatrix} \end{array}$$

Based on this information, by applying Cagman and Deli [21] approach under IFSS environment, from its ideal value $B = ((1, 0))_{2\times3}$ then the measurement values of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are obtained as 0.6833 and 0.6833. Clearly, from it we are unable to select the best one. On the other hand, by utilizing proposed operator with equal priorities levels we get relative score values as $S(\mathcal{A}^{(1)}) = 0.2622$ and $S(\mathcal{A}^{(2)}) = 0.2260$ which implies $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)}$ and hence $\mathcal{A}^{(1)}$ is preferred over $\mathcal{A}^{(2)}$.

8.4.4 Comparative study

A comparative analysis with some of the existing approaches [6, 21, 98, 142, 161, 168, 171] under IFSS environment has been conducted on the considered problem. The ranking order of the alternative based on these approaches is summarized in Table 8.5. It is

Table 8.5: Comparison with existing approaches under IFSS environment

	Overall values of alternatives				Ranking
	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
Arora and Garg [6]	0.3324	0.3054	0.2517	0.3486	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
Selvachandran, Maji, Faisal and Salleh [171]	0.7300	0.7232	0.7566	0.7377	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Muthukumar and Krishnan [142]	0.0185	0.0188	0.0161	0.0185	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \sim \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Sarala and Suganya [168]	0.6565	0.6792	0.6025	0.6552	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Khalid and Abbas [98]	0.5080	0.5040	0.5640	0.5280	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
Rajarajeswari and Dhanalakshmi [161]	0.3260	0.2778	0.2275	0.3370	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
Cagman and Deli [21]	0.3260	0.2778	0.2275	0.3370	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$

noted from this table that the final ranking of the proposed approach is different from these approaches which are quite obvious. This is mainly due to the changeable decision environments. For instance, in [6], weighted averaging and geometric aggregation operators were introduced based on the algebraic sum and product and do not take into account the information related to the dependency factor between the pairs of the IFSNs. On the other hand, the proposed method considers the consistency of the priority levels of different parameters and the experts for MCDM problems. In [21, 98, 142, 161, 168, 171], authors have used the similarity or distance measures to solve the MCDM problem. In these, measures have been computed from their ideal alternatives without considering the interaction between them and hence results obtained are inconsistent. For instance, from the table, it is seen that result corresponding to the approach [142] is unable to distinguish between the alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(4)}$ while the proposed one can easily distinguish them.

It is noted from the study that the computational procedure of the proposed approach is different from the existing distance/similarity or aggregation methods [6, 21, 98, 142, 161, 168, 171], but the proposed result in this work is more rational to reality in the decision process due to the consideration of the consistent priority degree between the pairs of the arguments. In addition, the advantages of the proposed operator are that it is based on the interactive operational laws defined in Definition 8.2.1, while the aggregation operators developed by Arora and Garg [6] are based on the traditional operations. Further, based on the comparison in Examples 8.4.1-8.4.3, we concluded that our proposed approach have benefits over the existing aggregation operators [6], information measures [21, 98, 142, 161, 168, 171], when prioritized relationship between the parameters is considered.

In the end, it is concluded that proposed operators consider the decision makers' risk parameter λ, which provide, the more choices to the decision makers to avail their desirable alternatives depending upon the different ranking values for the different parametric value of λ. Also, the corresponding studies under the IF set environment can be considered as a special case of the proposed operators by taking the number of parameters to be one.

8.5 Conclusion

This chapter develops some new aggregation operators, namely SPIFSIA and GSPIFSIA under the IFSS environment by considering the priority levels to show the relationship between the parameters and the experts. In these proposed operators, the factors of the degree of hesitancy have been brought into account during their operational laws. The prominent feature of these operators is studied. Furthermore, we have utilized these operators to develop some approaches to solve the MCDM problem with IFSS information. Finally, the effectiveness, feasibility, and superiority of the approach have been evidenced by a numerical example and some comparison is given. Also, the raking corresponding to the different attitude parameter λ will give the different choices to the decision makers to choose their desirable alternatives. From the existing approaches, our proposed approach also delivers a benefit over the existing approaches when the prioritized relationship between the parameters and the experts is considered.

Chapter 9

Bonferroni mean aggregation operators under intuitionistic fuzzy soft set environment and their applications to decision - making[1]

In this chapter, we developed some new Bonferroni mean (BM) and weighted BM averaging operator for aggregating the different preferences of the decision maker. Some of its desirable properties have also been discussed in details. Further, a DM method based on proposed operators has been presented and then illustrated with a numerical example. A comparative analysis between the proposed and the existing measures under the IFSS environment has been performed in terms of counter-intuitive cases for showing the validity of it.

9.1 Introduction

The extensive literature related to aggregation operators (AOs) for solving the DM problems are summarized in Section 1.1.4 of Chapter 1. However, from this literature, researchers have highlighted the importance of each factor or its ordered position but cannot

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reflect the interrelationships of the individual data. On the other hand, in our real-life situation, there always exist a situation in which a relationship between the different criteria such as prioritization, support, and impact each other pays a dominant role during an aggregation process. For example, when a decision maker makes a judgment based on consideration of risk and costs in a project, he should assign a higher priority to risk than cost. As an important tool, the AO plays a requisite role in aggregating the evaluations knowledge under the fuzzy environment. Though some operators have been proposed to aggregate uncertain and fuzzy information by taking all the input arguments as autonomous during the process. However, different criteria are not always free as some sort of relationship exists among them. So, combining the information with some classical AOs which considering the interrelationships between the argument is required. To address this, Bonferroni [16] suggested a mean-type operator named as Bonferroni mean (BM), which encapsulates the interconnection among the criteria and preferences. Due to this characteristic of BM operator, researchers are paying more attention to it. For example, Yager [242] introduced the BM operator with its interpretation and suggested its generalization. Xu and Yager [234] pointed out that BM operator can be used for aggregating not only crisp numbers but can be used for fuzzy arguments also and introduced BM operator for IFS theory. Kaur and Garg [97] put forward BM operator under cubic IFS environment. Li et al. [109] generalized BM operator and geometric mean and hence, proposed generalized Bonferroni mean and investigated their desirable properties.

Since, IFSSs have the great powerful ability to model the imprecise and ambiguous information in the real-world application than the other existing theories such as IFSs, IVIFSs. In the present chapter, motivated by the concept of the Bonferroni mean and by taking the advantages of the IFSS to express the uncertainty, we propose some new AOs called the intuitionistic fuzzy soft Bonferroni mean (IFSBM), as well as weighted intuitionistic fuzzy soft Bonferroni mean (WIFSBM) operator to aggregate the preferences of decision makers. Various desirable properties of these operators have also been investigated in details. The major advantages of the proposed operator are that they have considered the interrelationships of aggregated values. Further, we examine the properties and develop some special cases of proposed work. Some of the existing studies have been deduced from the proposed operator which signifies that the proposed operators are more generalized than the others. Finally, a DM approach has been given for ranking the different alternatives based on the proposed operators.

9.2 Aggregation operators based on Bonferroni mean

In this section, some new AOs have been presented for solving the DM problems. Under it, we shall propose the averaging operators namely, the intuitionistic fuzzy soft bonferroni mean(IFSBM) and weighted intuitionistic fuzzy soft bonferroni mean(WISFBM) operators for the collection of IFSNs $\mathcal{A}_{ik}(i = 1, 2, \ldots, n; k = 1, 2. \ldots, m)$, denoted by Ω, such that ξ_k and η_i respectively be the weight vector of the parameters and the experts such that $\eta_i, \xi_k > 0$ and $\sum\limits_{i=1}^{n} \eta_i = 1$, $\sum\limits_{k=1}^{m} \xi_k = 1$.

9.2.1 Intuitionistic fuzzy soft Bonferroni mean operator

Definition 9.2.1. Let $\mathcal{A}_{ik}(i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ be a collection of IFSNs. A IFSBM is a mapping IFSBM : $\Omega^n \to \Omega$ defined as

$$\text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\frac{1}{mn(m-1)(n-1)} \bigoplus_{\substack{k,l=1 \\ k \neq l}}^{m} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{jl}^{q} \right) \right)^{\frac{1}{p+q}} \tag{9.1}$$

where $p, q \geq 0$ be any two real numbers. Then, IFSBM is called as an intuitionistic fuzzy soft Bonferroni mean averaging operator.

Theorem 9.2.1. Let $\mathcal{A}_{ik} = (\zeta_{ik}, \vartheta_{ik}), (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ be a collection of IFSNs. Then, the aggregated value obtained by applying Definition 9.2.1 is again IFSN and is given by

$$\begin{aligned} &\text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ &= \left(\begin{array}{l} \left(1 - \left(\prod\limits_{\substack{k,l=1 \\ k \neq l}}^{m} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \zeta_{ik}^{p} \zeta_{jl}^{q})^{\frac{1}{mn(m-1)(n-1)}} \right) \right)^{\frac{1}{p+q}}, \\ 1 - \left(1 - \prod\limits_{\substack{k,l=1 \\ k \neq l}}^{m} \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \vartheta_{ik})^{p} (1 - \vartheta_{jl})^{q} \right)^{\frac{1}{mn(m-1)(n-1)}} \right)^{\frac{1}{p+q}} \end{array} \right) \end{aligned} \tag{9.2}$$

Proof. We will prove the Eq. (9.2) by mathematical induction on n and m. Since for all $i,k;\mathcal{A}_{ik}$ is an IFSN so we have $0 \leq \zeta_{ik}, \vartheta_{ik} \leq 1$ and $\zeta_{ik} + \vartheta_{ik} \leq 1$. Then the following steps of the mathematical induction have been followed.

Step 1: For $n = 2$, we have

$$\mathrm{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{2m}) = \left(\frac{1}{2m(m-1)} \bigoplus_{\substack{k,l=1 \\ k \neq l}}^{m} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{2} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{jl}^{q}\right) \right)^{\frac{1}{p+q}}$$

Now, by addition law of IFSNs, defined in Definition 2.5.2 of Chapter 2, we have

$$\begin{aligned}
& \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{2} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{jl}^{q}\right) \\
= \; & \left(\mathcal{A}_{1k}^{p} \otimes \mathcal{A}_{2l}^{q}\right) \oplus \left(\mathcal{A}_{2k}^{p} \otimes \mathcal{A}_{1l}^{q}\right) \\
= \; & \left(\zeta_{1k}^{p}\zeta_{2l}^{q}, 1-(1-\vartheta_{1k})^{p}(1-\vartheta_{2l})^{q}\right) \oplus \left(\zeta_{2k}^{p}\zeta_{1l}^{q}, 1-(1-\vartheta_{2k})^{p}(1-\vartheta_{1l})^{q}\right) \\
= \; & \left(1-(1-\zeta_{1k}^{p}\zeta_{2l}^{q})(1-\zeta_{2k}^{p}\zeta_{1l}^{q}), \left(1-(1-\vartheta_{1k})^{p}(1-\vartheta_{2l})^{q}\right)\left(1-(1-\vartheta_{2k})^{p}(1-\vartheta_{1l})^{q}\right)\right) \\
= \; & \left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^{2}(1-\zeta_{ik}^{p}\zeta_{jl}^{q}), \prod_{\substack{i,j=1 \\ i \neq j}}^{2}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{jl})^{q}\right)\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left(\frac{1}{2m(m-1)} \bigoplus_{\substack{k,l=1 \\ k \neq l}}^{m} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{2} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{jl}^{q}\right) \right)^{\frac{1}{p+q}} \\
= \; & \left(\left(1-\left(\prod_{\substack{k,l=1 \\ k \neq l}}^{m} \prod_{\substack{i,j=1 \\ i \neq j}}^{2} (1-\zeta_{ik}^{p}\zeta_{jl}^{q})^{\frac{1}{2m(m-1)}}\right)\right)^{\frac{1}{p+q}}, \right. \\
& \left. 1-\left(1-\prod_{\substack{k,l=1 \\ k \neq l}}^{m} \prod_{\substack{i,j=1 \\ i \neq j}}^{2} \left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{jl})^{q}\right)^{\frac{1}{2m(m-1)}}\right)^{\frac{1}{p+q}} \right)
\end{aligned}$$

Similarly, for $m = 2$, we have

$$\begin{aligned}
& \mathrm{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{n2}) \\
= \; & \left(\frac{1}{2n(n-1)} \bigoplus_{\substack{k,l=1 \\ k \neq l}}^{2} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{jl}^{q}\right) \right)^{\frac{1}{p+q}}
\end{aligned}$$

$$= \left(\left(1-\left(\prod_{\substack{k,l=1\\k\neq l}}^{2}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\zeta_{ik}^{p}\zeta_{jl}^{q}\right)^{\frac{1}{2n(n-1)}}\right)\right)^{\frac{1}{p+q}},\right.$$

$$\left. 1-\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{2}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{jl})^{q}\right)^{\frac{1}{2n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

Thus, the result holds for $n=2$ and $m=2$.

Step 2: If Eq. (9.2) holds for $m=k_1, n=k_2+1$ and $m=k_1+1, n=k_2$ then, for $m=k_1+1, n=k_2+1$,

$$\bigoplus_{\substack{k,l=1\\k\neq l}}^{k_1+1}\bigoplus_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{jl}^{q}\right)$$

$$= \left(\bigoplus_{\substack{k,l=1\\k\neq l}}^{k_1}\bigoplus_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{jl}^{q}\right)\right)\oplus\left(\bigoplus_{k=1}^{k_1}\bigoplus_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{j(k_1+1)}^{q}\right)\right)\oplus\left(\bigoplus_{k=1}^{k_1}\bigoplus_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(\mathcal{A}_{i(k_1+1)}^{p}\otimes\mathcal{A}_{jk}^{q}\right)\right) \tag{9.3}$$

Now, for it we can firstly prove

$$\bigoplus_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{j(k_1+1)}^{q}\right) = \begin{pmatrix} 1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{k_2+1}(1-\zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}), \\ \prod\limits_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right) \end{pmatrix} \tag{9.4}$$

by using mathematical induction on k_2.

Step 2a: For $k_2=1$, we have

$$\bigoplus_{\substack{i,j=1\\i\neq j}}^{2}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{j(k_1+1)}^{q}\right) = \left(\mathcal{A}_{1k}^{p}\otimes\mathcal{A}_{2(k_1+1)}^{q}\right)\oplus\left(\mathcal{A}_{2k}^{p}\otimes\mathcal{A}_{1(k_1+1)}^{q}\right)$$

$$= \left(1-\prod_{\substack{i,j=1\\i\neq j}}^{2}(1-\zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}),\quad \prod_{\substack{i,j=1\\i\neq j}}^{2}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right)\right)$$

Step 2b: Assume the result is true for $k_2=z$,

$$\bigoplus_{\substack{i,j=1\\i\neq j}}^{z+1}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{j(k_1+1)}^{q}\right) = \begin{pmatrix} 1-\prod\limits_{\substack{i,j=1\\i\neq j}}^{z+1}(1-\zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}), \\ \prod\limits_{\substack{i,j=1\\i\neq j}}^{z+1}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right) \end{pmatrix}$$

then for $k_2 = z+1$, we have

$$\begin{aligned}
&\bigoplus_{\substack{i,j=1\\ i\neq j}}^{z+2} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{j(k_1+1)}^{q}\right)\\
= &\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{z+1} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{j(k_1+1)}^{q}\right)\right) \oplus \left(\bigoplus_{i=1}^{z+1} \left(\mathcal{A}_{(z+2)k}^{p} \otimes \mathcal{A}_{i(k_1+1)}^{q}\right)\right) \oplus \left(\bigoplus_{i=1}^{z+1} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{(z+2)(k_1+1)}^{q}\right)\right)\\
= &\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{z+1} \left(1 - \zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}\right), \prod_{\substack{i,j=1\\ i\neq j}}^{z+1} \left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right)\right)\\
&\oplus \left(1 - \prod_{i=1}^{z+1} \left(1 - \zeta_{(z+2)k}^{p}\zeta_{i(k_1+1)}^{q}\right), \prod_{i=1}^{z+1} \left(1-(1-\vartheta_{(z+2)k})^{p}(1-\vartheta_{i(k_1+1)})^{q}\right)\right)\\
&\oplus \left(1 - \prod_{i=1}^{z+1} \left(\zeta_{ik}^{p}\zeta_{(z+2)(k_1+1)}^{q}\right), \prod_{i=1}^{z+1} \left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{(z+2)(k_1+1)})^{q}\right)\right)\\
= &\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{z+2} \left(1 - \zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}\right), \prod_{\substack{i,j=1\\ i\neq j}}^{z+2} \left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right)\right)
\end{aligned}$$

Hence, Eq. (9.4) holds for $k_2 = z+1$.

Therefore, we have

$$\begin{aligned}
\bigoplus_{k=1}^{k_1}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k_2+1} \left(\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{j(k_1+1)}^{q}\right) = &\left(1 - \prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\ i\neq j}}^{k_2+1} \left(1 - \zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}\right),\right.\\
&\left.\prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\ i\neq j}}^{k_2+1} \left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right)\right)
\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}
\bigoplus_{k=1}^{k_1}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k_2+1} \left(\mathcal{A}_{i(k_1+1)}^{p} \otimes \mathcal{A}_{jk}^{q}\right) = &\left(1 - \prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\ i\neq j}}^{k_2+1} \left(1 - \zeta_{i(k_1+1)}^{p}\zeta_{jk}^{q}\right),\right.\\
&\left.\prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\ i\neq j}}^{k_2+1} \left(1-(1-\vartheta_{i(k_1+1)})^{p}(1-\vartheta_{jk})^{q}\right)\right) \qquad (9.5)
\end{aligned}$$

Thus, from Eqs. (9.3), (9.4), and (9.5), we get

$$\begin{aligned}
&\bigoplus_{\substack{k,l=1\\k\neq l}}^{k_1+1}\bigoplus_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(\mathcal{A}_{ik}^{p}\otimes\mathcal{A}_{jl}^{q}\right)\\
=&\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{k_1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-\zeta_{ik}^{p}\zeta_{jl}^{q}\right),\prod_{\substack{k,l=1\\k\neq l}}^{k_1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{jl})^{q}\right)\right)\\
&\oplus\left(1-\prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-\zeta_{ik}^{p}\zeta_{j(k_1+1)}^{q}\right),\prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{j(k_1+1)})^{q}\right)\right)\\
&\oplus\left(1-\prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-\zeta_{i(k_1+1)}^{p}\zeta_{jk}^{q}\right),\prod_{k=1}^{k_1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-(1-\vartheta_{i(k_1+1)})^{p}(1-\vartheta_{jk})^{q}\right)\right)\\
=&\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{k_1+1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-\zeta_{ik}^{p}\zeta_{jl}^{q}\right),\prod_{\substack{k,l=1\\k\neq l}}^{k_1+1}\prod_{\substack{i,j=1\\i\neq j}}^{k_2+1}\left(1-(1-\vartheta_{ik})^{p}(1-\vartheta_{jl})^{q}\right)\right)
\end{aligned}$$

Hence the result given in Eq. (9.2), is true for all positive integers m, n. □

The above proposed AO has been illustrated with a following example to describe the "attractiveness of a house" under the IFSS environment.

Example 9.2.1. Let $\mathcal{X} = \{u_1, u_2, u_3, u_4\}$ represent the set of four experts which give their preferences to describe the "attractiveness of a house" over the set of parameters $\mathcal{E}$ such that $\mathcal{E}$={e_1("expensive"), e_2("beautiful"), e_3("in good location")} and corresponding to it the intuitionistic fuzzy soft set $(\mathcal{F}, \mathcal{E})$ which describes the "attractiveness of the houses" to the person have been evaluated by the decision-maker and gave their preferences as follows:

$$(\mathcal{F},\mathcal{E}) = \begin{array}{c} \\ u_1\\ u_2\\ u_3\\ u_4\end{array}\begin{array}{c}\begin{array}{ccc} e_1 & e_2 & e_3\end{array}\\ \begin{bmatrix}(0.8,0.1) & (0.5,0.3) & (0.4,0.5)\\ (0.4,0.3) & (0.3,0.5) & (0.7,0.2)\\ (0.7,0.1) & (0.8,0.2) & (0.5,0.1)\\ (0.3,0.5) & (0.5,0.2) & (0.6,0.1)\end{bmatrix}\end{array}$$

For the sake of simplicity, consider $p = q = 1$, be two real numbers. Then, we have

$$\begin{aligned}
&\prod_{\substack{k,l=1\\k\neq l}}^{3}\prod_{\substack{i,j=1\\i\neq j}}^{4}\left(1-\zeta_{ik}\zeta_{jl}\right)^{\frac{1}{3.4.(3-1)(4-1)}}\\
=\ &(1-(0.8)(0.3))^{\frac{1}{72}}\times(1-(0.8)(0.8))^{\frac{1}{72}}\times(1-(0.8)(0.5))^{\frac{1}{72}}\times\\
&\ldots\ldots\times(1-(0.6)(0.3))^{\frac{1}{72}}\times(1-(0.6)(0.8))^{\frac{1}{72}}\\
=\ &0.6899
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{\substack{k,l=1\\k\neq l}}^{3}\prod_{\substack{i,j=1\\i\neq j}}^{4}\left(1-(1-\vartheta_{ik})(1-\vartheta_{jl})\right)^{\frac{1}{3.4.(3-1)(4-1)}}\\
=\ &(1-(1-0.1)(1-0.5))^{\frac{1}{72}}\times(1-(1-0.1)(1-0.2))^{\frac{1}{72}}\times\\
&\ldots\ldots\times(1-(1-0.1)(1-0.5))^{\frac{1}{72}}\times(1-(1-0.1)(1-0.2))^{\frac{1}{72}}\\
=\ &0.4180
\end{aligned}$$

Therefore, by using Eq. (9.2), we get $\text{IFSBM}^{1,1}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{43}) = ((1-0.6899)^{\frac{1}{2}}, 1-(1-0.4180)^{\frac{1}{2}}) = (0.5569, 0.2371)$.

Further, from the Theorem 9.2.1, we have observed that proposed IFSBM operator satisfies the certain properties such as idempotency, boundedness, monotonicity which can be read as follows:

Property 9.2.1. (Idempotency:) If $\mathcal{A}_{ik} = \mathcal{A} = (\zeta, \vartheta)$, for all i, k then

$$\text{IFSBM}^{p,q}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) = \mathcal{A}$$

Proof. If for all $i, k, \mathcal{A}_{ik} = \mathcal{A}$, then

$$\begin{aligned}
&\text{IFSBM}^{p,q}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\
=\ &\left(\begin{array}{c}
\left(1-\left(\displaystyle\prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\zeta^{p}\zeta^{q}\right)^{\frac{1}{mn(m-1)(n-1)}}\right)\right)^{\frac{1}{p+q}},\\
1-\left(1-\displaystyle\prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta)^{p}(1-\vartheta)^{q}\right)^{\frac{1}{mn(m-1)(n-1)}}\right)^{\frac{1}{p+q}}
\end{array}\right)
\end{aligned}$$

$$
= \left(\begin{array}{c} \left(1-\left(\prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\zeta^{p+q}\right)^{\frac{1}{mn(m-1)(n-1)}}\right)\right)^{\frac{1}{p+q}}, \\ 1-\left(1-\prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\vartheta)^{p+q}\right)^{\frac{1}{mn(m-1)(n-1)}}\right)^{\frac{1}{p+q}} \end{array} \right)
$$

$$
\begin{aligned}
&= \left(\left(1-(1-\zeta^{p+q})\right)^{\frac{1}{p+q}}, 1-\left(1-(1-(1-\vartheta)^{p+q})\right)^{\frac{1}{p+q}}\right) \\
&= \left((\zeta^{p+q})^{\frac{1}{p+q}}, 1-((1-\vartheta)^{p+q})^{\frac{1}{p+q}}\right) \\
&= (\zeta, \vartheta) = \mathcal{A}
\end{aligned}
$$

Hence the result. □

Property 9.2.2. (Boundedness:) Let $\mathcal{A}^- = \left(\min_k \min_i \{\zeta_{ik}\},\ \max_k \max_i \{\zeta_{ik}\}\right)$ and $\mathcal{A}^+ = \left(\max_k \max_i \{\zeta_{ik}\},\ \min_k \min_i \{\zeta_{ik}\}\right)$, then

$$
\mathcal{A}^- \leq \text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \mathcal{A}^+
$$

Proof. For the sake of simplicity, we denote $\delta = \frac{1}{mn(m-1)(n-1)}$. Now, for all i, k, we have

$$
\begin{aligned}
& \min_k \min_i \{\zeta_{ik}\} \leq \zeta_{ik} \leq \max_k \max_i \{\zeta_{ik}\} \\
\Leftrightarrow\ & \left(\min_k \min_i \{\zeta_{ik}\}\right)^{p+q} \leq \zeta_{ik}^p \zeta_{jl}^q \leq \left(\max_k \max_i \{\zeta_{ik}\}\right)^{p+q} \\
\Leftrightarrow\ & \left(1-\left(\max_k \max_i \{\zeta_{ik}\}\right)^{p+q}\right)^{\delta} \leq \left(1-\zeta_{ik}^p \zeta_{jl}^q\right)^{\delta} \leq \left(1-\left(\min_k \min_i \{\zeta_{ik}\}\right)^{p+q}\right)^{\delta} \\
\Leftrightarrow\ & \prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(\max_k \max_i \{\zeta_{ik}\}\right)^{p+q}\right)^{\delta} \leq \prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\zeta_{ik}^p \zeta_{jl}^q\right)^{\delta} \\
& \qquad \leq \prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(\min_k \min_i \{\zeta_{ik}\}\right)^{p+q}\right)^{\delta} \\
\Leftrightarrow\ & 1-\left(\max_k \max_i \{\zeta_{ik}\}\right)^{p+q} \leq \prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\zeta_{ik}^p \zeta_{jl}^q\right)^{\delta} \leq 1-\left(\min_k \min_i \{\zeta_{ik}\}\right)^{p+q} \\
\Leftrightarrow\ & \left(\min_k \min_i \{\zeta_{ik}\}\right)^{p+q} \leq 1-\prod_{\substack{k,l=1\\k\neq l}}^{m}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\zeta_{ik}^p \zeta_{jl}^q\right)^{\delta} \leq \left(\max_k \max_i \{\zeta_{ik}\}\right)^{p+q}
\end{aligned}
$$

Therefore, we have

$$\min_k \min_i\{\zeta_{ik}\} \leq \left(1 - \prod_{\substack{k,l=1 \\ k\neq l}}^{m} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \zeta_{ik}^{p}\zeta_{jl}^{q}\right)^{\delta}\right)^{\frac{1}{p+q}} \leq \max_k \max_i\{\zeta_{ik}\} \tag{9.6}$$

Furthermore,

$$\begin{aligned}
& \min_k \min_i\{\vartheta_{ik}\} \leq \vartheta_{ik} \leq \max_k \max_i\{\vartheta_{ik}\} \\
\Leftrightarrow\ & \left(1 - \max_k \max_i\{\vartheta_{ik}\}\right)^{p+q} \leq (1-\vartheta_{ik})^p(1-\vartheta_{jl})^q \leq \left(1 - \min_k \min_i\{\vartheta_{ik}\}\right)^{p+q} \\
\Leftrightarrow\ & 1 - \left(1 - \min_k \min_i\{\vartheta_{ik}\}\right)^{p+q} \leq 1 - (1-\vartheta_{ik})^p(1-\vartheta_{jl})^q \leq 1 - \left(1 - \max_k \max_i\{\vartheta_{ik}\}\right)^{p+q} \\
\Leftrightarrow\ & \left(1 - \left(1 - \min_k \min_i\{\vartheta_{ik}\}\right)^{p+q}\right)^{\delta} \leq \left(1 - (1-\vartheta_{ik})^p(1-\vartheta_{jl})^q\right)^{\delta} \\
& \qquad \leq \left(1 - \left(1 - \max_k \max_i\{\vartheta_{ik}\}\right)^{p+q}\right)^{\delta} \\
\Leftrightarrow\ & 1 - \left(1 - \min_k \min_i\{\vartheta_{ik}\}\right)^{p+q} \leq \prod_{\substack{k,l=1 \\ k\neq l}}^{m} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - (1-\vartheta_{ik})^p(1-\vartheta_{jl})^q\right)^{\delta} \\
& \qquad \leq 1 - \left(1 - \max_k \max_i\{\vartheta_{ik}\}\right)^{p+q} \\
\Leftrightarrow\ & \left(1 - \max_k \max_i\{\vartheta_{ik}\}\right)^{p+q} \leq 1 - \prod_{\substack{k,l=1 \\ k\neq l}}^{m} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - (1-\vartheta_{ik})^p(1-\vartheta_{jl})^q\right)^{\delta} \\
& \qquad \leq \left(1 - \min_k \min_i\{\vartheta_{ik}\}\right)^{p+q}
\end{aligned}$$

Therefore, we have

$$\min_k \min_i\{\vartheta_{ik}\} \leq 1 - \left(1 - \prod_{\substack{k,l=1 \\ k\neq l}}^{m} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - (1-\vartheta_{ik})^p(1-\vartheta_{jl})^q\right)^{\delta}\right)^{\frac{1}{p+q}} \leq \max_k \max_i\{\vartheta_{ik}\} \tag{9.7}$$

Take $\beta \equiv \mathrm{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = (\zeta_\beta, \vartheta_\beta)$, $\mathcal{A}^- = (\min_k \min_i\{\zeta_{ik}\}, \max_k \max_i\{\zeta_{ik}\})$ and $\mathcal{A}^+ = (\max_k \max_i\{\zeta_{ik}\}, \min_k \min_i\{\zeta_{ik}\})$ then from Eq. (9.6) and (9.7), we have $\min_k \min_i\{\zeta_{ik}\} \leq \zeta_\beta \leq \max_k \max_i\{\zeta_{ik}\}$ and $\min_k \min_i\{\vartheta_{ik}\} \leq \vartheta_\beta \leq \max_k \max_i\{\vartheta_{ik}\}$. So by definition of score function, we get

$$\begin{aligned} S(\beta) &= \zeta_\beta - \vartheta_\beta \leq \max_k \max_i \{\zeta_{ik}\} - \min_k \min_i \{\vartheta_{ik}\} = S(\mathcal{A}^+) \\ \text{and } S(\beta) &= \zeta_\beta - \vartheta_\beta \geq \min_k \min_i \{\zeta_{ik}\} - \max_k \max_i \{\vartheta_{ik}\} = S(\mathcal{A}^-) \end{aligned}$$

Now, based on it, the following three cases arises:

(Case 1:) If $S(\beta) < S(\mathcal{A}^+)$ and $S(\beta) > S(\mathcal{A}^-)$, then, we have

$$\mathcal{A}^- < \text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) < \mathcal{A}^+.$$

(Case 2:) If $S(\beta) = S(\mathcal{A}^+)$, i.e., $\zeta_\beta - \vartheta_\beta = \max_k \max_i \{\zeta_{ik}\} - \min_k \min_i \{\vartheta_{ik}\}$, then, by above inequalities, we have $\zeta_\beta = \max_k \max_i \{\zeta_{ik}\}$ and $\vartheta_\beta = \min_k \min_i \{\vartheta_{ik}\}$. Therefore, $H_\beta = \zeta_\beta + \vartheta_\beta = \max_k \max_i \{\zeta_{ik}\} + \min_k \min_i \{\vartheta_{ik}\}$ and $\text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}^+$.

(Case 3:) If $S(\beta) = S(\mathcal{A}^-)$, i.e., $\zeta_\beta - \vartheta_\beta = \min_k \min_i \{\zeta_{ik}\} - \max_k \max_i \{\vartheta_{ik}\}$, then by above inequalities, we have $\zeta_\beta = \min_k \min_i \{\zeta_{ik}\}$ and $\vartheta_\beta = \max_k \max_i \{\vartheta_{ik}\}$. Therefore, we have $H_\beta = \zeta_\beta + \vartheta_\beta = \min_k \min_i \{\zeta_{ik}\} + \max_k \max_i \{\vartheta_{ik}\}$ and $\text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12},$ $\ldots, \mathcal{A}_{nm}) = \mathcal{A}^-$.

Hence, by combining all these cases, we have

$$\mathcal{A}^- \leq \text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \mathcal{A}^+$$

□

Property 9.2.3. (Monotonicity:) Let $\mathcal{A}'_{ik}$ be an another collection of IFSNs such that $\mathcal{A}_{ik} \leq \mathcal{A}'_{ik}$ for all k, i, then

$$\text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \leq \text{IFSBM}^{p,q}(\mathcal{A}'_{11}, \mathcal{A}'_{12}, \ldots, \mathcal{A}'_{nm})$$

Proof. Proof of this property is similar to Property 9.2.2, so we omit here. □

Property 9.2.4. (Commutativity:) Let $\mathcal{A}_{ik}(i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ be a collection of IFSNs. Then

$$\text{IFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \text{IFSBM}^{p,q}(\dot{\mathcal{A}}_{11}, \dot{\mathcal{A}}_{12}, \ldots, \dot{\mathcal{A}}_{nm})$$

where $(\dot{\mathcal{A}}_{11}, \dot{\mathcal{A}}_{12}, \ldots, \dot{\mathcal{A}}_{nm})$ is any permutation of $(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$.

Proof. Since $(\dot{\mathcal{A}}_{11}, \dot{\mathcal{A}}_{12}, \ldots, \dot{\mathcal{A}}_{nm})$ be any permutation of $(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$, then $\mathcal{A}_{ik}^{p} \otimes \mathcal{A}_{jl}^{q} = \dot{\mathcal{A}}_{ik}^{p} \otimes \dot{\mathcal{A}}_{jl}^{q}$. Hence, by definition of IFSBM^{p,q}, we get the desired result. □

9.2.2 Weighted intuitionistic fuzzy soft Bonferroni mean operator

From the IFSBM AO, it has been examined that they have considered only the preferences of decision makers over the different parameters. However, in real-life situations, it is necessary to consider the weights of the parameters and the experts during the analysis, so that the decision-maker may get the more closeable result according to their desired goals. For this, in the following, we have proposed the weighted intuitionistic fuzzy soft BM operator(WIFSBM), for the collection of IFSNs.

Definition 9.2.2. A WIFSBM operator for a collection of IFSNs $\mathcal{A}_{ik}, (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ is a mapping WIFSBM : $\Omega^n \to \Omega$ defined for any real numbers $p, q > 0$ as

$$\begin{aligned} &\mathrm{WIFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ =& \left(\frac{1}{mn(m-1)(n-1)} \bigoplus_{\substack{k,l=1 \\ k\neq l}}^{m} \bigoplus_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\left(\xi_k(\eta_i \mathcal{A}_{ik})\right)^p \otimes \left(\xi_l(\eta_j \mathcal{A}_{jl})\right)^q \right) \right)^{\frac{1}{p+q}} \end{aligned} \tag{9.8}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^T$ be the weight vectors of parameters and the experts respectively such that $\xi_k > 0$, $\sum\limits_{k=1}^{m} \xi_k = 1, \eta_i > 0$ and $\sum\limits_{i=1}^{n} \eta_i = 1$.

Theorem 9.2.2. The aggregated value of IFSNs $\mathcal{A}_{ik} = (\zeta_{ik}, \vartheta_{ik}); (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ by using WIFSBM operator is still IFSN and is given by:

$$\begin{aligned} &\mathrm{WIFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \\ &\left(\left(1 - \left(\prod_{\substack{k,l=1 \\ k\neq l}}^{m} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - (1 - (1-\zeta_{ik})^{\xi_k \eta_i})^p (1 - (1-\zeta_{jl})^{\xi_l \eta_j})^q \right)^{\frac{1}{mn(m-1)(n-1)}} \right) \right)^{\frac{1}{p+q}}, \right. \\ &\left. \quad 1 - \left(1 - \prod_{\substack{k,l=1 \\ k\neq l}}^{m} \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - (1 - \vartheta_{ik}^{\xi_k \eta_i})^p (1 - \vartheta_{jl}^{\xi_l \eta_j})^q \right)^{\frac{1}{mn(m-1)(n-1)}} \right)^{\frac{1}{p+q}} \right) \end{aligned} \tag{9.9}$$

Proof. Similar to Theorem 9.2.1, so we omit here. □

Example 9.2.2. Consider a data as given in Example 9.2.1. In it, let $\xi = (0.3, 0.5, 0.2)^T$ and $\eta = (0.1, 0.2, 0.4, 0.3)^T$ be the weight vectors of different parameters and the experts, respectively. Then by using Eq. (9.9) corresponding to $p = q = 1$, we have

$$\begin{aligned}
&\text{WIFSBM}^{1,1}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{43}) \\
= &\left(\left(1 - \left(\prod_{\substack{k,l=1\\k\neq l}}^{3} \prod_{\substack{i,j=1\\i\neq j}}^{4} \left(1 - (1-(1-\zeta_{ik})^{\xi_k \eta_i})(1-(1-\zeta_{jl})^{\xi_l \eta_j})\right)^{\frac{1}{3.4.(3-1)(4-1)}} \right) \right)^{\frac{1}{2}}, \right. \\
&\left. 1 - \left(1 - \prod_{\substack{k,l=1\\k\neq l}}^{3} \prod_{\substack{i,j=1\\i\neq j}}^{4} \left(1 - (1-\vartheta_{ik}^{\xi_k \eta_i})(1-\vartheta_{jl}^{\xi_l \eta_j})\right)^{\frac{1}{3.4.(3-1).(4-1)}} \right)^{\frac{1}{2}} \right) \\
= &\left((1-0.9960)^{\frac{1}{2}}, 1-(1-0.9817)^{\frac{1}{2}} \right) \\
= &(0.0635, 0.8647)
\end{aligned}$$

9.2.3 Special cases of the proposed operator

In the following, we will discuss some special cases of WIFSBM operator by taking different values of p and q.

Case 1. If $q \to 0$, then, the proposed operator reduces to weighted intuitionistic fuzzy soft mean operator as follows:

$$\begin{aligned}
&\lim_{q\to 0} \text{WIFSBM}^{p,q}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
= &\lim_{q\to 0} \begin{pmatrix} \left(1 - \left(\prod_{\substack{k,l=1\\k\neq l}}^{m} \prod_{\substack{i,j=1\\i\neq j}}^{n} \left(1 - (1-(1-\zeta_{ik})^{\xi_k \eta_i})^p(1-(1-\zeta_{jl})^{\xi_l \eta_j})^q\right)^{\frac{1}{mn(m-1)(n-1)}} \right) \right)^{\frac{1}{p+q}}, \\
1 - \left(1 - \prod_{\substack{k,l=1\\k\neq l}}^{m} \prod_{\substack{i,j=1\\i\neq j}}^{n} \left(1 - (1-\vartheta_{ik}^{\xi_k \eta_i})^p(1-\vartheta_{jl}^{\xi_l \eta_j})^q\right)^{\frac{1}{mn(m-1)(n-1)}} \right)^{\frac{1}{p+q}} \end{pmatrix} \\
= &\begin{pmatrix} \left(1 - \left(\prod_{k=1}^{m} \prod_{i=1}^{n} \left(1 - (1-(1-\zeta_{ik})^{\xi_k \eta_i})^p\right)^{\frac{(m-1)(n-1)}{mn(m-1)(n-1)}} \right) \right)^{\frac{1}{p}}, \\
1 - \left(1 - \prod_{k=1}^{m} \prod_{i=1}^{n} \left(1 - (1-\vartheta_{ik}^{\xi_k \eta_i})^p\right)^{\frac{(m-1)(n-1)}{mn(m-1)(n-1)}} \right)^{\frac{1}{p}} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(1-\left(\prod_{k=1}^{m}\prod_{i=1}^{n}\left(1-(1-(1-\zeta_{ik})^{\xi_k\eta_i})^p\right)^{\frac{1}{mn}}\right)\right)^{\frac{1}{p}},\right.\\
&\qquad\left.1-\left(1-\prod_{k=1}^{m}\prod_{i=1}^{n}\left(1-(1-\vartheta_{ik}^{\xi_k\eta_i})^p\right)^{\frac{1}{mn}}\right)^{\frac{1}{p}}\right)\\
&= \left(\frac{1}{mn}\bigoplus_{k=1}^{m}\bigoplus_{i=1}^{n}\left(\xi_k(\eta_i\mathcal{A}_{ik})\right)^p\right)^{\frac{1}{p}}\\
&= \mathrm{WIFSBM}^{p,0}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})
\end{aligned}
$$

Case 2. If $p=2$ and $q\to 0$, then the proposed operator reduces to weighted intuitionistic fuzzy soft square mean operator as follows:

$$
\begin{aligned}
&\mathrm{WIFSBM}^{2,0}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\
&= \left(\left(1-\left(\prod_{k=1}^{m}\prod_{i=1}^{n}\left(1-(1-(1-\zeta_{ik})^{\xi_k\eta_i})^2\right)^{\frac{1}{mn}}\right)\right)^{\frac{1}{2}},\right.\\
&\qquad\left.1-\left(1-\prod_{k=1}^{m}\prod_{i=1}^{n}\left(1-(1-\vartheta_{ik}^{\xi_k\eta_i})^2\right)^{\frac{1}{mn}}\right)^{\frac{1}{2}}\right)\\
&= \left(\frac{1}{mn}\bigoplus_{k=1}^{m}\bigoplus_{i=1}^{n}\left(\xi_k(\eta_i\mathcal{A}_{ik})\right)^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Case 3. If $p=1$ and $q\to 0$, then Eq. (9.9) reduces to weighted intuitionistic fuzzy soft average operator as follows:

$$
\begin{aligned}
&\mathrm{WIFSBM}^{1,0}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\
&= \left(\left(1-\left(\prod_{k=1}^{m}\prod_{i=1}^{n}\left(1-(1-(1-\zeta_{ik})^{\xi_k\eta_i})\right)^{\frac{1}{mn}}\right)\right),\right.\\
&\qquad\left.1-\left(1-\prod_{k=1}^{m}\prod_{i=1}^{n}\left(1-(1-\vartheta_{ik}^{\xi_k\eta_i})\right)^{\frac{1}{mn}}\right)\right)\\
&= \left(1-\left(\prod_{k=1}^{m}\prod_{i=1}^{n}(1-\zeta_{ik})^{\xi_k\eta_i}\right)^{\frac{1}{mn}},1-\left(1-\prod_{k=1}^{m}\prod_{i=1}^{n}(\vartheta_{ik}^{\xi_k\eta_i})^{\frac{1}{mn}}\right)\right)\\
&= \left(\frac{1}{mn}\bigoplus_{k=1}^{m}\bigoplus_{i=1}^{n}\xi_k(\eta_i\mathcal{A}_{ik})\right)
\end{aligned}
$$

Case 4. If $p = q = 1$, then we have

$$\begin{aligned}
&\mathrm{WIFSBM}^{1,1}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
&= \left(\frac{1}{mn(m-1)(n-1)} \bigoplus_{\substack{k,l=1 \\ k \neq l}}^{m} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((\xi_k(\eta_i \mathcal{A}_{ik})) \otimes (\xi_l(\eta_j \mathcal{A}_{jl})) \right) \right)^{\frac{1}{2}} \\
&= \left(\left(1 - \left(\prod_{\substack{k,l=1 \\ k \neq l}}^{m} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \zeta_{ik})^{\xi_k \eta_i})(1 - (1 - \zeta_{jl})^{\xi_l \eta_j})\right)^{\frac{1}{mn(m-1)(n-1)}} \right) \right) \right)^{\frac{1}{2}}, \\
&1 - \left(1 - \prod_{\substack{k,l=1 \\ k \neq l}}^{m} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \vartheta_{ik}^{\xi_k \eta_i})(1 - \vartheta_{jl}^{\xi_l \eta_j}) \right)^{\frac{1}{mn(m-1)(n-1)}} \right)^{\frac{1}{2}} \Bigg)
\end{aligned}$$

The reduced operator $\mathrm{WIFSBM}^{1,1}$ is called the weighted intuitionistic fuzzy soft interrelated square mean.

9.3 Proposed approach

In this section, based on the proposed operator, a DM method has been presented for ranking the different alternatives under the IFSS environment.

The description of the MCDM problem is given in Section 2.6 of Chapter 2. In it, the rating values corresponding to each alternative $\mathcal{A}^{(b)}(b = 1, 2, \ldots, t)$ is assessed by an expert $u_i(i = 1, 2, \ldots, n)$ under the parameters $e_j(j = 1, 2, \ldots, m)$ and represent them in terms of IFSNs $\mathcal{A}_{ik} = (\zeta_{ik}, \vartheta_{ik})$. Then, the following steps of the proposed approach are stated based on the proposed operators, to find the best alternative(s) among the all.

Step 1: Arrange the information related to each alternative in terms of intuitionistic fuzzy soft matrix $\mathcal{M} = (\mathcal{A}_{ik})$ for $i = 1, 2, \ldots, n; k = 1, 2, \ldots, m$.

Step 2: Obtain the normalized decision matrices $\mathcal{R} = (r_{ik})$ corresponding to each alternative, where r_{ik} is given according to Eq. (2.34) of Chapter 2.

Step 3: Aggregate the IFSNs $r_{ik}(i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ for each alternative $\mathcal{A}^{(b)}(b = 1, 2, \ldots, t)$ into collective decision matrix $r_b = (\zeta^{(b)}, \vartheta^{(b)})$ by the proposed IFSBM or WIFSBM operator(given in Eqs. (9.2) and (9.9) respectively).

Step 4: Calculate the score function of the aggregated value $r^{(b)} = (\zeta^{(b)}, \vartheta^{(b)})$ by using Eq.(9.10) as

$$\mathcal{S}(r^{(b)}) = \zeta^{(b)} - \vartheta^{(b)} \tag{9.10}$$

If for any two indices b_1 and b_2, we get $\mathcal{S}(r_{b_1}) = \mathcal{S}(r_{b_2})$ then compute the accuracy degree corresponding to such indices by using Eq. (9.11) as

$$\mathcal{H}(r^{(b)}) = \zeta^{(b)} + \vartheta^{(b)} \tag{9.11}$$

Step 5: Rank the alternatives $\mathcal{A}^{(b)}(b = 1, 2, \ldots, t)$ based on its score values and hence select the best one(s).

9.4 Numerical Example

The above-mentioned approach has been illustrated with a practical example of DM which can be read as:

9.4.1 Case study

Monsoon rains in river catchments have caused flooding in the northeastern state of Assam, India in July 2016. The Brahmaputra river and its tributaries overflow as a result of monsoon rains which effected over 80,000 people and 4,000 hectares of crop land according to Assam State Disaster Management Agency(ASDMA). The worst affected areas of Assam are $\{\text{Lakhimpur}(\mathcal{A}^{(1)}), \text{Dhemaji}(\mathcal{A}^{(2)}), \text{Nagaon}(\mathcal{A}^{(3)}), \text{Jorhat}(\mathcal{A}^{(4)})\}$. A team of five experts, denoted by $\{u_1, u_2, u_3, u_4, u_5\}$ from different departments of ASDMA with weight vector $\eta = (0.2, 0.15, 0.2, 0.3, 0.15)^T$, will inspect the flood affected areas on the different parameters {crop loss (e_1), loss of livestock (e_2), loss of life (e_3), damage of houses (e_4), ecological damage (e_5)} having weight vector $\xi = (0.2, 0.1, 0.3, 0.15, 0.25)^T$. Based on their recommendations, the government has to allocate funds according to the level of damage. In order to fulfill it, they have evaluated these and gave their preferences in terms of IFSNs. The main procedure steps for obtaining the best alternative(s) by utilizing the developed approach are summarized as follows:

Table 9.1: Rating values of each alternative in terms of IFSNs

	For alternative $\mathcal{A}^{(1)}$					For alternative $\mathcal{A}^{(2)}$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.3, 0.4)$	$(0.5, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.6, 0.2)$	$(0.4, 0.3)$	$(0.5, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.7, 0.2)$
u_2	$(0.6, 0.1)$	$(0.6, 0.2)$	$(0.2, 0.4)$	$(0.5, 0.1)$	$(0.7, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.3)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0.4, 0.1)$
u_3	$(0.5, 0.1)$	$(0.7, 0.2)$	$(0.5, 0.4)$	$(0.2, 0.2)$	$(0.4, 0.2)$	$(0.5, 0.3)$	$(0.5, 0.1)$	$(0.5, 0.3)$	$(0.3, 0.2)$	$(0.6, 0.2)$
u_4	$(0.2, 0.4)$	$(0.5, 0.1)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.7, 0.3)$	$(0.4, 0.2)$	$(0.5, 0.1)$	$(0.5, 0.2)$
u_5	$(0.6, 0.1)$	$(0.3, 0.4)$	$(0.4, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.4, 0.2)$	$(0.5, 0.2)$	$(0.3, 0.3)$	$(0.6, 0.1)$	$(0.4, 0.2)$
	For alternative $\mathcal{A}^{(3)}$					For alternative $\mathcal{A}^{(4)}$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.4, 0.3)$	$(0.5, 0.4)$	$(0.5, 0.2)$	$(0.6, 0.1)$	$(0.4, 0.2)$	$(0.3, 0.4)$	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.4, 0.3)$	$(0.2, 0.3)$
u_2	$(0.5, 0.1)$	$(0.3, 0.2)$	$(0.3, 0.2)$	$(0.4, 0.2)$	$(0.3, 0.2)$	$(0.5, 0.1)$	$(0.4, 0.2)$	$(0.4, 0.2)$	$(0.6, 0.1)$	$(0.2, 0.6)$
u_3	$(0.5, 0.3)$	$(0.5, 0.1)$	$(0.4, 0.2)$	$(0.2, 0.2)$	$(0.5, 0.4)$	$(0.2, 0.1)$	$(0.4, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.5, 0.2)$
u_4	$(0.5, 0.1)$	$(0.4, 0.5)$	$(0.3, 0.2)$	$(0.7, 0.2)$	$(0.3, 0.2)$	$(0.7, 0.2)$	$(0.5, 0.1)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.7, 0.1)$
u_5	$(0.7, 0.1)$	$(0.4, 0.6)$	$(0.4, 0.2)$	$(0.3, 0.1)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.5, 0.4)$	$(0.4, 0.2)$	$(0.3, 0.2)$	$(0.7, 0.1)$

Step 1: The experts give their rating values of each alternative $\mathcal{A}_b (b = 1, 2, 3, 4)$ under the consideration of the parameter set $\{e_1, e_2, e_3, e_4, e_5\}$ in terms of IFSNs and summarized in Table 9.1.

Step 2: Since all the preferences given by an expert are of same type, so there is no need of normalizing the data.

Step 3: Aggregate the preferences of these values corresponding to each alternative by utilizing the proposed aggregation operator WIFSBM, as given in Eq. (9.9), for $p = q = 1$, we get the respective values, denoted by $r^{(b)}; (b = 1, 2, 3, 4)$ are $r^{(1)} = (0.0273, 0.9290)$, $r^{(2)} = (0.0270, 0.9318)$, $r^{(3)} = (0.0228, 0.9312)$ and $r^{(4)} = (0.0273, 0.9265)$.

Step 4: The overall preference values of these aggregated number are computed by using the score function and get $\mathcal{S}(r^{(1)}) = -0.9017$, $\mathcal{S}(r^{(2)}) = -0.9048$, $\mathcal{S}(r^{(3)}) = -0.9084$ and $\mathcal{S}(r^{(4)}) = -0.8992$.

Step 5: According to the score values, the ordering of the alternatives is $\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$, in which '$\succ$' means "preferred to". From this, it has been seen that the best alternative is $\mathcal{A}^{(4)}$, i.e., $\mathcal{A}^{(4)}$ is the most damaged area than the other areas.

In order to see the impact of p and q on the DM, an analysis has been conducted by taking different values of p and q for the considered problem. Based on these different values, the proposed AO has been applied and their corresponding score values, as well as the ranking order of the alternatives, are summarized in Table 9.2. From this, it has been examined

Table 9.2: Behavior on ranking order of the alternatives by WIFSBM operator for different p and q

Value of		Rating Values				Ranking
p	q	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
1	0	-0.9071	-0.9099	-0.9131	-0.9028	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
2	0	-0.8941	-0.9011	-0.9031	-0.8842	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
10	0	-0.8058	-0.8548	-0.8569	-0.7958	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
0	1	-0.9051	-0.9075	-0.9135	-0.9048	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
0	2	-0.8911	-0.8959	-0.9018	-0.8887	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
1	1	-0.9017	-0.9048	-0.9084	-0.8992	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
2	2	-0.8807	-0.8895	-0.8922	-0.8760	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$

that as the value of p or q increases, the score values of each alternative by WIFSBM operator will also increase. Further, it has been concluded that the ranking order of given alternatives is same in each case and found that the most suitable alternative is $\mathcal{A}^{(4)}$, i.e., $\mathcal{A}^{(4)}$ is the most damaged area. In the special cases where at least one of two parameters p and q take the value of zero, the WIFSBM operator cannot capture the interrelationship of individual arguments. So, for convenience, the values of the parameters can be set as $p = q = 1$ or $p = q = 2$, which makes the computational procedure quite easy and also reflect the correlation among the criteria. Thus, the decision-maker can easily select the desirable alternative according to the actual need of the DM process.

In order to compare the performance of the proposed approach with some existing approaches under IFSS environment, we conducted a comparative analysis based on the different approaches, as given by the authors [6, 21, 98, 142, 161, 168, 171]. The results corresponding to it have been summarized in the Table 9.3.

Table 9.3: Comparison with existing approaches under IFSS and IFS environment

	Method	Rating Values				Ranking
		$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
Arora and Garg [6]	Aggregation operator	0.3324	0.3054	0.2517	0.3486	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
Selvachandran, Maji, Faisal and Salleh [171]	Distance measure	0.7300	0.7232	0.7566	0.7377	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Muthukumar and Krishnan [142]	Similarity measure	0.0185	0.0188	0.0161	0.0185	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} = \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
Sarala and Suganya [168]	Similarity measure	0.6565	0.6792	0.6025	0.6552	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)}$
Khalid and Abbas [98]	Distance measures	0.5108	0.5070	0.5705	0.5100	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
Rajarajeswari and Dhanalakshmi [161]	Similarity measures	0.3260	0.2778	0.2275	0.3370	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
Cagman and Deli [21]	Similarity measures	0.3260	0.2778	0.2275	0.3370	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$

9.4.2 Superiority of the proposed operator over the existing approaches

In this section, we have presented some counterexamples which show that the existing approaches under the IFSS environment fail to rank the given alternatives while the proposed approach can overcome their shortcoming.

Example 9.4.1. Consider the DM problem in which there are two alternatives denoted by $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, which are evaluated by the two judges u_1 and u_2 under the set of parameters denoted by e_1, e_2, e_3 with weight vector $(0.34, 0.33, 0.33)^T$. Assume that the weight vector of two judges are 0.54 & 0.46 and the preferences corresponding to each alternative as evaluated by these judges are represented in the form of IFSNs as follows.

$$(\mathcal{A}^{(1)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.6, 0.4) & (0.5, 0.3) & (0.7, 0.2) \\ (0.4, 0.5) & (0.7, 0.2) & (0.5, 0.4) \end{bmatrix} \end{array}$$

and

$$(\mathcal{A}^{(2)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.5, 0.2) & (0.6, 0.2) & (0.5, 0.4) \\ (0.3, 0.2) & (0.6, 0.2) & (0.5, 0.3) \end{bmatrix} \end{array}$$

Now, if we utilize an AO as proposed by Liu and Li [119] then the score values corresponding to alternative $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are: 0.2697 and 0.2697 respectively. Thus we get $\mathcal{A}^{(1)} = \mathcal{A}^{(2)}$. But $\mathcal{A}^{(1)} \neq \mathcal{A}^{(2)}$. Thus the existing operator is unable to distinguish between the alternatives. On the other hand, if we apply proposed AO to this problem, then their respective score values are $\mathcal{S}(\mathcal{A}^{(1)}) = -0.6916$ and $\mathcal{S}(\mathcal{A}^{(2)}) = -0.6804$. Thus, we get $\mathcal{A}^{(2)} \succ \mathcal{A}^{(1)}$ and hence $\mathcal{A}^{(2)}$ is preferred than $\mathcal{A}^{(1)}$.

Example 9.4.2. Consider two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are evaluated by considering set of parameters $\{e_1, e_2, e_3\}$ and two judges $\{u_1, u_2\}$ whose preference values are summarized under IFSS environment as follows:

$$(\mathcal{A}^{(1)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.5, 0.4) & (0.7, 0.1) & (0.6, 0.2) \\ (0.3, 0.6) & (0.8, 0.1) & (0.6, 0.3) \end{bmatrix} \end{array}$$

and

$$(\mathcal{A}^{(2)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.7, 0.1) & (0.6, 0.3) & (0.3, 0.4) \\ (0.5, 0.4) & (0.6, 0.1) & (0.8, 0.2) \end{bmatrix} \end{array}$$

If we apply distance measure $d(.)$ as proposed by Khalid and Abbas [98] under IFSS environment, then their measure values of the alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ from the ideal alternative $B = ((1, 0))_{2\times 3}$ are $d(\mathcal{A}^{(1)}, B) = 0.4167$ and $d(\mathcal{A}^{(2)}, B) = 0.4167$. Thus, it has been seen that by their approach it is unable to distinguish between them. On the other hand, if we apply the proposed aggregation IFSBM operator then the score values corresponding to the alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are 0.3058 & 0.3809. Hence we conclude that the alternative $\mathcal{A}^{(2)}$ is better than the alternative $\mathcal{A}^{(1)}$.

Example 9.4.3. Consider the DM problem in which there are two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, which are evaluated by two judges $\{u_1, u_2\}$ under the set of parameters $\{e_1, e_2, e_3\}$. The preference values corresponding to each alternative are summarized under IFSS environment as follows:

$$(\mathcal{A}^{(1)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.7, 0.1) & (0.8, 0.2) & (0.6, 0.2) \\ (0.3, 0.5) & (0.5, 0.4) & (0.7, 0.3) \end{bmatrix} \end{array}$$

and

$$(\mathcal{A}^{(2)}, \mathcal{E}) \quad = \quad \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{bmatrix} (0.8, 0.2) & (0.7, 0.3) & (0.5, 0.3) \\ (0.6, 0.2) & (0.3, 0.4) & (0.6, 0.2) \end{bmatrix} \end{array}$$

If we apply existing distance measure $d_1(.)$, as proposed by Cagman and Deli [21] under IFSS environment to find the measure values of alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ from the ideal value $B = ((1, 0))_{2\times 3}$ then we get: $d_1(\mathcal{A}^{(1)}, B) = 0.6833$ and $d_1(\mathcal{A}^{(2)}, B) = 0.6833$. As both have same distances so it is not possible to choose the best. Now, if we utilize the proposed operator on them then the relative score values of the alternatives are $\mathcal{S}(\mathcal{A}^{(1)}) = 0.3202$ and $\mathcal{S}(\mathcal{A}^{(2)}) = 0.3233$. Since the $S(\mathcal{A}^{(2)})$ is better than $S(\mathcal{A}^{(1)})$ and hence $\mathcal{A}^{(2)}$ is preferred over $\mathcal{A}^{(1)}$.

9.5 Conclusion

The intuitionistic fuzzy soft set is an important tool for dealing with the uncertainty and fuzziness by considering the parametrization factor during the DM process. The aim of this chapter is to present an AO named as Bonferroni Mean operator whose remarkable characteristic is their efficiency to capture the relationships between individual arguments. For this, we have presented two Bonferroni mean operators i.e., the IFSBM operator and the WIFSBM operator, to aggregate the different preferences of experts given over different parameters under IFSS environment. An approach for solving the DM problems has been presented by taking different values of parameters p and q, which makes the proposed operators more flexible and offers the various choices to the decision-maker for assessing the decisions. A comparative study with some existing operators shows that the proposed operators and their corresponding techniques provide a more stable, practical and optimistic nature to the decision-maker during the aggregation process. Also, the superiority of the proposed work over the existing methods is discussed by considering some examples.

Chapter 10

Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application[1]

In this chapter, we develop some new power AOs based on the Archimedean t-norm operators for IFSNs. The developed AOs are named as intuitionistic fuzzy soft (IFS) power averaging and geometric operators, weighted IFS power averaging and geometric operators, weighted and ordered weighted IFS power averaging and geometric operators. The salient features of these operators are also discussed in details. Further, these operators are extended to its generalized version and called generalized IFS power averaging or geometric aggregation operators (AOs). Finally, the applicability of these developed operators is explained with a numerical example related to DM problems. A comparative analysis of existing approaches has been done for showing the validity of the proposed work.

10.1 Introduction

The basic definition related to Archimedean t-norm and t-conorm are given in Section 2.4 of Chapter 2, while the literature related to AOs are defined in Section 1.1.4 of Chapter 1. The theories presented in these sections have the advantage of being capable of framing the

[1]The content of this chapter is published as "Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multi criteria decision-making", *International Journal of Intelligent Systems*, 34(2), pp. 215-246, 2019 (**SCI: Impact Factor 7.229**)

descriptions of the real-world situation, but these operators do not consider the supported aggregation. However, the power aggregation (PA) operator allows the input data to support each other during the aggregation process. Moreover, the PA operator can remove the impact of inconsistent data of decision-makers giving power weight to all individuals.

Considering the importance of the relationship among the attributes, prioritized AOs are one of the great significance of the decision problem. PA operator allows the input data to support each other during the aggregation process. In our real-life situation, there always exist a situation in which a relationship between the different criteria such as prioritization, support, and impact each other pays a dominant role during an aggregation process. For example, when a decision maker makes a judgment based on consideration of risk and costs in a project, he should assign a higher priority to risk than cost. For handling such types of situations and to incorporate into the DM analysis, Yager [238] proposed the power averaging (PA) operator which highlights the support of input values during the aggregation process. Wei [212] investigated the prioritized AO under hesitant fuzzy environment. Xu and Yager [233] proposed some geometric operations to introduce weighted and ordered weighted power geometric operators. Zhou et al. [265] developed a power averaging operator for measuring the performance of the different alternatives. Simultaneously, selecting the appropriate operational laws during the aggregation phase is crucial. For it, AT and AC and families, which is an important tool for some sets to generate general operational laws. They contain various ATs and ACs such as Algebraic, Einstein, Hamacher and so on, which is very useful to achieve the intersection and unions of the fuzzy evaluations given in Table 2.1 of Chapter 2.

Therefore, in this chapter, by keeping the advantages of PA operators and by taking the specific priority degrees of different attributes, we develop the power AOs under the intuitionistic fuzzy soft set environment. For it, different operators named as IF soft power averaging (IFSPA), weighted IFSPA (WIFSPA), ordered weighted IFSPA (OWIFSPA), IF soft power geometric (IFSPG), weighted IFSPG (WIFSPG), ordered weighted IFSPG (OWIFSPG), generalized weighted IFSPA (GWIFSPA), generalized ordered weighted IFSPA (GOWIFSPA), generalized weighted IFSPG(GWIFSPG) and generalized ordered weighted IFSPG (GOWIFSPG) operators are proposed. The salient properties of these operators have also been studied in details. Then, we developed a new approach to MCDM problems under the IFSS environment based on the proposed AOs. The feasibility, as well as the superiority of the proposed approach, has been illustrated with a numerical example.

10.2 Intuitionistic fuzzy soft Power aggregation operators

In the following, we shall develop some IF soft power averaging and geometric AOs based on the generalized t-norm operations of IFSNs and PA. For it, a strict AT and AC can be expressed in the form of continuous functions $p, q : [0,1] \to [0,1]$ respectively as $T(x,y) = q^{-1}(q(x)+q(y))$ and $S(x,y) = p^{-1}(p(x)+p(y))$ where q (or p) is a decreasing (or increasing) function with $q(1) = 0$; $p(0) = 0$ and $q(x) = p(1-x)$.

10.2.1 Operational laws for IFSSs

Definition 10.2.1. Let $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})\,; (j = 1,2)$ and $\mathcal{A} = (\zeta, \vartheta)$ be three IFSNs; then we have

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \left(p^{-1}\left(p(\zeta_{11}) + p(\zeta_{12})\right), q^{-1}\left(q(\vartheta_{11}) + q(\vartheta_{12})\right)\right)$.

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \left(q^{-1}\left(q(\zeta_{11}) + q(\zeta_{12})\right), p^{-1}\left(p(\vartheta_{11}) + p(\vartheta_{12})\right)\right)$.

(iii) $\lambda\mathcal{A} = \left(p^{-1}\left(\lambda p(\zeta)\right), q^{-1}\left(\lambda q(\vartheta)\right)\right)$, $\lambda > 0$.

(iv) $\mathcal{A}^{\lambda} = \left(q^{-1}\left(\lambda q(\zeta)\right), p^{-1}\left(\lambda p(\vartheta)\right)\right)$, $\lambda > 0$.

Theorem 10.2.1. If $\mathcal{A}_{1j} = (\zeta_{1j}, \vartheta_{1j})\,; (j = 1,2)$ and $\mathcal{A} = (\zeta, \vartheta)$ be IFSNs; then for $\lambda > 0; \mathcal{A}_{11} \oplus \mathcal{A}_{12}$, $\mathcal{A}_{11} \otimes \mathcal{A}_{12}$, $\mathcal{A}^{\lambda}$ and $\lambda\mathcal{A}$ are also IFSNs.

Proof. By using basic definition of p and q, we get $0 \leq q^{-1}\left(q\left(\zeta_{11}\right) + q\left(\zeta_{12}\right)\right) \leq 1$ and $0 \leq p^{-1}\left(p\left(\vartheta_{11}\right) + p\left(\vartheta_{12}\right)\right) \leq 1$. Therefore,

$$
\begin{aligned}
& q^{-1}\left(q\left(\zeta_{11}\right) + q\left(\zeta_{12}\right)\right) + p^{-1}\left(p\left(\vartheta_{11}\right) + p\left(\vartheta_{12}\right)\right) \\
& \leq q^{-1}\left(q\left(1-\vartheta_{11}\right) + q\left(1-\vartheta_{12}\right)\right) + p^{-1}\left(p\left(\vartheta_{11}\right) + p\left(\vartheta_{12}\right)\right) \\
& \leq 1 - p^{-1}\left(p\left(\vartheta_{11}\right) + p\left(\vartheta_{12}\right)\right) + p^{-1}\left(p\left(\vartheta_{11}\right) + p\left(\vartheta_{12}\right)\right) \\
& \leq 1
\end{aligned}
$$

Therefore, $\mathcal{A}_{11} \otimes \mathcal{A}_{12}$ is an IFSN. Similarly, we can prove $\mathcal{A}_{11} \oplus \mathcal{A}_{12}$, $\mathcal{A}^{\lambda}$ and $\lambda\mathcal{A}$ are IFSNs. $\square$

Theorem 10.2.2. Let $\mathcal{A}_{1j}; (j = 1, 2)$ and $\mathcal{A}$ be three IFSNs, then

(i) $\mathcal{A}_{11} \oplus \mathcal{A}_{12} = \mathcal{A}_{12} \oplus \mathcal{A}_{11}$.

(ii) $\mathcal{A}_{11} \otimes \mathcal{A}_{12} = \mathcal{A}_{12} \otimes \mathcal{A}_{11}$.

(iii) $\lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) = \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12}$, for $\lambda > 0$.

(iv) $(\mathcal{A}_{11} \otimes \mathcal{A}_{12})^{\lambda} = \mathcal{A}_{11}^{\lambda} \otimes \mathcal{A}_{12}^{\lambda}$, for $\lambda > 0$.

(v) $\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} = (\lambda_1 + \lambda_2)\mathcal{A}$, for $\lambda_1, \lambda_2 > 0$.

(vi) $\mathcal{A}^{\lambda_1} \otimes \mathcal{A}^{\lambda_2} = \mathcal{A}^{\lambda_1+\lambda_2}$, for $\lambda_1, \lambda_2 > 0$.

Proof. We will prove the parts (iii) and (v). Proofs of others can be proceeded likewise.

(iii) For any real number $\lambda > 0$,

$$\begin{aligned}
& \lambda(\mathcal{A}_{11} \oplus \mathcal{A}_{12}) \\
= \;& \lambda\left(p^{-1}\left(p\left(\zeta_{11}\right)+p\left(\zeta_{12}\right)\right), q^{-1}\left(q\left(\vartheta_{11}\right)+q\left(\vartheta_{12}\right)\right)\right) \\
= \;& \left(p^{-1}\left(\lambda p\left(p^{-1}\left(p\left(\zeta_{11}\right)+p\left(\zeta_{12}\right)\right)\right)\right), q^{-1}\left(\lambda q\left(q^{-1}\left(q\left(\vartheta_{11}\right)+q\left(\vartheta_{12}\right)\right)\right)\right)\right) \\
= \;& \left(p^{-1}\left(\lambda\left(p\left(\zeta_{11}\right)+p\left(\zeta_{12}\right)\right)\right), q^{-1}\left(\lambda\left(q\left(\vartheta_{11}\right)+q\left(\vartheta_{12}\right)\right)\right)\right) \\
= \;& \begin{pmatrix} p^{-1}\left(p\left(p^{-1}\left(\lambda p\left(\zeta_{11}\right)\right)\right)+p\left(p^{-1}\left(\lambda p\left(\zeta_{12}\right)\right)\right)\right), \\ q^{-1}\left(q\left(q^{-1}\left(\lambda q\left(\vartheta_{11}\right)\right)\right)+q\left(q^{-1}\left(\lambda q\left(\vartheta_{12}\right)\right)\right)\right) \end{pmatrix} \\
= \;& \left(p^{-1}\left(\lambda p\left(\zeta_{11}\right)\right), q^{-1}\left(\lambda q\left(\vartheta_{11}\right)\right)\right) \oplus \left(p^{-1}\left(\lambda p\left(\zeta_{12}\right)\right), q^{-1}\left(\lambda q\left(\vartheta_{12}\right)\right)\right) \\
= \;& \lambda\mathcal{A}_{11} \oplus \lambda\mathcal{A}_{12}
\end{aligned}$$

(v) For real number $\lambda_1, \lambda_2 > 0$,

$$\begin{aligned}
\lambda_1\mathcal{A} \oplus \lambda_2\mathcal{A} &= \left(p^{-1}\left(\lambda_1 p\left(\zeta\right)\right), q^{-1}\left(\lambda_1 q\left(\vartheta\right)\right)\right) \oplus \left(p^{-1}\left(\lambda_2 p\left(\zeta\right)\right), q^{-1}\left(\lambda_2 q\left(\vartheta\right)\right)\right) \\
&= \begin{pmatrix} p^{-1}\left(p\left(p^{-1}\left(\lambda_1 p\left(\zeta\right)\right)\right)+p\left(p^{-1}\left(\lambda_2 p\left(\zeta\right)\right)\right)\right), \\ q^{-1}\left(q\left(q^{-1}\left(\lambda_1 q\left(\vartheta\right)\right)\right)+q\left(q^{-1}\left(\lambda_2 q\left(\vartheta\right)\right)\right)\right) \end{pmatrix} \\
&= \left(p^{-1}\left(\lambda_1 p\left(\zeta\right)+\lambda_2 p\left(\zeta\right)\right), q^{-1}\left(\lambda_1 q\left(\vartheta\right)+\lambda_2 q\left(\vartheta\right)\right)\right) \\
&= (\lambda_1 + \lambda_2)\mathcal{A}
\end{aligned}$$

□

Theorem 10.2.3. For three IFSNs $\mathcal{A}, \mathcal{A}_{11}$ and $\mathcal{A}_{12}$, $\lambda > 0$ be real number, we have

(i) $(\mathcal{A}^c)^\lambda = (\lambda\mathcal{A})^c$.

(ii) $\lambda(\mathcal{A}^c) = (\mathcal{A}^\lambda)^c$.

(iii) $\mathcal{A}_{11}^c \oplus \mathcal{A}_{12}^c = (\mathcal{A}_{11} \otimes \mathcal{A}_{12})^c$.

(iv) $\mathcal{A}_{11}^c \otimes \mathcal{A}_{12}^c = (\mathcal{A}_{11} \oplus \mathcal{A}_{12})^c$.

where $\mathcal{A}^c = (\vartheta, \zeta)$ denotes the complement of IFSN $\mathcal{A} = (\zeta, \vartheta)$.

Proof. By using operations stated in Definition 10.2.1, we have

(i) $(\mathcal{A}^c)^\lambda = \left(q^{-1}\left(\lambda q\left(\vartheta\right)\right), p^{-1}\left(\lambda p\left(\zeta\right)\right)\right) = (\lambda\mathcal{A})^c$.

(ii) $\lambda(\mathcal{A}^c) = \left(p^{-1}\left(\lambda p\left(\vartheta\right)\right), q^{-1}\left(\lambda q\left(\zeta\right)\right)\right) = \left(\mathcal{A}^\lambda\right)^c$.

(iii) $\mathcal{A}_{11}^c \oplus \mathcal{A}_{12}^c = \left(p^{-1}\left(p\left(\zeta_{11}\right) + p\left(\zeta_{12}\right)\right), q^{-1}\left(q\left(\vartheta_{11}\right) + q\left(\vartheta_{12}\right)\right)\right) = (\mathcal{A}_{11} \otimes \mathcal{A}_{12})^c$.

(iv) $\mathcal{A}_{11}^c \otimes \mathcal{A}_{12}^c = \left(q^{-1}\left(q\left(\vartheta_{11}\right) + q\left(\vartheta_{12}\right)\right), p^{-1}\left(p\left(\zeta_{11}\right) + p\left(\zeta_{12}\right)\right)\right) = (\mathcal{A}_{11} \oplus \mathcal{A}_{12})^c$.

which completes the proof. □

Based on these operations, we have defined some new weighted averaging and geometric AOs as follows:

10.2.2 Intuitionistic fuzzy soft power averaging operator

Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, be a collection of IFSNs such that $\vartheta_{ij} \neq 0$ for all i, j, then we define the intuitionistic fuzzy soft power averaging (IFSPA) operator on IFSNs as follows.

Definition 10.2.2. For IFSNs $\mathcal{A}_{ij} \in \Omega$, IFSPA operator is a mapping from Ω^n to Ω and is defined as:

$$\text{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigoplus_{j=1}^{m} \left(\xi_j \bigoplus_{i=1}^{n} (\eta_i \mathcal{A}_{ij})\right) \tag{10.1}$$

where $\xi_j = \frac{(1+T_j)}{\sum\limits_{j=1}^{m}(1+T_j)}$, $\eta_i = \frac{(1+R_i)}{\sum\limits_{i=1}^{n}(1+R_i)}$ and $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n} \text{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m} \text{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ and $\text{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$ refers the support for $\mathcal{A}_{ij}$ from $\mathcal{A}_{kj}$.

Theorem 10.2.4. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the value obtained with IFSPA operator is still an IFSN and given by:

$$\begin{aligned} &\mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = &\left(p^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i p(\zeta_{ij})\right)\right), q^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i q(\vartheta_{ij})\right)\right)\right) \end{aligned} \tag{10.2}$$

Proof. To prove the result given in Eq. (10.2), we have applied principle of mathematical induction on n, m as follows.

Step 1: For $n = 1$, we have

$$\begin{aligned} &\mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1m}) \\ = &\bigoplus_{j=1}^{m} \xi_j \mathcal{A}_{1j} \\ = &\left(p^{-1}\left(\sum_{j=1}^{m} \xi_j p\left(\zeta_{1j}\right)\right), q^{-1}\left(\sum_{j=1}^{m} \xi_j q\left(\vartheta_{1j}\right)\right)\right) \\ = &\left(p^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{1} \eta_i p\left(\zeta_{ij}\right)\right)\right), q^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{1} \eta_i q\left(\vartheta_{ij}\right)\right)\right)\right) \end{aligned}$$

Similarly, for $m = 1$, we have

$$\begin{aligned} &\mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{n1}) = \bigoplus_{i=1}^{n} \eta_i \mathcal{A}_{i1} \\ = &\left(p^{-1}\left(\sum_{j=1}^{1} \xi_j \left(\sum_{i=1}^{n} \eta_i p\left(\zeta_{ij}\right)\right)\right), q^{-1}\left(\sum_{j=1}^{1} \xi_j \left(\sum_{i=1}^{n} \eta_i q\left(\vartheta_{ij}\right)\right)\right)\right) \end{aligned}$$

Thus the result holds for $n = 1$ and $m = 1$.

Step 2: Assume the result is true for $m = k_1, n = k_2 + 1$ and $m = k_1 + 1, n = k_2$ then for $m = k_1 + 1, n = k_2 + 1$, we have

$$\begin{aligned} &\bigoplus_{j=1}^{k_1+1} \left(\xi_j \bigoplus_{i=1}^{k_2+1} (\eta_i \mathcal{A}_{ij})\right) \\ = &\bigoplus_{j=1}^{k_1} \left(\xi_j \bigoplus_{i=1}^{k_2+1} (\eta_i \mathcal{A}_{ij})\right) \oplus \xi_{k_1+1} \left(\bigoplus_{i=1}^{k_2+1} (\eta_i \mathcal{A}_{i(k_1+1)})\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(p^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\zeta_{ij}\right)\right)\right), q^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\vartheta_{ij}\right)\right)\right)\right) \\
&\oplus \left(p^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\zeta_{i(k_1+1)}\right)\right)\right), q^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\vartheta_{i(k_1+1)}\right)\right)\right)\right) \\
&= \left(p^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\zeta_{ij}\right)\right) + \xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\zeta_{i(k_1+1)}\right)\right)\right),\right. \\
&\qquad \left. q^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\vartheta_{ij}\right)\right) + \xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\vartheta_{i(k_1+1)}\right)\right)\right)\right) \\
&= \left(p^{-1}\left(\sum_{j=1}^{k_1+1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\zeta_{ij}\right)\right)\right), q^{-1}\left(\sum_{j=1}^{k_1+1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\vartheta_{ij}\right)\right)\right)\right)
\end{aligned}
$$

Thus, the result holds for $m = k_1 + 1, n = k_2 + 1$. Hence, by mathematical induction, the result is true for all positive integers m, n. $\square$

It can be easily proved that the IFSPA operator has the following properties:

Property 10.2.1. (Idempotency:) If $\mathcal{A}_{ij} = \mathcal{A} = (\zeta, \vartheta)$ for all i, j then

$$\text{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}.$$

Proof. If $\mathcal{A}_{ij} = \mathcal{A}$, for all i, j, then by using Eq. (10.2), we have

$$
\begin{aligned}
& \text{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
&= \left(p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p(\zeta)\right)\right), q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q(\vartheta)\right)\right)\right) \\
&= \left(p^{-1}\left(p(\zeta)\right), q^{-1}\left(q(\vartheta)\right)\right) \\
&= (\zeta, \vartheta) \\
&= \mathcal{A}
\end{aligned}
$$

$\square$

Property 10.2.2. If $\mathcal{A}_{ij}$ and $\mathcal{A}$ are IFSNs, then

$$\text{IFSPA}(\mathcal{A}_{11} \oplus \mathcal{A}, \mathcal{A}_{12} \oplus \mathcal{A}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{A}) = \text{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathcal{A}$$

Proof. As $\mathcal{A}_{ij}$ and $\mathcal{A}$ are the IFSNs, then for all i, j, we have

$$\mathcal{A}_{ij} \oplus \mathcal{A} = \left(p^{-1}\left(p\left(\zeta_{ij}\right) + p\left(\zeta_{\mathcal{A}}\right)\right), q^{-1}\left(q\left(\zeta_{ij}\right) + q\left(\zeta_{\mathcal{A}}\right)\right)\right)$$

Therefore,

$$\begin{aligned}
& \mathrm{IFSPA}(\mathcal{A}_{11} \oplus \mathcal{A}, \mathcal{A}_{12} \oplus \mathcal{A}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{A}) \\
= & \left(\begin{array}{l} p^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i p\left(p^{-1}\left(p\left(\zeta_{ij}\right) + p\left(\zeta_{\mathcal{A}}\right)\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i q\left(q^{-1}\left(q\left(\vartheta_{ij}\right) + q\left(\vartheta_{\mathcal{A}}\right)\right)\right)\right)\right) \end{array}\right) \\
= & \left(\begin{array}{l} p^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(p\left(\zeta_{ij}\right) + p\left(\zeta_{\mathcal{A}}\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(q\left(\vartheta_{ij}\right) + q\left(\vartheta_{\mathcal{A}}\right)\right)\right)\right) \end{array}\right) \\
= & \left(\begin{array}{l} p^{-1}\left(p\left(p^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(p\left(\zeta_{ij}\right)\right)\right)\right) + p\left(\zeta_{\mathcal{A}}\right)\right)\right), \\ q^{-1}\left(q\left(q^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(q\left(\vartheta_{ij}\right)\right)\right)\right) + q\left(\vartheta_{\mathcal{A}}\right)\right)\right) \end{array}\right) \\
= & \left(\begin{array}{l} p^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(p\left(\zeta_{ij}\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i \left(q\left(\vartheta_{ij}\right)\right)\right)\right) \end{array}\right) \oplus \left(\zeta_{\mathcal{A}}, \vartheta_{\mathcal{A}}\right) \\
= & \ \mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathcal{A}
\end{aligned}$$

□

Property 10.2.3. For a collection of IFSNs $\mathcal{A}_{ij}$ and real number $\lambda > 0$,

$$\mathrm{IFSPA}(\lambda\mathcal{A}_{11}, \lambda\mathcal{A}_{12}, \ldots, \lambda\mathcal{A}_{nm}) = \lambda\ \mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$$

Proof. If $\mathcal{A}_{ij} = \left(\zeta_{ij}, \vartheta_{ij}\right); i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$ are the IFSNs then for any real number $\lambda > 0$, we have $\lambda\mathcal{A}_{ij} = \left(p^{-1}\left(p\left(\lambda\zeta_{ij}\right)\right), q^{-1}\left(q\left(\lambda\vartheta_{ij}\right)\right)\right)$. Therefore,

$$
\begin{aligned}
&\mathrm{IFSPA}(\lambda\mathcal{A}_{11}, \lambda\mathcal{A}_{12}, \ldots, \lambda\mathcal{A}_{nm}) \\
= &\left(\begin{array}{l} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(p\left(p^{-1}\left(\lambda p\left(\zeta_{ij}\right)\right)\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(q\left(q^{-1}\left(\lambda q\left(\vartheta_{ij}\right)\right)\right)\right)\right)\right) \end{array}\right) \\
= &\left(\begin{array}{l} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(\lambda p\left(\zeta_{ij}\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(\lambda q\left(\vartheta_{ij}\right)\right)\right)\right) \end{array}\right) \\
= &\left(\begin{array}{l} p^{-1}\left(\lambda p\left(p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p\left(\zeta_{ij}\right)\right)\right)\right)\right), \\ q^{-1}\left(\lambda q\left(q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q\left(\vartheta_{ij}\right)\right)\right)\right)\right) \end{array}\right) \\
= &\ \lambda\ \mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})
\end{aligned}
$$

□

Property 10.2.4. If $\mathcal{A}_{ij}, \mathcal{B}_{ij}$ for all i, j are IFSNs, then

$$
\begin{aligned}
&\mathrm{IFSPA}(\mathcal{A}_{11} \oplus \mathcal{B}_{11}, \mathcal{A}_{12} \oplus \mathcal{B}_{12}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{B}_{nm}) \\
= &\ \mathrm{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathrm{IFSPA}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm})
\end{aligned}
$$

Proof. Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$ and $\mathcal{B}_{ij} = (\theta_{ij}, \varphi_{ij})$ be two IFSNs for all i, j. Then, we have $\mathcal{A}_{ij} \oplus \mathcal{B}_{ij} = \left(p^{-1}\left(p\left(\zeta_{ij}\right) + p\left(\theta_{ij}\right)\right), q^{-1}\left(q\left(\vartheta_{ij}\right) + q\left(\varphi_{ij}\right)\right)\right)$. Therefore,

$$
\begin{aligned}
&\mathrm{IFSPA}(\mathcal{A}_{11} \oplus \mathcal{A}_{11}, \mathcal{A}_{12} \oplus \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{A}_{nm}) \\
= &\left(\begin{array}{l} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p\left(p^{-1}\left(p\left(\zeta_{ij}\right) + p\left(\theta_{ij}\right)\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q\left(q^{-1}\left(q\left(\vartheta_{ij}\right) + q\left(\varphi_{ij}\right)\right)\right)\right)\right) \end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(p\left(\zeta_{ij}\right)+p\left(\theta_{ij}\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(q\left(\vartheta_{ij}\right)+q\left(\varphi_{ij}\right)\right)\right)\right) \end{pmatrix} \\
&= \begin{pmatrix} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(p\left(\zeta_{ij}\right)+p\left(\theta_{ij}\right)\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i\left(q\left(\vartheta_{ij}\right)+q\left(\varphi_{ij}\right)\right)\right)\right) \end{pmatrix} \\
&= \begin{pmatrix} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p\left(\zeta_{ij}\right)\right)+\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p\left(\theta_{ij}\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q\left(\vartheta_{ij}\right)\right)+\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q\left(\varphi_{ij}\right)\right)\right) \end{pmatrix} \\
&= \begin{pmatrix} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p\left(\zeta_{ij}\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q\left(\vartheta_{ij}\right)\right)\right) \end{pmatrix} \oplus \begin{pmatrix} p^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i p\left(\theta_{ij}\right)\right)\right), \\ q^{-1}\left(\sum_{j=1}^{m}\xi_j\left(\sum_{i=1}^{n}\eta_i q\left(\varphi_{ij}\right)\right)\right) \end{pmatrix} \\
&= \text{IFSPA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) \oplus \text{IFSPA}(\mathcal{B}_{11},\mathcal{B}_{12},\ldots,\mathcal{B}_{nm})
\end{aligned}
$$

□

10.2.3 Special cases of IFSPA operator

By assigning various values to the generator q, the proposed IFSPA operator is particularized as follows:

(i) If $q(t) = -\log(t)$ is to be taken, then Eq. (10.2) reduces to intuitionistic fuzzy soft Archimedean weighted averaging operator [6].

$$
\text{IFSPA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) = \left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\zeta_{ij})^{\eta_i}\right)^{\xi_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\vartheta_{ij}^{\eta_i}\right)^{\xi_j}\right)
$$

(ii) If $q(t) = \log\left(\dfrac{2-t}{t}\right)$, then Eq. (10.2) reduces to intuitionistic fuzzy soft Einstein weighted averaging operator.

$$\text{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$$
$$= \left(\frac{\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1+\zeta_{ij})^{\eta_i}\right)^{\xi_j} - \prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1-\zeta_{ij})^{\eta_i}\right)^{\xi_j}}{\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1+\zeta_{ij})^{\eta_i}\right)^{\xi_j} + \prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1-\zeta_{ij})^{\eta_i}\right)^{\xi_j}}, \frac{2\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\vartheta_{ij}^{\eta_i}\right)^{\xi_j}}{\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(2-\vartheta_{ij})^{\eta_i}\right)^{\xi_j} + \prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\vartheta_{ij}^{\eta_i}\right)^{\xi_j}} \right)$$

(iii) If $q(t) = \log\left(\frac{\delta+(1-\delta)t}{t}\right)$, $\delta \in (0,\infty)$, then Eq. (10.2) reduces to intuitionistic fuzzy soft Hammer weighted averaging operator.

$$\text{IFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$$
$$= \left(\frac{\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1+(\delta-1)\zeta_{ij})^{\eta_i}\right)^{\xi_j} - \prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1-\zeta_{ij})^{\eta_i}\right)^{\xi_j}}{\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1+(\delta-1)\zeta_{ij})^{\eta_i}\right)^{\xi_j} + (\delta-1)\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1-\zeta_{ij})^{\eta_i}\right)^{\xi_j}}, \frac{\delta\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\vartheta_{ij}^{\eta_i}\right)^{\xi_j}}{\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}(1+(\delta-1)(1-\vartheta_{ij}))^{\eta_i}\right)^{\xi_j} + (\delta-1)\prod\limits_{j=1}^{m}\left(\prod\limits_{i=1}^{n}\vartheta_{ij}^{\eta_i}\right)^{\xi_j}} \right)$$

10.2.4 Weighted and ordered weighted IFSPA operator

Definition 10.2.3. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, WIFSPA operator is a mapping from Ω^n to Ω and is defined as:

$$\text{WIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigoplus_{j=1}^{m}\left(\phi_j \bigoplus_{i=1}^{n}(\psi_i \mathcal{A}_{ij})\right) \tag{10.3}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n}\kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m}\rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n}\text{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m}\text{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ and $\rho_j, \kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m}\rho_j = 1$ and $\sum\limits_{i=1}^{n}\kappa_i = 1$. Then, WIFSPA is called weighted intuitionistic fuzzy soft power averaging operator.

Theorem 10.2.5. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the value obtained with

WIFSPA operator is still an IFSN and given by:

$$\begin{aligned}&\text{WIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})\\ =\ &\left(p^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i p(\zeta_{ij})\right)\right), q^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i q(\vartheta_{ij})\right)\right)\right)\end{aligned} \tag{10.4}$$

Proof. Similar to that of Theorem 10.2.4. □

Definition 10.2.4. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, OWIFSPA operator is a mapping from Ω^n to Ω and is defined as:

$$\text{OWIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigoplus_{j=1}^{m}\left(\phi_j \bigoplus_{i=1}^{n}\left(\psi_i \mathcal{A}_{\sigma(i)\varsigma(j)}\right)\right) \tag{10.5}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n}\kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m}\rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n}\text{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m}\text{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ and $\rho_j, \kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m}\rho_j = 1$ and $\sum\limits_{i=1}^{n}\kappa_i = 1$ and σ, ς are permutations of $(1, 2, \ldots, n)$ and $(1, 2, \ldots, m)$ with condition $\mathcal{A}_{\sigma(i)j} \geq \mathcal{A}_{\sigma(i-1)j}$ and $\mathcal{A}_{i\varsigma(j)} \geq \mathcal{A}_{i\varsigma(j-1)}$ for any i, j. Then, OWIFSPA is called ordered weighted intuitionistic fuzzy soft power averaging operator.

Theorem 10.2.6. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the value obtained with OWIFSPA operator is still an IFSN and given by:

$$\begin{aligned}&\text{OWIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})\\ =\ &\left(p^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i p\left(\zeta_{\sigma(i)\varsigma(j)}\right)\right)\right), q^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i q\left(\vartheta_{\sigma(i)\varsigma(j)}\right)\right)\right)\right)\end{aligned} \tag{10.6}$$

Proof. Similar as Theorem 10.2.4, so we omit here. □

Clearly, the proposed WIFSPA and OWIFSPA operators also possess the properties defined in Properties 10.2.1- 10.2.4.

10.2.5 Intuitionistic fuzzy soft power geometric operator

Let $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, be a collection of IFSNs such that $\zeta_{ij} \neq 0$ for all i, j, then we define the intuitionistic fuzzy soft power geometric (IFSPG) operator on IFSNs as follows.

Definition 10.2.5. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, IFSPG operator is a mapping from Ω^n to Ω and is defined as:

$$\text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigotimes_{j=1}^{m} \left(\bigotimes_{i=1}^{n} (\mathcal{A}_{ij})^{\eta_i} \right)^{\xi_j} \tag{10.7}$$

where $\xi_j = \frac{(1+T_j)}{\sum\limits_{j=1}^{m}(1+T_j)}$, $\eta_i = \frac{(1+R_i)}{\sum\limits_{i=1}^{n}(1+R_i)}$ and $R_i = \sum\limits_{\substack{k=1 \\ k\neq i}}^{n} \text{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1 \\ l\neq j}}^{m} \text{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ and $\text{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$ means the support for $\mathcal{A}_{ij}$ from $\mathcal{A}_{kj}$ and $\text{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ is the support of $\mathcal{A}_j$ from $\mathcal{A}_l$.

Theorem 10.2.7. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the value obtained with IFSPG operator is still an IFSN and given by:

$$\begin{aligned} & \text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = & \left(q^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i q(\zeta_{ij}) \right) \right), p^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{n} \eta_i p(\vartheta_{ij}) \right) \right) \right) \end{aligned} \tag{10.8}$$

Proof. To prove the result given in Eq. (10.8), we have applied mathematical induction on n, m and the following are the steps to be followed.

(Step 1:) For $n = 1$, we have

$$\begin{aligned} & \text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{1m}) \\ = & \bigotimes_{j=1}^{m} (\mathcal{A}_{1j})^{\xi_j} \\ = & \left(q^{-1} \left(\sum_{j=1}^{m} \xi_j q(\zeta_{1j}) \right), p^{-1} \left(\sum_{j=1}^{m} \xi_j p(\vartheta_{1j}) \right) \right) \\ = & \left(q^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{1} \eta_i q(\zeta_{ij}) \right) \right), p^{-1} \left(\sum_{j=1}^{m} \xi_j \left(\sum_{i=1}^{1} \eta_i p(\vartheta_{ij}) \right) \right) \right) \end{aligned}$$

Similarly, for $m = 1$, we have

$$\begin{aligned} & \text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{21}, \ldots, \mathcal{A}_{n1}) \\ = & \bigotimes_{i=1}^{n} (\mathcal{A}_{i1})^{\eta_i} \\ = & \left(q^{-1} \left(\sum_{j=1}^{1} \xi_j \left(\sum_{i=1}^{n} \eta_i q(\zeta_{ij}) \right) \right), p^{-1} \left(\sum_{j=1}^{1} \xi_j \left(\sum_{i=1}^{n} \eta_i p(\vartheta_{ij}) \right) \right) \right) \end{aligned}$$

Thus the result holds for $n = 1$ and $m = 1$.

(Step 2:) Assume the result is true for $m = k_1, n = k_2 + 1$ and $m = k_1 + 1, n = k_2$ then for $m = k_1 + 1, n = k_2 + 1$, we have

$$\begin{aligned}
&\bigotimes_{j=1}^{k_1+1}\left(\bigotimes_{i=1}^{k_2+1}(\mathcal{A}_{ij})^{\eta_i}\right)^{\xi_j}\\
&= \bigotimes_{j=1}^{k_1}\left(\bigotimes_{i=1}^{k_2+1}(\mathcal{A}_{ij})^{\eta_i}\right)^{\xi_j} \otimes \left(\bigotimes_{i=1}^{k_2+1}\left(\mathcal{A}_{i(k_1+1)}\right)^{\eta_i}\right)^{\xi_{k_1+1}}\\
&= \begin{pmatrix} q^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\zeta_{ij}\right)\right)\right), \\ p^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\vartheta_{ij}\right)\right)\right)\end{pmatrix} \otimes \begin{pmatrix} q^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\zeta_{i(k_1+1)}\right)\right)\right), \\ p^{-1}\left(\xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\vartheta_{i(k_1+1)}\right)\right)\right)\end{pmatrix}\\
&= \begin{pmatrix} q^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\zeta_{ij}\right)\right) + \xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\zeta_{i(k_1+1)}\right)\right)\right), \\ p^{-1}\left(\sum_{j=1}^{k_1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\vartheta_{ij}\right)\right) + \xi_{k_1+1}\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\vartheta_{i(k_1+1)}\right)\right)\right)\end{pmatrix}\\
&= \left(q^{-1}\left(\sum_{j=1}^{k_1+1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i q\left(\zeta_{ij}\right)\right)\right), p^{-1}\left(\sum_{j=1}^{k_1+1}\xi_j\left(\sum_{i=1}^{k_2+1}\eta_i p\left(\vartheta_{ij}\right)\right)\right)\right)
\end{aligned}$$

Thus the result holds for all positive integers m, n. □

Based on the Theorem 10.2.7, it has been seen that the following properties are satisfied by proposed IFSPG operator.

Property 10.2.5. If $\mathcal{A}_{ij}, \mathcal{B}_{ij}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$ and $\mathcal{A}$ are the IFSNs then for any real number $\lambda > 0$, we have

(P1) If all $\mathcal{A}_{ij}$ are identical, i.e., $\mathcal{A}_{ij} = \mathcal{A}$ for all i, j, then $\text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \mathcal{A}$.

(P2) $\text{IFSPG}(\mathcal{A}_{11} \oplus \mathcal{A}, \mathcal{A}_{12} \oplus \mathcal{A}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{A}) = \text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \mathcal{A}$

(P3) $\text{IFSPG}(\lambda\mathcal{A}_{11}, \lambda\mathcal{A}_{12}, \ldots, \lambda\mathcal{A}_{nm}) = \lambda\ \text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm})$

(P4) $\text{IFSPG}(\mathcal{A}_{11} \oplus \mathcal{B}_{11}, \mathcal{A}_{12} \oplus \mathcal{B}_{12}, \ldots, \mathcal{A}_{nm} \oplus \mathcal{B}_{nm})$

$= \text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \oplus \text{IFSPG}(\mathcal{B}_{11}, \mathcal{B}_{12}, \ldots, \mathcal{B}_{nm})$

Further, by assigning the various values of generator q, the proposed IFSPG operator can be particularized as follows.

(i) If we take $q(t) = -\log(t)$, then Eq. (10.8) reduces to intuitionistic fuzzy soft Archimedean weighted geometric operator [6].

$$\text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \zeta_{ij}^{\eta_i} \right)^{\xi_j}, 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \vartheta_{ij})^{\eta_i} \right)^{\xi_j} \right)$$

(ii) If we take $q(t) = \log\left(\dfrac{2-t}{t}\right)$, then Eq. (10.8) becomes intuitionistic fuzzy soft Einstein weighted geometric operator.

$$\begin{aligned} &\text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = &\left(\frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\zeta_{ij}^{\eta_i}\right)^{\xi_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\zeta_{ij})^{\eta_i}\right)^{\xi_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\zeta_{ij}^{\eta_i}\right)^{\xi_j}}, \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\vartheta_{ij})^{\eta_i}\right)^{\xi_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\vartheta_{ij})^{\eta_i}\right)^{\xi_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\vartheta_{ij})^{\eta_i}\right)^{\xi_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\vartheta_{ij})^{\eta_i}\right)^{\xi_j}} \right) \end{aligned}$$

(iii) If $q(t) = \log\left(\dfrac{\delta + (1-\delta)t}{t}\right)$, $\delta \in (0, \infty)$, then Eq. (10.8) reduces to intuitionistic fuzzy soft Hammer weighted geometric operator.

$$\begin{aligned} &\text{IFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ = &\left(\begin{array}{c} \dfrac{\delta \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\zeta_{ij}^{\eta_i}\right)^{\xi_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+(\delta-1)(1-\zeta_{ij}))^{\eta_i}\right)^{\xi_j} + (\delta-1)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\zeta_{ij}^{\eta_i}\right)^{\xi_j}}, \\ \dfrac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+(\delta-1)\vartheta_{ij})^{\eta_i}\right)^{\xi_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\vartheta_{ij})^{\eta_i}\right)^{\xi_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+(\delta-1)\vartheta_{ij})^{\eta_i}\right)^{\xi_j} + (\delta-1)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\vartheta_{ij})^{\eta_i}\right)^{\xi_j}} \end{array} \right) \end{aligned}$$

10.2.6 Weighted and ordered weighted IFSPG operator

Definition 10.2.6. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, WIFSPG operator is a mapping from Ω^n to Ω and is defined as:

$$\text{WIFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \bigotimes_{j=1}^{m} \left(\bigotimes_{i=1}^{n} (\mathcal{A}_{ij})^{\psi_i} \right)^{\phi_j} \tag{10.9}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n}\kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m}\rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n}\mathrm{Sup}(\mathcal{A}_{ij},\mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m}\mathrm{Sup}(\mathcal{A}_j,\mathcal{A}_l)$ and $\rho_j,\kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m}\rho_j = 1$ and $\sum\limits_{i=1}^{n}\kappa_i = 1$. Then, WIFSPG is called weighted intuitionistic fuzzy soft power geometric operator.

Theorem 10.2.8. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij},\vartheta_{ij})$, the value obtained with WIFSPG operator is still an IFSN and given by:

$$\begin{aligned}&\mathrm{WIFSPG}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\ =\ &\left(q^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i q(\zeta_{ij})\right)\right), p^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i p(\vartheta_{ij})\right)\right)\right)\end{aligned} \tag{10.10}$$

Proof. Similar to Theorem 10.2.7, so we omit here. □

Definition 10.2.7. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, OWIFSPG operator is a mapping from Ω^n to Ω and is defined as:

$$\mathrm{OWIFSPG}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) = \bigotimes_{j=1}^{m}\left(\bigotimes_{i=1}^{n}\left(\mathcal{A}_{\sigma(i)\varsigma(j)}\right)^{\psi_i}\right)^{\phi_j} \tag{10.11}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n}\kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m}\rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n}\mathrm{Sup}(\mathcal{A}_{ij},\mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m}\mathrm{Sup}(\mathcal{A}_j,\mathcal{A}_l)$ and $\rho_j,\kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m}\rho_j = 1$ and $\sum\limits_{i=1}^{n}\kappa_i = 1$ and σ,ς are permutations of $(1,2,\ldots,n)$ and $(1,2,\ldots,m)$ with condition $\mathcal{A}_{\sigma(i)j} \geq \mathcal{A}_{\sigma(i-1)j}$ and $\mathcal{A}_{i\varsigma(j)} \geq \mathcal{A}_{i\varsigma(j-1)}$ for any i,j. Then, OWIFSPG is called ordered weighted intuitionistic fuzzy soft power geometric operator.

Theorem 10.2.9. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij},\vartheta_{ij})$, the value obtained with OWIFSPG operator is still an IFSN and given by:

$$\begin{aligned}&\mathrm{OWIFSPG}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\ =\ &\left(q^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i q\left(\zeta_{\sigma(i)\varsigma(j)}\right)\right)\right), p^{-1}\left(\sum_{j=1}^{m}\phi_j\left(\sum_{i=1}^{n}\psi_i p\left(\vartheta_{\sigma(i)\varsigma(j)}\right)\right)\right)\right)\end{aligned} \tag{10.12}$$

Proof. Similar to Theorem 10.2.7, so we omit here. □

Clearly, the proposed WIFSPG and OWIFSPG operators also possess the properties defined in Properties 10.2.1- 10.2.4.

10.3 Generalized weighted intuitionistic fuzzy soft power aggregation operators

In this section, some generalized averaging and geometric power AOs namely, GWIFSPA, GOWIFSPA, GWIFSPG and GOWIFSPG operators are presented to aggregate the collection of IFSNs Ω.

Definition 10.3.1. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, GWIFSPA operator is a mapping from Ω^n to Ω and is defined as:

$$\mathrm{GWIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\bigoplus_{j=1}^{m} \left(\phi_j \bigoplus_{i=1}^{n} \left(\psi_i \mathcal{A}_{ij}^{\lambda} \right) \right) \right)^{\frac{1}{\lambda}} \tag{10.13}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n} \kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m} \rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1 \\ k\neq i}}^{n} \mathrm{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1 \\ l\neq j}}^{m} \mathrm{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ and $\rho_j, \kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m} \rho_j = 1$ and $\sum\limits_{i=1}^{n} \kappa_i = 1$.

Remark 10.3.1. In particular, if we take $\lambda = 1$, then the GWIFSPA operator transformed to WIFSPA operator.

Theorem 10.3.1. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the value obtained with GWIFSPA operator is still an IFSN and given by:

$$\begin{aligned} &\mathrm{GWIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\ &= \left(\begin{array}{l} q^{-1}\left(\frac{1}{\lambda} k \left(p^{-1} \left(\sum\limits_{j=1}^{m} \phi_j \left(\sum\limits_{i=1}^{n} \psi_i p\left(q^{-1}\left(\lambda q\left(\zeta_{ij}\right)\right)\right) \right) \right) \right) \right), \\ p^{-1}\left(\frac{1}{\lambda} l \left(q^{-1} \left(\sum\limits_{j=1}^{m} \phi_j \left(\sum\limits_{i=1}^{n} \psi_i q\left(p^{-1}\left(\lambda p\left(\vartheta_{ij}\right)\right)\right) \right) \right) \right) \right) \end{array} \right) \end{aligned} \tag{10.14}$$

Definition 10.3.2. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, GOWIFSPA operator is a mapping from Ω^n to Ω and is defined as:

$$\mathrm{GOWIFSPA}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \left(\bigoplus_{j=1}^{m} \left(\phi_j \bigoplus_{i=1}^{n} \left(\psi_i \mathcal{A}_{\sigma(i)\varsigma(j)}^{\lambda} \right) \right) \right)^{\frac{1}{\lambda}} \tag{10.15}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n}\kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m}\rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n}\mathrm{Sup}(\mathcal{A}_{ij},\mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m}\mathrm{Sup}(\mathcal{A}_j,\mathcal{A}_l)$ and $\rho_j,\kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m}\rho_j = 1$ and $\sum\limits_{i=1}^{n}\kappa_i = 1$ and σ,ς are permutations of $(1,2,\ldots,n)$ and $(1,2,\ldots,m)$ with condition $\mathcal{A}_{\sigma(i)j} \geq \mathcal{A}_{\sigma(i-1)j}$ and $\mathcal{A}_{i\varsigma(j)} \geq \mathcal{A}_{i\varsigma(j-1)}$ for any i,j.

Remark 10.3.2. In particular, if we take $\lambda = 1$, then the GOWIFSPA operator transformed to OWIFSPA operator.

Theorem 10.3.2. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij},\vartheta_{ij})$, the value obtained with GOWIFSPA operator is still an IFSN and given by:

$$\begin{aligned}&\mathrm{GOWIFSPA}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm})\\ =&\left(\begin{array}{c} q^{-1}\left(\frac{1}{\lambda}k\left(p^{-1}\left(\sum\limits_{j=1}^{m}\phi_j\left(\sum\limits_{i=1}^{n}\psi_i p\left(q^{-1}\left(\lambda q\left(\zeta_{\sigma(i)\varsigma(j)}\right)\right)\right)\right)\right)\right)\right), \\ p^{-1}\left(\frac{1}{\lambda}l\left(q^{-1}\left(\sum\limits_{j=1}^{m}\phi_j\left(\sum\limits_{i=1}^{n}\psi_i q\left(p^{-1}\left(\lambda p\left(\vartheta_{\sigma(i)\varsigma(j)}\right)\right)\right)\right)\right)\right)\right)\end{array}\right)\end{aligned} \tag{10.16}$$

Definition 10.3.3. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, GWIFSPG operator is a mapping from Ω^n to Ω and is defined as:

$$\mathrm{GWIFSPG}(\mathcal{A}_{11},\mathcal{A}_{12},\ldots,\mathcal{A}_{nm}) = \frac{1}{\lambda}\left(\bigotimes_{j=1}^{m}\left(\bigotimes_{i=1}^{n}(\lambda\alpha_{ij})^{\psi_i}\right)^{\phi_j}\right) \tag{10.17}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum\limits_{i=1}^{n}\kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum\limits_{j=1}^{m}\rho_j(1+T_j)}$, $R_i = \sum\limits_{\substack{k=1\\k\neq i}}^{n}\mathrm{Sup}(\mathcal{A}_{ij},\mathcal{A}_{kj})$, $T_j = \sum\limits_{\substack{l=1\\l\neq j}}^{m}\mathrm{Sup}(\mathcal{A}_j,\mathcal{A}_l)$ and $\rho_j,\kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum\limits_{j=1}^{m}\rho_j = 1$ and $\sum\limits_{i=1}^{n}\kappa_i = 1$.

Remark 10.3.3. In particular, if we take $\lambda = 1$, then the GWIFSPG operator transformed to WIFSPG operator.

Theorem 10.3.3. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij},\vartheta_{ij})$, the value obtained with

GWIFSPG operator is still an IFSN and given by:

$$\begin{aligned}
&\mathrm{GWIFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
&= \left(\begin{array}{l} p^{-1}\left(\frac{1}{\lambda} l \left(q^{-1} \left(\sum_{j=1}^{m} \phi_j \left(\sum_{i=1}^{n} \psi_i q \left(p^{-1} \left(\lambda p \left(\zeta_{ij} \right) \right) \right) \right) \right) \right) \right), \\ q^{-1}\left(\frac{1}{\lambda} k \left(p^{-1} \left(\sum_{j=1}^{m} \phi_j \left(\sum_{i=1}^{n} \psi_i p \left(q^{-1} \left(\lambda q \left(\vartheta_{ij} \right) \right) \right) \right) \right) \right) \right) \end{array} \right)
\end{aligned} \tag{10.18}$$

Definition 10.3.4. For the IFSNs $\mathcal{A}_{ij} \in \Omega$, GOWIFSPG operator is a mapping from Ω^n to Ω and is defined as:

$$\mathrm{GOWIFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) = \frac{1}{\lambda} \left(\bigotimes_{j=1}^{m} \left(\bigotimes_{i=1}^{n} \left(\lambda \mathcal{A}_{\sigma(i)\varsigma(j)} \right)^{\psi_i} \right)^{\phi_j} \right) \tag{10.19}$$

where $\psi_i = \frac{\kappa_i(1+R_i)}{\sum_{i=1}^{n} \kappa_i(1+R_i)}$, $\phi_j = \frac{\rho_j(1+T_j)}{\sum_{j=1}^{m} \rho_j(1+T_j)}$, $R_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathrm{Sup}(\mathcal{A}_{ij}, \mathcal{A}_{kj})$, $T_j = \sum_{\substack{l=1 \\ l \neq j}}^{m} \mathrm{Sup}(\mathcal{A}_j, \mathcal{A}_l)$ and $\rho_j, \kappa_i > 0$ are the weight vectors corresponding to the parameters and the experts respectively with condition $\sum_{j=1}^{m} \rho_j = 1$ and $\sum_{i=1}^{n} \kappa_i = 1$ and σ, ς are permutations of $(1, 2, \ldots, n)$ and $(1, 2, \ldots, m)$ with condition $\mathcal{A}_{\sigma(i)j} \geq \mathcal{A}_{\sigma(i-1)j}$ and $\mathcal{A}_{i\varsigma(j)} \geq \mathcal{A}_{i\varsigma(j-1)}$ for any i, j.

Remark 10.3.4. In particular, if we take $\lambda = 1$, then the GOWIFSPG operator transformed to OWIFSPG operator.

Theorem 10.3.4. For a collection of IFSNs $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$, the value obtained with GOWIFSPG operator is still an IFSN and given by:

$$\begin{aligned}
&\mathrm{GOWIFSPG}(\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots, \mathcal{A}_{nm}) \\
&= \left(\begin{array}{l} p^{-1}\left(\frac{1}{\lambda} l \left(q^{-1} \left(\sum_{j=1}^{m} \phi_j \left(\sum_{i=1}^{n} \psi_i q \left(p^{-1} \left(\lambda p \left(\zeta_{\sigma(i)\varsigma(j)} \right) \right) \right) \right) \right) \right) \right), \\ q^{-1}\left(\frac{1}{\lambda} k \left(p^{-1} \left(\sum_{j=1}^{m} \phi_j \left(\sum_{i=1}^{n} \psi_i p \left(q^{-1} \left(\lambda q \left(\vartheta_{\sigma(i)\varsigma(j)} \right) \right) \right) \right) \right) \right) \right) \end{array} \right)
\end{aligned} \tag{10.20}$$

Clearly, the proposed GWIFSPA, GOWIFSPA, GWIFSPG and GOWIFSPG operators also possess the same properties that WIFSPA and WIFSPG operators have.

10.4 Proposed decision-making approach

In this section, an approach has been presented to solve DM problems with proposed AOs under IFSS environment. Assume a set of z alternatives $\mathcal{A} = \{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(z)}\}$, which are to be inspected by the experts $\mathcal{X} = \{u_1, u_2, \ldots, u_n\}$ having weights $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)^T$ such that $\kappa_i > 0$ and $\sum_{i=1}^{n} \kappa_i = 1$. The evaluation has been done over the set of parameters $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ whose weight vector is $\rho = (\rho_1, \rho_2, \ldots, \rho_m)^T$ such that $\rho_j > 0$ and $\sum_{j=1}^{m} \rho_j = 1$. Consider alternative ratings on the parameter e_j, $(j = 1, 2, \ldots, m)$ to be examined by the expert $u_i, (i = 1, 2, \ldots, n)$ in terms of IFSNs denoted by $\mathcal{A}_{ij} = (\zeta_{ij}, \vartheta_{ij})$ such that $0 \leq \zeta_{ij}, \vartheta_{ij} \leq 1$ and $\zeta_{ij} + \vartheta_{ij} \leq 1$. Then, following are the steps to be adopted to choose the supreme alternative(s) in accordance to the proposed operators.

Step 1: Arrange the information corresponding to each alternative $\mathcal{A}^{(b)}; b = 1, 2, \ldots, z$ in the form of IFSNs $\mathcal{A}_{ij}$ and their matrix is summarized as follows:

$$(\mathcal{A}^{(b)}, \mathcal{E}) = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array} \\ \begin{pmatrix} \mathcal{A}_{11}^{(b)} & \mathcal{A}_{12}^{(b)} & \cdots & \mathcal{A}_{1m}^{(b)} \\ \mathcal{A}_{21}^{(b)} & \mathcal{A}_{22}^{(b)} & \cdots & \mathcal{A}_{2m}^{(b)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{n1}^{(b)} & \mathcal{A}_{n2}^{(b)} & \cdots & \mathcal{A}_{nm}^{(b)} \end{pmatrix} \end{array}$$

Step 2: Compute the support R_i for each expert $u_i; i = 1, 2, \ldots, n$ by using

$$R_i^{(b)} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \text{Sup}\left(\mathcal{A}_{ij}^{(b)}, \mathcal{A}_{kj}^{(b)}\right) \tag{10.21}$$

where $\text{Sup}\left(\mathcal{A}_{ij}^{(b)}, \mathcal{A}_{kj}^{(b)}\right) = 1 - d\left(\mathcal{A}_{ij}^{(b)}, \mathcal{A}_{kj}^{(b)}\right)$.

Step 3: By using appropriate weighted averaging or geometric operations over expert $u_i; i = 1, 2, \ldots, n$ and hence find the support $T_j (j = 1, 2, \ldots, m)$ for each parameter as:

$$T_j^{(b)} = \sum_{\substack{l=1 \\ l \neq j}}^{m} \text{Sup}\left(\mathcal{A}_j^{(b)}, \mathcal{A}_l^{(b)}\right) \tag{10.22}$$

Step 4: Utilize WIFSPA or WIFSPG operator to aggregate the different preference of alternative $\mathcal{A}^{(b)}; b = 1, 2, \ldots z$ by using R_i and T_j into the collective one $r^{(b)} = \left(\zeta^{(b)}, \vartheta^{(b)}\right)$. For example, by WIFSPA operator, the collective values are computed as

$$\begin{aligned} r^{(b)} &= \text{WIFSPA}\left(\mathcal{A}_{11}^{(b)}, \mathcal{A}_{12}^{(b)}, \ldots, \mathcal{A}_{nm}^{(b)}\right) \\ &= \left(1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \zeta_{ij}^{(b)}\right)^{\psi_i}\right)^{\phi_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\vartheta_{ij}^{(b)}\right)^{\psi_i}\right)^{\phi_j}\right) \end{aligned} \quad (10.23)$$

while by WIFSPG operator, we get

$$\begin{aligned} r^{(b)} &= \text{WIFSPG}\left(\mathcal{A}_{11}^{(b)}, \mathcal{A}_{12}^{(b)}, \ldots, \mathcal{A}_{nm}^{(b)}\right) \\ &= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\zeta_{ij}^{(b)}\right)^{\psi_i}\right)^{\phi_j}, 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \vartheta_{ij}^{(b)}\right)^{\psi_i}\right)^{\phi_j}\right) \end{aligned} \quad (10.24)$$

Step 5: Compute the score values of $r^{(b)}; b = 1, 2, \ldots, z$ as

$$\mathcal{S}\left(r^{(b)}\right) = \zeta^{(b)} - \vartheta^{(b)}.$$

If for any two positive integers b_1 and b_2, we have $\mathcal{S}\left(r^{(b_1)}\right) = \mathcal{S}\left(r^{(b_2)}\right)$ then we compute the accuracy function as

$$\mathcal{H}\left(r^{(b)}\right) = \zeta^{(b)} + \vartheta^{(b)}.$$

Step 6: Rank the alternative $\mathcal{A}^{(b)}(b = 1, 2, \ldots, z)$ by using descending value of $\mathcal{S}(r^{(b)})$ and hence find the desired one(s).

10.5 Illustrative Example

The above developed approach is illustrate with a numerical example as follows.

10.5.1 A case study

Goods and Services Tax (GST) is an indirect tax which is designed to make India an integrated common market. While GST promises to the user in an era of the unified indirect tax regime, integrating India into a single homogenous market, it comes with certain complications inherited from the legacy tax regime. With the government gearing up to enforce the GST in Punjab from July 1, the issue of traders having limited computer knowledge and poor connectivity. In order to counter this, the state government has planned to train more than 2,000 youths as 'GST Mitra' to cater the traders. Punjab GST Mitra Scheme, which has to be started as a pilot project from Patiala, proposes to assist taxpayers in furnishing the details of outward supplies, inward supplies and returns, filing claims or refunds, filing any other applications, etc., in GST Regime. It aims to create a group of Tax professionals available in the locality or at the doorstep of the taxpayer, at affordable costs throughout the State of Punjab.

The poor internet connectivity in far-flung areas has emerged as a big stumbling block in the success of 'GST Mitra' scheme. In order to provide the online services to run this scheme, state government is planning to give contract combinedly to private mobile service provider along with state-owned BSNL. For this purpose, an expert committee of five members u_1, u_2, u_3, u_4 and u_5, with weights $\kappa = (0.3, 0.2, 0.25, 0.15, 0.1)^T$ is formed by the government short-listed the four internet service provider, namely, Bharti Airtel ($\mathcal{A}^{(1)}$, Relience Communications ($\mathcal{A}^{(2)}$), Vodafone India ($\mathcal{A}^{(3)}$) and Mahanagar Telecom Nigam ($\mathcal{A}^{(4)}$) under the parameters: Customer Services (e_1), Bandwidth (e_2), Pacakage Deal (e_3), Total Cost (e_4), Internet speed (e_5) having weights $\rho = (0.3, 0.2, 0.3, 0.1, 0.1)^T$. The experts have evaluated these different alternative $\mathcal{A}^{(b)}(b = 1, 2, 3, 4)$ under the set of parameters $e_j(j = 1, 2, 3, 4, 5)$ and given their decision preferences in terms of IFSNs summarized in Table 10.1. The procedure of selecting the best internet service provider by using proposed approach has been listed as follows:

Step 1: The rating values towards the each alternative are summarized in Table 10.1.

Step 2: By Eq. (10.21), the values of $R_i^{(b)}; b = 1, 2, 3, 4$ are computed as

$$R_i^{(1)} = \begin{bmatrix} 3.3500 & 2.9000 & 3.2500 & 3.1000 & 3.2000 \\ 3.5500 & 3.3500 & 3.4000 & 3.4000 & 3.5000 \\ 3.7000 & 3.8500 & 3.8000 & 3.8000 & 3.8500 \\ 3.6000 & 3.3000 & 3.6500 & 3.6000 & 3.4500 \\ 3.6500 & 3.3500 & 3.4500 & 3.5500 & 3.6000 \end{bmatrix}, R_i^{(2)} = \begin{bmatrix} 2.7500 & 3.4500 & 3.3500 & 3.4000 & 3.2500 \\ 2.8000 & 3.4500 & 3.4000 & 3.5000 & 3.4500 \\ 3.3000 & 3.6000 & 3.6500 & 3.4500 & 3.6000 \\ 3.6000 & 3.6500 & 3.7500 & 3.8000 & 3.8000 \\ 3.6000 & 3.6000 & 3.4500 & 3.7500 & 3.6000 \end{bmatrix}$$

Table 10.1: Rating values of each alternative in terms of IFSNs

	For alternative $\mathcal{A}^{(1)}$					For alternative $\mathcal{A}^{(2)}$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.5, 0.3)$	$(0.6, 0.2)$	$(0.5, 0.2)$	$(0.4, 0.3)$	$(0.6, 0.2)$	$(0.4, 0.5)$	$(0.5, 0.5)$	$(0.6, 0.4)$	$(0.6, 0.3)$	$(0.5, 0.3)$
u_2	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.8, 0.2)$	$(0.7, 0.2)$	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.8, 0.2)$	$(0.7, 0.3)$
u_3	$(0.4, 0.4)$	$(0.5, 0.4)$	$(0.6, 0.3)$	$(0.5, 0.3)$	$(0.6, 0.4)$	$(0.6, 0.3)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.7, 0.3)$	$(0.6, 0.1)$
u_4	$(0.4, 0.5)$	$(0.5, 0.4)$	$(0.6, 0.3)$	$(0.4, 0.3)$	$(0.5, 0.3)$	$(0.8, 0.2)$	$(0.7, 0.1)$	$(0.8, 0.1)$	$(0.7, 0.2)$	$(0.6, 0.3)$
u_5	$(0.7, 0.2)$	$(0.6, 0.1)$	$(0.6, 0.2)$	$(0.5, 0.1)$	$(0.5, 0.2)$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.7, 0.3)$	$(0.7, 0.2)$	$(0.6, 0.4)$
	For alternative $\mathcal{A}^{(3)}$					For alternative $\mathcal{A}^{(4)}$				
	e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
u_1	$(0.7, 0.2)$	$(0.6, 0.1)$	$(0.6, 0.2)$	$(0.5, 0.2)$	$(0.7, 0.1)$	$(0.7, 0.2)$	$(0.7, 0.1)$	$(0.8, 0.1)$	$(0.9, 0.1)$	$(0.8, 0.2)$
u_2	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.7, 0.2)$	$(0.6, 0.2)$	$(0.6, 0.3)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.6, 0.3)$	$(0.6, 0.2)$	$(0.5, 0.3)$
u_3	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.6, 0.3)$	$(0.5, 0.3)$	$(0.4, 0.4)$	$(0.4, 0.3)$	$(0.5, 0.4)$	$(0.5, 0.2)$
u_4	$(0.4, 0.3)$	$(0.6, 0.1)$	$(0.4, 0.1)$	$(0.4, 0.3)$	$(0.4, 0.2)$	$(0.7, 0.1)$	$(0.7, 0.2)$	$(0.6, 0.3)$	$(0.5, 0.2)$	$(0.5, 0.1)$
u_5	$(0.2, 0.3)$	$(0.2, 0.4)$	$(0.3, 0.4)$	$(0.3, 0.5)$	$(0.2, 0.5)$	$(0.8, 0.2)$	$(0.9, 0.1)$	$(0.8, 0.1)$	$(0.7, 0.2)$	$(0.7, 0.3)$

$$R_i^{(3)} = \begin{bmatrix} 3.3000 & 3.1000 & 3.3000 & 3.2000 & 2.9000 \\ 3.5000 & 3.3500 & 3.4000 & 3.5000 & 2.6500 \\ 3.5500 & 3.4000 & 3.5500 & 3.3000 & 3.0000 \\ 3.5500 & 3.4000 & 3.6000 & 3.5500 & 3.1000 \\ 3.0500 & 3.4000 & 3.4000 & 3.2500 & 2.7000 \end{bmatrix}, R_i^{(4)} = \begin{bmatrix} 3.6500 & 3.5500 & 3.3000 & 3.6000 & 3.5000 \\ 3.4000 & 3.3500 & 2.9000 & 3.4500 & 3.1000 \\ 3.3000 & 3.5000 & 3.2000 & 3.5000 & 3.3000 \\ 3.0500 & 3.5500 & 3.2000 & 3.5000 & 3.5000 \\ 3.3500 & 3.5500 & 3.6000 & 3.4500 & 3.4500 \end{bmatrix}$$

Step 3: By using Eq. (10.22) and the averaging operator, we get the values of $T_j^{(b)}; b = 1, 2, 3, 4$ as

$$T_j^{(1)} = \begin{bmatrix} 3.8592 \\ 3.8465 \\ 3.8852 \\ 3.7968 \\ 3.8198 \end{bmatrix}, T_j^{(2)} = \begin{bmatrix} 3.8196 \\ 3.8202 \\ 3.8328 \\ 3.8367 \\ 3.7693 \end{bmatrix}, T_j^{(3)} = \begin{bmatrix} 3.7372 \\ 3.7778 \\ 3.8358 \\ 3.6350 \\ 3.8219 \end{bmatrix}, T_j^{(4)} = \begin{bmatrix} 3.9134 \\ 3.9168 \\ 3.9240 \\ 3.8512 \\ 3.8926 \end{bmatrix}$$

On the other hand, the values of $T_j^{(b)}; b = 1, 2, 3, 4$ by using geometric operations are obtained as

$$T_j^{(1)} = \begin{bmatrix} 3.7611 \\ 3.8599 \\ 3.8348 \\ 3.7865 \\ 3.8135 \end{bmatrix}, T_j^{(2)} = \begin{bmatrix} 3.7752 \\ 3.8658 \\ 3.8268 \\ 3.8399 \\ 3.8204 \end{bmatrix}, T_j^{(3)} = \begin{bmatrix} 3.8125 \\ 3.8017 \\ 3.8509 \\ 3.7137 \\ 3.8299 \end{bmatrix}, T_j^{(4)} = \begin{bmatrix} 3.8829 \\ 3.9142 \\ 3.9265 \\ 3.9094 \\ 3.9225 \end{bmatrix}$$

Step 4: By Eq. (10.23), we get the collective value $r^{(b)}; b = 1, 2, 3, 4$ of each alternative as $r^{(1)} = (0.5746, 0.2443)$, $r^{(2)} = (0.6684, 0.2222)$, $r^{(3)} = (0.5898, 0.1766)$ and $r^{(4)} = (0.6612, 0.1848)$, while by Eq. (10.24), these values are obtained as$r^{(1)} = (0.5482, 0.2744)$, $r^{(2)} = (0.6390, 0.2696)$, $r^{(3)} = (0.5427, 0.2018)$ and $r^{(4)}$= (0.6135, 0.2134).

Step 5: The score values of collection $r^{(b)}$ obtained during the Step 3 are computed as $\mathcal{S}\left(r^{(1)}\right) = 0.3303$, $\mathcal{S}\left(r^{(2)}\right) = 0.4462$, $\mathcal{S}\left(r^{(3)}\right) = 0.4132$ and $\mathcal{S}\left(r^{(4)}\right) = 0.4764$. While these score values for the collective information obtained through WIFSPG operator are given as $\mathcal{S}\left(r^{(1)}\right) = 0.2738$, $\mathcal{S}\left(r^{(2)}\right) = 0.3694$, $\mathcal{S}\left(r^{(3)}\right) = 0.3409$ and $\mathcal{S}\left(r^{(4)}\right) = 0.4000$.

Step 6: The ranking order of $\mathcal{A}^{(b)}; b = 1, 2, 3, 4$ are: $\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$ by WIFSPA as well as WIFSPG operator based approach.

However, if we apply other proposed operators such as IFSPA, IFSPG, OWIFSPA and OWIFSPG, on this considered data for different generator, say $q(t) = -\log(t), q(t) = \log\left(\frac{2-t}{t}\right)$ etc. Then their corresponding aggregated results of each alternative $\mathcal{A}^{(b)}; b = 1, 2, 3, 4$ are summarized in Table 10.2. From this, we conclude that the best alternative found with different operators are slightly different. So, based on decision maker preference regarding optimistic and pessimistic viewpoint, they can select the best one accordingly.

To analyze the effect of λ on DM problem, an analysis has been performed by taking different values of λ. Based on these different values, the proposed AOs GWIFSPA, GWIFSPG, GOWIFSPA, and GOWIFSPG have been applied to each alternative then their score values together with the ranking order are summarized in Table 10.3. From this, we conclude that the results obtained with different values of λ are different and their corresponding ranking is also not same, which will help the expert to study the impact of the alternative values on the final decision process. Also, it has been observed that the best alternative obtained for different values of λ is different. Thus, the decision-maker can easily assess the decision according to the actual need of the DM process.

Table 10.2: Ordering of the ranking obtained with different operators

Generators	Operators	Overall Score values of				Ranking
		$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
$q(t) = -\log(t)$	IFSPA	0.3461	0.4793	0.3392	0.4836	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
	IFSPG	0.2875	0.4161	0.2397	0.4066	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
	WIFSPA	0.3303	0.4462	0.4132	0.4764	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	WIFSPG	0.2738	0.3694	0.3409	0.4000	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	OWIFSPA	0.3836	0.5148	0.4072	0.5351	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	OWIFSPG	0.3494	0.4827	0.2971	0.4748	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
$q(t) = \log\left(\frac{2-t}{t}\right)$	IFSPA	0.3386	0.4728	0.3271	0.4747	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
	IFSPG	0.2963	0.4265	0.2559	0.4177	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
	WIFSPA	0.3229	0.4379	0.4051	0.4676	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	WIFSPG	0.2823	0.3827	0.3532	0.4114	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	OWIFSPA	0.4092	0.5390	0.4249	0.5506	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	OWIFSPG	0.3626	0.4987	0.3695	0.4916	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$

10.5.2 Effect of the parameters and experts in DM process

From the above results, it is noticed that $\mathcal{A}^{(4)}$ is the best alternative. However, if we are interested to know which expert gives his highest preference and which is the strongest parameter of this alternative, an analysis has been conducted which is as follows:

(i) If the various choices over the parameters $e_j, j = 1, 2, 3, 4, 5$ for each expert $u_i, i = 1, 2, 3, 4, 5$ are aggregated then the results obtained are shown in Table 10.4. From this, it has been observed that $\mathcal{A}^{(4)}$ is highly recommended by the experts u_5 and u_1 while $\mathcal{A}^{(2)}$ is best according to all other experts.

(ii) If the different preferences given by experts $u_i, i = 1, 2, 3, 4, 5$ are aggregated then the computed values are listed in Table 10.5. In this, we have seen that $\mathcal{A}^{(3)}$ and $\mathcal{A}^{(4)}$ has e_2 as their strongest parameter while $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are best over the parameters e_4 and e_5 respectively.

Table 10.3: Ranking of the alternatives at different values of λ

		When $q(t) = -\log(t)$					When $q(t) = \log\left(\frac{2-t}{t}\right)$				
		$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	Ranking order	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	Ranking order
$\lambda \to 0$	GWIFSPA	0.3199	0.4352	0.4002	0.4647	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.3236	0.4398	0.4040	0.4695	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.2952	0.4026	0.3692	0.4320	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.2939	0.4044	0.3667	0.4308	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPA	0.3758	0.5084	0.3731	0.5176	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.4104	0.5402	0.4245	0.5529	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.3666	0.5044	0.3400	0.4873	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.3745	0.5134	0.3844	0.5143	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
$\lambda = 0.5$	GWIFSPA	0.3250	0.4406	0.4068	0.4704	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.3216	0.4373	0.4021	0.4668	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.2851	0.3873	0.3562	0.4169	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.2939	0.4023	0.3671	0.4297	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPA	0.3796	0.5115	0.4012	0.5306	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.4082	0.5388	0.4221	0.5505	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.3528	0.4941	0.3270	0.4714	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.3739	0.5129	0.3844	0.5126	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
$\lambda = 1$	GWIFSPA	0.3303	0.4462	0.4132	0.4764	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.3229	0.4379	0.4051	0.4676	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.2738	0.3694	0.3409	0.4000	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.2823	0.3827	0.3532	0.4114	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPA	0.3836	0.5148	0.4072	0.5351	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.4092	0.5390	0.4249	0.5506	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.3494	0.4827	0.2971	0.4748	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.3626	0.4987	0.3695	0.4916	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
$\lambda = 5$	GWIFSPA	0.3798	0.4897	0.4589	0.5258	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.4024	0.5032	0.4737	0.5406	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.1812	0.2224	0.1888	0.2754	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.1532	0.1797	0.1321	0.2450	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
	GOWIFSPA	0.4312	0.5616	0.4745	0.5653	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.4788	0.5769	0.4881	0.6060	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.2621	0.3539	0.1836	0.3474	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.2326	0.3141	0.1265	0.3165	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
$\lambda = 10$	GWIFSPA	0.4425	0.5340	0.5020	0.5780	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.4990	0.5708	0.5395	0.6222	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.1027	0.1110	0.0396	0.1883	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.0675	0.0666	-0.0372	0.1552	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
	GOWIFSPA	0.5109	0.5792	0.5148	0.6344	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.5422	0.6193	0.5506	0.6739	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.1757	0.2234	0.0350	0.2559	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.1387	0.1847	-0.0413	0.2223	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
$\lambda = 15$	GWIFSPA	0.4924	0.5664	0.5357	0.6169	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.5553	0.6077	0.5813	0.6709	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.0535	0.0521	-0.0485	0.1393	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$	0.0206	0.0157	-0.1162	0.1101	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
	GOWIFSPA	0.5374	0.6032	0.5471	0.6645	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.5863	0.6426	0.5899	0.7113	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.1202	0.1440	-0.0514	0.2014	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.0799	0.1070	-0.1190	0.1666	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
$\lambda = 20$	GWIFSPA	0.5293	0.5900	0.5623	0.6463	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.5888	0.6293	0.6078	0.7006	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GWIFSPG	0.0214	0.0178	-0.1025	0.1089	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$	-0.0072	-0.0120	-0.1596	0.0843	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$
	GOWIFSPA	0.5660	0.6208	0.5721	0.6877	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$	0.6125	0.6494	0.6145	0.7326	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
	GOWIFSPG	0.0798	0.0935	-0.1044	0.1640	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$	0.0409	0.0602	-0.1618	0.1306	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$

(iii) The element-wise weighted results of the alternative $\mathcal{A}^{(4)}$ are given in Table 10.6. From this table, it has been observed that expert u_1 and u_5 give their highest preference to $\mathcal{A}^{(4)}$ over the parameter e_3 while e_1 is preferred by the other experts.

Table 10.4: Score values obtained by aggregating different parameters e_j's (j=1,2,3,4,5)

	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$
u_1	0.2891	0.0963	0.4729	0.6396
u_2	0.5809	0.5868	0.5703	0.3805
u_3	0.1608	0.4237	0.3955	0.1383
u_4	0.1280	0.6114	0.2769	0.4670
u_5	0.4534	0.5776	-0.1425	0.6632

Table 10.5: Score values obtained by aggregating preferences of experts u_i's (i=1,2,3,4,5)

	e_1	e_2	e_3	e_4	e_5
$\mathcal{A}^{(1)}$	0.1586	0.1988	0.1919	0.2438	0.1182
$\mathcal{A}^{(2)}$	0.0857	0.1421	0.1076	0.0517	0.1673
$\mathcal{A}^{(3)}$	0.2011	0.2897	0.2291	0.2508	0.2013
$\mathcal{A}^{(4)}$	0.1724	0.1745	0.1411	0.0922	0.1600

Table 10.6: Score values of weighted alternative $\mathcal{A}^{(4)}$

	e_1	e_2	e_3	e_4	e_5
u_1	-0.8352	-0.8639	-0.7865	-0.9101	-0.9368
u_2	-0.8137	-0.9075	-0.8857	-0.9468	-0.9638
u_3	-0.8898	-0.9426	-0.9036	-0.9670	-0.9561
u_4	-0.8147	-0.8848	-0.8825	-0.9526	-0.9394
u_5	-0.8180	-0.8258	-0.7720	-0.9445	-0.9539

10.5.3 Comparative study

To compare the proposed approach with some existing approaches under the IFSS environment, a comparative analysis has been performed with different approaches given in [6, 21, 98, 142, 161, 168, 171]. The obtained results have been described in Table 10.7. From this table, it is noticed that the resulting values evaluated by utilizing the

Table 10.7: Comparison with existing approaches under IFSS environment

Authors	Rating values of				Ranking
	$\mathcal{A}^{(1)}$	$\mathcal{A}^{(2)}$	$\mathcal{A}^{(3)}$	$\mathcal{A}^{(4)}$	
Arora and Garg [6]	0.3334	0.4378	0.4096	0.4756	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
Selvachandran, Maji, Faisal and Salleh [171]	0.7507	0.6569	0.7609	0.6558	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \mathcal{A}^{(3)}$
Muthukumar and Krishnan [142]	0.0208	0.0241	0.0211	0.0239	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
Sarala and Suganya [168]	0.7122	0.7734	0.7391	0.7892	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
Khalid and Abbas [98]	0.4400	0.3545	0.4365	0.3710	$\mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
Rajarajeswari and Dhanalakshmi [161]	0.3030	0.4010	0.3955	0.4205	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$
Cagman and Deli [21]	0.3030	0.4010	0.3955	0.4205	$\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$

prevailing methods coincide with the proposed approach outcomes. However, these operators do not consider the relationship between the parameters as well as between the experts by the power weighting, but the proposed operators consider such relationships. In the proposed operators, the weight of each input argument depends upon the other input arguments. From these facts, we can conclude that the proposed method is more reasonable and valid for intuitionistic fuzzy soft set information. Moreover, to solve DM problem[21, 98, 142, 161, 168, 171], we have to assume an ideal extreme alternative which increases the complexity and in turn leads to additional overheads but with the proposed operators we don't need any ideal alternative which decreases the cost, computation time and complexity of the experts. Thus, the proposed method is suitable for solving the DM problem than the existing methods.

10.5.4 Validity test

Wang and Triantaphyllou [210] proposed the following testing criteria to validate MCDM method:

Test criterion 1: "The best alternative should never be changed on replacing a non-optimal alternative by other worse alternative without changing the relative importance of each decision criteria."

Test criterion 2: "The transitivity property should be followed by an efficient DM method."

Test criterion 3: "The ranking of DM problem should never be changed by applying same DM method to the smaller problems which are decomposed from the original problem."
The applicability of test criteria to proposed DM method is discussed below:

Validity test with criterion 1

For testing the validity of our approach under test criterion 1, the alternative $\mathcal{A}^{(3)}$ (non-optimal alternative) is to be substituted with any arbitrary alternative $\left(\mathcal{A}^{(3)}\right)'$ (worse alternative) then the preference values by each decision-maker under different parameters for the alternative $\mathcal{A}^{(3)}$ is replaced with the matrix given in Table 10.8. If the proposed

Table 10.8: Intuitionistic fuzzy soft matrix for transformed $\left(\mathcal{A}^{(3)}\right)'$

	e_1	e_2	e_3	e_4	e_5
u_1	$(0.5, 0.4)$	$(0.4, 0.3)$	$(0.4, 0.5)$	$(0.3, 0.4)$	$(0.4, 0.3)$
u_2	$(0.6, 0.2)$	$(0.5, 0.4)$	$(0.5, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.4)$
u_3	$(0.4, 0.3)$	$(0.3, 0.5)$	$(0.4, 0.4)$	$(0.3, 0.5)$	$(0.4, 0.4)$
u_4	$(0.2, 0.5)$	$(0.4, 0.2)$	$(0.1, 0.4)$	$(0.2, 0.5)$	$(0.1, 0.3)$
u_5	$(0.1, 0.5)$	$(0.1, 0.6)$	$(0.2, 0.5)$	$(0.1, 0.6)$	$(0.1, 0.7)$

WIFSPA operator applied to modified decision matrices then the aggregated value for each alternative is obtained as: $r^{(1)} = (0.5746, 0.2443)$; $r^{(2)} = (0.6684, 0.2222)$; $(r^{(3)})' = (0.3809, 0.3824)$ and $r^{(4)} = (0.6612, 0.1848)$ and hence the score values are $S\left(r^{(1)}\right) = 0.3303$, $S\left(r^{(2)}\right) = 0.4462$, $S\left((r^{(3)})'\right) = -0.0015$ and $S\left(r^{(4)}\right) = 0.4764$. Thus, the ranking order the alternatives is $\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(1)} \succ \left(\mathcal{A}^{(3)}\right)'$ and hence the best alternative $\mathcal{A}^{(4)}$ is similar as obtained with the original DM problem which indicates that the best alternative is same by replacing a non-optimal alternative with other worse alternative, which validates the *test criterion 1.*

Validity test with criteria 2 and 3

For testing the validity of our approach with test criteria 2 and 3, the decomposed smaller subproblems are taken as $\{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)}\}$, $\{\mathcal{A}^{(2)}, \mathcal{A}^{(3)}, \mathcal{A}^{(4)}\}$ and $\{\mathcal{A}^{(3)}, \mathcal{A}^{(4)}, \mathcal{A}^{(1)}\}$. Then by applying the proposed WIFSPA operator, the corresponding ranking is $\mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$, $\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$ and $\mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$ respectively and finally the combined ordering is $\mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(1)}$ which is identical with the actual DM problem and thus follows transitivity. Thus, the proposed method validates the *test criteria 2 and 3*.

10.6 Conclusion

In this chapter, we have presented some series of the power AO under the IF soft set environment. The existing operators and measures under IFSS environment to solve DM problem assume that the input argument is independent of the attributes and hence their results are sometimes undesirable to find the best alternatives. In order to resolve it, this chapter presents some series of AOs namely IF soft power averaging and geometric AOs, weighted and ordered weighted IF soft power averaging/geometric operators to aggregate the different preferences of the parameters and the experts under IFSS environment. The characteristics of these operators are also discussed in details. Then, we solve the DM problem based on these operators to show the superiority of the approach and compare their results with existing studies. Finally, an effect of the decision parameter λ has been made to show their effects on the ranking order of the alternatives. From the existing approaches, our proposed approach also delivers a benefit over the existing approaches when the prioritized relationship between the parameters and the experts is considered.

Chapter 11

Generalized and Group-based Generalized intuitionistic fuzzy soft sets with applications in decision-making[1]

In this chapter, we presented the concept of generalized IFSS (GIFSS), as well as the group-based generalized intuitionistic fuzzy soft set (GGIFSS) in which the evaluation of the object is done by the group of experts rather than a single expert. Based on this information, a new weighted averaging and geometric AO has been proposed by taking the intuitionistic fuzzy parameter. Finally, a DM approach based on the proposed operator is being built to solve the problems under the intuitionistic fuzzy environment. An illustrative example of the selection of the optimal alternative has been given to show the developed method. Comparison analysis between the proposed and the existing operators have been performed in term of counter-intuitive cases for showing the superiority of the approach.

11.1 Introduction

During the DM model, the hesitancy of expert is occurring at the membership grades, since the IFS allows an expert to record his hesitancy factor. However, it indicates only

[1]The content of this chapter is published as "Generalized and Group-based Generalized intuitionistic fuzzy soft sets with applications in decision-making", *Applied Intelligence*, 48(2), 343 - 356, 2018 (**SCI: Impact Factor 2.882**).

an expert preference during the decision process which is completely made by his views hence may lead to unfair decisions. Furthermore, the hesitancy is a feature of one's own perception, and therefore, it needs to be supported by other independent observer/expert. For addressing this issue, the concept of the generalized intuitionistic fuzzy soft set (GIFSS) was introduced by [41] by incorporating the idea of generalized preference of the other senior expert. In contrast to an IFSS, GIFSS provides the framework for assessing the credibility of the information in the original IFSS so as to make up for any distortion in the information provided. The major advantage of adding the generalized parameter into the analysis is to reduce the possibility of errors caused by the imprecise information by taking the moderator's opinion on the same. For example, a patient might report the symptoms to a doctor as per his perceptions, and the real information might get distorted in its reporting. Unless this misrepresentation is factored in by a doctor, it would lead to the wrong diagnosis. For this, it may be more judicious to have a junior and/or senior doctor moderate the severity of the symptoms of a patient through a generalization parameter, which indicates the credibility of the information presented to him.

The GIFSS has a wide scope in the different areas of DM such as medical diagnosis, plot selection process, object recognition, supplier selection problem, etc. In such type of DM problems, the possibility of error in the judgment of the domain experts cannot be ruled out. So, in such situations there is a need for a generalization parameter, indicating an expert's level of trust in the credibility of information, significantly making the approach quite close to real-life situations. This helps in removing the individual bias from the input data and brings more reliability to the final decision. Although, both of generalized FSS, as well as GIFSS, have a generalization parameter to express the uncertainties, still they have some limitations as it is hard for a single senior/junior expert (or moderator) to give a suitable generalization parameter with more fuzzy information according to his/her own knowledge.

To overcome the shortcomings of the existing work, in the present chapter, the concept of the GIFSS has been extended to group-based generalized intuitionistic fuzzy soft set (GGIFSS) by incorporating the ideas of more than one experts' opinion which will reduce the impact of the single expert's preferences. Further, in order to rank the different objects, an AO is necessary to aggregate all the preferences of the decision maker, which are in the form of IFNs, into a collective one. Thus, by considering the fact that the GIFSS, as well as GGIFSS, have a powerful tool to deal with the ambiguity and vagueness of the data, the present chapter will introduce some new averaging and geometric AOs denoted by GWA and GWG as well as GGWA and GGWG operators for the GIFSS and GGIFSS respectively. The advantages of these operators are that they are capable of facilitating the descriptions of the real-world situation with the help of their parameterizations property. Further, some of its desirable properties have been investigated in detail. Meanwhile, we also provide a DM method based on the proposed operators under the IFSS environment and then illustrate with a numerical example.

11.2 Generalized and Group-based Generalized Intuitionistic Fuzzy Soft Set and its operator

In this section, the concept of generalized and group generalized IFSS has been presented in a family of all GIFSSs and IFSs, denoted by Ω_s and Ω respectively, along with their AOs.

Definition 11.2.1. Generalized intuitionistic fuzzy soft set (GIFSS) [1]: A GIFSS, $\mathcal{F}_\beta$ over the soft universe $(\mathcal{X}, \mathcal{E})$ is defined as a mapping $\mathcal{F}_\beta : \mathcal{E} \to K^U \times IF$ and the generalization parameter $\beta : \mathcal{E} \to IF = (\zeta_\beta, \vartheta_\beta)$. Mathematically, it is defined as

$$\mathcal{F}_\beta(e) = (\mathcal{F}(e), \beta(e)) \tag{11.1}$$

where $\mathcal{F}(e) \in K^U$ and $\beta(e) \in IF$ for $e \in \mathcal{E}$. Here, $\beta(e) = (\zeta_\beta(e), \vartheta_\beta(e))$ indicates the senior person/expert assessment on the elements of $\mathcal{E}$ in $\mathcal{F}(e)$ such that $\zeta_\beta \in [0,1], \vartheta_\beta \in [0,1]$ and $\zeta_\beta + \vartheta_\beta \leq 1$ for $e \in \mathcal{E}$.

Example 11.2.1. Assume that a person wants to purchase a mobile from the set of possible five alternatives, say $\mathcal{X} = \{m_1, m_2, m_3, m_4, m_5\}$ under the set on certain parameters $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ and evaluate these alternatives with their rating values in the form of

IFNs denoted by $\mathcal{F}(e)$ as follows:

$$
\begin{aligned}
\mathcal{F}(e_1) &= \left\{\left(\frac{m_1}{(0.2,0.6)},\frac{m_2}{(0.3,0.5)},\frac{m_3}{(0.6,0.3)},\frac{m_4}{(0.5,0.4)},\frac{m_5}{(0.7,0.2)}\right)\right\},\\
\mathcal{F}(e_2) &= \left\{\left(\frac{m_1}{(0.4,0.5)},\frac{m_2}{(0.1,0.7)},\frac{m_3}{(0.3,0.7)},\frac{m_4}{(0.4,0.3)},\frac{m_5}{(0.2,0.6)}\right)\right\},\\
\mathcal{F}(e_3) &= \left\{\left(\frac{m_1}{(0.1,0.7)},\frac{m_2}{(0.2,0.5)},\frac{m_3}{(0.6,0.4)},\frac{m_4}{(0.4,0.4)},\frac{m_5}{(0.5,0.2)}\right)\right\},\\
\mathcal{F}(e_4) &= \left\{\left(\frac{m_1}{(0.3,0.5)},\frac{m_2}{(0.6,0.2)},\frac{m_3}{(0.5,0.3)},\frac{m_4}{(0.7,0.3)},\frac{m_5}{(0.8,0.2)}\right)\right\}
\end{aligned}
$$

Further, assume that a single senior expert β who gives his opinion, in the form of IFNs, for each parameter towards the assessment of the five possible alternatives in $\mathcal{F}(e)$ as $\beta(e_1) = (0.4, 0.5), \beta(e_2) = (0.7, 0.3), \beta(e_3) = (0.6, 0.3)$ and $\beta(e_4) = (0.3, 0.5)$. Then, the GIFSS $\mathcal{F}_\beta$ for the considered problem over the universal set $(\mathcal{X}, \mathcal{E})$ is given as

$$
\begin{aligned}
\mathcal{F}_\beta(e_1) &= \left\{\left(\frac{m_1}{(0.2,0.6)},\frac{m_2}{(0.3,0.5)},\frac{m_3}{(0.6,0.3)},\frac{m_4}{(0.5,0.4)},\frac{m_5}{(0.7,0.2)}\right),\frac{\beta}{(0.4,0.5)}\right\},\\
\mathcal{F}_\beta(e_2) &= \left\{\left(\frac{m_1}{(0.4,0.5)},\frac{m_2}{(0.1,0.7)},\frac{m_3}{(0.3,0.7)},\frac{m_4}{(0.4,0.3)},\frac{m_5}{(0.2,0.6)}\right),\frac{\beta}{(0.7,0.3)}\right\},\\
\mathcal{F}_\beta(e_3) &= \left\{\left(\frac{m_1}{(0.1,0.7)},\frac{m_2}{(0.2,0.5)},\frac{m_3}{(0.6,0.4)},\frac{m_4}{(0.4,0.4)},\frac{m_5}{(0.5,0.2)}\right),\frac{\beta}{(0.6,0.3)}\right\},\\
\mathcal{F}_\beta(e_4) &= \left\{\left(\frac{m_1}{(0.3,0.5)},\frac{m_2}{(0.6,0.2)},\frac{m_3}{(0.5,0.3)},\frac{m_4}{(0.7,0.3)},\frac{m_5}{(0.8,0.2)}\right),\frac{\beta}{(0.3,0.5)}\right\}
\end{aligned}
$$

Definition 11.2.2. [1] Let $\mathcal{F}_\beta(e) = (\mathcal{F}(e), \beta(e))$ and $\mathcal{G}_\gamma(e) = (\mathcal{G}(e), \gamma(e))$ be two GIFSSs over the soft universe $(\mathcal{X}, \mathcal{E})$ with two generalized parameters β and γ respectively. Then

(i) $\mathcal{F}_\beta \subseteq \mathcal{G}_\gamma$, if

  (a) $\beta \leq \gamma$

  (b) $\mathcal{F}(e)$ is an intuitionistic fuzzy subset of $\mathcal{G}(e), \forall e \in \mathcal{E}$.

(ii) $\mathcal{F}_\beta = \mathcal{G}_\gamma$, if

  (a) $\mathcal{G}(e)$ is an intuitionistic fuzzy subset of $\mathcal{F}(e), \forall e \in \mathcal{E}$

  (b) $\mathcal{F}(e)$ is an intuitionistic fuzzy subset of $\mathcal{G}(e), \forall e \in \mathcal{E}$.

Definition 11.2.3. [1] The complement of GIFSS $\mathcal{F}_\beta$, denoted by $\mathcal{F}^c_\beta$, is defined as $\mathcal{F}^c_\beta(e) = (\mathcal{F}^c(e), \beta^c(e)), \forall e \in \mathcal{E}$.

11.2.1 Aggregation operator for GIFSS

In this section, we have presented a weighted averaging and geometric AOs for a collection of generalized intuitionistic fuzzy soft set (GIFSS) $\mathcal{F}_{\beta_i}(e) = (\mathcal{F}_i(e), \beta_i(e))$, $(i = 1, 2, \ldots, n)$ and are denoted by GWA and GWG respectively.

Definition 11.2.4. Let $\mathcal{F}_{\beta_i}(e) = (\mathcal{F}_i(e), \beta_i(e))$, $(i = 1, 2, \ldots, n)$ be any GIFSS over $(\mathcal{X}, \mathcal{E})$, where $\mathcal{F}_i(e) = \{\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}\} \in K^{\mathcal{X}}$ and $\beta_i(e) = (\zeta_{\beta_i}, \vartheta_{\beta_i})$ is an expert assessment on $\mathcal{X}$ for the i^{th} alternative $\mathcal{F}_i(e)$. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the normalized weight vector of $\mathcal{F}_i(e)$ such that $\omega_j > 0$ and $\sum\limits_{j=1}^{m} \omega_j = 1$ and let GWA : $\Omega_s^n \to \Omega_s$, IFWA : $\Omega^n \to \Omega$. If

$$\mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \beta_i(e) \otimes \mathrm{IFWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \tag{11.2}$$

then GWA is called GIFSS weighted averaging operator.

Theorem 11.2.1. If $\mathcal{A}_{ij} = (\zeta_{\mathcal{A}_{ij}}, \vartheta_{\mathcal{A}_{ij}})$, $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ be an IFN, then the aggregated value by GWA operator is given by

$$\begin{aligned} & \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \\ = & \left(\zeta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j}\right), \vartheta_{\beta_i} + \prod_{j=1}^{m} \vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \vartheta_{\beta_i} \cdot \prod_{j=1}^{m} \vartheta_{\mathcal{A}_{ij}}^{\omega_j}\right) \end{aligned} \tag{11.3}$$

Proof. We will prove the result by using mathematical induction on m. For it, firstly, for $m = 2$, we have

$$\begin{aligned} \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}) &= \beta_i(e) \otimes \mathrm{IFWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}) \\ &= (\zeta_{\beta_i}, \vartheta_{\beta_i}) \otimes \left(1 - (1 - \zeta_{\mathcal{A}_{i1}})^{\omega_1} (1 - \zeta_{\mathcal{A}_{i2}})^{\omega_2}, \vartheta_{\mathcal{A}_{i1}}^{\omega_1} \vartheta_{\mathcal{A}_{i2}}^{\omega_2}\right) \\ &= \left(\zeta_{\beta_i} . \left(1 - (1 - \zeta_{\mathcal{A}_{i1}})^{\omega_1} (1 - \zeta_{\mathcal{A}_{i2}})^{\omega_2}\right), \vartheta_{\beta_i} + \vartheta_{\mathcal{A}_{i1}}^{\omega_1} \vartheta_{\mathcal{A}_{i2}}^{\omega_2} - \vartheta_{\beta_i} \cdot \vartheta_{\mathcal{A}_{i1}}^{\omega_1} \vartheta_{\mathcal{A}_{i2}}^{\omega_2}\right) \\ &= \left(\zeta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{2} \left(1 - \zeta_{\mathcal{A}_{ij}}\right)^{\omega_j}\right), \vartheta_{\beta_i} + \prod_{j=1}^{2} \vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \vartheta_{\beta_i} \cdot \prod_{j=1}^{2} \vartheta_{\mathcal{A}_{ij}}^{\omega_j}\right) \end{aligned}$$

Thus, Eq. (11.3) is true for $m = 2$. Assume the result holds for $m = t$, i.e.,

$$\begin{aligned} & \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{it}) \\ = & \left(\zeta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{t} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j}\right), \vartheta_{\beta_i} + \prod_{j=1}^{t} \vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \vartheta_{\beta_i} \cdot \prod_{j=1}^{t} \vartheta_{\mathcal{A}_{ij}}^{\omega_j}\right) \end{aligned}$$

Then, for $m = t+1$, by using the operational laws between the IFNs, we have

$$\begin{aligned}
& \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{i(t+1)}) \\
= \; & \beta_i(e) \otimes \mathrm{IFWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{i(t+1)}) \\
= \; & (\zeta_{\beta_i}, \vartheta_{\beta_i}) \otimes \left(1 - \prod_{j=1}^{t+1} \left(1 - \zeta_{\mathcal{A}_{ij}}\right)^{\omega_j}, \prod_{j=1}^{t+1} \vartheta_{\mathcal{A}_{ij}}^{\omega_j}\right) \\
= \; & \left(\zeta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{t+1} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j}\right), \vartheta_{\beta_i} + \prod_{j=1}^{t+1} \vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \vartheta_{\beta_i} \cdot \prod_{j=1}^{t+1} \vartheta_{\mathcal{A}_{ij}}^{\omega_j}\right)
\end{aligned}$$

It shows that the result holds for $m = t+1$, hence by mathematical induction, it holds for all positive integer m. □

Further, from this operator, it has been observed from the GWA operator that it is bounded, idempotent and monotonic which can be demonstrated as follows.

Property 11.2.1. (Idempotency:) If $\mathcal{A}_{ij} = \mathcal{A}_i$ for all j then, $\mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \beta_i \otimes \mathcal{A}_i$

Proof. Since $\mathcal{A}_{ij} = \mathcal{A}_i$ for all j i.e., $\zeta_{\mathcal{A}_{ij}} = \zeta_{\mathcal{A}_i}$ and $\vartheta_{\mathcal{A}_{ij}} = \vartheta_{\mathcal{A}_i}$, then from Eq. (11.3), we have

$$\begin{aligned}
\mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) &= \left(\zeta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_i})^{\omega_j}\right), \vartheta_{\beta_i} + \prod_{j=1}^{m} \vartheta_{\mathcal{A}_i}^{\omega_j} - \vartheta_{\beta_i} \cdot \prod_{j=1}^{m} \vartheta_{\mathcal{A}_i}^{\omega_j}\right) \\
&= \left(\zeta_{\beta_i} \cdot \left(1 - (1 - \zeta_{\mathcal{A}_i})^{\sum_{j=1}^{m} \omega_j}\right), \quad \vartheta_{\beta_i} + \vartheta_{\mathcal{A}_i}^{\sum_{j=1}^{m} \omega_j} - \vartheta_{\beta_i} \cdot \vartheta_{\mathcal{A}_i}^{\sum_{j=1}^{m} \omega_j}\right) \\
&= \Big(\zeta_{\beta_i} \cdot \big(1 - (1 - \zeta_{\mathcal{A}_i})\big), \quad \vartheta_{\beta_i} + \vartheta_{\mathcal{A}_i} - \vartheta_{\beta_i} \cdot \vartheta_{\mathcal{A}_i}\Big) \\
&= \Big(\zeta_{\beta_i} \cdot \zeta_{\mathcal{A}_i}, \quad \vartheta_{\beta_i} + \vartheta_{\mathcal{A}_i} - \vartheta_{\beta_i} \cdot \vartheta_{\mathcal{A}_i}\Big) \\
&= (\zeta_{\beta_i}, \vartheta_{\beta_i}) \otimes (\zeta_{\mathcal{A}_i}, \vartheta_{\mathcal{A}_i}) \\
&= \beta_i \otimes \mathcal{A}_i
\end{aligned}$$

Hence, the result. □

Property 11.2.2. (Boundedness:) Let $\mathcal{A}_i^- = \big(\zeta_{\beta_i \otimes \mathcal{A}_{ij}}^{\min}, \vartheta_{\beta_i \otimes \mathcal{A}_{ij}}^{\max}\big)$ and $\mathcal{A}_i^+ = \big(\zeta_{\beta_i \otimes \mathcal{A}_{ij}}^{\max}, \vartheta_{\beta_i \otimes \mathcal{A}_{ij}}^{\min}\big)$ then,

$$\mathcal{A}_i^- \leq \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \leq \mathcal{A}_i^+$$

Proof. Since $\mathcal{A}_{ij} = (\zeta_{\mathcal{A}_{ij}}, \vartheta_{\mathcal{A}_{ij}})$ and $\beta_i = (\zeta_{\beta_i}, \vartheta_{\beta_i})$ be IFNs, so for all i, j, $\beta_i \otimes \mathcal{A}_{ij} = \big(\zeta_{\beta_i} \cdot \zeta_{\mathcal{A}_{ij}},\ \ 1 - \big(1 - \vartheta_{\beta_i}\big)\big(1 - \vartheta_{\mathcal{A}_{ij}}\big)\big)$. Denote $\zeta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} = \zeta^{\min}_{\beta_i} \cdot \zeta^{\min}_{\mathcal{A}_{ij}}$, $\zeta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} = \zeta^{\max}_{\beta_i} \cdot \zeta^{\max}_{\mathcal{A}_{ij}}$, $\vartheta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} = 1 - (1 - \vartheta^{\min}_{\beta_i})(1 - \vartheta^{\min}_{\mathcal{A}_{ij}})$ and $\vartheta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} = 1 - (1 - \vartheta^{\max}_{\beta_i})(1 - \vartheta^{\max}_{\mathcal{A}_{ij}})$. Now, for $\mathcal{A}_{ij} = (\zeta_{\mathcal{A}_{ij}}, \vartheta_{\mathcal{A}_{ij}})$ for all i, j, we have, $\zeta^{\min}_{\mathcal{A}_{ij}} \leq \zeta_{\mathcal{A}_{ij}} \leq \zeta^{\max}_{\mathcal{A}_{ij}} \Leftrightarrow (1 - \zeta^{\max}_{\mathcal{A}_{ij}}) \leq (1 - \zeta_{\mathcal{A}_{ij}}) \leq (1 - \zeta^{\min}_{\mathcal{A}_{ij}})$ $\Leftrightarrow (1 - \zeta^{\max}_{\mathcal{A}_{ij}})^{\sum_{j=1}^{m} \omega_j} \leq \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j} \leq (1 - \zeta^{\min}_{\mathcal{A}_{ij}})^{\sum_{j=1}^{m} \omega_j} \Leftrightarrow 1 - (1 - \zeta^{\min}_{\mathcal{A}_{ij}}) \leq 1 - \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j} \leq 1 - (1 - \zeta^{\max}_{\mathcal{A}_{ij}})$ i.e., $\zeta^{\min}_{\mathcal{A}_{ij}} \leq 1 - \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j} \leq \zeta^{\max}_{\mathcal{A}_{ij}}$. Also, $\zeta^{\min}_{\beta_i} \leq \zeta_{\beta_i} \leq \zeta^{\max}_{\beta_i}$ which implies that $\zeta^{\min}_{\beta_i} \cdot \zeta^{\min}_{\mathcal{A}_{ij}} \leq \zeta_{\beta_i} \cdot \Big(1 - \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j}\Big) \leq \zeta^{\max}_{\beta_i} \cdot \zeta^{\max}_{\mathcal{A}_{ij}}$, i.e.,

$$\zeta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} \leq \zeta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{m} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j}\right) \leq \zeta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} \tag{11.4}$$

Furthermore, $\vartheta^{\min}_{\mathcal{A}_{ij}} \leq \vartheta_{\mathcal{A}_{ij}} \leq \vartheta^{\max}_{\mathcal{A}_{ij}} \Leftrightarrow \vartheta^{\min}_{\mathcal{A}_{ij}} \leq \prod_{j=1}^{m} \vartheta^{\omega_j}_{\mathcal{A}_{ij}} \leq \vartheta^{\max}_{\mathcal{A}_{ij}} \Leftrightarrow 1 - \vartheta^{\max}_{\mathcal{A}_{ij}} \leq 1 - \prod_{j=1}^{m} \vartheta^{\omega_j}_{\mathcal{A}_{ij}} \leq 1 - \vartheta^{\min}_{\mathcal{A}_{ij}}$. Similarly, $1 - \vartheta^{\max}_{\beta_i} \leq 1 - \vartheta_{\beta_i} \leq 1 - \vartheta^{\min}_{\beta_i}$ and hence we get $1 - (1 - \vartheta^{\min}_{\beta_i})(1 - \vartheta^{\min}_{\mathcal{A}_{ij}}) \leq 1 - (1 - \vartheta_{\beta_i})\big(1 - \prod_{j=1}^{m} \vartheta^{\omega_j}_{\mathcal{A}_{ij}}\big) \leq 1 - (1 - \vartheta^{\max}_{\beta_i})(1 - \vartheta^{\max}_{\mathcal{A}_{ij}})$, i.e.,

$$\vartheta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} \leq \vartheta_{\beta_i} + \prod_{j=1}^{m} \vartheta^{\omega_j}_{\mathcal{A}_{ij}} - \vartheta_{\beta_i} \cdot \Big(\prod_{j=1}^{m} \vartheta^{\omega_j}_{\mathcal{A}_{ij}}\Big) \leq \vartheta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} \tag{11.5}$$

Let $\alpha \equiv \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = (\zeta_\alpha, \vartheta_\alpha)$ then from the Eqs. (11.4) and (11.5), we get

$$\zeta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} \leq \zeta_\alpha \leq \zeta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} \quad ; \quad \vartheta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} \leq \vartheta_\alpha \leq \vartheta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}}$$

Take $\mathcal{A}^-_i = (\zeta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}}, \vartheta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}})$ and $\mathcal{A}^+_i = (\zeta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}}, \vartheta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}})$ and thus, by definition of score function, we get $\mathcal{S}(\alpha) = \zeta_\alpha - \vartheta_\alpha \leq \zeta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} - \vartheta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} = \mathcal{S}(\mathcal{A}^+_i)$ and $\mathcal{S}(\alpha) = \zeta_\alpha - \vartheta_\alpha \geq \zeta^{\min}_{\beta_i \otimes \mathcal{A}_{ij}} - \vartheta^{\max}_{\beta_i \otimes \mathcal{A}_{ij}} = \mathcal{S}(\mathcal{A}^-_i)$ and hence by order relation between two these numbers, we have $\mathcal{A}^-_i \leq \mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \leq \mathcal{A}^+_i$.

□

Property 11.2.3. (Monotonicity:) Let $\mathcal{A}'_{ij}$ be an another collection of IFSNs such that $\mathcal{A}_{ij} \leq \mathcal{A}'_{ij}$ for all i, j, then

$$\mathrm{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \leq \mathrm{GWA}(\mathcal{A}'_{i1}, \mathcal{A}'_{i2}, \ldots, \mathcal{A}'_{im})$$

Proof. Proof follows from property 11.2.2, so we omit here. □

Property 11.2.4. If the preferences of the senior expert β towards the alternative $\mathcal{F}_i(e) = \{\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}\} \in K^{\mathcal{X}}$ over the universal set $(\mathcal{X}, \mathcal{E})$ is given by $\beta_i(e) = (1, 0)$ for all $e \in \mathcal{E}$, then

$$\text{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \text{IFWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im})$$

Property 11.2.5. If the preferences of the senior expert β towards the alternative $\mathcal{F}_i(e) = \{\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}\} \in K^{\mathcal{X}}$ over the universal set $(\mathcal{X}, \mathcal{E})$ is given by $\beta_i(e) = (0, 1)$ for all $e \in \mathcal{E}$, then

$$\text{GWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = (0, 1)$$

Proof. Omitted. □

Definition 11.2.5. Let $\mathcal{F}_{\beta_i}(e) = (\mathcal{F}_i(e), \beta_i(e))$, $(i = 1, 2, \ldots, n)$ be any GIFSS over $(\mathcal{X}, \mathcal{E})$, where $\mathcal{F}_i(e) = \{\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}\} \in K^{\mathcal{X}}$ and $\beta_i(e) = (\zeta_{\beta_i}, \vartheta_{\beta_i})$ is an expert assessment on $\mathcal{X}$ for the i^{th} alternative $\mathcal{F}_i(e)$. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the normalized weight vector of $\mathcal{F}_i(e)$ and let GWG : $\Omega_s^n \to \Omega_s$, IFWG : $\Omega^n \to \Omega$. If

$$\text{GWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \;=\; \beta_i(e) \otimes \text{IFWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \tag{11.6}$$

then GWG is called GIFSS weighted geometric operator.

Theorem 11.2.2. If $\mathcal{A}_{ij} = (\zeta_{\mathcal{A}_{ij}}, \vartheta_{\mathcal{A}_{ij}})$, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$ be collection of IFNs, then the aggregated value by GWG operator is given by

$$\begin{aligned} & \text{GWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \qquad (11.7) \\ = & \left(\zeta_{\beta_i} \cdot \prod_{j=1}^{m} \zeta_{\mathcal{A}_{ij}}^{\omega_j}, \;\; \vartheta_{\beta_i} + \left(1 - \prod_{j=1}^{m} (1 - \vartheta_{\mathcal{A}_{ij}})^{\omega_j}\right) - \vartheta_{\beta_i} \cdot \left(1 - \prod_{j=1}^{m} (1 - \vartheta_{\mathcal{A}_{ij}})^{\omega_j}\right)\right) \end{aligned}$$

Proof. Follows from Theorem 11.2.1, so we omit here. □

Further, as similar to GWA operator, it has been observed that GWG operator also satisfies the property of boundedness, idempotent and monotonicity. Also, we have

Property 11.2.6. If the preferences of the senior expert β towards the alternative $\mathcal{F}_i(e)$ over $(\mathcal{X}, \mathcal{E})$ is given as $\beta_i(e) = (1, 0)$ for all $e \in \mathcal{E}$, then

$$\text{GWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \text{IFWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im})$$

Property 11.2.7. If the preferences of the senior expert β towards the alternative $\mathcal{F}_i(e)$ over $(\mathcal{X}, \mathcal{E})$ is given as $\beta_i(e) = (0, 1)$ for all $e \in \mathcal{E}$, then

$$\mathrm{GWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = (0, 1)$$

11.2.2 Aggregation operator for GGIFSS

In the above section, we have extended the above AOs by considering the experience of more than one experts on the elements of $\mathcal{X}$ in $\mathcal{F}(e)$ for aggregating the various preferences of the decision-makers in a more effective way. For it, we have presented a weighted averaging and geometric AOs for a collection of group generalized intuitionistic fuzzy soft set (GGIFSS) $\mathcal{F}_{\mathcal{G}_i}(e) = (\mathcal{F}_i(e), \mathcal{G}_i(e))$, $(i = 1, 2, \ldots, n)$ and are denoted by GGWA and GGWG respectively.

Definition 11.2.6. A group generalized intuitionistic fuzzy soft set (GGIFSS), $\mathcal{F}_{\mathcal{G}}$, over the soft universe $(\mathcal{X}, \mathcal{E})$ is defined as the mapping $\mathcal{F}_{\mathcal{G}} : \mathcal{E} \to K^{\mathcal{X}} \times IF$ then, for every $e \in \mathcal{E}$, we have

$$\mathcal{F}_{\mathcal{G}}(e) = (\mathcal{F}(e), \mathcal{G}_{\beta}(e)) \tag{11.8}$$

where $\mathcal{F}(e) \in K^{\mathcal{X}}$ and $\mathcal{G}_{\beta}(e) \in IF$. Here $\mathcal{G} = \{\beta_1, \beta_2, \ldots, \beta_l\}$ is an intuitionistic fuzzy subset of set of the parameter $\mathcal{E}$ and $\mathcal{G}_{\beta}(e)$ denotes the opinion of experts on the elements of $\mathcal{X}$ in $\mathcal{F}(e)$.

The above defined GGIFSS has been explained with a following example.

Example 11.2.2. Consider the data as given in Example 11.2.1 and their relative rating values for evaluating the different preferences of the houses with respect of the parameters $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ are denoted by $\mathcal{F}(e)$ for $e \in \mathcal{E}$. Further, if we assume that there are three senior experts, denoted by $\mathcal{G} = \{\beta_1, \beta_2, \beta_3\}$, which gives their opinion toward the assessment of the rating values of the alternatives in $\mathcal{F}(e)$ in the form of IFNs, then for any $e \in \mathcal{E}$, the GGIFSS $\mathcal{F}_{\mathcal{G}}(e)$ is given as:

$$\mathcal{F}_{\mathcal{G}}(e_1) = \left\{ \left(\frac{m_1}{(0.2,0.6)}, \frac{m_2}{(0.3,0.5)}, \frac{m_3}{(0.6,0.3)}, \frac{m_4}{(0.5,0.4)}, \frac{m_5}{(0.7,0.2)} \right), \left(\frac{\beta_1}{(0.4,0.5)}, \frac{\beta_2}{(0.6,0.3)}, \frac{\beta_3}{(0.5,0.3)} \right) \right\},$$

$$\mathcal{F}_{\mathcal{G}}(e_2)=\left\{\left(\frac{m_1}{(0.4,0.5)},\frac{m_2}{(0.1,0.7)},\frac{m_3}{(0.3,0.7)},\frac{m_4}{(0.4,0.3)},\frac{m_5}{(0.2,0.6)}\right),\left(\frac{\beta_1}{(0.7,0.3)},\frac{\beta_2}{(0.4,0.5)},\frac{\beta_3}{(0.5,0.2)}\right)\right\},$$
$$\mathcal{F}_{\mathcal{G}}(e_3)=\left\{\left(\frac{m_1}{(0.1,0.7)},\frac{m_2}{(0.2,0.5)},\frac{m_3}{(0.6,0.4)},\frac{m_4}{(0.4,0.4)},\frac{m_5}{(0.5,0.2)}\right),\left(\frac{\beta_1}{(0.6,0.3)},\frac{\beta_2}{(0.7,0.1)},\frac{\beta_1}{(0.6,0.2)}\right)\right\},$$
$$\mathcal{F}_{\mathcal{G}}(e_4)=\left\{\left(\frac{m_1}{(0.3,0.5)},\frac{m_2}{(0.6,0.2)},\frac{m_3}{(0.5,0.3)},\frac{m_4}{(0.7,0.3)},\frac{m_5}{(0.8,0.2)}\right),\left(\frac{\beta_1}{(0.3,0.5)},\frac{\beta_2}{(0.4,0.6)},\frac{\beta_3}{(0.2,0.5)}\right)\right\}$$

where $\beta_1(e_1)=(0.4,0.5)$, $\beta_2(e_1)=(0.6,0.3)$ and $\beta_3(e_1)=(0.5,0.3)$ are the assessment of the experts towards the alternative under e_1 parameter. Similarly for the others assessment.

Definition 11.2.7. Let $\mathcal{F}_{\mathcal{G}}(e)=(\mathcal{F}(e),\mathcal{G}(e))$ and $\mathcal{C}_{\mathcal{D}}(e)=(\mathcal{C}(e),\mathcal{D}(e))$ be two GGIFSS over the universe $(\mathcal{X},\mathcal{E})$ then

(i) $\mathcal{F}_{\mathcal{G}}\subseteq\mathcal{C}_{\mathcal{D}}$, if

- $\mathcal{G}(e)$ is intuitionistic fuzzy subset of $\mathcal{D}(e)$ for all $e\in\mathcal{E}$,
- $\mathcal{F}(e)$ is intuitionistic fuzzy subset of $\mathcal{C}(e)$ for all $e\in\mathcal{E}$.

(ii) $\mathcal{F}_{\mathcal{G}}=\mathcal{C}_{\mathcal{D}}$, if

- $\mathcal{F}_{\mathcal{G}}(e)$ is intuitionistic fuzzy subset of $\mathcal{C}_{\mathcal{D}}(e)$ for all $e\in\mathcal{E}$,
- $\mathcal{C}_{\mathcal{D}}(e)$ is intuitionistic fuzzy subset of $\mathcal{F}_{\mathcal{G}}(e)$ for all $e\in\mathcal{E}$.

Definition 11.2.8. The complement of $\mathcal{F}_{\mathcal{G}}$, denoted by $\mathcal{F}_{\mathcal{G}}^c$, is defined as $\mathcal{F}_{\mathcal{G}}^c(e)=(\mathcal{F}^c(e),\mathcal{G}^c(e))$, for all $e\in\mathcal{E}$.

Definition 11.2.9. Let $\mathcal{F}_{\mathcal{G}_i}(e)=(\mathcal{F}_i(e),\mathcal{G}_i(e))$, $(i=1,2,\ldots,n)$ be any GGIFSSs over $(\mathcal{X},\mathcal{E})$, where $\mathcal{F}_i(e)=\{\mathcal{A}_{i1},\mathcal{A}_{i2},\ldots,\mathcal{A}_{im}\}\in K^{\mathcal{X}}$ and $\mathcal{G}_i(e)=\{\beta_{i1},\beta_{i2},\ldots,\beta_{il}\}$ be the experts assessment on $\mathcal{X}$ for alternative $\mathcal{F}_i(e)$. Assume that $\omega=(\omega_1,\omega_2,\ldots,\omega_m)^T$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_l)^T$, respectively, are the normalized weight vectors of $\mathcal{F}_i(e)$, $\mathcal{G}_i(e)$ and let GGWA : $\Omega_S^n\to\Omega_S$, IFWA : $\Omega^n\to\Omega$. If

$$\mathrm{GGWA}(\mathcal{A}_{i1},\mathcal{A}_{i2},\ldots,\mathcal{A}_{im})=\mathrm{IFWA}(\beta_{i1},\beta_{i2},\ldots,\beta_{il})\otimes\mathrm{IFWA}(\mathcal{A}_{i1},\mathcal{A}_{i2},\ldots,\mathcal{A}_{im})\quad(11.9)$$

where Ω_S, Ω be the families of GGIFSS and IFSs respectively, then GGWA is called as GGIFSS weighted averaging operator.

Theorem 11.2.3. If $\mathcal{A}_{ij} = (\zeta_{\mathcal{A}_{ij}}, \vartheta_{\mathcal{A}_{ij}})$ and $\beta_{ik} = (\zeta_{\beta_{ik}}, \vartheta_{\beta_{ik}})$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, l)$ are IFNs then the aggregated value by using GGWA operator defined in Definition 11.2.9 is given by

$$\text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \begin{pmatrix} \left(1 - \prod_{k=1}^{l}(1-\zeta_{\beta_{ik}})^{\xi_k}\right) \cdot \left(1 - \prod_{j=1}^{m}(1-\zeta_{\mathcal{A}_{ij}})^{w_j}\right), \\ \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} + \prod_{j=1}^{m}\vartheta_{\mathcal{A}_{ij}}^{w_j} - \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} \cdot \prod_{j=1}^{m}\vartheta_{\mathcal{A}_{ij}}^{w_j} \end{pmatrix} \tag{11.10}$$

Proof. We will prove the result by using mathematical induction on m. Then, the following steps of the mathematical induction have been followed:

Step 1: For $m = 2$, we have

$$\begin{aligned} \text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}) &= \text{IFWA}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{il}) \otimes \text{IFWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}) \\ &= \left(1 - \prod_{k=1}^{l}(1-\zeta_{\beta_{ik}})^{\xi_k}, \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k}\right) \\ &\otimes \left(1 - (1-\zeta_{\mathcal{A}_{i1}})^{\omega_1}(1-\zeta_{\mathcal{A}_{i2}})^{\omega_2}, \vartheta_{\mathcal{A}_{i1}}^{\omega_1}\vartheta_{\mathcal{A}_{i2}}^{\omega_2}\right) \\ &= \begin{pmatrix} \left(1 - \prod_{k=1}^{l}(1-\zeta_{\beta_{ik}})^{\xi_k}\right) \cdot \left(1 - (1-\zeta_{\mathcal{A}_{i1}})^{\omega_1}(1-\zeta_{\mathcal{A}_{i2}})^{\omega_2}\right), \\ \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} + \vartheta_{\mathcal{A}_{i1}}^{\omega_1}\vartheta_{\mathcal{A}_{i2}}^{\omega_2} - \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} \cdot \vartheta_{\mathcal{A}_{i1}}^{\omega_1}\vartheta_{\mathcal{A}_{i2}}^{\omega_2} \end{pmatrix} \\ &= \begin{pmatrix} \left(1 - \prod_{k=1}^{l}(1-\zeta_{\beta_{ik}})^{\xi_k}\right) \cdot \left(1 - \prod_{j=1}^{2}(1-\zeta_{\mathcal{A}_{ij}})^{\omega_j}\right), \\ \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} + \prod_{j=1}^{2}\vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} \cdot \prod_{j=1}^{2}\vartheta_{\mathcal{A}_{ij}}^{\omega_j} \end{pmatrix} \end{aligned}$$

Thus, results holds for $m = 2$.

Step 2: If Eq. (11.10) holds for $m = t$, i.e.,

$$\text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{it}) = \begin{pmatrix} \left(1 - \prod_{k=1}^{l}(1-\zeta_{\beta_{ik}})^{\xi_k}\right) \cdot \left(1 - \prod_{j=1}^{t}(1-\zeta_{\mathcal{A}_{ij}})^{\omega_j}\right), \\ \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} + \prod_{j=1}^{t}\vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \prod_{k=1}^{l}\vartheta_{\beta_{ik}}^{\xi_k} \cdot \prod_{j=1}^{t}\vartheta_{\mathcal{A}_{ij}}^{\omega_j} \end{pmatrix}$$

then for $m = t + 1$, we have

$$
\begin{aligned}
& \text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{i(t+1)}) \\
= \; & \text{IFWA}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{il}) \otimes \text{IFWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{i(t+1)}) \\
= \; & \left(1 - \prod_{k=1}^{l} (1 - \zeta_{\beta_{ik}})^{\xi_k}, \prod_{k=1}^{l} \vartheta_{\beta_{ik}}^{\xi_k}\right) \otimes \left(1 - \prod_{j=1}^{t+1} \left(1 - \zeta_{\mathcal{A}_{ij}}\right)^{\omega_j}, \prod_{j=1}^{t+1} \vartheta_{\mathcal{A}_{ij}}^{\omega_j}\right) \\
= \; & \left(\begin{array}{l} \left(1 - \prod_{k=1}^{l} (1 - \zeta_{\beta_{ik}})^{\xi_k}\right) \cdot \left(1 - \prod_{j=1}^{t+1} (1 - \zeta_{\mathcal{A}_{ij}})^{\omega_j}\right), \\ \prod_{k=1}^{l} \vartheta_{\beta_{ik}}^{\xi_k} + \prod_{j=1}^{t+1} \vartheta_{\mathcal{A}_{ij}}^{\omega_j} - \prod_{k=1}^{l} \vartheta_{\beta_{ik}}^{\xi_k} . \prod_{j=1}^{t+1} \vartheta_{\mathcal{A}_{ij}}^{\omega_j} \end{array} \right)
\end{aligned}
$$

Thus, Eq. (11.10) holds for $m = t+1$ and hence, by the principle of mathematical induction, result given in Eq. (11.10) holds for all positive integer m.

$\square$

Now, for a collection of IFSNs $\mathcal{F}_{\mathcal{G}_i}(e)$ over the $(\mathcal{X}, \mathcal{E})$, the proposed GGWA operator satisfies the idempotent, boundedness and monotonicity property, which are stated as follows:

Property 11.2.8. Let$\mathcal{F}_{\mathcal{G}_i}(e) = (\mathcal{F}_i(e), \mathcal{G}_i(e))$ be any GGIFSSs over $(\mathcal{X}, \mathcal{E})$ where $\mathcal{F}_i(e) = \{\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}\} \in K^{\mathcal{X}}$ and $\mathcal{G}_i(e) = \{\beta_{i1}, \beta_{i2}, \ldots, \beta_{il}\}$ be the experts assessment on $\mathcal{X}$ for alternative $\mathcal{F}_i(e)$. Then, we have

(P1) (Idempotency:) If $\mathcal{A}_{ij} = \mathcal{A}_i$ and $\beta_{ik} = \beta_i$ for all j, k then, we have

$$\text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \beta_i \otimes \mathcal{A}_i$$

(P2) (Boundedness:) If $\mathcal{A}^-$ and $\mathcal{A}^+$ be the lower and bound of the collections of IFSNs $\mathcal{F}_i(e)$, then we have

$$\mathcal{A}^- \leq \text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \leq \mathcal{A}^+$$

(P3) (Monotonicity:) If $\mathcal{F}'_{\mathcal{G}_i}(e) = (\mathcal{F}'_i(e), \mathcal{G}_i(e))$ be the another collection of IFSNs where $\mathcal{F}'_i(e) = \{\mathcal{A}'_{i1}, \mathcal{A}'_{i2}, \ldots, \mathcal{A}'_{im}\} \in K^{\mathcal{X}}$ and $\mathcal{G}_i(e) = \{\beta_{i1}, \beta_{i2}, \ldots, \beta_{il}\}$ be the experts assessment on $\mathcal{X}$ for alternative $\mathcal{F}'_i(e)$ such that $\mathcal{A}_{ij} \leq \mathcal{A}'_{ij}$ for all i, j then,

$$\text{GGWA}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \leq \text{GGWA}(\mathcal{A}'_{i1}, \mathcal{A}'_{i2}, \ldots, \mathcal{A}'_{im})$$

Definition 11.2.10. Let $\mathcal{F}_{\mathcal{G}_i}(e) = (\mathcal{F}_i(e), \mathcal{G}_i(e))$, $(i = 1, 2, \ldots, n)$ be any GGIFSSs over $(\mathcal{X}, \mathcal{E})$, where $\mathcal{F}_i(e) = \{\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}\} \in K^{\mathcal{X}}$ and $\mathcal{G}_i(e) = \{\beta_{i1}, \beta_{i2}, \ldots, \beta_{il}\}$ be the experts assessment on $\mathcal{X}$ for alternative $\mathcal{F}_i(e)$. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_l)^T$, respectively, are the normalized weight vectors of $\mathcal{F}_i(e)$, $\mathcal{G}_i(e)$ and let GGWG : $\Omega_S^n \to \Omega_S$, IFWG : $\Omega^n \to \Omega$. If

$$\begin{aligned} &\mathrm{GGWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \mathrm{IFWG}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{il}) \otimes \mathrm{IFWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) \\ = &\left(\prod_{k=1}^{l} \zeta_{\beta_{ik}}^{\xi_k} \prod_{j=1}^{m} \zeta_{\mathcal{A}_{ij}}^{\omega_j}, \quad \left(1 - \prod_{k=1}^{l} (1 - \vartheta_{\beta_{ik}})^{\xi_k}\right) + \left(1 - \prod_{j=1}^{m} (1 - \vartheta_{\mathcal{A}_{ij}})^{\omega_j}\right) \right. \\ &\left. - \left(1 - \prod_{k=1}^{l} (1 - \vartheta_{\beta_{ik}})^{\xi_k}\right) \cdot \left(1 - \prod_{j=1}^{m} (1 - \vartheta_{\mathcal{A}_{ij}})^{w_j}\right) \right) \end{aligned} \tag{11.11}$$

where Ω_S, Ω is the family of GGIFSS and IFSs, then GGWG is called as GGIFSS weighted geometric operator.

The proposed GGWG operator also follows the same properties as that of GGWA operator. In addition to these, we have

Property 11.2.9. If the expert assessment towards the alternatives are represented as $\beta_{ik} = (1, 0)$ for $k = 1, 2, \ldots, l$ then, we have

$$\mathrm{GGWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = \mathrm{IFWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im})$$

Property 11.2.10. If the expert assessment towards the alternatives $\mathcal{F}_i(e)$ is taken as $\beta_{ik} = (0, 1)$ for $k = 1, 2, \ldots, l$, then

$$\mathrm{GGWG}(\mathcal{A}_{i1}, \mathcal{A}_{i2}, \ldots, \mathcal{A}_{im}) = (0, 1)$$

11.3 Proposed DM method based on the operators

In this section, a DM method by using above defined AOs for GIFSSs have been presented.

Consider a DM problem with a collection of alternatives, $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(n)}$, and the parameters $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. A committee of the experts has been constituted which are going to evaluate each alternative $\mathcal{A}^{(i)} (i = 1, 2, \ldots, n)$ against the parameter $e_j (j = 1, 2, \ldots, m)$ and given their rating in terms of IFNs denoted by $\mathcal{A}_{ij} = (\zeta_{\mathcal{A}_{ij}}, \vartheta_{\mathcal{A}_{ij}})$.

Let $\{\beta_1, \beta_2, \ldots, \beta_l\}$ be the group of senior persons/experts which furnish their assessment towards the preferences of the each alternatives $\mathcal{A}_{ij}$ in terms of IFNs $\beta_k = (\zeta_{\beta_k}, \vartheta_{\beta_k})$, $(k = 1, 2, \ldots, l)$. Then, in the following, we develop an approach based on the proposed operators with intuitionistic fuzzy soft information, which involve the following steps:

Step 1: Create a framework of the experts' assessment related to each alternative in the form IFNs and hence construct their corresponding decision matrix $[\mathcal{F}]_{n\times m}$.

Step 2: Obtain the generalization parameter matrix $[\mathcal{G}_\beta]_{n\times l}$ by incorporating the preferences of the senior persons/experts' committee on each alternative.

Step 3: Add the matrix $[\mathcal{G}_\beta]$ into assessment matrix $[\mathcal{F}]$ for i^{th} alternative $\mathcal{A}^{(i)}$ and construct the new matrix $[\mathcal{F}_\mathcal{G}]_{n\times(m+l)}$.

Step 4: Compute the net aggregation evaluation of the matrix $\mathcal{F}_\mathcal{G}$ by using GGWA or GGWG operator, given in Eq. (11.10) or Eq. (11.11), of alternative $\mathcal{A}^{(i)}$ and its results are denoted by $r^{(i)}$, $i = 1, 2, \ldots, n$.

Step 5: Rank the alternatives based on the descending values of their score values $r^{(i)}$.

11.4 Illustrative Example

The presented approach has been demonstrated with a numerical example to show the superiority of it.

11.4.1 A case study

Consider a university which wants to appoint an outstanding professor in their respective department from the six suitable candidates denoted by, $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$, $\mathcal{A}^{(3)}$, $\mathcal{A}^{(4)}$, $\mathcal{A}^{(5)}$ and $\mathcal{A}^{(6)}$. These candidates have been screened out with respect to the five different parameters as $\mathcal{E}$ ={"qualification" (e_1), "teaching experience"(e_2), "research experience"(e_3), "number of publications"(e_4), "teaching capability"(e_5) } whose weight vector is ω =(0.19, 0.17, 0.19, 0.23, 0.22). The evaluation of these candidates against the parameters is measured by the screening committee and arranged their rating values in the form of the IFNs

Table 11.1: Intuitionistic fuzzy assessment matrix $[\mathcal{F}]_{6\times5}$

	e_1	e_2	e_3	e_4	e_5
$\mathcal{A}^{(1)}$	$(0.92, 0)$	$(0.92, 0)$	$(1, 0)$	$(0.73, 0.21)$	$(0.98, 0)$
$\mathcal{A}^{(2)}$	$(0.52, 0.29)$	$(0.56, 0.19)$	$(0.92, 0.03)$	$(0.67, 0.19)$	$(0.54, 0.31)$
$\mathcal{A}^{(3)}$	$(0.78, 0.06)$	$(0.24, 0.47)$	$(0.75, 0.16)$	$(0.33, 0.48)$	$(0.67, 0.24)$
$\mathcal{A}^{(4)}$	$(0.52, 0.39)$	$(0.35, 0.59)$	$(0.88, 0.03)$	$(0.55, 0.26)$	$(0.96, 0)$
$\mathcal{A}^{(5)}$	$(0.38, 0.33)$	$(0.72, 0.10)$	$(0.96, 0)$	$(0.13, 0.49)$	$(0.96, 0)$
$\mathcal{A}^{(6)}$	$(0.02, 0.82)$	$(0.28, 0.47)$	$(0.81, 0.12)$	$(0.29, 0.48)$	$(0.67, 0.22)$

and are summarized in Table 11.1. Further, assume that there are three senior experts from the university denoted by $\mathcal{G} = \{\beta_1, \beta_2, \beta_3\}$, who provides their assessments toward the candidates $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)}, \mathcal{A}^{(4)}, \mathcal{A}^{(5)}$ and $\mathcal{A}^{(6)}$, which reflect their attitude with the descriptions, in the form of IFNs and are their rating values are summarized in Table 11.2. Also, assume that the weight vector of these three senior experts is $\xi = (0.4, 0.3, 0.3)^T$.

Table 11.2: The generalization parameter matrix $\mathcal{G}_\beta$ given by the experts

	β_1	β_2	β_3
$\mathcal{A}^{(1)}$	$(0.92, 0)$	$(0.96, 0)$	$(0.90, 0)$
$\mathcal{A}^{(2)}$	$(0.71, 0.16)$	$(0.82, 0.1)$	$(0.74, 0.15)$
$\mathcal{A}^{(3)}$	$(0.62, 0.20)$	$(0.65, 0.20)$	$(0.65, 0.27)$
$\mathcal{A}^{(4)}$	$(0.78, 0)$	$(0.84, 0.07)$	$(0.62, 0.30)$
$\mathcal{A}^{(5)}$	$(0.75, 0.20)$	$(0.80, 0)$	$(0.68, 0.21)$
$\mathcal{A}^{(6)}$	$(0.82, 0.08)$	$(0.96, 0)$	$(0.9, 0)$

Then, the following steps have been executed for finding the most desirable candidate for the post of an outstanding professor, in their respective department, by using GGWA operator as follows:

Step 1: Construct the description matrix $[\mathcal{F}]_{6\times5}$ of the alternatives in terms of IFNs, as shown in Table 11.1.

Step 2: Evaluation matrix of the three experts' toward the alternatives is assessed in terms of decision matrix $[\mathcal{G}]_{6\times3}$, as shown in Table 11.2.

Step 3: Create a new generalized IFSS matrix $[\mathcal{F}_{\mathcal{G}}]_{6\times 8}$ as shown in Table 11.3.

Table 11.3: GGIFSS assessment matrix $[\mathcal{F}_{\mathcal{G}}]_{6\times 8}$

	e_1	e_2	e_3	e_4	e_5	β_1	β_2	β_3
$\mathcal{A}^{(1)}$	$(0.92, 0)$	$(0.92, 0)$	$(1, 0)$	$(0.73, 0.21)$	$(0.98, 0)$	$(0.92, 0)$	$(0.96, 0)$	$(0.90, 0)$
$\mathcal{A}^{(2)}$	$(0.52, 0.29)$	$(0.56, 0.19)$	$(0.92, 0.03)$	$(0.67, 0.19)$	$(0.54, 0.31)$	$(0.71, 0.16)$	$(0.82, 0.1)$	$(0.74, 0.15)$
$\mathcal{A}^{(3)}$	$(0.78, 0.06)$	$(0.24, 0.47)$	$(0.75, 0.16)$	$(0.33, 0.48)$	$(0.67, 0.24)$	$(0.62, 0.20)$	$(0.65, 0.20)$	$(0.65, 0.27)$
$\mathcal{A}^{(4)}$	$(0.52, 0.39)$	$(0.35, 0.59)$	$(0.88, 0.03)$	$(0.55, 0.26)$	$(0.96, 0)$	$(0.78, 0)$	$(0.84, 0.07)$	$(0.62, 0.30)$
$\mathcal{A}^{(5)}$	$(0.38, 0.33)$	$(0.72, 0.10)$	$(0.96, 0)$	$(0.13, 0.49)$	$(0.96, 0)$	$(0.75, 0.20)$	$(0.80, 0)$	$(0.68, 0.21)$
$\mathcal{A}^{(6)}$	$(0.02, 0.82)$	$(0.28, 0.47)$	$(0.81, 0.12)$	$(0.29, 0.48)$	$(0.67, 0.22)$	$(0.82, 0.08)$	$(0.96, 0)$	$(0.9, 0)$

Step 4: Aggregate these different preferences of the decision-maker corresponding to the weight vectors ω and ξ by using GGWA operator, as given in Eq. (11.10), and hence, we get $r^{(1)} = (0.9305, 0)$, $r^{(2)} = (0.5253, 0.2758)$, $r^{(3)} = (0.3874, 0.3942)$, $r^{(4)} = (0.5951, 0)$, $r^{(5)} = (0.6058, 0)$ and $r^{(6)} = (0.4541, 0.3427)$.

Step 5: Score value of each candidate is obtained as $\mathcal{S}(r^{(1)}) = 0.9305$, $\mathcal{S}(r^{(2)}) = 0.2496$, $\mathcal{S}(r^{(3)}) = -0.0068$, $\mathcal{S}(r^{(4)}) = 0.5951$, $\mathcal{S}(r^{(5)}) = 0.6058$ and $\mathcal{S}(r^{(6)}) = 0.1114$ and thus ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(6)} \succ \mathcal{A}^{(3)}$. Therefore, the candidate $\mathcal{A}^{(1)}$ is the best one for the required post.

11.4.2 Comparison of results with those obtained through GWA

As it has been observed from the above results that the best candidate is $\mathcal{A}^{(1)}$ for the desired post. However, if we are interested to know the recommendation of an individual expert, then an analysis has been conducted for the considered problem by using GWA operator as given in Eq. (11.3), for each expert.

(i) If only expert-1 i.e., β_1 has been taken into the account during the analysis, then by following the proposed approach steps we get the score values of each alternative are $\mathcal{S}(r^{(1)}) = 0.9200$, $\mathcal{S}(r^{(2)}) = 0.1972$, $\mathcal{S}(r^{(3)}) = -0.0033$, $\mathcal{S}(r^{(4)}) = 0.6072$, $\mathcal{S}(r^{(5)}) = 0.4073$ and $\mathcal{S}(r^{(6)}) = 0.0166$. Thus, their ranking is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(6)} \succ \mathcal{A}^{(3)}$.

(ii) If only expert-2 i.e., β_2 has been taken into an account, then the score values of their corresponding aggregated values are $\mathcal{S}(r^{(1)}) = 0.9600$, $\mathcal{S}(r^{(2)}) = 0.3239$, $\mathcal{S}(r^{(3)}) = 0.0149$, $\mathcal{S}(r^{(4)}) = 0.5839$, $\mathcal{S}(r^{(5)}) = 0.6477$ and $\mathcal{S}(r^{(6)}) = 0.1396$. Therefore, ranking order of the candidate is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(6)} \succ \mathcal{A}^{(3)}$.

(iii) Finally, if only β_3 expert has been taken into an account, then their corresponding score value of each candidate is $\mathcal{S}(r^{(1)}) = 0.9000$, $\mathcal{S}(r^{(2)}) = 0.2264$, $\mathcal{S}(r^{(3)}) = -0.0394$, $\mathcal{S}(r^{(4)}) = 0.1827$, $\mathcal{S}(r^{(5)}) = 0.3406$ and $\mathcal{S}(r^{(6)}) = 0.1094$ and hence ranking is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(6)} \succ \mathcal{A}^{(3)}$.

As each expert has their own preferences and parameters due to their own knowledge, experience, etc., the ranking based on expert-1 preference is, obviously, different from the expert-2 or expert-3. But the best ($\mathcal{A}^{(1)}$) and worst ($\mathcal{A}^{(3)}$) candidates found by the GWA or GGWA operator are same.

11.4.3 Comparison of results with those obtained through IFSS

The existing approaches on the IFSS environment have been utilized here for finding the best alternative(s) and their results are summarized as follows:

(i) If we utilize the distance measure(l^s_{IFS}) as proposed by Cagman and Deli [21] on the considered problem, then corresponding to each alternative, the index values are obtained from their ideal alternative $\mathcal{B} = (1, 0)$ as $l^s_{IFS}(\mathcal{A}^{(1)}, \mathcal{B}) = 0.0660$, $l^s_{IFS}(\mathcal{A}^{(2)}, \mathcal{B}) = 0.2800$, $l^s_{IFS}(\mathcal{A}^{(3)}, \mathcal{B}) = 0.3640$, $l^s_{IFS}(\mathcal{A}^{(4)}, \mathcal{B}) = 0.3010$, $l^s_{IFS}(\mathcal{A}^{(5)}, \mathcal{B}) = 0.2770$ and $l^s_{IFS}(\mathcal{A}^{(6)}, \mathcal{B}) = 0.5040$. Thus, their ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$ and hence, it has been concluded that the best candidate is $\mathcal{A}^{(1)}$.

(ii) If we compute the best alternative based on the similarity measure(S) for IFSS as proposed by Rajarajeswari and Dhanalakshmi [161], then we have $S(\mathcal{A}^{(1)}, \mathcal{B}) = 0.8680$, $S(\mathcal{A}^{(2)}, \mathcal{B}) = 0.4400$, $S(\mathcal{A}^{(3)}, \mathcal{B}) = 0.4240$, $S(\mathcal{A}^{(4)}, \mathcal{B}) = 0.4940$, $S(\mathcal{A}^{(5)}, \mathcal{B}) = 0.5900$ and $S(\mathcal{A}^{(6)}, \mathcal{B}) = 0.4640$. Therefore, based on their measurement values the ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(6)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)}$.

(iii) If we apply distance measure(d_H) as proposed by Khalid and Abbas[98] to find the best candidate of the considered data from the ideal alternative $\mathcal{B} = (1, 0)$ then, their corresponding measurement values are $d_H(\mathcal{A}^{(1)}, \mathcal{B}) = 0.0900$, $d_H(\mathcal{A}^{(2)}, \mathcal{B}) = 0.3580$, $d_H(\mathcal{A}^{(3)}, \mathcal{B}) = 0.4460$, $d_H(\mathcal{A}^{(4)}, \mathcal{B}) = 0.3480$, $d_H(\mathcal{A}^{(5)}, \mathcal{B}) = 0.3700$ and $d_H(\mathcal{A}^{(6)}, \mathcal{B}) = 0.5860$. Therefore, their ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$ and hence $\mathcal{A}^{(1)}$ is the best candidate.

(iv) If we compute distance measure(d_{nH}) as given by Sarala and Suganya [168] to the considered data from the ideal alternative $\mathcal{B} = (1, 0)$ then, we get $d_{nH}(\mathcal{A}^{(1)}, \mathcal{B}) = 0.0660$, $d_{nH}(\mathcal{A}^{(2)}, \mathcal{B}) = 0.2800$, $d_{nH}(\mathcal{A}^{(3)}, \mathcal{B}) = 0.3640$, $d_{nH}(\mathcal{A}^{(4)}, \mathcal{B}) = 0.3010$, $d_{nH}(\mathcal{A}^{(5)}, \mathcal{B}) = 0.2770$ and $d_{nH}(\mathcal{A}^{(6)}, \mathcal{B}) = 0.5040$ and thus, their ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$.

(v) If we apply weighted similarity measure(W_{IFSS}) as proposed by Mutukumar and Krishnan [142], then we get their measurement values are $W_{IFSS}(\mathcal{A}^{(1)}, \mathcal{B}) = 0.2161$, $W_{IFSS}(\mathcal{A}^{(2)}, \mathcal{B}) = 0.2956$, $W_{IFSS}(\mathcal{A}^{(3)}, \mathcal{B}) = 0.3106$, $W_{IFSS}(\mathcal{A}^{(4)}, \mathcal{B}) = 0.2773$, $W_{IFSS}(\mathcal{A}^{(5)}, \mathcal{B}) = 0.2450$ and $W_{IFSS}(\mathcal{A}^{(6)}, \mathcal{B}) = 0.3308$. The ranking order corresponding to it is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$ and hence, $\mathcal{A}^{(1)}$ is the best alternative.

11.4.4 Comparison of results with those obtained through IFNs

On the other hand, if we ignore the additional judgements of the experts matrix $[\mathcal{G}_\beta]$ and evaluate the performance of the alternative by using decision matrix $[\mathcal{F}]$ only, then the following approaches as proposed by the authors [206, 209, 227, 231] have been utilized and obtained the following results.

(i) By aggregating the preference values of the considered problem using IFWA operator [227] approach, we get the overall score values for each alternative are $\mathcal{S}(r^{(1)}) = 1.0000$, $\mathcal{S}(r^{(2)}) = 0.5327$, $\mathcal{S}(r^{(3)}) = 0.3824$, $\mathcal{S}(r^{(4)}) = 0.7785$, $\mathcal{S}(r^{(5)}) = 0.8097$, $\mathcal{S}(r^{(4)}) = 0.1596$ and thus ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$.

(ii) If we utilize IFWG operator as proposed by Xu and Yager [231] to aggregate the different preferences of the decision-maker, then we get their corresponding score values are $\mathcal{S}(r^{(1)}) = 0.8459$, $\mathcal{S}(r^{(2)}) = 0.4165$, $\mathcal{S}(r^{(3)}) = 0.1980$, $\mathcal{S}(r^{(4)}) = 0.3486$,

$\mathcal{S}(r^{(5)}) = 0.2637$ and $\mathcal{S}(r^{(6)}) = -0.2313$. Thus, the ranking order of the alternative is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$.

(iii) If we utilize Wang and Liu [206] approach, then the score values corresponding to their approach are $\mathcal{S}(r^{(1)}) = 1.000$, $\mathcal{S}(r^{(2)}) = 0.5197$, $\mathcal{S}(r^{(3)}) = 0.3606$, $\mathcal{S}(r^{(4)}) = 0.7630$, $\mathcal{S}(r^{(5)}) = 0.7850$, $\mathcal{S}(r^{(6)}) = 0.1134$ and hence the ranking order is $\mathcal{A}^{(1)} \succ \mathcal{A}^{(5)} \succ \mathcal{A}^{(4)} \succ \mathcal{A}^{(2)} \succ \mathcal{A}^{(3)} \succ \mathcal{A}^{(6)}$.

(iv) If we utilize Wang and Liu [209] approach, then we get score values of each alternative are $\mathcal{S}(r^{(1)}) = 0.8544$, $\mathcal{S}(r^{(2)}) = 0.4277$, $\mathcal{S}(r^{(3)}) = 0.2266$, $\mathcal{S}(r^{(4)}) = 0.3876$, $\mathcal{S}(r^{(5)}) = 0.3277$, $\mathcal{S}(r^{(6)}) = -0.1809$. Thus, the best one is $\mathcal{A}^{(1)}$.

11.4.5 Superiority of the proposed approach

In this section, we have presented some counterexamples which show that the existing operators under the IFS, as well as IFSS, environment fails to rank the given alternatives while the proposed approach can overcome their shortcoming.

Example 11.4.1. Consider a DM problem by considering the two alternatives, namely $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which have been evaluated by the expert over the set of three parameters, say $\{e_1, e_2, e_3\}$ whose weight vector is $\omega = (0.24, 0.40, 0.36)^T$. Then, the objective of the problem is to find the best alternative under the given set. In order to do so, an expert evaluated these alternatives and gave their preferences in terms of IFNs which are summarized as follows:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}^{(1)} \\ \mathcal{A}^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 \quad & \quad\mathcal{G}_2\quad & \quad\mathcal{G}_3 \end{array} \\ \begin{bmatrix} (0.510, 0.361) & (0.66, 0.25) & (0.53, 0.29) \\ (0.532, 0.400) & (0.36, 0.52) & (0.76, 0.12) \end{bmatrix} \end{array}$$

Now, if we utilize IFWG operator [231] for aggregating these preferences then, the score values corresponding to these alternatives are 0.2949 and 0.2949 respectively. Therefore, we can see that both the alternatives have same score value hence we can't choose the best one. This is due to the fact that IFWG operator completely ignores the judgments of the senior experts' and thus they have completely made by his own views while evaluating the alternatives and hence lead to an unfair decision.

On the other hand, if we utilized the proposed GWA operator by adding preference of one senior expert

$$\beta = \begin{pmatrix} (0.9, 0.1) \\ (0.5, 0.4) \end{pmatrix}$$

towards the two different alternatives then, we get their relative score values as $\mathcal{S}(\mathcal{A}^{(1)}) = 0.1654$ and $\mathcal{S}(\mathcal{A}^{(2)}) = -0.2813$. Thus $\mathcal{S}(\mathcal{A}^{(1)}) > \mathcal{S}(\mathcal{A}^{(2)})$ and hence $\mathcal{A}^{(1)} \succ \mathcal{A}^{(2)}$ where $\succ$ represents the "preferred to".

Example 11.4.2. Consider a DM problem which consists of two different alternatives denoted by $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ under the set of the different parameters/criteria denoted by $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ whose weight vector is $\omega = (0.24, 0.40, 0.36)^T$. A decision-maker has evaluated these alternatives and give their preferences with respect to these three criteria in the form of the intuitionistic fuzzy decision matrix $\mathcal{M}$ as follows:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}^{(1)} \\ \mathcal{A}^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathcal{G}_1 \quad & \quad \mathcal{G}_2 \quad & \quad \mathcal{G}_3 \quad \end{array} \\ \begin{bmatrix} (0.361, 0.51) & (0.25, 0.66) & (0.29, 0.53) \\ (0.4, 0.532) & (0.52, 0.36) & (0.12, 0.76) \end{bmatrix} \end{array}$$

Then, the target of this problem is to choose the best alternatives among the two. For it, the existing IFWA operator [227] has been utilized for aggregating these different preferences and hence, the relative score values corresponding to the alternative $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are -0.2949 and -0.2949. Hence, it is unable to justify their approach to find the best alternative which leads to an unfair decision.

On the other hand, if we apply the proposed GWA operator by considering the preference of one senior expert β, towards the assessments of the rating values of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, as

$$\beta = \begin{pmatrix} (0.9, 0.1) \\ (0.5, 0.4) \end{pmatrix}$$

then, we get the relative score values corresponding to their alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ as -0.3528 and -0.5254 respectively. Thus, from it, it has been concluded that the alternative $\mathcal{A}^{(1)}$ is better than the alternative $\mathcal{A}^{(2)}$.

Example 11.4.3. Consider an another DM problem with two alternatives $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are evaluated under the set of the different parameters e_1, e_2 and e_3. An expert has evaluated these alternatives under the set of the three parameters e_1, e_2 and e_3 and gave their preferences in terms of IFNs, which are represented in the form of decision matrix $\mathcal{M}$ as follows:

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{A}^{(1)} \\ \mathcal{A}^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & \quad e_2 & \quad e_3 \end{array} \\ \begin{bmatrix} (0.5, 0.4) & (0.7, 0.2) & (0.6, 0.2) \\ (0.4, 0.5) & (0.8, 0.1) & (0.6, 0.3) \end{bmatrix} \end{array}$$

Then, the target of this problem is to choose the best alternative among them. For it, if we utilize the existing distance measure, $d(\cdot)$, as proposed by Khalid and Abbas [98], then their corresponding distance measures for the alternative $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ from its positive ideal measure $\mathcal{G}^+ = ((1,0))_{2\times 3}$ are $d(\mathcal{A}^{(1)}, \mathcal{G}^+) = 0.4$ and $d(\mathcal{A}^{(2)}, \mathcal{G}^+) = 0.4$ respectively. Thus, from it, we have analyzed that their measure is also unable to distinguish between these two alternatives. On the other hand, if we apply the proposed GWA operator by adding preference of one senior expert β as given in above example then, we get their respective score values as $\mathcal{S}(\mathcal{A}^{(1)}) = 0.7316$ and $\mathcal{S}(\mathcal{A}^{(2)}) = 0.0670$. From it, we can conclude that $\mathcal{A}^{(1)}$ is preferred to $\mathcal{A}^{(2)}$.

11.4.6 Advantages of the proposed approach

Based on the comparative studies, the following advantages have been obtained which are summarized as follows.

(i) The results computed under IFNs [54, 56, 102, 177, 206, 209, 227, 231] have completely ignored the judgments of the expert's matrix $[\mathcal{G}_\beta]$ and thus they have completely made by his own views and hence lead to an unfair decision.

(ii) The results computed by the existing approaches [21, 98, 142, 161, 168] under the IFSS environment did not consider the degree of the expert's opinion $\beta_k(k = 1, 2, \ldots, l)$'s during the analysis.

(iii) The influence of each expert on the ranking of the alternative has been judged by using GWA operator and found that the on expert-1 preference, a ranking is different from the expert-2 or expert-3 since each expert has their own knowledge and experience during the judgment of the alternative. But the best alternative remains same as found by the GWA or GGWA operators.

(iv) On the other hand, it has been seen the effect of the generalized parameter $\beta(e)$ on the ranking of the candidate. For example, as far as for the candidate concerned, $\mathcal{A}^{(3)}$ is leading to $\mathcal{A}^{(6)}$, $\mathcal{A}^{(4)}$ is leading to $\mathcal{A}^{(2)}$ with the IFSS approaches, but with the consideration of the generalized parameters, $\mathcal{A}^{(3)}$ precedes $\mathcal{A}^{(6)}$ and $\mathcal{A}^{(4)}$ precedes $\mathcal{A}^{(2)}$. Also, it has been observed that the best candidate has remained $\mathcal{A}^{(1)}$ but their results are no longer suitable from a practical point of view due to non-consideration of the parameters of the expert opinion.

11.5 Conclusion

In this chapter, we have presented the concept of generalized IFSS and group-based generalized IFSS based on the preferences of the group of experts' whose rating values are summarized in the form of IFNs. To do so, we first study few properties of GIFSS and then an averaging and the geometric aggregating operator named as GWA and GWG have been proposed. Further, the concept of GIFSS has been extended to GGIFSS and hence their corresponding averaging and geometric AO have been proposed by embedding more number of senior experts which reduce the impact of single senior expert's preference during the DM process. The proposed approach has been illustrated with a numerical example and compared the study with the state-of-art literature. From the study, it has been concluded that without considering the generalization parameter of the expert's opinion in the decision process, the results are no longer suitable from a practical point of view. Furthermore, the proposed GGIFSS reduces the impact of unfair arguments on the final results given by a single expert.

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