A STUDY

OF

MOCK THETA FUNCTIONS

A dissertation submitted in
partial fulfilment of the requirements
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UNDER THE GUIDANCE OF

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "A ***study of mock theta functions****" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Meenakshi Rana.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Inderpal Singh

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Date: INDERPAL SINGH

Abstract

In this thesis, we studied some mock theta functions combinatorially defined by S. Ramanujan, three months before his death in 1920. We studied partitions with n copies of n defined by Agarwal and Andrews in 1987 (A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan identities for partitions with "N copies of N", J. Combin. Theory, Ser. A, 45(1) (1987), 40-49) and lattice paths defined by A.K. Agarwal and Bressoud in 1989 (A.K. Agarwal and D.M. Bressoud, Lattice paths and multiple basic hypergeometric series, Pacific J. Math., 136(2) (1989), 209-228).

Using these combinatorial objects we discussed combinatorial interpretation of four mock theta functions out of 17 mock theta functions defined by S. Ramanujan. Ramanujan asserted that these functions share certain properties with theta functions, which have been investigated at great length by Carl Gustav Jacob Jacobi (1804-1851). Theta functions are essentially elliptic analogs of exponential function. Jacobi proved a variety of identities and found expressions for these special functions in terms of infinite series and infinite products. Our main results are contained in Chapters 2-3.

Contents

Chapter 1

Introduction

The theory of partition is an important branch of additive number theory. The concept of partition of non negative integers also belongs to combinatorics. This theory was established by Euler in the 18th century and has been further developed by many of the other great mathematicians - prominant among them are Cayley, Gauss, Gordan, Gupta, Hardy, Jacobi, Ramanujan, Schur, Andrews, Stanley, Sylvester and MacMohan.

Constructive Partition theory is a rich subject, with many classical and important results which influenced the development of Enumerative Combinatorics in the twentieth century.

Partition Theory originated as a part of the Number Theory, where numerous applications has emerged. Textbooks, such as [19], traditionally had at least one section devoted to Partition Theory. Later, Partition Theory was considered as a part of Combinatorial Analysis, a subject which evolved into modern day combinatorics and Discreate Mathematics.

The method of proving partition identities "constructively" was pioneered by Sylvester in [21]. In this thesis, Our goal is to study direct combinatorial (bijective or involutive) proofs of some partition identities, in particular, we studied mock theta functions defined by S. Ramanujan in Lost notebook [11].

Historically, most partition identities were first proved analytically, and only much later combinatorially. So in this thesis we emphasize on the combinatorial interpretation of the partition identities. In Chapters 2 and 3 we study combinatorial interpretation of some mock theta functions using n-coloured partitions and lattice paths respectively.

1.1 Basic Definitions and Notations

Definition 1.1.1. [12] A partition of a positive integer n is a finite non - increasing sequence of positive integers $a_1 \geq a_2 \geq \cdots \geq a_r$ such that there sum is n. The a_i are called the parts of the partition. We denote by $p(n)$ the number of partitions of n.

For example the partitions of 5 are,

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

Note that 0 has one partition. We set $p(0) = 1$ and $p(n) = 0$ for n negative integer.

Definition 1.1.2. The partition function $p(n)$, is the number of partitions of n.

For example $p(5)$=7.

Definition 1.1.3. A Composition is an ordered partitions.

For example 2+3 and 3+2 are two different compositions of 5, where as they are considered to be same part in case of partition.

Definition 1.1.4. [3] The Ferrers graph of a partition α is a graphical representataion of α, we define it as below;

The Ferrers graph of a partition $t_1, \cdots, t_i$ of n is a set of i rows of equi - spaced dots aligned to the left, where the j^{th} row has t_j dots.

Note that the number of rows is equal to the number of parts in the partition while the number of columns is equal to the largest part of the partition.

The Ferrers graph of the partition 4+3+3+2+1 of 13 is

```
. . . .
. . .
. . .
. .
.
```

If the graph is read vertically by columns

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \\ \cdot & & & & \end{array}$$

then this represents the partition 5+4+3+1 of 13. This new partition is called the conjugate of the given partition.

We see that in the new partition the number of parts is equal to the largest part in the given partition.

Thus we have following theorems:

Theorem 1.1.1. The number of partitions of n into r parts is equal to the number of partitions of n in which the largest part is r.

Theorem 1.1.2. The number of partitions of n into at most r parts is equal to the number of partitions of n in which the largest part does not exceed r.

Definition 1.1.5. A partition is said to be self conjugate if it is identical with its conjugate.

For example, 3+2+1 is a self conjugate partition of 6.

The following result is given by Sylvester:

Theorem 1.1.3. The number of partitions of n with distinct parts is equal to the number of self-conjugate partitions of n.

Definition 1.1.6. [6] The generalized basic hypergeometric series is defined as

$$r\phi_s \left(\begin{array}{ccccc} a_1 & a_2 & \cdots & a_r & ; z \\ b_1 & b_2 & \cdots & b_s & \end{array} \right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n [(-1)^n q^{\frac{1}{2}n(n-1)}]^{1+s-r} z^n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n (q;q)_n}$$

where $q \neq 0$, $r > s+1$.

Definition 1.1.7. [16] The rank of a partition is the largest part minus the number of parts.

In terms of the Ferrers graph, rank of partition π of n is the number of columns minus the number of rows.

For example, if π=5+4+3+1 then its rank is 5 - 4 = 1.

Definition 1.1.8. [13] The crank c(π)of a partition is defined by

$$c(\pi) = \begin{cases} \ell(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

where $\ell(\pi)$ denotes the largest part of π, $\omega(\pi)$ denotes the number of ones in π and $\mu(\pi)$ denotes number of parts of π larger than $\omega(\pi)$.

Partition of 5	$\ell(\pi)$	$\omega(\pi)$	$\mu(\pi)$	$c(\pi)$
5	5	0	1	5
4+1	4	1	1	0
3+2	3	0	2	3
3+1+1	3	2	1	-1
2+2+1	2	1	2	1
2+1+1+1	2	3	1	-2
1+1+1+1+1	1	5	0	-5

1.2 Generating function

Definition 1.2.1. The sum of series whose general coefficient is $f(n)$ is called the generating function of $f(n)$ and is said to enumerate $f(n)$.

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.2.1}$$

where $\mid q \mid < 1$ and $(q;q)_n$ is a rising q-factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

for any constant a.

If n is a positive integer, then obviously

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots .$$

The generating function for partition into odd parts is given by

$$\sum_{n=0}^{\infty} p(O,n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})}.$$

The generating function for partition into distinct parts is given by

$$\sum_{n=0}^{\infty} p(D,n)q^n = \prod_{n=1}^{\infty} (1+q^n).$$

Using the above generating functions Euler proved the following theorem:

Theorem 1.2.1. The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Analytic Proof

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(D,n)q^n &= (1+q)(1+q^2)(1+q^3)... \\
&= \frac{(1-q^2)(1-q^4)(1-q^6)}{(1-q)(1-q^2)(1-q^3)}... \\
&= \frac{1}{(1-q)(1-q^3)...} \\
&= \sum_{n=0}^{\infty} p(O,n)q^n.
\end{aligned}
$$

Combinatorial Proof

Given a partition of n into distinct parts replace each even parts by its two halves and repeat this process till no even parts are left. Finally arrange the parts in non-increasing order. This will be a partition of n into odd parts.

Conversely, for a partition into odd parts go on adding two equal parts until there are no repetitions. Finally arrange the parts in non-increasing order. This gives rise to a partition into distinct parts.

We now illustrate the proof by an example.

$$
\begin{aligned}
26 &= 10+8+5+3 \\
&\to 5+5+4+4+5+3 \\
&\to 5+5+2+2+2+2+5+3 \\
&\to 5+5+1+1+1+1+1+1+1+1+5+3 \\
&\to 5+5+5+3+1+1+1+1+1+1+1+1,
\end{aligned}
$$

conversely

$$
\begin{aligned}
26 &= 5+5+5+3+1+1+1+1+1+1+1+1 \\
&\to 10+5+3+2+2+2+2 \\
&\to 10+5+3+4+4 \\
&\to 10+5+3+8 \\
&\to 10+8+5+3.
\end{aligned}
$$

1.3 Mock - Theta functions

Srinivasa Ramanujan is regarded as the most influential Indian Mathematician of the twentieth Century. Early in 1920, three months before his death, he wrote his last letter to Hardy. In the course of it he said: "I discovered very interesting functions recently which I call 'mock theta functions', they enter into mathematics as beautifully as the ordinary θ - functions".

In the 1980's, G. Andrews and F.G. Garvan conjectured that two families of Ramunajan's theta functions are genuinely different from Jacobi's theta functions, not just the same functions in disguise. They linked Ramanujan's functions to partitions of a given integer - the ways of writing an integer as a sum of smaller integers. Building on the work by Andrews and Garvan, D. Hickerson proved that five identities in each of the two families are equivalent, at the same time confirming that these mock theta functions are truly *mock theta functions.*

Definition 1.3.1. [20] A mock - theta function is a function defined by a q - series convergent when $|q| < 1$, for which we can calculate asymptotic formulae when q tends to a 'rational point' $e^{2\pi ri/s}$ of the unit circle of the same degree of precision as those furnished for the ordinary θ - function by theory of linear transformation.

Definition 1.3.2. [7] The order of a mock - theta function is $(2r+1)$ if it is expressible in terms of a ${}_{r+1}\phi_r$ series, on a single base q^k, $k \leq r+1$. There may be in the definition of the mock - theta functions on additive term with ${}_{r+1}\phi_r$ consisting of a θ - products, which do not affect the order. It is understood that ${}_{r+1}\phi_r$ is expressed in terms of the lowest possible order θ - function.

Ramanujan separated his 17 functions into three classes. First containing 4 functions of order 3, second containing 10 functions of order 5 and the third containing 3 functions of order 7.

Mock - theta functions of order 3 are:

$$F(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.3.1}$$

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n}, \tag{1.3.2}$$

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}, \tag{1.3.3}$$

and

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}. \tag{1.3.4}$$

Mock - theta functions of order 5 are:

Group - A

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}, \tag{1.3.5}$$

$$\Phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q; q^2)_n, \tag{1.3.6}$$

$$\Psi_0(q) = \sum_{n=0}^{\infty} q^{(n+1)(n+2)/2}(-q; q)_n, \tag{1.3.7}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \tag{1.3.8}$$

and

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n}. \tag{1.3.9}$$

Group - B

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n}, \tag{1.3.10}$$

$$\Phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n, \tag{1.3.11}$$

$$\Psi_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}(-q;q)_n, \tag{1.3.12}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \tag{1.3.13}$$

and

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}. \tag{1.3.14}$$

And the mock - theta functions of order 7 are:

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}, \tag{1.3.15}$$

$$\mathcal{F}_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n}, \tag{1.3.16}$$

and

$$\mathcal{F}_2(q) = \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}. \tag{1.3.17}$$

1.4 Coloured partitions

Definition 1.4.1. [4] An n - colour partition of a positive integer is a partition in which a part of size n, can come in n different colours denoted by subscripts: $n_1, n_2, ..., n_n$ and parts satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 ...$

For example, the n-coloured partitions of 3 are

$$3_1, \quad 3_2, \quad 3_3$$

$$2_1 + 1_1, \quad 2_2 + 1_1$$

$$1_1 + 1_1 + 1_1.$$

Definition 1.4.2. [4] An $(n+t)$ - colour partition is a partition in which a part of size n, $n \geq 0$ can come in $n+t$ different colours, $t \geq 0$, denoted by subscripts: $n_1, n_2, ..., n_{n+t}$.

For example, the partitions of 2 with "$n+1$ copies of n" are

$$\begin{aligned}
&2_1, \quad 2_1 + 0_1, \quad 1_1 + 1_1, \quad 1_1 + 1_1 + 0_1, \\
&2_2, \quad 2_2 + 0_1, \quad 1_2 + 1_1, \quad 1_2 + 1_1 + 0_1, \\
&2_3, \quad 2_3 + 0_1, \quad 1_2 + 1_2, \quad 1_2 + 1_2 + 0_1.
\end{aligned}$$

Note: Zeros are permitted if and only if t is greater than or equal to one. Also, in no partition are zeros permitted to repeat.

Definition 1.4.3. [4] The weighted difference of any two parts m_i, n_j, $m \geq n$ is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

In Chapter 2, We studied some mock theta functions combinatorially given by (1.3.3), (1.3.6), (1.3.8) and (1.3.11) using n- coloured partitions.

1.5 Lattice paths

Agarwal and Bressoud [5] studied a new class of weighted lattice paths and used them to interpret several q - series combinatorially.

Definition 1.5.1. We describe the lattice paths as paths of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:

northeast : from (i, j) to $(i+1, j+1)$,

southeast : from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$,

horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed along x-axis.

All our lattice paths are either empty or terminate with a southeast step: from $(i, 1)$ to $(i+1, 0)$.

The following terminology will be used in describing lattice paths:

Peak: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

Valley: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

Mountain: A section of the path which start on either the x-axis or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

Plain: A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

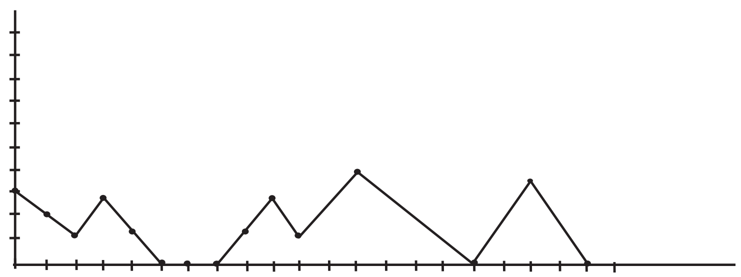
Range: A section of the path which starts either on the y-axis or at a vertex preceded by a horizontal step, which ends either at the end of the path or at a vertex followed by a horizontal step, and which does not contain any horizontal step, and which does not contain any horizontal steps. Every range includes at least one mountain and may have more than one.

The **Height** of a vertex is its y-coordinate.

The **Weight of a vertex** is its x-coordinate.

The **Weight of a path** is the sum of the weights of its peaks.

Example: The following path has five peaks, three valleys, three mountains, two ranges and one plain.



In Chapter 3, we studied some mock theta functions combinatorially given by (1.3.3), (1.3.6), (1.3.8) and (1.3.11) using weighted lattice paths and also observed a bijection between coloured partition and lattice paths.

Chapter 2

Number Theoretic interpretation of some mock theta functions using n - coloured partitions

2.1 Introduction

Out of 17 functions of Ramanujan given by equations (1.3.1) to (1.3.17), a very few mock theta functions are interpreted combinatorially.

G. N. Watson in 1937 proved most of the assertions about the fifth order mock theta functions and N.J. Fine [18] has also interpreted a mock theta function $\Psi(q)$, given in equation (1.3.3) combinatorially as;

Theorem 2.1.1.

$$\Psi(q) = \sum_{n=1}^{\infty} \beta(n) q^n,$$

where $\beta(n)$ denotes the number of partitions of n of the form,

$$n = n_1.1 + n_3.3 + n_5.5 + ...n_{2k-1}.(2k-1); n_j > 0.$$

In this Chapter we give combinatorial interpretation of some mock theta functions given recently by Agarwal in [1], using n - coloured partitions.

2.2 Main Results

In this section we give our main theorems:

Theorem 2.2.1. For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-colour partitions of ν such that even parts appear with even subscripts and odd with odd. For some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then

$$\sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu} = \Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \tag{2.2.1}$$

Theorem 2.2.2. For $\nu \geq 0$, let $A_2(\nu)$ denote the number of n-colour partitions of ν such that even parts appear with even subscripts and odd with odd greater than 1, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^{\nu} = F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}. \tag{2.2.2}$$

Theorem 2.2.3. For $\nu \geq 0$, let $A_3(\nu)$ denote the number of n-colour partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the type $(2k-1)_1$ or $(2k)_2$, the minimum part is 1_1 or 2_2 and the weighted difference of any two consecutive part is 0. Then

$$\sum_{\nu=0}^{\infty} A_3(\nu)q^{\nu} = \Phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n. \tag{2.2.3}$$

Theorem 2.2.4. For $\nu \geq 1$, let $A_4(\nu)$ denote the number of n-colour partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, the minimum part is 1_1 and the weighted difference of any two consecutive part is 0. Then

$$\sum_{\nu=0}^{\infty} A_4(\nu)q^{\nu} = \Phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n. \tag{2.2.4}$$

2.3 Proof of Theorems

In this section we give proofs of the theorems.

Proof of the Theorem 2.2.1

Let $A_1(m,\nu)$ denote the number of partitions of ν enumerated by $A_1(\nu)$ into m parts. Let for $|q| < 1$ and $|z| < |q|^{-1}$, $f_1(z,q)$ is defined by

$$f_1(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_1(m,\nu)z^m q^{\nu}. \tag{2.3.1}$$

We shall first prove that

$$A_1(m,\nu) = A_1(m-1,\nu-2m+1) + A_1(m,\nu-2m+1). \tag{2.3.2}$$

To prove (2.3.2), we split the partitions enumerated by $A_1(m,\nu)$ into two classes, viz.,

(i) those which contain 1_1 as a part, and
(ii) those which contain $k_k (k > 1)$ as a part.

We now transform the partitions in class (i) by deleting the part 1_1 and then subtracting 2 from all the remaining parts without disturbing the subscripts. The transformed partition will be of the type enumerated by
$A_1(m-1,\nu-2m+1)$.

Conversely, if we have a partition enumerated by $A_1(m-1, \nu-2m+1)$, we add 2 to each part and then add 1_1 as a part. The resulting partition will be a partition of class (i).

This shows that the partitions in class (i) are in one-to-one correspondence with the partitions enumerated by $A_1(m-1, \nu-2m+1)$.

Next we transform the partitions in class (ii) by first replacing the part k_k by $(k-1)_{k-1}$ and then subtracting 2 from each of remaining parts. The transformed partition is a partition enumerated by $A_1(m, \nu-2m+1)$.

Since this transformation is also reversible, we see that the number of partitions in class (ii) is $A_1(m, \nu-2m+1)$.

This completes the proof of (2.3.2).

On substituting for $A_1(m, \nu)$ from (2.3.2) in (2.3.1) and then simplifying,

we get

$$f_1(z,q) = zqf_1(zq^2,q) + q^{-1}f_1(zq^2,q). \tag{2.3.3}$$

Setting

$$f_1(z,q) = \sum_{n=0}^{\infty} \alpha_n(q)z^n, \tag{2.3.4}$$

in (2.3.3) and then comparing the coefficients of z^n in the resulting expression, we obtain

$$\alpha_n(q) = \frac{q^{2n-1}}{1-q^{2n-1}}\alpha_{n-1}(q). \tag{2.3.5}$$

Iterating (2.3.5) n-times and noting that $\alpha_0(q) = 1$, we are lead to

$$\alpha_n(q) = \frac{q^{n^2}}{(q;q^2)_n}, \tag{2.3.6}$$

and so

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}z^n}{(q;q^2)_n}. \tag{2.3.7}$$

$$\sum_{\nu=0}^{\infty} A_1(\nu) q^\nu = \sum_{\nu=0}^{\infty} (\sum_{m=0}^{\infty} A_1(m,\nu)) q^\nu$$

$$= f_1(1,q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$$

$$= \Psi(q).$$

This completes the proof of the Theorem 2.2.1.

Proof of the Theorem 2.2.2.

Let $A_2(m,\nu)$ denote the number of partitions of ν enumerated by $A_2(\nu)$ into m parts. Let for $|q| < 1$ and $|z| < |q|^{-1}$, $f_2(z,q)$ is defined by

$$f_2(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_2(m,\nu) z^m q^\nu. \tag{2.3.8}$$

The proof is similar to that of Theorem 2.2.1, the q - functional used in this case is

$$f_2(z,q) = zq^2 f_2(zq^4, q) + q^{-1} f_2(zq, q). \tag{2.3.9}$$

Setting

$$f_2(z,q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n, \tag{2.3.10}$$

in (2.3.8) and then comparing the coefficients of z^n in the resulting expression, we obtain

$$\alpha_n(q) = \frac{q^{4n-2}}{1-q^{n-1}}\alpha_{n-1}(q). \tag{2.3.11}$$

Iterating (2.3.11) n-times and noting that $\alpha_0(q) = 1$, we are led to

$$\alpha_n(q) = \frac{q^{2n^2}}{(q;q^2)_n}, \tag{2.3.12}$$

and so

$$f_2(z,q) = \sum_{n=0}^{\infty} \frac{q^{2n^2} z^n}{(q;q^2)_n}. \tag{2.3.13}$$

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^\nu = \sum_{\nu=0}^{\infty}(\sum_{m=0}^{\infty} A_2(m,\nu))q^\nu$$

$$= f_2(1,q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}$$

$$= F_0(q).$$

This completes the proof of the Theorem 2.2.2

Proof of the Theorem 2.2.3

Let $A_3(m,\nu)$ denote the number of partitions of ν enumerated by $A_3(\nu)$ into m parts. Let for $|q| < 1$ and $|z| < |q|^{-1}$, $f_3(z,q)$ is defined by

$$f_3(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_3(m,\nu) z^m q^\nu. \tag{2.3.14}$$

We shall first prove that

$$A_3(m,\nu) = A_3(m-1,\nu-2m+1) + A_3(m-1,\nu-4m+2). \qquad (2.3.15)$$

To prove (2.3.15), we split the partitions enumerated by $A_3(m,\nu)$ into two classes, viz.,
(i) those which contain 1_1 as a part, and
(ii) those which contain 2_2 as a part.

We now transform the partitions in class (i) by deleting the part 1_1 and then subtracting 2 from all the remaining parts without disturbing the subscripts. The transformed partition will be of the type enumerated by
$A_1(m-1,\nu-2m+1)$.

Conversely, if we have a partition enumerated by $A_1(m-1,\nu-2m+1)$, we add 2 to each part and then add 1_1 as a part. The resulting partition will be a partition of class (i).

This shows that the partitions in class (i) are in one-to-one correspondence with the partitions enumerated by $A_1(m-1,\nu-2m+1)$.

Next we transform the partitions in class (ii) by deleting the part 2_2 and then subtracting 4 from all the remaining parts without disturbing the subscripts. The transformed partition will be of the type enumerated by
$A_2(m-1,\nu-4m+2)$.

Conversely, if we have a partition enumerated by $A_2(m-1,\nu-4m+2)$, we add 4 to each part and then add 2_2 as a part. The resulting partition will be a partition of class (ii). This shows that the partitions in class (i) are in one-to-one correspondence with the partitions enumerated by $A_2(m-1,\nu-4m+2)$. Thus the number of partitions in class (ii) is $A_2(m-1,\nu-4m+2)$.

This leads to the identity

$$A_3(m,\nu) = A_3(m-1,\nu-2m+1) + A_3(m-1,\nu-4m+2). \qquad (2.3.16)$$

Using (2.3.14) one can easily obtain the following q - functional equation

$$f_3(z,q) = zqf_3(zq^2,q) + zq^2f_3(zq^4,q). \qquad (2.3.17)$$

Setting

$$f_3(z,q)=\sum_{n=0}^{\infty}\beta_n(q)z^n,$$

and noting that

$$f_3(0,q)=1,$$

we can easily check by then comparing the coefficients of z^n in the resulting expression, we obtain

$$\beta_n(q)=q^{n^2}(-q;q^2)_n. \tag{2.3.18}$$

and therefore

$$f_3(z,q)=\sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n z^n. \tag{2.3.19}$$

$$\sum_{\nu=0}^{\infty}A_3(\nu)q^\nu=\sum_{\nu=0}^{\infty}(\sum_{m=0}^{\infty}A_3(m,\nu))q^\nu$$

$$=f_3(1,q)$$

$$=\sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n$$

$$=\Phi_0(q).$$

This completes the proof of the Theorem 2.2.3.

Proof of the Theorem 2.2.4

The partitions enumerated by $A_4(m, \nu)$ are precisely those partitions which belong to class (i) of the previous case.
Therefore,

$$A_4(z, \nu) = A_3(m-1, \nu - 2m + 1). \tag{2.3.20}$$

Using Equations (2.3.15) and (2.3.20), one can easily obtain the following q-functional equation:

$$f_4(z, q) = f_3(z, q) - zq^2 f_3(zq^4, q). \tag{2.3.21}$$

Setting

$$f_4(z, q) = \sum_{n=0}^{\infty} \gamma_n(q) z^n$$

and then comparing the coefficients of z^n on each of (2.3.21), we see that

$$\gamma_n = \beta_n(q) - \beta_{n-1}(q) q^{4n-2}$$

$$= q^{n^2}(-q; q^2)_{n-1}.$$

This implies that

$$f_4(z, q) = \sum_{n=1}^{\infty} q^{n^2}(-q; q^2)_{n-1} z^n.$$

$$\sum_{\nu=0}^{\infty} A_4(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} (\sum_{m=0}^{\infty} A_4(m, \nu)) q^{\nu}$$

$$= f_4(1, q)$$

$$= \sum_{n=1}^{\infty} q^{n^2}(-q; q^2)_{n-1}$$

$$= \sum_{n=0}^{\infty} q^{(n+1)^2} (-q; q^2)_n$$

$$= \Phi_1(q).$$

This completes the proof of the Theorem 2.2.4.

Chapter 3

Number Theoretic interpretation of some mock theta functions using Lattice Paths

3.1 Introduction

Recently Agarwal in [2] extended combinatorial interpretation of the mock theta functions given in Chapter 2 using Lattice paths.

So in this Chapter we give the combinatorial interpretation of mock theta functions discussed in Chapter 2 using lattice paths. In the fourth section of the Chapter a bijection is established between n - coloured partition and lattice path.

3.2 Main Results

Theorem 3.2.1 For $\nu \geq 1$, let $B_1(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0 and no plain. Then

$$\sum_{\nu=1}^{\infty} B_1(\nu) q^{\nu} = \Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \tag{3.2.1}$$

Theorem 3.2.2 For $\nu \geq 0$, let $B_2(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0, no plain and the height of each peak is ≥ 2. Then

$$\sum_{\nu=0}^{\infty} B_2(\nu) q^{\nu} = F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}. \tag{3.2.2}$$

Theorem 3.2.3 For $\nu \geq 0$, let $B_3(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2. Then

$$\sum_{\nu=0}^{\infty} B_3(\nu) q^{\nu} = \Phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n. \tag{3.2.3}$$

Theorem 3.2.4 For $\nu \geq 1$, let $B_4(\nu)$ denote the number of lattice paths of weight ν which start from (0,0), have no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2 and the weight of the first peak is 1. Then

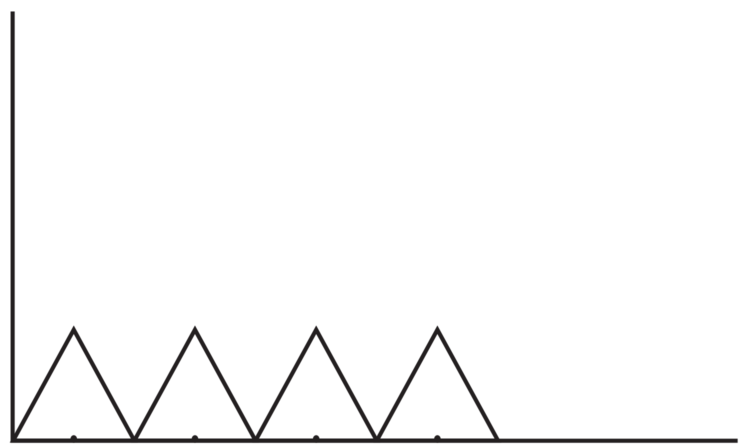
$$\sum_{\nu=0}^{\infty} B_4(\nu) q^{\nu} = \Phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n. \tag{3.2.4}$$

3.3 Proof of Theorems

In this section we give the proofs of our main theorems:

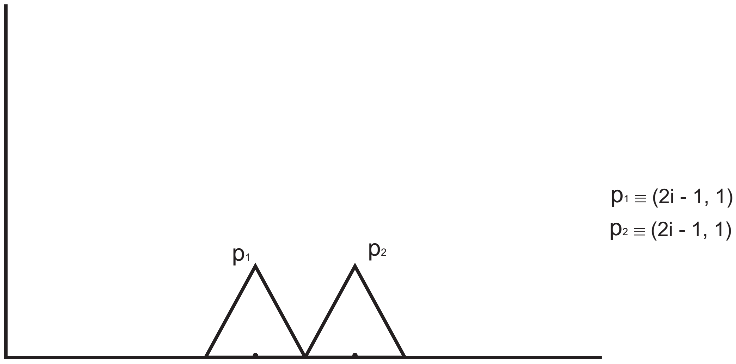
Proof of the Theorem 3.2.1

In $\frac{q^{m^2}}{(q;q^2)_m}$ the factor q^{m^2} generates the lattice path of m peaks starting at $(0, 0)$ and terminating at $(2m, 0)$. For m=4, the path begins as



Graph A

In the Graph A we consider two successive peaks say $(i)^{th}$ and $(i + 1)^{th}$, and denote them by p_1 and p_2, respectively.

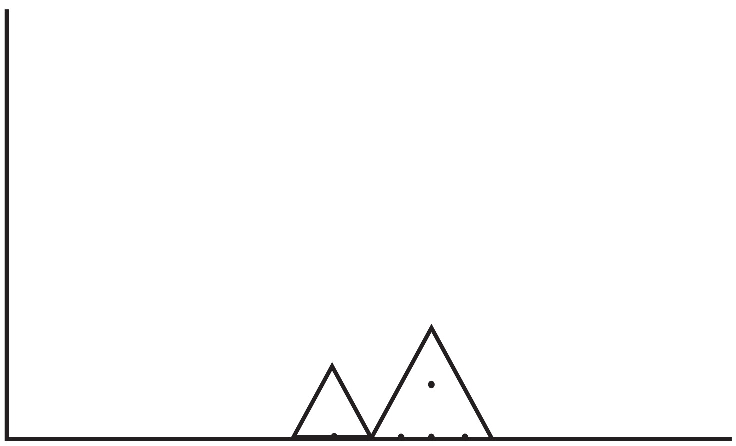


Graph B

The factor $1/(q;q^2)_m$ generates non-negative multiples of $(2i-1)$, $1 \leq i \leq m$, say,

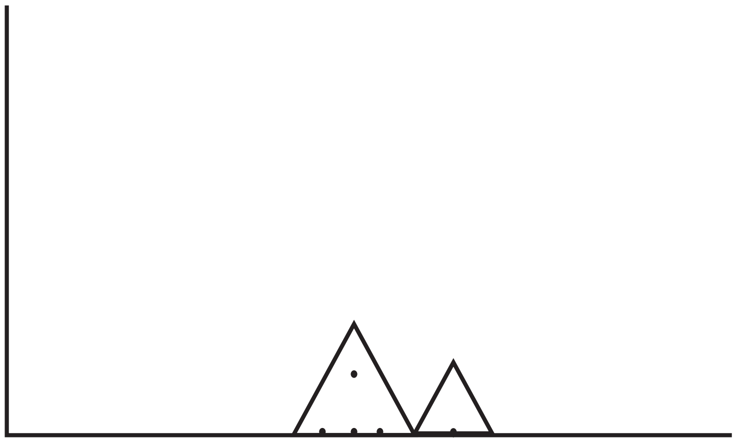
$$b_1 \times 1, \quad b_2 \times 3, \cdots, b_m \times (2i-1).$$

This is encoded by having the i^{th} peak grow to height $b_{m-i+1} + 1$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. The graph B now changes to graph C or graph D depending on whether $b_{m-i} > b_{m-i+1}$ or $b_{m-i} < b_{m-i+1}$. Note that if $b_{m-i} = b_{m-i+1}$, then the new graph will look like Graph B.



Graph C

and



Graph D

Every lattice path starting at $(0, 0)$ with all valleys at height 0, no plain is uniquely generated in this manner. This proves that extreme right-hand side of Equation (3.2.1) generates $B_1(\nu)$.

This completes the proof of the Theorem 3.2.1

Proof of the Theorem 3.2.2

The case k=2 is treated in exactly the same manner as the the first case except that the extra factor q^{m^2} increases the height of the each peak by 1. This increases the weight of the path by 1+3+5+...(2m-1)=m^2. Thus the minimum height of each peak is 2.

Proof of the Theorem 3.2.3

In the case k=3, we observe that $b_1, b_2, ..., b_m$ are 0 or 1 since the factor $(-q; q^2)_m$ generates non negative multiples of distinct (2i - 1), $1 \leq i \leq m$.

Proof of the Theorem 3.2.4

The case k=4 is treated exactly the same manner as Case k=3 except that in this the height of the first is not increased since the factor $(-q; q^2)_m$ generates only m-1 non negative multiples of distinct (2i - 1), $1 \leq i \leq m-1$

3.4 Second Proof (bijective)

We now establish a bijection(ie 1-1 correspondence) between the n-colour partitions enumerated by $A_1(\nu)$ given in Theorem 2.2.1 and the lattice paths enumerated by $B_1(\nu)$ given in Theorem 3.2.1. We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak. Thus, if we denote the two peaks in Graph C (or Graph D) by B_x and A_y,($A \geq B$) respectively, then

$$B = (2i-1) + 2(b_m + b_{m-1} + \cdots + b_{m-i+2}) + b_{m-i+1}$$

$$x = b_{m-i+1} + 1$$

$$A = (2i-1) + 2(b_{n+1} + b_n + \cdots + b_{n-i+1}) + b_{m-i}$$

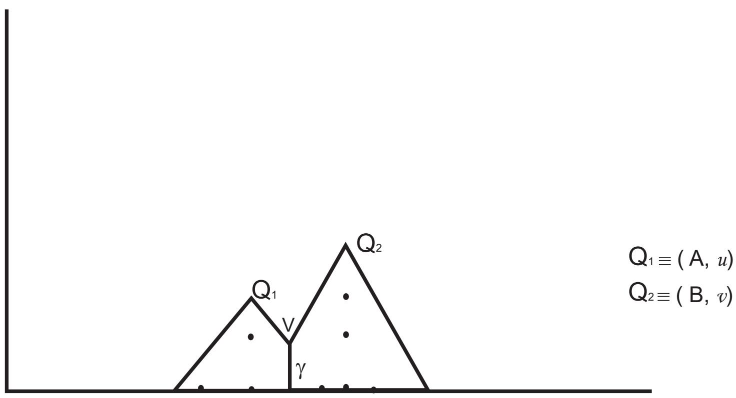
$$y = b_{m-i} + 1$$

The weighted difference of these two parts is $((A_y - B_x)) = A - B - x - y = 0$. Further, if we look at the n-colour part B_x, we find that the parity of both B and x is determined by b_{m-i+1}. If b_{m-i+1} is even then both B and x are odd and if b_{m-i+1} is odd then both B and x are even and since $x \geq 2$, we conclude that the even parts appear with even subscripts and odd with odd. And since there is no plain in front of the first mountain, the first peak corresponds to a part of the type k_k

To see the reverse implication we consider two consecutive n-colour parts of a partition enumerated by $B_1(\nu)$, say, C_u and D_v. Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the associated lattice path.

If there were a plain between Q_1 and Q_2, its length would be is $D - C - v - u$ which is the weighted difference between the two parts C_u and D_v and is therefore equal to zero showing that there are no plains between any two peaks and since there is a part of type k_k there is no plain in the beginning of the path either. We thus conclude that the associated path has no plain . Finally we prove by contradiction that there can not be a valley above height 0.

Suppose there is a valley V of height γ ($\gamma > 0$) between the peaks Q_1 and Q_2.



Graph E

In this case there is a descent of $u-\gamma$ from Q_1 to V and an ascent of $v-\gamma$ from V to Q_2. This implies that $D = C+(u-\gamma)+(v-\gamma)$, or $D-C-u-v = -2\gamma$. But since the weighted difference is zero, therefore γ=0. This completes the proof of the Theorem 3.2.1

Similarly we can prove bijectively other theorems.

3.5 Concluding Remarks

We have only studied mock theta functions combinatorially in the theory of partitions which represent only a small part of discussion on mock theta functions. The mock theta functions have made several appearances in the literature in the recent years [6,8,9,10,11]. The results discussed in this thesis are related to only third and fifth order mock theta functions, although mock theta functions of higher order 6, 8 and 10 has also been discussed in [14], [15] and [17].

The extensive treatment of mock theta functions by Ramanujan in his "Lost" Notebook [11] suggests that these functions possess a rich mathematical structure and are still far from understood.

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