Rogers - Ramanujan Type Identities
and
Split (n+t) - Color Partitions

A

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CERTIFICATE

I hereby Certify that the dissertation entitled, "**Rogers—Ramanujan Type Identities And Split (n+t) - Color Partitions**", which is being submitted by **Miss. Diksha Chawla** (Roll No. 301303001), in the partial fulfillment of the requirements for the award of the degree of **Master of Science** in "Mathematics and Computing", to the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of **Dr. Meenakshi Rana** during the period from January 2015 to July 2015.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/ Institute for the award of any degree

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

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ABSTRACT

The present dissertation contains a detail study of investigations carried out by various authors and by us on existence of combinatorial interpretations of certain q – series. The whole work is divided into four chapters.

Chapter 1 is introductory which includes elementary definitions and results which will be required for the later chapters. It also includes a theorem in classical ordinary partitions proved by Agarwal [“On a generalized partition theorem”, J. Indian Math. Soc.50(1986), pp. 185-190.].

In Chapter 2, the main theorem of Chapter 1 is extended to colored partitions proved by Agarwal and Rana [“Two new combinatorial interpretations of a fifth order mock theta function”, The Ind. Math. Soc., Special Centenary Volume, 1907-2007,(2008), 11 - 24]. This chapter is further devoted to the study of (n+t) – colored partitions.

In Chapter 3, a new class of partitions which has been defined by Agarwal and Sood [“Split (n+t) – color partitions and Gordon – McIntosh eight order mock theta functions”, Electronic J. Comb.No. 2, 21(2014), p2.46], called split (n+t) – color partitions. The purpose of this new class of partitions is to interpret combinatorially some q - series which have not been interpreted by previous tools. This class also extends the color partitions.

Finally in Chapter 4, we give some new combinatorial interpretations using split (n+t) – color partitions introduced in Chapter 3. We had given four generalized theorems which further give rise to four Rogers – Ramanujan Type Identities as a particular cases.

Towards the end, references of various publications cited in the present dissertation have been reported.

Contents

Chapter 1

Introduction

1.1 Introduction

Partition theory is a subject of enduring interest as well as a major research area. It is concerned with the number of ways that a whole number can be partitioned into whole number parts. The theory is an important branch of additive number theory and studies various enumerations and asymptotic problems related to integer partitions, q–series, special functions etc. The theory has found its numerous applications in particle physics, statistical mechanics, conformal field theory and lie algebra.

G. W. Leibnitz (1674)[26] was among the first mathematicians who paid attention in the development stages in this area of mathematics, but the greatest contributions in the early stages of partition theory were due to L. Euler (1748)[19]. Over a centuries, a great number of mathematicians like Rogers(1894)[29], Hardy and Wright (1978)[25], MacMahon (1960)[27] and Ramanujan (1919)[28] had devoted their time in the search of new identities in partition theory.

In the theory of partitions we often come across a number of identities which states that for each positive integer n, the partitions of n with parts restricted to certain residue classes are equinumerous with the partitions of n on which certain difference conditions are imposed. Among the most striking results of this type are the Rogers–Ramanujan Identities [28, 29]. Many more Rogers–Ramanujan type identities are listed in Slater's compendium [32] and Chu and Zhang compendium[18]. These identities are interpreted by many combinatorial tools introduced in the theory of partitions. Some of these tools are ordinary partitions[2, 3, 20, 21, 34, 35], n–color partitions[1, 9, 13], $(n+t)$–color partitions[4, 5, 12], lattice paths[6, 11, 12, 13], F–partitions[7, 8, 30, 31, 33].

In this thesis we will study about the combinatorial interpretations of Rogers–Ramanujan type identities using some of the above mentioned tools. Before we proceed further let us recall some basic notations and definitions.

1.2 Basic Definitions

Definition 1.2.1. A ***Partition*** of positive integer n is a representation of n as a sum of finite non increasing sequence of positive integers where the order of summands (or parts) is irrelevant. The number of partition of n is denoted by $p(n)$.

Example. $p(5) = 7$, the relevant partition of 5 are

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

Remark 2.1 We take $p(n) = 0$ for all negative values of n and $p(0) = 1$.

Generating Function: Leonhard Euler was asked to solve a problem: "How many partitions there are of 50 into seven distinct summands?" The correct answer, 522, is not likely to be obtained by writing out all the ways of adding seven distinct positive integers to get 50.

To solve this problem Euler introduced generating function.

The generating function for $p(n)$, the total number of partitions of n is given by

$$\begin{aligned} \sum_{n=0}^{\infty} p(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(q;q)_\infty} \\ &= 1 + q^1 + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots . \end{aligned}$$

where $|q| < 1$ and $(q;q)_\infty$ is q - rising factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{i+n})}.$$

If n is a positive integer, then

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots,$$

and when $a = q$

$$(q;q)_n = \prod_{i=1}^{n}(1-q^i)$$

and

$$(a,q)_0 = 1.$$

Definition 1.2.2. Let $p(D,n)$ and $p(O,n)$ denote the number of partitions of n into distinct and odd summands respectively.

The generating function of partition of n into distinct summands is given by

$$\begin{aligned}\sum_{n=0}^{\infty} p(D,n)q^n &= \prod_{n=1}^{\infty}(1+q^n) = (-q;q)_\infty \\ &= 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \cdots. \qquad (*)\end{aligned}$$

The generating function of partition of n into odd summands is given by

$$\begin{aligned}\sum_{n=0}^{\infty} p(O,n)q^n &= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}} = \frac{1}{(q;q^2)_\infty} \\ &= 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \cdots. \qquad (**)\end{aligned}$$

The equation (*) and (**) leads to a well known identity known as ***Euler's Identity*** which states that the number of partitions of n into distinct summands is equal to the number of partitions of n into odd summands that is

$$p(D,n) = p(O,n), \qquad \text{for all } n.$$

Definition 1.2.3. The ***Ferrers graph*** named after NORMAN MACLEOD FERRERS is a partition $\pi = t_1, t_2, \ldots, t_i$ of n is a set of rows of equi-spaced dots aligned on the left where the *jth* row has t_j dots..

Example. The Ferrers graph of the partition $\pi = 6 + 4 + 3$ of 13 as following,

. . . .
. . .

by reading this graph horizontally, we see that first row has 6 dots, second row has 4, third row has 3.

Definition 1.2.4. A ***Conjugate of a Partition*** π is denoted by π^c, is obtained by interchanging rows and columns in the Ferrers graph.

Example. The conjugate of above example is

. . .
. . .
. . .
. .
.
.

and reading this graph in usual manner, we get $\pi^c = 3 + 3 + 3 + 2 + 1 + 1$ is the conjugate partition of 13.

A partition is said to be ***Self Conjugate*** if it is identical with its conjugate.

Example. Consider the partition $\pi = 3 + 2 + 1$ of 6 then the Ferrers graph is

. . .
. .
.

now if we read this graph vertically we get the same partition, that is, $\pi^c = 3 + 2 + 1$. So this partition is a self conjugate partition. In others words, for a self conjugate partition, $\pi = \pi^c$.

Definition 1.2.5. A ***Restricted Partition*** of n defined as the partition of n with specific conditions on either/both the number of summands or/and the size of summands.

Let P denote the set of all partitions and let $p(N, M, n)$ denote the number of partitions of n into at most M summands each $\leq$ N, then the generating function for this is given by

$$\sum_{n=0}^{MN} p(N,M,n)q^n = \frac{(q,q)_{(N+M)}}{(q,q)_M(q,q)_N}.$$

Remark 1: If $n = NM$; then $p(N, M, n) = 1$.
Remark 2: If $n > MN$; then $p(N, M, n) = 0$.

1.3 Special Identities

1.3.1 Rogers–Ramanujan Identities

The two celebrated Rogers–Ramanujan Identities were first discovered and proved by ***Leonard James Rogers***(1894) [29]. They were subsequently rediscovered by ***Srinivasa Ramanujan*** some time before (1913) [28]. Ramanujan had no proof, but rediscovered Rogers's paper in 1917 and then published a joint new proof (Rogers and Ramanujan, 1919) and then the identities are known as Rogers–Ramanujan Identities(RRI).

The following two identities are RRI:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{1.2}$$

These identities were combinatorially interpreted by MacMohan in 1916 [27] given as:

Theorem 1.3.1.1. First Rogers–Ramanujan Identity: The partitions of n into parts that differs from each other by at least 2 are equinumerous with the partitions into summands of the form $5m + 1$ and $5m + 4$.

Theorem 1.3.1.2. Second Rogers–Ramanujan Identity: The partitions of n into parts each greater than 1 which differ from each other by atleast 2 are equinumerous with the partitions into summands of the form $5m + 2$ and $5m + 3$.

1.3.2 Göllnitz–Gordon Identities

The Göllnitz–Gordon Identities, given below, are due to H.Göllnitz [20] and were included in his 1961 unpublished honors thesis. However, essentially no one knew about the results until Gordon (1965) [23] independently rediscovered them.

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+1})(1-q^{8n+4})(1-q^{8n+7})}, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+3})(1-q^{8n+4})(1-q^{8n+5})}, \tag{1.4}$$

Theorem 1.3.2.1. First Göllnitz–Gordon Identity: The number of partitions of n into summands differing by at least 2, among which no two consecutive even summands are appear equals the number of partition of n into summands $\equiv 1, 4$ *or* $7(mod 8)$.

Theorem 1.3.2.2. Second Göllnitz–Gordon Identity: The number of partitions of n into summands differing by at least 2, among which no two consecutive even summands are appear and with each summands ≥ 3 equals the number of partition of n into summands $\equiv 3, 4$ *or* $5(mod 8)$.

Another identity given below;

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{8n+2})(1-q^{8n+3})(1-q^{8n+7})}. \tag{1.5}$$

has the combinatorial interpretation as the number of partitions of n into summands differing by at least 2 among which no consecutive odd numbers appear and with each summands being at least equal to 2 is equal to the number of partitions of n into summands $\equiv 2, 3$ *or* $7(mod 8)$,
is due to Göllnitz[20].

1.4 Main Theorem

In [3], Agarwal proved a generalized theorem using ordinary partitions which unify the Göllnitz–Gordon Identities and Göllnitz Identity as a particular case. The theorem is stated as:

Theorem 1.4.1. Given a positive integer k, let $A_k(\nu)$ denote the number of partitions of ν in which summands $\geq k$, the summands differ by at least 2, consecutive odd integers are not allowed if k is even and consecutive even integers are not allowed if k is odd. Then

$$\sum_{n=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}. \tag{1.6}$$

Identities (1.3), (1.4) and (1.5) are the particular cases for $k = 1, 3$ *and* 2 of Theorem 1.4.1 respectively.

In the next chapter, n–color partition has been introduced which arise in Baxter's solution of the hard hexagon model [16][Chap. 14]. These partitions are used in giving combinatorial interpretations of several q–series identities.

Chapter 2

Colored Partitions

2.1 n–Color Partition and Theorems

The n–color partitions were defined by Agarwal and Andrews [1] in 1987 which extends the ordinary partitions and which are further extended to $(n+t)$–color partitions.

Definition 2.1.1. A ***n-Color Partition*** (also called a partition with "n copies of n") of a positive integer n is a partition in which a summands of size n, can come in n-different colors denoted by subscripts: $n_1, n_2, \cdots, n_n$ and the summands satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < \cdots$.

Tab. 2.1: **Partitions of some integers**

Integer	**Colored partitions**	**Ordinary partitions**
1	1_1	1
2	$2_1, 2_2, 1_1 1_1$	$2, 1+1$
3	$3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1$	$3, 2+1, 1+1+1$

Generating Function: Let $P(\nu)$ denote the number of n-color partitions of ν, then

$$\begin{aligned}\sum_{\nu=0}^{\infty} P(\nu) q^{\nu} &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \\ &= 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + 86q^7 + \cdots .\end{aligned}$$

Definition 2.1.2. The ***Weighted Difference*** of two parts $m_i,\ n_j,\ m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.

To further unify the Göllnitz–Gordon partition functions appearing in Theorem 1.3.2.1 and Theorem 1.3.2.2 and the Göllnitz partition function of Theorem 1.3.2.3, Agarwal and Rana [13] interpreted the right hand side of Theorem 1.4.1 of Chapter 1 using colored partition as follows:

Theorem 2.1.1. Let $B_k(\nu)$ denote the number of n–color partitions of ν such that summands are greater than or equal to k, summands used are of the type $(2l-1)_1$ and $(2l)_2$ if k is odd, $(2l-1)_2$ and $(2l)_1$ if k is even, and the weighted difference between two summands are non–negative and even. Then

$$A_k(\nu) = B_k(\nu), \tag{2.1}$$

where $A_k(\nu)$ is defined in Theorem 1.4.1 of Chapter 1.

The above results leads an infinite family of combinatorial identities. In some particular cases we get even 3–way identities. We discuss three such cases and obtain two new combinatorial versions of Theorem 2.1.1.

2.2 Particular Cases

Corollary 1. For $k = 1$, in view of (1.3) Theorem 2.1.1 reduces to

$$A_1(\nu) = B_1(\nu) = C_1(\nu), \tag{$*$}$$

where $C_1(\nu)$ is the number of partitions of ν into summands which are congruent to 1, 4, 7(mod8).
The case $A_1(\nu) = C_1(\nu)$ is the first Göllnitz–Gordon Identity as given in (1.3). The other two

$$A_1(\nu) = B_1(\nu),$$
$$B_1(\nu) = C_1(\nu)$$

are new two–way combinatorial versions of it.

Corollary 2. For $k = 2$, in view of (1.5) Theorem 2.1.1 reduces to

$$A_2(\nu) = B_2(\nu) = C_2(\nu), \tag{$**$}$$

where $C_2(\nu)$ is the number of partitions of ν into summands which are congruent to 2, 3, 7(mod8). The case $A_2(\nu) = C_2(\nu)$ is the Göllnitz Identity as given in (1.5). The other two

$$A_2(\nu) = B_2(\nu),$$
$$B_2(\nu) = C_2(\nu)$$

are new two–way combinatorial versions of it.

Corollary 3. For $k = 3$, in view of (1.4) Theorem 2.1.1 reduces to

$$A_3(\nu) = B_3(\nu) = C_3(\nu), \qquad (***)$$

where $C_3(\nu)$ is the number of partitions of ν into summands which are congruent to 3, 4, 5(mod8). The case $A_3(\nu) = C_3(\nu)$ is the second Göllnitz–Gordon Identity as given in (1.4). The other two

$$A_3(\nu) = B_3(\nu),$$
$$B_3(\nu) = C_3(\nu)$$

are new two–way combinatorial versions of it.

2.3 $(n + t)$–color partitions and theorems

In [4], Agarwal and Andrews extended n–color partitions to $(n + t)$–color partitions which further interpreted several q–series combinatorially which were not possible to interpret by n–color partitions. Before stated some of the results let us recall the definition of $(n + t)$–color partitions.

Definition 2.3.1 A partition with ***"$n + t$ copies of n"***, $t \geq 0$ is a partition in which a summands of $n, (n \geq 0)$, can come in $n + t$ different colors, denoted by the subscripts, $n_1, n_2, n_3, \cdots, n_{n+t}$.

Example. Partitions of 2 with "$n + 1$ copies of n" are,

$$2_1,\quad 2_1 0_1,\quad 1_1 1_1,\quad 1_1 1_1 0_1,$$
$$2_2,\quad 2_2 0_1,\quad 1_2 1_1,\quad 1_2 1_1 0_1,$$
$$2_3,\quad 2_3 0_1,\quad 1_2 1_2,\quad 1_2 1_2 0_1.$$

Note that zeros are permitted if and only if $t \geq 1$. Also in no partition are zeros permitted to repeat.

In [5], Agarwal proved some generalized theorems given below using $(n+1)$–color partition and $(n + 2)$–color partition respectively.

Theorem 2.3.1. For $k \geq -3$, let $B_k(\nu)$ denote the number of partitions of ν with "$(n + 1)$ copies of n" such that weighted difference of each pair of summands is greater than k, the summands are nonnegative, and for some i, i_{i+1} is a summand. Then

$$\sum_{\nu=0}^{\infty} B_k(\nu)q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{\frac{\nu(\nu+1)(k+3)}{2}}}{(q;q)_\nu (q;q^2)_{\nu+1}}. \tag{2.2}$$

Theorem 2.3.2. For $k \geq -3$, let $C_k(\nu)$ denote the number of partitions of ν with "$(n+2)$ copies of n" into nonnegative summands such that weighted difference of each pair of summands is greater than k, and for some i, i_{i+2} is a summand. Then

$$\sum_{\nu=0}^{\infty} C_k(\nu)q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+\frac{(\nu+1)(k+3)}{2}]}}{(q;q)_\nu (q;q^2)_{\nu+1}}. \tag{2.3}$$

Before proving the above Theorems 2.3.1–2.3.2, we first prove the Theorem 2.3.3, given below, using n–color partitions given by Agarwal in [1].

Theorem 2.3.3. For $k \geq -3$, let $A_k(\nu)$ denote the number of partitions with "n copies of n" such that weighted difference of each pair of summands m_i, n_j is greater than k. Then

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^\nu = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+\frac{(k+3)(\nu-1)}{2}]}}{(q;q)_\nu (q;q^2)_\nu}. \tag{2.4}$$

proof. Let $A_k(m,\nu)$ denote the number of partitions enumerated by $A_k(\nu)$ with the added restriction that there be exactly m summands. We shall first prove that

$$\begin{aligned} A_k(m,\nu) =& A_k(m,\nu-m) + A_k(m-1,\nu-km-3m+k+2) \\ &+ A_k(m,\nu-2m+1) - A_k(m,\nu-3m+1). \end{aligned} \tag{2.5}$$

To prove (2.5) we split the partitions enumerated by $C_k(\nu)$ into three classes:
(i) those that do not contain j_j as a summand,
(ii) those that contain 1_1 as a summand, and
(iii) those that contain $j_j (j > 1)$ as a summand.
We now transform the partitions in class (i) by deleting 1 from each summand, ignoring the subscripts. Obviously, this transformation will not disturb the inequalities between the summands and so the transformed partition will be of the type enumerated by $A_k(m,\nu-m)$. Next we transform the partitions in class (ii) by deleting the summand 1_1, and then subtracting $(k+3)$ from all the remaining summands ignoring the subscripts. The transformed partition will be of the type enumerated by $A_k(m-1,\nu-km-3m+k+2)$. Here we note that k cannot be less than -3. Finally, we

transform the partitions in class (*iii*) by replacing k_k by $(k-1)_{k-1}$ and then subtracting 2 from all the remaining summands. This will produce a partition of $\nu - 1 - 2(m-1) = \nu - 2m + 1$ into m summands. It is important to note here that by this transformation we get only those partitions of $\nu - 2m + 1$ into m summands which contain $(k-1)_{k-1}$ as a summand. Therefore the actual number of partitions which belong to class (*iii*) is $A_k(m, \nu-2m+1) - A_k(m, \nu-3m+1)$, where $A_k(m, \nu-3m+1)$ is the number of partitions of $\nu-2m+1$ into m summands which are free from the summands like k_k. The above transformations clearly establish a bijection between the partitions enumerated by $A_k(m, \nu)$ and those enumerated by $A_k(m, \nu-m) + A_k(m-1, \nu-km-3m+k+2) + A_k(m, \nu-2m+1) - A_k(m, \nu-3m+1)$. Thus identity (2.5) is established. Let

$$f_k(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,\nu) z^m q^\nu. \tag{2.6}$$

Then (2.6) implies that

$$\begin{aligned}
f_k(z,q) =&\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}(A_k(m,\nu-m) + A_k(m-1,\nu-km-3m+k+2) \\
&+ A_k(m,\nu-2m+1) - A_k(m,\nu-3m+1))z^m q^\nu \qquad (2.7)\\
=&\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,\nu-m)(zq)^m q^{\nu-m} \\
&+zq\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_k(m-1,\nu-km-3m+k+2)(zq^{k+3})^{m-1} q^{\nu-m(k+3)+k+2} \\
&+\frac{1}{q}\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,\nu-2m+1)(zq^2)^m q^{\nu-2m+1} \\
&-\frac{1}{q}\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,\nu-3m+1)(zq^3)^m q^\nu - 3m + 1 \\
=&f_k(zq,q) + zqf_k(zq^{k+3},q) + \frac{1}{q}f_k(zq^2,q) - \frac{1}{q}f_k(zq^3,q).
\end{aligned}$$

Setting $f_k(z,q) = \sum_{\nu=0}^{\infty} \lambda_{k,\nu}(q) z^\nu$, and then comparing the coefficients of z^ν on each side of (2.3) we see that

$$\lambda_{k,\nu}(q) = \frac{\lambda_{k,\nu-1}(q) q^{(\nu-1)(k+3)+1}}{(1-q^\nu)(1-q^{2\nu-1})} \tag{2.8}$$

Iterating (2.4) ν times and observing that $\lambda_{k,0}(q) = 1$ we find that

$$\lambda_{k,\nu}(q) = \frac{q^{\nu\left[1+\frac{(k+3)(\nu-1)}{2}\right]}}{(q,q)_\nu (q,q^2)_\nu}. \tag{2.9}$$

Therefore

$$f_k(z,q)=\sum_{\nu=0}^{\infty}\frac{q^{\nu\left[1+\frac{(k+3)(\nu-1)}{2}\right]}z^{\nu}}{(q,q)_{\nu}(q,q^2)_{\nu}}. \tag{2.10}$$

Now

$$\sum_{\nu=0}^{\infty}A_k(\nu)q^{\nu}=\sum_{\nu=0}^{\infty}\left[\sum_{m=0}^{\infty}A_k(m,\nu)\right]q^{\nu}=f_k(1,q)$$
$$=\sum_{\nu=0}^{\infty}\frac{q^{\nu\left[1+\frac{(k+3)(\nu-1)}{2}\right]}}{(q;q^2)_{\nu}(q;q)_{\nu}}.$$

This completes the proof of the Theorem 2.3.3.

Now we provide the interpretations of Theorem 2.3.1–2.3.2 using $(n+t)$–color partitions with the help of above proved result.

Proof of Theorem 2.3.1. Let $A_k(m,\nu)$ and $B_k(m,\nu)$ denote respectively the number of partitions of ν enumerated by $A_k(\nu)$ and $B_k(\nu)$ with the added restriction that there be exactly m summands. From equation (2.2) we have if

$$f_k(z,q)=\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}A_k(m,\nu)z^mq^{\nu}, \tag{2.11}$$

then

$$f_k(z,q)=\sum_{\nu=0}^{\infty}\frac{q^{\nu\left[1+\frac{(k+3)(\nu-1)}{2}\right]}z^{\nu}}{(q,q)_{\nu}(q,q^2)_{\nu}}; \tag{2.12}$$

and so

$$f_k(z,q)-f_k(zq,q)=zq\sum_{\nu=0}^{\infty}\frac{q^{\frac{\nu(\nu+1)(k+3)}{2}}(zq)^n\nu}{(q,q)_{\nu}(q,q^2)_{\nu+1}}. \tag{2.13}$$

Setting

$$g_k(z,q)=\sum_{\nu=0}^{\infty}\frac{q^{\frac{\nu(\nu+1)(k+3)}{2}}z^{\nu}}{(q,q)_{\nu}(q,q^2)_{\nu+1}}$$
$$=\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}E_k(m,\nu)z^mq^{\nu},$$

we see by coefficient comparison in (2.13) that

$$A_k(m,\nu)-A_k(m,\nu-m)=E_k(m-1,\nu-m). \tag{2.14}$$

Equation (2.14) shows that $E_k(m,\nu)$ equals the number of partitions of $\nu+m+1$ with "n copies of n" into $m+1$ summands such that the weighted difference of each pair of summands m_i, n_j is greater than k and for some i, i_i is a summand. If we subtract 1 from each summand of a partition enumerated by $E_k(m,\nu)$ ignoring the subscripts, we see that the resulting partition is enumerated by $B_k(m+1,\nu)$. This implies that

$$E_k(m,\nu) = B_k(m+1,\nu). \tag{2.15}$$

Hence

$$\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty} B_k(m+1,\nu) z^m q^\nu = \sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{q^{\frac{\nu(\nu+1)(k+3)}{2}} z^m}{(q,q)_\nu (q,q^2)_{\nu+1}}.$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} B_k(\nu) q^\nu &= \sum_{\nu=0}^{\infty}\left[\sum_{m=0}^{\infty} B_k(m,\nu)\right] q^\nu = g_k(1,q)\\ &= \sum_{\nu=0}^{\infty} \frac{q^{\frac{\nu(\nu+1)(k+3)}{2}}}{(q,q)_\nu (q,q^2)_{\nu+1}}.\end{aligned}$$

This completes the proof of Theorem 2.3.1.

Proof of Theorem 2.3.2. Let $C_k(m,\nu)$ denote the number of partitions of ν counted by $C_k(\nu)$ with the added restriction that there be exactly m summands. Equation (2.13) can be written as

$$f_k(z,q) - f_k(zq,q) = zqh_k(z,q), \tag{2.16}$$

where

$$h_k(z,q) = \sum_{\nu=0}^{\infty} \frac{q^{\nu\left[1+\frac{(\nu+1)(k+3)}{2}\right]} z^\nu}{(q,q)_\nu (q,q^2)_\nu}. \tag{2.17}$$

Setting

$$h_k(z,q) = \sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty} F_k(m,\nu) z^m q^\nu, \tag{2.18}$$

we see by coefficient comparison in (2.16) that

$$A_k(m,\nu) - A_k(m,\nu-m) = F_k(m-1,\nu-1). \tag{2.19}$$

Equation (2.19) shows that $F_k(m,\nu)$ equals the number of partitions of $n+1$ with "n copies of n" into $m+1$ summands such that the weighted difference of each pair of summands m_i, n_j is greater than k and for some i, i_i is a summand. If we replace this summand i_i by $(i-1)_{i+1}$, we see that the resulting partition is enumerated by $C_k(m+1,\nu)$. This implies that

$$F_k(m,\nu) = C_k(m+1,\nu). \tag{2.20}$$

Hence

$$\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty} C_k(m+1,\nu) z^m q^\nu = \sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{q^{\nu\left[1+\frac{(\nu+1)(k+3)}{2}\right]} z^\nu}{(q,q)_\nu (q,q^2)_{\nu+1}}.$$

Now

$$\sum_{\nu=0}^{\infty} C_k(\nu) q^n = \sum_{\nu=0}^{\infty}\left[\sum_{m=0}^{\infty} C_k(m,\nu)\right] q^\nu = h_k(1,q)$$
$$= \sum_{\nu=0}^{\infty} \frac{q^{\nu\left[1+\frac{(\nu+1)(k+3)}{2}\right]}}{(q,q)_\nu (q,q^2)_{\nu+1}}.$$

This completes the proof of Theorem 2.3.2.

2.4 Particular Case

For $k = 0$, the Theorem 2.3.3 leads to

$$\sum_{\nu=0}^{\infty} \frac{q^{\frac{\nu(3\nu-1)}{2}}}{(q;q)_\nu (q;q^2)_\nu} = \frac{1}{(q;q)_\infty} \prod_{\nu=1}^{\infty} (1-q^{10\nu})(1-q^{10\nu-6})(1-q^{10\nu-4}) \quad (2.21)$$

which is combinatorially interpreted as

Theorem 2.4.1. The number of partitions with "n copies of n" of n such that each pair of summands m_i, r_j satisfies $|m-r| > i+j$ equals the number of ordinary partitions of n into summands $\neq 0, \pm 4(mod 10)$.

This identity (2.21) is listed in Slater's compendium [32][I(46), p.156]

Example. For $\nu = 6$, we have 8 relevant partitions of each kind, viz., $6_1, 6_2, 6_3, 6_4, 6_5, 6_6, 5_1 + 1_1, 5_2 + 1_1$ of the first kind and 51, 33, 321, 3111, 222, 2211, 21111, 111111 of the second kind.

In the next chapter, a new class of partitions, known as Split $(n + t)$–color partitions has been introduced. This new class has been used to interpret Rogers-Ramanujan type identities which are not interpreted earlier.

Chapter 3

Split Colored Partitions

3.1 Split $(n+t)$–Color Partitions and Theorems

In his last letter dated 12 January, 1920 to G.H. Hardy, S. Ramanujan listed 17 functions which he called mock theta functions. He separated these 17 functions into three classes. First containing 4 functions of order 3, second containing 10 functions of order 5 and the third containing 3 functions of order 7. Watson [36] found three more functions of order 3 and two more of order 5 appear in the lost notebook [28]. Mock theta functions of order 6 and 8 have also been studied in [15] and [22], respectively. For the definitions of mock theta functions and their order the reader is referred to [24]. Very recently Bringmann and Ono [17] redefined the mock theta functions and used their ideas in solving the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the series

$$f(q) = 1 + \sum_{n=1}^{\infty} \alpha(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n}, \tag{3.1}$$

where $f(q)$ is the first mock theta function of order 3 in Ramanujan's list of 17 mock theta functions. In 2004 [9], Agarwal interpreted of following four mock theta functions (first is of order 3 and the remaining three of order 5) of S Ramanujan;

$$\psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m}, \tag{3.2}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m}, \tag{3.3}$$

$$\phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q;q^2)_m, \tag{3.4}$$

$$\phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q;q^2)_m, \tag{3.5}$$

combinatorially using tools defined in Chapter 2.

Recently, in (2014) [14] split $(n+t)$–color partitions were introduced by Agarwal and Sood which generalizes Agarwal and Andrews $(n+t)$–color partitions. They used this new combinatorial tool to find combinatorial interpretations of two basic functions of Gordon –McIntosh[22]. Before stating the main results, let us define the new class of partitions called split $(n+t)$–color partitions.

Definition 3.1.1. A ***Split*** $(n+t)$–***Color Partition*** is a partition in which if m_i be a part in an $(n+t)$–color partition of a non negative integer ν, split the color 'i' into two parts - 'the green part' and 'the red part' and denote them by 'g' and 'r', respectively, such that $1 \leq g \leq i$, $0 \leq r \leq i-1$ and $i = g + r$. So, such $(n+t)$–color partition in which subscript of each part splits in this manner is called a split $(n+t)$–color partition.

Example: Split $(n+1)$–color partitions of 2 are,

$$2_1,\ \ 2_2,\ \ 2_3,\ \ 2_{1+1},\ \ 2_{2+1},\ \ 1_1 1_1.$$

The following two Gordon–McIntosh basic functions from [22]

$$V_0(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}, \tag{3.6}$$

and

$$V_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_{(n-1)}}{(q;q^2)_n} \tag{3.7}$$

are combinatorially interpreted by Agarwal and Sood [14] given as,

Theorem 3.1.1. For $\nu \geq 1$, let $A_1(\nu)$ denote the number of split n–color partitions of ν such that
(i) the summands and their subscripts have the same parity,
(ii) the red part of the subscripts can not exceed 1,
(iii) the least part is either $k_k (k \geq 1)$ or $k_{(k-1)+1} (k \geq 2)$, and
(iv) the weighted difference of any two consecutive parts is 0.
Then

$$V_0(q) = 1 + 2\sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu}. \tag{3.8}$$

Remark: In conditions (i) and (iv) we consider the whole subscript i, not its parts g and r, separately.

Theorem 3.1.2. For $\nu \geq 1$, let $A_2(\nu)$ denote the number of split n–color partitions of ν such that
(i) the summands and their subscripts have the same parity,
(ii) the red part of the subscripts can not exceed 1,
(iii) the least part is $k_k (k \geq 1)$, and
(iv) the weighted difference of any two consecutive parts is 0.
Then

$$V_1(q) = \sum_{n=1}^{\infty} A_2(\nu) q^{\nu}. \tag{3.9}$$

Remark: As in Theorem 3.1.1 here also, in conditions (i) and (iv) we consider the whole subscript i, not its parts g and r, separately.

In the proof we use the notation $A_i(m,\nu)(1 \leq i \leq 2)$ to denote the number of split n–color partitions enumerated by $A_i(\nu)$ into m summands with $A_1(0,0) = 1$. Also, we shall write

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n z^n}{(q;q^2)_n}, \tag{3.10}$$

and

$$f_2(z,q) = \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_{n-1} z^n}{(q;q^2)_n}, \tag{3.11}$$

where $|q| < 1, |z| < |q|^{-1}$.

Proof of Theorem 3.1.1. We split the partitions enumerated by $A_1(m,\nu)$ into three classes:
(i) those that contain 1_1 as a summand,
(ii) those that contain 2_{1+1} as a summand, and
(iii) those that contain $k_k(k > 1)$ or $k_{(k-1)_1}(k > 2)$ as a summand.

We transform the partitions in class (i) by deleting 1_1 and subtracting 2 from all the remaining summands, ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1(m-1, \nu-2m+1)$. Next we transform the partitions in class (ii) by deleting the summand 2_{1+1}, and then subtracting 4 from all the remaining summands ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1(m-1, \nu-4m+2)$. Finally, we transform the partitions in class (iii) by replacing k_k by $(k-1)_{k-1}$ or by replacing $k_{(k-1)+1}$ by $(k-1)_{(k-2)+1}$, as the case may be, and then subtracting 2 from all the remaining summands.The transformed partition will be of the type enumerated by $A_1(m, \nu-2m+1)$.

The above transformations clearly establish a bijection between the partitions enumerated by $A_1(m,\nu)$ and those enumerated by $A_1(m-1,\nu-2m+1)+A_1(m-1,\nu-4m+2)+A_1(m,\nu-2m+1)$. This leads to the identity.

$$A_1(m,\nu)=A_1(m-1,\nu-2m+1)+A_1(m-1,\nu-4m+2)+A_1(m,\nu-2m+1). \tag{3.12}$$

Let

$$h_1(z,q)=\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}A_1(m,\nu)z^m q^\nu. \tag{3.13}$$

where $|q|<1,|z|<|q|^{-1}$.

Substituting for $A_1(m,\nu)$ from (3.12) into (3.13) and then simplifying we arrive at the q–functional equation

$$h_1(z,q)=zqh_1(zq^2,q)+zq^2h_1(zq^4,q)+\frac{1}{q}h_1(zq^2,q). \tag{3.14}$$

Setting

$$h_1(z,q)=\sum_{n=0}^{\infty}\alpha_n(q)z^n, \qquad \alpha_0(q)=1, \tag{3.15}$$

we may easily check by coefficient comparison in (3.14) that

$$\alpha_n(q)=\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}. \tag{3.16}$$

Thus

$$h_1(z,q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n z^n}{(q;q^2)_n}=f_1(z,q). \tag{3.17}$$

Now

$$\begin{aligned}1+\sum_{\nu=1}^{\infty}A_1(\nu)q^\nu &=\sum_{\nu=0}^{\infty}\left[\sum_{m=0}^{\infty}A_1(m,\nu)\right]q^\nu\\ &=h_1(1,q)\\ &=f_1(1,q)\\ &=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}\\ &=1+\sum_{\nu=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}.\end{aligned}$$

Using penultimate equation in (3.6), we arrive at (3.9). This completes the proof of Theorem(3.1.1).

Proof of Theorem 3.1.2. We split the partitions enumerated by $A_2(m,\nu)$ into two classes:
(i) those that contain 1_1 as a summand,
(ii) those that contain $k_k (k>1)$ as a summand.

It is easy to see that there are $A_1(m-1,\nu-2m+1)$ in class (i) and $\frac{1}{2}(A_1(m-1,\nu-4m+2)+A_1(m,\nu-2m+1))$ in class (ii), so we get the identity

$$A_2(m,\nu) = A_1(m-1,\nu-2m+1)+\frac{1}{2}(A_1(m-1,\nu-4m+2)+A_1(m,\nu-2m+1)). \tag{3.18}$$

Let

$$h_2(z,q) = \sum_{\nu=1}^{\infty}\sum_{m=0}^{\infty} A_2(m,\nu)z^m q^\nu. \tag{3.19}$$

Translating (3.18) into a q–functional equation, we get

$$h_2(z,q) = zqh_1(zq^2,q) + \frac{1}{2}(zq^2h_1(zq^4,q) + \frac{1}{q}h_1(zq^2,q)). \tag{3.20}$$

Setting

$$h_2(z,q) = \sum_{n=1}^{\infty} \beta_n(q)z^n, \tag{3.21}$$

and using (3.15), we may easily check by coefficient comparison in (3.20) that

$$\beta_n(q) = q^{2n-1}\alpha_{n-1}(q) + \frac{1}{2}(q^{4n-2}\alpha_{n-1}(q) + q^{2n-1}\alpha_n(q)). \tag{3.22}$$

Substituting for $\alpha_n(q)$ from (3.16) into (3.22),we get

$$\beta_n(q) = \frac{q^{n^2}(-q;q^2)_{n-1}}{(q;q^2)_n}. \tag{3.23}$$

Thus

$$h_2(z,q) = \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_{n-1}z^n}{(q;q^2)_n} = f_2(z,q). \tag{3.24}$$

Hence

$$\begin{aligned}\sum_{\nu=1}^{\infty} A_2(\nu)q^\nu &= \sum_{\nu=1}^{\infty}\left[\sum_{m=0}^{\infty} A_2(m,\nu)\right] q^\nu \\ &= h_2(1,q) \\ &= f_2(1,q) \\ &= \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_{n-1}}{(q;q^2)_n} \\ &= V_1(q).\end{aligned}$$

This proves Theorem (3.1.2).

3.2 Conclusion

It is interesting to study this new tool, split $(n + t)$–color partitions. It shows a new path for a possible combinatorial interpretations of many more q–series. Agarwal and Sood in [14] posed an open problem: "Is it possible to find Rogers–Ramanujan type identities using split $(n+t)$–color partitions?" In Chapter 4, we address to this problem to some extent. We use split $(n+t)$–color partitions and interpret four generalized basic series combinatorially.

Chapter 4

New Combinatorial Interpretations of generalized q–series using Split $(n+t)$–Color Partitions

We use split $(n+t)$–color partitions defined in Chapter 3 to interpret combinatorially four generalized q– series, which in conjunction with the following four q–series identities from Chu and Zhang's compendium [18][I(25), I(27), I(29), I(113)], leads to new four combinatorial identities.

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_nq^{n^2}}{(q^4,q^4)_n(-q,q^2)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^2,-q^3;q^5]_\infty, \tag{4.1}$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_nq^{2n(n+1)}}{(q^4,q^4)_n(q,q^2)_{n+1}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^3,q^9;q^{12}]_\infty, \tag{4.2}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_nq^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_{n+1}}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^5,-q^5;q^5]_\infty, \tag{4.3}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_nq^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q,-q^4;q^5]_\infty. \tag{4.4}$$

It is important to mention here that, when two q–rising factorials $(q;q^2)_n$ and $(-q;q^2)_n$ appear together in a q–series then it is difficult to interpret them by using colored partitions.

Let $S=\{-1,1,3,5,\ldots\}$, for $|q|<1$, $j\in S$ and $1\le i\le 4$, we define $g_i^j(q)$ by

$$g_1^j(q)=\sum_{n=0}^{\infty}\frac{(-q,q^2)_nq^{n[1+\frac{(n-1)(j+3)}{2}]}}{(q^4,q^4)_n(q,q^2)_n}, \tag{4.5}$$

$$g_2^j(q)=\sum_{n=0}^{\infty}\frac{(-q,q^2)_nq^{\frac{n(n+1)(j+3)}{2}}}{(q^4,q^4)_n(q,q^2)_{n+1}}, \tag{4.6}$$

$$g_3^j(q) = \sum_{n=0}^{\infty} \frac{(-q,q^2)_n q^{n[1+\frac{(n+1)(j+3)}{2}]}}{(q^4,q^4)_n (q,q^2)_{n+1}}, \tag{4.7}$$

$$g_4^j(q) = \sum_{n=0}^{\infty} \frac{(-q,q^2)_n q^{n[1+\frac{(n+1)(j+3)}{2}]}}{(q^4,q^4)_n (q,q^2)_n}. \tag{4.8}$$

In next section, we provide the combinatorial proofs of (4.5) – (4.8) with the help of split $(n+t)$–color partitions. In the further section, we obtain combinatorial interpretations of Rogers–Ramanujan type identities (4.1)–(4.4) as particular cases of Theorems (4.1.1)–(4.1.4) given in next section.

4.1 Main Results

Theorem 4.1.1. For $j \in S$, let $P_1^j(\nu)$ represent the number of split n–color partitions of ν such that
(i) the value of red part can either be 0 or 1,
(ii) if m_i is the least or only summand of partition, then $m - i \equiv 0(mod 4)$,
(iii) the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1)(mod 4)$.
Then

$$\sum_{\nu=0}^{\infty} P_1^j(\nu) q^\nu = g_1^j(q), \tag{4.9}$$

where conditions (ii) and (iii) are allowed for the whole subscripts i irrespective of green and red part separately.

Theorem 4.1.2. For $j \in S$, let $P_2^j(\nu)$ represent the number of split $(n+1)$–color partitions of ν such that
(i) the value of red part can either be 0 or 1,
(ii) the smallest summand is of the form i_{i+1} where $g = i+1$; $r = 0$,
(iii) the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1)(mod 4)$.
Then

$$\sum_{\nu=0}^{\infty} P_2^j(\nu) q^\nu = g_2^j(q), \tag{4.10}$$

where conditions (ii) and (iii) are allowed for the whole subscripts i irrespective of green and red part separately.

Theorem 4.1.3. For $j \in S$, let $P_3^j(\nu)$ represent the number of split $(n+2)$–color partitions of ν such that
(i) the value of red part can either be 0 or 1,
(ii) the smallest summand is of the form i_{i+2} where $g = i+2$; $r = 0$,
(iii) the weighted difference among any two consecutive summands is greater

than j and is congruent to $(j+1)(mod 4)$.
Then

$$\sum_{\nu=0}^{\infty} P_3^j(\nu)q^\nu = g_3^j(q), \tag{4.11}$$

where conditions (ii) and (iii) are allowed for the whole subscripts i irrespective of green and red part separately.

Theorem 4.1.4. For $j \in S$, let $P_4^j(\nu)$ represent the number of split n–color partitions of ν such that
(i) the value of red part can either be 0 or 1,
(ii) if m_i is the least or only summand of partition, then $m \geq (j+4)$ and $m - i \equiv (j+3)(mod 4)$,
(iii) the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1)(mod 4)$.
Then

$$\sum_{\nu=0}^{\infty} P_4^j(\nu)q^\nu = g_4^j(q), \tag{4.12}$$

where conditions (ii) and (iii) are allowed for the whole subscripts i irrespective of green and red part separately.

Proof of theorem(4.1.1). Let $P_1^j(m,\nu)$ denote the number of partitions of ν enumerated by $P_1^j(\nu)$ into m parts. We split the partitions enumerated by $P_1^j(m,\nu)$ into four classes:

(i) those that do not involve k_k or $k_{(k-1)+1}$ as a summand,
(ii) those that involve 1_1 as a summand,
(iii) those that involve 2_{1+1} as a summand,
(iv) those that involve $k_k (k \geq 2)$ and $k_{(k-1)+1} (k \geq 3)$ as a summand.

We transform the partitions which lie in class (i) by subtracting 4 from subsequent summands, ignoring the subscripts. The transformed partitions are enumerated by $P_1^j(m, \nu - 4m)$. In the class (ii) delete the summand 1_1 and then subtract $(j+3)$ from remaining summands ignoring the subscripts, we get partitions enumerated by $P_1^j(m-1, \nu - m(j+3) + j + 2)$. Next, in class (iii) delete 2_2 and then subtract $(j+5)$ from the remaining summands ignoring the subscripts. The transformed partitions are enumerated by $P_1^j(m-1, \nu - m(j+5) + j + 3)$. In the last class (iv) replace k_k by $(k-1)_{k-1}$ and $k_{(k-1)+1}$ by $(k-1)_{(k-2)+1}$ and subtract 2 from the remaining summands ignoring the subscripts. This will results in partition enumerated by $P_1^j(m, \nu - 2m + 1)$. It should be noted here that we are obtaining only those partitions here of $\nu - 2m + 1$ which involve a summand of the type k_k

and $k_{(k-1)+1}$. So the number of partition in class (iv) enumerated actually is $P_1^j(m, \nu - 2m + 1) - P_1^j(m, \nu - 6m + 1)$.

Since the transformation defined above are reversible and they give one to one correspondence between the partitions enumerated by $P_1^j(m, \nu)$ and those by

$P_1^j(m, \nu - 4m) + P_1^j(m-1, \nu - m(j+3) + j + 2) + P_1^j(m-1, \nu - m(j+5) + j + 3) + P_1^j(m, \nu - 2m + 1) - P_1^j(m, \nu - 6m + 1)$.

This will give rise to the following recurrence relation:

$$\begin{aligned} P_1^j(m, \nu) =& P_1^j(m, \nu - 4m) + P_1^j(m-1, \nu - m(j+3) + j + 2) \\ &+ P_1^j(m-1, \nu - m(j+5) + j + 3) \\ &+ P_1^j(m, \nu - 2m + 1) - P_1^j(m, \nu - 6m + 1). \end{aligned} \tag{4.13}$$

For $|q| < 1$ and $|z| < |q|^{-1}$, let $f_1^j(z, q)$ is defined by

$$f_1^j(z, q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} P_1^j(m, \nu) z^m q^\nu. \tag{4.14}$$

Substitute $P_1^j(m, \nu)$ from (4.13) in (4.14) and then analyzing we get

$$\begin{aligned} f_1^j(z, q) =& f_1^j(zq^4, q) + zq f_1^j(zq^{j+3}, q) + zq^2 f_1^j(zq^{j+5}, q) \\ &+ q^{-1} f_1^j(zq^2, q) - q^{-1} f_1^j(zq^6, q). \end{aligned} \tag{4.15}$$

Setting

$$f_1^j(z, q) = \sum_{n=0}^{\infty} \beta(n, q) z^n \tag{4.16}$$

in (4.15) and then examining the coefficients of z^n in the above expression we get

$$\beta(n, q) = \frac{q^{1+(j+3)(n-1)}(1 + q^{2n-1})}{(1 - q^{4n})(1 - q^{2n-1})} \beta(n-1, q). \tag{4.17}$$

Iterating (4.17) n times and note that $\beta(0, q) = 1$, we find that

$$\beta(n, q) = \frac{(-q; q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^4; q^4)_n (q; q^2)_n}. \tag{4.18}$$

Therefore,

$$f_1^j(z, q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^4; q^4)_n (q; q^2)_n} z^n \tag{4.19}$$

$$= g_1^j(z,q) \tag{4.20}$$

and

$$\begin{aligned}\sum_{\nu=0}^{\infty} P_1^j(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} P_1^j(m,\nu)\right) q^\nu \\ &= g_1^j(1,q) \\ &= g_1^j(q).\end{aligned}$$

This proves Theorem 4.1.1.

Proof of Theorem 4.1.2.

Let $D^j(\nu)$ represent the number of split n–color partitions of ν enumerated by $P_1^j(\nu)$ with some constraint that the least part is of the type k_k and let $D^j(m,\nu)$ represent the number of split n–color partitions of ν enumerated by $D^j(\nu)$ into m parts. Further let

$$h^j(q) = \sum_{\nu=0}^{\infty} D^j(\nu)q^\nu, \tag{4.21}$$

and

$$h^j(z,q) = \sum_{\nu,m=0}^{\infty} D^j(m,\nu)z^m q^\nu. \tag{4.22}$$

With the help of (4.13),we have

$$\begin{aligned}D^j(m,\nu) =& P_1^j(m-1,\nu-m(j+3)+j+2) \\ &+ \frac{1}{2}[P_1^j(m-1,\nu-m(j+5)+j+3) \\ &+ P_1^j(m,\nu-2m+1) - P_1^j(m,\nu-6m+1)].\end{aligned} \tag{4.23}$$

Transforming (4.23) into a q–functional equation, we get

$$\begin{aligned}h^j(z,q) =& zqf_1^j(zq^{j+3},q) + \frac{1}{2}zq^2 f_1^j(zq^{j+5},q) \\ &+ \frac{1}{2}q^{-1}f_1^j(zq^2,q) - \frac{1}{2}q^{-1}f_1^j(zq^6,q).\end{aligned} \tag{4.24}$$

Setting

$$h^j(z,q) = \sum_{n=0}^{\infty} \gamma(n,q)z^n, \tag{4.25}$$

and then examining the coefficients of z^n in the above expression (4.24) we get

$$\begin{aligned}2\gamma(n,q) =& 2q^{(j+3)(n-1)+1}\beta(n-1,q) + q^{(j+5)(n-1)+2}\beta(n-1,q)\\ &+ q^{2n-1}\beta(n,q) + q^{6n-1}\beta(n,q)\end{aligned} \tag{4.26}$$

Replacing $\beta(n,q)$ from (4.18) in (4.26) and then analyzing, we get

$$\gamma(n,q) = \frac{(-q;q^2)_{n-1}q^{n(1+\frac{(j+3)(n-1)}{2})}}{(q^4;q^4)_{n-1}(q;q^2)_n}. \tag{4.27}$$

Thus

$$h^j(z,q) = \sum_{\nu=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)[1+\frac{(j+3)n}{2}]}}{(q^4;q^4)_n(q;q^2)_{n+1}} z^{n+1} = zqg_2^j(zq,q). \tag{4.28}$$

Define $Q^j(m,\nu)$ by

$$g_2^j(z,q) = \sum_{m,\nu=0}^{\infty} Q^j(m,\nu)z^m q^\nu.$$

By examining the coefficients of (4.28), we get

$$D^j(m+1,\nu+m+1) = Q^j(m,\nu).$$

If each summand is subtracted by 1 which is enumerated by $D^j(m+1,\nu+m+1)$ ignoring the subscripts, we have the final partition enumerated by $P_2^j(m+1,\nu)$. Thus

$$Q^j(m,\nu) = P_2^j(m+1,\nu)$$

and so

$$\sum_{m,\nu=0}^{\infty} P_2^j(m+1,\nu)z^m q^\nu = g_2^j(z,q).$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} P_2^j(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} P_2^j(m,\nu)\right)q^\nu\\ &= \sum_{m,\nu=0}^{\infty} P_2^j(m+1,\nu)q^\nu\\ &= g_2^j(1,q)\\ &= g_2^j(q).\end{aligned}$$

This proves Theorem 4.1.2.

Proof of Theorem 4.1.3.

If we write equation (4.28) as

$$h^j(z,q) = zqg_3^j(z,q). \tag{4.29}$$

Define $R^j(m,\nu)$ by

$$g_3^j(z,q) = \sum_{m,\nu=0}^{\infty} R^j(m,\nu)z^m q^\nu. \tag{4.30}$$

By examining the coefficients of (4.29), we get

$$D^j(m+1,\nu+1) = R^j(m,\nu). \tag{4.31}$$

If summand k_k is replaced by $(k-1)_{k+1}$ which is enumerated by $D^j(m+1,\nu+1)$, we have the final partition enumerated by $P_3^j(m+1,\nu)$. Thus

$$R^j(m,\nu) = P_3^j(m+1,\nu)$$

and

$$\sum_{m,\nu=0}^{\infty} P_3^j(m+1,\nu)z^m q^\nu = g_3^j(z,q). \tag{4.32}$$

Now

$$\begin{aligned}
\sum_{\nu=0}^{\infty} P_3^j(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} P_3^j(m,\nu)\right)q^\nu \\
&= \sum_{m,\nu=0}^{\infty} P_3^j(m+1,\nu)q^\nu \\
&= g_3^j(1,q) \\
&= g_3^j(q).
\end{aligned}$$

This proves Theorem 4.1.3.

Proof of Theorem 4.1.4.

Let $P_4^j(m,\nu)$ denote the number of partitions of ν enumerated by $P_4^j(\nu)$ into m parts. We split the partitions enumerated by $P_4^j(m,\nu)$ into four classes:

(i) those that do not involve $k_{(k-(j+3))+0}$ or $k_{(k-(j+4))+1}$ as a summand,
(ii) those that involve $(j+4)_1$ as a summand,
(iii) those that involve $(j+5)_{1+1}$ as a summand,
(iv) those that involve $k_{(k-(j+3))+0}(k \geq (j+5))$ and $k_{(k-(j+4))+1}(k \geq (j+6))$ as a summand.

We transform the partitions which lie in class (i) by subtracting 4 from subsequent summands ignoring the subscripts. The transformed partitions are enumerated by $P_4^j(m, \nu - 4m)$. In the class (ii) delete the summand $(j+4)_1$ and then subtract $(j+3)$ from remaining summands ignoring the subscripts, we get partitions enumerated by $P_4^j(m-1, \nu - m(j+3) - 1)$. Next, in class (iii) delete $(j+5)_{1+1}$ and then subtract $(j+5)$ from the remaining summands ignoring the subscripts. The transformed partitions are enumerated by $P_4^j(m-1, \nu - m(j+5))$. In the last class (iv) replace $k_{(k-(j+3))+0}$ by $(k-1)_{(k-(j+4))+0}$ and $k_{(k-(j+4))+1}$ by $(k-1)_{(k-(j+5))+1}$ and subtract 2 from the remaining summands ignoring the subscripts. This will results in partition enumerated by $P_4^j(m, \nu - 2m + 1)$. It should be noted here that we are obtaining those partitions here of $\nu - 2m + 1$ which involve a summand of the type $k_{(k-(j+3))+0}$ and $k_{(k-(j+4))+1}$. So the number of partition in class (iv) enumerated actually is $P_4^j(m, \nu - 2m + 1) - P_4^j(m, \nu - 6m + 1)$.

Since the transformations given above are reversible so we get the following recurrence relation:

$$\begin{aligned} P_4^j(m,\nu) = &P_4^j(m, \nu - 4m) + P_4^j(m-1, \nu - m(j+3) - 1) \\ &+ P_4^j(m-1, \nu - m(j+5)) \\ &+ P_4^j(m, \nu - 2m + 1) - P_4^j(m, \nu - 6m + 1). \end{aligned} \tag{4.33}$$

For $|q| < 1$ and $|z| < |q|^{-1}$, let $f_2^j(z,q)$ is defined by

$$f_2^j(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} P_4^j(m,\nu) z^m q^\nu. \tag{4.34}$$

Using (4.33) and (4.34) we get the following q–functional equation

$$\begin{aligned} f_2^j(z,q) = &f_2^j(zq^4, q) + zq^{j+4} f_2^j(zq^{j+3}, q) + zq^{j+5} f_2^j(zq^{j+5}, q) \\ &+ q^{-1} f_2^j(zq^2, q) - q^{-1} f_2^j(zq^6, q). \end{aligned} \tag{4.35}$$

Setting

$$f_2^j(z,q) = \sum_{n=0}^{\infty} \beta(n,q) z^n \tag{4.36}$$

in (4.35) and then examining the coefficients of z^n in the above expression we get

$$\beta(n,q) = \frac{q^{1+(j+3)(n)}(1+q^{2n-1})}{(1-q^{4n})(1-q^{2n-1})}\beta(n-1,q). \tag{4.37}$$

Iterating above n–times and noting $\beta(0,q) = 1$, we find

$$\beta(n,q) = \frac{(-q;q^2)_n q^{n(1+\frac{(j+3)(n+1)}{2})}}{(q^4;q^4)_n(q;q^2)_n}. \tag{4.38}$$

Therefore

$$f^j(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(1+\frac{(j+3)(n+1)}{2})}}{(q^4;q^4)_n(q;q^2)_n} z^n = g_1^j(z,q) \tag{4.39}$$

and

$$\begin{aligned}\sum_{\nu=0}^{\infty} P_4^j(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} P_4^j(m,\nu)\right)q^\nu \\ &= g_4^j(1,q) \\ &= g_4^j(q).\end{aligned}$$

This proves Theorem 4.1.4.

4.2 Particular Cases

The Identity (4.1), in conjunction with Theorem 4.1.1, for $j = -1$, leads to the following theorem:

Theorem 4.2.1. Let $E_1(\nu)$ denote the number of n–color partitions of ν such that parts are distinct and first two copies of parts $\equiv 5(mod 10)$ are used and only first copy of parts $\equiv \pm 1(mod 10)$ are used and let $F_1(\nu)$ denote the number of n–color partitions of ν such that first two copies of parts $\equiv \pm 2(mod 10)$ are used. Further let

$$A_1(\nu) = \sum_{k=0}^{\nu} E_1(k)F_1(\nu-k),$$

then

$$A_1(\nu) = P_1^{-1}(\nu), \text{ for all } \nu.$$

Example. We can verify Theorem 4.1.1 by showing that

$$P_1^{-1}(10) = A_1(10) = 18.$$

The relevant n–color partitions corresponding to $P_1^{-1}(10)$ are

10_{10}, 10_{9+1}, 10_6, 10_{5+1}, 10_2, 10_{1+1}, $9_7 1_1$, $9_{6+1} 1_1$, $9_3 1_1$, $9_{2+1} 1_1$,

$8_4 2_2$, $8_{3+1} 2_2$, $8_4 2_{1+1}$, $8_{3+1} 2_{1+1}$, $7_1 3_3$, $7_1 3_{2+1}$, $6_2 3_1 1_1$, $6_{1+1} 3_1 1_1$

and $A_1(10) = 18$, since

$$A_1(10) = \sum_{k=0}^{10} E_1(k) F_1(10-k),$$

where the relevant partitions corresponding to $E_1(\nu)$ and $F_1(\nu)$ are given in the table below:

ν	$E_1(\nu)$	partitions enumerated by $E_1(\nu)$	$F_1(\nu)$	partitions enumerated by $F_1(\nu)$
0	1	empty partition	1	empty partition
1	1	1_1	0	-
2	0	-	2	2_1, 2_2
3	0	-	0	-
4	0	-	3	$2_1 2_1$, $2_2 2_1$, $2_2 2_2$
5	2	5_1, 5_2	0	—
6	2	$5_1 1_1$, $5_2 1_1$	4	$2_1 2_1 2_1$, $2_2 2_1 2_1$, $2_2 2_2 2_1$, $2_2 2_2 2_2$
7	0	-	0	—
8	0	-	7	8_1, 8_2, $2_1 2_1 2_1 2_1$, $2_2 2_1 2_1 2_1$, $2_2 2_2 2_1 2_1$, $2_2 2_2 2_2 2_1$, $2_2 2_2 2_2 2_2$
9	1	9_1	0	—
10	2	$5_2 5_1$, $9_1 1_1$	10	$8_1 2_1$, $8_2 2_1$, $8_1 2_2$, $8_2 2_2$, $2_1 2_1 2_1 2_1 2_1$, $2_2 2_1 2_1 2_1 2_1$, $2_2 2_2 2_1 2_1 2_1$, $2_2 2_2 2_2 2_1 2_1$, $2_2 2_2 2_2 2_2 2_1$, $2_2 2_2 2_2 2_2 2_2$

Therefore,

$$\begin{aligned} A_1(10) =& E_1(10)F_1(0) + E_1(9)F_1(1) + \cdots + E_1(0)F_1(10) \\ =& 18. \end{aligned}$$

The Identity (4.2), in conjunction with Theorem 4.1.2, for $j = 1$, leads to the following theorem:

Theorem 4.2.2. Let $E_2(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm 1, \pm 5 (mod 12)$ and let $F_2(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 2, \pm 4 (mod 12)$. Further let

$$A_2(\nu) = \sum_{k=0}^{\nu} E_2(k) F_2(\nu - k),$$

then

$$A_2(\nu) = P_2^1(\nu), \text{ for all } \nu.$$

Example. We can verify Theorem 4.1.2 by showing that

$$P_2^1(10) = A_2(10) = 8.$$

The relevant $(n+1)$–color partitions corresponding to $P_2^1(10)$ are

10_{11}, $10_7 0_1$, $10_{6+1} 0_1$, $10_3 0_1$ $10_{2+1} 0_1$, $9_4 1_2$, $9_{3+1} 1_2$, $8_1 2_3$
and $A_2(10) = 8$, since

$$A_2(10) = \sum_{k=0}^{10} E_2(k) F_2(10 - k),$$

where the relevant partitions corresponding to $E_2(\nu)$ and $F_2(\nu)$ are given in the table below:

ν	$E_2(\nu)$	partitions enumerated by $E_2(\nu)$	$F_2(\nu)$	partitions enumerated by $F_2(\nu)$
0	1	empty partition	1	empty partition
1	1	1	0	-
2	0	-	1	2
3	0	-	0	-
4	0	-	2	4, 2+2
5	1	5	0	-
6	1	5+1	2	4+2, 2+2+2
7	1	7	0	-
8	1	7+1	4	8, 4+4, 4+2+2, 2+2+2+2
9	0	-	0	-
10	0	-	5	10, 8+2, 4+4+2, 4+2+2+2, 2+2+2+2+2

Therefore,

$$\begin{aligned} A_2(10) =& E_2(10)F_2(0) + E_2(9)F_2(1) + \cdots + E_2(0)F_2(10) \\ =& 8. \end{aligned}$$

The Identity (4.3), in conjunction with Theorem 4.1.3, for $j = -1$, leads to the following theorem:

Theorem 4.2.3. Let $E_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm1, \pm3 (mod 10)$ and let $F_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm2, \pm4 (mod 10)$. Further let

$$A_3(\nu) = \sum_{k=0}^{\nu} E_3(k) F_3(\nu - k),$$

then

$$A_3(\nu) = P_3^{-1}(\nu), \text{ for all } \nu.$$

Example. We can verify Theorem 4.1.3 by showing that

$$P_3^{-1}(10) = A_3(10) = 12.$$

The relevant $(n+2)$–color partitions corresponding to $P_3^{-1}(10)$ are

10_{12}, $10_8 0_2$, $10_4 0_2$, $10_{7+1} 0_2$, $10_{3+1} 0_2$, $9_5 1_3$, $9_1 1_3$, $9_{4+1} 1_3$,

$8_2 2_4$, $8_{1+1} 2_4$, $7_3 3_1 0_2$, $7_{2+1} 3_1 0_2$,

and $A_3(10) = 12$, since

$$A_3(10) = \sum_{k=0}^{10} E_3(k) F_3(10 - k),$$

where the relevant partitions corresponding to $E_3(\nu)$ and $F_3(\nu)$ are given in the table below:

ν	$E_3(\nu)$	partitions enumerated by $E_3(\nu)$	$F_3(\nu)$	partitions enumerated by $F_3(\nu)$
0	1	empty partition	1	empty partition
1	1	1	0	-
2	0	-	1	2
3	1	3	0	-
4	1	$3+1$	2	4, 2+2
5	0	-	0	-
6	0	-	3	6, 4+2, 2+2+2
7	1	7	0	-
8	1	$7+1$	5	8, 6+2, 4+4, 4+2+2, 2+2+2+2
9	1	9	0	-
10	2	7+3, 9+1	6	8+2, 6+2+2, 6+4, 4+4+2, 4+2+2+2, 2+2+2+2+2

Therefore,

$$\begin{aligned} A_3(10) =& E_3(10)F_3(0) + E_3(9)F_3(1) + \cdots + E_3(0)F_3(10) \\ =& 12. \end{aligned}$$

The Identity (4.4), in conjunction with Theorem 4.1.4, for $j = -1$, leads to the following theorem:

Theorem 4.4.4. Let $E_4(\nu)$ denote the number of n–color partitions of ν into distinct parts such that first two copies of parts $\equiv 5(mod 10)$ are used and only first copy of parts $\equiv \pm 3(mod 10)$ are used and let $F_4(\nu)$ denote the number of n–color partitions of ν such that first two copies of parts $\equiv \pm 4(mod 10)$ are used. Further let

$$A_4(\nu) = \sum_{k=0}^{\nu} E_4(k)F_4(\nu - k),$$

then

$$A_4(\nu) = P_4^{-1}(\nu), \text{ for all } \nu.$$

Example. We can verify Theorem 4.1.4 by showing that

$$P_4^{-1}(10) = A_4(10) = 6.$$

The relevant n–color partitions corresponding to $P_4^{-1}(10)$ are

10_8, 10_{7+1}, 10_4, 10_{3+1}, $7_3 3_1$, $7_{2+1} 3_1$

and $A_4(10) = 6$, since

$$A_4(10) = \sum_{k=0}^{10} E_4(k)F_4(10-k),$$

where the relevant partitions corresponding to $E_4(\nu)$ and $F_4(\nu)$ are given in the table below:

ν	$E_4(\nu)$	partitions enumerated by $E_4(\nu)$	$F_4(\nu)$	partitions enumerated by $F_4(\nu)$
0	1		1	empty partition
1	0	-	0	
2	0	-	0	-
3	1	3_1	0	-
4	0	-	2	4_1, 4_2
5	2	5_1, 5_2	0	-
6	0	-	2	6_1, 6_2
7	1	7_1	0	-
8	2	$4_1 3_1$, $5_2 3_1$	3	$4_1 4_1$, $4_2 4_1$, $4_2 4_2$
9	0	0	0	-
10	2	$5_1 5_2$, $7_1 3_1$	4	$6_1 4_1$, $6_1 4_2$, $6_2 4_1$, $6_2 4_2$

Therefore,

$$\begin{aligned} A_4(10) =& E_4(10)F_4(0) + E_4(9)F_4(1) + \cdots + E_4(0)F_4(10) \\ =& 6. \end{aligned}$$

4.3 Conclusion

The study of this thesis concludes the combinatorial interpretations of some q–series identities, generalized q–series and split $(n+t)$–colored partitions. And to apply these tools to explore some new q–series. With the help of these tools we obtain some new combinatorial interpretations of generalized q–series which further give rise to Rogers–Ramanujan type identities.

Now we can say there is advances in enumerations of q–series using this new tool. So we further believe that many more q–series identities can be explored with this tool and using the techniques given by us in Chapter 4.

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