STRUCTURE AND OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED WHITE DWARF MODELS OF STARS

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CERTIFICATE

This is to certify that the thesis "Mathematical Modelling of Rotationally and Tidally Distorted Pulsating White Dwarfs Stars" submitted by Mr. Tarun Kumar of M.Sc (Mathematics and Computing), Thapar University, Patiala, was carried out by me under supervisions of Dr. A.K. Lal. He has not submitted this material for credit towards any other degree at Thapar University Patiala or any other University.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

In the present thesis we have studied the possibility of using Mohan, Saxena and Agarwal (1990) approach to obtain the effect of rotation and tidal forces on the equilibrium structures and the eigenfrequencies of the radial and nonradial modes of oscillations of theoretical models of the stars. Mathematical models of rotationally and tidally distorted stars have been developed in general and used to compute the equilibrium structures and eigenfrequencies of small adiabatic modes of oscillations of certain rotationally white dwarfs. Effects of tidal distortion, occurring due to the presence of companion stars in binary system, on the equilibrium structure and periods of oscillations of rotating stars, have also been discussed. The study is expected to help in better understanding the limitations and scope of Mohan, Saxena and Agarwal (1990) approach to determine the structure and pulsations of theoretical models of stars.

Thesis consists of four chapters. Chapter I is introductory in nature. in this chapter we briefly discuss the astrophysical significance of the problem of determining the equilibrium structure and periods of small adiabatic oscillations of rotationally and tidally distorted stellar models of stars. A brief summary of literature available on the subject is also presented in this chapter and summary of the work presented in the succeeding chapter of the thesis also appears in this chapter

In chapter II we use the approach of Mohan, Saxena and Agarwal (1990) and Lal et al (2000) to determine the equilibrium structure of rotationally and tidally distorted white dwarfs models of stars under the assumptions that angular velocity of rotation is not too large. Numerical results have been computed for certain rotationally and tidally distorted white dwarfs models for the parameters $\frac{1}{\phi_0^2}$ as $0.01, 0.05, 0.2, 0.4, 0.6$ and 0.8 and analyses the result.

In chapter III we analyze the effect of rotation and tidal distortion on the eigenfrequencies of small adiabatic barotropic mode of oscillations of stars. The eigenvalued boundary value problems which determine the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted white

dwarf models have been formulated using Mohan, Saxena and Agarwal (1990) approach and computed the eigenfrequencies of various pseudo-radial modes of oscillations rotationally and tidally distorted white dwarfs stars with central degeneracy parameters $\frac{1}{\phi_0^2}$ as $0.01, 0.05, 0.2, 0.4, 0.6, 0.8$. Certain conclusions are drawn.

Chapter IV is devoted to study the effect of rotation and tidal distortion on the eigenfrequencies of small adiabatic barotropic mode of oscillations of stars. The eigenvalued boundary value problems which determine the eigenfrequencies of small adiabatic nonradial modes of oscillations of rotationally and tidally distorted white dwarf stars have been formulated using Mohan, Saxena and Agarwal (1990) approach. The eigenfrequencies of various nonradial modes of oscillations rotationally and tidally distorted white dwarfs stars with parameters $\frac{1}{\phi_0^2}$ as $0.05, 0.2, 0.6, 0.8$ have been computed using the approach earlier discussed by Mohan and Saxena (1985). Conclusions based on the present study are also summarized in this chapter.

CONTENTS

CHAPTER – I

INTRODUCTION

This chapter is introductory in nature. In section 1.1 we first explain in brief the astrophysical significance of the theoretical study of the problem of determining the effects of rotation and/or tidal distortions on the equilibrium structures and the periods of small adiabatic oscillations of gaseous spheres. A brief survey of the literature available on the subject is presented in Section 1.2. In section1.3 we present how Kippenhahn and Thomas (48) averaging technique to derive system of differential equations which determine the equilibrium structures of rotationally and tidally distorted gaseous spheres. The concepts of Roche equipotentials and Roche limits are then introduced in section 1.4. In section 1.5 we present how Mohan et al (91,92) used Kippenhahn and Thomas (39) approach in conjunction with certain result on Roche equipotentials to obtain the system of differential equation governing the equilibrium structures of rotationally and tidally distorted stellar models .A brief summary of the work presented in the succeeding chapters of this thesis is finally presented in section 1.6.

1.1 ASTROPHYSICAL SIGNIFICANCE OF DETERMINING THE EFFECTS OF ROTATIONAL AND TIDAL DISTORTIONS ON THE EQUILIBRIUM STRUCTURES AND THE PERIODS OF OSCILLATIONS OF GASEOUS SPHERES

The theoretical model of a star is essentially a self gravitating gaseous sphere in hydrostatic and thermal equilibrium. Theoretical studies of the problems of the equilibrium structure of gaseous spheres have often been carried out to understand the nature of the internal structures, responsible for various observed phenomena of the stars. Whereas some of the stars are observed as single stars others are observed in groups of two or more stars. Observations also show that some of the stars are rotating about their axes of rotation. Many of the stars in binary and multiple systems are also known to be rotating about their axes as well as revolving around each other. Thus if we assume the equilibrium model of a single nonrotating star as a gaseous sphere, the equilibrium model of a rotating star will be rotationally distorted gaseous sphere. Similarly, the equilibrium model of a star appearing in a binary or a multiple system will be a tidally distorted gaseous sphere if it is not rotating and a rotationally and tidally distorted gaseous sphere if the star is rotating as well.

The brightness of certain observed stars varies with time which is called variable stars. In some of these variable stars, the variations in luminosity are periodic. It is reasonable to assume in the case of such regular variable stars that these are pulsating gaseous spheres in which the variation in luminosity is being caused by the periodic contraction and expansion of the gaseous mass. The regular variable stars gained importance in astrophysics when it was discovered that there exists a definite relation between the periods of pulsation and the luminosities of such stars. This relationship has been utilized to determine the distance of stars. Such an important use of the regular variable stars motivated theoretical astrophysicists to investigate the problems of small oscillations of the equilibrium models of the variable stars so as to have a clear idea of the mechanism which could possibly be sustaining pulsations in these stars. Such investigations are also expected to help us in better understanding the nature of the internal structure of the stars. In most of these theoretical studies, the variable star is represented by a gaseous sphere undergoing radial and nonradial oscillations.

Observations show that some of the variable stars are also rotating stars. The theoretical models of such stars can be regarded as rotationally distorted gaseous spheres performing small oscillations about their equilibrium configuration. Similarly some of the variable stars have also been observed in binary and multiple stellar systems. The theoretical model of such stars can be regarded as rotationally and tidally distorted gaseous spheres performing small oscillations about their equilibrium configuration.

Analytic studies of the problems of rotating stars and stars in binary systems have engaged the attention of astrophysicist since long with a view to analyze and understand the observational behavior of such stars. In a binary system of stars, the two stars normally rotate about their own axis as well as revolve about their common center of mass. In a majority of binary stars, one star called primary, is generally more massive compared to its companion star. Contact binaries also been observed in which outermost surfaces of the two stars just touch each other Keeping this in view an attempt has been made in the present thesis to investigate certain aspects of the problems of equilibrium structures and small oscillations of rotationally and tidally distorted gaseous spheres which still need investigations.

1.2 BRIEF SURVEY OF THE LITERATURE.

Most of the theoretical studies about the equilibrium structures and oscillations of the stars have been carried out in literature by assuming the star to be an undistorted spherical gaseous sphere. Extensive literature is now available on this subject (see for instance Chandrasekhar (14), Rosseland (109), Eddington (34), Mentzel et al. (80), Cox and Giuli (26), Kippenhahn and Weigert (49), Clement (20), Kopal (53,55), Tassoul (133), Cox (25), Bohm-Vitense (6), and Unno et al (138).

It is observed that brightness of some of the stars varies with time. These are stars called variable stars. Whereas variation in brightness of some of the stars is regular and periodic in others it is not so. An important class of regular variable stars is Cepheid variables. The regular variable stars, gained importance in astrophysics in the year 1912, when Miss Leavitt discovered that there exists a definite relation between the periods of pulsation and the luminosities of such stars and the relationship could be utilized to determine the distance of these stars. In most of the theoretical studies of such stars, the variable star is represented by a gaseous sphere, both in hydrostatic and thermal equilibrium, undergoing small periodic oscillations. These oscillations can be radial as well as nonradial.

In case of a regular variable star which is a nonrotating and exists in isolation it may be reasonable to represent it as a gaseous sphere performing radial and/or nonradial oscillations. Some of the variable stars are also observed to be rotating stars. In such cases the rotation of the star will effect the equilibrium structure as well as the modes of its oscillations. However, if the pulsating star is a member of the binary or a multiple system, then not only its equilibrium structure but also its modes of oscillations will get affected by the rotational and tidal forces. Mathematical models of such stars will obviously have to be rotationally and tidally distorted gaseous spheres performing pseudo-radial or nonradial or some entirely different types of oscillations. The mathematical study of the problem of equilibrium structure and periods of small adiabatic modes of oscillations of such stars becomes quite complex.

The analytic study of the problem of oscillations of gaseous spheres was initiated by Ritter. He was perhaps the first to suggest in the year 1879 that the periodic variations in the luminosity of a variable star may be due to radial oscillations. Extensive studies have since been made to analyze the problems of small adiabatic radial modes of oscillations of gaseous spheres. Investigators such as Prasad (103), Hurley et al (44), Prasad and Mohan (104) studied the problems of oscillations of gaseous spheres with no rotation. Later on Gurm (38), Clement (18), Kochar and Trehan (50), Lee and Saio (66), Das et al (29), Lee and Baraffe (65), Soofi et al (113), Dintrans and Rieutord (32) and Townsend (137) studied the effect of rotation on stellar pulsations.

In the case of regular variables, the high symmetry of their observed properties favors the hypothesis of purely radial oscillations. However, purely radial oscillations may not be able to explain many other phenomena observed in the case of certain variables stars. Ledoux and Walraven (64) pointed out that the dynamical instability leading to explosions in the stars might be easier to reach for some modes of nonradial oscillations. Chandrasekhar and Lebovitz (16) were of the view that it might be possible to explain variability of Beta Canis major type stars on the basis of resonance between the radial and nonradial modes of oscillations. Dalsgaard and Gough (28) suggested that certain observed phenomena in the outer layer of sun could be explained on the basis of certain nonradial modes of oscillations of the sun. Smith (127) studied zero-age main sequence B star and found that this star is pulsating nonradially.

Theoretical studies of the problem of nonradial oscillations commenced with Kelvin's investigation of the oscillatory modes of an incompressible gas sphere. But the proper formulation of the problem was given by Perkeris (101) who derived the fourth order linear differential equation governing the adiabatic nonradial modes of oscillations of a compressible self- gravitating gaseous sphere. Since then, the theoretical studies of the problem of nonradial oscillations of spherical models have been carried out by many investigators. Several authors such as Cowling (23), Kopal

(51), Ledoux (63), Owen (99), Hansen et al (41), Saio (113), Clement (21), McDermott et al. (78), Bhatia (4), Lee and Strohmayer (67) and Savonije (115) have made significant contributions to the studies of the problems of nonradial pulsations of stars.

Cox and Cahn (24) calculated representative radial and nonradial pulsation modes of five Wolf- Rayet star models. Chandrasekhar and Ferrari (15) have proposed a complete theory of the nonradial oscillations of a static spherical symmetric distribution of matter described in terms of energy density and isotropic pressure on the premise that the oscillations are excited by incident gravitational waves. Rosenwald and Rabaey (108) have given an application of the continuous orthonormalization and adjoint methods to the computation of stars eigenfrequencies and eigenfrequency sensitivities. This method integrates an eight-order nonlinear system of ordinary differential equations which define the linear adiabatic nonradial oscillatory modes of the sun. Telting and Schrijvers (136) used a model of a nonradially, adiabatically pulsating rotating star to generate time series of absorption line profiles. Clement (22) also discussed normal modes of oscillations for rotating stars using a new numerical method for computing nonradial eigenfunctions. This technique for calculating the normal modes of spherical stellar models is generalized to two dimensions..

The theoretical investigation related to the problems of determining the equilibrium structures and stability of rotating and self gravitating objects, possibly begun with the work of Newton. He was the first to realize the importance of the law of gravitation for explaining the figures of celestial bodies. Later on Maclaurin, Clairaut, Laplace, Legendre, Jacobi, Poincare etc. contributed ideas, necessary for the development of the general theory of rotating bodies. Maclaurin, Jacobi, Kelvin and Jeans investigated in detail the problem of structure and stability of rotating liquid masses assuming uniform rotation.

In the year 1923, Edward Arthur Milne developed a technique for constructing the first detailed model for a slowly rotating star in pure radiative equilibrium. Later on in the year 1933, the technique of Milne was generalized and applied to slightly distorted polytrope by Chandrasekhar. The effect of uniform rotation on slow rotating

Cowling star obeying simple Kramer's opacity has been studied by Sweet and Roy (132). Much of the work on the effect of rotation on stellar interiors is summarized in the review article of Authors such as Kruszewski (58), Limber (68), Roberts (106), James (46), Hurley and Roberts (43), Roxburgh and Durney (111), Martin (77), Sackmann and Anand (112), Linnell (69,70), Endal and Sofia (37), Smith (136), Lubow (74), Kopal (54), Durney (33), , Deupree (30), Einsel and Spurzem (36), Maeder and Zahn (81), Reyniers and Smeyers (105) and Sood and Singh (128) have also investigated the problems of equilibrium structures of rotating stars. Meynet and Meader (81) studied the effects of rotation on the equilibrium structure and evolution of massive stars. Mender et al (79) investigated the theoretical models of low mass pre- main sequence rotating stars. Lovekin and Deupree (72) have studied the radial and nonradial modes of oscillations of rapidly rotating stars using the 2D stellar models and 2D finite differences integration of the linearized pulsation equation.

Equilibrium structures of stars which appear in binary and multiple systems are likely to be affected by both the rotational as well as the tidal effects of the companion stars. Attempts have been made in literature to determine the effects of rotation and tidal distortions on the equilibrium structure and modes of oscillations of the stars in binary and multiple systems. In a series of papers Chandrasekhar (11,12,13) developed a first order analysis which he applied to the study of the rotational problem, the tidal problem and the binary star problem. The method, however, was found unsuitable when the separation between the components is only a few times the undisturbed radius of the primary. Monaghan (93) modified it to get more accurate results near the surface.

The method of Monaghan and Roxburgh (9) to study the structure of the primary component of a synchronous close binary was further extended by Naylor and Anand (95). Kippenhahn and Thomas (48) suggested a practical way of analyzing the effects of rotation and tidal distortions on the equilibrium structures of stars by approximating the actual equipotentials surfaces of the star by Roche equipotentials.

Chan and Chau (9 developed a method which allows an efficient and accurate investigation of the structure and evolution of a rotationally and tidally distorted star in close binary systems. Tassoul and Tassoul (134) considered the meridional circulation in rotating stars and mean steady motions in rotationally and tidally distorted stars. Later, Tassoul and Tassoul (135) extended the earlier work to study the reflection effects in close binaries when there is meridional circulation in rotating stars. Nelson et al (96) have discussed the evolution of rotationally and tidally distorted low-mass close binary systems. Marten and Smeyers (76) investigated the problem of linear adiabatic oscillations of a uniformly and synchronously rotating component of a binary system. Lopezorti et al (71) analyzed the equilibrium configurations of close binary systems by expanding the auto gravitational, centrifugal and tidal potentials in Clairaut coordinates. Lal et al (61) have discussed the equilibrium structures of rotationally and tidally distorted primary component of binary stars taking into account the effect of mass variation inside the star.

Mohan and Singh (88) have used the Kippenhahn and Thomas (48) averaging technique in conjunction with certain results of Kopal (52) on Roche equipotential to study the effect of rotation and tidal distortions on the small adiabatic oscillations of stars in binary system. Sepenisky et al (119) investigated the existence and properties of equipotential surfaces and Lagrangian points in non-synchronous, eccentric binary star and planetary systems under the assumption of quasi-static equilibrium. Hachisu (40) formulated a new-three dimensional method for obtaining structure of a rapidly rotating star and multiple stellar system including binaries. Rocca (107) studied effect of slow uniform rotation on the tidal effects in close binary system. Deupree and Karkas (31) studied the structure and evolution of close binary stars using the two dimensional stellar structure algorithm. They have calculated a series of solar composition stellar evolution sequences of binary..

Orlov (97) have generalized the Roche model as is applied in the case of double star. In this model the point nuclei of the Roche model has been substituted by polytropic gas nuclei of finite dimensions. Plavec (102) presented tables of Roche model for the use of investigators in close binary systems. Kopal and Ali (56) studied the integrability of the Roche coordinates. Avni and Schiller (3) studied the Roche potential systems where the stellar rotation axis is not aligned with the orbital revolution axis. Eggleton (35) computed the effective radii of Roche lobes and compared the results with the earlier results available in literature. Mochnacki (83) accurately integrated Roche model for close binary system in synchronous rotation to give volume, radii, surface area, mean gravities and mean inverse gravities in normalized form. Seidov (118) derived the exact analytical formula for the potential and mass ratio as a function of Lagrangian point's position in the classical Roche model of the close binary stars. Csataryova and Skopal (27) derive approximate analytical formulas for the basic parameters of the Roche lobe, its radius and the position of the L_1 point for asynchronously rotating component in a binary system. Lal et al (62) studied the validity of series expression being used for determining the position of a point on a Roche equipotential in case of rotating stars and stars in binary systems.

The simple hypothesis of the pulsating model of a regular variable star is made all the more complicated by the fact that some of the variable stars observed to be rotating stars or stars in binary or multiple systems. The eigenfrequencies of small oscillations of such stars are expected to be influenced by rotation and tidal effects of companion stars.

Most of the authors have studied pulsations of stars having solid body rotation. The influence of uniform rotation on the global structure of the white dwarf models has been considered by Chandrasekhar (17), Suda(131) and Lal, A.K et al. (60). The most detailed models of uniformly rotating white dwarf are due to Anand(1), Roxburg (110),Monaghan (93), Anand et al. (2).Some of the authors such as Ostriker and Tassoul (98),Shapiro and Teukolsky (119) have noted the stability analysis of uniformly rotating white dwarf stars. Ostriker and Bodenheimer (92), Smart and Monaghan (124), and Blinnikov (10) extensively analysed the models of zero-temperature white dwarf in non-uniform rotation, Hachisu et al.(40) studied the fate of merging double white dwarfs and presented a numerical method, Lal el at (60) presented a method for computing equilibrium structure of differentially rotating and tidally distored white dwarf models of stars.

Adiabatic radial pulsation of zero-temperature white dwarfs have been discussed by various authors such as Sauvenier-Goffin (114) and Schatzman (117) using the equation of state of completely degenerate stellar matter. Singh and Das (121) studied the radial oscillations of cold and hot white dwarfs for different values of the central degeneracy parameters $\frac{1}{\phi_0^2}$ and temperature T. Harper and Rose (42) obtained the solution of the Pekeris equation for nonradial oscillations for white dwarf, hot white dwarfs and 10 M_0 models. Singh (87) evaluated the fundamental frequencies of non radial modes of oscillations of complete degenerate configurations using variational method for various values of $\frac{1}{\phi_0^2}$ and uniform temperature $0.2\times10^7 K$ and $10^8 K$.

Kopal (52) introduced a system of coordinates, which he called Roche coordinates, to study the problems of rotating stars and stars in binary system. Mohan and Singh (87) considered the use of Roche coordinates in solving the problems of small adiabatic oscillations of rotationally and tidally distorted stellar models. Mohan and Saxena (85,86) used the Kippenhahn and Thomas (48) averaging technique in conjunction with Kopal's results on Roche equipotentials to determine the combined effects of rotation and tidal distortions on the equilibrium structures and oscillations of the polytropic models of the stars. This approach is presented in detail by Saxena (116). Later this approach was also used by Mohan and Agarwal (84) to study the effects of rotation and tidal distortions on the structure and periods of small adiabatic oscillations of composite models of stars. The technique was subsequently formalized by Mohan et al (91,92) and used to study the problems of equilibrium structures and oscillations of rotationally and tidally distorted main sequence stars. Lal (59) studied in detail the equilibrium structures and periods of oscillations of differentially rotating stellar models. Later on Singh and Sharma (123) also studied the oscillations of differentially rotating stars in binary system.

The approximation of exact equipotential surface of rotationally and tidally distorted stars by corresponding Roche equipotential used in this method may not be very much justified for the white dwarfs stars. However still, it will be of interest to see how the equilibrium structures and eigenfrequencies of pseudo - radial and nonradial modes of oscillations of white dwarfs stars are affected by rotation and tidally distorted with the present approach. In fact, Lal et al (60) applied this technique to study the equilibrium structures of differentially rotating and tidally distorted white dwarf models of stars using this approach.

1.3 AVERAGING TECHNIQUE OF KIPPENHAHN AND THOMAS

In order to study the effects of rotation and tidal distortions on the equilibrium structure of gaseous spheres, Kippenhahn and Thomas (48) developed the concept of topologically equivalent spherical surfaces corresponding to actual equipotential surfaces of a rotationally and tidally distorted model. They define on these equivalent spherical surfaces, quantities such as $\overline{f}, \overline{g}$ etc. which denote certain averages of the quantities f, g, respectively on the actual equipotential surfaces. If ψ denotes the total potential (gravitation, rotation and tidal forces) of a rotationally and tidally distorted model at an arbitrary point $P(x, y, z)$ then $\psi(x, y, z)$ = constant, is an equipotential surface. Let V_ψ be the volume enclosed by the equipotential surface. ψ = constant and S_ψ the surface area of this equipotential surface. For any function $f(x, y, z)$ they define $\overline{f}$ as its mean value over the equipotential surfaces ψ = constant by the relation

$$\bar{f} = \frac{1}{S_\psi} \int_{\psi = const.} f \, d\sigma \tag{1.1}$$

where $d\sigma$ denotes the surface element of the equipotential surface ψ = constant. Clearly $\bar{f}$ is a function of equipotential surface ψ only and can be obtained as equation (1.1) for each equipotential surface ψ = constant. Kippenhahn and Thomas also define a variable r_ψ in analogy with the radius of sphere by the relation

$$V_\psi = \frac{4}{3} \pi r_\psi^3 \tag{1.2}$$

Also by definition

$$S_\psi = \int_{\psi = constant} d\sigma \tag{1.3}$$

Obviously, in general, S_ψ is not equal to $4\pi r_\psi^2$. Kippenhahn and Thomas (58) define a function $g(x, y, z)$ by the relation

$$g = \frac{d\psi}{dn} \tag{1.4}$$

This g corresponds to the force of gravity of a sphere. The distance dn between two neighboring surfaces ψ = constant and $\psi + d\psi$ = constant is, in general, not constant (i.e. not same at all points of the surface). They used (1.4) to compute the mean values $\bar{g}$ and $\overline{g^1}$ with the help of relations

$$\left.\begin{aligned} \bar{g} &= \frac{1}{S_\psi} \int_{\psi=const} \frac{d\psi}{dn} d\sigma \\ \overline{g^{-1}} &= \frac{1}{S_\psi} \int_{\psi=const} (\frac{d\psi}{d\,n})^{-1} d\sigma \end{aligned}\right\} \tag{1.5}$$

Both $\bar{g}$ and $\overline{g^{-1}}$ are functions of ψ alone and represent the value of g and g^{-1} respectively over the topologically equivalent spherical surface. The volume dV_ψ between the surface ψ = constant and $\psi + d\psi$ = constant is given by

$$dV_\psi = \int_{\psi=const} dn\, d\sigma = \int_{\psi=const} (\frac{d\psi}{dn})^{-1} dn \; = S_\psi \overline{g^{-1}}\, d\psi \tag{1.6}$$

Kippenhahn and Thomas (1970) also defined nondimensional parameters u, v and w as

$$u = \frac{S_\psi}{4\pi r_\psi^2},\; v = \frac{\bar{g}\, r_\psi^2}{GM_\psi},\; w = \frac{\overline{g^{-1}}\, GM_\psi}{r_\psi^2} \tag{1.7}$$

where M_ψ is the mass enclosed by equipotential surface ψ=constant.

We may thus regard the equipotential surface ψ = constant to be topologically equivalent to a sphere of radius r_ψ for which various functions are defined by the above relations. (It may be noticed that if ψ is the gravitational potential of a sphere then the surface ψ = constant is spherical surface with $r_\psi = r$ for which u =1 and $g = GM_\psi / r_\psi^{\,2}$ is constant on these spheres and therefore v and w are constants and equal to 1).

Equations (1.1) to (1.7) are purely mathematical definitions, which have been applied by Kippenhahn and Thomas (58) to gravitational fields of gaseous spheres distorted by rotational and tidal forces. In hydrostatic equilibrium the equipotential surfaces are also surface of equipressure and equidensity. Therefore, on an equipotential surface the pressure P_ψ and the density ρ_ψ are also constant. Using these concepts, Kippenhahn and Thomas (48) obtained the equations governing the equilibrium structure of a rotationally and tidally distorted stellar model in the following manner

From equation (1.2) the mass dM_ψ between the equipotential surface ψ = constant and $\psi + d\psi$ = constant is given by

$$dM_\psi = dV_\psi \, \rho_\psi = 4\pi r_\psi^2 \rho_\psi \, dr_\psi \tag{1.8}$$

Thus, we get

$$\frac{dM_\psi}{dr_\psi} = 4\pi r_\psi^2 \rho_\psi \tag{1.9}$$

From equation (1.6) and (1.8) we have

$$d\psi = \frac{d\psi}{dV_\psi} dV_\psi = \left(\frac{dV_\psi}{d\psi}\right)^{-1} \frac{dM_\psi}{\rho_\psi} = \frac{dM_\psi}{S_\psi \, \overline{g^{-1}} \, \rho_\psi} \tag{1.10}$$

Using relations (1.7), we get

$$d\psi = \frac{G M_\psi \, dM_\psi}{4\pi r_\psi^4 \rho_\psi \, u \, w} \tag{1.11}$$

The conditions for hydrostatic equilibrium, $dP_\psi / d\psi = -\rho_\psi$, can now be written with equation (1.7) in the form

$$\frac{dP_\psi}{dM_\psi} = -\frac{G M_\psi}{4\pi r_\psi^4} f_p \tag{1.12}$$

where

$$f_P = \frac{1}{u\,w} = \frac{4\pi r_\psi^4}{G M_\psi} \frac{1}{S_\psi \, \overline{g^{-1}}}$$

The factor f_P is a function of ψ only. If ψ is known the equipotential surface can be determined, and then consequently values of $S_\psi, r_\psi, \bar{g}$ and $\overline{g^{-1}}$ for each equipotential surface can be obtained simply from the geometry of the equipotentials.

The mass M_ψ which depends on the density distribution ρ_ψ can be determined by integrating the equation (1.9). Similarly the other structure equations derived by Kippenhahn and Thomas (48), which includes the effects of rotation and tidal distortions on the equilibrium structure of gaseous spheres are as follows.

For chemically homogenous spheres, the nuclear energy generation rate ε depends only upon density ρ_ψ and the temperature T_ψ and are, therefore, constant on equipotential surface. Thus, if L_ψ is the energy which passes per second through the equipotential surface $\psi =$ constant, then

$$\frac{dL_\psi}{dM_\psi} = \varepsilon \tag{1.13}$$

Using equation (1.8), it can be written as

$$\frac{dL_\psi}{dr_\psi} = 4\pi r_\psi^2 \rho_\psi \varepsilon \tag{1.14}$$

If the energy is transported by radiation, then the energy transport equation is

$$F_\psi = -\frac{4acT_\psi^3}{3\kappa}\frac{d\psi}{dn}\frac{dT_\psi}{dM_\psi}\frac{4\pi r_\psi^4 u w}{GM_\psi} \tag{1.15}$$

where F_ψ is the radioactive flux on the equipotential surface $\psi =$constant. By integrating F_ψ over the equipotential surface ψ = constant, we get

$$\begin{aligned} L_\psi &= \int_{\psi = constant} F_\psi \, d\sigma \\ &= -\frac{4acT_\psi{}^3}{3\kappa}\frac{dT_\psi{}^3}{dM_\psi} u w \frac{4\pi r_\psi{}^4}{GM_\psi} \int_{\psi = constant} (\frac{d\psi}{dn}) \, d\sigma \\ &= -\frac{64\pi^2 a c T_\psi{}^3 r_\psi{}^4}{3\kappa} u^2 v w \frac{dT_\psi}{dM_\psi} \end{aligned} \tag{1.16}$$

So that

$$\frac{dT_\psi}{dM_\psi} = -\frac{3\kappa L_\psi}{64\pi^2 a c T_\psi{}^3 r_\psi{}^4} f_T \tag{1.17}$$

Using equation (1.8), this equation can be expressed as

$$\frac{dT_\psi}{dr_\psi} = -\frac{3\kappa\,\rho_\psi\,L_\psi}{16\pi\,a\,c\,T_\psi^3\,r_\psi^2}\,f_T \tag{1.18}$$

where

$$f_T = \frac{1}{u^2\,v\,w}$$

Equations (1.9), (1.12), (1.13) and (1.17) which are the four basic equations governing the equilibrium structure of a gaseous sphere distorted by rotational and tidal forces may now be collected together and written as.

$$\frac{dM_\psi}{dr_\psi} = 4\pi\,r_\psi^2\,\rho_\psi \tag{1.19a}$$

$$\frac{dP_\psi}{dM_\psi} = -\frac{G\,M_\psi}{4\,\pi\,r_\psi^4}\,f_p \tag{1.19b}$$

$$\frac{dL_\psi}{dM_\psi} = \varepsilon \tag{1.19c}$$

and

$$\frac{d\,T_\psi}{d\,M_\psi} = -\frac{3\,\kappa\,L_\psi}{64\,\pi^2 a\,c\,T_\psi^{\,3} r_\psi^4}\,f_T \tag{1.19d}$$

where

$$f_P = \frac{1}{u\,w} \qquad \text{and} \qquad f_T = \frac{1}{u^2\,v\,w}.$$

These reduce to the normal equations used for determining the equilibrium structures of spherical models of stars when distortion parameters u, v, w are set one each. The boundary conditions which the above equations has to satisfy are

$$M_\psi = 0,\;\; L_\psi = 0, \quad \text{at the center } \; r_\psi = 0 \tag{1.20a}$$

$$M_\psi = M_0,\, L_\psi = L_{\psi S}$$

$$P_\psi = 0,\, T_\psi = 0 \;\; or \;\; P_\psi = P_{\psi s},\, T_\psi = T_{\psi s}$$

at the free surface $r_\psi = R_\psi$ (1.20b)

where M_0 is the total mass of the model and $L_{\psi s}, P_{\psi s}, T_{\psi s}$ are the values of L_ψ, P_ψ, T_ψ respectively, on the outermost equipotential surface.

1.4 ROCHE EQUIPOTENTIAL OF DISTORTED STARS

Roche equipotentials have often been used to analyze the problems of rotationally and tidally distorted models of stars. In order to introduce the concept of Roche equipotential, we assume two components of a close binary system known as primary and secondary star. The primary star is supposed to be much more massive than the secondary which is assumed as a point mass causing tidal effects on the more massive primary component. Both the components of binary system are assumed to be rotating about their axes as well as revolving about their common center of mass. Following Kopal (52), Mohan and Singh (87), and Mohan et al (190,91) certain results on Roche equipotential which are of practical interest to the present study, are summarized below:

Suppose that M_0 and M_1 are the masses of the two components of a close binary system separated by distance *D*. Further suppose that the primary component of this system of mass M_0 is much larger than its companion star of mass $M_1 (M_0 \geq M_1)$ which can be regarded as a point mass. Next suppose that the position of the two components is referred to a rectangular system of Cartesian coordinates with origin at the center of gravity of mass M_0, the $x-$ axis along the line joining the mass centers of two components and $z-$axis perpendicular to the plane of the orbit of the two components (Fig. 1.1). Then the total potential ψ due to the gravitational and disturbing force acting at an arbitrary point $P(x, y, z)$, which is not inside any of these two gaseous spheres is given by:

$$\psi = \frac{GM_0}{r} + \frac{GM_1}{r_2} + \frac{\Omega^2}{2}[(x - \frac{M_1 D}{M_0 + M_1})^2 + y^2] \quad (1.21)$$

where $r^2 = x^2 + y^2 + z^2$ and $r_2^2 = (D-x)^2 + y^2 + z^2$ represent the squares of the distances of *P* from the center of gravity of the two components, Ω denotes the angular velocity of rotation of the system about an axis perpendicular to the *xy*- plane and passing through the center of gravity of the system and *G* the constant of gravitation.

The first, second and third term on the right hand side of equation (1.21) respectively represent the potential which arises due to the mass M_0 of the primary component, the disturbing potential of its companion of mass M_1 and the potential arising from the centrifugal force respectively. Equation (1.21) strictly holds at points which are outside the components of binary system.

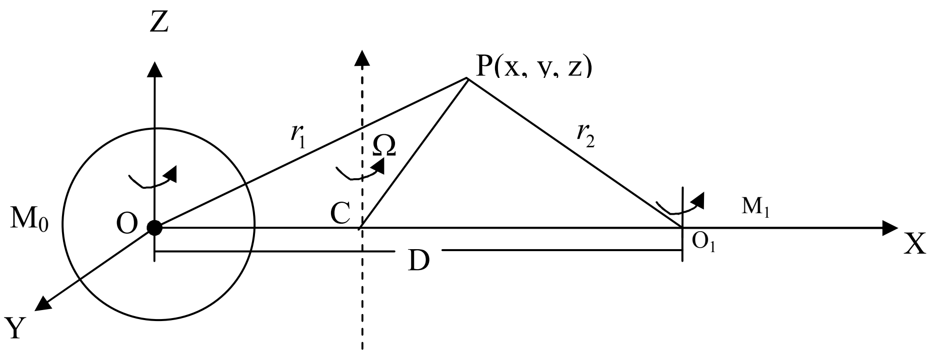

Fig 1.1: Axis of reference for a binary system

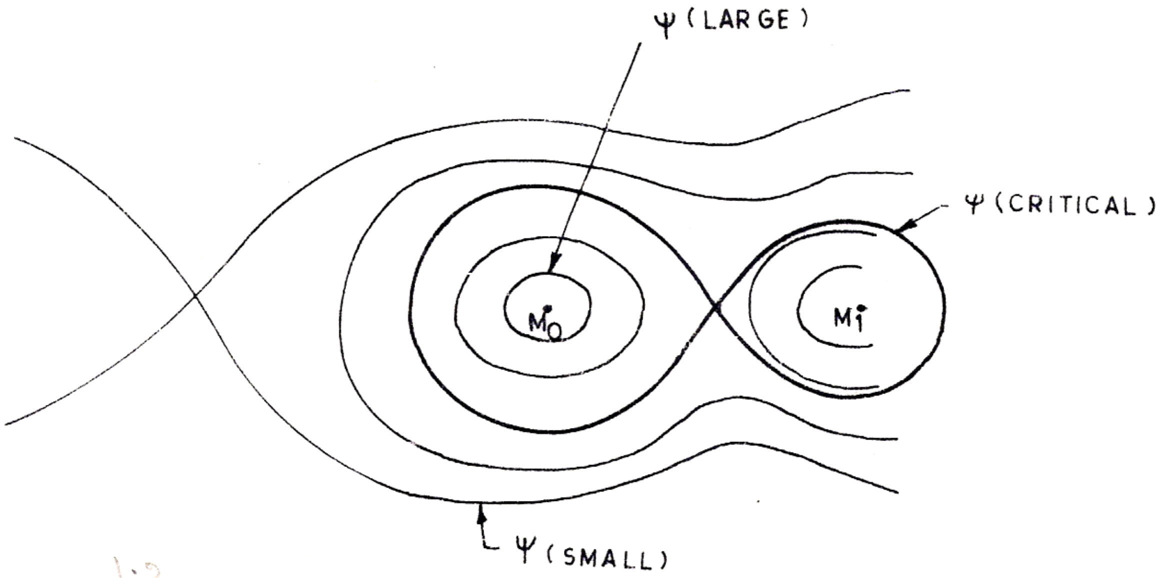

Fig 1.2: Roche equipotential surfaces (two dimensional)

In case we assume Roche model for the primary and a point mass for the secondary component, equation (1.20) holds everywhere. In Roche model, it is assumed that the total mass of a star is concentrated at its center and this point mass is surrounded by an evanescent envelope in which density varies inversely as the square of the distance from its center. Also if we assume that the angular velocity Ω is identical with Keplerian angular velocity, that is,

$$\Omega^2 = \Omega_k^2 = G\frac{M_0 + M_1}{D^3} \tag{1.22}$$

then we get a relation of the type

$$n = \frac{q+1}{2} \tag{1.23}$$

Equation (1.21) can be expressed in nondimensional form as

$$\psi^* = \frac{1}{r^*} + q[\frac{1}{\sqrt{1 - 2\lambda r^* + r^{*2}}} - \lambda r^*] + nr^{*2}(1 - \nu^2) \tag{1.24}$$

where

$$\psi^* = \frac{D\psi}{GM_0} - \frac{M_1^2}{2M_0(M_0 + M_1)}$$

is the nondimensional form of total potential ψ and $r^* = r/D$ is nondimensional form of r. Also $\lambda = \sin\theta\cos\phi$, $\mu = \sin\theta\sin\phi$ and $\nu = \cos\theta$ (r, θ, ϕ being the polar spherical coordinate of the point *P*. Moreover,

$$q = \frac{M_1}{M_0} \tag{1.25}$$

is a nondimensional parameter representing the ratio of mass of the secondary over primary and $2n$ represent the square of the normalized angular velocity Ω.The equation (1.21) reduces to the potential of a purely rotating spherical model if $q = 0$. For $n = 0$, it reduces to the potential of a non-rotating spherical model distorted by the tidal effects of the companion alone.

The surfaces generated by setting ψ = constant on the left hand side of equation (1.21) are referred to as Roche equipotentials. Roche equipotentials in nondimensional form may be represented by ψ^*= constant where ψ^* is same as defined in equation (1.24). The form of Roche-equipotential depends entirely upon the values of ψ . If ψ is large the corresponding equipotentials consist of two separate ovals, closed around each of the two mass points (Fig. 1.2). For specified values of M_0, M_1, Ω and D the right hand side of equation (1.21) can be large only if r and r_2 becomes small. Therefore, large values of ψ correspond to equipotentials which differ little from spheres surrounding each of the two mass centers. With decreasing values of ψ, these spherical equipotential surfaces become oval shaped and get elongated in the direction of the center of gravity of the system until for a certain critical value of ψ, which is characteristic of each mass ratio, both oval shaped surfaces unite at a single point on the x-axis to form a dumbbell like configuration. These limiting values of ψ are called Roche limits. For certain mass ratios Kopal (52) computed the numerical values of Roche limits in the case of synchronous binary stars for values of q ranging from zero to one.

Defining a non-dimensional variable r_0 by the relation

$$r_0 = \frac{1}{\psi^* - q} \tag{1.26}$$

Kopal (52) has also shown that on the surface of Roche equipotentials, (r, θ, ϕ) are connected through the relation

$$r^* = r_0 [1 + C_3 r_0^3 + C_4 r_0^4 + C_5 r_0^5 + C_6 r_0^6 + C_7 r_0^7 + C_8 r_0^8 + C_9 r_0^9 + ...] \tag{1.27}$$

where

$$C_3 = q P_2 + n(1 - v^2),\ \mathrm{C}_4 = \mathrm{q}\,\mathrm{P}_3,\ \mathrm{C}_5 = \mathrm{q}\,\mathrm{P}_4$$
$$C_6 = q\, p_5 + 3C_3^{\,2},\ \mathrm{C}_7 = q P_6 + 7q C_3^{\,2} P_3$$
$$C_8 = q P_7 + 8q C_3 P_4 + 4q^2 P_3^2$$
$$C_9 = q P_8 + 9q C_3 P_5 + 9q^2 P_3 P_4$$

Here, $P_j = P_j(\lambda)$ are the Legendre polynomials and terms up to second order of smallness in n and q have been retained in equation (1.27). This relation helps to obtain the shape of a Roche equipotentials ψ^* = constant.

The volume enclosed by the equipotential surface ψ^* = constant is given by

$$V_\psi = \frac{2}{3} \int_{-1}^{1} \int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^3}{\mu}\, d\lambda\, dv \tag{1.28}$$

Kopal has shown that the explicit expression of V_ψ in terms of r_0 defined by equation (1.26), can be represented as

$$V_\psi = \frac{4}{3}\pi D^3 r_0^3 [1 + 2n r_0^3 + (\frac{12}{5}q^2 + \frac{8}{5}n q + \frac{32}{5} n^2) r_0^6 + \frac{15}{7} q^2 r_0^8 + 2q^2 r_0^{10} + ...] \tag{1.29}$$

where terms up to second order of smallness in *n* and *q* are retained.

Following the approach of Kopal (63), Mohan and Singh (103) obtained the explicit expressions for the surface area S_ψ and the values of averages or parameters r_ψ, $\bar{g}$ and $\overline{g^{-1}}$ on the Roche equipotential $\psi^* =$ constant. These are

$$\begin{aligned} S_\psi &= 2\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^2}{\mu} d\lambda\, d\nu \\ &= 4\pi D^2 r_0^2 [1 + \frac{4n}{3} r_0^3 + (\frac{7}{5}q^2 + \frac{14}{15}n q + \frac{56}{15} n^2) r_0^6 + \frac{9}{7}q^2 r_0^8 + \frac{11}{9}q^2 r_0^{10} + ...] \end{aligned} \tag{1.30}$$

$$r_\psi = (\frac{3V_\psi}{4\pi})^{1/3}$$

$$= D r_0 [1 + \frac{2n}{3} r_0^3 + (\frac{4}{5}q^2 + \frac{8}{15}n q + \frac{76}{45}n^2)\, r_0^6 + \frac{5}{7}q^2 r_0^8 + \frac{2}{3}q^2 r_0^{10} + ...] \tag{1.31}$$

$$\begin{aligned} \bar{g} &= \frac{2}{S_\psi}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} (\frac{d\psi}{dn})\frac{r^2}{\mu} d\lambda\, d\nu \\ &= \frac{GM_\psi}{D^2 r_0^2}[1 - \frac{8n}{3} r_0^3 - (3q^2 + 2n q + \frac{40}{9}n^2) r_0^6 - \frac{51}{14}q^2 r_0^8 - \frac{13}{3}q^2 r_0^{10} + ...] \end{aligned} \tag{1.32}$$

and

$$\begin{aligned} \overline{g^{-1}} &= \frac{2}{S_\psi}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} (\frac{d\psi}{dn})^{-1}\frac{r^2}{\mu} d\lambda\, d\nu \\ &= \frac{D^2 r_0^2}{GM_\psi}[1 + \frac{8n}{3} r_0^3 + (\frac{31}{5}q^2 + \frac{62}{15}n q + \frac{584}{45}n^2) r_0^6 + \frac{101}{14}q^2 r_0^8 + \frac{25}{3}q^2 r_0^{10} + ...] \end{aligned} \tag{133}$$

Inverting the relation (1.39) they also obtain

$$r_0 = r_\psi^* [1 - \frac{2n}{3} r_\psi^{*3} - [\frac{4}{5}q^2 + \frac{8}{15}n q - \frac{4}{45}n^2] r_\psi^{*6} - \frac{5}{7}q^2 r_\psi^{*8} - \frac{2}{3}q^2 r_\psi^{*10} - ...] \tag{1.34}$$

Where $r_\psi^* = r_\psi / D$, r_ψ^* being the nondimensional form r_ψ. In all the above expressions terms up to second order of smallness in n and q have been retained.

1.5 MOHAN, SAXENA AND AGARWAL'S APPROACH FOR COMPUTING THE EFFECTS OF ROTATIONAL AND TIDAL DISTORTIONS ON THE EQUILIBRIUM STRUCTURES OF ROTATIONALLY AND TIDALLY DISTORTED GASEOUS SPHERES

Mohan, Saxena and Agarwal (91,92) used the concept of Roche equipotentials proposed by Kopal in conjunction with Kippenhahn and Thomas's averaging approach to explicitly obtain equations governing the equilibrium structures and periods of radial and nonradial oscillations of rotationally and/or tidally distorted stars and applied these to analyze the problems of rotating stars and stars in binary systems. In this section we briefly review their approach.

In order to determine the inner structure of a rotationally and tidally distorted gaseous sphere, the system of equations (1.19) has to be integrated numerically subject to the
boundary conditions (1.20) specified therein. Therefore, the evaluation of the actual equipotential surface of a rotationally and tidally distorted gaseous sphere is complicated. Kippenhahn and Thomas (48) proposed that for evaluation of the distortion parameters u, v, w, f_P, f_T etc., the actual equipotential surface may be replaced by Roche equipotential surface (It may be noted that this approximation is reasonably valid for most of the models of the actual stars. In fact as far back as 1933, Chandrasekhar had shown that for stars whose central density bears to the mean density a ratio of 100 or more, the Roche model of a rotating configuration will represent the actual equipotential surfaces of the star within an error of less than one percent).

Once the equipotential surfaces of a rotationally and tidally distorted star are approximated by the Roche equipotentials, the results obtained by Kopal (63, 64) and Mohan and Singh (87,88) may be used to evaluate explicitly the values of the distortion parameters u, v, w, f_P and f_T appearing in stellar structure equations (1.12) and (1.18). Using equations (1.7), (1.12), (1.1.18) and (1.29 – 1.34) the explicit expressions of the distortions parameters u, v, w, f_P and f_T on the equipotential surface as obtained by Mohan et al (108) are

$$u = 1 - (\frac{1}{5}q^2 + \frac{2}{15}nq + \frac{4}{45}n^2) r_\psi^{*6} - \frac{1}{7}q^2 r_\psi^{*8} - \frac{1}{9}q^2 r_\psi^{*10} + ... \qquad (1.35a)$$

$$v = 1 - \frac{4n}{3} r_\psi^{*3} - (\frac{7}{5}q^2 + \frac{14}{15}nq + \frac{68}{45}n^2) r_\psi^{*6} - \frac{31}{14}q^2 r_\psi^{*8} - 3q^2 r_\psi^{*10} - ... \qquad (1.35b)$$

$$w = 1 + \frac{4n}{3} r_\psi^{*3} + (\frac{23}{5}q^2 + \frac{16}{15}nq + \frac{212}{45}n^2) r_\psi^{*6} + \frac{81}{14}q^2 r_\psi^{*8} + 7q^2 r_\psi^{*10} + ... \qquad (1.35c)$$

$$f_P = 1 - \frac{4n}{3} r_\psi^{*3} - (\frac{22}{5}q^2 + \frac{44}{15}nq + \frac{128}{45}n^2) r_\psi^{*6} - \frac{79}{14}q^2 r_\psi^{*8} - \frac{62}{9}q^2 r_\psi^{*10} - ... \qquad (1.35d)$$

and

$$f_T = 1 - (\frac{14}{5}q^2 + \frac{28}{15}nq + \frac{56}{45}n^2) r_\psi^{*6} - \frac{46}{14}q^2 r_\psi^{*8} - \frac{34}{9}q^2 r_\psi^{*10} - ... \qquad (1.35e)$$

where $r_\psi^{\bullet} = r_\psi / D$ is the nondimensional form of r_ψ and terms up to second order of smallness in n and q are retained.

The values of M_ψ, P_ψ, L_ψ etc. on the various equipotential surfaces of a rotationally and tidally distorted gaseous spheres may now be obtained by solving the system of differential equations (1.19) with boundary conditions (1.20) and using the values of distortion parameters f_P and f_T as given in (1.35).

It may be noted that while approximating the equipotential surfaces of a rotationally and tidally distorted model by Roche equipotentials, the structure of the star is not approximated by the structure of a Roche model. In the case of no distortion (*n=q=0),* equation (1.35) gives $u = v = w = f_P = f_T = 1$ and the system of differential equations (1.19) reduce to the equations governing the equilibrium structure of the original undistorted star and not of the Roche model.

Usual methods for stellar structure equations such as Henyey et al (48) method can be now used to integrate the system of differential equation (1.19) governing the equilibrium structure of a rotationally and tidally distorted gaseous sphere. At every step, the values of the distortion parameters u, v, w, f_P and f_T have to be computed using (1.35).

In case the thermal properties are not considered important and only hydrostatic equilibrium of a rotationally and tidally distorted gaseous spheres is to be investigated then we need only to integrate equation (1.9) and (1.12) subject to the boundary conditions

At the center $r_\psi = 0$, $M_\psi = 0$ (1.36a)

and at the free surface $r_\psi = R_\psi$,

$$\begin{aligned} &M_\psi = M_0, P_\psi = 0 \\ &\rho_\psi = 0 \;\; or \;\; P_\psi = P_{\psi s}, \rho_\psi = \rho_{\psi s} \end{aligned} \tag{1.36b}$$

In case the star is being distorted by rotational forces alone (or tidal forces alone) we may set $q=0$ ($n=0$) in (1.43) and still use the above approach to determine the equilibrium structure of corresponding purely rotationally distorted or purely tidally distorted model. For obtaining the structure of the primary component of a synchronous binary system we may set $n = (q+1)/2$.

Mohan and Saxena (99) found it more convenient to work with r_0 in place of M_ψ or r_ψ as independent variable by using (1.26) which is connected with variable r_ψ through relation (1.34). Saxena (133) expressed the system of differential equations governing the equilibrium structure of a rotationally and tidally distorted stellar model as

$$\frac{dM_\psi}{dr_0} = 4\pi D^3 \rho_\psi r_0^2 f_1, \tag{1.37a}$$

$$\frac{dP_\psi}{dr_0} = -\frac{GM_\psi}{Dr_0^2} \rho_\psi f_2, \tag{1.37b}$$

$$\frac{dL_\psi}{dr_0} = 4\pi \varepsilon D^3 \rho_\psi r_0^2 f_1, \tag{1.37c}$$

and

$$\frac{dT_\psi}{dr_0} = -\frac{3\kappa L_\psi}{16\pi D ac T_\psi^3} \frac{\rho_\psi}{r_0^2} f_3.$$

(1.37d) Here f_1, f_2 and f_3 are functions of n, q and r_0 incorporating the effects of rotation and tidal distortions on the equilibrium structure equations of a stellar model. The explicit expressions for these distortion parameters as given by Saxena (133) are

$$f_1 = 1 + 4n r_0^3 + (\frac{36}{5} q^2 + \frac{24}{5} n q + \frac{96}{5} n^2) r_0^6 + \frac{55}{7} q^2 r_0^8 + \frac{26}{3} q^2 r_0^{10} + ... \qquad (1.38a)$$

$$f_2 = 1 - (\frac{2}{5} q^2 + \frac{4}{15} n q + \frac{16}{15} n^2) r_0^6 - \frac{9}{14} q^2 r_0^8 - \frac{8}{9} q^2 r_0^{10} + ... \qquad (1.38b)$$

and

$$f_3 = 1 + \frac{4n r_0^3}{3} + (\frac{6}{5} q^2 + \frac{4}{5} n q + \frac{224}{45} n^2) r_0^6 + \frac{24}{14} q^2 r_0^8 + \frac{20}{9} q^2 r_0^{10} + ... \qquad (1.38c)$$

In these above expressions terms up to second order of smallness in n, q and up to r_0^{10} in r_0 are retained. The boundary conditions governing the system of differential equations (1.45) are:

At the center $r_0 = 0, \quad M_\psi = 0, \; L_\psi = 0$ (1.39a)

and at the free surface $r_0 = r_{0s}$,

$$M_\psi = M_0, \; L_\psi = L_{\psi s}$$
$$P_\psi = 0, \; \rho_\psi = 0, \; T_\psi = 0 \;\; or \;\; P_\psi = P_{\psi s}, \; \rho_\psi = \rho_{\psi s}, T_\psi = T_{\psi s} \qquad (1.39b)$$

where r_{0s} being the value of r_0 at the free surfaces. In fact ,

$$r_{0s} = \frac{1}{\psi_s^* - q}$$

(1.40) where $\psi_s^\bullet$ is the nondimensional form of the total potential ψ on the outermost equipotential surface of a rotationally and tidally distorted stellar model. In the case of no distortion $f_p = f_T = 1$ and the above equations reduce to the usual equations governing the equilibrium structure of an undistorted gaseous sphere.

1.6 THE PRESENT WORK

The present thesis deals with the possibility of using Mohan, Saxena and Agarwal (91) approach to obtain the effect of rotation and tidal forces on the equilibrium structures and the eigenfrequencies of the radial and nonradial modes of oscillations of theoretical models of the stars. Mathematical models of rotationally and tidally distorted stars have been developing in general and used to compute the equilibrium structures and eigenfrequencies of small adiabatic modes of oscillations of certain rotationally white dwarfs. Effects of tidal distortion, occurring due the presence of companion's stars in binary system, on the equilibrium structure and periods of oscillations rotating stars have also been discussed.

The approximation of exact equipotential surface of rotationally and tidally distorted stars by corresponding Roche equipotential used by Mohan, Saxena and Agarwal (91) may not be very much justified for the white dwarfs stars. However still, it will be of interest to see how the equilibrium structures and eigenfrequencies of pseudo- radial and nonradial modes of oscillations of white dwarfs stars are affected by rotation and tidally distortion with the present approach. In fact, Lal et al (61) applied this technique to study the equilibrium structures of differentially rotating and tidally distorted white dwarf models of stars using this approach but did not study the equilibrium structures and oscillations of rotationally and tidally distortrd white dwarf models of stars. The study is expected to help in better understanding the limitations and scope of Mohan, Saxena and Agarwal (91) approach to determine the structure and pulsations of theoretical models of stars.

Chapter wise summary of the work presented in the subsequent chapters of this is as follow. This thesis is divided into four chapters.

Chapter I is introductory in nature. In this chapter we briefly discuss the astrophysical significance of the problem of determining the equilibrium structure and periods of small adiabatic oscillations of rotationally and tidally distorted stellar models. A brief summary of literature available on the subject is also presented in this chapter and summary of the work presented in the succeeding chapter of the thesis also appears in this chapter.

In chapter II we use the approach of Mohan, Saxena and Agarwal (91) and Lal et al (60) to determine the equilibrium structure of rotationally and tidally distorted white dwarfs models of stars under the assumptions that angular velocity of rotation is not too large. Numerical results have been computed for certain rotationally and tidally distorted white dwarfs models for the parameters $\frac{1}{\phi_0^2}$ as $0.01, 0.05, 0.2, 0.4, 0.6$ and 0.8 and analyze the result.

In chapter III we analyze the effect of rotation and tidal distortion on the eigenfrequencies of small adiabatic barotropic mode of oscillations of stars. The eigenvalued boundary value problems which determine the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted white dwarf models have been formulated using Mohan, Saxena and Agarwal (91) approach and computed the eigenfrequencies of various pseudo-radial modes of oscillations rotationally and tidally distorted white dwarfs stars with central degeneracy parameters $\frac{1}{\phi_0^2}$ as $0.01, 0.05, 0.2, 0.4, 0.6, 0.8$. Certain conclusions are drawn.

Chapter IV is devoted to study the effect of rotation and tidal distortion on the eigenfrequencies of small adiabatic barotropic mode of oscillations of stars. The eigenvalued boundary value problems which determine the eigenfrequencies of small adiabatic nonradial modes of oscillations of rotationally and tidally distorted white dwarf stars have been formulated using Mohan, Saxena and Agarwal (91) approach. The eigenfrequencies of various nonradial modes of oscillations rotationally and tidally distorted white dwarfs stars with parameters $\frac{1}{\phi_0^2}$ as $0.05, 0.2, 0.6, 0.8$ have been computed using the approach earlier discussed by Mohan and Saxena (1985). Conclusions based on the present study are also summarized in this chapter.

CHAPTER-II

EQUILIBRIUM STRUCTURES OF ROTATIONALLY AND TIDALLY DISTORTED WHITE DWARF MODELS

White dwarf stars are supported by electron degeneracy and they are found to the lower left of the main sequence of HR (Hertsprung - Russell) diagram. A white dwarf also called a degenerate dwarf is a small star composed mostly of electron-degenerate matter. They are very dense, faint luminosity and its mass is comparable to that of sun. Its volume is comparable to that of earth. These stars are believed to be the representative of the final evolutionary stages of many types of stars. Their large densities and low luminosities imply that the electron gas is highly degenerate throughout the mass. Most of the pressure that supports these stars against gravity is believed to be the degeneracy pressure of the electron. This degeneracy pressure is so large that the structure of these stars can be computed to good accuracy on the assumption that the internal temperature is zero (Chandrasekhar 14,chap 11). Further details regarding the static structure of white dwarf can be found in the classic review by Chandrasekhar 17,chap 11), Schatzman (117) and in more recent reviews by , Cox and Giuli (26, Chap 25), Ostriker (97). An attempt has been made by Lal et al. (60) to use Roche approximation for determining the equilibrium structures of differentially rotating and tidally distorted white dwarf models of stars.

In this chapter we use the methodology of Lal et. al (60) to compute the equilibrium structures of rotationally and tidally distorted white dwarf stars whose results will be further used in the subsequent chapter for studying the effect of rotation and tidal distortion on the eigenfrequencies of pseudo-radial modes of oscillations of white dwarf stars.

This chapter is organized as follows: In section 2.1 we present a brief introduction to the white dwarf stars. Following the approach of Lal. et al (60), a differential equation determining the equilibrium structure of rotationally and tidally distorted white dwarf stars is formulated in section 2.2. The equilibrium structures of rotationally and tidally distorted white dwarf models of stars with certain central degeneracy parameters $\frac{1}{\phi^2}$ have been computed and presented in section 2.3.

2.1 WHITE DWARF MODELS OF STARS

White dwarfs models have been extensively studied in literature as representative models of low mass stars in their last stage of evolution (Chandrasekhar (17) , chapter XI). In the case of completely degenerate white dwarf model, the equation of state can be written as (Chandrasekhar ((17), eqns (16), (17) and (18) chapter XI)

$$P = A f(x), \quad \rho = B x^3 \tag{2.1}$$

Where $A = 6.01 \times 10^{22}$, $B = 9.82 \times 10^5 \mu_e$, $f(x) = x\left(2x^2 - 3\right)\sqrt{x^2 + 1} + 3\sinh^{-1} x$

and $x = \frac{p_0}{mc}$. The symbols x, μ_e, P, ρ used above denote the relativistic constant, is relativistic constant, mean molecular weight per electron, pressure and density, respectively.

The equilibrium structure of a undistorted white dwarf model can be shown to be governed by the nonlinear differential equation

$$\frac{1}{\eta^2}\frac{d}{d\eta}(\eta^2\frac{d\phi}{d\eta}) = -(\phi^2 - \frac{1}{\phi_0^2})^{\frac{3}{2}} \tag{2.2}$$

This has to be solved subject to the boundary conditions

$$\phi = 1, \ \frac{d\phi}{d\eta} = 0 \text{ at the centre } \eta = 0, \tag{2.3}$$

and $\phi = \frac{1}{\phi_0}$ at the surface $\eta = \eta_1$

The exact treatment of the differential equation (2.2) provides much more quantative information. The boundary condition (2.3) combined with a particular value of ϕ_0 determines ϕ completely and therefore the mass of the configuration as well. Once the solution to the differential equation (2.2) satisfying conditions (2.3) is obtained, other physical parameters of the white dwarf model can also be obtained.

Equation (2.2) does not admit of a homology constant, and hence each mass has a density distribution characteristic of itself which cannot be inferred from the density distribution in a configuration of a different mass. This is most fundamental difference between the white dwarfs and the polytropic models. Chandrasekhar (18) and other investigators have numerically solved the equation (2.2) to satisfy the boundary conditions (2.3) for the various values of $\frac{1}{\phi_0}$ varying from 0 to 1 and used these to determine the values of various physical parameters of white dwarf stars.

2.2 EQUILIBRIUM STRUCTURES OF ROTATINALLY AND TIDALLY DISTORTED WHITE DWARF MODELS

In this section we investigate the problems of determining the equilibrium structures of rotationally and tidally distorted stellar models which are primary component of binary stars. If primary component of binary star is supposed to be a white dwarf stars, then such a white dwarf is subject to rotation and tidal distortion and its structure becomes a rotationally and tidally distorted white dwarf model. Following the approach of section 1.5, we may approximate the equipotential surfaces of such a distorted model by modified Roche equipotential surfaces.

Let P_ψ denote the pressure and ρ_ψ the density on the equipotential surface ψ^* = constant of the distorted model. Then the value of the density and the pressure on the equivalent surface of the corresponding topologically equivalent spherical model will also be ρ_ψ and P_ψ, respectively. We assume that the distorted model is also a completely degenerate white dwarf model so that ρ_ψ and P_ψ are also connected through the white dwarf relations of the type

$$P_\psi = A\, f\left(x\right) \text{ and } \rho_\psi = B\, x^3 \tag{2.4}$$

Following Lal et al.(2000) the equation governing the equilibrium structure of a rotationally and tidally distorted white dwarf model can be written as

$$\frac{1}{r_\psi^2}\,\frac{d}{dr_\psi}\left[\frac{r_\psi^2}{\rho_\psi}\,uw\,\frac{dp_\psi}{dr_\psi}\right] = -\,4\,\pi\,\mathrm{G}\,\rho_\psi \tag{2.5}$$

Using relations (2.4) and substituting $(x^2+1) = \phi_0^2\phi_\psi^2$, it reduces to

$$\frac{\alpha^2}{r_\psi^2}\frac{d}{dr_\psi}[r_\psi^2\, uw\,\frac{d\phi_\psi}{dr_\psi}] = -(\phi_\psi^2 - \frac{1}{\phi_0^2})^{\frac{3}{2}} \tag{2.6}$$

Where

$$\alpha^2 = \frac{2A}{\pi G B^2 \phi_0^2}$$

It may be noted that the approximation of equipotential surfaces by Roche equipotentials does not basically alter the structures of the white dwarf model because in the absence of any distortion ($u = w = 1$), equation (2.6) reduces to the usual structure equation (2.2) of white dwarf given in the earlier section.

To obtain the equilibrium structure of a rotationally distorted model, (2.6) has to be integrated numerically subject to the boundary conditions

$$\phi_\psi = 1, \qquad \frac{d\phi_\psi}{dr_\psi} = 0 \text{ at the centre } r_\varphi = 0 \tag{2.7}$$

and $\phi_\psi = \frac{1}{\phi_0}$ at the surface $r_\psi = R_\psi$

The values of r_ψ on the outermost equipotential surface of the distorted white dwarf is given by

$$r_\psi = \alpha\,\eta_u \tag{2.8}$$

where η_u is the value of η when ϕ equals $\frac{1}{\phi_0}$ for the undistorted model.

Using the values of r_ψ, u, w presented in section 1.5 and retaining the terms upto second order of smallness in n , q and upto order r_0^{10} in r_0, the differential equation (2.8) governing the equilibrium structure of a rotationally and tidally distorted white dwarf can be written explicitly in the nondimensional form as

$$\frac{d}{dr_0}[A(r_0, n, q)\frac{d\phi_\psi}{dr_0}] = -\frac{\eta_u^2}{K^2}\left(\phi_\psi^2 - \frac{1}{\phi_0^2}\right)^{3/2} r_0^{\,2} B(r_0, n, q) \tag{2.9}$$

where

$$A(r_0,n,q) = \frac{r_0^2}{f_2} = r_0^{\,2}[1-(\frac{3}{5}q^2+\frac{4}{15}n^2+\frac{2}{5}nq\,)r_0^6-\frac{6}{7}q^2r_0^8-\frac{10}{9}q^2r_0^{10}+...]$$

$$B(r_0,n,q) = f_1 = [1+4nr_0^{\,3}+(\frac{36}{5}q^2+\frac{96}{5}n^2+\frac{24}{5}n\,q+)\,r_0^{\,6}\;+\frac{55}{7}q^2\;r_0^{\,8}+\frac{26}{3}q^2r_0^{10}+...]$$

$$K = \frac{\eta_u\alpha}{D} \text{ and } \quad r_0 = \frac{1}{\psi^*-q}$$

Equation (2.9) has to be solved subject to the boundary conditions (2.7) which now become

$$\text{at the centre: } r_0 = 0\,,\ \phi_\psi = 1\,,\quad \frac{d\phi_\psi}{dr_0} = 0 \tag{2.10}$$

$$\text{and at the surface: } \quad r_0 = r_{0s}\,,\ \phi_\psi = \frac{1}{\phi_0}$$

r_{0s} being the value of r_0 at the outer surface (r_0 and r_{0s} are both nondimensional quantities). Equation (2.9) subject to the boundary condition (2.10) determines the equilibrium structure of a rotationally and tidally distorted white dwarf model.

In order to determine the numerical solution of the second – order nonlinear differential equation (2.9) subject to the boundary conditions (2.10), we integrate equation (2.9) for certain choice of the values of η_u, n, and q. The integration may be continued till $\phi_\psi = \frac{1}{\phi_0}$. The value of r_{0s} of r_0 for which $\phi_\psi = \frac{1}{\phi_0}$ determines the outermost free surface of the topologically equivalent spherical model. Once the solutions of equation (2.9) are obtained, we know the values of ϕ_ψ for various values of the nondimensional independent variable r_0 varying from zero to r_{os}. The pressure P_ψ and the density ρ_ψ on various equipotentials of the distorted model may now be obtained through the relations (2.4) in the same manner as is done for undistorted white dwarf model.

2.3 ANALYSIS OF THE RESULTS

Numerical solution equation (2.9) may be performed by the use of fourth-order Runge-Kutta method for the specified values of the input parameters. Since the centre and the surface of the star are singularities of (2.9), for starting numerical integration a series, solution should preferably developed near the centre . Such a solution for the present case is given by

$$\phi_\psi = 1 - \frac{\eta_u^2}{6}Q^2 r_0^2 + \frac{\eta_u^2}{40}Q^4 r_0^4 - \frac{2n\eta_u^2}{15}Q^3 r_0^4 + \frac{\eta_u^2}{40}Q^4 r_0^5 - \frac{\eta_u^6 Q^5\left(5q^2+14\right)}{5040}r_0^6$$
$$+\frac{6n\eta_u^4}{140}Q^4 r_0^7 + \left(\frac{\eta_u^8 Q^6\left(339q^2+280\right)}{1088640} - \frac{20n^2\eta_u^2}{72}Q3r_0^5\right)r_0^8 + \cdots \qquad (2.11)$$

where $Q^2 = \left(1 - \frac{1}{\phi_0^2}\right)$

Numerical integration have been performed to obtain the inner structures of certain rotationally and tidally distorted white dwarf model for the central degeneracy parameters $\frac{1}{\phi_0^2}$ as $0.01, 0.05, 0.2, 0.4, 0.6, 0.8$ and values of n and q listed in table 2.1. After obtaining the starting values of ϕ_ψ from the series solution (2.11) at $r_o = 0.005$ numerical integration of equation (2.9) is carried forward using Ruge-Kutta metod of fourth order for a step length 0.005 and is continued till ϕ_ψ equals to $\frac{1}{\phi_0^2}$ .The results of numerical integration are presented in certain in table (4.2 to 4.4) for values of ϕ_ψ at various points in the stars.

Results given in the second and third columns of the Tables (2.2), (2.3), (2.4) are the values of ϕ_ψ for the respective undistorted and uniformly rotating white dwarfs models. Entries in the subsequent columns of these tables are the values of ϕ_ψ for certain rotationally and tidally distorted white dwarfs models listed in table 2.1. Our results show that for $\frac{1}{\phi_o^2} = 0.01$ the values of ϕ_ψ for rotationally and tidally distorted models are larger compared to their corresponding values for the undistorted model. These are also larger than their corresponding values for the uniformly rotating model 3. A similar behavior in the values of ϕ_ψ is noticed in the case of for rotationally and tidally distorted white dwarfs model computed for values of $\frac{1}{\phi_o^2}$ as 0.01, 0.4, 0.8. It is also observed that starting with rotationally and tidally distorted white dwarfs model computed for $\frac{1}{\phi_o^2} = 0.01$, the values ϕ_ψ at corresponding points gradually increase as $\frac{1}{\phi_o^2}$ approaches 0.8.

Table 2.1: Values of r_{os} for various types of rotationally and tidally white dwarf models for different values of central degeneracy parameters

Model No	Values of			Values of $1/\phi_0^2$					
	n	q	k	0.01	0.05	0.2	0.4	0.6	0.8
Undistorted Model									
1	0.00	0.00	1.00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
Rotationally distorted model									
2	0.01	0.00	1.0	.997799	.997499	.997199	.997049	.996949	.996899
3	0.10	0.00	1.0	.977599	.974400	.971599	.970399	.969750	.969299
Tidally distorted model									
4	0.00	0.10	0.5	.500000	.500000	.499950	.499950	.499900	.499900
5	0.00	0.20	0.5	.499950	.499950	.499900	.499850	.499850	.499850
Rotationally and Tidally distorted model									
6	0.1	0.1	0.5	.498650	.498400	.498200	.498100	.498050	.498000
7	0.2	0.5	0.5	.496800	.496350	.495900	.495700	.495600	.495550
8	0.2	0.1	0.5	.497200	.496850	.496450	.496250	.496200	.496100
9	0.2	0.2	0.5	.497200	.496750	.496400	.496200	.496100	.496050
Synchronous									
10	0.525	0.05	0.5	.492800	.491700	.490750	.490350	.490100	.489950
11	0.55	0.1	0.5	.492400	.491300	.490300	.489850	.489600	.489450
12	0.6	0.2	0.5	.491600	.490400	.489300	.488800	.488550	.488400

Table 2.2: Values of ϕ_ψ for certain rotationally and tidally distorted white dwarf models For $1/\phi_0^2 = 0.01$

$x = r_o/r_{os}$	Model 1	Model 3	Model 5	Model 7	Model 10
0.1	.954824	.956731	.953953	.954491	.955170
0.2	.839097	.845175	.838536	.840207	.842386
0.3	.693859	.703355	.693636	.696306	.699669
0.4	.552370	.563213	.552401	.555449	.559256
0.5	.430365	.440618	.430552	.433445	.436992
0.6	.331093	.339570	.331362	.333774	.336641
0.7	.252189	.258386	.252495	.254282	.256305
0.8	.189751	.193612	.190067	.191202	.192389
0.9	.140045	.141758	.140353	.140866	.141322
1.0	.099998	.0998542	.100289	.100221	.10103

Table 2.3: Values of ϕ_ψ for certain rotationally and tidally distorted white dwarf models for $1/\phi_o^2 = 0.4$, $\eta_u = 3,5245$

$x = r_o/r_{os}$	Model 1	Model 3	Model 5	Model 7	Model 10
0.1	.990550	.991072	.990463	.990617	.990812
0.2	.963655	.965644	.963594	.964156	.964874
0.3	.923495	.927333	.923303	.924554	.925935
0.4	.875476	.880948	.875491	.877032	.878988
0.5	.824865	.831253	.824914	.826710	.828970
0.6	.775844	.782188	.775916	.777700	.779908
0.7	.731166	.736539	.731250	.732765	.734590
0.8	.692249	.695975	.692331	.693386	.694603
0.9	.659463	.661236	.654534	.660030	.660564
1.0	.632445	.632340	.632497	.632432	.632350

Table 2.4: Values of ϕ_ψ for certain rotationally and tidally distorted white dwarf models for $1/\phi_0^2 = 0.8$, $\eta_u = 4.0446$

$x = r_o/r_{os}$	Model 1	Model 3	Model 5	Model 7	Model 10
0.1	.997604	.997747	.997600	.997641	.997917
0.2	.990673	.991210	.990675	.990826	.991022
0.3	.980073	.981138	.980081	.980381	.980768
0.4	.966994	.968567	.967008	.967451	.968020
0.5	.952725	.954631	.952745	.953281	.953963
0.6	.938435	.940395	.938458	.939009	.939701
0.7	.925027	.926739	.925051	.925533	.926125
0.8	.913086	.914302	.913108	.913449	.913858
0.9	.902419	.903471	.902899	.903062	.903249
1.0	.894423	.894393	.894433	.894411	.894394

Chapter III

EIGENVALUED BOUNDARY VALUE PROBLEM DETERMINING THE EIGENFREQUENCIES OF SMALL ADIABATIC PSEUDO – RADIAL MODES OF OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED WHITEDWARF MODELS OF STARS

In order to determine the effects of rotation and tidal distortions on the eigen-frequencies of stars in binary systems, Mohan and Singh (88) formulated an eigenvalued boundary value problem which determines the periods of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted Roche model. Mohan et al (59) used this approach to formulate eigenvalue problem which determine the eigenfrequencies of small adiabatic pseudo-radial and nonradial modes of oscillation of rotationally and tidally distorted gaseous spheres in general. The approach adopted by them was later used by Lal (70) to set up the eigenvalue problems which determine the eigenfrequencies of small adiabatic pseudo radial oscillation of differential rotating and tidally distorted stellar model.

In this chapter we consider the use of averaging approach discussed in chapter I to study the effects of rotational and tidal distortion on the eigenfrequencies of small adiabatic pseudo-radial and nonradial modes of oscillations of white dwarfs models of stars. Assuming that during the oscillations of fluid elements on an equipotential surface oscillate in unison, the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of the actual rotating star can be obtained from the topologically equivalent spherical model developed on the basis of averaging technique of Kippenhahn and Thomas (58).

The basic assumptions involved in this approach are that the rotational and tidal forces causing distortions in the spherical shape of the model are no too large so that the deviation of the actual shape of the distorted model from its original undistorted configuration of spherical symmetry is not unduly large.

This chapter is organized as follows: In section 3.1 we have discussed eigenvalued boundary value problem determining the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted stellar model. The procedure of computing eigenvalues of radial modes of oscillations is discussed in section 3.2. In section 3.3 we discussed eigenvalued boundary value problem determining the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted white dwarfs models. Numerical computation of eigenfrequencies of pseudo-radial modes of oscillations of distorted white dwarfs models of stars is next discussed in section 3.4. Analysis of the result is finally discussed in section 3.5.

3.1 EIGENVALUED BOUNDARY VALUE PROBLEM DETERMINING EIGENFREQUENCIES OF SMALL ADIABATIC PSEUDO – RADIAL MODES OF OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED STELLAR MODELS

Following Roseland (109) the equation of radial motion of a self-gravitating gaseous sphere expressed as

$$\ddot{r} = -g - \frac{1}{p}\frac{dp}{dr} \tag{3.1}$$

In analogy to the above equation, governing radial motion of the topologically equivalent spherical model of a rotationally and tidally distorted gaseous sphere can be written as

$$\ddot{r}_\psi = -\overline{g} - \frac{1}{p_\psi}\frac{dp_\psi}{dr_\psi} \tag{3.2}$$

Similarly, the equation of continuity of the motion of this equivalent spherical model is

$$p_\psi\, r_\psi^2\, dr_\psi = \text{constant} = p_{0\psi}\, r_{0\psi}^2\, dr_{0\psi} \tag{3.3}$$

where the subscript ‘o’ refers to the equilibrium values of a variable. Now assuming the changes take place during the motion are adiabatic, we can also write

$$p_\psi = kp_\psi^\gamma \tag{3.4}$$

Where k is constant and γ is the ratio of specific heat during oscillations. The radius r_ψ may be expressed as

$$r_\psi = r_{0\psi}(1 + r_{1\psi}) \tag{3.5}$$

Where $r_{1\psi}$, the relative variation in $r_{0\psi}$ may be regarded as small quantity whose squares and higher power can be neglected. For small harmonic oscillations the time dependence of $r_{1\psi}$ may be taken as

$$r_{1\psi} = \eta \cos(\sigma t) \tag{3.6}$$

where σ is the eigenfrequency of small pseudo-radial oscillations and η the relative amplitude of pulsation of the fluid elements on the equipotential surface $\psi =$constant $r_{o\psi}$.

Now combining equation (3.2) to (3.6) we get after simplifications

$$\frac{d^2\eta}{dr_{0\psi}^2} + \frac{4-\mu}{r_{0\psi}}\frac{d\eta}{dr_{0\psi}} + \left[\frac{\rho_{0\psi}}{\gamma p_{0\psi}}\sigma^2 - (3-\frac{4}{\gamma})\frac{\mu}{r_{0\psi}^2}\right]\eta = 0 \tag{3.7}$$

Where

$$\mu = -\frac{r_{0\psi}}{p_{0\psi}}\frac{dp_{0\psi}}{dr_{0\psi}}$$

If we agree to consider r_ψ , p_ψ and ρ_ψ as equilibrium values of $r_{o\psi}$, $p_{o\psi}$ and $\rho_{o\psi}$ on equipotential surface ψ =constant then the subscript 'o' used in $r_{0\psi}$, p_ψ , $\rho_{0\psi}$ may be dropped. Further, if we use r_0 defined by equation (1.26) as the independent variable, then equation (3.7) with the help of (1.31) may be expressed as

$$\begin{aligned} A(n,q,r_0)\frac{d^2\eta}{dr_0^2} + \left[\frac{4-\mu}{r_0}B(n,q,r_0) - C(n,\ q,r_0)\right]\frac{d\eta}{dr_0} \\ + \left[\frac{D^2\sigma^2\rho_\psi}{\gamma P_\psi} - \left(3-\frac{4}{\gamma}\right)\frac{\mu}{r_0^2}E(n,q,r_0)\right]\eta = 0 \end{aligned} \tag{3.8}$$

where

$$A(n,q,r_0) = 1 - \frac{16}{3}nr_0^3 - (\frac{56}{5}q^2 + \frac{104}{45}n^2 + \frac{112}{15}nq)r_0^6 - \frac{90}{7}q^2r_0^8 - \frac{44}{3}q^2r_0^{10} - ...$$

$$B(n,q,r_0) = 1 - \frac{10}{3}nr_0^3 - (\frac{32}{5}q^2 + \frac{188}{45}n^2 + \frac{64}{15}nq)r_0^6 - \frac{50}{7}q^2r_0^8 - 8q^2r_0^{10} - ...$$

$$C(n,q,r_0) = \frac{1}{r_0}[8nr_0^3 + (\frac{168}{5}q^2 + \frac{104}{15}n^2 + \frac{112}{5}nq)r_0^6 + \frac{360}{7}q^2r_0^8 + \frac{220}{3}q^2r_0^{10} + ...]$$

$$E(n,q,r_0) = 1 - \frac{4}{3}nr_0^3 - (\frac{8}{5}q^2 + \frac{92}{45}n^2 + \frac{16}{15}nq)r_0^6 - \frac{10}{7}q^2r_0^8 - \frac{4}{3}q^2r_0^{10} - ...$$

and

$$\mu = -\frac{r_\psi}{P_\psi}\frac{dP_\psi}{dr_0}\frac{dr_0}{dr_\psi} = -F(n,q,r_0)\frac{r_0}{P_\psi}\frac{dP_\psi}{d\,r_0}$$

Where

$$F(n,q,r_0)=1-2nr_0^3-(\frac{24}{5}q^2+\frac{72}{15}n^2+\frac{16}{5}nq)r_0^6-\frac{40}{7}q^2r_0^8-\frac{20}{3}q^2r_0^{10}-...$$

In the absence of any distortion i.e. ($n=q=0,\ \rho_\psi=\rho, p_\psi=p$ and $r_o=x$), the above equation reduces to

$$\frac{d^2\eta}{dr_0^2}+\frac{4-\mu}{x}\frac{d\eta}{dx}+\left[\frac{D^2\sigma^2\rho_\psi}{\gamma P_\psi}-\left(3-\frac{4}{\gamma}\right)\frac{\mu}{x^2}\right]\eta=0$$

with

$$\mu=-\frac{x}{P}\frac{dP}{dx}$$

which is the usual equation determining the eigenfrequencies of small adiabatic radial modes of oscillations of gaseous sphere (Cf Roseland (109), with($\gamma=0$)

Equation (3.8) forms an eigenvalue problem in the eigenfrequency of oscillation σ. As usual, this eigenvalue problem is of Sturm-Liouville type having singularities both

at the centre and the surface of the model. It has to be solved subject to the boundary conditions which require η being finite at the centre as well as at the free surface.

In reality equation (3.8) determines the periods of small adiabatic pseudo-radial modes of oscillations of the topologically equivalent spherical model. However, since equipotential surfaces of the actual rotationally and tidally distorted model are also the surfaces of equipressure and equidensity, the values of pressure and density on the equipotential surfaces of the rotationally and tidally distorted star are same as on the corresponding equipotential surfaces of the equivalent spherical model. Hence the eigenfrequencies of the radial modes of oscillations determined by solving the eigenvalue problem for the topologically equivalent spherical model are indeed the eigenfrequencies of the radial modes of oscillation of the undistorted model which have got influenced by the rotational and tidal effects. However, the values of the eigenfunction η obtained on solving (3.8) for the equivalent spherical model are not the actual values of amplitudes of pulsation η for the distorted model but rather some averages of the true values of eigenfunctions η on the rotationally and tidally distorted model.

We may thus use equation (3.8) to determine the periods of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted stellar model. The effect of rotation and tidal distortion appear through the expressions $A(n,q,r_0)$, $B(n,q,r_0)$, $C(n,q,r_0)$, $E(n,q,r_0)$, $F(n,q,r_0)$ and its influence is incorporated through the values evaluating ρ_ψ and P_ψ for the equilibrium model**.** The present method in fact incorporates the effect of distortional force both while computing the equilibrium structure (in computing the values of P_ψ, ρ_ψ etc.) as well as in the coefficients *A, B* and *C* of the equation (3.8) which determines the periods of small adiabatic pseudo-radial modes of oscillations.

3.2 COMPUTATION OF EIGENVALUES OF RADIAL OSCILLATIONS

The eigenvalue problem (3.8) together with the boundary conditions which require η to be finite both at the centre as well as the free surface of the star may be solved numerically in the usual manner as is done in the case of undistorted models. For convenience in numerical work, it is sometimes found convenient to set

$$\eta = \frac{\varsigma}{r_0} \quad \text{and} \quad r_0 = x\, r_{OS} \tag{3.9}$$

(r_{OS} being the value of r_0 on the outermost surface) in equation (3.8) and treat x as the independent variable and ζ as the dependent variable. With these substitutions, x is now zero at the centre and one at the free surface. The boundary condition η = finite at the centre is now replaced by $\varsigma = 0$ at the centre whereas the boundary condition η= finite at the free surface becomes ς = finite at $x=1$. Using (3.9), equation (3.8) gets transformed in terms of the variables ς and x as

$$A^*(n,q,x)\frac{d^2\varsigma}{dx^2} + B^*(n,q,x)\frac{d\varsigma}{dx} + C^*(n,q,x)\varsigma = 0 \tag{3.10}$$

Where

$$A^*(n,q,x) = A(n,q,x)$$

$$B^*(n,q,x) = \frac{4-\mu}{x} B(n,q,x) - r_{0S} C(n,q,x) - \frac{2}{x} A(n,q,x)$$

$$C^*(n,q,x) = \frac{r_{0S}^2 D^2 \rho_\psi}{\gamma P_\psi} \sigma^2 - (3 - \frac{4}{\gamma}) \frac{\mu}{x^2} E(n,q,x) - \frac{1}{x} B^*(n,q,x)$$

The boundary conditions now becomes

$$\begin{aligned} &\varsigma = 0 \text{ at the centre } \mathrm{x} = 0 \\ \text{and} \quad &\varsigma = \text{finite at the free surface } \mathrm{x} = 1 \end{aligned} \tag{3.11}$$

For computing an eigenvalue σ, equation (3.10) has to be solved numerically subject to the specified boundary conditions (3.11). Since the centre and the free surface of the star are the singularities of this differential equation, it is advisable to write series solutions of (3.10) near the singularities to start numerical integrations. If we assume ς to be normalized to have value one at the free surface, we can assume a series solutions of the type

$$\varsigma = \sum_{J=0}^{\infty} a_J x^J \tag{3.12}$$

near the centre x = 0 and

$$\varsigma = 1 + \sum_{J=0}^{\infty} b_J (1-x)^J \tag{3.13}$$

near the surface x =1, to start the integration of (3.10) near these two singularities.

For obtaining an eigenfrequency of pseudo-radial mode of oscillation, the equation (3.10) has to be integrated numerically for trial values of σ till a value of σ is obtained for which both the boundary conditions are satisfied. One way to achieve this objective could be to integrate equation (3.10) numerically from the surface towards the centre using say fourth-order Runge-Kutta method. Starting values near the surface may be obtained from series solution (3.13). Similarly we can integrate equation (3.10) numerically outwards from the centre starting from a point near the centre. The starting values near the centre may be obtained from the series solution (3.12). Trials with different values of σ may be continued till a value of σ is found for which the value $\varsigma/(d\varsigma/dx)$ from the inward and outward integrations match to desired accuracy at some suitably selected point inside the model For determining the eigenfrequencies it is recommended that ρ_ψ, P_ψ be first converted into suitable nondimensional forms keeping in view the physical nature of the model under investigation.

3.3 EIGENVALUED BOUNDARY VALUE PROBLEM DETERMINING EIGENFREQUENCIES OF SMALL ADIABATIC PSEUDO – RADIAL MODES OF OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED WHITE DWARF MODELS OF STARS

The eigenvalued boundary value problem which determines the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted gaseous sphere has been formulated in section 3.1. In order to use this formulation to determine the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted white dwarf model, we have to use the values of ρ_ψ and P_ψ for the appropriate rotationally and tidally distorted white dwarf model in this eigenvalue problem.

On substituting in equation (3.8) the values of P_ψ and ρ_ψ as defined by relations (2.1) of chapter II, we get the eigenvalued boundary value problem determining the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted white dwarfs models of stars..

$$H_1^\bullet \frac{d^2\zeta}{dx^2} + H_2^\bullet \frac{d\zeta}{dx} + H_3^\bullet \zeta = 0 \qquad (3.14)$$

Where

$$H_1^\bullet = 1 - \frac{16}{3} n\left(x r_{0s}\right)^3 - \left(\frac{56}{5} q^2 + \frac{104}{45} n^2 + \frac{112}{15} nq\right)\left(x r_{0s}\right)^6 - \frac{90}{7} q^2 \left(x r_{0s}\right)^8 - \frac{44}{3} q^2 \left(x r_{0s}\right)^{109} - \ldots$$

$$\begin{aligned} H_2^\bullet &= H_2\, r_{0s} - \frac{2}{x} H_1 \\ &= \frac{1}{x}[(2 - \frac{32}{3} n\left(x r_{0s}\right)^3 - (\frac{184}{5} q^2 + \frac{856}{45} n^2 + \frac{368}{15} nq)\left(x r_{0s}\right)^6 - \frac{380}{7} q^2 \left(x r_{0s}\right)^{38} - \frac{228}{3} q^2 \left(x r_{0s}\right)^{10} - \ldots) \\ &\quad + 8(\phi_\psi - \frac{1}{\phi_0^2})^{\frac{3}{2}} \frac{d\phi_\psi}{dr_0} x \frac{1}{F(\phi_\psi)} H_1^\bullet] \end{aligned}$$

$$H_3^\bullet = -\frac{1}{x} H_2^\bullet + r_{0s}^2 \left[\, \omega^2 H_3^{\bullet\bullet} - H_4^{\bullet\bullet} \right]$$

where

$$H_3^{\bullet\bullet} = \frac{8\eta_u^{\,2}[(\phi_\psi^2 - \frac{1}{\phi_0^2})(1-\frac{1}{\phi_0^2})]^{\frac{3}{2}} K}{3\gamma r_{os}^{\,3} F(\phi_\psi)} \left(\frac{\bar{\rho}}{\rho_c}\right)$$

$$H_4^{\bullet\bullet} = -(3-\frac{4}{\gamma})\frac{(8(\phi_\psi - \frac{1}{\phi_0^2})^{\frac{3}{2}}\frac{d\phi_\psi}{dr_0})}{F(\phi_\psi)\left(x r_{0s}\right)}[1-\frac{10}{3}n\left(x r_{0s}\right)^3 - (\frac{32}{5}q^2 + \frac{188}{45}n^2 + \frac{64}{15}nq)\left(x r_{0s}\right)^6$$
$$-\frac{50}{7}q^2\left(x r_{0s}\right)^8 - 8q\left(x r_{0s}\right)^{10} - ...]$$

$$F(\phi_\psi) = \left[\phi_\psi(\phi_\psi^2 - \frac{1}{\phi_0^2})^{\frac{3}{2}}(2\phi_\psi^2 - \frac{5}{\phi_0^2}) + \frac{3}{\phi_0^4}\log(\phi_0 + (\phi_\psi^2 - \frac{1}{\phi_0^2})^{\frac{1}{2}}\right] \text{ and } \omega^2 = \frac{D^3 r_{os}^3 \sigma^2}{GM}$$

ω being the nondimensional form of the eigenfrequency σ and M is the total mass of the star. In the above expressions, values of the parameters ξ_u, $\bar{\rho}$, ρ_c and K are to be taken for the original undistorted white dwarfs model.

Equation (3.14) is the general equation in nondimensional form which determines the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of rotationally and tidally distorted white dwarfs model when terms up to second order of smallness in n, q and up to r_0^{10} in r_0 are retained. For numerical evaluation of the eigenfrequencies, the second order differential equation (3.14) is to be solved numerically subject to the boundary conditions which require η to be finite at points corresponding to the centre $(r_0 = 0)$ and the free surface $(r_0 = r_{OS})$ of the model.

Now using the equation (3.12) the series solution at centre $x = 0$ for the equation (3.14) can be written as

$$\varsigma(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + ... \qquad (3.15)$$

where

$$a_0 = 0$$

$$a_1 = 1$$

$$a_2 = -\frac{1}{4}(3-\frac{4}{\gamma})\frac{8\phi_0^4(\phi_\psi - \frac{1}{\phi_0^2})^{\frac{3}{2}}\frac{d\phi_\psi}{dr_0}}{F(\phi_\psi)}a_1$$

$$a_3 = \frac{1}{10}\left[-(4-\frac{4}{\gamma})\frac{8\phi_0^4(\phi_\psi - \frac{1}{\phi_0^2})^{\frac{3}{2}}\frac{d\phi_\psi}{dr_0}}{F(\phi_\psi)}a_2 - \frac{8\eta_u{}^2\phi_0^4[(\phi_\psi^2 - \frac{1}{\phi_0^2})(1-\frac{1}{\phi_0^2})]^{\frac{3}{2}}K}{3\gamma r_{os}\,F(\phi_\psi)}\omega^2\left(\frac{\overline{\rho}}{\rho_c}\right)\right]$$

$$a_4 = \frac{1}{18}\left[-(5-\frac{4}{\gamma})\frac{8\phi_0^4(\phi_\psi - \frac{1}{\phi_0^2})^{\frac{3}{2}}\frac{d\phi_\psi}{dr_0}}{F(\phi_\psi)}a_3 - \frac{8\eta_u{}^2\phi_0^4[(\phi_\psi^2 - \frac{1}{\phi_0^2})(1-\frac{1}{\phi_0^2})]^{\frac{3}{2}}K}{3\gamma r_{os}\,F(\phi_\psi)}\omega^2\left(\frac{\overline{\rho}}{\rho_c}\right)a_2\right]$$

and the series solution at surface $x = 1$ for the equation (3.14) becomes

$$\xi(x) = 1 - c_1 z + c_2 z^2 - c_3 z^3 + c_4 z^4 - a_5 z^5 + ... \qquad (3.16)$$

Where

$$z = 1 - x$$

$$c_1 = \left[1 - c_0\left\{1 + \frac{16}{3}nr_{0s}^3\right\} - \left\{(3-\frac{4}{\gamma})\left\{1 + 2nr_{0s}^3\right\}\right\}\right]$$

$$c_2 = -\frac{1}{2}\left[1 + c_0(16nr_{0s}^3) + (3-\frac{4}{\gamma})(-1 + 4nr_{0s}^3)\right]$$

$$c_3 = \frac{1}{6}\left[2 - c_0(32nr_{0s}^3) - (3-\frac{4}{\gamma})(2 + 4nr_{0s}^3)\right]$$

$$c_4 = -\frac{1}{24}\left[6 + c_0(32nr_{0s}^3) - (3-\frac{4}{\gamma})(6)\right]$$

where

$$c_0 = \frac{\eta_u^2\,\phi_0\,\phi_\psi\,(1-\frac{1}{\phi_0^2})^{\frac{3}{2}}\,\omega^2 k}{3\gamma r_{0s}\,(\phi_\psi^2-\frac{1}{\phi_0^2})^{\frac{1}{2}}\,\frac{d\phi_\psi}{dx}}(\frac{\bar{\rho}}{\rho_c})$$

3.4 NUMERICAL COMPUTATION OF THE EIGENFREQUENCIES OF PSEUDO – RADIAL MODES OF OSCILLATIONS OF DISTORTED WHITE DWARF MODELS OF STARS

For determining the eigenfrequencies of small adiabatic pseudo- radial oscillations of rotationally and tidally distorted white dwarfs model, equation (3.14) is to be integrated numerically subject to the boundary conditions which require ζ being finite at points corresponding to the center and the free surface of the model. The numerical integration can be performed using the approach suggested in section 3.2. The values of ϕ_ψ and $d\phi_\psi/dx$ needed at various points are to be taken from the numerical solution of the equation (2.9) of chapter II. Computations are started with some trial value of ω^2. For this chosen value of ω^2, series solution is first developed at a point close to the center. This solution is then used to carry integration of the pulsation equation (3.14) outwards using fourth order Runge-Kutta method. Using the same numerical value of ω^2, series solution is also developed at points near the surface which is then used to carry integration of the equation (3.14) inwards. The value of $\zeta/(d\zeta/dx)$ obtained from the outward and inward integrations of (3.14), is then matched at some preselected point in the interior of the model. Process is iteratively continued with different choices of the value of ω^2, till a value of ω^2 is found for which the two solutions agree to specified accuracy. To start integrations from points near the centre and the surface, series solutions were developed at x = 0.01 and x = 0.99. Outward and inward integrations were performed using a step length x = 0.01. Trials with different values of ω^2 were continued till the absolute difference in the value $\zeta/(d\zeta/dx)$ at the preselected point in the interior of the model from the outward and inward integrations was found to be less than 0.0005.

Computations have been performed to determine the eigenvalues of the fundamental, first and second mode of pseudo-radial modes of oscillations of rotationally and tidally distorted white dwarf models central degeneracy parameters varying from 0.01 to 0.8 for which equilibrium structures were earlier obtained in Chapter II for the same values of distortion parameters n and q. The results thus computed are presented in Table (4.1 - 4.3). The behavior of eigenfunctions of various modes of pseudo – radial oscillations of certain rotationally and tidally distorted white dwarf models of stars have been shown in figures (1-8).

3.5 ANALYSIS OF THE RESULT

The results shown in tables (4.1 -4.3) present the eigenfrequencies of the fundamental, the first and the second pseudo radial modes of oscillations of rotationally and tidally distorted white dwarf models for $\gamma = 5/3$ with the parameters $1/\phi_0^2$ as 0.01, 0.05, 0.2, 0.4, 0.6, 0.8.. The eigenfunctions of each radial modes of rotationally and tidally distorted white dwarf modes decreases aw compared to the respective value of undistorted white dwarf models. However, in the case of tidally distorted models (model no. 4, 5) these increases. The amount of increase or decrease in the values varies from model to model. For synchronously rotating white dwarf models 10, 11, 12, it is noted that the eigenvalues of each mode decreases as compared to no synchronously rotating white dwarfs. On comparing the eigenvalues of each modes for rotationally distorted white dwarf s (model 2,3) tidally distorted white dwarf (4,5), it observed that the effect of tidal distortion on the eigenvalues of white dwarf are larger as compared to the rotation.

Table 4.1: Eigenfrequencies $\omega^2 (= r_{0s}^3 R^3 \sigma^2 / GM_0)$ for the fundamental mode (ω_0^2), first mode (ω_1^2) and second mode (ω_2^2) of pseudo radial oscillations of rotationally and tidally distorted White dwarf models of stars

Model No	$\frac{1}{\phi_0^2}$ =0.01, η_u =5.3571			$\frac{1}{\phi_0^2}$ =0.05, η_u =4.4601		
	ω_0^2	ω_1^2	ω_2^2	ω_0^2	ω_1^2	ω_2^2
1	1.14796	23.91368	67.14943	1.19889	25.05952	69.22273
2	1.09147	23.34240	65.5139	1.13833	24.45769	67.53965
3	0.59783	17.77704	49.33752	0.63259	18.77889	51.3553
4	1.20629	24.27611	67.98250	1.23137	25.25409	69.65751
5	1.19927	24.23092	67.84085	1.23073	25.25409	69.66010
6	1.12645	23.51115	65.81441	1.15672	24.51207	67.57122
7	0.99026	22.32983	62.29744	1.01017	23.25427	63.91472
8	1.05289	22.77100	63.69118	1.07421	23.70502	65.32169
9	1.04196	22.69519	63.46256	1.06904	23.66714	65.18337
10	0.80903	20.20584	56.33113	0.83300	21.13036	58.04941
11	0.78843	19.98102	55.66195	0.80582	20.85409	57.26742
12	0.74039	19.46880	54.14061	0.77886	20.32908	55.73754

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Table4.2 Eigenfrequencies $\omega^2 (= r_{0s}^3\ R^3\ \sigma^2 / GM_0)$ for the fundamental mode (ω_0^2), first mode (ω_1^2) and second mode (ω_2^2) of pseudo radial oscillations of rotationally and tidally distorted White dwarf models of stars

Model No	$1/\phi_0^2 =$ 0.2, η_u =3.7271			$1/\phi_0^2 =$ 0.4, η_u =3.5245		
	ω_0^2	ω_1^2	ω_2^2	ω_0^2	ω_1^2	ω_2^2
1	1.23691	25.89313	70.61967	1.25852	26.27435	71.28601
2	1.17591	25.28490	68.94305	1.19863	25.67228	69.63181
3	0.66197	19.54530	52.84912	0.67707	19.88826	53.52542
4	1.25607	26.0036	70.85648	1.26798	26.32674	71.39117
5	1.24656	25.94374	70.68317	1.26720	26.31975	71.33820
6	1.25054	26.06069	70.98097	1.91918	25.56717	69.30035
7	1.03069	23.96640	65.09593	1.11069	24.71210	66.62264
8	1.09574	24.42284	66.50335	1.11397	24.78241	67.14215
9	1.082057	24.32962	66.23849	1.09904	24.68163	66.86155
10	0.850305	21.80389	59.22663	0.85967	22.10162	59.76120
11	0.82325	21.52802	58.45663	0.83654	21.8545	59.06165
12	0.77515	21.00852	56.96083	0.78916	21.34127	57.58994

Table 4.3: Eigenfrequencies $\omega^2 (= r_{0s}^3 R^3 \sigma^2 / GM_0)$ for the fundamental mode (ω_0^2), first mode (ω_1^2) and second mode (ω_2^2) of pseudo radial oscillations of rotationally and tidally distorted White dwarf models of stars.

Model No	$\frac{1}{\phi_0^2} = 0.6,\ \eta_u = 3.6038$			$\frac{1}{\phi_0^2} = 0.8,\ \eta_u = 4.0446$		
	ω_0^2	ω_1^2	ω_2^2	ω_0^2	ω_1^2	ω_2^2
1	1.22645	26.20238	71.05616	1.50471	28.05036	75.13154
2	1.166485	25.57295	69.34393	1.43732	27.43418	73.45769
3	0.654153	19.80469	53.30536	0.82470	21.33668	56.75472
4	1.23587	26.25465	71.15958	1.50156	28.05756	75.14538
5	1.22794	26.20522	71.01085	1.50155	28.03527	75.05905
6	1.15467	25.45618	68.98497	1.41546	27.24346	72.93895
7	1.01192	24.20187	65.39642	1.25374	25.96424	69.32995
8	1.06986	24.61688	66.70717	1.35203	26.91506	71.89514
9	1.06307	24.56668	66.54012	1.31485	26.35543	70.51402
10	0.83384	22.02610	59.54586	1.07160	23.30388	62.30549
11	0.80391	21.72409	58.72555	1.01640	24.43218	62.54562
12	0.75845	21.21802	57.2697	0.96549	22.93185	61.11709

BEHAVIOUR OF EIGENFUNCTION OF CERTAIN ROTATIONALLY AND TIDALLY DISTORTE MODELS OF WHITE DWARFS MODELS OF STARS.

Model 1: $\frac{1}{\phi_0^2} = 0.01,\ \eta_u = 5.3571,\ n = 0.0,\ q = 0.0,\ k = 1.0$

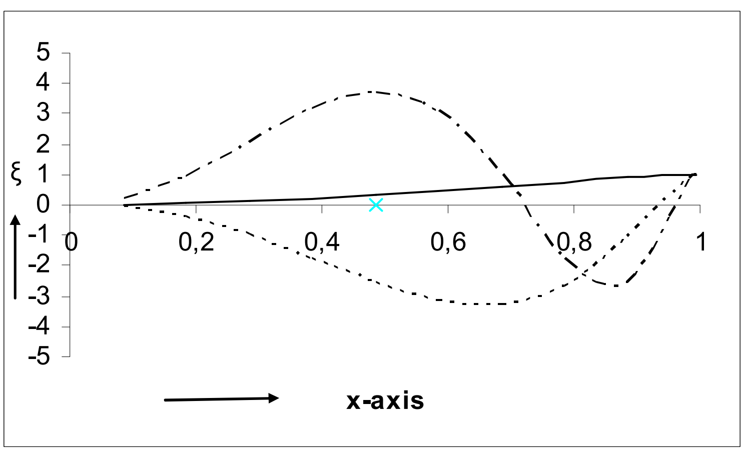
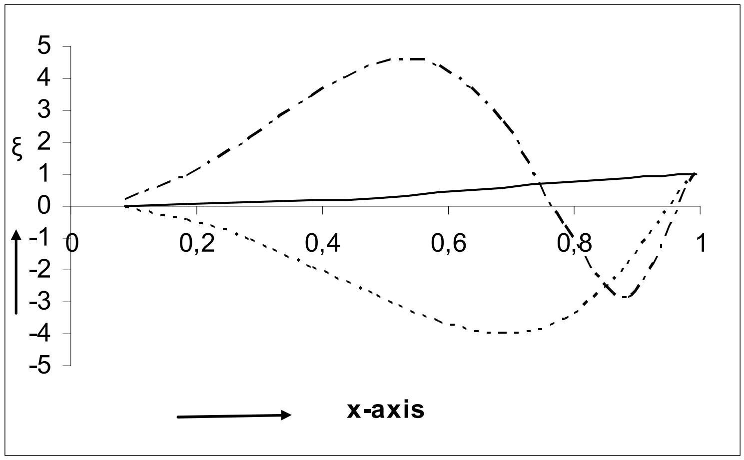


Fig. 1

Model 3: $\frac{1}{\phi_0^2} = 0.01,\ \eta_u = 5.3571,\ n = 0.1,\ q = 0.0,\ k = 1.0$



.Fig. 2

Model 4: $\frac{1}{\phi_0^2} = 0.01,\ \eta_u = 5.3571,\ n = 0.0,\ q = 0.1,\ k = 0.5$

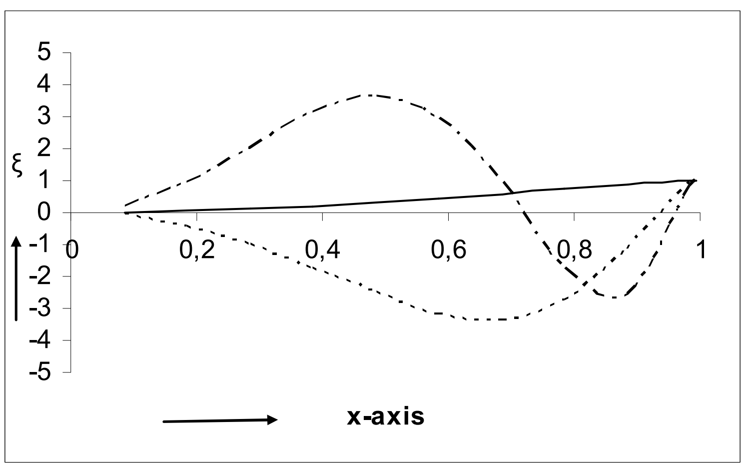


Fig. 3

Model 10: $\frac{1}{\phi_0^2} = 0.01,\ \eta_u = 5.3571,\ n = 0.525,\ q = 0.05,\ k = 0.5$

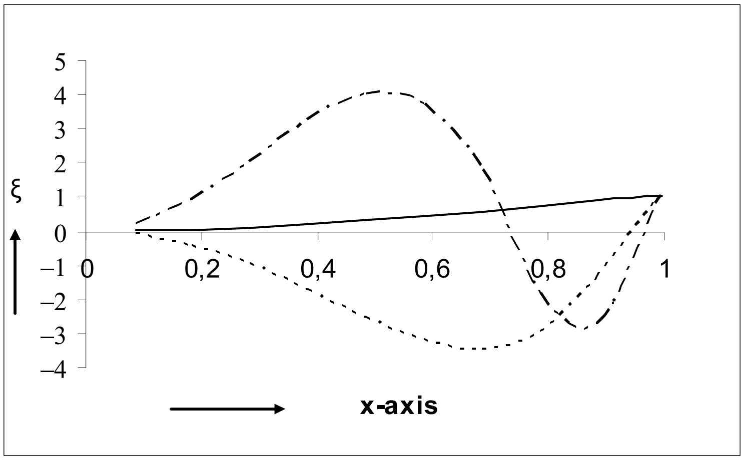


Fig. 4

Model 1: $\frac{1}{\phi_0^2} = 0.4,\ \eta_u = 3.5245,\ n = 0.0,\ q = 0.0,\ k = 1.0$

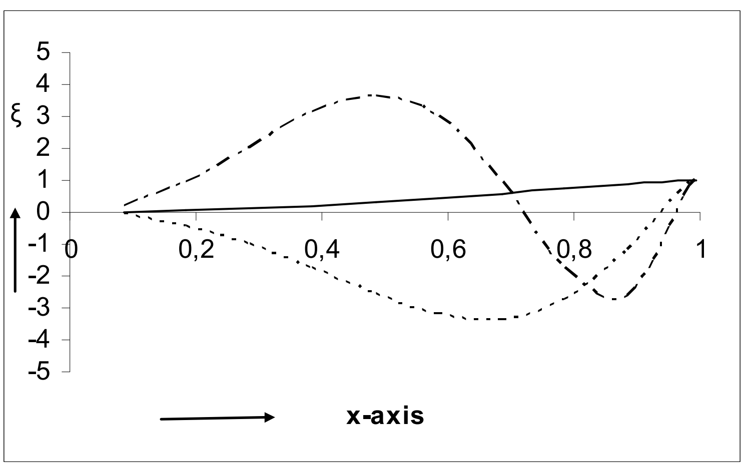


Fig. 5

Model 3: $\frac{1}{\phi_0^2} = 0.4\ \eta_u = 3.5245,\ n = 0.1,\ q = 0.0,\ k = 1.0$

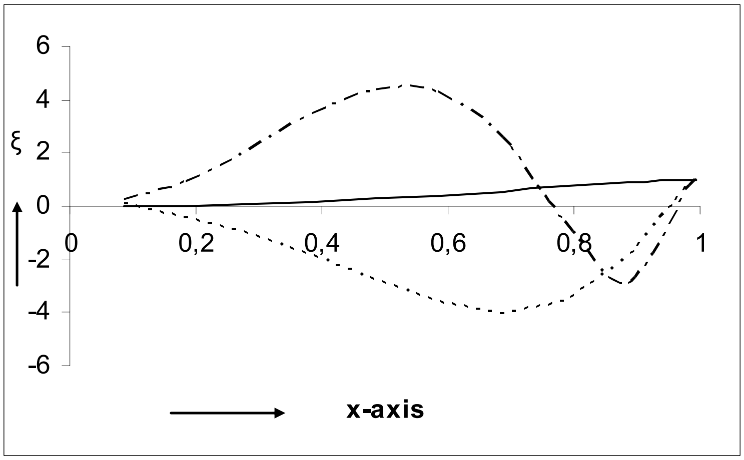


Fig. 6

Model 4: $\frac{1}{\phi_0^2} = 0.4,\ \eta_u = 3.5245,\ n = 0.0,\ q = 0.1,\ k = 0.5$

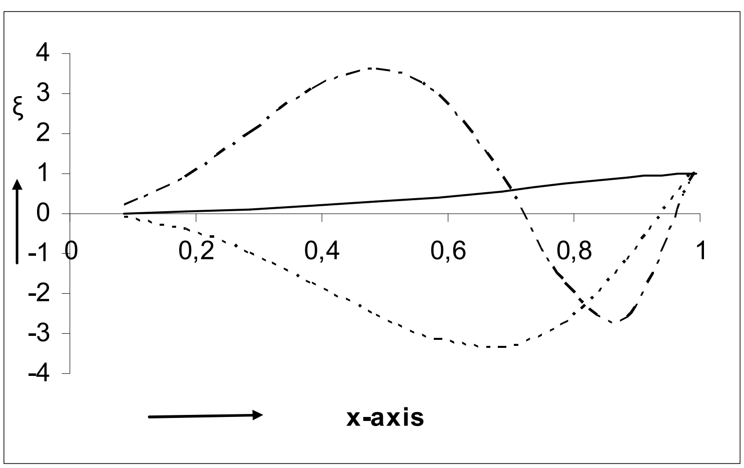


Fig. 7

Model 10: $\frac{1}{\phi_0^2} = 0.4,\ \eta_u = 3.5245,\ n = 0.525,\ q = 0.05,\ k = 0.5$

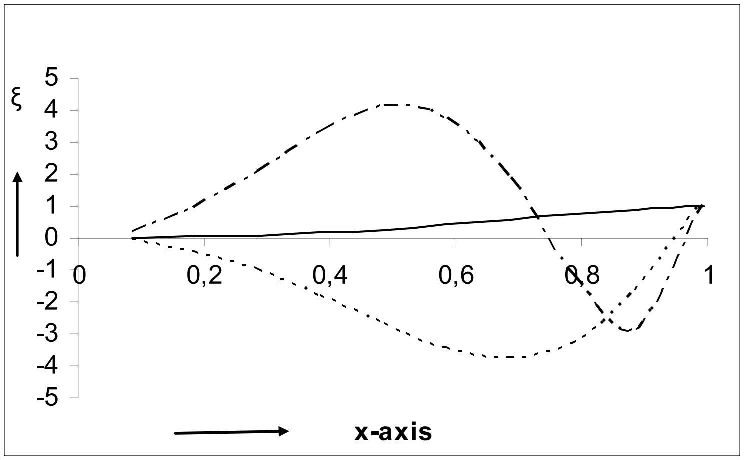


Fig. 8

CHAPTER-IV

EIGENVALUED BOUNDARY VALUE PROBLEM DETERMINING THE EIGENFREQUENCIES OF SMALL ADIABATIC NONRADIAL MODES OF OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED WHITE DWARF MODELS OF STARS

In this chapter we formulate an eigenvalued boundary value problem which may be used to determine the eigenfrequencies of nonradial oscillations of rotationally and tidally distorted gaseous spheres. The basic assumptions utilized in formulating this eigenvalued boundary value problem are same as were used in setting up the eigenvalued boundary value problem of pseudo-radial oscillations the distorting forces causing rotational and tidal distortions are not large and terms beyond second order of smallness in n and q can be neglected. To determine the eigenfrequencies of the distorted model we first set up the eigenvalue problem which determines the eigenfrequencies of nonradial oscillations of the corresponding topologically equivalent spherical model. For this purpose we follow the usual approach adopted by Ledoux and Walraven (64), page 511) to set up the eigenvalued boundary value problem determining the eigenfrequencies of nonradial oscillations of spherical models and then studied the effect of rotation and tidal distortion on the eigenfrequencies of small adiabatic nonradial modes oscillations of white dwarfs model of stars.

The organization of this chapter is as follows: In section 4.1 we have discussed eigenvalued boundary valued problem determining the eigenfrequencies of small adiabatic nonradial modes of oscillations of rotationally and tidally distorted stellar models. In section 4.2 we have discussed eigenvalued boundary valued problem determining the eigenfrequencies of small adiabatic nonradial modes of oscillations of rotationally and tidally distorted white dwarfs models of stars. Numerical computation of the eigenfrequencies of nonradial modes of oscillations of distorted white models of stars is discussed in section 4.3. Analysis of result is discussed in section 4.4..

4.1 EIGENVALUED BOUNDARY VALUE PROBLEM DETERMINING THE EIGENFREQUENCIES OF SMALL ADIABATIC NONRADIALMODES OF OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED STELLAR MODELS

Mohan et al (92) formulated an eigenvalued boundary value problem to determine the eigenfrequencies of the nonradial modes of oscillations of rotationally and tidally distorted gaseous spheres. As in the radial case since the values of the physical parameters ρ_ψ and P_ψ on the equipotential surfaces of the distorted model are same as those on the corresponding equipotential surfaces of a topologically equivalent spherical model, we may use this topological equivalent spherical model to compute the eigenfrequencies of nonradial modes of oscillations of rotationally and tidally distorted gaseous spheres as well. Following Mohan et al (92), the eigenvalue problem determining the eigenfrequencies of nonradial modes of oscillations of rotationally and tidally distorted gaseous spheres which force as well may be expressed in an explicit form convenient for computational work as

$$\left.\begin{aligned} &\frac{d\zeta}{dx} + B_1\zeta + \left(B_2 + \frac{1}{\sigma^2}B_3\right)\eta + \frac{1}{\sigma^2}B_3\,\phi = 0 \\ &\frac{d\eta}{dx} + \left(E_1\sigma^2 + E_2\right)\zeta + E_3\eta + E_4\,\phi + \frac{d\phi}{dx} = 0 \\ &\frac{\mathrm{d}^2\phi}{dx^2} + F_1\frac{d\phi}{dx} + F_2\zeta + F_3\eta + F_4\,\phi = 0 \end{aligned}\right\} \tag{4.1}$$

where

$$B_1 = \frac{l+1}{x} + \frac{1}{\gamma P_\psi}\frac{dP\psi}{dx}$$

$$\begin{aligned} B_2 &= \frac{2\pi G\rho_c}{Dx}\frac{\rho_\psi}{\gamma P_\psi} r_\psi^2 \frac{dr_\psi}{dx} \\ &= \frac{2\pi G\rho_c}{\gamma P_\psi} D^2\,\rho_\psi\, r_{0s}^3\, x[1+4n(xr_{0s})^3 + (\frac{36}{5}q^2 + \frac{864}{45}n^2 + \frac{24}{5}nq)(xr_{0s})^6 \\ &\quad + \frac{55}{7}q^2(xr_{0s})^8 + \frac{26}{3}q^2(xr_{0s})^{10} + \ldots] \end{aligned}$$

$$B_3 = -\frac{l(l+1)}{D\,x}\frac{dr_\psi}{dx}2\pi G\rho_c$$

$$= -\frac{l(l+1)}{x}2\pi G\rho_c r_{0s}\,[1+\frac{8}{3}n(xr_{0s})^3 + (\frac{28}{5}q^2 + \frac{532}{45}n^2 + \frac{56nq}{15})(xr_{0s})^6$$

$$+\frac{45}{7}q^2(xr_{0s})^8 + \frac{22}{3}q^2(xr_{0s})^{10} + ...]$$

$$E_1 = -\frac{1}{2\pi G\rho_c}\frac{D\,x}{r_\psi^2}\frac{dr_\psi}{dx}$$

$$= -\frac{1}{2\pi G\rho_c\, r_{0s}\, x}[1+\frac{4}{3}n(xr_{0s})^3 + (4q^2 + \frac{56}{9}n^2 + \frac{8}{3}nq)(xr_{0s})^6$$

$$+\,5q^2(xr_{0s})^8 + 6q^2(xr_{0s})^{10} + ...]$$

$$E_2 = \frac{1}{2\pi G\rho_c}\frac{A_\psi}{\rho_\psi}\frac{dP_\psi}{dx}\frac{D\,x}{r_\psi^2}$$

$$= \frac{1}{2\pi G\rho_c D^2}\frac{1}{\rho_\psi}(\frac{1}{\rho_\psi}\frac{d\rho_\psi}{dx} - \frac{1}{\gamma P_\psi}\frac{dP_\psi}{dx})\frac{dP_\psi}{dx}\frac{1}{x\,r_{0s}^3}[1-4n(xr_{0s})^3$$

$$-(\frac{36}{5}q^2 + \frac{144}{45}n^2 + \frac{72}{15}nq)(xr_{0s})^6 - \frac{55}{7}q^2(xr_{0s})^8 - \frac{26}{3}q^2(xr_{0s})^{10} - ...]$$

$$E_3 = \frac{l}{x} + A_\psi\frac{dr_\psi}{dx} = \frac{l}{x} + (\frac{1}{\rho_\psi}\frac{d\rho_\psi}{dx} - \frac{1}{\gamma P_\psi}\frac{dP_\psi}{dx})\,,\, E_4 = \frac{l}{x}$$

$$F_1 = \frac{2l}{x} - \frac{d^2r_\psi/dx^2}{dr_\psi/dx} + \frac{2}{r_\psi}\frac{dr_\psi}{dx}$$

$$= \frac{1}{x}[\,2(l+1) - 4n(xr_{0s})^3 - (24q^2 + 32n^2 + 16nq)(xr_{0s})^6$$

$$-\,40q^2(xr_{0s})^8 - 60q^2(xr_{0s})^{10} - ...]$$

$$F_3 = -\frac{4\pi G\rho_\psi^2}{\gamma P_\psi}(\frac{d\,r_\psi}{d\,x})^2$$

$$= -\frac{4\pi G r_{0s}^2 D^2 \rho_\psi^2}{\gamma P_\psi}[1 + \frac{16}{3}n(xr_{0s})^3 + (\frac{56}{5}q^2$$

$$+ \frac{1384}{45}n^2 + \frac{112}{15}nq)(xr_{0s})^6 + \frac{90}{7}q^2(xr_{0s})^8 + \frac{44}{3}q^2(xr_{0s})^{10} + ...]$$

$$F_4 = \frac{l(l+1)}{x^2} - \frac{l}{x}(\frac{d^2r_\psi}{dx^2})/(\frac{dr_\psi}{dx}) + \frac{2l}{x}(\frac{1}{r_\psi}\frac{dr_\psi}{dx}) - \frac{l(l+1)}{r_\psi^2}(\frac{dr_\psi}{dx})^2$$

$$= -\frac{l}{x^2}\{[8n(xr_{0s})^3 + (\frac{168}{5}q^2 + \frac{2412}{45}n^2 + \frac{336}{15}nq)(xr_{0s})^6 + \frac{360}{7}q^2(xr_{0s})^8 + \frac{220}{3}q^2(xr_{0s})^{10} + ...]$$

$$+ l[4n(xr_{0s})^3 + (\frac{48}{5}q^2 + \frac{972}{45}n^2 + \frac{96}{15}nq)(xr_{0s})^6 + \frac{80}{7}q^2(xr_{0s})^8 + \frac{40}{3}q^2(xr_{0s})^{10} + ...]\}$$

Here σ is the eigenfrequency of oscillations, $x = r_0/r_{0s}$ and

$$\zeta = \frac{r_\psi^2 \delta r_\psi}{D^3 x^{l+1}},\ \eta = \frac{\mathrm{P}_\psi'}{2\,\pi\,\mathrm{G}\,\rho_\mathrm{c}\,D^2 x^l \rho_\psi} \text{ and } \phi = \frac{\psi_g'}{2\pi\,\mathrm{G}\rho_\mathrm{c}\,D^2 x^l}$$

δr_ψ being an average of the amplitudes of Lagrangian variations in the radial direction and P_ψ', ψ_g' the amplitudes of Lagrangian variation in pressure and gravitational potential on the equipotential surface $\psi^* =$ constant

The eigenvalue problem (4.1) determining the eigenfrequencies of nonradial modes of oscillations of rotationally and tidally distorted gaseous sphere has to be solved subject to the boundary conditions at the centre and the free surface. Boundary conditions at the centre require $\delta r_\psi = 0$, $P_\psi'/\rho_\psi = 0$ and $\psi_g' = 0$ for $r_\psi = 0$. These requirements lead to the analytic conditions

$$\eta + \phi = \frac{\sigma^2}{2\pi G\rho_c\, l\, r_{OS}}\zeta, \qquad \frac{d\phi}{dx} = 0 \tag{4.2}$$

at the centre $x = 0$.

If the pressure P_ψ on the free surface $(r_\psi = R_\psi)$ is taken to be zero, then δP_ψ the Lagrangian variation in pressure should also be zero at the outer surface. This leads to the condition

$$2\pi G\rho_c\, r_\psi^2\, \rho_\psi\, \frac{d\, r_\psi}{d\, x}\eta + D\frac{d\, P_\psi}{d\, x}\zeta = 0$$

or

$$2\pi G\, \rho_c \rho_\psi D^2 r_{0s}^3\, [1 + 4n\, r_{0s}^3 + (\frac{36}{5}q^2 + \frac{864}{45}n^2 + \frac{72}{15}nq)\, r_{0s}^6$$
$$+\frac{55}{7}q^2 r_{0s}^8 + \frac{26}{3}q^2 r_{0s}^{10} + ...]\eta + \frac{dP_\psi}{dx}\zeta\ = 0. \tag{4.3a}$$

The condition requiring gravitational potential to be continuous across the free surface gives

$$\frac{d\phi}{dx} + \left[l + \frac{(l+1)}{r_\psi}\frac{dr_\psi}{dx}\right]\phi + \frac{2D\,\rho_\psi}{\rho_c r_\psi^2}\frac{dr_\psi}{dx}\zeta = 0 \tag{4.3b}$$

or

$$\frac{d\theta_\psi}{dx} + \phi\{l + (l+1)\ [1\ +\ 2n\, r_{0s}^3\ +\ (\frac{24}{5}q^2 + \frac{396}{45}n^2 + \frac{48}{15}nq)\, r_{0s}^6$$
$$+\frac{40}{7}q^2 r_{0s}^8 + \frac{20}{3}q^2 r_{0s}^{10} + ...]\}\ + \frac{2\rho_\psi}{\rho_c\, r_{0\,s}}[1 + \frac{4}{3}n\, r_{0\,s}^3\ + (4\,q^2 + \frac{56}{9}n^2$$
$$+\frac{8}{3}n\,q)\, r_{0\,s}^6\ + 5\,q^2 r_{0s}^8\ + 6\,q^2 r_{0s}^{10}\ + ...]\zeta\ = 0$$

at the surface $x = 1$

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4.2 EIGENVALUED BOUNDARY VALUE PROBLEM DETERMINING THE EIGENFREQUENCIES OF SMALL ADIABATIC NONRADIAL MODES OF OSCILLATIONS OF ROTATIONALLY AND TIDALLY DISTORTED WHITE DWARF MODELS OF STARS

System of equation (4.1) with the boundary conditions (4.2 – 4.3) constitutes the eigenvalued boundary value problem which determine the eigenfrequencies of nonradial modes of oscillations of rotationally and tidally distorted gaseous spheres. In order to use this eigenvalue problem to determine the eigenfrequencies of nonradial modes of oscillations of white dwarf models the values of P_ψ and ρ_ψ appearing in these equations are to be taken from relation (2.1) of Chapter II. Using these, the system of differential equations (4.4) governing the nonradial modes of oscillations of rotationally and tidally distorted white dwarf models may be expressed as

$$\left.\begin{aligned}&\frac{d\zeta}{dx}+B_1\zeta+(B_2+\frac{B_3}{\omega^2})\eta+\frac{B_3}{\omega^2}\phi=0\\&\frac{d\eta}{dx}+(E_1\omega^2+E_2)\zeta+E_3\eta+E_4\phi+\frac{d\phi}{dx}=0\\&\frac{\mathrm{d}^2\phi}{dx^2}+F_1\frac{d\phi}{dx}+F_2\zeta+F_3\eta+F_4\phi=0\end{aligned}\right\}\qquad(4.4)$$

Where

$$B1=\frac{l+1}{x}+\frac{8\phi_\psi^2(\phi_\psi^2-\frac{1}{\phi_0})^{\frac{3}{2}}}{\gamma F(\phi_\psi)}\frac{d\phi_\psi}{dx}$$

$$B_2=\frac{4\eta_u^2}{\gamma k^2}\frac{(\phi_\psi^2-\frac{1}{\phi_0^2})(1-\frac{1}{\phi_0^2})^{\frac{3}{2}}}{F(\phi_\psi)}[1+4n(xr_{0s})^3+(\frac{36}{5}q^2+\frac{864}{45}n^2+\frac{24}{5}nq)(xr_{0s})^6$$
$$+\frac{55}{7}q^2(xr_{0s})^8+\frac{26}{3}q^2(xr_{0s})^{10}+...]$$

$$B_3=\frac{3}{2}\frac{1}{k^3}\frac{l(l+1)}{x}\frac{\rho_0}{\rho}r_{0s}{}^4(\frac{\rho_c}{\overline{\rho}})[1+\frac{8}{3}n(xr_{0s})^3+(\frac{28}{5}q^2+\frac{532}{45}n^2+\frac{56}{15}nq)(xr_{0s})^6$$
$$+\frac{45}{7}q^2(xr_{0s})^8+\frac{22}{3}q^2(xr_{0s})^{10}+...]$$

$$E_1=\frac{-2K^3}{3xr_{0s}^4}(\frac{\overline{\rho}}{\rho_c})[1+\frac{4}{3}n(xr_{0s})^3+(4q^2+\frac{56}{9}n^2+\frac{8}{3}nq)(xr_{0s})^6+5q^2(xr_{0s})^8+6q^2(xr_{0s})^{10}+...]$$

$$E_2=\frac{2k^2}{\eta_u^2}\frac{1}{(\frac{1}{\phi_0^2})(1-\frac{1}{\phi_0^2})^{\frac{3}{2}}}[\frac{3\phi_\psi}{(\phi_\psi^2-\frac{1}{\phi_0^2})}-\frac{8(\phi_\psi^2-\frac{1}{\phi_0^2})^{\frac{3}{2}}}{\gamma F(\phi_\psi)}](\frac{d\theta_\psi}{dx})^2\frac{1}{xr_{0s}^3}[1-4n(xr_{0s})^3$$
$$-(\frac{36}{5}q^2+\frac{144}{45}n^2+\frac{72}{15}nq)(xr_{0s})^6-\frac{55}{7}q^2(xr_{0s})^8-\frac{26}{3}q^2(xr_{0s})^{10}-...]$$

$$E_3=\frac{l}{x}+(\frac{d\phi_\psi}{dx})[\frac{3\phi_\psi}{(\phi_\psi^2-\frac{1}{\phi_0^2})}-\frac{8(\phi_\psi^2-\frac{1}{\phi_0^2})^{\frac{3}{2}}}{\gamma F(\phi_\psi)}],\quad E_4=\frac{l}{x}$$

$$F_1 = \frac{1}{x}[2(l+1) - 4n(xr_{0s})^3 - (24q^2 + 32n^2 + 16nq)(xr_{0s})^6 - 40q^2(xr_{0s})^8 - 60q^2(xr_{0s})^{10} - \ldots]$$

$$F_2 = \frac{2(\phi_\psi^2 - \frac{1}{\phi_0^2})^{\frac{3}{2}}}{xr_{0s}(1-\frac{1}{\phi_0^2})^{\frac{3}{2}}}\left[\frac{3\phi_\psi}{(\phi_\psi^2 - \frac{1}{\phi_0^2})} - \frac{8(\phi_\psi^2 - \frac{1}{\phi_0^2})^{\frac{3}{2}}}{\gamma F(x)}\right]\left(\frac{d\phi_\psi}{dx}\right)[1 + \frac{4}{3}n(xr_{0s})^3 + (4q^2 + \frac{56}{9}n^2 + \frac{8}{3}nq)(xr_{0s})^6 + 5q^2(xr_{0s})^8 + 6q^2(xr_{0s})^{10} + \ldots]$$

$$F_3 = -\frac{8}{\gamma}\frac{\eta_u^{\ 2}}{F(x)K^2}r_{0s}^2[1 + \frac{16}{3}n(xr_{0s})^3 + (\frac{56}{5}q^2 + \frac{1384}{45}n^2 + \frac{112}{15}nq)(xr_{0s})^6 + \frac{90}{7}q^2(xr_{0s})^8 + \frac{44}{3}q^2(xr_{0s})^{10} + \ldots]$$

$$F_4 = -\frac{l}{x^2}\{[8n(xr_{0s})^3 + (\frac{168}{5}q^2 + \frac{2412}{45}n^2 + \frac{336}{15}nq)(xr_{0s})^6 + \frac{360}{7}q^2(xr_{0s})^8 + \frac{220}{3}q^2(xr_{0s})^{10} + \ldots] + l[4n(xr_{0s})^3 + (\frac{48}{5}q^2 + \frac{972}{45}n^2 + \frac{96}{15}nq)(xr_{0s})^6 + \frac{80}{7}q^2(xr_{0s})^8 + \frac{40}{3}q^2(xr_{0s})^{10} + \ldots]\}$$

and

$$\omega^2 = \frac{Dr_{0s}^{\ 3}\sigma^2}{GM_0}$$

ω being the nondimensional form of the eigenfrequency σ. As mentioned in the radial case, values of the parameters ξ_u, ρ_c, $\bar{\rho}$ and K are to be taken from the original undistorted white dwarf model.

The boundary conditions (4.2) at the centre ($x=0$) for the case of distorted white dwarf model become

$$\eta + \phi = \frac{2K^3\omega^2}{3l\,r_{0s}^4}\left(\frac{\bar{\rho}}{\rho_c}\right)\zeta, \qquad \frac{d\phi}{dx} = 0 \tag{4.5}$$

On substituting the values of P_ψ and ρ_ψ from equation (2.1) in the boundary conditions (3.10) at the free surface ($x=1$), the boundary conditions at the free surface in the case of white dwarfs models become

$$\eta r_{0s}^3[1+4n(xr_{0s})^3+(\frac{36}{5}q^2+\frac{864}{45}n^2+\frac{72}{15}nq)(xr_{0s})^6+\frac{55}{7}q^2(xr_{0s})^8$$
$$+\frac{26}{3}q^2(xr_{0s})^{10}+...]+2\frac{K^2}{\xi_u^2}\frac{d\phi_\psi}{dx}\zeta=0 \tag{4.6a}$$

and

$$\frac{d\theta_\psi}{dx}+\phi\{l+(l+1)\,[1+2n(xr_{0s})^3+(\frac{24}{5}q^2+\frac{396}{45}n^2+\frac{48}{15}nq)(xr_{0s})^6$$
$$+\frac{40}{7}q^2(xr_{0s})^8+\frac{20}{3}q^2(xr_{0s})^{10}+...]\}=0 \tag{4.6b}$$

at the surface $x=1$. The system of differential equations (4.4) together with the boundary conditions (4.5 – 4.6) constitutes the eigenvalued boundary value problem determine the eigenfrequencies of nonradial modes of oscillations of white dwarfs models of stars.

4.3 NUMERICAL COMPUTATION OF THE EIGENFREQUENCIES OF NONRADIAL MODES OF OSCILLATIONS OF DISTORTED WHITE DWARF MODELS OF STARS

In order to determine the eigenfrequencies of nonradial modes of oscillation of rotationally and tidally distorted gaseous spheres, system of differential equations (4.8) has to solved subject to the boundary conditions (4.2) at the centre and the boundary conditions (4.3) at the free surface.

For determining the eigenfrequencies, it is recommended that in all the above expressions ρ_ψ, P_ψ be first converted into suitable nondimensional forms keeping in view the physical nature of the model under investigation.The numerical method which has been used to solve the eigenvalued boundary value problem of this section is discussed in an appendix to this chapter

The eigenfrequencies of the nonradial modes of oscillations of some of these rotationally and tidally distorted white dwarf models have also been computed using Chebyshev polynomial expansion technique earlier used by Mohan et al (92). The essential details of the method are given in Saxena (116) and are presented for ready reference as an appendix to this chapter. The boundary condition (4.6) was used as the discriminate condition and $\zeta=1$ at the centre was used as the normalization condition. The values of ϕ_ψ and $(d\,\phi_\psi / d\,x)$ needed at various points in the interior of the model were obtained from the solutions of the structure equation (3.30) of these models earlier obtained in Chapter II. For central degeneracy parameter 0.05, 0.2, 0.6 and 0.8 we used 13,14,13 and 13 collocations points. However, for determining the eigenfrequencies of certain higher modes of nonradial oscillations, the number of collocation points was increased to achieve the desired accuracy of 0.0001 in getting the discriminant condition to be satisfied for the central degeneracy parameters $\frac{1}{\phi_0^2} = 0.05, 0.2, 0.6, 0.8$. The number of collocation points used in determining a specific mode of nonradial oscillation of a distorted white dwarf model was same as used in determining this mode for the corresponding undistorted model. The numerical results computed are presented in table (4.1 - 4.3). The number of nodes appearing in the eigenfunctions ζ and η are also shown in parenthesis in these tables.

4.4 ANALYSIS OF THE RESULT

The results presented in table 4.1 show the eigenfrequencies of nonradial modes of oscillations of various types of rotationally and tidally distorted white dwarf stars with central degeneracy parameters 0.05, 0.2, 0.6, and 0.8. In all the models it is noticed that the eigenvalues of g_1, g_2, g_3 , f, p_1, p_2, p_3 modes decrease as compared to the respective values of undistorted white dwarf models

It has been observed from the table 4.1-4.3 the that the percentage change in the values of p -mode is more than the values of g-mode of rotationally and tidally distorted white dwarf models for $\frac{1}{\phi_0^2} = 0.05, 0.2, 0.6, 0.8$. In model (4, 5), there is not any appreciable difference in the present result as compared to the corresponding result of undistorted model. The magnitude of the eigenfrequencies decreases in all the models considered as soon as the values of central degeneracy parameters increases from 0.05 to 0.8

Table 4.3 Eigenfrequencies of the nonradial modes of oscillations of rotationally and tidally distorted white dwarf model of ($\frac{1}{\phi_0^2} = 0.01$)

MODEL NO	g_3	g_2	g_1	f	p_1	p_2	p_3
1	16.911 (3-3)	22.208 (2-2)	27.685 (1-1)	60.245 (0-0)	236.683 (1-1)	350.783 (2-2)	488.474 (3-3)
2	16.739	22.005	27.325	59.761	234.478	364.741	482.624
3	15.110	20.105	23.993	55.418	214.580	309.914	431.564
4	16.822	22.157	27.660	60.216	236.404	350.126	487.027
5	16.813	22.148	27.638	60.198	236.322	349.964	486.776
6	16.603	21.901	27.200	59.599	233.602	345.006	479.609
7	16.294	21.555	26.534	58.821	230.088	338.413	470.077
8	16.387	21.650	26.748	59.002	230.883	339.997	472.395
9	16.372	21.633	26.713	58.967	230.730	339.712	472.000
10	15.663	20.815	25.257	57.054	221.980	323.560	449.308
11	15.599	20.745	25.130	56.905	221.285	322.243	447.478
12	15.462	20.579	24.851	56.572	219.759	319.369	443.551

Table 4.4 Eigenfrequencies of the nonradial modes of oscillations of rotationally and tidally distorted white dwarf model of ($\frac{1}{\phi_0^2} = 0.2$)

MODEL NO	g_3	g_2	g_1	f	p_1	p_2	p_3
1	2.898 (3-3)	4.654 (2-2)	8.518 (1-1)	17.843 (0-0)	84.382 (1-1)	179.425 (2-2)	314.000 (3-3)
2	2.862	4.595	8.397	17.649	83.441	177.456	310.668
3	2.528	4.033	7.261	15.638	75.111 (1-3)	159.159	255.500 (3-8)
4	2.899	4.650	8.510	17.820	84.345	179.360	314.000
5	2.899	4.646	8.503	17.819	84.311	179.285	314.000
6	2.849	4.576	8.358	17.549	83.170	176.891	309.500 (3-4)
7	2.787	4.4671	8.313	17.166	81.643	173.673	304.000
8	2.804	4.502	8.205	17.272	82.000	174.435	305.500
9	2.806	4.498	8.196	17.262	81.964	174.354	305.500
10	2.660	4.255	7.702	16.382	78.261	166.657	291.733
11	2.649	4.233	7.658	16.310	77.963	165.950	290.500
12	2.618	4.185	7.561	16.153	77.318 (1-3)	164.592	288.252

Table 4.5 Eigenfrequencies of the nonradial modes of oscillations of rotationally and tidally distorted white dwarf model of ($\frac{1}{\phi_0^2} = 0.6$)

MODEL NO	g_3	g_2	g_1	f	p_1	p_2	p_3
1	1.573 (3-8)	2.334 (2-6)	3.756 (1-3)	6.423 (0-0)	-	-	-
2	1.552	2.304	3.704	6.325	-	-	-
3	1.363	2.026	3.221	5.482	-	-	-
4	1.567	2.333	3.751	6.419	-	-	-
5	1.565	2.332	3.747	6.414	-	-	-
6	1.541	2.296	3.685	6.297	-	-	-
7	1.504	2.243	3.590	6.133	-	-	-
8	1.515	2.259	3.619	6.178	-	-	-
9	1.513	2.256	3.615	6.171	-	-	-
10	1.430	2.136	3.404	5.796	-	-	-
11	1.423	2.125	3.385	5.764	-	-	-
12	1.407	2.102	3.345	5.698	-	-	-

Table 4.6 Eigenfrequencies of the nonradial modes of oscillations of rotationally and tidally distorted white dwarf model of ($\frac{1}{\phi_0^2} = 0.8$)

MODEL NO	g_3	g_2	g_1	f	p_1	p_2	p_3
1	-	1.485 (2-2)	2.411 (1-1)	3.980 (0-0)	-	-	-
2	-	1.466	2.376	3.915	-	-	-
3	-	1.290	2.054	3.334	-	-	-
4	-	1.484	2.408	3.978	-	-	-
5	-	1.483	2.407	3.974	-	-	-
6	-	1.461	2.366	3.895	-	-	-
7	-	1.427	2.303	3.778	-	-	-
8	-	1.437	2.323	3.815	-	-	-
9	-	1.436	2.320	3.807	-	-	-
10	-	1.361	2.179	3.549	-	-	-
11	-	1.354	2.165	3.527	-	-	-
12	-	1.338	2.139	3.482	-	-	-

APPENDIX

A Computational method for determining the eigenfrequencies of nonradial oscillations using Chebyshev polynomial expansions

The eigenvalued boundary value problem which determines the effects of rotation and tidal distortions on the eigenfrequencies of nonradial modes of oscillations of gaseous spheres is governed by the set of differential equations (3.8) together with the boundary conditions (3.9 – 3.10). This eigenvalued boundary value problem does not yield analytical solutions even for simple density distribution laws. In this appendix we present a method based on the use of Chebyshev polynomial expansions which can be used to solve such problems.

Hurley et al (52) were perhaps the first to use Chebyshev polynomial expansion to obtain the eigenfrequencies of various nonradial modes of oscillations of polytropic models. Later on, the same technique with slight modifications was used by Singh (140), Saxena (133) and Lal (70) to compute the eigenfrequencies of nonradial oscillations of various stellar models. Following this approach the Chebyshev polynomial expansion may be explained as follows:

Chebyshev polynomials are orthogonal polynomials related to the trigonometric functions. A simple way to define a Chebyshev polynomial $T_n(x)$ of order n is given by the relation $T_n(x) = \cos(n\theta)$, where $\theta = \cos^{-1} x$.

In Chebyshev polynomial expansion technique, the unknown functions are represented as linear combinations of Chebyshev polynomials containing a number of unspecified expansion parameters. The substitution of these expansions of unknown functions in the set of differential equations converts the problem of solving the set of linear differential equation into the problem of solving a set of linear simultaneous algebraic equations.

In order to solve the system of differential equations (3.8) with the boundary conditions (3.9 – 3.10), the system of equations together with the boundary conditions are transformed using the substitution $x = (z+1)/2$ $(-1 \le z \le 1)$ so that the range of integration is renormalized from (0, 1) to (-1, 1). The above substitution transforms the set of equations (3.8) to the form

$$\left.\begin{aligned}
&\frac{d\zeta}{dz}+\frac{1}{2}B_1[\zeta+(B_2+\frac{1}{\sigma^2}B_3)\eta+\frac{1}{\sigma^2}B_3\,\phi]=0,\\
&\frac{d\eta}{dz}+\frac{1}{2}[(E_1\sigma^2+E_2)\zeta+E_3\eta+E_4\,\phi]+\frac{d\phi}{dz}=0,\\
&\text{and}\\
&\frac{\mathrm{d}^2\phi}{dz^2}+\frac{1}{2}F_1\frac{d\phi}{dx}+\frac{1}{4}[F_2\zeta+F_3\eta+F_4\,\phi]=0
\end{aligned}\right\}\quad(4.15)$$

where B_1, B_2, B_3, E_1, E_2 etc are the same functions as defined in (4.8) with x replaced by $(z+1)/2$. Now we assume Chebyshev polynomial expansions for $\frac{d\zeta}{dz}, \frac{d\eta}{dz}$ and $\frac{d^2\phi}{dz^2}$ truncated after 'm' terms as

$$\frac{d\zeta}{dz}=\sum_{j=2}^{m+1}a_j\,T_{j-2},\quad \frac{d\eta}{dz}=\sum_{j=2}^{m+1}b_j\,T_{j-2}\,,\quad \frac{d^2\phi}{dz^2}=\sum_{j=3}^{m+2}d_j\,T_{j-3}\quad(4.16)$$

where a_j, b_j $(j=2,3,\ldots,m+1)$ and d_j $(j=3,4,\ldots,m+2)$ are unknown expansion parameters and $T_j(z)=T_j(\cos\theta)=\cos(j\theta)$.

Integrating the above expression analytically, we get

$$\left.\begin{aligned}
\zeta&=a_1+a_2T_1+a_3\frac{T_2}{4}+\sum_{j=4}^{m+1}\frac{a_j}{2}(\frac{T_{j-1}}{j-1}-\frac{T_{j-3}}{j-3})\\
\eta&=b_1+b_2T_1+b_3\frac{T_2}{4}+\sum_{j=4}^{m+1}\frac{b_j}{2}(\frac{T_{j-1}}{j-1}-\frac{T_{j-3}}{j-3})\\
\frac{d\phi}{dz}&=d_2+d_3T_1+d_4\frac{T_2}{4}+\sum_{j=5}^{m+2}\frac{d_j}{2}(\frac{T_{j-2}}{j-2}-\frac{T_{j-4}}{j-4})\\
\phi&=d_1+d_2T_1+d_3\frac{T_2}{4}+\frac{d_4}{24}(T_3-3T_1)+\frac{d_5}{48}(T_4-8T_2)\\
&\quad+\sum_{j=6}^{m+2}\frac{d_j}{4}(\frac{T_{j-1}}{(j-1)(j-2)}-\frac{2T_{j-3}}{(j-2)(j-4)}+\frac{T_{j-5}}{(j-4)(j-5)})
\end{aligned}\right\}\quad(4.17)$$

where a_1, b_1, d_1 and d_2 are constants of integration.

Substituting above expressions in the system of differential equations (4.15), we get in their place three algebraic equations which contain $a_1, a_2, ..., a_{m+1}$, $b_1, b_2, ..., b_{m+1}$ and $d_1, d_2, ..., d_{m+1}$, that is, in all $3m+4$ unknown expansion parameters. To evaluate these unknown parameters, the above equations may now be forced to be satisfied at m collocation points in the range of integration. Zeros of Chebyshev polynomials of order m are often preferred as the collocation points. On satisfying the above equations at m collocation points in the range of integration we get $3m$ linear algebraic equations in $3m+4$ unknown expansion parameters. Now, if we add to these $3m$ equations, the four linear equations which can be obtained from the four boundary conditions (3.9 – 3.10) by converting them to Chebyshev polynomial expansion forms, we get a set of $3m+4$ linear homogeneous algebraic equations in as many unknowns.

In order to determine the eigenfrequencies an approach similar to one followed by Singh (140) may now be used. In this approach we withhold one of the boundary conditions and replace it by a normalization condition which assigns a non – zero value to one of the unknown eigenfunction ζ or η at the center or the surface of the model. The withhold boundary condition is now used as a discriminant condition for determining the value of the nondimensional eigenfrequency σ. Trials with different values of σ are made till a value of σ is found which satisfies the withheld condition to the desired accuracy. (In our subsequent investigations in this thesis, we have used $\zeta = 1$ at the center as normalization condition and the boundary condition (3.10a) at the surface as the discriminant condition. The set of linear algebraic simultaneous equations have been solved using Gaussian elimination method with pivotal condensation).

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