Modeling and Simulation of Traffic Flow

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CERTIFICATE

This is to certify that the thesis entitled ***"Modeling and Simulation of Traffic Flow ",*** *being presented in partial fulfillment of the requirements for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, is a bonafide work carried out under the supervision of* ***Sapna Sharma.***

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

The present thesis contained the brief study of Car Following Model, Continuum modeling and Lattice Hydrodynamic Models. The basics of traffic flow theory are studied briefly. Traffic flow modeling is classified and some earlier important work in the field is discussed concisely. The 1st chapter of the thesis introduce the traffic flow modeling and various models describing the traffic flow phenomenon. In 2 nd chapter we have discussed about Car Following Models and its types. These models describes drivers behaviour only in the presence of interaction with other vehicles. The 3 rd chapter contained Continuum Models and its various types. In chapter 4 we studied about Lattice hydrodynamic model.

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Chapter 1

Mathematical Modeling and Traffic Flow

1.1 Traffic Flow

Traffic flow is the continuous movements of vehicles along a road at a particular time. It is the study of interaction between travelers and infrastructure. The main purpose of the study is to develop the optimal road network with maximum flow of traffic and minimum traffic congestion.

1.2 Mathematical Modeling

A mathematical modeling is a description of a system using mathematical concepts. Mathematical models usually describes a system by a set of variables and equations that establish relationship between these variables. The Traffic flow models are used to improve road safety. In other words, traffic flow modes was developed to understand and express the properties of traffic flow on highway. Modeling of traffic flow has been a key tool to simulate the behaviour of transportation system. There are mainly three types of methodology used in literature namely:

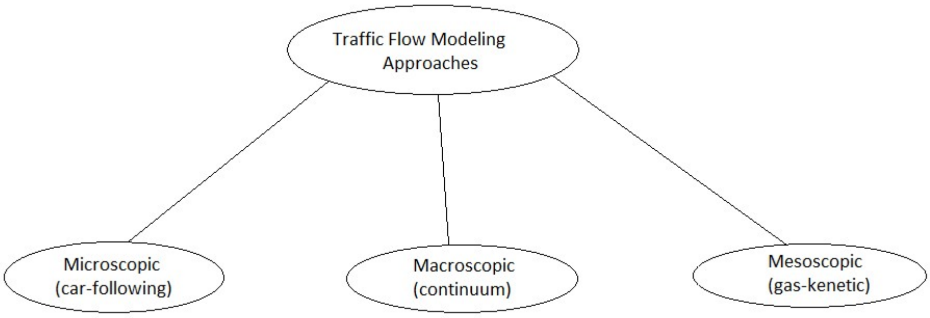


Figure 1.1: General Classification of Traffic Flow Modeling Approaches

(1) Microscopic Approach :-

In this approach motion of individual vehicles and there interaction with other vehicles are modeled. These are created using ordinary differential equations with each vehicle having its own equation and the solution of large system of equations may provide the desired description of the flow conditions on road.

(2) Macroscopic Approach:-

This approach is based on the assumption that a traffic stream on a single lane can be considered as continuum of moving particles. Instead of describing the individual behaviour of each vehicle we look from a global perspective. The flow is governed by a single or a system of partial differential equation in density and velocity.

(3) Mesoscopic Approach:-

This model describes traffic flow at a medium detail level. Vehicles and driver behaviour are not distinguished nor described individually, but rather in more aggregate terms the behaviour rules are described at an individual level.

1.3 Fundamental Variables of Traffic Flow

1.3.1 Speed:

Speed in a traffic flow is defined as the distance covered per unit time. If time is kept as reference then it is called time mean speed or otherwise it is space mean speed.

Time Mean Speed and Space Mean Speed:

Time mean speed is defined as the average speed of all the vehicles passing a point on a highway over some specified time period. Space mean speed is define as the average speed of all the vehicles occupying a given section of a highway over some specified time period. Time mean speed is a point measurement while space mean speed is a measure relating to length of highway or lane.

1.3.2 Density:

It is defined as the number of vehicles per unit length of road. If l is the length of road and n is the number of vehicles then density is given

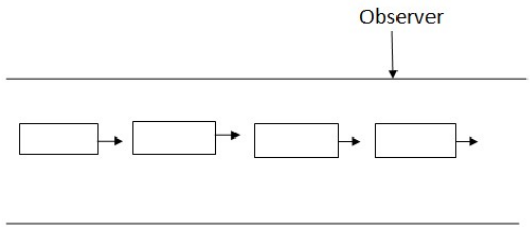
$$\rho = \frac{n}{l}$$

1.3.3 Flow:

It is defined as an average number of vehicles passing per unit time through a reference point. All the fundamental variables discussed above depends upon position of vehicles and time

1.4 Flow Equals Density Times Velocity

We will show that there is a close relationship among these three variables. We first consider one of the simplest possible traffic situations. Suppose that on some road, traffic is moving at a constant velocity u_0 with a constant density ρ_0, as shown in fig.



Constant flow of cars

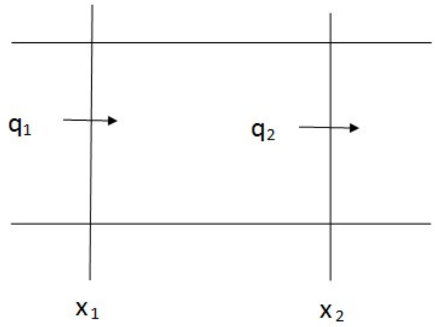
Since each car moves at the same speed, the distance between cars remains constant. Hence the traffic density does not change. What is the flow of cars ?

To answer that, consider an observer measuring the traffic flow (the number of cars per hour that pass the observer). In τ hours each car has moved $u_0\tau$ distance (moving at a constant velocity, the distance travelled equals the velocity multiplied by the time) and thus the number of cars that pass the observer in τ hours is the number of cars in $u_0\tau$ distance,

Since ρ_0 is the number of cars per mile and there are $u_0\tau$ miles, then $\rho_0 u_0 \tau$ is the number of cars passing the observer in τ hours. Thus the number of cars per hour which we have called the traffic flow, q is

$$q = \rho_0 u_0$$

1.5 Conservation of cars



Let x_1 and x_2 be two positions and let N_1 is the number of vehicles passing through x_1 and N_2 is the number of vehicles passing through x_2

During time interval $\Delta t = t_2 - t_1$, N_1 vehicles passed x_1 and N_2 vehicles passed x_2

Now,

$$q_1 = \frac{N_1}{\Delta t} \quad \text{and} \quad q_2 = \frac{N_2}{\Delta t}$$

$$N_1 = q_1 \Delta t \quad \text{and} \quad N_2 = q_2 \Delta t$$

change of vehicles,

$$\Delta N = N_2 - N_1 = (q_2 - q_1)\Delta t \tag{1.1}$$

Let ρ_1 and ρ_2 be the densities, then

$$\rho_1 = \frac{M_1}{\Delta x}$$

Where M_1 is the number of vehicles

similarly,

$$\rho_2 = \frac{M_2}{\Delta x}$$

$$M_1 = \rho_1 \Delta x \quad \text{and} \quad M_2 = \rho_2 \Delta x$$

Now,

$$\Delta M = M_1 - M_2 = \rho_1 \Delta x - \rho_2 \Delta x$$

$$\Delta M = (\rho_1 - \rho_2)\Delta x \qquad (1.2)$$

Since vehicles neither be created nor destroyed inside the section, the change of vehicles should be same in the section during the same time interval.

Therefore, $\Delta N = \Delta M$ that is, $(q_2 - q_1)\Delta t = (\rho_1 - \rho_2)\Delta x$

$$(q_2 - q_1)\Delta t = -(\rho_2 - \rho_1)\Delta x$$

$$\Delta q \Delta t = -\Delta \rho \Delta x$$

$$\Delta q \Delta t + \Delta \rho \Delta x = 0$$

dividing both sides by $\Delta x \Delta t$

$$\frac{\Delta q}{\Delta x} + \frac{\Delta \rho}{\Delta t} = 0$$

Let $\Delta x \to 0$ and $\Delta t \to 0$, the above equation becomes a partial differential equation.

$$\frac{\partial q}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

$$\text{or} \quad q_x + \rho_t = 0$$

$$\text{Where} \quad q_x = \frac{\partial q}{\partial x} \quad \text{and} \quad \rho_t = \frac{\partial \rho}{\partial t}$$

1.6 Velocity Density Relationship

Traffic density and car velocity are related by one equation, conservation of vehicles:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$

If velocity and initial density are known, above equation can be used to predict future density. So we can think of velocity to be function of density. The choice of such function depends upon different models. On the basis of some experiments and observations we assume that at any point along the road the velocity of car only depends upon the density.

$$v = v(\rho)$$

If there are no cars or density of cars on the road is very low, then car can move at maximum speed v_{max}.
Now, if the density of road increases so, due to the presence of other cars vehicle will move slow. As the density keep on increasing velocity will keep on decreasing. So we can say,

$$\frac{dq}{d\rho} = v'(\rho) \leq 0$$

and at a certain density car will not move. This maximum density ρ_{max} represent the bumper to bumper traffic.

$$v(\rho_{max}) = 0$$

So, different scientists purposed different models to represent the relationship between velocity and density. Here are some well known models given
(1) Greenshield linear relationship (1935)

$$v = v_{max}(1 - \frac{\rho}{\rho_{max}})$$

(2) Greenberg relation(1959)

$$v = v_{max} \log(\frac{\rho_{max}}{\rho})$$

(3) Underwood Relation (1961)

$$v = v_{max} \exp \frac{-\rho}{\rho_{max}}$$

v_{max} and ρ_{max} are freeway velocity and jam density respectively.

1.7 Fundamental Diagram

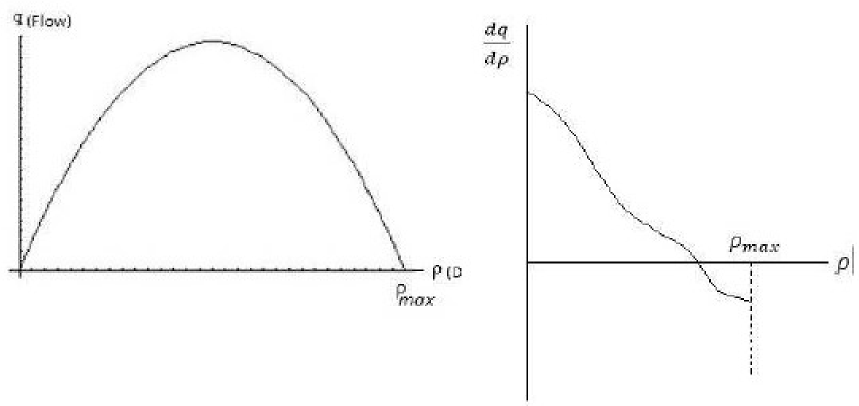
Assume a road is homogeneous such that the car velocity is dependent only on the density of vehicles on road. Since the traffic flow is equal to density time velocity, so the flow only depends upon the density of vehicles along the road.

$$q = \rho v(\rho)$$

This traffic flow has some properties. The traffic flow may be zero in two ways:

(1) If there is no traffic on the road that means $\rho = 0$, and

(2) If the velocity is zero it means no vehicle is moving i.e the case of $\rho = \rho_{max}$ So, for all other values of density between 0 and ρ_{max} traffic flow is positive. The relationship between traffic flow and density is shown in figure:



This flow density relationship is called **Fundamental Diagram of Traffic Flow**

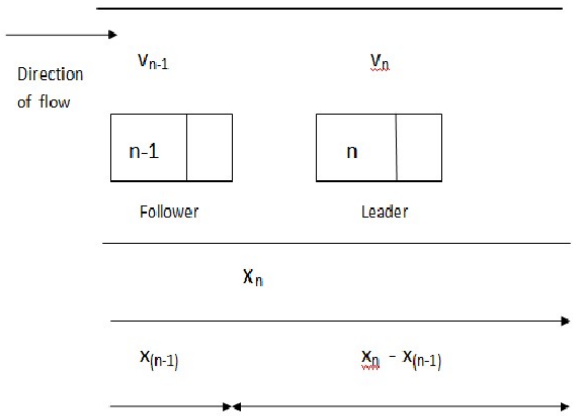
Chapter 2

Car Following Models

Car following models are the most important representatives of microscopic traffic flow models. Car following models describe the drivers behavior only in the presence of interactions with other vehicles. Car following theories describe how one vehicle follows another vehicle in an uninterrupted flow. Various models were formulated to represent how a driver reacts to the changes in the relative positions of the vehicle ahead. Models like Pipes, Forbes , General Motors and optimal velocity model are worth discussing.

2.1 Notation

Before going into the details, various notations used in car-following models are discussed here with the help of figure



The leader vehicle is denoted as n and the following vehicle as (n+1). Two characteristics at an instant t are of importance; location and speed. Location and speed of the lead vehicle at time instant t are represented by x_n^t and v_n^t respectively. Similarly, the location and speed of the follower are denoted by x_{n+1}^t and v_{n+1}^t, respectively. The following vehicle is assumed to accelerate at time $t + \Delta T$ and not at t, where ΔT is the interval of time required for a driver to react to a changing situation. The gap between the leader and the follower vehicle is therefore, $x_n^t - x_{n+1}^t$.

2.1.1 Pipes Model

Pipes (1953) developed a car following model on the assumption that drivers maintain a constant distance headway. His work was followed by forbes (1958) who assumes that drivers maintain a constant time headway

The basic assumption of this model is " A good rule for following another vehicle at a safe distance is to allow yourself at least the length of a car between your vehicle and the vehicle ahead for every ten miles per hour of speed at which you are traveling " According to Pipes car-following model, the minimum safe distance headway increases linearly with speed. A disadvantage of this model is that at low speeds, the minimum headways proposed by the theory are considerably less than the corresponding field measurements.

2.1.2 Forbe's Model

In this model, the reaction time needed for the following vehicle to perceive the need to decelerate and apply the brakes is considered i.e, the time gap between the rear of the leader and the front of the follower should always be equal to or greater than the reaction time. Therefore, the minimum time headway is equal to the reaction time (minimum time gap) and the time required for the lead vehicle to traverse a distance equivalent to its length. A disadvantage of this model is that, similar to Pipes model, there is a wide difference in the minimum distance headway at low and high speed.

2.1.3 General Motor's Model

The General Motors model is the most popular of the car-following theories because of the following reasons:

1. Agreement with field data ; the simulation models developed based on General motors car following models shows good correlation to the field data.

2. Mathematical relation to macroscopic model; Greenbergs logarithmic model for speed density relationship can be derived from General motors car following model.

2.1.4 Optimal Velocity Model

The optimal velocity model is a traffic flow model proposed by Bando, the governing equations are given by:

$$\frac{dx_n}{dt} = v_n \tag{2.1}$$

$$\frac{dv_n}{dt} = a[V(x_{n+1}) - x_n] \tag{2.2}$$

with positions x_n and v_n of cars and a is sensitivity denoting the speed of the next car. This model is having the following features:

(1) A car will keep the maximum speed with enough the distance to the next car.

(2) A car tries to run with optimal velocity determined by the distance to the next car.

Chapter 3

Continuum Modeling of Traffic Flow

In continum model, traffic flow is described in terms of average density, average velocity and average flow.

3.1 LWR Model

The research on traffic was started in 1950. To describe the dynamical properties on a homogeneous and unidirectional highway, ligthhill(1955), Whithiam(1955) and Richards(1956) independently proposed a continum model, which is known as LWR model. This model is based on the assumption that the number of vehicles is conserved between any two points if there are no entrances (sources) or exits (sinks). This model has been used to analyse a number of traffic flow problems. In LWR Model, the relationship between fundamental variables : flow, speed, density is supplemented by continuty equation

$$\partial_t \rho + \partial_x(\rho v) = 0$$

Above equation is not consistent in self but needs an additional relation which is supplemented by the equation of traffic flow

$$q = \rho v$$

and the relationship between the mean velocity and traffic density under steady state uniform flow

$$v = v_e(\rho)$$

Where $v_e(\rho)$ is the equilibrium velocity, x and t represent the space and time respectively. Using LWR modeling approach exibits a wide range of phenomena such as shock formation and rarefaction waves.

3.1.1 Drawbacks:

(1) LWR models could not explain amplifications of small disturbances on heavy traffic.
(2) This model failed to explain the traffic flow instabilities, such as cluster formation in initially homogeneous traffic conditions.
This was only because of the fact that all vehicles travels at $v_e(\rho)$ equilibrium velocity. Therefore non equilibrium models were needed.

3.2 Payne Whithiam Model

The first non equilibrium model was given by Payne whithiam in 1970 known as, PW model

$$\partial_t(\rho) + \partial_x(\rho v)$$

$$\partial_t v + v\partial_x v + \frac{c^2(\rho)\partial_x \rho}{\rho} = \frac{v(\rho) - v}{\tau}$$

Where v_x is the equilibrium velocity, τ is the relaxation time and c is the sound speed .
The relaxation time is the time taken by driver to adjust its velocity due to front stimuli. This model is able to explain the cluster formation.

3.2.1 Drawback:

In this model there exit a characteristic speed which is greater than flow velocity. This means that the future condition of traffic flow will be affected by the traffic condition behind the flow. But in traffic flow vehicles only travels in one direction and respond only to front stimuli. This is known as wrong way travel problem.

3.3 Zhang Model

The wrong way problem was the major flaw in the PW model, that is why new higher model was developed by Zhang in 1998.

$$\partial_t \rho + \partial_x(\rho v) = 0$$

$$\partial_t + v\partial_x v + 2\beta c(\rho)\partial_x v + \frac{c^2(\rho)\partial_x \rho}{\rho} = \frac{V(\rho) - v}{\tau} + \mu(\rho)\partial_{xx} v$$

Where

$$\mu(\rho) = 2\beta\tau c^2(\rho)$$

$$c(\rho) = \rho V(\rho)$$

is the viscosity coefficient and it is dimensionless parameter.

Chapter 4

Lattice Hydrodynamic Model

Lattice :

Lattice is a regular rearrangement of an array of points in one dimentional plane. Consider a road of lenght L and x_i represents the position of vehicle at point i . Let dx is the distance of 2 succesive positions. Then x° “ $\frac{x_i}{dx_j}$ which means space becomes dimensionless.

Hydrodynamic :

It is the study of liquid in motion. It looks at the ways different forces affect the movement of liquids. A series of equations explain how the conservation law of mass, energy and momentum apply to liquids. Hydrodynamic modeling consists in deriving evolution equations for the macroscopic quantities, mass,

In 1998, Nagatani introduced a lattice hydrodynamic model which is the simplified version of continuum model and also incorporates some ideas of car following model. A lattice model is proposed by considering drivers characteristics. In this model the effects of drivers characteristics is examined by linear stability analysis. Here jamming transitions are analysed through mkdv equations. Lattice hydrodynamic modeling is very helpful to describe traffic jam in term of kink antikink density waves. The main motivation behind the lattice hydrodynamic approach is to check analytically that kink-antikink types of density waves exit in the traffic flow or not.

4.1 Need for Lattice Hydrodynamic Modeling

Traffic flow is divided into three distinct regions: the stable region out- side the coexisting curve, the metastable region between the coexisting and neutral stability lines, and the unstable region enclosed by the co- existing curve. Different nonlinear wave equations have been derived to describe the corresponding density waves, among which the Burgers,Korteweg-de-Vries (KdV) and Modified Korteweg-de-Vries (MKdV) equations depict the density waves appearing in the three distinct regions, respectively. In 1993 Kerner and Konhauser have observed a single-pulse density wave in numerical simulation of hydrodynamic model. The nonlinear stability analysis was not able to analyze the density wave of traffic congestion. Lately Kurtze and Hong have derived KdV equation from nonlinear stability analysis and found that soliton is the solution of KdV equation, but the single-pulse density wave as observed in was not consistent with the soliton, but it was asymmetric kink antikink soliton. Komatsu and Sasa derived MKdV equation through nonlinear analysis and density wave obtained from the solution of MKdV equation is type of symmetric kink-antikink.

Due to the complexity of the partial differential equation of hydrodynamic model, it was very difficult to derive modified KdV equation through nonlinear analysis. Also, it was not easy to get the evolution equation from the analytical method, which can describe the traffic jam phase transitions.

To overcome the complexity of macroscopic model and explain the nonlinear traffic waves at hydrodynamic level, Nagatani in 1998, developed a new model called lattice hydrodynamic model.

4.2 Literature Survey

(1) Nagatani in 1998 firstly introduced the lattice hydrodynamic model for traffic flow which is the simplified version of continuum model and also follows car following model. Two simplified versions of continuum models are presented. One is the continuum model described in terms of partial differential equations and the other is the lattice version given by difference

equations. The MKdV equation is derived near the critical points from the continuum models. Coexisting curves are derived from the MKdV equation. It was shown that the MKdV equation obtained by the lattice model agrees with that of the optimal velocity model.
(2) Nagatni proposed the lattice models of traffic to describe the jamming transition in traffic flow on a highway in terms of thermodynamic terminology of phase transitions and critical phenomena. MKdV and TDGL equations are derived and the connection between them has been shown. The basic idea of this model is that the jamming transitions can be formulated exactly in terms of terminology of the phase transition and critical phenomena.
(3) In this series of lattice hydrodynamic modeling the backward looking effect is considered by Hong and Cheng because other models deals only with forward looking effect. A new optimal velocity function has been derived to check the backward looking effect. The parameter r in optimal velocity function corresponds to the relative role of the backward looking optimal velocity function. The stability region increases corresponding to the increasing value of r. Due to the backward looking effect, large cluster disappears gradually, and the KdV soliton density waves occur, which could be derived from the KdV equation.
(4) An extended lattice model is presented by Tian considering the optimal current difference effect. This model is an extension of Lis model which not only studies the optimal and local current but the relative current as well. It has been shown that the stability of traffic flow is enhanced by introducing the new consideration.
Non lane based car following model incorporating the lateral effect of lane width was presented by Jin and then Peng proposed first non lane based lattice model by considering lateral effect of lane width. As in real traffic flow vehicles not always moves in the centre of road which is an assumption of car following models. The off-central-line effect results in lateral separations between the leaders and followers. The region of stability increases with the effect of lateral separation.
(5) As in real traffic flow the road is not always single lane, so to incorporate the real traffic phenomena Nagatani introduce first lattice model for two lane traffic flow. If the density at site j-1 on the first lane is higher than that at site j on the second lane, the lane changing oc-

curs from the first lane to the second lane and will be proportional to their density difference $\gamma|\rho_0^2 V'(\rho_0)|(\rho_{1,j-1}(t) - \rho_{2,j}(t))$ and If the density at site j on the second lane is higher than that at site j+1 on the first lane, the lane changing occurs from the second lane to the first lane and will be proportional to their density difference $\gamma|\rho_0^2 V'(\rho_0)|(\rho_{2,j}(t) - \rho_{2,j+1}(t))$. The results shows that with increasing rate of the lane changing the critical point, the coexisting curve and the neutral stability line decreases.

(6) In this process of lattice hydrodynamic approach one more important aspect of density difference effect on two lane road is considered by Gupta and Redhu . As the drivers adjust their velocity according to the observed headway and the density at any site depends on the forward and backward lattice. The results shows that the density difference between the leading and following lattices stabilizes the flow. Larger the values of the reaction coefficient larger the stability region.

(7) In real traffic flow, driver adjusts his run velocity according to the observed headway and estimate his individual running behavior. So, by extending Pengs model , Gupta consider DAESRF on a two lane system and shown that the anticipation coefficient corresponds to drivers behavior in sensing relative flux increases significantly the stability of traffic flow in two-lane system and negative value of anticipation coefficient, which corresponds to drivers delay response in sensing relative flux effect, reduces the stable region and increases congestion.

(8) One and most important aspect in traffic flow is passing which was firstly introduced by Nagatani . This approach is further extended by Gupta and Redhu by considering drivers anticipation effect with passing. Corresponding to larger values of anticipation parameter the stable region increases but due to passing coefficient stable region reduces. There exist two regions kink jam and no jam on phase plane for smaller values of passing while another is chaotic jam exists for larger rate of passing.

(9) In real traffic situations drivers play an important role and in this direction many models comes into existence as an extension of Nagatanis model.But effect of drivers characteristics has been incorporated by Sharma on two lane system. In traffic situations driver observe the traffic situation and make the decision to adjust the velocity at some delay time. In the study

it has been shown that the aggressive drivers help in reducing traffic jams on two lane system for any value of anticipation coefficient. While timid drivers having negative impact on traffic situations.

(10) To make the traffic models more realistic lattice approach has been extended towards curved road by for single lane. In this model the effect of angle, friction coefficient and radius of curvature on stability of traffic flow has been investigated. Enlarging the angle going into curved road reduces the jams and reducing the radius of curvature leads to stabilisation of flow. Increasing friction coefficient make the traffic situations worse. In this series Zhou introduce same model for two lane system and conclude that lane changing helps in reducing traffic jams on curved road.

4.3 A New Lattice Hydrodynamic Model for Circular Road

The very first lattice hydrodynamic model was presented by Nagatani comprises the idea of car following model and as well as macroscopic model to analyze the density wave of traffic flow on a unidirectional simple road and is given by

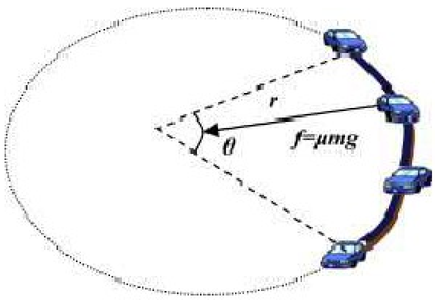
$$\partial_t \rho + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0 \tag{4.1}$$

$$\partial_t(\rho_j v_j) = a[\rho_0 V(\rho_{j+1}) - \rho_j v_j] \tag{4.2}$$

where j represent the $j\,th$ site on one dimensional lattice. ρ_j and v_j represent the local density and velocity respectively at the $j\,th$ site at time t. ρ_0 is the average density ; a is the sensitivity of driver.

4.4 Models

The traffic flow on a curved road is different from unidirectional non curved road. So, we consider a situation in which vehicles run on a single lane curved road under closed boundary conditions, as shown in the Fig.



Where the force $f = \mu mg$ acting on the running vehicle is called central force. Where m is the mass of vehicle, Where is the friction coefficient and g is the acceleration due to gravity. The total length of the curved section is $= r\theta$. Where r is the radius of curved road section and Where θ is angular distance. Based on the lattice optimal current difference model, we propose a new mathematical model to analyze the traffic flow on a curved section road as follows:

$$\partial_t \rho_j + \frac{1}{\eta r}(\rho_j \omega_j - \rho_{j-1}\omega_{j-1}) = 0 \tag{4.3}$$

$$\partial_t(\rho_j \omega_j) = a[\rho_0 v(\rho_{j+1}(t+\alpha\tau)) - \rho_j \omega_j] \tag{4.4}$$

Where η is average angular headway, and ω_j is the angular velocity at site j. Now differentiate (4.3) w.r.t to t

$$\partial_t^2 \rho_j + \frac{1}{\eta r}[\partial_t(\rho_j \omega_j) - \partial_t(\rho_{j-1}\omega_{j-1})] = 0 \tag{4.5}$$

Now replace j by j-1, in (4.4), we get

$$\partial_t(\rho_{j-1}\omega_{j-1}) = a[\rho_0 v(\rho_j(t+\alpha\tau)) - \rho_{j-1}\omega_{j-1}] \tag{4.6}$$

Put (4.4) and (4.6) in equation (4.5)

$$\partial_t^2(\rho_j) + \frac{1}{\eta r}[a\rho_0 V(\rho_{j+1}(t+\alpha\tau)) - a\rho_j\omega_j - a\rho_0 V(\rho_j(t+\alpha\tau)) + a\rho_{j-1}\omega_{j-1}] = 0 \tag{4.7}$$

Now we can write

$$\partial_t^2(\rho_j) = \frac{\rho_j(t+2\tau) - 2\rho_j(t+\tau) + \rho_j(t)}{\tau^2} \tag{4.8}$$

Put this value in above equation, we get

$$\rho_j(t+2\tau) - \rho_j(t+\tau) + \tau\rho_0^2[V(\rho_{j+1}(t)) - V(\rho_j(t))] + \alpha\tau\rho_0^2[\tilde{\Delta}(\rho_{j+1}(t))V'(\rho_{j+1}(t)) - \tilde{\Delta}(\rho_j(t))V'(\rho_j(t))] = 0 \tag{4.9}$$

which is the required density equation.

4.5 Linear stability

Now we discuss the Linear stability of our new traffic model to investigate the effect of friction coefficient and radius on traffic flow of circular road.

Assume traffic flow is uniform on the curved road, under these conditions density and velocity are taken as ρ_0 and $V(\rho_0)$. Hence, the steady-state solution of the homogeneous uniform traffic flow is given

$$\rho = \rho_0 \tag{4.10}$$

$$v = V(\rho_0) \tag{4.11}$$

Now we give small perturbation $\rho_j(t)$ to the steady state on site j. Then

$$\rho_j(t) = \rho_0 + y_j(t) \tag{4.12}$$

Now we put this perturbed density profile into (4.9) and we get,

$$y_j(t+2\tau) - y_j(t+\tau) + \tau\rho_0^2[y_{j+1}(t) - y_j(t)] + \alpha\tau\rho_0^2 V'(\rho_0)\Delta[y_{j+1}(t) - y_j(t)] = 0 \tag{4.13}$$

Now put

$$y_j(t) = \exp(ikj + zt) \tag{4.14}$$

Put this value in above equation, we get

$$[e^{2z\tau} - e^{z\tau}] + \tau\rho_0^2 V'(\rho_0)[e^{ik} - 1] + \alpha\tau\rho_0 V'(\rho_0)[e^{z\tau} - 1][e^{ik} - 1] = 0 \tag{4.15}$$

Now put

$$z = z_1(ik) + z_2(ik)^2 \tag{4.16}$$

in above equation, we get

$$[z_1(ik)\tau + z_2(ik)^2\tau + \frac{3}{2}z_1^2(ik)^2\tau^2] + \tau\rho_0^2 V'(\rho_0)[(ik) + \frac{(ik)^2}{2}] + \alpha\tau\rho_0^2 V'(\rho_0)z_1(ik)^2\tau = 0 \tag{4.17}$$

Now compare the cofficient of ik,we get

$$z_1\tau + \tau\rho_0^2 V'(\rho_0) = 0 \tag{4.18}$$

$$z_1 = -\rho_0^2 V'(\rho_0) \tag{4.19}$$

compare the cofficient of $(ik)^2$, we get

$$z_2 = -\frac{3}{2}[-\rho_0^2 V'(\rho_0)]^2\tau - \frac{\rho_0^2}{2}V'(\rho_0) - \alpha\rho_0^2 V'(\rho_0)[-\rho_0^2 V'(\rho_0)]\tau \tag{4.20}$$

If z_2 is a negative value, the uniform steady-state ow becomes unstable for long wavelengths. When z_2 is a positive value, the uniform flow is stable.

Put $z_2 = 0$, we get

$$-\frac{3}{2}[\rho_0^4 V'^2(\rho_0)] + \alpha\rho^4 V'^2(\rho_0)]\tau = \frac{\rho_0^2}{2}V'(\rho_0) \tag{4.21}$$

On solving this we get,

$$\tau = -\frac{1}{(3 - 2\alpha)\rho_0^2 V'(\rho_0)} \tag{4.22}$$

The neutral stability conditions is given by:

$$\tau = -\frac{1}{(3 - 2\alpha)\rho_0^2 V'(\rho_0)} \tag{4.23}$$

The stability condition for homogeneous uniform traffic flow is

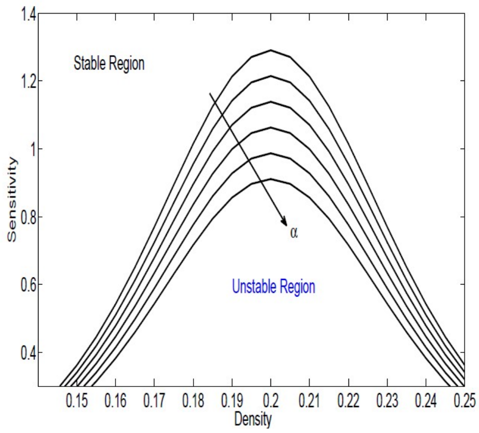


Figure 4.1: Phase Diagram with $\rho_0 = \rho_c = 0.2$

$$\tau > -\frac{1}{(3-2\alpha)\rho_0^2 V'(\rho_0)} \tag{4.24}$$

The instability condition for homogeneous uniform traffic flow is

$$\tau < -\frac{1}{(3-2\alpha)\rho_0^2 V'(\rho_0)} \tag{4.25}$$

4.5.1 Numerical Simulation

In order to check impact of circular road on traffic flow as well as the validity of theoretical results, numerical simulation is carried out under periodic boundary conditions

$$\rho_j(1) = \rho_j(0) = \begin{cases} \rho_0; & j \neq \frac{N}{2}, \frac{N}{2}+1, \\ \rho_0 - \sigma, & j = \frac{N}{2} \\ \rho_0 + \sigma, & j = \frac{N}{2} \end{cases}$$

Where σ is initial disturbance and N is total no of sites

shows the spatiotemporal evolution of density waves at time $t = 9.9 \times 10^4 s$ for $\alpha = 0.1, 0.3, 0.5$ while other parameters remain fixed $(\mu = 0.5, R = 20)$. It is clear from the figs 4.2$(a)-(c)$ that due to increasing value of anticipation parameter α the density waves dies out and the flow is stable for $\alpha = 0.5$. It can be explained by the fact that the motion of vehicles on circular road is different than the straight road and more the radius of circular road more the congestion will be. In this situation the driver anticipation helps in reducing the congestion. Thus the higher anticipation coefficient makes the flow stable.

4.3 represents the density profiles for different α and other parameters remain fixed. It is clear from the figure that the amplitude of density waves is higher for lower value of α and corresponding increase in α leads to stability of flow. The reason behind it is that while moving on circular road skillful drivers observes the traffic situation quickly and avoids the jamming situations.

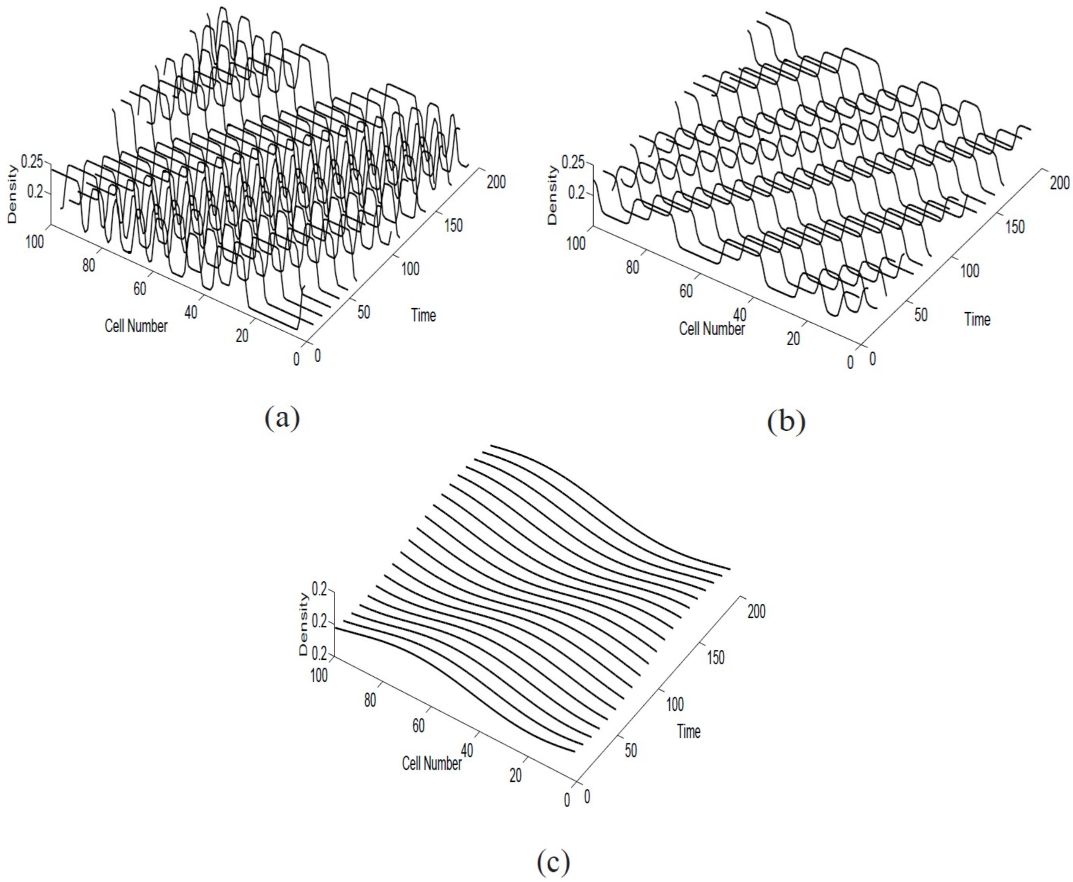


Figure 4.2: Spatiotemporal Evolution of density waves at time $t = 99000 - 100000$ for $R = 10, \mu = 0.3$ for $\alpha = 0.1, 0.3, 0.5$

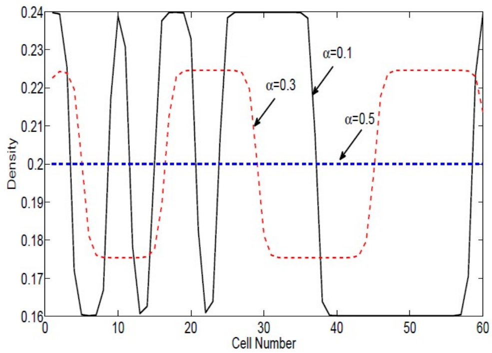


Figure 4.3: Density Profiles for $\rho_0 = \rho_c = 0.2, R = 20, \mu = 0.5$ when $\alpha = 0.1, 0.3, 0.5$

4.6 Conclusion

We can conclude that state of traffic flow on circular road is more complex than on normal road. The complexity of flow can be minimised if more skillfull drivers are present on road as we have shown the results that higher values of anticipation parameter makes the flow stable. Because skillfull drivers observes the traffic conditions and take the decision within less time and move fast. This results in stability of flow. Thus our assumption of anticipation effect for circular road on traffic flow is reasonable and the simulation results are in good agreement with analytical results.

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