

# **A Study on Rayleigh Wave Propagation in Elastic continua**

*Thesis submitted in partial fulfillment of the requirements for the award of the  
degree of*

**Master of Science**

in

**Mathematics and Computing**

submitted by

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**July 2017**

# Declaration

I hereby certify that the work, which is being presented in the thesis, titled **A Study on Rayleigh Wave Propagation in Elastic continua**, in partial fulfillment of the requirements for the award of the degree of **Master of Science** in Mathematics and Computing, and submitted to Thapar University is an authentic record of my own work carried out under the supervision of **Dr. Satish Kumar**. I have also cited the reference about the text(s)/figure(s) from where they have been taken.

The matter presented in this thesis has not been submitted elsewhere for the award of any other degree or diploma from any institution.



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**Dolcy**

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.



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# Abstract

The thesis aims to study the propagation of Rayleigh waves in a homogeneous isotropic elastic half-space under free conditions and underlying a liquid half-space of inviscid liquid. Secular equations are obtained for the propagation of Rayleigh waves. Variation of phase velocity with the wave number has been depicted by graphs.

There are mainly two types of waves generated in a solid, Body waves and Surface waves. Body waves include P-waves and S-waves and Surface waves include Love waves and Rayleigh waves. Surface waves are generated by the constructive interference of P and S waves. These waves usually propagate in the half-space. They have wide range applications in seismology and non-destructive evaluation.

Chapter-1 deals with the brief historical introduction of the theory of elasticity. Continuum mechanics is the branch of the mechanics that deals with the kinematic behavior of continuous matter. Summary of Hooke's law has been discussed. Types of elastic waves with their subtypes have been discussed in this chapter. The basic governing equations of elasticity have been presented and expressions for speeds of primary and secondary waves have been derived using Helmholtz decomposition .

Chapter-2 deals with the Rayleigh waves in the homogeneous isotropic elastic half-space. In this chapter, formulation of the problem is done and secular equation is obtained from the basic governing equation of elasticity in the absence of body forces. The value of Rayleigh phase velocity is obtained.

Chapter-3 deals with the study of propagation of rayleigh waves in an elastic half-space underlying a half-space liquid. In this chapter, effect of fluid loading on phase velocity of Rayleigh waves has been studied. The secular equation is derived analytically. Effect of liquid loading is shown graphically.

In the present work I have reviewed the following work in the field of wave propagation in elastic solid.

1. (Chapter-5 Waves in infinite media) of the textbook

Graff, K. F.(1991): Wave motion in elastic solids.Dover Publications, New York.

2. Particular case of Resarch Article:

Sharma, J. N., Satish Kumar, and Y. D. Sharma. "Propagation of Rayleigh surface waves in microstretch thermoelastic continua under inviscid fluid loadings." *Journal of Thermal Stresses* 31 (2008): 18-39.

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At last, I would like to thank my parents, my friends and my senior scholar Richa Goyal for always being there for me and staying calm at times required.

**Dolcy**



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# Chapter 1

## Introduction

### 1.1 Theory of elasticity

Solid mechanics is the branch of continuum mechanics that studies about the behavior of solid materials, especially their motion and their changes under the action of forces, temperature changes, phase changes, and other external or internal agents. Solid bodies are basically divided into two categories: plastic body and an elastic body. Elastic bodies are the bodies that return to its normal shape after the removal of external forces acting on it. This elastic behavior is exhibited by bodies over small loads only. Thus the theory of elasticity is the mathematical analysis of the elastic behavior of a solid body. Plastic bodies are the bodies that do not return to its original shape and size upon the removal of the external forces.

Before moving on to the modern conceptions it is necessary to study the research of seventeenth and eighteenth centuries. In seventeenth and eighteenth century experimental knowledge of the behavior of strained bodies was studied and their related principles were formulated[1].

Robert Hooke (1678) gave the relation between the applied forces and deformation occurred in the body. This law assumes linear relationship between stress and strain. Thus Hookes law of elasticity basically states that extension of the body is proportional to the load applied to it. As solids obey Hookes law up to a certain limit, that limit is known as an elastic limit. It is the maximum change that a body will undergo without being permanently changed. When elastic limit is exceeded it becomes a plastic body. Glass is a brittle material that usually breaks near its elastic limit. Many other scholars also published similar discoveries as given by Hookes. Swiss mathematician and mechanics, Euler(1703-1783) proposed that a linear relation between stress and strain is of the form  $\sigma = E\epsilon$ , where E is the coefficient of elasticity generally named after the scientist Thomas Young. In 1821 Louis Navier gave three-dimensional equations of elasticity. According to him elastic reactions arise from the changes in the force which are acting between the molecules, which are taking place due to changes in the molecular arrangement.

Navier assumed the isotropic material to be characterized by two elastic constants. Poisson[2] used his theory to carry out the research on the propagation of waves through an isotropic elastic solid. He showed that the motion sent out by the quicker wave was longitudinal. While the motion sends out by the slower wave was transverse. Further, Cauchy a mathematician and engineer also studied the theory of elasticity. Cauchy studied elasticity in a different manner by considering the notion of pressure on the plane. He gave the concept of stress in the form of three-dimensional theory. Cauchy had discovered most of the elements of the pure theory of elasticity. According to him, stress at a point is completely resolved if the deformations per unit area across all plane elements through point were known. Cauchy formalized the stress concept in three dimensional theory. Cauchy derived the equations for motion in the terms of components of stress and discussed for the elastic solids. Besides this, equation of stress with six components were also introduced (three longitudinal and three shears) in the terms of derivatives of displacements.

## 1.2 Generalised Hooke's law

Hooke's law states that the force needed to extend spring by some distance  $x$  is directly proportional to that distance[3]. Stress is that quantity that is directly proportional to the force causing a change (deformation) and strain is the measure of a degree of change. Further stress has two components; normal and shear stress. Normal stress is defined as the deforming force acting on per unit area normal to the surface of the body while tangential stress is the deforming force acting per unit area tangential to the surface of the body. In three-dimensional axes, stress is resolved into three parts. One normal stress and a shearing stress which can be further resolved into two components parallel to the direction of coordinates. Thus nine stress components are used to describe the stress state of any cubic body. These are  $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}, \sigma_{yx}, \sigma_{yy}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ , where the first suffix indicates the direction of the normal to the plane under consideration while the second suffix refers to the the component of stress vector. In the similar way, a strain is also resolved into two components longitudinal and shear strain. The longitudinal strain is the change in the length per unit original length while shear strain is the change in the shape of the body. The nine strain components are defined as  $\epsilon_{xx}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}, \epsilon_{yx}, \epsilon_{yy}, \epsilon_{zx}, \epsilon_{zy}, \epsilon_{zz}$ . Modified form of Hookes law was introduced which states that stress components are the functions of strain components .This is the generalized form of Hookes law.

In its most generalized form Hooke's law is defined by

$$\tau_{ij} = c_{ijkl}e_{kl}$$

;  $c_{ijkl}$  is the fourth order tensor giving different coefficients of elasticity, where  $\tau_{ij}$  is stress tensor,  $e_{kl}$  is strain tensor defined by

$$e_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

A medium is said to be elastically homogenous if elastic properties are same at all points of medium. In general case  $c_{ijkl}$  will have 81 constants but if body is assumed to be isotropic i.e. its elastic properties are independent of orientation of axes of coordinates, then number of elastic constants reduces to two. Further, if medium is both isotropic and homogenous then these elastic coefficients are constants. The two constants in the isotropic case are called Lamé constants  $\lambda$  and  $\mu$  and Hooke's law takes the form

$$\sigma_{ij} = \lambda \delta_{ij} u_{r,r} + \mu (u_{i,j} + u_{j,i})$$

where  $(i,j,r=1,2,3)$   $\delta_{ij}$  is Kronecker delta and  $\sigma_{ij}$  is the stress tensor.

## 1.3 Elastic waves

A wave is an oscillation along with a transfer of energy from one point to another. Mainly there are two types of waves; mechanical and electromagnetic waves. Elastic waves are the mechanical waves that need medium for its propagation. The small disturbances in the elastic waves differ from the other ordered motion of the medium particle.

### 1.3.1 Types of Elastic Waves

1. Body Waves
2. Surface Waves

### 1.3.1.1 Body Waves

Body waves are the waves that can travel deep into the medium along which paths are controlled by the material properties in term of density. These properties vary according to temperature and material phase. There are two types of body waves.

1. Longitudinal waves
2. Transverse waves

**Longitudnal waves(primary waves)** : The waves in which the displacement of the medium is in the direction of wave propagation are the longitudinal waves. The particles are oscillating back and forth about the equilibrium position. It travels in the form of compressions and rarefactions as the two components in which they are traveling. The P-waves are also known as compressional waves. P-wave is a type of a body wave which is the fastest kind of the seismic wave. These waves can move through the solids as well as through the liquid layers of water. Vibrations in the solid are along or parallel to the direction of wave energy. Examples of primary waves are sound waves.

**Tranverse waves(secondary waves)** : The transverse waves are the waves in which particle vibrate perpendicular to the direction of wave propagation. The particles in the transverse wave are not moving along with the wave but are oscillating up and down about their individual positions. The transverse wave which is also known as secondary wave is slower than the primary wave. As the secondary waves move as the shear waves, so they can move the rock particles up and down or side to side perpendicular in the direction in which wave is traveling. Examples of sound waves are Light waves, waves in a guitar string.

#### *Differnce between longitudinal and transverse waves*

In Longitudinal waves, the particles are moving left and right which results in the oscillation of the other particles also. While the transverse waves are the waves in which the motion of the medium is at the right angle.

The waves in which boundary interactions are not possible are basically involved in infinite media[4]. The governing equation in the absence of body forces

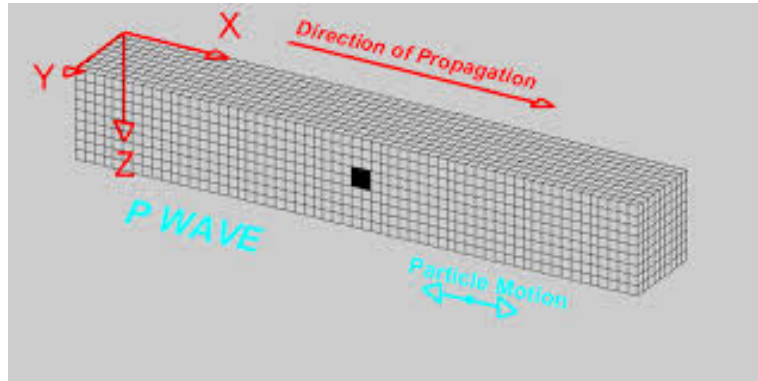


Figure 1.1: Primary waves

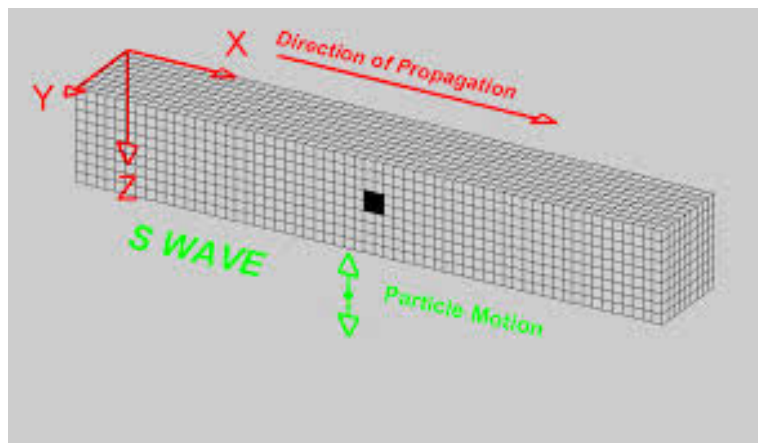


Figure 1.2: Secondary waves

for an elastic solids

$$(\lambda + \mu)\nabla(\nabla.\vec{u}) + \mu\nabla^2\vec{u} = \rho\ddot{\vec{u}} \quad (1.1)$$

Here  $\vec{u}$  is displacement vector and  $\vec{u} = (u, v, w)$ ,  $\lambda$  and  $\mu$  are Lamé's constant and  $\rho$  is density. In two-dimensional problems we take

$$\vec{u} = (u, 0, w)$$

and

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial z}$$

According to Helmholtz theorem, any vector field is uniquely separable into a divergence-free part (solenoid part) and a curl free part (irrotational part) as

$$\vec{u} = \nabla\phi + \nabla \times \psi \quad (1.2)$$

such that

$$\nabla.\psi = 0$$

Here we take

$$\vec{\psi} = (0, \psi, 0)$$

$$\vec{u} = (u, 0, w)$$

$$\frac{\partial}{\partial y} = 0$$

If the vector operation of divergence is applied then using eq.(1.2) in eq.(1.1) we get

$$(\lambda + 2\mu)\nabla^2\Phi = \rho\ddot{\Phi}$$

,

$$\mu\nabla^2\Psi = \rho\ddot{\Psi}$$

In general wave equation is as

$$\nabla^2\phi = \frac{1}{c_1^2}\ddot{\phi}$$

$$\nabla^2\Psi = \frac{1}{c_2^2}\ddot{\Psi}$$

Here,

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$c_2 = \sqrt{\frac{\mu}{\rho}}$$

Here  $c_1$  is the velocity of the longitudinal waves and  $c_2$  is the velocity of shear waves in the elastic half-space.

### 1.3.1.2 Surface waves

It is a mechanical wave that propagates along the interconnection between differing media. They propagate on the surface along a circular wave front[1][5]. In seismology, different types of surface waves are encountered. On seismograms these waves form the principal phase. These waves usually have larger amplitudes and longer periods. They display a characteristic dispersion and polarization at the surface of water and air. These waves are generated by shallow earthquakes. There are two types of surface elastic waves

1. Rayleigh surface waves
2. Love Waves

**Rayleigh surface waves** : In 1887 Lord Rayleigh showed that there exist a wave that propagates along the surfaces and the motion associated with it decayed exponentially. These types of waves are known as Rayleigh Waves. Rayleigh waves have both longitudinal as well as transverse motion. In solids, these waves cause the surface particles to move in ellipses in planes normal to the surface and parallel to the direction of propagation. In seismology, they are known as ground roll. Rayleigh waves do not show dispersion in ideal, elastic, homogeneous solids.

**Love waves** : In these waves, particle motion is transverse and parallel to the surface. In opposition to Rayleigh waves, Love waves cannot propagate in homogeneous half-space. Propagation of Love waves takes place if the S-wave velocity increases with the distance from the surface of the medium. Love waves can propagate in a homogeneous isotropic layer on a homogeneous isotropic half-space. The velocity is dependent on frequency.

## 1.4 Applications of Rayleigh surface waves

1. Seismology

## Rayleigh Wave

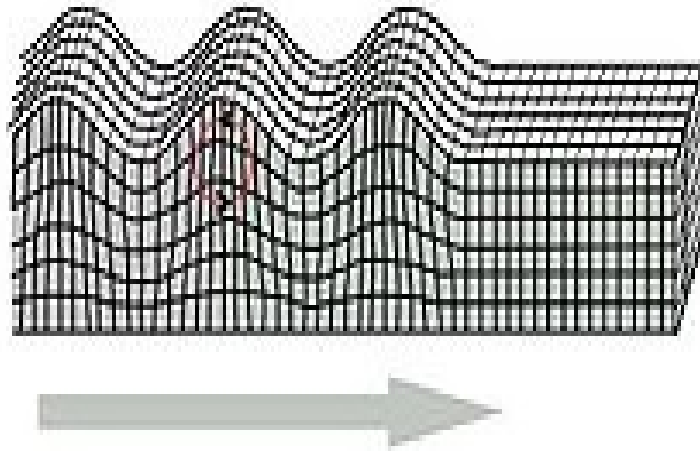


Figure 1.3: Rayleigh waves

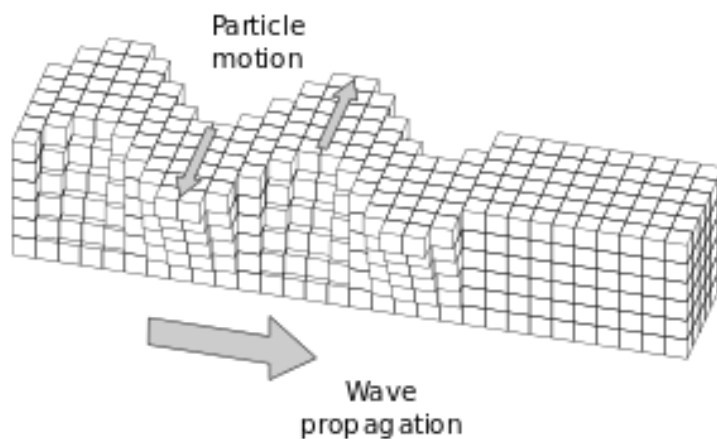


Figure 1.4: Love waves

## 2. Non-destructive evaluation

**Seismology** : The interest of seismologists on Rayleigh waves developed due to the fact that Rayleigh waves were investigated with noise on seismic records[6][1] . To enhance the signal strength noise has to be removed. For multilayered media, Rayleigh waves make it possible to determine seismic velocities. Several results have been obtained in soils on large and small scales.

**Non-destructive evaluation** : In concrete structure, non-destructive techniques use different physical phenomena like ultrasonic, electronic methods etc[7].

Rayleigh waves satisfy many non-destructive testing requirements. The basic features of Rayleigh waves is that whole energy propagates near the free surface, parallel to it. As the surface waves propagate radially on the surface, the contraction of this wave by geometrical spreading is shorter than that of body waves with a spherical wavefront. Due to these features, Rayleigh waves are easy to excite and to record. The detection and characterisation of surface cracks with Rayleigh waves were developed for metallic materials with ultrasonic techniques.

# Chapter 2

## PROPAGATION OF RAYLEIGH WAVES IN A HOMOGENOUS ISOTROPIC ELAS- TIC HALF-SPACE

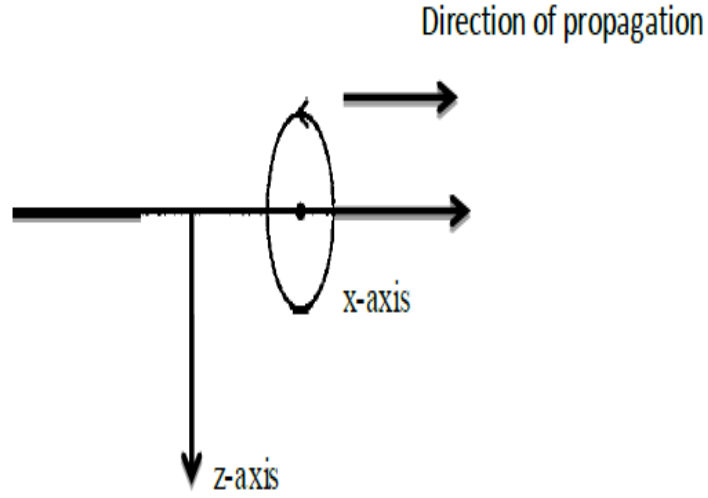
### 2.1 Introduction

Rayleigh waves are known as surface seismic waves which are responsible for destruction and loss of human lives in earthquakes. They are an important part of seismology as they cause a vertical shifting of the earth during an earthquake. Rayleigh[8] had discussed the propagation of surface waves in an isotropic elastic solid half-space and named these waves as Rayleigh wave. Besides Rayleigh, many other scientists have also discussed, the propagation of Rayleigh waves. The ratio of the horizontal and vertical component of these waves which was approximate to their observed value was found by a A.E.H Love[3]. Sengupta[9] studied the propagation of wave in an elastic medium and discussed the effect of gravity on the Rayleigh waves. In this section we have studied about the Rayleigh wave with free boundary conditions in a homogeneous isotropic elastic half-space. Secular equations giving the information about the Rayleigh wave in elastic half-space has been derived analytically.

### 2.2 Formulation of the problem

When an Elastic wave encounters a body between two media energy is reflected as well as transmitted from and across the boundary[10]. Consider the wave propagating in a homogenous isotropic elastic solid half-space. Assume a Cartesian coordinate system where the x-plane is along the surface of the medium and z-axis is pointing vertically downwards in positive direction. Due to shear, the transverse particle motion has components in vertical as well as parallel to the horizontal plane. Origin of the cartesian coordinate system (x,y,z) is taken at the top of the solid half-space.  $\vec{u} = (u, v, w)$  represents the displacement components.

As the wave motion is invariant w.r.t z-axis. These field quantities are independent of y coordinate. The surface of half-space is free of stresses.



The basic governing equation of elastic stress in the absence of body forces is

$$(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2\vec{u} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \quad (2.1)$$

The constitutive relation is given as

$$\tau_{ij} = \lambda u_{r,r}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \quad (2.2)$$

where

$$\vec{u} = (u, v, w)$$

is the displacement vector.  $\lambda$  and  $\mu$  are Lamé's constant of classical elasticity,  $\rho$  is the density

Here we assume that

$$\vec{u} = (u, 0, w)$$

By using Helmholtz decomposition, we have

$$\vec{u} = \nabla\phi + \nabla \times \vec{\psi} \quad (2.3)$$

where

$$\nabla\vec{\psi} = 0$$

Thus we get displacement components as:-

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} \quad (2.4)$$

and

$$w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \quad (2.5)$$

where  $\phi$  is scalar potential function representing longitudinal waves and  $\psi$  is vector potential function representing shear waves in an solid half-space.

On substituting these values of the eq.(2.4) (2.5) in eq.(2.1), we have

we get

$$\nabla^2\phi = \frac{1}{c_1^2} \frac{\partial^2\phi}{\partial t^2} \quad (2.6)$$

$$\nabla^2\psi = \frac{1}{c_2^2} \frac{\partial^2\psi}{\partial t^2} \quad (2.7)$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}$$

,

$$c_2^2 = \frac{\mu}{\rho}$$

are longitudinal and shear velocities.

using eq.(2.4) and eq.(2.5) the stress - displacement equations in the form of the potential functions are

$$\tau_{xx} = \lambda\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\psi}{\partial z\partial x}\right) \quad (2.8)$$

$$\tau_{yy} = \lambda\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) \quad (2.9)$$

$$\tau_{zz} = \lambda\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial z\partial x}\right) \quad (2.10)$$

$$\tau_{xz} = \mu \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right) \quad (2.11)$$

## 2.3 Boundary conditions

As we have assumed that the surface of solids is free of stresses ,so the boundary conditions are ;

at  $z = 0$

$$\tau_{zz} = 0$$

$$\tau_{xz} = 0$$

## 2.4 Formal solution of the problem

We assume the solution of above problem as follows

$$\phi = \phi(z)e^{i\xi(x-ct)} \quad (2.12)$$

$$\psi = \psi(z)e^{i\xi(x-ct)} \quad (2.13)$$

where  $c$  is non-dimensional phase velocity

$$c = \frac{\omega}{\xi}$$

$\xi$  is the wave number

$\omega$  is the circular frequency.

Now using the above solution of (2.12)in the (2.6)

$$\frac{d^2 \phi(z)}{dz^2} - \xi^2 \left( 1 - \frac{c^2}{c_1^2} \right) \phi(z) = 0$$

$$\frac{d^2 \phi(z)}{dz^2} - \alpha^2 \phi(z) = 0$$

where

$$\alpha^2 = \xi^2 \left( 1 - \frac{c^2}{c_1^2} \right)$$

Therefore,

$$\phi(z) = A_1 e^{\alpha z} + A_2 e^{-\alpha z}$$

where  $A_1$  and  $A_2$  are arbitrary constants

$$\phi = (A_1 e^{\alpha z} + A_2 e^{-\alpha z}) e^{i\xi(x-ct)} \quad (2.14)$$

Similarly using (2.7) in (2.13) we obtain

$$\frac{d^2 \psi(z)}{dz^2} - \xi^2 \left(1 - \frac{c^2}{c_1^2}\right) \psi(z) = 0$$

$$\frac{d^2 \psi(z)}{dz^2} - \beta^2 \psi(z) = 0$$

where

$$\beta^2 = \xi^2 \left(1 - \frac{c^2}{c_2^2}\right)$$

$$\psi = (B_1 e^{\beta z} + B_2 e^{-\beta z}) e^{i\xi(x-ct)} \quad (2.15)$$

If  $c > \alpha$  and  $c > \beta$ , radicals  $x$  and  $t$  are real, so that amplitudes  $\phi(z)$  and  $\psi(z)$  are oscillating functions for  $z \rightarrow \infty$ . The terms containing  $A_2$  and  $B_2$  decrease exponentially as  $z \rightarrow \infty$ . However, the terms with  $A_1$  and  $B_1$  increase to infinity, which is physically implausible. Thus, we must put  $A_1 = B_1 = 0$ .

$$\phi = A_2 e^{-\alpha z} e^{i\xi(x-ct)} \quad (2.16)$$

$$\psi = B_2 e^{-\beta z} e^{i\xi(x-ct)} \quad (2.17)$$

On applying boundary conditions,  $\tau_{zz} = 0$ , at  $z = 0$

we get

$$(\lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2)A_2 - (2\mu i\xi\beta)B_2 = 0 \quad (2.18)$$

$\tau_{zx} = 0$  at  $z = 0$

we get

$$(2\alpha i\xi)A_2 + (\beta^2 + \xi^2)B_2 \quad (2.19)$$

Eq.(2.18) (2.19) represents a system of linear homogenous equations. The equations possess the non trivial solution if determinant is equal to zero .

$$\begin{vmatrix} \lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2 & -i2\mu\beta\xi \\ 2\alpha i\xi & (\beta^2 + \xi^2) \end{vmatrix} = 0$$

Thus the equation is given as

$$4\alpha\beta\mu\xi^2 - (\beta^2 + \xi^2)(\lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2) = 0 \quad (2.20)$$

This equation is known as dispersion equation/frequency equation involving phase velocity and wave number of Rayleigh wave.

## 2.5 Numerical example and graph

For the graphical representation of phase velocity with respect to wave number in the case of Rayleigh wave propagating in a homogenous isotropic elastic half-space, we take the following data for the aluminium epoxy material[11].

$$c_1 = 7205m/s, c_2 = 2937m/s, \rho = 1500kg/m^3$$

Fig 2.1 Represents the variation of phase velocity with wave number for the propagation of Rayleigh waves. It is observed that Rayleigh wave propagate in the velocity approximately equal to  $2790.1m/s$

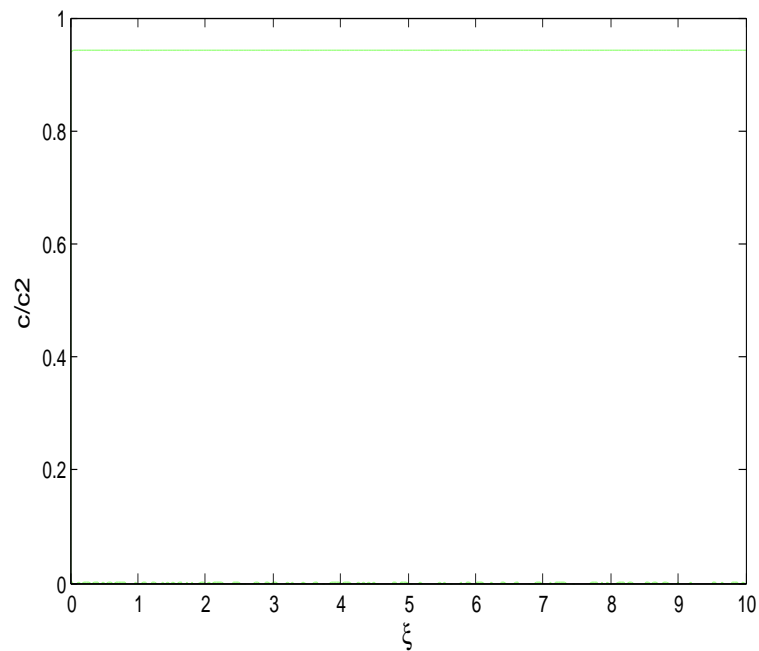


Figure 2.1: Variation of phase velocity with wave number

### **2.5.1 Conclusion**

It is observed that Rayleigh wave doesnot exhibit any dispersion. It is one of the property of the Rayleigh wave that is doesnot show dispersion in isotropic elastic half-space [4]. For the considered material it is found to be as  $2970.1m/s$

# Chapter 3

## PROPAGATION OF RAYLEIGH WAVES IN AN ELASTIC HALF SPACE UNDERLYING A LIQUID HALF-SPACE

### 3.1 Introduction

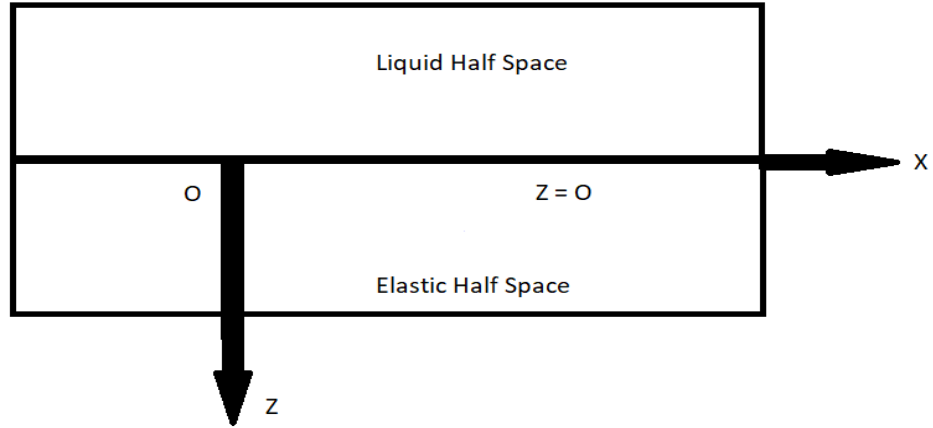
The study of Rayleigh waves becomes more interesting and application oriented when homogeneous elastic half-space is loaded with liquid. Interaction of elastic waves with fluid loaded solids can be used for non-destructive evaluation of solid structure. This type of phenomena have been studied for wide variety of solids extending from simple half-space to more complicated systems of anisotropic media. Wu and Zhu[12] studied the propagation of Lamb waves in a plate bordered with a layer of inviscid liquid. Further, Zhu and Wu[13] derived the dispersion equation of Lamb waves of a plate bordered with a viscous liquid layer. The exact characteristic equation for leaky waves propagating in isotropic elastic solids loaded with the viscous fluid including half-space and finite thickness plates totally immersed in fluids is derived by Nayfeh and Nagy[14]. Rayleigh Surface waves in microstretch thermal continua under inviscid loading has been studied by Sharma et al.[11]. Sharma and Pathania[15][16] also discussed the propagation of leaky surface waves in thermoelastic solids due to inviscid fluid loadings. Viscous fluid loading on the propagation of leaky Rayleigh wave in the presence of heat conduction effect is studied by Qi[17]. Here, in this chapter we have derived the secular equation for propagation of Rayleigh waves propagating in an elastic half-space underlying a half-space of inviscid liquid .

### 3.2 Formulation of the problem

Consider a homogeneous, isotropic elastic half-space of inviscid liquid. Introducing a cartesian coordinate where  $x$  axis is along the surface of the medium and  $z$ -axis is downwards in positive direction. All the particles lying along the line which is

parallel to the  $y$ -axis are equally displaced, so the solutions are independent of  $y$  coordinate.

The basic governing equations and constitutive relations for solid half-space are discussed in Ch-2 (eq.(2.1) to eq.(2.11))



In case of liquid medium we have equation of the form,

$$\lambda_L \nabla (\nabla \cdot \vec{u}_L) = \rho_L \ddot{\vec{u}}_L \quad (3.1)$$

$\lambda_L$  is Bulk modulus,  $\rho_L$  is density of liquid and

$$\vec{u}_L = (u_L, 0, w_L)$$

is displacement of fluid medium. The constitutive relations are

$$\tau_{ij}^L = \lambda_L \delta_{ij} (u_{k,k})_L \quad (3.2)$$

where  $\tau_{ij}^L$  is force stress tensor for liquid medium

By using Helmholtz decomposition

$$\vec{u}_L = \nabla \phi_L + \nabla \times \vec{\psi}_L, \quad (3.3)$$

$$\vec{\psi}_L = (0, \psi_L, 0)$$

where

$$\nabla \vec{\psi} = 0$$

We have,

$$u_L = \frac{\partial \phi_L}{\partial x} - \frac{\partial \psi_L}{\partial z} \quad (3.4)$$

$$w_L = \frac{\partial \phi_L}{\partial z} + \frac{\partial \psi_L}{\partial x} \quad (3.5)$$

$\phi_L, \vec{\psi}_L$  are scalar and vector potential functions. Since liquid doesnot support shear stress i.e  $\mu_L = 0$  and  $\psi_L = 0$  Then using eq.(3.3) (3.4) and (3.5) in eq. (3.1) we have

$$\nabla^2 \phi_L = \frac{\rho_L}{\lambda_L} \ddot{\phi}_L \quad (3.6)$$

Therefore

$$\nabla^2 \phi_L = \frac{1}{c_L^2} \ddot{\phi}_L \quad (3.7)$$

where,

$$c_L^2 = \frac{\rho_L}{\lambda_L}$$

### 3.3 Boundary Conditions

The following boundary conditions have to be satisfied at solid liquid interface at  $z = 0$

1) The normal component of stress in an elastic half-space is equal to the normal component of stress in liquid half-space at  $z = 0$

$$\tau_{zz} = \tau_{zz}^L \quad (3.8)$$

2)The shear component of stress in an elastic half-space at  $z = 0$

$$\tau_{zx} = 0 \quad (3.9)$$

3)The displacement of vertical z-axis of elastic half-space is equal to the vertical displacement component of liquid half space

$$w = w_L \quad (3.10)$$

### 3.4 Formal solution of the problem

Assuming solution of the eq.(3.6) be of the form

$$\Phi_L = \Phi_L(z)e^{i\xi(x-ct)} \quad (3.11)$$

substituting this solution in the eq.(3.6), we get

$$\frac{d^2\phi(z)}{dz^2} - \xi^2\left(1 - \frac{c^2}{c_L^2}\right)\phi_L(z) = 0 \quad (3.12)$$

$$\frac{d^2\phi(z)}{dz^2} - a^2\Phi_L(z) = 0$$

where

$$a^2 = \xi^2\left(1 - \frac{c^2}{c_L^2}\right)$$

$$\phi(z) = A_1e^{-az} + A_2e^{az}$$

where

$$a^2 = \xi^2 - \frac{\xi^2 c^2}{c_1^2}$$

Therefore

$$\Phi_L(z) = (D_1e^{az} + D_2e^{-az}) \quad (3.13)$$

Hence the solutions are

$$\Phi_L = (D_1e^{az} + D_2e^{-az})e^{i\xi(x-ct)}$$

In bounded elastic media, as only two waves are propagated. These two types of waves are P or SH waves. As in half-space problem, the boundary is also considered, the third type of wave exists whose effect is confined to the surface. Applying the conditions that variables are bounded within the solids.

We take solution as

$$\Phi_L = (D_1e^{az}e^{i\xi(x-ct)})$$

The stress-displacement for liquid half-space is

$$\tau_{zz}^L = \lambda_L \left( \frac{\partial u_L}{\partial x} + \frac{\partial w_L}{\partial z} \right)$$

$$\tau_{zz}^L = \lambda_L \left( \frac{\partial^2 \phi_L}{\partial x^2} + \frac{\partial^2 \phi_L}{\partial z^2} \right)$$

We obtained the following system of equation using boundary conditions (eq.(3.8) to (3.10))

$$[\lambda^2(\alpha^2 - \xi^2) + 2\mu\alpha^2]A_2 - [i2\mu\beta\xi]B_2 - [\lambda_L(a^2 - \xi^2)D_1] = 0 \quad (3.14)$$

$$-2i\xi\alpha A_2 + (\beta^2 + \xi^2)B_2$$

$$-\alpha A_2 + i\xi B_2 - aD_1 = 0 \quad (3.15)$$

These are homogenous eq.'s with unknown amplitudes  $A_2, B_2$  and  $D_1$  For non-trivial solution determinant of coefficients of unknown constants  $A_2, B_2$  and  $D_1$  is equal to zero

$$\begin{vmatrix} \lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2 & -i2\mu\beta\xi & -(a^2 + \xi^2)\lambda_L \\ 2i\xi\alpha & (\beta^2 + \xi^2) & 0 \\ -\alpha & i\xi & -a \end{vmatrix} = 0$$

Thus the equation is obtained as

$$a[4\mu\xi^2\alpha\beta - (\beta^2 + \xi^2)[\lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2]] - \alpha\lambda_L(a^2 - \xi^2)(\beta^2 - \xi^2) = 0 \quad (3.16)$$

This is the secular equation for the propagation of rayleigh waves in an elastic solid half-space underlying a liquid half-space.

**Validation :** When  $\lambda_L = 0$  eq.(3.16) reduces to

$$4\mu\xi^2\alpha\beta - (\beta^2 + \xi^2)[\lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2] = 0$$

This is the frequency equation (2.20) of Rayleigh wave in solid elastic half-space

as obtained in Chapter-2.

### 3.5 Numerical example and graph

For the graphical representation of phase velocity with respect to wave number in the case of Rayleigh wave propagating in an elastic half-space underlying a liquid half-space, we assume the half-space to be of aluminium epoxy material as taken in chapter-2. The physical data for which is :

$$c_1 = 7205m/s, c_2 = 2937m/s,$$

$$\rho = 1500kg/m^3,$$

The liquid taken for the numerical purpose is water, where  $c_L = 1.5km/s$  ,

$$\rho_L = 1000kg/m^3$$

Variation of phase velocity with wave number in the case of Rayleigh waves have been shown in figure (3.1)

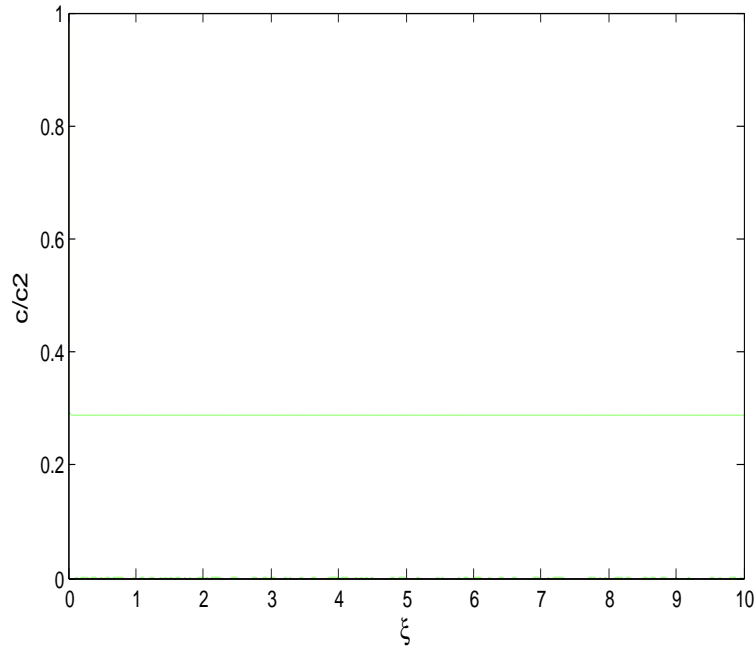


Figure 3.1: Variation of phase velocity with wave number

### 3.5.1 Conclusion

The Rayleigh phase speed in the absence of half-space is obtained in Chapter-2. Here, in the presence of liquid half-space the phase speed has decreased significantly. This phase speed in the presence of the liquid is  $851.7m/s$ .

Thus, under the effect of liquid loading phase speed decreases. This fact can be utilised for scanning the surface of the solids in the non-destructive testing of the material.

# References

- [1] O. Novotny, “Seismic surface waves,” *Bahia, Salvador: Instituto de Geociencias*, 1999.
- [2] S. D. Poisson, “Suite du mémoire sur la probabilité des résultats moyens des observations, inséré dans la connaissance des tems de l’année 1827,” *Connaissance des tems*, pp. 3–22, 1829.
- [3] A. E. H. Love, *A treatise on the mathematical theory of elasticity*. Cambridge university press, 2013.
- [4] K. F. Graff, *Wave motion in elastic solids*. Courier Corporation, 2012.
- [5] A. C. Eringen, “On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves,” *Journal of applied physics*, vol. 54, no. 9, pp. 4703–4710, 1983.
- [6] A. U. Vallina, *Principles of seismology*. Cambridge University Press, 1999.
- [7] G. Hevin, O. Abraham, H. Pedersen, and M. Campillo, “Characterization of surface cracks with rayleigh waves: a numerical model,” *NDT & E International*, vol. 31, no. 4, pp. 289–297, 1998.
- [8] L. Rayleigh, “On waves propagated along the plane surface of an elastic solid,” *Proceedings of the London Mathematical Society*, vol. 1, no. 1, pp. 4–11, 1885.
- [9] P. Sengupta and B. Ghosh, “Effect of couple-stresses on the propagation of waves in an elastic layer,” *pure and applied geophysics*, vol. 112, no. 2, pp. 331–338, 1974.
- [10] W. Rudin, “Principles of mathematical analysis (international series in pure & applied mathematics),” 1976.
- [11] J. Sharma, S. Kumar, and Y. Sharma, “Propagation of rayleigh surface waves in microstretch thermoelastic continua under inviscid fluid loadings,” *Journal of Thermal Stresses*, vol. 31, no. 1, pp. 18–39, 2007.
- [12] J. Wu and Z. Zhu, “The propagation of lamb waves in a plate bordered with layers of a liquid,” *The Journal of the Acoustical Society of America*, vol. 91, no. 2, pp. 861–867, 1992.
- [13] Z. Zhu and J. Wu, “The propagation of lamb waves in a plate bordered with a viscous liquid,” *The Journal of the Acoustical Society of America*, vol. 98, no. 2, pp. 1057–1064, 1995.
- [14] A. H. Nayfeh and P. B. Nagy, “Excess attenuation of leaky lamb waves due to viscous fluid loading,” *The Journal of the Acoustical Society of America*, vol. 101, no. 5, pp. 2649–2658, 1997.

- [15] J. Sharma and V. Pathania, “Thermoelastic waves in a plate bordered with layers of inviscid liquid,” *Journal of thermal stresses*, vol. 26, no. 2, pp. 149–166, 2003.
- [16] J. Sharma and V. Pathania, “Generalized thermoelastic lamb waves in a plate bordered with layers of inviscid liquid,” *Journal of sound and vibration*, vol. 268, no. 5, pp. 897–916, 2003.
- [17] Q. Qi, “Attenuated leaky rayleigh waves,” *The Journal of the Acoustical Society of America*, vol. 95, no. 6, pp. 3222–3231, 1994.