Optimality and Duality Results for Some Bilevel Programming Problems

A Thesis

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Doctor of Philosophy

in

Department of Mathematics

by

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Certificate

I hereby certify that the work, which is being presented in the thesis, entitled **Optimality and Duality Results for Some Bilevel Programming Problems**, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted to the institution is an authentic record of my own work carried out during the period **July 24, 2018** to **October 03, 2023** under the supervision of **Dr. Navdeep Kailey**, Assistant Professor, Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala.

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.........dedicated to maa

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Abstract

The work exhibited in this thesis is an endeavor to achieve various optimality and duality results for bilevel programming problems. The proposed work encapsulates these results which are weaved into five chapters. The present thesis is assembled into chapters as described below:

Chapter 1 is introductory and consists of definitions, notations and prerequisites of the present work. A brief account of the related work studied by various authors in the field and a summary of the thesis are also presented.

Chapter 2 presents a Wolfe type dual corresponding to a multiobjective bilevel problem. Duality results are developed and with the help of a non-trivial example weak duality theorem is demonstrated. Further we have studied a multi-objective bilevel problem where both the levels have multiple objectives. By using optimal value reformulation and a scalarization technique we reformulate the problem. We have developed sufficient optimality conditions for this model. We have proposed a Mond-Weir type dual corresponding to this model and developed the relevant duality theorems under ∂^*-pseudoconvex and ∂^*-quasiconvex assumptions.

In **Chapter 3**, we examined a bilevel problem with multiple objectives at both levels. With the aid of kth-objective weighted constraint scalarization and objective value function reformulation, the problem is converted into a single-level mathematical programming problem. The necessary optimality conditions are obtained and an illustrative example is given to validate our result.

In **Chapter 4**, we have considered a bilevel programming problem with uncertainty at the upper-level constraint. By using robust counterpart approach and optimal value reformulation we transform the robust counterpart bilevel problem into a single-level problem. We have developed the optimality conditions in terms of subdifferentials and convexifactors. Moreover we have considered a multi-objective robust bilevel problem and developed the necessary optimality conditions.

Chapter 5 is devoted to the development of relationship between a fractional multi-objective bilevel programming problem and its Mond-Weir type dual under ∂^*-pseudoconvex and ∂^*-quasiconvex assumptions. An example is given to validate the weak duality theorem.

List of Published/Communicated Papers

1. Saini, S., & Kailey, N. (2022). Sufficiency and duality of multi-objective bi-level programming problem under Guignard constraint qualification. Maejo International Journal of Science and Technology, 16(1), 13-24.

2. Saini, S., & Kailey, N. (2022). Necessary optimality conditions for nonsmooth multi-objective bilevel optimization problem under the optimistic perspective. Positivity, 26(4), 65-80.

3. Saini, S., Kailey, N., & Ahmad, I. (2023). Optimality conditions and duality results for a robust bi-level programming problem. RAIRO-Operations Research, 57(2), 525-539.

4. Saini, S., & Kailey, N. (2024). Duality Results in terms of convexifactors for a bilevel multiobjective optimization problem. FILOMAT, 38(6) 2015-2022.

5. Saini, S., & Kailey, N. Duality of multi-objective fractional bi-level programming problem and its application (Communicated)

6. Saini, S., & Kailey, N. Optimality conditions for a robust bilevel multiobjective programming problem (Communicated)

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List of Abbreviations

BPP	Bilevel programming problem
$\mathcal{B}.\mathcal{T}$	Bouligand tangent
$\mathcal{C}.cd$	Clarke coderivative
$\mathcal{C}.\mathcal{N}$	Clarke normal
$\mathcal{C}.sd$	Clarke subdifferential
$\mathcal{C}.\mathcal{T}$	Clarke tangent
$cnstr.\ fn$	Constraint function
CF	Convexifactor
CPP	Convex programming problem
$\mathcal{DM}$	decision maker
$dlty.$	Duality
eff. sol.	efficient solution
FP	Fractional programming problem
$\mathcal{F}.cd$	Fréchet coderivative
$\mathcal{F}.\mathcal{N}$	Fréchet normal
$\mathcal{F}.sd$	Fréchet subdifferential
LB	Linear bilevel programming problem
$\mathcal{LC}$	Lipschitz continuous
$l\mathcal{L}$	locally Lipschitz
ll	lower-level
LCF	lower convexifactor
MWD	Mond-Weir dual
$\mathcal{M}.cd$	Mordukhovich coderivative
$\mathcal{M}.\mathcal{N}$	Mordukhovich normal
$\mathcal{M}.sd$	Mordukhovich subdifferential
MP	Multi-objective programming problem
$\mathcal{N}.\ \&\ \mathcal{S}.\ opt.\ conds$	Necessary and sufficient optimality conditions
$obj.\ fn$	Objective function
$Opt.\ conds$	Optimality conditions
OVF	Optimal value function
$\mathcal{P}.\mathcal{K}.$	Painlevé-Kuratowski
UCF	upper convexifactor
ul	upper-level
USRCF	upper semi-regular convexifactor

sd	Subdifferential
WD	Wolfe dual

Chapter 1

INTRODUCTION

The discipline of mathematics, commonly referred to as optimization, is crucial to modern life. It is especially essential to avoid wasting now when natural resources like oil and gas are becoming depleted. This necessitates making the best use of these resources. The same holds for other finite resources, including time. The pursuit of optimal solutions has long permeated not only industrial but also ordinary life. Choosing a mode of transportation to get to the desired location or purchasing a specific product from the market are examples of daily decision-making scenarios. "What decision should one make at this point?" is a general query that arises at the moment. It is interesting how solving such challenges is similar to finding a solution of an optimization problem. Thus, optimization is an art of maximizing or minimizing a goal subject to certain restrictions on that goal. Optimization theory is a dynamic and fascinating area of study in modern mathematics. It is one of the cores of applied mathematics. *Opt.* and *dlty.* conditions play indispensable parts in analyzing and solving optimization problems. These notions are foundational for discovering optimal solutions in optimization models. For a given solution, we want to find out, if possible, whether the solution is local or global minimum or not; hence we must characterize a minimizing solution. Results that provide us with structural details regarding optimal solution's characteristics are called *opt. conds*. The development of efficient numerical approaches for the practical solution of a real world optimization program is made possible by the availability of the $\mathcal{N}$. & $\mathcal{S}$. *opt. cond.* John [73] was the first to develop the opt. cond. using the notion of Lagrange multipliers for constrained optimization problems. These criteria were independently determined by Karush [74] and, Kuhn and Tucker [80], with the additional restriction on the Lagrange multiplier linked with the obj. fn's gradient. One of the most crucial findings in optimization theory is the existence of Karush-Kuhn-Tucker (KKT) conditions. *KKT. conds.* laid the groundwork for several computational methods in mathematical programming and are the fundamental concept used to devise the *dlty.* theory. A mathematical programming program and a related mathematical programming program are closely correlated. The foremost program is the primal, and the second is the dual. John Von Neumann [100] initially introduced *dlty.* in linear programming problems.

Scalar (or single objective) programming problems are those mathematical programming problems that have only one obj. fn. Later, it was acknowledged that the single objective optimization program was not applicable in all cases. Making decisions, selecting options, and looking for compromises are inevitable in day-to-day life. A $\mathcal{DM}$ attempts to reduce or optimize not just one but numerous obj. fns in various planning and design situations. For instance, for manufacturing metal sheets one aims to simultaneously enhance process speed, minimize energy usage, and increase product strength. The clash between diverse interests and goals causes problems in this situation. A multi-objective programming program is a mathematical optimization model with more than one goals to be optimized. Optimizing all objectives in a multi-objective optimization problem to achieve simultaneously is generally impossible. The root of vector optimization can be traced to Edgeworth [50] and Pareto [102], who provided the concept of the Opt. theory in multi-objective optimization. However, this branch of optimization in mathematics began its development with the classic paper by Kuhn and Tucker [80]. Pareto Opt. is a key idea for determining reasonable trade-offs between conflicting goals.

In conventional linear programming programs, there is just one $\mathcal{DM}$ who attempts to determine an optimal course of action subject to a specific set of restrictions so that their objectives are best accomplished. Applying well-known optimization techniques to a problem with a sole $\mathcal{DM}$ may not be a close approximation of the real-world system's actual behavior. Due to this, in the previous few decades, there has been an upsurge in research in novel and more advanced mathematical programming techniques that can deal with the participation of multiple $\mathcal{DM}$s while yielding realistic models of real-world programs. Multi-level programming problems enable modeling a specific instance where the $\mathcal{DM}$s have different authority levels and a well-defined hierarchical relationship between them. A bilevel program usually has two competing decision-making parties: one is composed of *ul* $\mathcal{DM}$ (leader), and the other is composed of *ll*-problem $\mathcal{DM}$ (follower). The interaction between the two levels occurs in the following fashion:

- The set of *ll*-problem constraints is entirely determined by the choice made by the leader.
- Leader's objectives are mutually determined by the choices made at both the levels.
- For each choice made by the leader, the follower will select the best option based on his/her goals and the set of constraints.

Nearly every branch of mathematics and other disciplines like science, economics, and engineering rely greatly on convex functions. The cause of this is that they are particularly well adapted to extremum problems because, in the presence of convexity, necessary criteria for the existence of a minimum also become sufficient. It becomes essential to expand the study to address nonconvex problems, nevertheless, because not all real-world

situations can be characterized as convex models. Some of the nonconvex functions (quasiconvex and pseudoconvex) investigated in the literature maintain specific characteristics of convex functions, which in turn aids in the investigation of the *opt.cond*. Differentiable functions characterize convexity in terms of its gradient. However, the theory based on the differentiation concept fits poorly in the real-world environment. Thus, to handle such situations, the conceptual framework for differentiation needs to be expanded. So, the concept of subgradient is employed in place of the gradient when dealing with non-differentiable functions.
The current chapter is broken down into four sections. The first section contains vital preliminary information. The second segment examines several mathematical programming problems existing in the literature. The third section includes a overview of the mathematical program which is the primary concern of this thesis, and the final section provides a thesis framework.
"Sections, subsections, theorems, lemmas, propositions, remarks and equations etc., are numbered consecutively along the chapter number. For example, Section 2.4 represents the Section 4 of Chapter 2, Subsection 3.2.1 means the Subsection 1 of Section 2 in Chapter 3 and Theorem 4.1 means the Theorem 1 of Chapter 4."

1.1 Prerequisite

The foundational techniques from the variational analysis are introduced in this segment and will be used throughout the thesis. The books and the references therein provide further details on the subject matter that is briefly covered here. Until specified differently, the below stated notations are utilized across the thesis. $\mathbb{N}$ and $\mathbb{R}^r$ represents collection of natural numbers and $r-D$ Euclidean space, respectively. $\mathbb{R}^1=\mathbb{R}$ collection of all real numbers, $\mathbb{R}^{n_a}_+=\{x_a\in\mathbb{R}^{n_a}: x_{as}\geqq 0,\ s=1,2,\ldots,n\}$ the non-negative orthant of $\mathbb{R}^{n_a}$ and $\mathbb{R}_+$ collection of non-negative real numbers. All vectors will be considered as column vectors.
For $x_a, x_b\in\mathbb{R}^{n_a}$,

$$\begin{aligned}
&x_a\geqq x_b \Leftrightarrow x_a-x_b\in R^n_+ \;\Leftrightarrow\; x_{aj}\geq x_{bj}, \forall j=1,2,...n;\\
&x_a\geq x_b \Leftrightarrow x_a-x_b\in R^n_+\setminus\{0\} \;\Leftrightarrow\; x_{aj}\geq x_{bj}, \forall j=1,2,...n, x_{ai}>x_{bi}, \text{for some } i;\\
&x_a> x_b \Leftrightarrow x_a-x_b\in \text{int}R^n_+ \;\Leftrightarrow\; x_{aj}> x_{bj}, \forall j=1,2,...n;\\
&x_a\ngeqq x_b \Leftrightarrow x_a-x_b\notin R^n_+ \;\Leftrightarrow\; \text{negation of } x_a\geqq x_b;\\
&x_a\ngeq x_b \Leftrightarrow x_a-x_b\notin R^n_+\setminus\{0\} \;\Leftrightarrow\; \text{negation of } x_a\geq x_b;\\
&x_a\not> x_b \Leftrightarrow x_a-x_b\notin \text{int}R^n_+ \;\Leftrightarrow\; \text{negation of } x_a> x_b.
\end{aligned}$$

Let us start by remembering that while analyzing adjacent values revolving near a given point in a set-valued mapping, the concept of the $\mathcal{P}.\mathcal{K}.$ outer limit is a crucial tool in variational analysis. Let $\Phi : \mathbb{R}^{n_a} \rightrightarrows \mathbb{R}^{n_b}$ be a set-valued mapping. The $\mathcal{P}.\mathcal{K}.$ outer limit as $x_{\bar{x}} \to \bar{x}$ is designed as

$$\limsup_{x_{\bar{x}} \to \bar{x}} \Phi(x_{\bar{x}}) := \{v \in \mathbb{R}^{n_b} | \exists \text{ sequences } x_{\bar{x}}^k \to \bar{x},\ v_v^k \to v \text{ with } v_v^k \in \Phi(x_{\bar{x}}^k) \text{ as } k \to \infty\}.$$

Definition 1.1 *A map $\psi : \mathbb{R}^{n_a} \to \mathbb{R}$ is differentiable at a point x_a if $\exists\ \nabla\psi(x_a) \in \mathbb{R}^{n_a}$ such that*

$$\psi(u) = \psi(x_a) + \langle \nabla\psi(x_a), u - x_a \rangle + o(||u - x_a||),$$

where $\lim_{u \to x_a} \dfrac{o(||u - x_a||)}{||u - x_a||} = 0$. *Note that if $u = x_a + \xi v$, where $\xi > 0$, then $o(||u - x_a||) = o(\xi)$.*

If the function ψ is differentiable everywhere in $\mathbb{R}^{n_a}$ we call it a differentiable function. Continuously differentiable functions are those for which the mapping $u \mapsto \nabla\psi(u)$ is continuous. Where the gradient $(\nabla\psi(u))$ is characterized as:

$$\nabla\psi(u) = \Big(\frac{\partial\psi}{\partial u_1}, \frac{\partial\psi}{\partial u_2}, \ldots, \frac{\partial\psi}{\partial u_n}\Big),$$

here $\frac{\partial\psi}{\partial u_i}$, $i = 1, 2, \ldots, n$ is the partial derivative of ψ.
A function $T : B \subset \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ is termed as $l\mathcal{L}$ near $\hat{x_b} \in B$ if $\exists$ $n.b.d$ $\mathbb{B}$ of $\hat{x_b}$ and $\ell > 0$ such that

$$|T(x_b) - T(x_a)| \leq \ell||x_b - x_a||, \quad \forall x_b,\ x_a \in \mathbb{B}.$$

where ℓ, known as the Lipschitz constant, holds. When a function is $l\mathcal{L}$ continuous near every point on S, it is said to be $l\mathcal{L}$ continuous. If $\mathbb{B} = \mathbb{R}$ in the aforementioned inequality then it is said to as Lipschitz continuous($\mathcal{LC}$). On the relative interior of its domain, any convex function is $l\mathcal{L}$.

1.1.1 Cones

We start by defining a few basic concepts.

Definition 1.2 [12] *A convex set S of $\mathbb{R}^{n_a}$ is termed as a convex cone if $\forall s \in S$ and $\xi \geqq 0$, $\xi s \in S$.*

Definition 1.3 [123] $S^* := \{z \in \mathbb{R}^{n_a} : \langle z, s \rangle \leqq 0, \forall\ s \in S\}$ *is called negative polar cone* *of S.*

It's evident that the cone S^* is closed and convex.
For a set $S \subset \mathbb{R}^{n_a}$:

- The set of convex combination of S is termed as convex hull of S. Symbolized as co (S).

- A point s is said to be in the closure of the S if $S \cap \mathbb{B}(s) \neq \emptyset$. Symbolized as cl S.

- A point s is said to be in the interior of the S if $S \cap \mathbb{B}(s) \subset S$. Symbolized as int S.

For $s \in S$, we give the following definition of the $\mathcal{B}.\mathcal{T}$ cone and $\mathcal{C}.\mathcal{T}$ cone, respectively, to S at s:

$$\mathcal{T}_S(s) := \{d \in \mathbb{R}^{n_a} | \exists \{t_k\} \subseteq \mathbb{R}^{n_a}_+,\ \exists \{d_k\} \subseteq \mathbb{R}^{n_a} : t_k \downarrow 0,\ d_k \to d,\ s + t_k d_k \in S\ \forall k \in \mathbb{N}\}$$
$$\mathcal{T}^c_S(s) := \{d \in \mathbb{R}^{n_a} | \forall \{t_k\} \subseteq \mathbb{R}^{n_a} : t_k \downarrow 0,\ s_k \to s, \exists d_k \to d,\ s_k + t_k d_k \in S\ \forall k \in \mathbb{N}\}$$

Using [106, Theorem 6.28] and [33], one may express the regular and the $\mathcal{C}.\mathcal{N}$ cones in the form of the $\mathcal{B}.\mathcal{T}$ and the $\mathcal{C}.\mathcal{T}$ cone, respectively as:

$$\widehat{\mathcal{N}}_S(s) := \Big\{ s_* \in \mathbb{R}^{n_a} | \langle s_*, u \rangle \leq 0,\ \forall u \in \mathcal{T}_S(s)\}$$
$$\mathcal{N}^c_S(s) := \Big\{ s_* \in \mathbb{R}^{n} | \langle s_*, u \rangle \leq 0,\ \forall u \in \mathcal{T}^c_S(s)\}$$

Example 1.1 *Let $S = \{(x_a, x_b) \in \mathbb{R}^2 |\ x_b \leq |x_a|\}$ at $s = (0, 0)$ the $\mathcal{C}.\mathcal{T}$ and $\mathcal{C}.\mathcal{N}$ cone, as shown in Figure 1.1, are characterized respectively as*

$$\mathcal{T}^c_S(s) := \{(v_a, v_b) \in \mathbb{R}^2 |\ v_b \leq -|v_a|\}$$

,

$$\mathcal{N}^c_S(s) := \{(u_a, u_b) \in \mathbb{R}^2 |\ u_b \geq |u_a|\}$$

.

Let us further introduce two normal cones to S at s
1) *regular/Fréchet* ($\mathcal{F}.\mathcal{N}$) cone

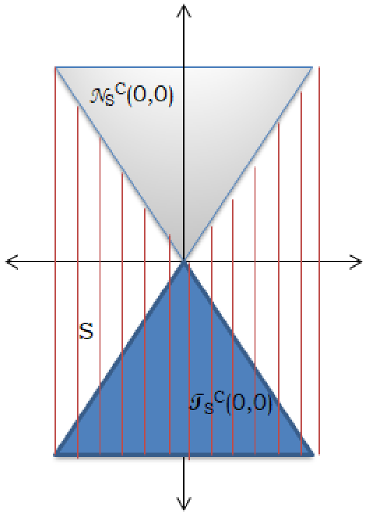

Figure 1.1: $\mathcal{C}.\mathcal{T}$ and $\mathcal{C}.\mathcal{N}$ cone

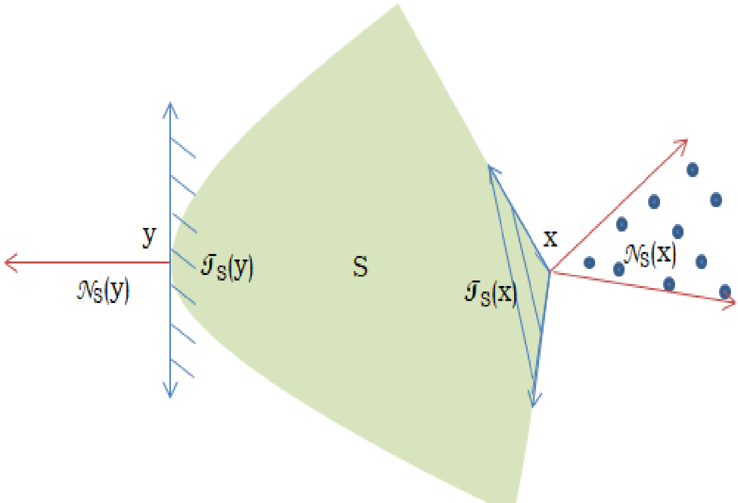

Figure 1.2: $\mathcal{B}.\mathcal{T}$ and normal cone

2) *basic/Mordukhovich* ($\mathcal{M}.\mathcal{N}$) cone:

$$\widehat{\mathcal{N}}_S(s) := \Big\{ s_* \in \mathbb{R}^n | \limsup_{x \to s, x \in S} \frac{\langle s_*, x - s \rangle}{||x - s||} \leq 0 \Big\},$$

$$\mathcal{N}_S(s) := \{ s_* \in \mathbb{R}^n | \exists \{s^k\} \subseteq S \exists \{s_*^k\} \subseteq \mathbb{R}^{n_a} : \ s^k \to s, \ s_*^k \to s_*, \\ s_*^k \in \widehat{\mathcal{N}}_S(s^k) \ \forall k \in \mathbb{N} \}.$$

Example 1.2 *Let* $S := \{(x_a, x_b) \in \mathbb{R}^2 : x_b \geq x_a, x_a \leq 0\} \bigcup \{(x_a, x_b) \in \mathbb{R}^2 : x_b \geq \sqrt{x_a}, x_a \geq 0\}$ *at* $s = (0,0)$ *the* $\mathcal{M}.\mathcal{N}$ *cone is characterized as* $\mathcal{N}_S(0,0) := \{(x_a, x_b) \in \mathbb{R}^2 : x_a \geq -x_b, x_a \geq 0, x_b \leq 0\}$ *see Figure 1.3.*

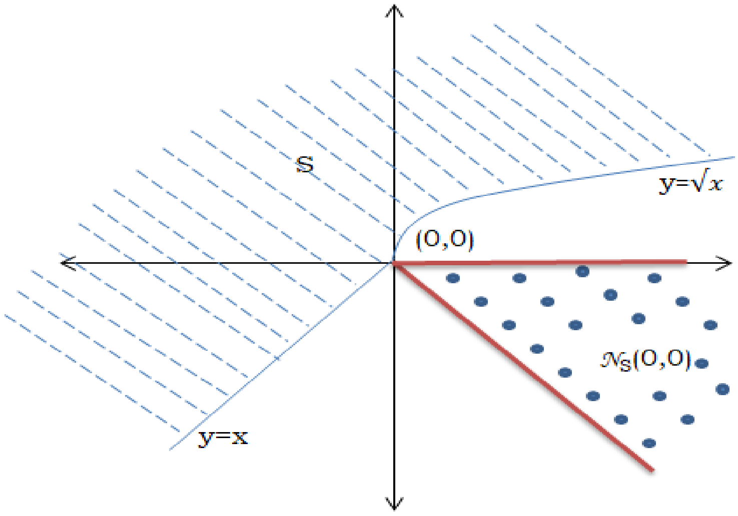

Figure 1.3: $\mathcal{M}.\mathcal{N}$ cone

Example 1.3 *Let* $S := \{(x_a, x_b) \in \mathbb{R}^2 : x_b \geq |x_a|\}$ *at* $s = (0,0)$ *the* $\mathcal{F}.\mathcal{N}$ *cone is characterized as* $\widehat{\mathcal{N}}_S(0,0) := \{(0,0)\}$ *the* $\mathcal{F}.\mathcal{N}$ *cone of the left arm points is represented by the right arm and the* $\mathcal{F}.\mathcal{N}$ *cone of the right arm points is represented by the left arm, see Figure 1.4.*

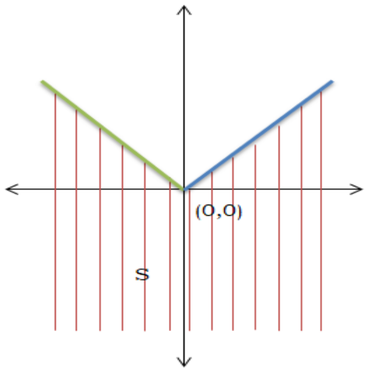

Figure 1.4: $\mathcal{F}.\mathcal{N}$ cone

Unlike the regular normal and $\mathcal{C}.\mathcal{N}$ cones, which are invariably convex sets, the $\mathcal{M}.\mathcal{N}$ cone typically exhibits nonconvexity. This prevents the basic cone from having a polar relation to any tangential approximation of S [97]. But one should keep in mind that the $\mathcal{M}.\mathcal{N}$ cone possesses all the calculus rules [97, 106]. Moreover, the above discussed normal cones possess the following relationship:

$$\widehat{\mathcal{N}}_S(s) \subseteq \mathcal{N}_S(s) \subseteq \mathcal{N}_S^c(s).$$

Each of the aforementioned normal cones corresponds with the normal cone in the sense of convex analysis when the set under examination is convex i.e.

$$\widehat{\mathcal{N}}_S(s) = \mathcal{N}_S(s) = \mathcal{N}^c_S(s) = \{v \in \mathbb{R}^{n_a} | \langle v, u - s\rangle \leq 0, \ \forall u \in S\}.$$

Let us now recollect certain properties of normal cones. A set S is known to be regular at a point $s \in S$ if

$$\widehat{\mathcal{N}}_S(s) = \mathcal{N}_S(s)$$

obviously one can note that this will happen when S is a convex set. Each of the aforementioned normal cones will be empty if $s \notin S$ and all will correspond with the origin $\{0\}$ of the space if $s \in \text{int}S$.

1.1.2 Subdifferentials

We now discuss a few generalized differential concepts for extended-real valued functions that might not accept a classical derivative or gradient.

Definition 1.4 [33] *Let $\Phi : \mathbb{R}^{n_a} \to \mathbb{R}$ be a $l\mathcal{L}$ map on $\mathbb{R}^{n_a}$. Then $\mathcal{C}$.sd of Φ at $x_a \in \mathbb{R}^{n_a}$ is designed as*

$$\partial^c \Phi(\bar{x}_a) := \Big\{\xi \in \mathbb{R}^{n_a} \text{ such that } \Phi^0\Big(\bar{x}_a, l\Big) \geq \Big\langle \xi, l \Big\rangle, \forall l \in \mathbb{R}^{n_a}\Big\},$$

where

$$\Phi^0(x_a, l) = \lim_{v_{x_a} \to x_a} \sup_{t \to 0^+} \frac{\Phi(v_{x_a} + tl) - \Phi(v_{x_a})}{t},$$
$$\text{where } v_{x_a} \in \mathbb{R}^{n_a} \text{ and } t > 0.$$

Definition 1.5 [39] *Let $\Phi : \mathbb{R}^{n_a} \to \mathbb{R}$ be $\mathcal{LC}$ near $\bar{x}_a$, its $\mathcal{M}$.sd at $\bar{x}_a$ is characterized as*

$$\partial \Phi(\bar{x}_a) = \limsup_{x_a \to \bar{x}_a} \widehat{\partial} \Phi(x_a),$$

where

$$\widehat{\partial}\Phi(x_a) := \Big\{l \in \mathbb{R}^{n_a} \text{ such that } \liminf_{x_s \to x_a} \frac{\Phi(x_s) - \Phi(x_a) - \langle l, \ x_s - x_a\rangle}{\| \ x_s - x_a \ \|} \geq 0\Big\},$$

represents $\mathcal{F}$.sd of Φ at x_a.

For a lower semicontinuous map the $\mathcal{M}.sd$ with respect to the epigraph's normal cone can be characterized as:

$$\partial\Phi(x_a) := \{l \in \mathbb{R}^{n_a} | (l, -1) \in \mathcal{N}_{\text{epi}\Phi}(x_a, \Phi(x_a))\}.$$

Also $\partial^c\Phi(\bar{x}_a) := \text{co}\partial\Phi(\bar{x}_a)$, as a result of the connection between the $\mathcal{M}.sd$ and $\mathcal{C}.sd$, we enjoy the *convex hull property* under the $\mathcal{L}$ continuity of Φ, as will be shown below with emphasis.:

$$\text{co}\partial(-\Phi)(\bar{x}_a) = -\text{co}\partial(\Phi)(\bar{x}_a).$$

In the instance that Φ is strictly differentiable at $\bar{x}_a$, i.e.

$$\lim_{l \to \bar{x}_a, x_a \to \bar{x}_a} \frac{\Phi(l) - \Phi(x_a) - \langle \nabla\Phi(\bar{x}_a), (l - x_a)\rangle}{||(l - x_a)||} = 0,$$

thus

$$\partial^c\Phi(\bar{x}_a) = \widehat{\partial}\Phi(\bar{x}_a) = \partial\Phi(\bar{x}_a) = \{\nabla\Phi(\bar{x}_a)\}.$$

Moreover, if the map under consideration is convex then the aforementioned *sd*'s coincide with the *sd* in the convex sense

$$\begin{aligned}
&\partial^c\Phi(\bar{x}_a) = \widehat{\partial}\Phi(\bar{x}_a) = \partial\Phi(\bar{x}_a) \\
&= \{l \in \mathbb{R}^{n_a} | (\Phi(x_a) - \Phi(\bar{x}_a)) \geq \langle l, (x_a - \bar{x}_a)\rangle, \ \forall x_a \in \mathbb{R}^{n_a}\}.
\end{aligned}$$

Every function whose $\mathcal{M}.sd$ is a singleton set and is $l\mathcal{L}$ in n.b.d of $\bar{x}_a$ is strictly differentiable at $\bar{x}_a$. Throughout this thesis we will follow the convention that if $\bar{x}_a \notin \text{dom } \Phi$, then

$$\partial^c\Phi(\bar{x}_a) = \widehat{\partial}\Phi(\bar{x}_a) = \partial\Phi(\bar{x}_a).$$

However, if Φ is $\mathcal{LC}$ in n.b.d $\bar{x}_a$ then the collection $\partial\Phi(\bar{x}_a)$ is closed, bounded and nonempty. If $\Phi : \mathbb{R}^{n_a} \to \mathbb{R}\bigcup\{\pm\infty\}$ the above studied *sd*'s in general follows the relation as characterized below

$$\widehat{\partial}\Phi(\bar{x}_a) \subseteq \partial\Phi(\bar{x}_a) \subseteq \partial^c\Phi(\bar{x}_a).$$

Before proceeding further let us recollect the chain and the sum rules of the *sd*'s. Let us consider two $l\mathcal{L}$ functions Φ_1 and Φ_2 near $(\bar{x}_a)$ and let $\alpha_1, \alpha_2 \in \mathbb{R}_+$ then we have

$$\partial(\alpha_1\Phi_1 + \alpha_2\Phi_2)(\bar{x}_a) \subseteq \alpha_1\partial\Phi_1(\bar{x}_a) + \alpha_2\partial\Phi_2(\bar{x}_a),$$

and in the above inclusion equality holds if Φ_1 and Φ_2 are continuously differentiable at

$\bar{x}_a$ [96].

1.1.3 Convexifactors

Demyanov [46] first suggested the idea of the convexifactor (CF) as a broader concept of upper convex and lower concave approximations. CF, sometimes called a convexificator, was described in [46] as a convex and compact set. However, in their subsequent research, Jeyakumar and Luc [72] proposed that one can define a convexificator using a closed, non-convex set rather than a compact and convex one. They were referred to as CFs by Dutta and Chandra [48]. Further research on CFs has been conducted by Dutta and Chandra [49], Li and Zhang [84], and others. Before starting with CFs, let us recollect the definition of $\mathcal{D}$. dd.

Let $\Phi : \mathbb{R}^{n_a} \to \mathbb{R} \bigcup \{\pm\infty\}$ and let $x_a \in \mathbb{R}^{n_a}$ where $\Phi(x_a)$ is finite. The *lower* and *upper* $\mathcal{D}$. dd of Φ at x_a in the direction l are designed, respectively, as

$$\Phi_d^-\Big(x_a, l\Big) = \liminf_{t \to 0^+} \frac{\Phi(x_a + tl) - \Phi(x_a)}{t}$$

and

$$\Phi_d^+\Big(x_a, l\Big) = \limsup_{t \to 0^+} \frac{\Phi(x_a + tl) - \Phi(x_a)}{t}.$$

Definition 1.6 [48, 72] *Let* $\Phi : \mathbb{R}^{n_a} \to \mathbb{R} \bigcup \{\pm\infty\}$

- Φ *is stated to admit an UCF* ($\partial^*\Phi(x_a)$) *at* x_a *iff* $\partial^*\Phi(x_a) \subseteq \mathbb{R}^{n_a}$ *is closed and for each* $l \in \mathbb{R}^{n_a}$,
$$\Phi_d^-(x_a, l) \leq \sup_{x_a^* \in \partial^*\Phi(x_a)} \Big\langle x_a^*, l \Big\rangle.$$

- Φ *is stated to admit a LCF* ($\partial_*\Phi(x_a)$) *at* x_a *iff* $\partial_*\Phi(x_a) \subseteq \mathbb{R}^{n_a}$ *is closed and for each* $l \in \mathbb{R}^{n_a}$,
$$\Phi_d^+(x_a, l) \geq \inf_{x_a^* \in \partial_*\Phi(x_a)} \Big\langle x_a^*, l \Big\rangle.$$

- Φ *admits a CF* ($\partial^*\Phi(x_a)$) *at* (x_a) *iff* $\partial^*\Phi(x_a)$ *is both an UCF and LCF of* Φ *at* x_a.

- Φ *is said to have an USRCF* ($\partial^*\Phi(x_a)$) *at* x_a *if* $\partial^*\Phi(x_a)$ *is an UCF at* x_a *and* $\forall\ l \in \mathbb{R}^{n_a}$,
$$\Phi_d^+(x_a, l) \leq \sup_{x_a^* \in \partial^*\Phi(x_a)} \Big\langle x_a^*, l \Big\rangle.$$

Example 1.4 *Let map* $H : \mathbb{R}^2 \to \mathbb{R}$, *be characterized by* $H(x_a, x_b) = |x_a| - |x_b|$. *Then,*

- *the* $\mathcal{C}.sd$ *is* $\partial^C H(0,0) = co\{(1,-1),\ (-1,1),\ (1,1),\ (-1,-1)\}$.

- *the* $\mathcal{M}.sd$ *is* $\partial^M H(0,0) = \{(\wp,1) \in \mathbb{R}^2,\ -1 \leq \wp \leq 1\} \cup \{(\wp,-1) \in \mathbb{R}^2,\ -1 \leq \wp \leq 1\}$.

- *the CF of* H *at* $(0,0)$ *is characterized as* $\partial^* H(0,0) = \{(1,-1),\ (-1,1)\}$.

Using the idea of CF, Dutta and Chandra [49] proposed ∂^*-pseudoconvex and ∂^*-quasiconvex functions.
For a $l\mathcal{L}$ function $\mathcal{C}.sd$, $\mathcal{M}.sd$ and $\mathcal{F}.sd$ are CF of Φ at $(x_a, \bar{x}_a)$.

Definition 1.7 [124] *Let* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *be a real-valued function and* $\bar{x}_a \in \mathbb{R}^{n_a}$. *Assume that* Φ *admits CF* $\partial^*\Phi(\bar{x}_a)$, Φ *is termed as*

- ∂^*-convex *at* $\bar{x}_a$ *iff* $\forall\ x_a \in \mathbb{R}^{n_a}$:

$$\langle \rho, x_a - \bar{x}_a \rangle \leq \Phi(x_a) - \Phi(\bar{x}_a), \qquad \forall \rho \in \partial^*\Phi(\bar{x}_a).$$

- ∂^*-quasiconvex *at* $\bar{x}_a$ *iff* $\forall\ x_a \in \mathbb{R}^{n_a}$:

$$\Phi(x_a) - \Phi(\bar{x}_a) \leq 0 \ \Rightarrow\ \langle \rho, x_a - \bar{x}_a \rangle \leq 0, \qquad \forall \rho \in \partial^*\Phi(\bar{x}_a).$$

- ∂^**-pseudoconvex at* $\bar{x}_a$ *iff* $\forall\ x_a \in \mathbb{R}^{n_a}$:

$$\Phi(x_a) - \Phi(\bar{x}_a) < 0 \ \Rightarrow\ \langle \rho, x_a - \bar{x}_a \rangle < 0, \qquad \forall \rho \in \partial^*\Phi(\bar{x}_a).$$

CFs are significant because they apply even when function is boundless or non-convex. Utilizing a non-convex set for its definition offers the advantage of potentially requiring a convexification process with a limited number of points, rendering it more versatile for various applications. Additionally, one can have CFs for the $l\mathcal{L}$ function weaker than the $\mathcal{C}.sd$ and $\mathcal{M}.sd$.

Lemma 1.1 [84] *Let* $\partial^*\Phi(x_a)$ *be a CF of* Φ *at* x_a. *Then,* $\forall \xi \in \mathbb{R}$, $\xi\partial^*\Phi(x_a)$ *is a CF of* $\xi\Phi$ *at* x_a.

Remark 1.1 [72] *Assume that the function* $\Phi,\ \Upsilon : \mathbb{R}^{n_a} \to \mathbb{R}$ *admit UCFs* $\partial^*\Phi(x_a)$ *and* $\partial^*\Upsilon(x_a)$ *at* x_a*, respectively, and that one of the CFs is upper regular at* x_a. *Then,* $\partial^*\Phi(x_a) + \partial^*\Upsilon(x_a)$ *is an UCF of* $\Phi + \Upsilon$ *at* x_a.

1.1.4 Coderivatives

Let us consider a set valued map $\Phi : \mathbb{R}^{n_a} \rightrightarrows \mathbb{R}^{n_b}$ and its graph

$$\text{gph}\Phi := \{(x_a, x_b) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} |\ x_b \in \Phi(x_a)\}.$$

The $\mathcal{M}.cd$ $D^*\Phi(\bar{x}_a, \bar{x}_b) : \mathbb{R}^{n_b} \rightrightarrows \mathbb{R}^{n_a}$, for $v_{x_a} \in \mathbb{R}^{n_b}$ of Φ at $(\bar{x}_a, \bar{x}_b) \in \text{gph}\Phi$ in terms of $\mathcal{N}_{\text{gph}}$ of Φ is expressed as follows.

$$D^*\Phi(\bar{x}_a, \bar{x}_b)(v_{x_a}) := \{x_a \in \mathbb{R}^{n_a} |(x_a, -v_{x_a}) \in \mathcal{N}_{\text{gph}\Phi}(\bar{x}_a, \bar{x}_b)\}.$$

It is a positively homogenous mapping. For a case where the considered map Φ is single-valued and $l\mathcal{L}$ near the point under consideration the $\mathcal{M}.cd$ is characterized in terms of the $\mathcal{M}.sd$ of the Lagrange scalarization $\langle v_{x_a}, \Phi\rangle(x_a) = \langle v_{x_a}, \Phi(x_a)\rangle$ as

$$D^*\Phi(\bar{x}_a)(v_{x_a}) = \partial\langle v_{x_a}, \Phi\rangle(\bar{x}_a) \text{ for } v_{x_a} \in \mathbb{R}^{n_b}.$$

In the same fashion the $\mathcal{F}.cd$, and *convexified* coderivative, also known as the $\mathcal{C}.cd$, for

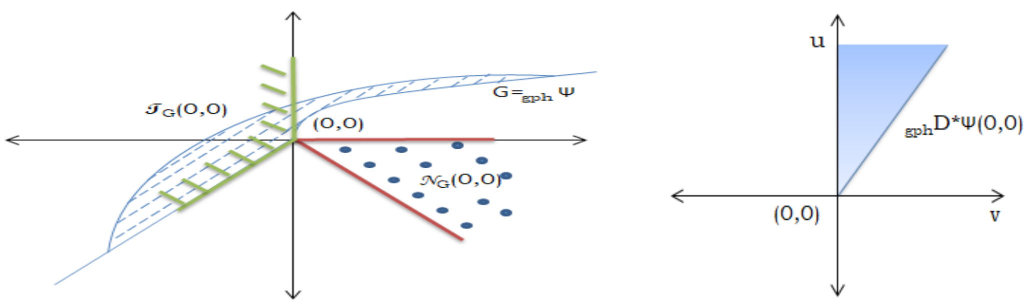

Figure 1.5: Coderivative

$v_{x_a} \in \mathbb{R}^{n_b}$ in terms of regular and $\mathcal{C}.\mathcal{N}$ cones, respectively, can be expressed as

$$\widehat{D}^*\Phi(\bar{x}_a, \bar{x}_b)(v_{x_a}) := \{x_a \in \mathbb{R}^{n_a} |(x_a, -v_{x_a}) \in \widehat{\mathcal{N}}_{\text{gph}\Phi}(\bar{x}_a, \bar{x}_b)\}$$
$$D^{c*}\Phi(\bar{x}_a, \bar{x}_b)(v_{x_a}) := \{x_a \in \mathbb{R}^{n_a} |(x_a, -v_{x_a}) \in \mathcal{N}^c_{\text{gph}\Phi}(\bar{x}_a, \bar{x}_b)\}.$$

1.1.5 Set-valued functions

Let $\Gamma : \mathbb{R}^{n_a} \rightrightarrows \mathbb{R}^{n_b}$ be a set-valued function. For a case when Γ is positively homogenous function, its outer norm and inner norm are respectively expressed as

$$||\Phi||^{+} := \sup_{x_a \in \mathbb{B}} \sup_{x_a \in \Gamma(x_a)} ||x_a||$$
$$||\Phi||^{-} := \sup_{x_a \in \mathbb{B}} \inf_{x_a \in \Gamma(x_a)} ||x_a||$$

Γ is termed as *inner semicompact* at $\bar{x}_a$ with $\Gamma(\bar{x}_a) \neq \emptyset$ if for each $\{x_{as}\} \to \bar{x}_a$ with $\Gamma(x_{a_s}) \neq \emptyset \quad \exists \{x_{b_s}\} \in \Gamma(x_{a_s})$ that includes a convergent subsequence as $s \to \infty$. In finite dimensional space if Φ is uniformly bounded near $\bar{x}_a$ states the map Γ is inner semicompact.

Remark 1.2 [43] *For map $\Gamma : R^{n_a} \rightrightarrows R^{n_b}$, the inner semicompactness of Γ holds whenever Γ contains nonempty values and is uniformly bounded near $\hat{u}_s$, i.e. a n.b.d $U_{\hat{u}_s}$ of $\hat{u}_s$ exists and a bounded set $A \subset R^{n_b}$ such that $\Gamma(u_s) \subset A$, $\forall u_s \in U_{\hat{u}_s}$.*

Map Γ is termed *Inner semicontinuous* at $(\bar{u}_a, \bar{u}_b) \in \text{gph}\Gamma$ if $\forall$ $\{u_{as}\} \to \bar{u}_a$ there is a convergent $\{u_{bs}\} \in \Gamma(u_{as}) \to \bar{u}_b$ as $s \to \infty$.
The phrases discussed above are linked. If $\Gamma(\bar{u}_a) = \bar{u}_b$ and Γ is inner semicompact at $\bar{u}_a$ implies that Γ is inner semicontinuous at $(\bar{u}_a, \bar{u}_b)$.

1.2 General Programming Problems

Suppose a $\mathcal{DM}$ wishes to set the selling price of a certain good subject to a set of restrictions in order to maximize her profit. Such type of problems where one decision maker wants to maximize or minimize her gain or loss, respectively, is called an optimization problem. A standard mathematical programming problem is characterized as:

$$\begin{aligned} &\min_{x_a \in \mathbb{R}^{n_a}} \ \mathcal{F}(x_a) \\ &\text{subject to } x_a \in F_S = \{x_a \in X : \mathcal{G}_j(x_a) \leqq 0\},\ k \in \{1, 2, \ldots, q\}, \end{aligned} \tag{1.1}$$

where $X \subset \mathbb{R}^{n_a}$, the map $\mathcal{F} : X \to \mathbb{R}$ is termed as *obj. fn*, the function $\mathcal{G}_k : X \to \mathbb{R}^m$ represents the restrictions on the problem, also usually called as *const. fn*, the set F_S is known as the *feasible set* and an element $x_a \in F_S$ is termed as *feasible point*.
If every function of the program (1.1) is linear, then the program is refered as linear programming problem (LPP). Nonlinear programming problems are those in which atleast one function of program is nonlinear.

For $x_a \in F_S$ the collection of active constraints is given by

$$I_{\mathcal{G}}(x_a) := \{k \in \{1, 2, \ldots, q\} |\ \mathcal{G}_k(x_a) = 0\}. \tag{1.2}$$

The solution $x_a \in F_S$ is called a global minimum of (1.1) if

$$\mathcal{F}(x_a) \leq \mathcal{F}(x_s) \quad \forall x_s \in F_S$$

and is known as local solution of(1.1) if $\exists$ n.b.d $\mathbb{B}_{x_a}$ of x_a so as

$$\mathcal{F}(x_a) \leq \mathcal{F}(x_s) \quad \forall x_s \in F_S \cap \mathbb{B}_{x_a}.$$

The above-discussed procedure is not generally used in practice to locate the optimal solution since one has to compare infinite points here. Thus the $\mathcal{N}$. *opt. conds.* are developed to arrive at a more usable formulation.

1.2.1 Single objective Programming Problem

In the considered problem (1.1) if there is sole obj. fn then the mathematical programming program is termed as the scalar objective programming problem. In this segment we will explore the *opt. conds.* Developing the *opt. conds.* in optimization theory is vital to the subject. *Opt. conds.* enable one to develop algorithms and numerical methods to locate the optimal solution. In 18th century, Lagrange designed a classical approach to compute the required minimum in case of equality-constrained optimization problems. The method is coined as the method of Lagrange multipliers. F. John extended the Lagrange multipliers rule to the problems with inequality constraints. The F. John $\mathcal{N}$. *opt. cond.* for (1.1) are stated as below:
If x_a is an optimal solution of (1.1) then $\exists$ $(\mu, \nu) \in \mathbb{R} \times \mathbb{R}^q$ such that

$$\begin{aligned} \mu \nabla \mathcal{F}(x_a) + \nu^T \nabla \mathcal{G}(x_a) &= 0, \\ \nu^T \mathcal{G}(x_a) &= 0, \\ (\mu, \nu) &\geq 0. \end{aligned}$$

Here, scalars μ and $\nu_k,\ k = 1, 2, \ldots, q$ are called Lagrange multipliers. The issue with these F. John *opt. conds.* is that if $\mu = 0$, then the obj. fn does not have any influence on the determination of the problem's minimal. In this case, regardless of the obj. fn, the $\mathcal{N}$. *opt. cond.* give the exact same optimal solution which does not make any sense. Thus, some restrictions are placed on the constraints in order to prevent such situations. These restrictions are known as constraint qualifications (CQs) in the literature. Three

assumptions: convexity, differentiability, or a combination of both, are generally required to meet the traditional constraint criteria.
We state below few CQs:

- Slater constraint qualification (SCQ) satisfies if there exists an element $x_a \in F_S$ such that $\mathcal{G}_i(x_a) < 0, \ \forall k = 1, 2, \ldots, q$.
- Mangasarian Fromovitz (MFCQ) satisfies if $\{\exists l \in \mathbb{R}^{n_a} : \ \nabla \mathcal{G}_i(x_a)^T l < 0\}$ for every active inequality constraint at x_a.
- Abadie constraint qualification (ACQ) satisfies at a point x_a if Linearized cone $(L(x_a))$= Tangent cone $(T(x_a))$ where $L(x_a)$ and $T(x_a)$ are designed as:

$$L(x_a) := \{l \in \mathbb{R}^{n_a} : \ \nabla \mathcal{G}_k(x_a)^T l \leq 0, \quad \forall i \in I_{\mathcal{G}}(x_a)\}$$
$$T(x_a) := \{l \in \mathbb{R}^{n_a} | \ \exists \{x_k\} \subseteq X, \ t_k \downarrow 0 : \ x_k \to x_a, \ \frac{x_k - x_a}{t_k} \to l\}.$$

- Guignard constraint qualification (GCQ) satifies at a point x_a if the negative polar of Linearized cone $(L(x_a)^*)$= the negative polar of Tangent cone $(T(x_a)^*)$.

Under Lipschitz assumptions, the traditional multipliers rule was expanded in order to substitute a few generalized gradients for the standard gradient.

1.2.2 Multi-objective Programming Problems

In the problem under consideration (1.1), there is sole objective to accomplish; however, this is not commonly the case in real life. We are again visiting our pricing problem if the $\mathcal{DM}$ maximizing her profit wants to minimize the production cost simultaneously. That means she has two objectives in hand to optimize at the same time. So, if someone seeks to maximize/minimize more than one objective at once, we say that they are working with a *multi-objective programming problem* (MP). The general MP in *r-D* Euclidean space is designed as:

$$(MP) \qquad \min \ \mathcal{F}(x_a) = \{\mathcal{F}_1(x_a), \mathcal{F}_2(x_a), \ldots, \mathcal{F}_r(x_a)\}$$
$$\text{subject to } x_a \in F_S = \{x_a \in X : \mathcal{G}(x_a) \leq 0\},$$

It is uncommon for such problems to have an ideal solution, which is one that concurrently minimizes all of the obj. fns. The various aims often conflict with one another. Finding a singular solution is challenging because these problems rarely have viable solutions which concurrently min/max all goals. In MP problems, the idea of an optimal solution is inextricably linked to the $\mathcal{DM}$s perspective. A good choice is grounded in the idea that no other option could be superior in a particular area of consideration. An eff. sol. is one such point. A non-inferior, nondominated, or Pareto-optimal solution is another name for an eff. sol. With the help of Koopman's [78] ground breaking work, efficiency was introduced into operations research. Later, the idea of weak efficiency was presented. These two ideas of *opt.* in MP are discussed below.

Definition 1.8 [120] *A point $x_a \in F_S$ is termed as weak eff. sol. (weak minimum) of MP, if $\nexists\ x_s \in F_S$ such that*

$$\mathcal{F}(x_s) < \mathcal{F}(x_a).$$

Definition 1.9 [120] *A point $x_a \in F_S$ is termed as an eff. sol. of MP, if $\nexists\ x_s \in F_S$ such that*

$$\mathcal{F}(x_s) \leq \mathcal{F}(x_a).$$

A r-D vector function $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_r\}$ is termed convex at x_a if for each $s = 1, 2, \dots, r$, $\mathcal{F}_j$ is convex at x_a.
Kuhn and Tucker [80] and Arrow et al. [7] made early attempts to formulate the *opt. conds*. The following scalar parametric program was first presented by Geoffrion [58]:

$$(EP) \qquad \min\ \mu^T \mathcal{F}(x_a) = \sum_{j \in K} \mu_j \mathcal{F}_j(x_a)$$
$$\text{subject to} \quad x_a \in F_S,$$

where $\mu_j > 0,\ (j \in K)$ (often normalized according to $\sum\limits_{j \in K} \mu_j = 1$) Kaul et al. [75] devised the $KKT.$ *conds.* in terms of eff. sol. of MP:

Theorem 1.1 (KKT necessary condition) *Suppose that x_a is an eff. sol. for (MP) at which the KTCQ is satisfied. Then there exist a vector $\mu_{x_a} \neq 0 \in R^r$ and $v_{x_a} \in R^n$ such that*

$$\nabla {\mu_{x_a}}^T f(x_a) + \nabla {v_{x_a}}^T g(x_a) = 0,$$
$${v_{x_a}}^T g(x_a) = 0,$$
$$v_{x_a} \geqq 0,\ \mu_{x_a} \geqq 0,\ \sum_{i=1}^{r} \mu_{x_{ai}} = 1.$$

In [51], a summary of ongoing advancements in MP has been published.

1.2.3 Convex Programming Problems

If the maps $\mathcal{F}$ and $\mathcal{G}$ in (1.1) are convex then the problem is termed as convex programming problem (CPP). The convexity of the involved functions of a problem ensures that the *KKT opt. conds.* is sufficient. Every critical point of the convex programming problem is a global minimum, and every local minimum is also a global minimum. These properties of convexity make it a vital concept. The *KKT. conds.* for the CPPs are given as:

$$\begin{aligned} \nabla \mathcal{F}(x_a) + \sum_{i=1}^{n} \nu_{x_{ai}} \nabla \mathcal{G}_i(x_a) &= 0, \\ \nu_{x_a}^T \mathcal{G}(x_a) &= 0, \\ \mathcal{G}(x_a) &\leq 0, \\ \nu_{x_a} &\geq 0. \end{aligned}$$

CPP plays the base role in development of concepts and algorithms in modern optimization problems.

1.2.4 Fractional Programming Problems

In their discussion of a stock-cutting issue in the paper industry [59], Gilmore and Gomory demonstrate that it is preferable to reduce the ratio of wasted raw material to that used rather than just reducing the amount of waste. There are many models where the cost-to-time ratio must be minimized. In their discussion of a routing problem for ships or aircraft, Dantzig et al. [34] identify a network cycle that minimizes the cost-to-time ratio. This prompted the investigation of a fresh set of mathematical programming problems. Thus, if in the considered problem (1.1) $\mathcal{F}$ is expressed as a ratio of two numerical functions then a special name is given to such problems called as *fractional programming problems* (FP). Neumann [100] was the first to address FP. Charnes and Cooper [28] proposed a methodology for managing a linear FP problem 1962. Dinkelbach's algorithm was subsequently offered for nonlinear FP algorithms by Dinkelbach [47]. A general FP problem is sated below:

$$\begin{aligned} (FP) \qquad\qquad & \min \quad \frac{\mathcal{F}(x_a)}{\mathcal{H}(x_a)} \\ & \text{subject to} \quad \mathcal{G}(x_a) \leq 0,\ x_a \in X, \end{aligned}$$

in order to have ratio defined the denominator function is considered to be strictly greater than zero for each $x_a \in X \subset \mathcal{F}$. Here, if $\mathcal{F}$ and $\mathcal{H}$ are linear as well as $\mathcal{G}$ are linear then (FP) is referred as *linear FP problem* else is known as *nonlinear FP problem.*

1.2.5 Robust Programming Problems

A common assumption of many optimization models is that the problem data is already known. In practice, nonetheless, this is not always the state. Realistic data is commonly ambiguous due to measurement inaccuracy, randomness, or other factors. As discussed by [16] the solution of a mathematical programming program is extremely sensitive to data perturbations, neglecting data uncertainty may produce solutions that are not optimal or even infeasible. In this type of situation where the $\mathcal{DM}$ does not have any information about the uncertain parameter's probability distribution but knows a deterministic data set in uncertain space to which these uncertain parameters belong, the robust counterpart optimization problem computes the best feasible solution that can be implemented regardless of the degree of data uncertainty present in the original set [29]. Uncertainty in bilevel problems can take many forms, such as decision uncertainty (uncertainty in the *ul* $\mathcal{DM}$'s decision or uncertainty in the *ll* $\mathcal{DM}$'s response) and data uncertainty (uncertainty in the *ul* problem or uncertainty in the *ll* problem). Recent studies have concentrated on a variety of real-world applications that model robust bilevel problems, including the location of renewable energy sources [87], electric vehicle charging stations [140], supply distribution [110], and resource recovery planning [135] among others. Goerigk et al. [60] recently investigated bilevel combinatorial problem with convex uncertainty. With the aid of the generalized Farkas Lemma, Chuong and Jeyakumar [32] deduced a strong *dlty.* between an affinely adjustable robust linear BPP and its dual. The complexity of robust BPP with ambiguous *ll*-problem objectives was examined in [21]. Beck and Schmidt [13] have studied the impact of choice uncertainty at the *ul* and *ll*. By transforming the program into a BPP, Swain and Ojha [125] explored the robust equivalents of the uncertain mean-variance problems under the box and ellipsoidal uncertainty. In [31], Chuong and Jeyakumar considered the below stated robust bilevel model:

$$\begin{aligned} \min_{a^u, a^l} \quad & F_u(a^u, a^l) \\ \text{subject to} \quad & \tilde{A}_i a^u + \tilde{B}_i a^l \geq \tilde{a}_i \qquad \forall (\tilde{A}_i, \tilde{B}_i, \tilde{a}_i) \in U_i^L, i \in \{1, 2, \ldots, m\} \\ & a^u \in \mathbb{R}^{n_{a^u}}, \; a^l \in S(a^u), \end{aligned}$$

the uncertain parameters $\tilde{A}_j$ designs the j^{th} row of the matrix $\tilde{A}$ and $S(a^u)$ represents the collection of optimal points of the robust *ll*-program

$$\begin{aligned} \min_{a^l} \quad & d^T a^l \\ \text{subject to} \quad & \tilde{P}_k a^u + \tilde{Q}_k a^l \geq \tilde{b}_k \qquad \forall (\tilde{Q}_k, \tilde{b}_k) \in U_k^F, \; k \in \{1, 2, \ldots, l\}. \end{aligned}$$

The uncertainty sets on the first and second level are given by U_i^L and U_k^F respectively, the matrix $\tilde{P}$ is supposed to be certain.

1.3 Bilevel Programming Problems

Often in our day-to-day life, we encounter situations where we have to make a decision that depends on the decision of some other. When deciding, we frequently consider the (best, i.e., optimal) response of another $\mathcal{DM}$ whose choice depends on our own. Moreover, the reaction of the other $\mathcal{DM}$ affects our outcome (or our obj. fn and feasible set in more mathematical terminology) as well. Until now, we have studied various mathematical programming problems; some had single-objective, and some had multiple objectives, whereas others had a particular type of obj. fns. But none of the above-discussed problems presented an additional optimization problem within the constraints of the mathematical programming program. This type of structure is called a hierarchical structure. Hierarchical optimization issues are frequently considered multilevel programming problems in academic literature. A particular case of multilevel problems with just two $\mathcal{DM}$s is the primary focus of this thesis, called a bilevel programming problem (BPP). For better acquaintance, let us go back to our pricing problem while settling the price of certain goods in order to maximize revenue if the $\mathcal{DM}$ also considers the reaction of its consumers. Then the situation at hand becomes the BPP.
BPP has its roots in the Stackelberg game (see Stackelberg [126, 127]), and Bracken and McGill's [18] work was the first to investigate it in the context of mathematical optimization. Although the terms "bilevel" and "multilevel programming" were first used by W. Candler and R. Norton [27]. Formally an optimization problem with a feasible set that is determined by resolving a parameterized problem is known as a BPP.
Consider an extensive network of roadways operated by the government or a commercial association that desires to maximize revenue by optimizing tolls. A high toll would avert users from utilizing the network, limiting revenues owing to low volume. In contrast, a low toll would increase volume but may not be optimal for maximizing profits. In such a case, the government cannot design its optimization problem while ignoring highway users; instead, a two-level hierarchical optimization problem must be constructed. The leader problem in such a program is the government, which aims to maximize revenues within a set of restrictions, with the highway user's optimization problem representing an extra constraint. One can learn about bilevel problems in the transportation policy literature by reading Migdalas [91] Brotcorne et al. [19], and Sinha et al. [114]. BP has applications in nearly every sector, for instance, in engineering, bilevel problems are ordinary in structural optimization (Christiansen et al. [30], Sobieszczanski-Sobieski et

al. [119]). Bilevel problems are widespread in defense (Brown et al. [20], Wein [131], Ramamoorthy et al. [104, 105]). BPP's are typical in the corporate world while creating appropriate tax policies (Labb*é* et al. [81], Sinha et al. [116]), supply chain and location problems (K*ücü*kaydin et al. [79], Sun et al. [122]), agribusiness management (Whittaker et al. [133], Bostian et al. [17]). One might consult research articles and books for further information on BPP approaches and applications (Sinha et al. [115], Dempe [37], Bard [11]).
If all the functions involved at both the levels are linear then the BPP is known as linear BPP (LB). It can be formulated as:

$$\begin{aligned} \min_{a^u} \quad & C_1 a^u + D_1 a^l \\ \text{subject to} \quad & A_1 a^u + B_1^l \leq b_1, \\ & a^u \geqq 0, \end{aligned}$$

where a^l solves

$$\begin{aligned} \min_{a^l} \quad & C_2 a^u + D_2 a^l \\ \text{subject to} \quad & A_2 a^u + B_2 a^l \leq b_2, \\ & a^l \geqq 0. \end{aligned}$$

"Where for each $i = 1, 2$,

- a^u is the *ul*-decision variable,
- a^l is the *ll*-decision variable,
- C_i is the row vector of dimension n_1,
- d_i is the row vector of dimension n_2,
- $b_i \in \mathbb{R}^{m_i}$,
- A_i is a matrix of dimension $m_i \times n_1$,
- B_i is a matrix of dimension $m_i \times n_2$."

Ben-Ayed [15] and Wen and Hsu [132] presented the initial evaluations of LB programs. A BPP is generally characterized as:

$$\begin{aligned} \min_{a^u} \quad & \mathcal{F}^u(a^u, a^l) \\ \text{subject to} \quad & \mathcal{G}^u(a^u, a^l) \leq 0, \\ & a^u \in X, a^l \in S(a^u), \end{aligned}$$

Here, the set-valued mapping $S : X \subset \mathbb{R}^{n_a} \rightrightarrows \mathbb{R}^{n_b}$ is expressed as:

$$S(a^u) := \underset{a^l}{\operatorname{argmin}}\{\mathcal{F}^l(a^u, a^l|\ \mathcal{G}^l(a^u, a^l) \leq 0\},$$

designs the collection of optimal points of the followers parametric programming problem

$$\min_{a^l}\{\mathcal{F}^l(a^u, a^l)|\ \mathcal{G}^l(a^u, a^l) \leq 0\}.$$

for some $a^u \in X$ chosen by leader. The variables $a^u \in \mathbb{R}^{n_a}$ are the *ul*'s variables and $a^l \in \mathbb{R}^{n_b}$ are *ll*'s variables. The obj. fns are characterized by $\mathcal{F}^u, \mathcal{F}^l : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}$ and the cnstr. fns by $\mathcal{G}^u : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}^{p_u}$, $\mathcal{G}^l : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}^{p_l}$. The *ul* cnstrs. $\mathcal{G}^u_i(a^u, a^l) \leq 0,\ i = \{1, 2, \ldots, p_u\}$ are called *coupling cnstrs.* if they depend on the *ll*-problem variable a^l. Also, a unique name is given to all the leader's variables that appear in the follower's cnstrs. , known as *linking variables.* Before proceeding further let us discuss some basic terminology:

- *Constraint region* of the BPP

$$\Phi := \{(a^u, a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} :\ \mathcal{G}^u(a^u, a^l) \leq 0,\ \mathcal{G}^l(a^u, a^l) \leq 0\}.$$

- *Feasible region* of the follower problem for each fixed $a^u \in \mathbb{R}^{n_a}$:

$$S(a^u) := \{a^l \in \mathbb{R}^{n_b} :\ \mathcal{G}^l(a^u, a^l) \leq 0\}.$$

- *Projection* of S on $\mathbb{R}^{n_a}$:

$$P := \{a^u \in X :\ \exists a^l \in \mathbb{R}^{n_b} \text{ s.t. } \mathcal{G}^u(a^u, a^l) \leq 0,\ \mathcal{G}^l(a^u, a^l) \leq 0\}.$$

- *Rational reaction* set of follower for $a^l \in P$:

$$O(a^u) := \{a^l \in S(a^u) :\ a^l \in \arg\min_{z \in S(a^u)} \mathcal{F}^l(a^u, z)\}.$$

- *Inducible* set of the BPP:

$$\overline{S} := \{(a^u, a^l) : \ (a^u, a^l) \in S, a^l \in O(a^u)\}.$$

- *optimal value function* (OVF) of the follower's program:

$$V(a^u) := \inf_{a^l \in S(a^u)} \{\mathcal{F}^l(a^u, a^l) | \mathcal{G}^l(a^u, a^l) \leq 0\}.$$

Even though all the functions $\mathcal{F}^u$, $\mathcal{F}^l$, $\mathcal{G}^u$, $\mathcal{G}^l$, involved are, the value function $V(.)$ is typically neither differentiable nor convex nor continuous. It is necessary to convert BPPs into one-level optimization problems to study, particularly for deriving *opt. cond.* and solving.

1.3.1 Pessimistic and Optimistic Approach for BPP

If for all $a^u \in X$ the *ll*-program produces a unique solution then BPP is said to be well defined. However, usually it is not true. The solution set is a set-valued function, which turns the *ul* $\mathcal{DM}$'s obj. fn into a set-valued function. BPPs can be formulated optimistically or pessimistically in order to prevent dealing with set-valued *obj. fn*s at the *ul*. If the *ll* $\mathcal{DM}$'s reaction is not unique, either the *ll* $\mathcal{DM}$ favors the *ul* $\mathcal{DM}$ in minimizing his/her *obj. fn*, or the leader is bound to bear the loss due to the follower's inadvertent choice. The former strategy is termed the optimistic approach, whereas the latter is the pessimistic one.

Optimistic approach for BPP

$$\begin{aligned} \min_{a^u, a^l} \quad & \mathcal{F}^u(a^u, a^l) \\ \text{subject to} \quad & \mathcal{G}^u(a^u, a^l) \leq 0, \\ & a^u \in X, \quad a^l \in S(a^u). \end{aligned}$$

Pessimistic approach for BPP

$$\begin{aligned} \min_{a^u} \quad & \mathcal{F}^u(a^u, a^l) \\ \text{subject to} \quad & \mathcal{G}^u(a^u, a^l) \leq 0, \\ & a^u \in X, \quad a^l \in \arg\max_{a^l \in S(a^u)} \{\mathcal{F}^l(a^u, a^l)\}. \end{aligned}$$

Let us consider an example to get better understanding of the two approaches:

Example 1.5

$$
\begin{aligned}
\min_{a^u} \quad & -a^u + 20a_1^l + a_2^l \\
\textit{subject to} \quad & a^u - 1 \leq 0, \\
& -a^u \leq 0, \\
& a^l \in \mathit{arg}\min \quad a^u(a_1^l + a_2^l) \\
& \qquad\qquad\qquad a_1^l + a_2^l - a^u = 0, \\
& \qquad\qquad\qquad a_1^l, a_2^l \geq 0.
\end{aligned}
$$

$$
S(a^u) = \begin{cases} (0,0) & a^u = 0, \\ (\lambda a_1^l, (1-\lambda)a_2^l) & a^u \in (0,1], \lambda \in [0,1]. \end{cases}
$$

At $a^u = 1$, *the reaction set* $S(1) = \{(a_1^l, a_2^l) \geq 0 : \ a_1^l + a_2^l = 1\}$ *which is a set. Any bilevel-feasible solution for* $x = 1$ *is unstable because the leader's ideal value shifts considerably depending on the follower's best response. The optimistic solution is* $(1, 0, 1)$ *with* $\mathcal{F}^u(1, 0, 1) = 0$ *and the pessimistic solution is* $(1, 1, 0)$ *with* $\mathcal{F}^u(1, 1, 0) = 19$.

The leader get affected by the follower's reaction, and the optimistic method typically models a cooperative relationship between the $\mathcal{DM}$s. In this entire thesis we are concerned with the optimistic technique.

1.3.2 Various Reformulation Techniques

In this subsection, we will study the various ways of reformulating the BPPs into a single level program:

- *KKT reformulation*: The predominant methodology is to utilize the KKT *opt. conds.* to substitute the follower's problem.

$$
\begin{aligned}
\nabla_{a^l} \mathcal{F}^l(a^u, a^l) + \sum_{j=1}^{p_l} \mu_j \nabla_{a^l} \mathcal{G}_j^l(a^u, a^l) &= 0 \\
\mu \mathcal{G}^l(a^u, a^l) &= 0 \\
\mathcal{G}^l(a^u, a^l) &\leq 0 \\
\mu &\geq 0.
\end{aligned}
$$

The reformulated problem so obtained is a programming problem with equilibrium *cnstrs.* or complementary *cnstrs.* (MPCC).

$$\begin{aligned} \min \quad & \mathcal{F}^u(a^u, a^l) \\ \text{subject to} \quad & \nabla_{a^l}\mathcal{F}^l(a^u, a^l) + \sum_{j=1}^{p_l} \mu_j \nabla_{a^l}\mathcal{G}^l_j(a^u, a^l) = 0 \\ & \mu\mathcal{G}^l(a^u, a^l) = 0 \\ & \mathcal{G}^l(a^u, a^l) \leq 0 \\ & \mu \geq 0. \end{aligned}$$

For the two problems to be equivalent, the ll-problem must satisfy specific requirements. Firstly, the follower's problem should be convex, meaning that the obj. and cnstr. fns should be convex. Secondly, let a CQ holds for follower's problem at any parameter $a^u \in X$. Without the first condition, the *KKT. conds.* are just necessary but not sufficient hence one gets a more extensive feasible region [92]. It follows that the BPP cannot be equal to the relevant MPCC if the ll's problem is not convex. If the latter condition does not hold for all $(a^u, a^l) \in X \times S(a^u)$ then the *KKT. conds.* might not fulfil the follower's problem. Thus without this condition, it may happen that the MPCC does not have a solution while BPPs have a global optimal solution. Moreover, at any feasible point of the reformulated problem, the MFCQ is violated [138].

- *optimal value function reformulation*: This technique of reformulation substitutes follower's problem with its OVF. i.e

$$\begin{aligned} \min_{a^u, a^l} \quad & \mathcal{F}^u(a^u, a^l) \\ \text{subject to} \quad & \mathcal{F}^l(a^u, a^l) - V(a^u) \leq 0, \\ & \mathcal{G}^u(a^u, a^l) \leq 0, \\ & \mathcal{G}^l(a^u, a^l) \leq 0, \\ & a^u \in X, \;\; a^l \in \mathbb{R}^{n_b}. \end{aligned}$$

The drawback of this reformulation is that the OVF is non-differentiable usually even when all the considered functions are [39]. Thus, the single-level mathematical programming problem obtained via this approach is nonsmooth. Despite this, many researchers have used this reformulation. Both problems are equivalent in terms of local and global solutions.

- Ψ-*reformulation*: In [54], Gadhi and Dempe proposed a new technique to derive a one-level problem. Hiriart-Urruty [64,65] introduced a special scalarization function in optimization; with the aid of this function, Gadhi and Dempe formulated a new type of reformulation called Ψ-reformulation. Let $\Delta_C : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ characterized by

$$\Delta_C(a^p) = \begin{cases} -d(a^p,\ \mathbb{R}^{n_a} \setminus C) & \text{if } a^p \in C, \\ d(a^p,\ C) & \text{if } a^p \in \mathbb{R}^{n_a} \setminus C, \end{cases}$$

where

$$d(a^p,\ C) = \inf \Big\{ \|a^p - a^d\|,\ a^d \in C \Big\}.$$

Ψ-reformulation

$$\min_{a^u, a^l} \quad \mathcal{F}^u(a^u, a^l)$$

$$\text{subject to} \ : \begin{cases} \mathcal{G}_i^u(a^u, a^l) \leq 0 & i = \{1, 2, \ldots, p_u\}, \\ \mathcal{G}_j^l(a^u, a^l) \leq 0 & j = \{1, 2, \ldots, p_l\}, \\ \Psi(a^u, a^l) \leq 0, \\ (a^u, a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}, \end{cases}$$

where

$$\Psi(a^u, a^l) = \max_{z \in S(a^u)} \ \psi(a^u, a^l, z)$$

and

$$\psi(a^u, a^l, z) = \min \Big\{ \mathcal{F}^l(a^u, a^l) - \mathcal{F}^l(a^u, z), \\ -\Delta_{(-\mathbb{R}_+^{p_l})} \big(\mathcal{G}_1^l(a^u, z), \ldots, \mathcal{G}_{p_l}^l(a^u, z) \big) \Big\}.$$

Via the supposition on the closed and boundedness of the feasible region of the *ll*-program, the two problems are equivalent in terms of local and global solutions [54].

1.3.3 Algorithms to solve BPP

Minimizing the computational overhead of the follower evolutionary algorithm and reducing the required number of evaluations have been the essential vital pursuits in this domain, and a variety of works have been proposed using different techniques such as hybridization with local search [94], the quadratic approximation method [118], the use of surrogates [68], multi-parametric programming [8], and the use of multicriteria optimization principle [53]. The idea of single-level reduction to transform the multiobjective BPP into a simple single-level problem has achieved some results. However, this method often needs to meet some extremely harsh conditions, which are difficult to apply to practical problems. For example, based on the theoretical progress of establishing the relationship between BPP and MP, Ruuska and Miettinen [107] transformed the multiobjective BPP into a MP and solved it. However, this method is only applicable to optimistic multi-objective BPPs. Pieume et al. [103] show how to construct two artificial MPs so that any point which is Pareto efficient solution to the two problems is an efficient solution to the multiobjective BPP, and some necessary and sufficient conditions are given for the results to be applicable. He and Lv [61] use the KKT optimality condition to replace the *ll* problem of the class of multiobjective BPP, and the perturbed Fischer–Burmeister function is used to smooth the complementary condition. On this basis, a particle swarm optimization algorithm is used to solve the smooth MP. Sinha et al. [117] introduces an approximated set-valued mapping to obtain an approximate *ll* Pareto optimum solution set for any *ul* vector, significantly reducing the number of evaluations for *ll*. The procedure's utility has been emphasized by incorporating it in a hierarchical evolutionary framework. Deb and Sinha [35] used a hybrid evolutionary-local-search algorithm to solve a BPP with multiple conflicting objectives at both levels. Said et al. [111] study the bilevel combinatorial optimization problem. They reduced the computational cost through a multi-population strategy and migration mechanism. They introduced an index in the *ll* optimization to measure the performance of the *ll* non-dominated solution in the *ul* problem. Wang et al. [130] proposed a new effective evolutionary algorithm for a nonlinear BPP whose leader's problem is nondifferentiable and the follower's problem is nonconvex. Ma and Wang [88] designed an algorithm based on the human evolutionary model for solving nonlinear BPPs. The working principle of the algorithm is to feed back the optimal solution of the *ll* problem to the *ul*. Li et al. [85] used nested particle swarm optimization (PSO) to solve BPP. In [6], authors used an ant colony optimization to handle the *ul* and differential evolution to handle the *ll* in a transportation routing problem. Recently, Liu et al. [86] designed an evolutionary algorithm embedded with a surrogate model to solve the complex nonlinear BPP. Abo-Elnaga and Nasr [3] proposed an algorithm consisting of two nested artificial multi-objective algorithms. One algorithm is for the *ul* problem, and the other is for the *ll* problem. Also, the proposed algorithm is enriched with a k-means cluster scheme in two phases. Cai et al. [23] used an interaction matrix to represent the variable interactions. Based on the interaction matrix, the variables are divided into different groups and optimized collaboratively. Zhang et al. [142–144] adopted the framework of alternately executing fixed *ul* vectors for *ll* optimization and fixed *ll* vectors for *ul* optimization. It simplifies the optimization process of the BPP, reduces the computational cost to a certain extent, and changes the nested structure of the BPP. Therefore, the solution obtained by the algorithm is only a rough approximation of the optimal solution of the BPP, and the accuracy could be better.

1.3.3.1 Computational Challenges

Since the *ul* and *ll* problems are nested within one another, BPPs are NP-hard problems. BPP's computational complexity has been studied extensively, and various polynomial solutions to common combinatorial problems have been put forth. The NP-hardness of problems involving linear BPP was initially established by Jeroslow [70]. Because they are NP-hard and non-convex, linear BPPs are computationally expensive to solve. It is well recognized that a polynomial method to solve a linear BPP to global *opt.* is unlikely to be developed due to the problem's difficulty.
Despite these challenges, academics have continued to work on creating *opt. conds.* that can use the mathematical characteristics of BPP to solve particular kinds of problems. It should be no surprise that dual problems for nonlinear BPPs with traits comparable to those shown in the case of linear BPP are challenging to formulate. Dual information is difficult to extract from the most widely used primal solution techniques, in contrast to the case of linear BPPs.

1.3.4 Optimality Conditions for BPP

First-order $\mathcal{N}$. *opt. cond.* for the linear BPP were established by Bard [10] under the assumptions that $S(a^u)$ is singleton for every $a^u \in X$ and appropriate regularity conditions holds.

Theorem 1.2 [10] *Let $\mathcal{F}^u$, $\mathcal{F}^l$ and $\mathcal{G}^l$ be once continuously differentiable. Then a necessary condition that (a^{u^*}, a^{l^*}) solves BPP is that $\exists\ v \in \mathbb{R}$ and a $u \in \mathbb{R}^{n_b}$ such that the following system of cnstrs. is satisfied at this point:*

$$
\begin{aligned}
\nabla_{a^u}\mathcal{F}^u(a^u, a^l) + u\nabla_{a^u}\mathcal{G}^l(a^u, a^l) = 0,\\
\nabla_{a^l}\mathcal{F}^u(a^u, a^l) + u\nabla_{a^l}\mathcal{G}^l(a^u, a^l) + v\nabla_{a^l}\mathcal{F}^l(a^u, a^l) = 0,\\
\mathcal{F}^l(a^u, a^l) - \mathcal{F}^l(a^u, a_t) \geq 0, \quad \forall a_t \in S(a^u),\\
\mathcal{G}^l(a^u, a^l) \geq 0,\\
u\mathcal{G}^l(a^u, a^l) = 0,\\
u \geq 0, \quad v \geq 0.
\end{aligned}
$$

Zhang [141] expanded the conventional method to investigate the nonsmooth problem data under the semi-Lipschitz property and established existence and opt. cond. for the programs in the form of a graph set of the *ll*'s solution multifunction. Dempe [36] and Outrata [101] determined the *opt. conds.* for the scenario in which the *ll*-problem's collection of optimal points is a singleton set. Subsequently, Ye and Zhu [137] established *opt. conds.* for BPPs without the prerequisite that the *ll*-problem's solution set is a singleton or the *ul*-problem is convex. In [38], *opt. conds.* for a BPP are obtained using $\mathcal{B}.\mathcal{T}$ cone and $\mathcal{M}$.*cd*. Using CFs, Suneja and Kohli [124] developed $\mathcal{S}$. *opt. cond.* for a BPP. Bilevel programs with extremal value functions were taken into consideration by Aboussoror et al. [4], who then deduced the global *opt. cond.* In [136], *opt. cond.* for a non-smooth multi-objective BPP were given by combining the OVF and *KKT cond.* of the *ll*-problem in the cnstrs. Dempe and Gadhi [40] studied the *opt. cond.* for vector BPP with a variable ordering structure. In [41], a multi-objective BPP is considered, by using scalarization and optimal value reformulation under nonsmooth generalized Guignard CQ; the $\mathcal{N}$. *opt. cond.* in terms of CFs are developed. For optimistic BPPs involving smooth functions, Mehlitz and Zemkoho [90] have developed first and second-order $\mathcal{S}$. *opt. cond.*. Dempe et al. [108] considered a nonsmooth semivectorial BPP and obtained the *opt. conds.*. Recently, El Idrissi et al. [52] developed the *opt. conds.* in terms of directional USRCFs for a bilevel problem with all the considered functions being lower semicontinuous. Gadhi and Ohda [57] investigated a multi-objective BPP and obtained $\mathcal{N}$. *opt. cond.*s in terms of tangential sd.s.

1.3.5 Duality for BPP

Neumann [100] initially introduced *dlty.* in LPP in 1947. He designed the dual pair below and presented the association in the primal and the dual programs.

Primal problem

$$(PP) \qquad \begin{aligned} &\min z(a^p) = c^T a^p \\ &\text{subject to } \; Aa^p \geq d, \\ &\qquad\qquad\quad a^p \geq 0. \end{aligned}$$

Dual Problem

$$(DP) \qquad \begin{aligned} &\max w(a^d) = b^T a^d \\ &\text{subject to } \; A^T a^d \leq c, \\ &\qquad\qquad\quad a^d \geq 0. \end{aligned}$$

The dual of the dual problem in the case of LPP's turns out to be the primal problem. The concept of *dlty.* was expanded to MP problems in the late 1970s. In the linear case, Isermann [67] established multi-objective *dlty.* In MP using CQ, Bector et al. [14] and

Singh [112, 113] addressed the below stated MWD

$$(MWD) \qquad \max \quad \mathcal{F}(a^d)$$
$$\text{subject to } \lambda^T \nabla \mathcal{F}(a^d) + \mu^T \nabla \mathcal{G}(a^d) = 0,$$
$$\mu^T \mathcal{G}(a^d) \geq 0, \quad \lambda > 0, \ \mu \geq 0.$$

and established *dlty.* relations in terms of eff. sol. of (MP) and (D). For recent developments on this topic one can follow [5].
For the convex primal programme, Wolfe [134] provided *dlty.* results. Several dual programs have been defined in the literature for nonlinear situations. Contrary to the linear scenario, in general the dual of a dual may not always be primal.
Wolfe gave this dual:

$$(WD) \qquad \max \quad \mathcal{F}(a^d) + \mu^T \mathcal{G}(a^d)$$
$$\text{subject to} \quad \nabla \mathcal{F}(a^d) + \mu^T \nabla \mathcal{G}(a^d) = 0,$$
$$a^d \in X, \mu \geq 0.$$

and established the relationship between the two problems assuming that the obj. and cnstr. fns are convex. Mangasarian [89] made a point that argues that weaker convexity assumptions do not support these *dlty.* connections. Later, Mond and Weir [95] introduced a dual, which satisfy the *dlty.* theorems under the pseudoconvex and quasiconvex suppositions which are weaker assumptions as compared to the convexity assumptions.

$$(MWD) \qquad \max \quad \mathcal{F}(a^d)$$
$$\text{subject to} \quad \nabla \mathcal{F}(a^d) + \mu^T \nabla \mathcal{G}(a^d) = 0,$$
$$\mu^T \mathcal{G}(a^d) \geq 0,$$
$$a^d \in X, \ \mu \geq 0.$$

Suneja and Kohli [124] discovered *dlty.* results connecting the BPP to the WD and MWD. Under appropriate assumptions, Aboussoror et al. [4] got the Fenchel-Lagrange *dlty.* results. By expanding the strategy in [4], Wang and Zhang [128] established the *dlty.* theory for the multi-objective BPP. The MWD and $\mathcal{S}$. *opt. cond.* were recently developed by Gadhi et al. [55] utilizing the Ψ function to reformulate the multi-objective

BPP. The pair of primal and dual relations explored in [55] are:

$$(P) \qquad R^n_+ - \min_{a^u, a^l} \mathcal{F}^u(a^u, a^l) = (\mathcal{F}^u_1(a^u, a^l), \mathcal{F}^u_2(a^u, a^l), \ldots, \mathcal{F}^u_n(a^u, a^l))$$

$$\begin{aligned} \text{subject to} \quad & \mathcal{G}^u_j(a^u, a^l) \geq 0 \qquad j \in J, \\ & \mathcal{G}^l_s(a^u, a^l) \leq 0 \qquad s \in S, \\ & \Psi(a^u, a^l) \leq 0, \\ & (a^u, a^l) \in R^{n_a} \times R^{n_b}, \end{aligned}$$

where $\Psi(a^u, a^l) = \max_{z \in Y(a^u)} \psi(a^u, a^l, z)$

and

$$\psi(a^u, a^l, z) = \min \Big\{ \mathcal{F}^l(a^u, a^l) - \mathcal{F}^l(a^u, z), \\ -\Delta_{(-R^{p_l}_+)}(\mathcal{G}^l_1(a^u, z), \mathcal{G}^l_2(a^u, z), \ldots, \mathcal{G}_{p_l}(a^u, z)) \Big\}$$

$$(D) \qquad R^n_+ - \max_{v, \pi^*} \mathcal{F}^u(v) = (\mathcal{F}^u_1(v), \mathcal{F}^u_2(v), \ldots, \mathcal{F}^u_n(v))$$

$$\begin{aligned} \text{subject to} \quad & \alpha^*_j \mathcal{G}^u_j(v) \geq 0, \ \beta^*_s \mathcal{G}^l_s(v) \geq 0, \ \tau^* \Psi(v) \geq 0, \quad j \in J, \ s \in S, \\ & 0 \in \sum_{i=1}^{n} \gamma^*_i \partial^* \mathcal{F}^u_i(v) \ + \ \sum_{j=1}^{p_u} \alpha^*_j \partial^* \mathcal{G}^u_j(v) \\ & + \ \sum_{s=1}^{p_l} \beta^*_s \partial^* \mathcal{G}^l_s(v) \ + \ \tau^* \partial^* \Psi(v), \\ & \pi^* = (\gamma^*_1, \ldots, \gamma^*_n, \alpha^*_1, \ldots, \alpha^*_{p_u}, \beta^*_1, \ldots, \beta^*_{p_l}, \tau^*) \geq 0, \\ & (\gamma^*_1, \ldots, \gamma^*_n) \neq (0, \ldots, 0). \end{aligned}$$

In [121], a non-smooth multi-objective BPP with equilibrium cnstrs. is examined and the $\mathcal{N}$. *opt*. *conds*. are developed further the MWD and WD are investigated.

1.4 Summary of Thesis

This thesis seeks to investigate the *opt*. and *dlty*. *conds*. associated with certain BPP under the assumptions of generalized sds. and CFs. The findings are discussed in detail in Chapters 2 to 5. The opening Chapter (Chapter-1) is designated as an introductory, and the summary of the remaining Chapters is outlined below:
In Chapter 2, we investigate the below stated pair of multi-objective BPP with n-objs.

and WD.

$$(P) \qquad \mathbb{R}^n_+ - \min_{a^u,a^l} \mathcal{F}^u(a^u,a^l) = \Big\{\mathcal{F}^u_1(a^u,a^l), \mathcal{F}^u_2(a^u,a^l), \ldots, \mathcal{F}^u_n(a^u,a^l)\Big\}$$
$$\text{subject to } \mathcal{G}^u_j(a^u,a^l) \leq 0, \ \forall j \in J,$$
$$a^l \in \Lambda(a^u),$$

where for each $a^u \in \mathbb{R}^{n_a}$, $\Lambda(a^u)$ is the collection of optimal solutions of the below stated parametric program

$$(P_{a^u}) \qquad \min_{a^l} \mathcal{F}^l(a^u,a^l)$$
$$\text{subject to } \mathcal{G}^l_s(a^u,a^l) \leq 0, \ \forall s \in S.$$

The Ψ-reformulated problem is characterized as:

$$(RP) \qquad \mathbb{R}^n_+ - \min_{a^u,a^l} \mathcal{F}^u(a^u,a^l) = \Big\{\mathcal{F}^u_1(a^u,a^l), \mathcal{F}^u_2(a^u,a^l), \ldots, \mathcal{F}^u_n(a^u,a^l)\Big\}$$
$$\text{subject to } : \begin{cases} \mathcal{G}^u_j(a^u,a^l) \leq 0 & j \in J, \\ \mathcal{G}^l_s(a^u,a^l) \leq 0 & s \in S, \\ \Psi(a^u,a^l) \leq 0, \\ (a^u,a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}. \end{cases}$$

Following is the corresponding WD:

$$(WD) \qquad \mathbb{R}^n_+ - \max_{v,\pi^*} \phi(v,\pi^*) = \Big\{\phi_1(v,\pi^*), \ldots, \phi_n(v,\pi^*)\Big\}$$
$$\text{subject to } \quad 0 \in \sum_{i=1}^{n} \lambda^*_i \partial^* \mathcal{F}^u_i(v) \ + \ \sum_{j=1}^{p_u} \mu^*_j \partial^* \mathcal{G}^u_j(v) \ + \ \sum_{s=1}^{p_l} \nu^*_s \partial^* \mathcal{G}^l_s(v) \ + \ \eta^* \partial^* \Psi(v),$$
$$\pi^* = (\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n, \ \mu^*_1, \mu^*_2, \ldots, \mu^*_{p_u}, \ \nu^*_1, \nu^*_2, \ldots, \ \nu^*_{p_l}, \ \eta^*) \geq 0,$$
$$(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n) \neq (0, \ldots, 0), \ \sum_{i=1}^{n} \lambda^*_i = 1,$$

where,

$$\phi_i(v,\pi^*) = \mathcal{F}^u_i(v) \ + \ \sum_{j=1}^{p_u} \mu^*_j \mathcal{G}^u_j(v) \ + \ \sum_{s=1}^{p_l} \nu^*_s \mathcal{G}^l_s(v) \ + \ \eta^* \Psi(v).$$

Here, $\mathcal{F}^u_i$, $\mathcal{F}^l$, $\mathcal{G}^u_j$, $\mathcal{G}^l_s$ $\mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}$, for each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p_u$ and $s = 1, 2, \ldots, p_l$ are $l\mathcal{L}$ continuous functions.

The standard *dlty.* results are proven under the nonsmooth ACQ and ∂^*-convex assumptions. In order to demonstrate our Weak *dlty.* Theorem, we use a non-trivial case. Further, we examined a bilevel framework that incorporates multiple *obj.* fns in Section 2.4. For this problem, we presented $\mathcal{S}$. *opt. cond.* and proposed a MWD pairing to this model and developed relevant *dlty.* theorems under ∂^*-pseudoconvex and ∂^*-quasiconvex assumptions.

In Chapter 3, we examined a BPP with multiple objectives at both levels. With the aid of kth-objective weighted constraint scalarization and OVF reformulation, the problem is converted to a single-level mathematical programming problem.

$$(Q): \begin{cases} \mathbb{R}^l_+ - \min\limits_{a^u, a^l} \mathcal{F}^u(a^u,\ a^l) = (\mathcal{F}^u_1, \mathcal{F}^u_2, \ldots, \mathcal{F}^u_l)(a^u,\ a^l) \\ \text{subject to} \begin{cases} a^u \in X,\ a^l \in \mathfrak{L}_{wef}(a^u), \\ (a^u,\ a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}, \end{cases} \end{cases}$$

where, for every $a^u \in \mathbb{R}^{n_a}$, $\mathfrak{L}_{wef}(a^u)$ is the collection of weak eff. sol. of the below characterized multi-objective optimization problem

$$(Q_{a^u}): \begin{cases} \mathbb{R}^k_+ - \min\limits_{a^l} \mathcal{F}^l(a^u,\ a^l) = (\mathcal{F}^l_1, \mathcal{F}^l_2, \ldots, \mathcal{F}^l_k)(a^u,\ a^l) \\ \text{subject to} \begin{cases} \mathcal{G}^l_j(a^u,\ a^l) \leq 0,\ \forall j \in J, \\ (a^u,\ a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}. \end{cases} \end{cases}$$

Here, $\mathcal{F}^u_1, \mathcal{F}^u_2, \ldots, \mathcal{F}^u_l,\ \mathcal{F}^l_1, \mathcal{F}^l_2, \ldots, \mathcal{F}^k_l,\ \ \mathcal{G}^l_1, \mathcal{G}^l_2, \ldots, \mathcal{G}^l_{p_l}$, are $l\mathcal{L}$ continuous functions from $\mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}$ and $\mathcal{G}^u_1, \mathcal{G}^u_2, \ldots, \mathcal{G}^u_{p_u}$, are $l\mathcal{L}$ continuous functions from $\mathbb{R}^{n_a} \to \mathbb{R}$.

The $\mathcal{N}$. *opt. cond.*s are obtained and an illustrative example is given to validate our result.

In Chapter 4, we have considered a BPP with some data uncertainity attached to its *ul* cnstrs. To deal with uncertainity we have used robust counterpart approach as the sets to which these uncertainities belong are known.

$$\begin{aligned} (Q) \qquad & \min_{a^u,\ a^l} \mathcal{F}^u(a^u, a^l) \\ & \text{subject to } \ \mathcal{G}^u_i(a^u, \nu_i) \leq 0, \quad \forall i \in I, \\ & \qquad a^l \in \Upsilon(a^u), \end{aligned}$$

where for some sequentially compact topological space Ω_i, $\nu_i \in \Omega_i$ is an uncertain parameter. For each $a^u \in \mathbb{R}^{n_a}$ the parametric optimization problem (Q_{a^u}) has the set of optimal solutions $\Upsilon(a^u)$

$$\begin{aligned} (Q_{a^u}) \qquad & \min_{a^l} \mathcal{F}^l(a^u, a^l) \\ & \text{subject to } \ \mathcal{G}^l_j(a^u, a^l) \leq 0, \quad \forall j \in J, \end{aligned}$$

The robust counterpart of (Q) is:

$$(RP') \qquad \min_{a^u, a^l} \mathcal{F}^u(a^u, a^l)$$
$$\text{subject to } \ \mathcal{G}^u_i(a^u, \nu_i) \leq 0, \quad \forall \nu_i \in \Omega_i, \ \forall i \in I,$$
$$a^l \in \Upsilon(a^u).$$

And the single level reformulated problem is:

$$(RBPP) \qquad \min_{a^u, a^l} \mathcal{F}^u(a^u, a^l)$$
$$\text{subject to } \ \mathcal{G}^u_i(a^u, \nu_i) \leq 0, \quad \forall \nu_i \in \Omega_i, \ \forall \ i \in I,$$
$$\mathcal{G}^l_j(a^u, a^l) \leq 0, \qquad \forall j \in J,$$
$$\mathcal{F}^l(a^u, a^l) \leq \varphi(a^u),$$

where,

$$\varphi(a^u) = \inf_{a^l}\{\mathcal{F}^l(a^u, a^l) : \ \mathcal{G}^l_j(a^u, a^l) \leq 0, \qquad \forall j \in J\}.$$

The $\mathcal{N}$. *opt. cond.* is obtained and an example is given to show its working. Moreover, in Section 4.6 following robust bilevel MWD dual is formulated corresponding to the considered problem.

$$(RBMWD) \qquad \max \quad \mathcal{F}^u(u, \ v)$$
$$\text{subject to} \qquad 0 \in conv\ \partial^*\mathcal{F}^u(u, \ v) + \sum_{i \in I_n} \alpha_i \ conv\ \partial_{a^u}\mathcal{G}^u_i(u, \ \nu_i) \times \{0\}$$
$$+ \sum_{j \in J_m} \beta_j \ conv\ \partial^*\mathcal{G}^l_j(u, \ v) \ + \ \gamma\Big(\partial\mathcal{F}^l(u, v)$$
$$-\partial\varphi(u) \times \{0\}\Big),$$
$$\mathcal{F}^l(u, v) - \varphi(u) \geq 0,$$
$$\alpha_i \mathcal{G}^u_i(u, \nu_i) \geq 0, \qquad \nu_i \in \Omega_i,$$
$$\beta_j \mathcal{G}^l_j(u, \ v) \geq 0,$$
$$\gamma \geq 0, \quad \alpha_i \geq 0, \quad i \in I_n, \quad \beta_j \geq 0, \quad j \in J_m.$$

The usual *dlty.* results under ∂^*-pseudoconvex and ∂^*-quasiconvex assumptions are obtained. Further in Section 4.7 a multi-objective robust BPP is considered and $\mathcal{N}$. *opt. conds.* corresponding to it is proved.

In Chapter 5, we have formulated a MWD corresponding to a multi-objective fractional

BPP given as:

$$(MOFBLPP) \qquad \min \qquad \Big(\frac{\mathcal{F}_1^u(a^u,a^l)}{\mathcal{F}_1'^u(a^u,a^l)}, \frac{\mathcal{F}_2^u(a^u,a^l)}{\mathcal{F}_2'^u(a^u,a^l)}, \ldots, \frac{\mathcal{F}_p^u(a^u,a^l)}{\mathcal{F}_p'^u(a^u,a^l)}\Big)$$
$$\text{subject to} \qquad \mathcal{G}_i^u(a^u,a^l) \leq 0, \quad i \in I, \; a^l \in \psi(a^u),$$

where, for each $a^u \in \mathbb{R}^{n_a}$, $\psi(a^u)$ is the collection of the optimal solutions to the below stated program

$$\begin{aligned} \min \qquad & \mathcal{F}^l(a^u,a^l) \\ \text{subject to} \qquad & \mathcal{G}_j^l(a^u,a^l) \leq 0, \quad j \in J. \end{aligned}$$

Associated MWD is characterized as:

$$\begin{aligned} (MWD) \quad \max \quad & \phi(c_1,c_2) = \Big(\frac{\mathcal{F}_1^u(c_1,c_2)}{\mathcal{F}_1'^u(c_1,c_2)}, \ldots, \frac{\mathcal{F}_p^u(c_1,c_2)}{\mathcal{F}_p'^u(c_1,c_2)}\Big) \\ \text{subject to} \quad & 0 \in \sum_{k=1}^{p_u} \lambda_k \mathrm{conv}(\partial^*\mathcal{F}_k^u(c_1,c_2) - \phi_k \partial^* \mathcal{F}_k'^u(c_1,c_2)) \\ & \quad + \sum_{i=1}^{m_1} \alpha_i \mathrm{conv}\partial^*\mathcal{G}_i^u(c_1,c_2) + \sum_{j=1}^{m_2} \beta_j \mathrm{conv}\partial^*\mathcal{G}_j^l(c_1,c_2) \\ & \quad + \tau \mathrm{conv}(\partial^*\mathcal{F}^l(c_1,c_2) - \partial^* V(c_1) \times \{0\}), \\ & \alpha_i \mathcal{G}_i^u(c_1,c_2) \geq 0, \quad \beta_j \mathcal{G}_j^l(c_1,c_2) \geq 0, \\ & \mathcal{F}^l(c_1,c_2) - V(c_1) \geq 0, \\ & \Big(\lambda_1, \ldots, \lambda_p, \alpha_1, \ldots, \alpha_{m_1}, \beta_1 \ldots, \beta_{m_2}, \tau\Big) \geq 0, \\ & (\lambda_1, \ldots, \lambda_p) \neq (0, \ldots, 0), \end{aligned}$$

where $\nu_k = \phi_k(c_1,c_2) = \dfrac{\mathcal{F}_k^u(c_1,c_2)}{\mathcal{F}_k'^u(c_1,c_2)}$, $k = 1, 2, \ldots, p$.
In Section 5.3 the relationship between the two models under ∂^*-pseudoconvex and ∂^*-quasiconvex assumptions in terms of weak and strong *dlty.* theorems are obtained. An example is given to validate the weak *dlty.* theorem.

Chapter 2

Sufficient optimality condition and duality for bilevel programming problems under generalized convexity[1]

2.1 Introduction

Multi-objective optimization is an optimization problem in which many frequently competing objectives must be maximized/minimized simultaneously. In classical optimization problems with certain restrictions, only one objective function is minimized or maximized. Yet, in real-world scenarios, numerous objectives must be optimized simultaneously. Engineering, economics, finance, and environmental science have multi-objective optimization dilemmas. For example, a product designer in engineering may need to optimize his product's cost and performance. Investors may wish to maximize their revenues while reducing their investment risks in finance.

Babahadda and Gadhi [9] used CFs and designed the $\mathcal{N}$. *opt. cond.* for BPP. Later on, for the model considered by Babahadda and Gadhi [9], Suneja and Kohli [124] developed $\mathcal{S}$. *opt. cond.*, formulated the WD and MWD corresponding to it, and proved the duality results. Gadhi and Dempe [54] considered a multi-objective BPP and developed the $\mathcal{N}$. *opt. cond.* in terms of $\mathcal{C}.sd$ by employing a special scalarization function developed in optimization by Hiriart-Urruty [65] to reformulate BPP into a single-level mathematical program. Later on, Lafhim et al. [82] called this reformulation Ψ-reformulation. Lafhim et al. [82] used Ψ-reformulation and established the KKT-type $\mathcal{N}$. *opt. cond.* in terms of CFs. Further, Gadhi et al. [55] developed $\mathcal{S}$. *opt. cond.*, established MWD for the same problem, and produced the duality results. We have developed the WD for this model and proved the relationship between the two models via weak and strong duality theorems. Moreover, we have given an example to validate the weak duality theorem. Dempe et al. [108] considered a multi-objective BPP where both levels have multiple objectives and

[1]First half part of this chapter has appeared as "*Duality Results in terms of convexifactors for a bilevel multiobjective optimization problem*", FILOMAT, 38 (2024) 2015-2022 and other half part as "*Sufficiency and duality of multi-objective bilevel programming problem under Guignard constraint qualification*", Maejo International Journal of Science and Technology, 16 (2022) 13-24.

used OVF reformulation and a scalarization technique to obtain $\mathcal{N}$. *opt. cond.*. We have developed $\mathcal{S}$. *opt. cond.* for the same model, obtained a MWD, and proved the duality results.

The present chapter is structured in the following manner. In Section 2.2, a bilevel model and Ψ-reformulation is discussed to reform the considered program into a single level program and by using *opt. conds.* the weak eff. sol. of a bilevel problem is obtained. In Section 2.3, WD is formulated and duality results are obtained. Section 2.4 is dedicated to the formulation of a single level problem of a multi-objective BPP via scalarization technique and optimal value reformulation. In Section 2.5, $\mathcal{S}$. *opt. cond.* for this problem are established. Further, in Section 2.6 the MWD is defined, and duality theorems are proved.

2.2 Bilevel Problem with upper-level Multi-objective

In this Section, a bilevel model has been considered whose leader $\mathcal{DM}$ has multiple objectives to optimize, and the follower $\mathcal{DM}$ has a single goal. At both optimization levels, all *obj.* and *cnstr.* fns are $l\mathcal{L}$. The bilevel model is:

$$(P) \qquad \mathbb{R}^n_+ - \min_{a^u, a^l} \mathcal{F}^u(a^u, a^l) = \Big\{\mathcal{F}^u_1(a^u, a^l), \mathcal{F}^u_2(a^u, a^l), \ldots, \mathcal{F}^u_n(a^u, a^l)\Big\}$$
$$\text{subject to } \mathcal{G}^u_j(a^u, a^l) \leq 0, \ \forall j \in J$$
$$a^l \in \Lambda(a^u),$$

where for every $a^u \in \mathbb{R}^{n_u}$, $\Lambda(a^u)$ is the collection of optimal solutions of the below stated program

$$(P_{a^u}) \qquad \min_{a^l} \mathcal{F}^l(a^u, a^l)$$
$$\text{subject to } \ \mathcal{G}^l_s(a^u, a^l) \leq 0, \ \forall s \in S.$$

Here, $\mathcal{F}^l : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, $\mathcal{G}^l_s : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, $s \in S = \{1, 2, \ldots, p_l\}$, $\mathcal{G}^u_j : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, $j \in J = \{1, 2, \ldots, p_u\}$, $\mathcal{F}^u_i : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, $i \in I = \{1, 2, \ldots, n\}$ are given $l\mathcal{L}$ continuous functions and $n_u, n_l, p_u, p_l, n \geq 1$ are integers.

Definition 2.1 [82] *The nonsmooth ACQ retains at* $(\bar{a}^u, \bar{a}^l) \in E$ *with respect to (UCFs)* $\partial^*\mathcal{G}^u_j(\bar{a}^u, \bar{a}^l);\ j \in J_0(\bar{a}^u, \bar{a}^l),\ \partial^*\mathcal{G}^l_s(\bar{a}^u, \bar{a}^l);\ s \in S_0(\bar{a}^u, \bar{a}^l)$ *and* $\partial^*\Psi(\bar{a}^u, \bar{a}^l)$ *if*

$$[C(\bar{a}^u, \bar{a}^l)]^o \subseteq K(E, (\bar{a}^u, \bar{a}^l)),$$

where

$$E = \{(a^u, a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} :\ G_j(a^u, a^l) \leq 0,\ \forall j \in J \ \textit{and}\ a^l \in \Lambda(a^u)\}$$

$$J_0(\bar{a}^u,\ \bar{a}^l) = \{j \in J :\ \mathcal{G}^u_j(\bar{a}^u,\ \bar{a}^l) = 0\},\ S_0(\bar{a}^u,\ \bar{a}^l) = \{s \in S :\ \mathcal{G}^l_s(\bar{a}^u,\ \bar{a}^l) = 0\}$$

and

$$C(\bar{a}^u,\ \bar{a}^l) = \left(\bigcup_{j \in J_0(\bar{a}^u,\ \bar{a}^l)} \partial^* \mathcal{G}^u_j(\bar{a}^u,\ \bar{a}^l) \right) \cup \left(\bigcup_{s \in S_0(\bar{a}^u,\ \bar{a}^l)} \partial^* \mathcal{G}^l_s(\bar{a}^u,\ \bar{a}^l) \right) \cup \partial^* \Psi(\bar{a}^u,\ \bar{a}^l).$$

Here, $K(E,\ (\bar{a}^u,\ \bar{a}^l))$ notifies the $\mathcal{B}.\mathcal{T}$ cone to E at $(\bar{a}^u,\ \bar{a}^l)$.

2.2.1 Ψ- Reformulation of (P)

To develop the WD of (P) firstly it's reframed as a sole level problem. As per [54], the reframed program (RP) corresponding to (P) is given as follows:
Let $a^u \in \mathbb{R}^{n_u}$ and let

$$Y(a^u) = \{a^l \in \mathbb{R}^{n_l} :\ \mathcal{G}^l_s(a^u, a^l) \leq 0,\ \ \forall s \in S\}$$

be the collection of all the points that satisfies all the follower's $cnstrs$. of (P_{a^u}).

$$(RP) \qquad \mathbb{R}^n_+ - \min_{a^u, a^l} \mathcal{F}^u(a^u, a^l) = \left\{ \mathcal{F}^u_1(a^u, a^l), \mathcal{F}^u_2(a^u, a^l), \ldots, \mathcal{F}^u_n(a^u, a^l) \right\}$$

$$\text{subject to } : \begin{cases} \mathcal{G}^u_j(a^u, a^l) \leq 0 & j \in J, \\ \mathcal{G}^l_s(a^u, a^l) \leq 0 & s \in S, \\ \Psi(a^u, a^l) \leq 0, \\ (a^u, a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}, \end{cases}$$

where

$$\Psi(a^u, a^l) = \max_{z \in Y(a^u)} \psi(a^u,\ a^l,\ z)$$

and

$$\psi(a^u,\ a^l,\ z) = \min \left\{ \mathcal{F}^l(a^u, a^l) - \mathcal{F}^l(a^u,\ z), -\Delta_{(-\mathbb{R}^{p_l}_+)} \big(\mathcal{G}^l_1(a^u,\ z), \ldots, \mathcal{G}^l_{p_l}(a^u,\ z) \big) \right\}.$$

Taking $a^u \in \mathbb{R}^{n_u}$ and suppose $Y(a^u)$ is closed and bounded.

$$V(a^u) = \inf_{a^l} \{\mathcal{F}^l(a^u, a^l) : \mathcal{G}^l_s(a^u, a^l) \leq 0,\ \ \forall s \in S\}.$$

represents the OVF corresponding to the follower program of (P_{a^u}).

Lemma 2.1 [55]

$$\left\{\begin{array}{c}(a^u,a^l)\in\mathbb{R}^{n_u}\times\mathbb{R}^{n_l}:\\ a^l\in Y(a^u)\ and\ \mathcal{F}^l(a^u,a^l)-V(a^u)<0\end{array}\right\}=\left\{\begin{array}{c}(a^u,a^l)\in\mathbb{R}^{n_u}\times\mathbb{R}^{n_l}:\\ a^l\in Y(a^u)\ and\ \Psi(a^u,a^l)<0\end{array}\right\}=\emptyset.$$

Lemma 2.2 [55]

$$\left\{\begin{array}{c}(a^u,a^l)\in\mathbb{R}^{n_u}\times\mathbb{R}^{n_l}:\\ a^l\in Y(a^u)\ and\ \mathcal{F}^l(a^u,a^l)-V(a^u)=0\end{array}\right\}=\left\{\begin{array}{c}(a^u,a^l)\in\mathbb{R}^{n_u}\times\mathbb{R}^{n_l}:\\ a^l\in Y(a^u)\ and\ \Psi(a^u,a^l)=0\end{array}\right\}.$$

"

Theorem 2.1 (Necessary optimality condition) [55] *Let $\bar{u}=(\bar{a}^u,\ \bar{a}^l)\in E$ be a local weak eff. sol. of (P). Assume that $\mathcal{F}_i^u;\ i\in I$ admit bounded* USRCF $\partial^*\mathcal{F}_i^u(\bar{u})$ *at $\bar{u}$, $\mathcal{G}_j^u;\ j\in J,\ \mathcal{G}_s^l;\ s\in S$, admit* UCFs $\partial^*\mathcal{G}_j^u(\bar{u}),\ \partial^*\mathcal{G}_s^l(\bar{u})$, *respectively at $\bar{u}$. If the nonsmooth ACQ holds at $\bar{u}$, then there exists $\bar{\pi}^*=(\lambda^*,\ \mu^*,\ \nu^*,\ \eta^*)\ \in\mathbb{R}_+^{\bar{n}},\ \lambda^*\neq 0_{\mathbb{R}^n}$,*

$$\begin{aligned}0\in\sum_{i=1}^{n}\lambda_i^*\partial^*\mathcal{F}_i^u(\bar{u})\ +\ \sum_{j=1}^{p_u}\mu_j^*\partial^*\mathcal{G}_j^u(\bar{u})\ +\ \sum_{s=1}^{p_l}\nu_s^*\partial^*\mathcal{G}_s^l(\bar{u})\ +\ \eta^*\partial^*\Psi(\bar{u})\\ \mu_j^*\mathcal{G}_j^u(\bar{u})=0,\ \nu_s^*\mathcal{G}_s^l(\bar{u})=0,\qquad j\in J,\ s\in S.\end{aligned}\tag{2.1}$$

Theorem 2.2 (Sufficient optimality condition) [55] *Let $\bar{u}\in E$ be a feasible solution of (P). Suppose that $\mathcal{F}_i^u,\ i\in I,\ \mathcal{G}_j^u,\ j\in J_0(\bar{u}),\ \mathcal{G}_s^l,\ s\in S_0(\bar{u})$, and Ψ are ∂^*-convex at $\bar{u}$ and that there exists $\bar{\pi}^*=(\lambda^*,\ \mu^*,\ \nu^*,\ \eta^*)\in\mathbb{R}_+^{\bar{n}},\ \lambda^*\neq 0_{\mathbb{R}^n}$, satisfying* (2.1). *Then, $\bar{u}$ is a weak eff. sol. of (P).*

"

Example 2.1 *We will look at the subsequent example of a multi-objective BPP:*

$$(P):\left\{\begin{array}{l}\mathbb{R}_+^2-\min\limits_{a^u,a^l}\ \mathcal{F}^u(a^u,a^l)=\left\{a^{u2}-a^l,\ a^u+a^l+1\right\}\\ \quad subject\ to:\left\{\begin{array}{l}\mathcal{G}_1^u(a^u,a^l)=-a^u-2a^l\leq 0,\\ \mathcal{G}_2^u(a^u,a^l)=-a^l\leq 0,\\ a^l\in\Lambda(a^u),\end{array}\right.\end{array}\right.$$

where for every $a^u\in\mathbb{R},\ \Lambda(a^u)$ is the collection of optimal solutions of the below stated program

$$(P_{a^u}):\left\{\begin{array}{l}\min\limits_{y}\ \mathcal{F}^l(a^u,a^l)=|a^u|+{a^l}^2\\ subject\ to:\mathcal{G}_1^l(a^u,a^l)=a^l\leq 0.\end{array}\right.$$

We have,

$$\mathcal{F}_1^u(a^u,a^l)=a_2^u-a^l\ and\ \mathcal{F}_2^u(a^u,a^l)=a^u+a^l+1.$$

We claim that $\bar{u}=(0,0)\in E$, with

$$\Lambda(a^u)=\{0\},\ E=\mathbb{R}_+\times\{0\},\ J_0(\bar{u})=\{1,2\},\ S_0(\bar{u})=\{1\}$$

and

$$\Psi(a^u, a^l) = \min\{a^l_2,\ 0\}.$$

- *The collections* $\partial^*\mathcal{F}^u_1(0,0) = \{(0,-1)\}$ *and* $\partial^*\mathcal{F}^u_2(0,0) = \{(1,1)\}$ *are bounded USRC of* $\mathcal{F}^u_1$ *and* $\mathcal{F}^u_2$*. Further,* $\mathcal{F}^u_1$ *and* $\mathcal{F}^u_2$ *are* ∂^**-convex functions at* $(0,0)$*.*

- *The collections* $\partial^*\mathcal{G}^u_1(0,0) = \{(-1,-2)\},\ \partial^*\mathcal{G}^u_2(0,0) = \{(0,-1)\},\ \partial^*\mathcal{G}^l_1(0,0) = \{(0,1)\}$ *and* $\partial^*\Psi(0,0) = \{(0,0)\}$ *are UCF of* $\mathcal{G}^u_1,\ \mathcal{G}^u_2,\ \mathcal{G}^l_1$ *and* Ψ*. Furthermore,* $\mathcal{G}^u_1,\ \mathcal{G}^u_2,\ \mathcal{G}^l_1$ *and* Ψ *are* ∂^**-convex functions at* $(0,0)$*.*

- *The nonsmooth ACQ retains at* $(0,0)$*. As*

  $$C(0,0) = \{(-1,-2),\ (0,-1),\ (0,1),\ (0,0)\} \text{ and } E = \mathbb{R}_+ \times \{0\}$$

  we obtain,

  $$[C(\bar{u})]^\circ = \mathbb{R}^+ \times \{0\} = K(E,\ \bar{u}).$$

- *For* $\pi^* = \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},\ 1\right)$*, the condition (2.1) is satisfied. Indeed,*

  $$\frac{1}{2}(0,-1) + \frac{1}{2}(1,1) + \frac{1}{2}(-1,-2) + \frac{1}{2}(0,-1) + \frac{3}{2}(0,1) + 1(0,0) = (0,0)$$

  by Theorem 2.2, it is clear that, for (P)*,* $(0,0)$ *is a weak eff. sol. .*

2.3 Wolfe Dual

Let $\bar{n} = n + p_u + p_l + 1$, $\mathcal{F}^u_i;\ i \in I$ admit bounded USRCF $\partial^*\mathcal{F}^u_i(v)$ at $v \in \mathbb{R}^{n_1+n_2}$ and $\mathcal{G}^u_j,\ j \in J,\ \mathcal{G}^l_s,\ s \in S$, admit UCFs $\partial^*\mathcal{G}^u_j(.)$ and $\partial^*\mathcal{G}^l_s(.)$ at v. We formulate the Wolfe dual program (WD) and prove *dlty.* results for (P) and (WD). (WD) of (P) is as follows:

$$(WD)\qquad \mathbb{R}^n_+ - \max_{v,\pi^*} \phi(v,\pi^*) = \Big\{\phi_1(v,\pi^*), \ldots, \phi_n(v,\pi^*)\Big\}$$

$$\text{subject to}\quad 0 \in \sum_{i=1}^{n} \lambda^*_i \partial^*\mathcal{F}^u_i(v) + \sum_{j=1}^{p_u} \mu^*_j \partial^*\mathcal{G}^u_j(v) + \sum_{s=1}^{p_l} \nu^*_s \partial^*\mathcal{G}^l_s(v) + \eta^*\partial^*\Psi(v),$$

$$\pi^* = (\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n,\ \mu^*_1, \mu^*_2, \ldots, \mu^*_{p_u},\ \nu^*_1, \nu^*_2, \ldots,\ \nu^*_{p_l},\ \eta^*) \geq 0,$$

$$(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n) \neq (0, \ldots, 0),\ \sum_{i=1}^{n} \lambda^*_i = 1,$$

where,

$$\phi_i(v,\pi^*) = \mathcal{F}_i^u(v) \ + \ \sum_{j=1}^{p_u} \mu_j^* \mathcal{G}_j^u(v) \ + \ \sum_{s=1}^{p_l} \nu_s^* \mathcal{G}_s^l(v) \ + \ \eta^* \Psi(v).$$

Remark 2.1 *According to the assumptions stated in Theorem 2.1, it may be concluded that the set $\tilde{E}$, which represents all the feasible points of the program (WD), is not empty.*

Theorem 2.3 (Weak Duality) *Let u be feasible for (P) and for any feasible solution $(v,\ \pi^*)$ of (WD), such that $\mathcal{F}_i^u,\ i \in I,\ \mathcal{G}_j^u,\ j \in J_0(u),\ \mathcal{G}_s^l,\ s \in S_0(u)$ and Ψ are ∂^*-convex at v, then $\mathcal{F}^u(u) \nleq \phi(v,\ \pi^*)$.*

Proof. Let, on the contrary $\mathcal{F}^u(u) \leq \phi(v,\ \pi^*)$
that is,

$$\Big\{\mathcal{F}_1^u(u), \mathcal{F}_2^u(u), \ldots, \mathcal{F}_n^u(u)\Big\} \leq \Big\{\phi_1(v,\ \pi^*), \phi_2(v,\ \pi^*), \ldots, \phi_n(v,\ \pi^*)\Big\}$$

Since $(\lambda_1^*, \lambda_2^*,\ \ldots,\ \lambda_n^*) \geq 0$ and $(\lambda_1^*, \lambda_2^*,\ \ldots,\ \lambda_n^*) \neq 0$, we obtain,

$$\sum_{i=1}^{n} \lambda_i^* \mathcal{F}_i^u(u) \leq \sum_{i=1}^{n} \lambda_i^* \phi_i(v,\ \pi^*). \tag{2.2}$$

Since v is feasible for dual so,

$$0 \in \sum_{i=1}^{n} \lambda_i^* \partial^* \mathcal{F}_i^u(v) \ + \ \sum_{j=1}^{p_u} \mu_j^* \partial^* \mathcal{G}_j^u(v) \ + \ \sum_{s=1}^{p_l} \nu_s^* \partial^* \mathcal{G}_s^l(v) \ + \ \eta^* \partial^* \Psi(v).$$

Therefore there exist $\xi_i \in \partial^* \mathcal{F}_i^u(v),\ \eta_j \in \partial^* \mathcal{G}_j^u(v),\ \theta_s \in \partial^* \mathcal{G}_s^l(v)$ and $\chi \in \partial^* \Psi(v)$, such that,

$$0 = \sum_{i=1}^{n} \lambda_i^* \xi_i \ + \ \sum_{j=1}^{p_u} \mu_j^* \eta_j \ + \ \sum_{s=1}^{p_l} \nu_s^* \theta_s \ + \ \eta^* \chi, \tag{2.3}$$

Since $\mathcal{F}_i^u,\ i \in I$ is ∂^*-convex at v, thus
$\langle \xi_i,\ u - v \rangle \leq \mathcal{F}_i^u(u) - \mathcal{F}_i^u(v), \quad \forall i \in I$
for $\lambda_i^* \geq 0$ and $\lambda_i^* \neq 0$, we have,
$\lambda_i^* (\mathcal{F}_i^u(u) - \mathcal{F}_i^u(v)) \geq \langle \lambda_i^* \xi_i,\ u - v \rangle, \quad \forall i \in I$. Therefore,

$$\sum_{i=1}^{n} \lambda_i^* \mathcal{F}_i^u(u) \geq \sum_{i=1}^{n} \lambda_i^* \mathcal{F}_i^u(v) + \Big\langle \sum_{i=1}^{n} \lambda_i^* \xi_i,\ u - v \Big\rangle.$$

From (2.3)

$$\sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(u) \geq \sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(v) + \Big\langle -\Big(\sum_{j=1}^{p_u}\mu_j^*\eta_j \ + \ \sum_{s=1}^{p_l}\nu_s^*\theta_s \ + \ \eta^*\chi\Big), \ u-v\Big\rangle,$$

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(u) \geq &\sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(v) - \Big\langle \sum_{j=1}^{p_u}\mu_j^*\eta_j, \ u-v\Big\rangle \\ &- \Big\langle \sum_{s=1}^{p_l}\nu_s^*\theta_s, \ u-v\Big\rangle \ - \ \langle \eta^*\chi, \ u-v\rangle.\end{aligned} \tag{2.4}$$

Since $\mathcal{G}_j^u; \ j \in J_0(u), \ \mathcal{G}_s^l; \ s \in S_0(u)$ and Ψ are ∂^*-convex at v and $\mu_j^* \geq 0, \ \nu_s^* \geq 0$ and $\eta^* \geq 0$, we have,

$$\begin{aligned}\sum_{j=1}^{p_u}\mu_j^*\big(\mathcal{G}_j^u(u) - \mathcal{G}_j^u(v)\big) &\geq \Big\langle \sum_{j=1}^{p_u}\mu_j^*\eta_j, \ u-v\Big\rangle, \quad \forall j \in J_0(\bar{u}), \\ \sum_{s=1}^{p_l}\nu_s^*\big(\mathcal{G}_s^l(u) - \mathcal{G}_s^l(v)\big) &\geq \Big\langle \sum_{s=1}^{p_l}\nu_s^*\theta_s, \ u-v\Big\rangle, \quad \forall s \in S_0(\bar{u}), \\ \eta^*\big(\Psi(u) - \Psi(v)\big) &\geq \langle \eta^*\chi, \ u-v\rangle.\end{aligned} \tag{2.5}$$

Putting (2.5) in (2.4)

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^*\mathcal{F}^u(u) \ \ &\geq \sum_{i=1}^{n}\lambda_i^*\mathcal{F}^u(v) + \sum_{j=1}^{p_u}\mu_j^*\big(\mathcal{G}_j^u(v) - \mathcal{G}_j^u(u)\big) \\ &+ \sum_{i=1}^{p_l}\nu_s^*\big(\mathcal{G}_s^l(v) - \mathcal{G}_s^l(u)\big) + \eta^*\big(\Psi(v) - \Psi(u)\big)\end{aligned}$$

since u is feasible for (P) therefore, by Lemma 2.1 and Lemma 2.2 it is feasible for (RP), therefore

$$\sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(u) > \sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(v) + \sum_{j=1}^{p_u}\mu_j^*\mathcal{G}_j^u(v) + \sum_{i=1}^{p_l}\nu_s^*\mathcal{G}_s^l(v) + \eta^*\Psi(v)$$

Since $\sum_{i=1}^{n}\lambda_i^* = 1, \ \lambda_i^* > 0$ and $\lambda_i^* \neq 0$, we obtain,

$$\sum_{i=1}^{n}\lambda_i^*\mathcal{F}_i^u(u) > \sum_{i=1}^{n}\lambda_i^*\phi_i(v, \ \pi^*)$$

it contradicts (2.2). Therefore, the outcome.

Theorem 2.4 (Strong Duality) *Let $\hat{u}$ be a weak eff. sol. of (P) where the nonsmooth ACQ retains. Then, there exists $\hat{\pi}^* = (\lambda^*,\ \mu^*,\ \nu^*,\ \eta^*) \in \mathbb{R}^{\bar{n}}_+,\ \lambda^* \neq 0_{\mathbb{R}^{\bar{n}}}$, such that $(\bar{u},\ \bar{\pi}^*)$ is a feasible point of (WD) and optimal values of objective functions are equal. Moreover, if $\mathcal{F}^u_i,\ i \in I,\ \mathcal{G}^u_j,\ j \in J_0(\hat{u}),\ \mathcal{G}^l_s,\ s \in S_0(\hat{u})$, and Ψ are ∂^*-convex at $\hat{u}$, then $(\hat{u}, \hat{\pi}^*)$ is a weak eff. sol. of (WD).*

Proof Let $\hat{u}$ be a weak eff. sol. of (P) where the nonsmooth ACQ retains.

- As per Theorem 2.1, there exists $\lambda^* \in (-\mathbb{R}^n_+)^* \setminus \{0\}$ and $(\mu^*,\ \nu^*,\ \eta^*) \in \mathbb{R}^{p_u+p_l+1}_+$ such that

$$0 \in \sum_{i=1}^{n} \lambda^*_i \partial^* \mathcal{F}^u_i(\hat{u}) \ + \ \sum_{j=1}^{p_u} \mu^*_j \partial^* \mathcal{G}^u_j(\hat{u}) \ + \ \sum_{s=1}^{p_l} \nu^*_s \partial^* \mathcal{G}^l_s(\hat{u}) \ + \ \eta^* \partial^* \Psi(\hat{u})$$

$$\mu^*_j \mathcal{G}^u_j(\hat{u}) = 0, \quad \nu^*_s \mathcal{G}^l_s(\hat{u}) = 0, \quad j \in J,\ s \in S.$$

  Since $\Psi(\hat{u}) = 0$, concludes that $(\hat{u},\ \hat{\pi}^*)$ is a feasible point of (WD).

- To show $(\hat{u},\ \hat{\pi}^*)$ is a weak eff. sol. of (WD) on the contrary let there exists a point $(\hat{v}, \hat{\pi_1}^*) \in \tilde{E}$ such that

$$\phi(\hat{v}, \hat{\pi_1}^*) - \mathcal{F}^u(\hat{u}) \in \text{int}(\mathbb{R}^n_+) \tag{2.6}$$

  As $(\hat{u},\ \hat{\pi}^*)$ is a feasible solution of (WD) and also $\hat{u}$ is a feasible point of (P), due to Theorem 2.3 we get contradiction to (2.6). Hence the result.

Example 2.2 *The WD of the problem (P) considered in Example 2.1 is*

$$(WD):\ \begin{cases} \mathbb{R}^2_+ - \max\limits_{\hat{a}^u, \hat{a}^l, \pi^*} \left\{\phi_1(\hat{a}^u,\ \hat{a}^l,\ \pi^*),\ \phi_2(\hat{a}^u,\ \hat{a}^l,\ \pi^*)\right\} \\ \textit{subject to}: \\ 0 \in \lambda^*_1 \partial^* \mathcal{F}^u_1(\hat{a}^u,\ \hat{a}^l) + \lambda^*_2 \partial^* \mathcal{F}^u_2(\hat{a}^u,\ \hat{a}^l) + \mu^*_1 \partial^* \mathcal{G}^u_1(\hat{a}^u,\ \hat{a}^l) \\ \quad + \mu^*_2 \partial^* \mathcal{G}^u_2(\hat{a}^u,\ \hat{a}^l) + \nu^*_1 \partial^* \mathcal{G}^l_1(\hat{a}^u,\ \hat{a}^l) + \eta^* \partial^* \Psi(\hat{a}^u,\ \hat{a}^l), \\ \pi^* = (\lambda^*_1,\ \lambda^*_2,\ \mu^*_1,\ \mu^*_2,\ \nu^*_1,\ \eta^*) \geq 0, \\ (\lambda^*_1,\ \lambda^*_2) \neq (0,\ 0),\ \lambda^*_1 + \lambda^*_2 = 1 \\ (\hat{a}^u,\ \hat{a}^l) \in \mathbb{R}^2. \end{cases}$$

We claim that $\hat{v} = (-1, 0)$ is a feasible point of (WD). Indeed, for $\pi^* = \left(\frac{1}{4},\ \frac{3}{4},\ \frac{1}{4},\ \frac{1}{2},\ \frac{1}{4},\ 1\right)$, one get

$$\frac{1}{4}(-2,-1) + \frac{3}{4}(1,1) + \frac{1}{4}(-1,-1) + \frac{1}{2}(0,-1) + \frac{1}{4}(0,1) + 1(0,0) = (0,0).$$

Which implies

$$0 \in \lambda_1^* \partial^* \mathcal{F}_1^u(-1,0) + \lambda_2^* \partial^* \mathcal{F}_2^u(-1,0) + \mu_1^* \partial^* \mathcal{G}_1^u(-1,0)$$
$$+\mu_2^* \partial^* \mathcal{G}_2^u(-1,0) + \nu_1^* \partial^* \mathcal{G}_1^l(-1,0) + \eta^* \partial^* \Psi(-1,0).$$

Here, the sets $\partial^* \mathcal{F}_1^u(\hat{v}) = \{(-2,-1)\}$ *and* $\partial^* \mathcal{F}_2^u(\hat{v}) = \{(1,1)\}$ *are bounded USRCF of* $\mathcal{F}_1^u$ *and* $\mathcal{F}_2^u$ *at* $\hat{v}$ *and the collections* $\partial^* \mathcal{G}_1^u(\hat{v}) = \{(-1,-2)\}$, $\partial^* \mathcal{G}_2^u(\hat{v}) = \{(0,-1)\}$, $\partial^* \mathcal{G}_1^l(\hat{v}) = \{(0,1)\}$ *and* $\partial^* \Psi(\hat{v}) = \{(0,0)\}$ *are UCFs of* $\mathcal{G}_1^u$, $\mathcal{G}_2^u$, $\mathcal{G}_1^l$ *and* Ψ *at* $(\hat{v})$. *Moreover,* $\mathcal{F}_1^u$, $\mathcal{F}_2^u$, $\mathcal{G}_1^u$, $\mathcal{G}_2^u$, $\mathcal{G}_1^l$ *and* Ψ *are* ∂^**-convex at* $(\hat{v})$.
We have the collection $E = \mathbb{R}^+ \times \{0\}$ *for any point* $(a^u, a^l) \in E$ *of* (P) *and* $(\hat{a}^u,\ \hat{a}^l,\ \pi^*) \in \tilde{E}$ *of* (WD),

$$\phi(\hat{a}^u,\ \hat{a}^l,\ \pi^*) - \mathcal{F}^u(a^u, a^l) \notin \mathbb{R}_+^2 / \{0\}.$$

Hence, the Theorem 2.3 holds for (P) *and* (WD).

2.4 BPP with both level Multi-objective

In the current section, our focus is on a bilevel model that encompasses multiple objectives at both the levels. Functions at ul are $l\mathcal{L}$, whereas the ll's $obj.\ fn$ and $cnstr.\ fns$ are convex continuous functions.

$$\begin{aligned}
(Q): \qquad & \mathbb{R}_+^n - \min_{a^u, a^l} \mathcal{F}^u(a^u, a^l) = (\mathcal{F}_1^u(a^u, a^l), \mathcal{F}_2^u(a^u, a^l), \ldots,\ \mathcal{F}_n^u(a^u, a^l)) \\
& \text{subject to } \ \mathcal{G}_j^u(a^u, a^l) \le 0, \quad \forall j \in J, \\
& \qquad\qquad (a^u, a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l},\ a^l \in \Upsilon_{\text{weff}}(a^u),
\end{aligned}$$

where, for every $a^u \in \mathbb{R}^{n_u}$, $\Upsilon_{\text{weff}}(a^u)$ is the collection of weak eff. sols. of the subsequent multi-objective program:

$$\begin{aligned}
(Q_{a^u}): \qquad & \mathbb{R}_+^m - \min_{a^l} \mathcal{F}^l(a^u, a^l) = (\mathcal{F}_1^l(a^u, a^l), \mathcal{F}_2^l(a^u, a^l), \ldots,\ \mathcal{F}_m^l(a^u, a^l)) \\
& \text{subject to } \ \mathcal{G}_i^l(a^u, a^l) \le 0, \quad \forall i \in I, \\
& \qquad\qquad (a^u, a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}.
\end{aligned}$$

where, $\mathcal{F}_k^u,\ \mathcal{G}_j^u : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$ are $l\mathcal{L}$ functions, $\forall\ k \in K = \{1, 2 \ldots, n\}$, $\forall\ j \in J = \{1, 2 \ldots, p_u\}$ and $\mathcal{G}_i^l,\ \mathcal{F}_s^l : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$ are given convex continuous functions, $\forall\ i \in I = \{1, 2 \ldots, p_l\}$, $\forall\ s \in S = \{1, 2 \ldots, m\}$; $m,\ n,\ n_u,\ n_l,\ p_u,\ p_l \ge 1$ are integers.

For a fixed $a^u \in \mathbb{R}^{n_u}$, let

$$\Phi(a^u) = a^l \in \mathbb{R}^{n_l} : \mathcal{G}_i^l(a^u, a^l) \leq 0, \forall i \in I,$$

where $a^l \in \Phi(a^u)$ is said to be a local weak eff. sol. of (Q_{a^u}). Let the set C be collection of feasible points of (Q),

$$C = \Big\{(a^u, a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} : \quad a^l \in \Upsilon_{\text{weff}}(a^u),\ \mathcal{G}_j^u(a^u, a^l) \leq 0, \quad \forall j \in J\Big\}.$$

2.4.1 Reformulation

To deal with the multiple objectives at the follower program weighted sum scalarization is used. Then with the aid of value function reformulation, the considered bilevel model is converted into a single-level mathematical program. A lemma is proved to show the equivalence of the bilevel model (Q) and the reformulated model (Q_e) in terms of the global optimal solution.
Let

$$S = \{z \in \mathbb{R}_+^m \text{ such that } z^T e = 1\},$$

Now for any $z \in S$, we consider the following scalarized problem:

$$(Q_{zx_u}) : \qquad \min_{a^l} z^T \mathcal{F}^l(a^u, a^l)$$
$$\text{subject to } a^l \in \Phi(a^u).$$

Due to convexity of the set $A_{a^u} = \{\mathcal{F}^l(a^u, a^l) : \ a^l \in \Phi(a^u)\} + \mathbb{R}_+^m$, for any $a^u \in \mathbb{R}^{n_u}$, by using [69], we have

$$\Upsilon_{\text{weff}}(a^u) = \bigcup_{z \in S} \Theta(z,\ a^u)$$

where $\Theta : S \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_l}$ is the solution-set-mapping of (Q_{zx_u}) defined by:

$$\Theta(z,\ a^u) = \{a^l \in \Phi(a^u) \text{ such that } z^T \mathcal{F}^l(a^u, a^l) \leq \Lambda(z,\ a^u)\}$$

and OVF $\Lambda : S \times \mathbb{R}^{n_u} \to \bar{\mathbb{R}}$ is characterized as

$$\Lambda(z,\ a^u) = \inf_{a^l \in \Phi(a^u)} z^T \mathcal{F}^l(a^u, a^l)$$

The set-valued mapping Θ is assumed to be closed at all points $(\bar{z},\ \bar{a}^u)$ where $\bar{z} \in S$. To obtain the $\mathcal{LC}$ of the scalarized problem's value function, one requires the inner semi-continuity of the new ll's solution set-valued mapping. Let the set of active *cnstrs.* is designed as:

$$I_0(a^u, a^l) = \{i \in I : \ \mathcal{G}_i^l(a^u, a^l) = 0\}$$
$$J_0(a^u, a^l) = \{j \in J : \ \mathcal{G}_j^u(a^u, a^l) = 0\}$$

and

$$I_0^{\neq}(a^u, a^l) = \{i \in I : \ \mathcal{G}_i^l(a^u, a^l) \neq 0\}$$
$$J_0^{\neq}(a^u, a^l) = \{j \in J : \ \mathcal{G}_j^u(a^u, a^l) \neq 0\}$$

It is said that (Q_{zx_u}) is lower level regular at $(\bar{a}^u,\ a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l},\ a^l \in \Theta(\bar{z},\ \bar{a}^u)$, if

$$\Big\{ \sum_{i \in I_0(\bar{a}^u,\ a^l)} \mu_i v_i = 0,\ \mu_i \geq 0 \Big\} \Rightarrow \Big\{ \mu_i = 0\ \forall i \in I_0(\bar{a}^u,\ a^l) \Big\}$$

whenever $(u_i,\ v_i) \in \partial^c \mathcal{G}_i^l(\bar{a}^u,\ a^l)$ with some $u_i \in \mathbb{R}^{n_u}$ as $i \in I_0(\bar{a}^u,\ a^l)$.

Lemma 2.3 *Let $(\bar{a}^u,\ \bar{a}^l) \in C$ be a global weak eff. sol. of (Q) and let $\bar{z} \in S$ such that $\bar{a}^l \in \Lambda(\bar{z},\ \bar{a}^u)$. Then, $(\bar{z},\ \bar{a}^u,\ \bar{a}^l)$ is a global optimal point of*

$$
\begin{aligned}
(Q_e): \quad & \min_{z, a^u, a^l} \Lambda(z,\ a^u,\ a^l) = \Delta_{-int\mathbb{R}_+^{n_u}}(\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\bar{a}^u,\ \bar{a}^l)), \\
& \textit{subject to } \mathcal{G}_j^u(a^u, a^l) \leq 0,\ j = 1, 2, \ldots, p_u, \\
& \qquad \mathcal{G}_i^l(a^u, a^l) \leq 0,\ i = 1, 2, \ldots, p_l, \\
& \qquad z^T \mathcal{F}^l(a^u, a^l) - \Lambda(z,\ a^u) \leq 0 \\
& \qquad z_l \geq 0,\ l = 1, \ldots, m,\ z^T e = 1.
\end{aligned}
$$

Proof. As $(\bar{a}^u,\ \bar{a}^l)$ is a global weak eff. sol. of (Q), therefore

$$\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\bar{a}^u,\ \bar{a}^l) \notin -\mathrm{int}\mathbb{R}_+^n, \quad \forall (a^u, a^l) \in C.$$

By [82, Proposition 2.1],

$$\Delta_{-\mathrm{int}\mathbb{R}_+^n}(\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\bar{a}^u,\ \bar{a}^l)) \geq 0, \quad \forall (a^u, a^l) \in C.$$

i.e. $(\bar{a}^u,\ \bar{a}^l)$ solves globally problem given below

$$\begin{aligned}(Q_1): \qquad & \min_{a^u,a^l} \Delta_{-\mathrm{int}\mathbb{R}^{n_u}_+}(\mathcal{F}^u(a^u,a^l) - \mathcal{F}^u(\bar{a}^u,\ \bar{a}^l)) \\ & \text{subject to } (a^u,a^l) \in C.\end{aligned}$$

Then, by [44], $(\bar{z},\ \bar{a}^u,\ \bar{a}^l)$ is a global optimal point of

$$\begin{aligned}(Q_e): \qquad & \min_{z,a^u,a^l} \Lambda(z,\ a^u,\ a^l) = \Delta_{-\mathrm{int}\mathbb{R}^{n_u}_+}(\mathcal{F}^u(a^u,a^l) - \mathcal{F}^u(\bar{a}^u,\ \bar{a}^l)), \\ & \text{subject to } \mathcal{G}^u_j(a^u,a^l) \leq 0,\ \forall j \in J, \\ & \mathcal{G}^l_i(a^u,a^l) \leq 0,\ \forall i \in I, \\ & z^T\mathcal{F}^l(a^u,a^l) \leq \Lambda(z,\ a^u), \\ & z \in S, \\ & (z,\ a^u,\ a^l) \in \mathbb{R}^m \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}.\end{aligned}$$

Thus, $(\bar{z},\ \bar{a}^u,\ \bar{a}^l)$ is a global optimal point of (Q_e).

Definition 2.2 [42] *The nonsmooth generalized GCQ holds at $(\bar{a}^u,\ \bar{a}^l)$ if for each $\bar{z} \in S$ such that $\bar{a}^l \in \Theta(\bar{z},\ \bar{a}^u)$. Then one has*

$$[T(\Gamma,\ (\bar{z},\ \bar{a}^u,\ \bar{a}^l))]^- \subseteq cl\left(\sum_{i\in\Pi(\bar{z},\bar{a}^u,\bar{a}^l)} cone\ \partial^*\varphi_i(\bar{z},\ \bar{a}^u,\ \bar{a}^l)\right),$$

where

$$\begin{aligned}\varphi_0(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= \bar{z}_k\mathcal{F}^l(\bar{a}^u,\ \bar{a}^l) - \Lambda(\bar{z}_k,\ \bar{a}^u), \\ \varphi_j(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= \mathcal{G}^u_j(\bar{a}^u,\ \bar{a}^l), \quad j = 1,2,\ldots,p_u, \\ \varphi_i(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= \mathcal{G}^l_i(\bar{a}^u,\ \bar{a}^l), \quad i = 1,2,\ldots,p_l, \\ \varphi_k(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= -\bar{z}_k, \quad k = 1,2,\ldots,l, \\ \varphi_{p_u+p_l+l+1}(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= \bar{z}^Te - 1, \\ \varphi_{p_u+p_l+l+2}(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= -\bar{z}^Te + 1, \\ \Pi(\bar{z},\ \bar{a}^u,\ \bar{a}^l) &= \{i \in \theta:\ \varphi_i(\bar{z},\ \bar{a}^u,\ \bar{a}^l) = 0\}, \\ \theta &= \{1,2,\ldots,n\}\end{aligned}$$

and Γ is the collection of feasible points of (Q_e) represented as

$$\Gamma = \{(z,\ a^u,\ a^l) \in \mathbb{R}^m \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}:\ \varphi_i(z,\ a^u,\ a^l) \leq 0,\ i \in \theta\}.$$

2.5 Sufficient Optimality Condition

In this section we present the $\mathcal{S}$. *opt. cond.* for (Q) by using the optimistic approach. The following $\mathcal{N}$. *opt. cond* for (Q) has been proved by Dempe et al. [42].

Theorem 2.5 (Necessary Condition) *Let $(\bar{a}^u,\ \bar{a}^l) \in C$ be a local weak eff. sol. of (Q) and let $\bar{z} \in S$ such that $\bar{a}^l \in \Lambda(\bar{z},\ \bar{a}^u)$. Assume that $\mathcal{F}_k^u,\ k \in K$ admit bounded USRCF $\partial^*\mathcal{F}_k^u(\bar{a}^u,\ \bar{a}^l)$ at $(\bar{a}^u,\ \bar{a}^l)$, that $\mathcal{G}_j^u,\ j \in J,\ \mathcal{G}_i^l,\ i \in I$, admit UCFs $\partial^*\mathcal{G}_j^u(\bar{a}^u,\ \bar{a}^l),\ \partial^*\mathcal{G}_i^l(\bar{a}^u,\ \bar{a}^l)$, respectively, at $(\bar{a}^u,\ \bar{a}^l)$. Suppose that the nonsmooth generalized GCQ holds at $(\bar{z},\ \bar{a}^u,\ \bar{a}^l)$, that the solution-set-mapping Θ is inner semicontinuous at $(\bar{z},\ \bar{a}^u,\ \bar{a}^l)$, and that for each vector $a^l \in \Theta(\bar{z}, \bar{a}^u)$, (Q_{zx_u}) is ll regular at $(\bar{a}^u,\ a^l)$. Then there exists $\bar{\pi}^* = (\lambda^*,\ \rho^*,\ \sigma^*,\ \tau^*) \in \mathbb{R}_+^{n+p_u+p_l+1}$, $\lambda^* \neq 0_{\mathbb{R}^n}$ such that*

$$
\begin{aligned}
0 \in & \sum_{k=1}^{n}\{0\} \times \lambda_k^*\partial^*\mathcal{F}_k^u(\bar{a}^u,\ \bar{a}^l) \ + \ \sum_{j=1}^{p_u}\{0\} \times \rho_j^*\partial^*\mathcal{G}_j^u(\bar{a}^u,\ \bar{a}^l) \\
& + \ \sum_{i=1}^{p_l}\{0\} \times \sigma_i^*\partial^*\mathcal{G}_i^l(\bar{a}^u,\ \bar{a}^l) \ + \ \tau^*\partial^c\Big((z^T\mathcal{F}^l)(\bar{z},\ \bar{a}^u,\ \bar{a}^l) \\
& - \ \Lambda(\bar{z},\ \bar{a}^u) \times \{0\}\Big), \\
& \rho_j^*\mathcal{G}_j^u(\bar{a}^u,\ \bar{a}^l) = 0,\ \sigma_i^*\mathcal{G}_i^l(\bar{a}^u,\ \bar{a}^l) = 0, \qquad j \in J,\ i \in I.
\end{aligned}
\tag{2.7}
$$

Where ∂^c stand for the $\mathcal{C}$.sd.

If the variable z is fixed to $\bar{z}$, we have the following $\mathcal{S}$. *opt. cond.*

Theorem 2.6 (Sufficient Condition) *Let $\bar{u} = (\bar{a}^u,\ \bar{a}^l) \in C$ is a feasible solution of (Q) and let $\bar{z} \in S$ such that $\bar{a}^l \in \phi(\bar{z},\ \bar{a}^u)$. Suppose that $\mathcal{F}_k^u,\ k \in K$, is ∂^*-pseudoconvex at $(\bar{a}^u,\ \bar{a}^l)$ and $\mathcal{G}_j^u,\ j \in J_0(\bar{u}),\ \mathcal{G}_i^l,\ i \in I_0(\bar{u})$ are ∂^*-quasiconvex at $(\bar{a}^u,\ \bar{a}^l)$ and $z^T\mathcal{F}^l(\bar{z},\ \bar{a}^u,\ \bar{a}^l) - \Lambda(\bar{z},\ \bar{a}^u) \times \{0\}$ is ∂^c-quasiconvex at $(\bar{z},\ \bar{a}^u,\ \bar{a}^l)$ and there exists $\bar{\pi}^* = (\lambda^*,\ \rho^*,\ \sigma^*,\ \tau^*) \in \mathbb{R}_+^{n+p_u+p_l+1}$, $\lambda^* \neq 0_{\mathbb{R}^n}$ such that (2.7) holds. Then $\bar{u}$ is a weak eff. sol. of (Q).*

Proof. To the contrary, suppose that $\bar{u}$ is not a weak eff. sol. of (Q) then there exist $u = (\tilde{a}^u,\ \tilde{a}^l) \in C$ such that

$$\mathcal{F}^u(u) - \mathcal{F}^u(\bar{u}) \in -\text{int}\mathbb{R}_+^n$$

then $\mathcal{F}_k^u(u) - \mathcal{F}_k^u(\bar{u}) < 0, \quad \forall k \in K$.
By (2.7) one gets $\xi_k \in \partial^*\mathcal{F}_k^u(\bar{u}),\ \mu_j \in \partial^*\mathcal{G}_j^u(\bar{u}),\ j \in J,\ \nu_i \in \partial^*\mathcal{G}_i^l(\bar{u}),\ i \in I$ and $\delta \in$

$\partial^c\Big((z^T\mathcal{F}^l)(\bar{z},\ \bar{a}^u,\ \bar{a}^l) - \Lambda(\bar{z},\ \bar{a}^u) \times \{0\}\Big)$ such that

$$0 = \sum_{k=1}^{n}\{0\} \times \lambda_k^*\xi_k \ + \ \sum_{j=1}^{p_u}\{0\} \times \rho_j^*\mu_j \ + \ \sum_{i=1}^{p_l}\{0\} \times \sigma_i^*\nu_i \ + \ \tau^*\delta, \tag{2.8}$$

and

$$\rho_j^*\mathcal{G}_j^u(\bar{u}) = 0,\ \sigma_i^*\mathcal{G}_i^l(\bar{u}) = 0, \qquad j \in J,\ i \in I. \tag{2.9}$$

Since $\mathcal{F}_k^u,\ k \in K$ is ∂^*-pseudoconvex at $\bar{u}$, we have

$$\langle \xi_k,\ u - \bar{u}\rangle < 0, \quad \forall k \in K.$$

As $\lambda^* \in \mathbb{R}_+^n \setminus \{0\}$, it follows that

$$\Big\langle \lambda_k^*\xi_k,\ u - \bar{u}\Big\rangle < 0, \quad \forall k \in K.$$

Therefore,

$$\Big\langle \sum_{j=1}^{p_u}\{0\} \times \rho_j^*\mu_j \ + \ \sum_{i=1}^{p_l}\{0\} \times \sigma_i^*\nu_i \ + \ \tau^*\delta,\ \{0\} \times (u - \bar{u})\Big\rangle > 0. \tag{2.10}$$

Since $u \in C$, we have

$$\mathcal{G}_j^u(u) \leq 0,\ \mathcal{G}_i^l(u) \leq 0,\ (z^T\mathcal{F}^l)(z,\ \tilde{a}^u,\ \tilde{a}^l) \leq \Lambda(z,\ \tilde{a}^u) \qquad \forall j \in J,\ \forall i \in I.$$

On the one hand, in case $j \in J_0^{\neq}(\bar{u})$ we get $\rho_j^* = 0$; in case $i \in I_0^{\neq}$ we get $\sigma_i^* = 0$. Therefore,

$$\begin{aligned} \Big\langle \rho_j^*\mu_j,\ u - \bar{u}\Big\rangle &= 0, \quad \forall j \in J_0^{\neq}(\bar{u}), \\ \Big\langle \sigma_i^*\nu_i,\ u - \bar{u}\Big\rangle &= 0, \quad \forall i \in I_0^{\neq}(\bar{u}). \end{aligned} \tag{2.11}$$

On the other hand, since $(z^T\mathcal{F}^l)(\bar{z},\ \bar{a}^u,\ \bar{a}^l) - \Lambda(\bar{z},\ \bar{a}^u) = 0$ and since

$$\mathcal{G}_j^u(\bar{u}) = \mathcal{G}_i^l(\bar{u}) = 0, \quad \forall j \in J_0(\bar{u}),\ i \in I_0(\bar{u}),$$

we have

$$\begin{aligned} \mathcal{G}_j^u(u) - \mathcal{G}_j^u(\bar{u}) &\leq 0, \quad \forall j \in J_0(\bar{u}), \\ \mathcal{G}_i^l(u) - \mathcal{G}_i^l(\bar{u}) &\leq 0, \quad \forall i \in I_0(\bar{u}), \\ \Big[(z^T\mathcal{F}^l)(z,\ \tilde{a}^u,\ \tilde{a}^l) - \Lambda(z,\ \tilde{a}^u)\Big] - \Big[(z^T\mathcal{F}^l)(\bar{z},\ \bar{a}^u,\ \bar{a}^l) - \Lambda(\bar{z},\ \bar{a}^u)\Big] &\leq 0. \end{aligned}$$

Since $\mathcal{G}_j^u$ and $\mathcal{G}_i^l$ are ∂^*-quasiconvex at $\bar{u}$ and $(z^T\mathcal{F}^l)(z,\ a^u,\ a^l)-\Lambda(z,\ a^u)$ is ∂^c-quasiconvex at $(\bar{z},\ \bar{u})$, we get

$$\begin{aligned} \langle \mu_j,\ u-\bar{u}\rangle &\leq 0, \quad \forall j \in J_0(\bar{u}), \\ \langle \nu_i,\ u-\bar{u}\rangle &\leq 0, \quad \forall i \in I_0(\bar{u}), \\ \langle \delta,\ (z-\bar{z},\ u-\bar{u})\rangle &\leq 0. \end{aligned}$$

As $\tau^* \geq 0,\ \rho_j^* \geq 0$ and $\sigma_i^* \geq 0$, one gets

$$\begin{aligned} \langle \{0\}\times \rho_j^*\mu_j,\ \{0\}\times(u-\bar{u})\rangle &\leq 0, \quad \forall j \in J_0(\bar{u}), \\ \langle \{0\}\times \sigma_i^*\nu_i,\ \{0\}\times(u-\bar{u})\rangle &\leq 0, \quad \forall i \in I_0(\bar{u}), \\ \langle \tau^*\delta,\ (z-\bar{z},\ u-\bar{u})\rangle &\leq 0. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), since $J = J_0^{\neq}(\bar{u}) \cup J_0(\bar{u})$ and $I = I_0^{\neq}(\bar{u}) \cup I_0(\bar{u})$, we obtain

$$\begin{aligned} \langle \{0\}\times \rho_j^*\mu_j,\ \{0\}\times(u-\bar{u})\rangle &\leq 0, \quad \forall j \in J, \\ \langle \{0\}\times \sigma_i^*\nu_i,\ \{0\}\times(u-\bar{u})\rangle &\leq 0, \quad \forall i \in I, \\ \langle \tau^*\delta,\ (z-\bar{z},\ u-\bar{u})\rangle &\leq 0. \end{aligned}$$

Summing these inequalities, we obtain a contradiction with (2.10). The proof is then complete.

2.6 Mond-Weir Dual

opt. conds. and duality results are the backbones of the optimization program. In mathematical optimization theory, the principle of duality states that optimization problems can be viewed from one of two viewpoints: the primal problem or the dual problem. WD and MWD are the two widely studied duals in optimization. MWD has an advantage over WD due to the weaker assumptions used. Here, we have formulated a MWD for a multi-objective BPP.

Let us suppose that $\mathcal{F}_k^u,\ k \in K$ admit bounded USRCF $\partial^* \mathcal{F}_k^u(v,\ w)$ at $(v,\ w) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}$ and $\mathcal{G}_j^u,\ j \in J,\ \mathcal{G}_i^l,\ i \in I$, admit UCFs $\partial^* \mathcal{G}_j^u(.,\ .)$ and $\partial^* \mathcal{G}_i^l(.,\ .)$ at $(v,\ w)$. Let (MWD) be a MWD of problem (Q):

$$
\begin{aligned}
(MWD): \quad & \mathbb{R}_+^n - \max\ \mathcal{F}^u(v,\ w) = (\mathcal{F}_1^u(v,\ w), \mathcal{F}_2^u(v,\ w), \ldots,\ \mathcal{F}_n^u(v,\ w)) \\
\text{subject to} \quad & 0 \in \sum_{k=1}^{n} \{0\} \times \lambda_k^* \partial^* \mathcal{F}_k^u(v,\ w) \ + \ \sum_{j=1}^{p_u} \{0\} \times \rho_j^* \partial^* \mathcal{G}_j^u(v,\ w) \\
& \quad + \sum_{i=1}^{p_l} \{0\} \times \sigma_i^* \partial^* \mathcal{G}_i^l(v,\ w) \ + \ \tau^* \partial^c \Big((z^T \mathcal{F}^l)(z,\ v,\ w) \\
& \quad - \ \Lambda(z,\ v) \times \{0\} \Big), \\
& \tau^*(z^T \mathcal{F}^l)(v,\ w) \geq \tau^* \Lambda(z,\ v), \\
& \rho_j^* \mathcal{G}_j^u(v,\ w) \geq 0,\ \sigma_i^* \mathcal{G}_i^l(v,\ w) \geq 0, \qquad j \in J,\ i \in I, \\
& (\lambda_1^*, \ldots,\ \lambda_n^*)\ \neq (0, \ldots, 0), \\
& \pi^* = (\lambda_1^*, \ldots,\ \lambda_n^*,\ \rho_1^*, \ldots,\ \rho_{p_u}^*,\ \sigma_1^*, \ldots, \sigma_{p_l}^*,\ \tau^*) \geq 0.
\end{aligned}
$$

Let $\tilde{C}$ represents the feasible set of dual.

Theorem 2.7 (Weak duality) *For any feasible point* $\hat{u} = (\hat{a}^u,\ \hat{a}^l)$ *of* (Q) *and for any feasible point* $(v,\ w,\ \pi^*)$ *of* (MWD)*, such that* $\mathcal{F}_k^u,\ k \in K$*, is* ∂^**-pseudoconvex at* $(v,\ w)$*,* $\mathcal{G}_j^u,\ j \in J_0(\hat{u}),\ \mathcal{G}_i^l,\ i \in I_0(\hat{u})$*, are* ∂^**-quasiconvex at* $(v,\ w)$*, and* $(z^T \mathcal{F}^l)(z,\ .,\ .) - \Lambda(z,\ .)$ *is* ∂^c*-quasiconvex at* $(z,\ v,\ w)$*, there exists* $k_0 \in K$ *such that* $\mathcal{F}_{k_0}^u(\hat{a}^u,\ \hat{a}^l) \geq \mathcal{F}_{k_0}^u(v,\ w)$*.*

Proof. By contrary, assume that there exist a feasible point $\hat{u}$ of (Q) and a feasible point $(v,\ w,\ \pi^*)$ such that

$$
\mathcal{F}_k^u(\hat{a}^u,\ \hat{a}^l) - \mathcal{F}_k^u(v,\ w) < 0, \quad \forall k \in K.
$$

Notice that $(\hat{a}^u,\ \hat{a}^l) \neq (v,\ w)$. As $(\lambda_1^*, \lambda_2^*, \ldots,\ \lambda_n^*) \geq 0$, $(\lambda_1^*, \lambda_2^*, \ldots,\ \lambda_n^*) \neq (0, \ldots,\ 0)$, we have

$$
\sum_{k=1}^{n} \lambda_k^* (\mathcal{F}_k^u(\hat{a}^u,\ \hat{a}^l) - \mathcal{F}_k^u(v,\ w)) < 0. \tag{2.13}
$$

Since $(v,\ w,\ \pi^*) \in \tilde{C}$, we obtain $\xi_k \in \partial^* \mathcal{F}_k^u(v,\ w)$, $\mu_j \in \partial^* \mathcal{G}_j^u(v,\ w)$, $\nu_i \in \partial^* \mathcal{G}_i^l(v,\ w)$ and $\delta \in \partial^c((z^T \mathcal{F}^l)(z,\ v,\ w) - \Lambda(z,\ v))$ such that

$$
\sum_{k=1}^{n} \{0\} \times \lambda_k^* \xi_k \ = -\sum_{j=1}^{p_u} \{0\} \times \rho_j^* \mu_j \ - \ \sum_{i=1}^{p_l} \{0\} \times \sigma_i^* \nu_i \ - \ \tau^* \delta, \tag{2.14}
$$

and

$$\rho_j^* \mathcal{G}_j^u(v,\ w) \geq 0,\ \sigma_i^* \mathcal{G}_i^l(v,\ w) \geq 0, \qquad j \in J,\ i \in I,$$
$$\tau^*((z^T \mathcal{F}^l)(z,\ v,\ w) - \Lambda(z,\ v)) \geq 0.$$

As $(\hat{a}^u,\ \hat{a}^l) \in C$, we have

$$\begin{aligned} \rho_j^* \mathcal{G}_j^u(\hat{a}^u,\ \hat{a}^l) &\leq \rho_j^* \mathcal{G}_j^u(v,\ w), \qquad j \in J, \\ \sigma_i^* \mathcal{G}_i^l(\hat{a}^u,\ \hat{a}^l) &\leq \sigma_i^* \mathcal{G}_i^l(v,\ w), \qquad i \in I, \\ \tau^*((z^T \mathcal{F}^l)(z,\ \hat{a}^u,\ \hat{a}^l) - \Lambda(z,\ \hat{a})) &\leq \tau^*((z^T \mathcal{F}^l)(z,\ v,\ w) - \Lambda(z,\ v)). \end{aligned}$$

Since $\mathcal{F}_k^u,\ k \in K$ is ∂^*-pseudoconvex at $(v,\ w)$, we get

$$\langle \xi_k,\ ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \rangle < 0,\ k \in K.$$

Therefore,

$$\Big\langle \sum_{k=1}^{n} \{0\} \times \lambda_k^* \xi_k,\ \{0\} \times ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \Big\rangle < 0. \tag{2.15}$$

Since $\mathcal{G}_j^u,\ j \in J$ and $\mathcal{G}_i^l,\ i \in I$ are ∂^*-quasiconvex at $(v,\ w)$, we obtain

$$\begin{aligned} \langle \mu_j,\ ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \rangle &\leq 0, \quad \forall j \in J_0(\hat{a}^u,\ \hat{a}^l), \\ \langle \nu_i,\ ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \rangle &\leq 0, \quad \forall i \in I_0(\hat{a}^u,\ \hat{a}^l). \end{aligned}$$

Then,

$$\begin{aligned} \Big\langle \rho_j^* \mu_j,\ ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \Big\rangle &\leq 0, \quad \forall j \in J_0(\hat{a}^u,\ \hat{a}^l), \\ \Big\langle \sigma_i^* \nu_i,\ ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \Big\rangle &\leq 0, \quad \forall i \in I_0(\hat{a}^u,\ \hat{a}^l). \end{aligned}$$

For $j \in J_0^{\neq}(\hat{a}^u,\ \hat{a}^l)$, we have $\rho_j^* = 0$ and for $i \in I_0^{\neq}(\hat{a}^u,\ \hat{a}^l)$, we have $\sigma_i^* = 0$. Then

$$\begin{aligned} \Big\langle \sum_{j=1}^{p_u} \{0\} \times \rho_j^* \mu_j,\ \{0\} \times ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \Big\rangle &\leq 0, \\ \Big\langle \sum_{i=1}^{p_l} \{0\} \times \sigma_i^* \nu_i,\ \{0\} \times ((\hat{a}^u,\ \hat{a}^l) - (v,\ w)) \Big\rangle &\leq 0. \end{aligned} \tag{2.16}$$

Let us prove that

$$\langle \tau^* \delta,\ ((z,\ \hat{a}^u,\ \hat{a}^l) - (z,\ v,\ w)) \rangle \leq 0. \tag{2.17}$$

Indeed, in case $\tau^* > 0$, we have $((z^T\mathcal{F}^l)(z,\ \hat{a}^u,\ \hat{a}^l) - \Lambda(z,\ \hat{a})) \leq ((z^T\mathcal{F}^l)(z,\ v,\ w) - \Lambda(z,\ v))$. Since $(z^T\mathcal{F}^l)(z,\ .,\ .) - \Lambda(z,\ .)$ is ∂^c-quasiconvex at $(z,\ v,\ w)$, we obtain $\langle \delta,\ ((z,\ \hat{a}^u,\ \hat{a}^l) - (z,\ v,\ w))\rangle \leq 0$. Then,

$$\langle \tau^*\delta,\ ((z,\ \hat{a}^u,\ \hat{a}^l) - (z,\ v,\ w))\rangle \leq 0.$$

In case $\tau^* = 0$, this implies

$$\langle \tau^*\delta,\ ((z,\ \hat{a}^u,\ \hat{a}^l) - (z,\ v,\ w))\rangle = 0.$$

From (2.14), (2.16) and (2.17), we get

$$\Big\langle \sum_{k=1}^{n}\{0\} \times \lambda_k^*\xi_k,\ ((\hat{a}^u,\ \hat{a}^l) - (v,\ w))\Big\rangle = \Big\langle -\sum_{j=1}^{p_u}\{0\} \times \rho_j^*\mu_j\ -\ \sum_{i=1}^{p_l}\{0\} \times \sigma_i^*\nu_i$$
$$-\ \tau^*\delta,\ ((z,\ \hat{a}^u,\ \hat{a}^l) - (z,\ v,\ w))\Big\rangle \geq 0.$$

Hence, we get a contradiction to (2.15). The proof is then complete.

Theorem 2.8 (Strong duality) *Let $\bar{u} = (\bar{a}^u,\ \bar{a}^l)$ be a weak eff. sol. of (Q) where the nonsmooth GCQ holds. Then there exists $\bar{\pi}^* = (\lambda^*,\ \rho^*,\ \sigma^*,\ \tau^*) \in \mathbb{R}_+^{n+p_u+p_l+1}$, $\lambda^* \neq 0_{\mathbb{R}^n}$, such that $(\bar{u},\ \bar{\pi}^*)$ is a feasible point of (MWD). Moreover, if $\mathcal{F}_k^u$, $k \in K$, is ∂^*-pseudoconvex at $\bar{u}$ and $\mathcal{G}_j^u$, $j \in J_0(\bar{u})$, $\mathcal{G}_i^l$, $i \in I_0(\bar{u})$ are ∂^*-quasiconvex at $\bar{u}$ and $(z^T\mathcal{F}^l)(z,\ .,\ .) - \Lambda(z,\ .)$ is ∂^c-quasiconvex at $(\bar{z},\ \bar{u})$, then $(\bar{u},\ \bar{\pi}^*)$ is a weak eff. sol. of (MWD).*

Proof. Let $\bar{u}$ be a weak eff. sol. of (Q) where the nonsmooth GCQ holds. According to Theorem (2.5), we have $\lambda^* \in (-\mathbb{R}_+^n)^- \setminus \{0\}$ and $(\rho^*,\ \sigma^*,\ \tau^*) \in \mathbb{R}_+^{p_u+p_l+1}$ such that

$$\begin{aligned} 0 \in &\sum_{k=1}^{n}\{0\} \times \lambda_k^*\partial^*\mathcal{F}_k^u(\bar{a}^u,\ \bar{a}^l)\ +\ \sum_{j=1}^{p_u}\{0\} \times \rho_j^*\partial^*\mathcal{G}_j^u(\bar{a}^u,\ \bar{a}^l) \\ &+\ \sum_{i=1}^{p_l}\{0\} \times \sigma_i^*\partial^*\mathcal{G}_i^l(\bar{a}^u,\ \bar{a}^l)\ +\ \tau^*\partial^c\Big((z^T\mathcal{F}^l)(\bar{z},\ \bar{a}^u,\ \bar{a}^l) \\ &-\ \Lambda(\bar{z},\ \bar{a}^u) \times \{0\}\Big), \\ &\rho_j^*\mathcal{G}_j^u(\bar{a}^u,\ \bar{a}^l) = 0,\ \sigma_i^*\mathcal{G}_i^l(\bar{a}^u,\ \bar{a}^l) = 0, \qquad j \in J,\ i \in I. \end{aligned} \tag{2.18}$$

Since $(\bar{z}^T\mathcal{F}^l)(\bar{z},\ \bar{a}^u,\ \bar{a}^l) - \Lambda(\bar{z},\ \bar{a}^u) = 0$, this implies that the point $(\bar{z},\ \bar{u},\ \bar{\pi}^*)$ is a feasible point of (D). Now to prove that $(\bar{z},\ \bar{u},\ \bar{\pi}^*)$ is a weak eff. sol. of (D), on the contrary, let there exists a point $(z_*,\ u_*, \pi_*^*) \in \tilde{C}$ such that

$$\mathcal{F}^u(\bar{u}) - \mathcal{F}^u(u_*) \in -\mathrm{int}(\mathbb{R}^n_+).$$

Since $(\bar{z},\ \bar{u},\ \bar{\pi}^*)$ is a feasible point of (D) and since $\bar{u}$ is a feasible solution of (Q), by Theorem 2.7, we have $\mathcal{F}^u(\bar{u}) - \mathcal{F}^u(u_*) \notin -\mathrm{int}(\mathbb{R}^n_+)$, a contradiction. The proof is then complete.

Chapter 3

Necessary optimality conditions for nonsmooth multi-objective BPP under the optimistic perspective [1]

3.1 Introduction

An enormous class of real-life optimization problems are multi-objective in nature, with numerous levels of decision-making. For instance, at a research institute, the department heads are the *ul* decision-makers, and the research coordinators within the institute are the *ll* $\mathcal{DM}'s$ representing their research group in the decision-making process of resource allocation. Each department has its objectives, and research groups have their own requirements and compete with each other for the limited available resources. Such a problem is referred to as a multi-objective BPP. Thus, a multi-objective BPP is one that optimizes multiple objectives appearing at two levels simultaneously.
Scalarization is a popular approach for finding solution of multi-objective optimization problems. The benefit of using scalarization is that once a scalar objective problem is obtained, the user can use all of the theoretical results and numerical algorithms that have already been established for the scalar objective case. The weighted-sum scalarization approach is used in the existing literature [42], which is simple to implement but cannot generate points in nonconvex portions of the efficient set. BPPs, in general, are nonconvex and have a disconnected feasible region. Thus, we applied kth-objective weighted-constraint scalarization (Burachik et al. [22]), as it applies to nonconvex problems, particularly to those with disconnected efficient and feasible sets.
This chapter is structured as: In Section 3.2, we examined a multi-objective BPP and used the kth-objective weighted-constraint scalarization technique to break down the nonconvex *ll* multi-objective optimization problem into several scalar optimization problems, which we then redefined into a sole-level program via reformulation. The $\mathcal{N}$. *opt. cond.* are developed in Section 3.3 via a function introduced by Hiriart-Urruty [64, 65]. Further in

[1]This chapter is published as "*necessary optimality conditions for nonsmooth multi-objective bilevel optimization problem under the optimistic perspective*", Positivity, 26 (2022) 1-16.

Section 3.4 an example is given to illustrate the applicability of obtained condition.

3.2 Multi-objective Bilevel Model

In this Section, we give a bilevel model whose both ul and ll programs are multi-objective. The kth-objective weighted-constraint scalarization technique is used to convert the non-convex follower multi-objective program into many scalar programs. The scalarized BPP is then reformulated. The multi-objective BPP is:

$$(Q) : \begin{cases} \mathbb{R}^l_+ - \min\limits_{a^u, a^l} \mathcal{F}^u(a^u,\ a^l) = (\mathcal{F}^u_1, \mathcal{F}^u_2, \ldots, \mathcal{F}^u_l)(a^u,\ a^l) \\ \text{subject to} \begin{cases} a^u \in X,\ a^l \in \mathfrak{L}_{wef}(a^u), \\ (a^u,\ a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}, \end{cases} \end{cases}$$

where, for every $a^u \in \mathbb{R}^{n_u},\ \mathfrak{L}_{wef}(a^u)$ is the collection of weak eff. sol. of the below defined multi-objective program

$$(Q_{a^u}) : \begin{cases} \mathbb{R}^k_+ - \min\limits_{a^l} \mathcal{F}^l(a^u,\ a^l) = (\mathcal{F}^l_1, \mathcal{F}^l_2, \ldots, \mathcal{F}^l_k)(a^u,\ a^l) \\ \text{subject to} \begin{cases} \mathcal{G}^l_j(a^u,\ a^l) \leq 0,\ \forall j \in J, \\ (a^u,\ a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}. \end{cases} \end{cases}$$

We define $d \in D^u = \{1, 2, \ldots, l\},\ i \in I = \{1, 2, \ldots, p\},\ s \in D^l = \{1, 2, \ldots, k\},\ j \in J = \{1, 2, \ldots, q\}; k,\ l,\ m,\ n,\ p,\ q \geq 1$ are integers, and $\mathcal{F}^u_d : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, $\mathcal{F}^l_s : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, $\mathcal{G}^u_i : \mathbb{R}^{n_u} \to \mathbb{R}$, and $\mathcal{G}^l_j : \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$, are $l\mathcal{L}$ functions with

$$\min_{s \in D^l} \min_{a^l \in K(a^u)} \mathcal{F}^l_s(a^u,\ a^l) > 0,\ \forall a^u \in X,$$

where

$$X = \{a^u \in \mathbb{R}^{n_u} :\ \mathcal{G}^u_i(a^u) \leq 0,\ \forall i \in I\}$$

and

$$K(a^u) = \{a^l \in \mathbb{R}^{n_l} :\ \mathcal{G}^l_j(a^u,\ a^l) \leq 0,\ \forall j \in J\}$$

are the collection of feasible points of (Q).
$\hat{a}^l \in \mathfrak{L}_{wef}(\hat{a}^u)$ iff there exists $s \in D^l$ such that

$$\mathcal{F}^l_s(\hat{a}^u,\ a^l) - \mathcal{F}^l_s(\hat{a}^u,\ \hat{a}^l) \geq 0,\ \forall a^l \in K(\hat{a}^u)$$

which means that

$$\nexists a^l \in K(\hat{a}^u) \text{ such that } \mathcal{F}_s^l(\hat{a}^u,\ a^l) < \mathcal{F}_s^l(\hat{a}^u,\ \hat{a}^l),\ \forall s \in D^l.$$

Definition 3.1 *A pair $(\hat{a}^u,\ \hat{a}^l) \in \Upsilon$ is known as local weak eff. sol. of (Q) if there exists n.b.d $U_{\hat{a}^u}$ of $\hat{a}^u$ and $V_{\hat{a}^l}$ of $\hat{a}^l$ such that*

$$\mathcal{F}^u(a^u,\ a^l) - \mathcal{F}^u(\hat{a}^u,\ \hat{a}^l) \notin -\text{int}\mathbb{R}_+^l,\ \forall (a^u,\ a^l) \in \Upsilon \cap (U_{\hat{a}^u} \times V_{\hat{a}^l}),$$

where

$$\Upsilon = \{(a^u,\ a^l) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} :\ a^u \in X \text{ and } a^l \in \mathfrak{L}_{wef}(a^u)\}$$

is the feasible set of (Q).

3.2.1 Reformulation

Let a ones vector of $\mathbb{R}^k$ is denoted by $e = (1,\ 1, \dots,\ 1)$, and let

$$\mathbb{Z}_S = \{z \in \mathbb{R}_+^k\ :\ z^T e = 1\}$$

$$\text{and } \mathbb{Z}_S^+ = \{z \in \mathbb{R}_+^k :\ z^T e = 1 \text{ and } z_s > 0,\ s \in D^l\}.$$

For any $s \in D^l$ and $z \in \mathbb{Z}_S$, the program $(Q_{zx_u}^s)$ is defined by

$$(Q_{zx_u}^s) : \begin{cases} \underset{a^l}{\text{Min}}\ \mathcal{F}_s^l(a^u,\ a^l) \\ \text{subject to } a^l \in K_s(z,\ a^u), \end{cases}$$

where

$$K_s(z,\ a^u) = \{a^l \in K(a^u) :\ h_{s,r}(z,\ a^u,\ a^l) \le 0,\ \forall r \in D^l\} \tag{3.1}$$

and for any $r \in D^l$,

$$h_{s,r}(z,\ a^u,\ a^l) = z_r f_r(a^u,\ a^l) - z_s f_s(a^u,\ a^l).$$

Remark 3.1 [45] *Let $s \in D^l$ and $z \in \mathbb{Z}_S$. In case $z_s = 0$, $K_s(z,\ a^u) = \emptyset$. For $s \in D^l$, and for each $(z,\ a^u) \in \mathbb{Z}_S \times X$, let*
$\mathfrak{J}_s(z,\ a^u) = \{a^l \in K(a^u) :\ a^l \text{ solves } (Q_{zx_u}^s)\} = \{a^l \in K_s(z,\ a^u) :\ \rho_s(z,\ a^u,\ a^l) \le 0\}$

and

$$\Lambda^{+}(a^u,\ a^l) = \{z \in \mathbb{Z}_S^{+}:\ a^l \in \mathfrak{J}_s(z,\ a^u),\ s \in D^l\} \tag{3.2}$$

where

$$\rho_s(z,\ a^u,\ a^l) = \mathcal{F}_s^l(a^u,\ a^l) - \Psi_s(z,\ a^u)$$

and

$$\Psi_s(z,\ a^u) = \inf_{a^l}\{\mathcal{F}_s^l(a^u,\ a^l):\ a^l \in K_s(z, a^u)\}.$$

Required nomenclature:
For any $r \in D^l$, let

$$\chi_r(z,\ a^u,\ a^l) = z_r\mathcal{F}_r^l(a^u,\ a^l) - z_1{\mathcal{F}^l}_1(a^u,\ a^l).$$

Let

$$\chi(z,\ a^u,\ a^l) = \Big(0,\ \chi_2(z,\ a^u,\ a^l), \ldots, \chi_k(z,\ a^u,\ a^l)\Big)$$

$$\rho(z,\ a^u,\ a^l) = \Big(\ \rho_1(z,\ a^u,\ a^l), \ldots, \rho_k(z,\ a^u,\ a^l)\Big)$$

$$\Theta = \Big\{(z,\ a^u,\ a^l) \in \mathbb{R}^k \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}:\ z \in \mathbb{Z}_S,\ a^u \in X,\ a^l \in K(a^u),\ \chi(z,\ a^u,\ a^l) = 0\Big\}$$

and

$$\Omega = \Big\{(z,\ a^u,\ a^l) \in \Theta:\ \rho(z,\ a^u,\ a^l) \in \mathbb{R}_-^k\Big\}.$$

Let us consider $\Pi \subseteq \mathbb{R}^{k+n_u+n_l}$, given as

$$\Pi = \rho^{-1}\Big(\mathbb{R}_-^k\Big) = \Big\{(z,\ a^u,\ a^l):\ \rho(z,\ a^u,\ a^l) \in \mathbb{R}_-^k\Big\}.$$

Assuming $k \geq 2$, let $(z,\ a^u,\ a^l) \in \mathbb{R}^k \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l}$ and let $s \in D^l$. Using $h_{s,s}(z,\ a^u,\ a^l) = 0$, one can easily see that

$$K_s(z,\ a^u) = \Big\{a^l \in K(a^u):\ h_{s,r}(z,\ a^u,\ a^l) \leq 0,\ \forall r \in D^l \setminus \{s\}\Big\}.$$

Proposition 3.1 [45] *Let $\hat{a}^u \in X$ and let $\hat{a}^l \in K(\hat{a}^u)$. If $\bigwedge^{+}(\hat{a}^u,\ \hat{a}^l) \neq \emptyset$, then $\bigwedge^{+}(\hat{a}^u,\ \hat{a}^l)$ is a singleton. Moreover, $\hat{a}^l \in \mathfrak{L}_{wef}(\hat{a}^u)$ if and only if $\hat{a}^l \in \bigcap_{r\in D^l}\mathfrak{J}_r(\hat{z},\ \hat{a}^u)$, where*

$$\hat{z}_s = \frac{1/\mathcal{F}_s^l(\hat{a}^u,\ \hat{a}^l)}{\sum_{r\in D^l} 1/\mathcal{F}_r^l(\hat{a}^u,\ \hat{a}^l)}.$$

Requisite regularities [97]

- (Q_{a^u}) is said to be *ll* regular at $(\hat{a}^u,\ \hat{a}^l) \in X \times K(\hat{a}^u)$ if

$$\left.\begin{array}{l} 0 \in \sum_{j\in J} \eta_j \partial_{a^l} \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) \\ \eta_j \geq 0,\ \eta_j \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) = 0,\ j \in J \end{array}\right\} \Longrightarrow [\eta_j = 0,\ j \in J]. \tag{3.3}$$

- $(Q^s_{zx_u})$, $s \in D^l$, is said to be *ll* regular at $(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \in \mathbb{Z}_S \times X \times K_s(\hat{z},\ \hat{a}^u)$ if, depending on k.

  - When $k \geq 2$,

$$\left.\begin{array}{l} 0 \in \sum_{j\in J} \eta_j \partial_{a^l} \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l)\ +\ \sum_{r\in D^l\setminus\{1\}} \delta_r \partial_{a^l} \chi_r(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \\ \eta_j \geq 0,\ \eta_j \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) = 0,\ j \in J,\ \delta_r \geq 0,\ r \in D^l \setminus \{1\} \end{array}\right\} \tag{3.4}$$

$$\Longrightarrow \left\{\begin{array}{l} \eta_j = 0,\ j \in J \\ \delta_r = 0,\ r \in D^l \setminus \{1\}. \end{array}\right. \tag{3.5}$$

  - When $k = 1$,

$$\left.\begin{array}{l} 0 \in \sum_{j\in J} \eta_j \partial_{a^l} \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) \\ \eta_j \geq 0,\ \eta_j \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) = 0,\ j \in J \end{array}\right\} \Longrightarrow [\eta_j = 0,\ j \in J]. \tag{3.6}$$

3.3 Necessary optimality conditions

Proposition 3.2 *Let $(\hat{a}^u,\ \hat{a}^l)$ be a local weak eff. sol. of (Q). Then, for every $\hat{z} \in \bigwedge^+(\hat{a}^u,\ \hat{a}^l)$, the point $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ is a local optimal point of*

$$(Q_{aux}) : \left\{\begin{array}{l} \min_{z,a^u,a^l} \phi(z,\ a^u,\ a^l) = \Delta_{-\mathrm{int}\mathbb{R}^l_+}(\mathcal{F}^u(a^u,\ a^l) - \mathcal{F}^u(\hat{a}^u,\ \hat{a}^l)), \\ \text{subject to } z \in \mathbb{Z}_S,\ a^u \in X \text{ and } a^l \in \mathfrak{J}_s(z,\ a^u),\ \forall s \in D^l. \end{array}\right.$$

Proof. Since $(\hat{a}^u,\ \hat{a}^l)$ is a local weak eff. sol. of (Q), there exists n.b.d $U_{\hat{a}^u}$ of $(\hat{a}^u,\ \hat{a}^l)$ such that

$$\mathcal{F}^u(a^u,\ a^l) - \mathcal{F}^u(\hat{a}^u,\ \hat{a}^l) \notin -\mathrm{int}\mathbb{R}^l_+ \qquad \forall (a^u,\ a^l) \in U_{\hat{a}^u} \cap \Omega.$$

Then, by [82, Proposition 2.1],

$$\Delta_{-\mathrm{int}\mathbb{R}^l_+}\Big(\mathcal{F}^u(a^u,\ a^l) - \mathcal{F}^u(\hat{a}^u,\ \hat{a}^l)\Big) \geq 0, \qquad \forall (a^u,\ a^l) \in U_{\hat{a}^u} \cap \Omega.$$

i.e $(\hat{a}^u,\ \hat{a}^l)$ solves locally the subsequent constrainted program:

$$(Q_1) : \begin{cases} \min\limits_{a^u, a^l} \Delta_{-\text{int}\mathbb{R}^l_+}\Big(\mathcal{F}^u(a^u,\ a^l) - \mathcal{F}^u(\hat{a}^u,\ \hat{a}^l)\Big) \\ (a^u,\ a^l) \in \Omega. \end{cases}$$

Therefore, due to [42], $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ is a local optimal point of

$$(Q_{aux}) : \begin{cases} \min\limits_{z, a^u, a^l} \phi(z,\ a^u,\ a^l) = \Delta_{-\text{int}\mathbb{R}^l_+}\Big(\mathcal{F}^u(a^u,\ a^l) - \mathcal{F}^u(\hat{a}^u,\ \hat{a}^l)\Big) \\ \text{subject to } z \in \mathbb{Z}_S,\ a^u \in X \text{ and } a^l \in \mathfrak{J}_s(z,\ a^u),\ \forall s \in D^l, \end{cases}$$

The collection $\mathbb{Z}_S$ of the new parameter of the followers program is considered as an additional constraints for the leaders program (Q_{aux}). Given the presumption of closedness of the set-valued mapping $\mathfrak{J}_s,\ s \in D^l$, the proposition above shows that (Q) is equivalent to (Q_{aux}) in terms of local optimal point. According to Remark 1.2, the set-valued mapping $\bigwedge^+$ is semicompact at any point $(a^u,\ a^l) \in X \times \mathbb{R}^{n_l}$ since it is uniformly bounded. With the aid of reformulation, the corresponding sole level program Q^* to (Q_{aux}) is:

$$(Q^*) : \begin{cases} \min\limits_{z, a^u, a^l} \phi(z,\ a^u,\ a^l) \\ \text{subject to } (z,\ a^u,\ a^l) \in \Omega. \end{cases} \tag{3.7}$$

Theorem 3.1 *Let $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ be a local optimal point of (Q^*). Let*

1. *the collection Θ is regular and semismooth at $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ so that*

$$D^*\rho(\hat{z},\ \hat{a}^u,\ \hat{a}^l)(\tau) \cap \Big(- bdN((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta)\Big) = \emptyset,\ \forall \tau \in \mathbb{R}^k_+ \setminus \{0\} \tag{3.8}$$

2. *$\mathcal{G}^u_i,\ i \in I$ are $\mathcal{L}$ continuous near $\hat{a}^u$ and the functions $\mathcal{F}^l_s,\ s \in D^l$ and $\mathcal{G}^l_j,\ j \in J$ are $\mathcal{L}$ continuous near $(\hat{a}^u,\ a^l)\ \forall\ a^l \in \bigcap\limits_{s \in D^l} \mathfrak{J}_s(\hat{z},\ \hat{a}^u)$;*

3. *the set valued mappings $\mathfrak{J}_s,\ s \in D^l$, are inner semicompact at $(\hat{z},\ \hat{a}^u)$;*

4. *$\forall\ a^l \in \bigcap\limits_{s \in D^l} \mathfrak{J}_s(\hat{z},\ \hat{a}^u)$ the problems $(Q^s_{zx_u}),\ s \in D^l$, are ll regular at $(\hat{z},\ \hat{a}^u,\ a^l)$.*

Then, there exist $\gamma \geq 0,\ \gamma \neq 0_{\mathbb{R}^l},\ \nu_s \geq 0,\ s \in D^l$ such that

$$\begin{aligned} 0 \in & \{0\} \times \sum_{d \in D^u} \gamma_d \partial \mathcal{F}^u_d(\hat{a}^u,\ \hat{a}^l)\ +\ \{0\} \times \sum_{s \in D^l} \nu_s \partial \mathcal{F}^l_s(\hat{a}^u,\ \hat{a}^l) \\ & - \sum_{s \in D^l} \nu_s co \partial \Psi_s(\hat{z},\ \hat{a}^u) \times \{0\}\ +\ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta). \end{aligned} \tag{3.9}$$

Proof. The proof presented here is based on the framework established in [56, Theorem 16], albeit with appropriate modifications. Let $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ be a local optimal solution of (Q^*). As $\Delta_{(-\mathrm{int}\mathbb{R}^l_+)}$ and $\mathcal{F}^u$ are $l\mathcal{L}$, then there exists $\eta \geq 0$ such that ϕ is $l\mathcal{L}$ with a Lipschitz constant $\eta \geq 0$ around $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$, from [98, Proposition 5.3], we have

$$0 \in \partial\phi(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Omega), \tag{3.10}$$

where

$$\Omega = \Pi \cap \Theta.$$

By virtue of the $l\mathcal{L}$ continuity of all the functions and inner semicompactness of $\mathfrak{J}_s,\ s \in D^l$ near $(\hat{z},\ \hat{a}^u)$, as the problems $(Q^s_{zx_u}),\ s \in D^l$, are ll regular at $(\hat{z},\ \hat{a}^u,\ a^l)$ for all $a^l \in \mathfrak{J}_s(\hat{z},\ \hat{a}^u)$. Therefore, Ψ_s and $\rho_s,\ s \in D^l$, are $\mathcal{LC}$ around $(\hat{z},\ \hat{a}^u)$ and $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ respectively, by using [99, Theorem 5.2].

- Since ρ is $l\mathcal{L}$ near some $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ and the assumptions of [63, Theorem 3.1] holds for Θ and the CQ (3.8), one can say that the mapping

  $$M(u) = \Big\{(z,\ a^u,\ a^l) \in \Theta:\ \rho(z,\ a^u,\ a^l)\ +\ u \in \mathbb{R}^k_-\Big\}$$

  is calm at $(0,\ (\hat{z},\ \hat{a}^u,\ \hat{a}^l))$.

- By [62, Theorem 4.1], we get

  $$N\Big((\hat{z},\ \hat{a}^u,\ \hat{a}^l), \Pi \cap \Theta\Big) \subseteq \bigcup_{\nu \in \mathbb{R}^k_+} D^*\rho(\hat{z},\ \hat{a}^u,\ \hat{a}^l)(\nu)\ +\ N\Big((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta\Big). \tag{3.11}$$

- From (3.10) and (3.11) since ρ is $l\mathcal{L}$ near $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$, we have $\nu \in \mathbb{R}^k_+$ such that

  $$0\ \in\ \partial\phi(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ \partial\langle\nu,\ \rho\rangle(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta).$$

  Using [97, Theorem 3.36], in conjunction with [97, Theorem 1.26] and [97, Corollary 1.81],

  $$0\ \in\ \partial\phi(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ \sum_{s \in D^l} \partial\langle\nu_s,\ \rho_s\rangle(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta).$$

  Then,

  $$0\ \in\ \partial\phi(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ \sum_{s \in D^l} \nu_s\partial\rho_s(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\ +\ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta).$$

- By chain rule [71, Theorem 4.3], there are $\gamma \in \partial\{.\}(0)$ such that

$$0 \in \partial(\gamma \circ \mathcal{F}^u)(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \ + \ \sum_{s\in D^l} \nu_s \partial\rho_s(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \ + \ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta),$$

equivalently,

$$0 \in \sum_{d\in D^u} \gamma_d \partial\mathcal{F}^u_d(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \ + \ \sum_{s\in D^l} \nu_s \partial\rho_s(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \ + \ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta).$$

Consequently

$$\begin{aligned} 0 \in & \{0\} \times \sum_{d\in D^u} \gamma_d \partial\mathcal{F}^u_d(\hat{a}^u,\ \hat{a}^l) \ + \ \{0\} \times \sum_{s\in D^l} \nu_s \partial\mathcal{F}^l_s(\hat{a}^u,\ \hat{a}^l) \\ & - \sum_{s\in D^l} \nu_s co\partial\Psi_s(\hat{z},\ \hat{a}^u) \times \{0\} \ + \ N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\ \Theta). \end{aligned}$$

Since $\Delta_{(-\text{int}\mathbb{R}^l_+)}(.)$ is a convex function and $\Delta_{(-\text{int}\mathbb{R}^l_+)}(0) = 0$, so $\forall\ v \in \mathbb{R}^l$

$$\Delta_{(-\text{int}\mathbb{R}^l_+)}(v) \geq \langle \gamma,\ v \rangle,$$

and $\forall v \in (-\mathbb{R}^l_+)$,

$$0 \geq \Delta_{(-\text{int}\mathbb{R}^l_+)}(v) = -d_{\mathbb{R}^l \setminus (-\text{int}\mathbb{R}^l_+)}(v) \geq \langle \gamma,\ v \rangle.$$

According to [82, Proposition 2.2], it is established that $\gamma \in \mathbb{R}^l_+$. It concludes our proof.

As a result of Theorem 3.1, we have Theorem 3.2, which states the $\mathcal{N}$. *opt*. *cond*. for the case $k \geq 2$.

Theorem 3.2 *Let $(\hat{z},\ \hat{a}^u,\ \hat{a}^l)$ be a local optimal point (Q^*). Let the assumptions of Theorem 3.1 holds. Furthermore, it is assumed that ll regularity (3.3) is satisfied at $\hat{a}^u$. Then, there exists*

$$\begin{aligned} & (a^*_d,\ b^*_d) \in \partial\mathcal{F}^u_d(\hat{a}^u,\ \hat{a}^l),\ ({a^u}^*_s,\ {a^l}^*_s) \in \partial\mathcal{F}^l_s(\hat{a}^u,\ \hat{a}^l),\ k^*_i \in \partial\mathcal{G}^u_i(\hat{a}^u), \\ & (p^*_j,\ q^*_j) \in \partial\mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l),\ (({a^u}^t_s)^*,\ ({a^l}^t_s)^*) \in \partial\mathcal{F}^l_s(\hat{a}^u,\ {a^l}_{s,t}), \\ & ((p^t_{js})^*,\ (q^t_{js})^*) \in \partial\mathcal{G}^l_j(\hat{a}^u,\ {a^l}_{s,t}),\ \nu \in \mathbb{R}^k_+,\ \alpha \in R^p_+,\ \gamma \in \mathbb{R}^l_+ \\ & \eta^{s,t},\ \eta \in \mathbb{R}^q_+,\ \delta^{s,t},\ \delta \in \mathbb{R}^{k-1}_+,\ {a^l}_{s,t} \in \mathfrak{J}_s(\hat{z},\ \hat{a}^u), \end{aligned}$$

and $\beta^{s,t} \geq 0,\ t \in \{1, 2, \ldots, n+k+1\}$ with $\sum_{t=1}^{n+k+1} \beta^{s,t} = 1,\ s \in D^l$ such that

1.

$$0=\sum_{d\in D^u}\gamma_d^* a_d^* \ + \ \sum_{i\in I}\alpha_i k_i \ + \ \sum_{j\in J}\eta_j p_j^* \ + \sum_{r\in D^l\setminus\{1\}}\delta_r(\hat z_r {a^u}_r^* - \hat z_1 {a^u}_1^*)$$
$$+ \sum_{s\in D^l}\nu_s {a^u}_s^* \ - \ \sum_{s\in D^l}\nu_s\Big(\sum_{t=1}^{n+k+1}\beta^{s,t}(({a^{ut}}_s)^* \ + \ \sum_{j\in J}\eta_j^{s,t}(p_{js}^t)^*$$
$$+ \sum_{r\in D^l\setminus\{s\}}\delta_r^{s,t}(\hat z_r({a^{ut}}_r)^* - \hat z_s({a^{ut}}_s)^*))\Big),$$

2. $0=\sum_{d\in D^u}\gamma_d^* b_d^* \ + \ \sum_{s\in D^l}\nu_s {a^l}_s^* \ + \ \sum_{j\in J}\eta_j q_j^* \ + \ \sum_{r\in D^l\setminus\{1\}}\delta_r(\hat z_r {a^l}_r^* - \hat z_1 {a^l}_1^*),$

3. $\forall s\in D^l,\ \forall t=1,2,\ldots,n+k+1$

$$0=({a^{lt}}_s)^* \ + \ \sum_{j\in J}\eta_j^{s,t}(q_{js}^t)^* \ + \ \sum_{r\in D^l\setminus\{s\}}\delta_r^{s,t}(\hat z_r({a^{lt}}_r)^* - \hat z_s({a^{lt}}_s)^*),$$

4. $\forall s\in D^l,\ \forall j\in J,\ \forall r\in D^l\setminus\{s\},\ \forall t=1,2,\ldots,n+k+1$

$$\eta_j^{s,t}\mathcal{G}_j^l(\hat a^u,\ {a^l}_{s,t})=0 \ \textit{and}\ \delta_r^{s,t}\Big(\hat z_r\mathcal{F}_r^l(\hat a^u,\ {a^l}_{s,t}) - \hat z_s\mathcal{F}_s^l(\hat a^u,\ {a^l}_{s,t})\Big)=0,$$

5. $\forall i\in I,\ \forall j\in J$

$$\alpha_i\mathcal{G}_i^u(\hat a^u)=0 \ \textit{and}\ \eta_j\mathcal{G}_j^l(\hat a^u,\hat a^l)=0.$$

Proof. By Theorem 3.1, $\exists\nu_s\geq 0,\ s\in D^l,\ \gamma_d\geq 0,\ \gamma\neq 0_{\mathbb{R}^l}$ such that

$$\begin{aligned}0\in&\{0\}\times\sum_{d\in D^u}\gamma_d\partial\mathcal{F}_d^u(\hat a^u,\ \hat a^l) \ + \ \{0\}\times\sum_{s\in D^l}\nu_s\partial\mathcal{F}_s^l(\hat a^u,\ \hat a^l)\\ &-\sum_{s\in D^l}\nu_s co\partial\Psi_s(\hat z,\ \hat a^u)\times\{0\} \ + \ N((\hat z,\ \hat a^u,\ \hat a^l),\ \Theta).\end{aligned} \tag{3.12}$$

As the mappings $\mathfrak{J}_s,\ s\in D^l$, are inner semicompact at $(\hat z,\ \hat a^u)$ and the point $(\hat a^u,\ {a^l}_s)$ is ll regular for all ${a^l}_s\in\mathfrak{J}_s(\hat z,\ \hat a^u),\ s\in D^l$, according to [109, Theorem 7], we obtain $\partial\Psi_s(\hat z,\ \hat a^u)$

$$\subseteq \bigcup_{a^l{}_s \in \mathfrak{J}_s(\hat{z}, \hat{a}^u)} \left\{ \begin{pmatrix} \sum_{r\in D^l\setminus\{s\}} \delta_r^s \nabla_z h_{s,r}(\hat{z},\ \hat{a}^u,\ a^l{}_s) \\ \tilde{a^u}{}^*_s \ + \ \sum_{j\in J} \eta_j^s p^*_{js} \ + \ \sum_{r\in D^l\setminus\{s\}} \delta_r^s(\hat{z}_r a^{u*}_r - \hat{z}_s a^{u*}_s) \end{pmatrix} : \left\{ \begin{array}{l} (\tilde{a^u}{}^*_s,\ \tilde{a^l}{}^*_s) \in \partial \mathcal{F}^l_s(\hat{a}^u,\ a^l{}_s),\ (p^*_{js},\ q^*_{js}) \in \partial \mathcal{G}^l_j(\hat{a}^u,\ a^l{}_s),\ s\in D^l,\ j\in J, \\ \eta_j^s \mathcal{G}^l_j(\hat{a}^u,\ a^l{}_s) = 0,\ j\in J,\ \eta_j^s \geq 0, \\ \delta_r^s(\hat{z}_r \mathcal{F}^l_r(\hat{a}^u,\ a^l{}_s) - \hat{z}_s \mathcal{F}^l_s(\hat{a}^u,\ a^l{}_s)) = 0,\ r\in D^l\setminus\{s\},\ \delta_r^s \geq 0, \\ \tilde{a^l}{}^*_s \ + \ \sum_{j\in J} \eta_j^s q^*_{js} \ + \ \sum_{r\in D^l\setminus\{s\}} \delta_r^s(\hat{z}_r \tilde{a^l}{}^*_r - \hat{z}_s \tilde{a^l}{}^*_s) = 0. \end{array} \right. \right\}$$

With the help of Carathéodory's Theorem to $co(\partial\Psi_s(\hat{z},\ \hat{a}^u)),\ s\in D^l$, we get $co(\partial\Psi_s(\hat{z},\ \hat{a}^u))$

$$\subseteq \left\{ \begin{pmatrix} \sum_{t=1}^{n+k+1} \beta^{s,t}\Big(\sum_{r\in D^l\setminus\{s\}} \delta_r^{s,t} \nabla_z h_{s,r}(\hat{z},\ \hat{a}^u,\ a^l{}_{s,t})\Big) \\ \sum_{t=1}^{n+k+1} \beta^{s,t}\Big((a^{ut}{}_s)^* \ + \ \sum_{j\in J} \eta_j^{s,t}(p^t_{js})^* \ + \ \sum_{r\in D^l\setminus\{s\}} \delta_r^{s,t}(\hat{z}_r(a^{ut}{}_r)^* - \hat{z}_s(a^{ut}{}_s)^*)\Big) \end{pmatrix} : \left\{ \begin{array}{l} a^l{}_{s,t} \in \mathfrak{J}_s(\hat{z},\ \hat{a}^u),\ \eta_j^{s,t},\ j\in J,\ \delta_r^{s,t},\ \beta^{s,t}, \\ \sum_{t=1}^{n+k+1} \beta^{s,t} = 1,\ r\in D^l\setminus\{s\},\ t=1,2,\ldots,n+k+1, \\ ((a^{ut}{}_s)^*,\ (a^{lt}{}_s)^*) \in \partial\mathcal{F}^l_s(\hat{a}^u,\ a^l{}_{s,t}),\ ((p^t_{js})^*,\ (q^t_{js})^*) \in \partial\mathcal{G}^l_j(\hat{a}^u,\ a^l{}_{s,t}),\ j\in J, \\ \eta_j^{s,t} \geq 0,\ \eta_j^{s,t}\mathcal{G}^l_j(\hat{a}^u,\ a^l{}_{s,t}) = 0,\ j\in J,\ t=1,2,\ldots,n+k+1, \\ \delta_r^s \geq 0, \delta_r^{s,t}(\hat{z}_r\mathcal{F}^l_r(\hat{a}^u, a^l{}_{s,t}) - \hat{z}_s\mathcal{F}^l_s(\hat{a}^u, a^l{}_{s,t})) = 0, \\ \qquad\qquad r\in D^l\setminus\{s\}, t=1,2,\ldots,n+k+1, \\ (a^{lt}{}_s)^* + \sum_{j\in J}\eta_j^{s,t}(q^t_{js})^* + \sum_{r\in D^l\setminus\{s\}} \delta_r^{s,t}(\hat{z}_r(a^{lt}{}_r)^* - \hat{z}_s(a^{lt}{}_s)^*) = 0, \\ \qquad\qquad t=1,2,\ldots,n+k+1. \end{array} \right. \right\} \quad (3.13)$$

Assume

$$\Gamma(z,\ a^u,\ a^l) = [\mathcal{G}^u(a^u),\ \mathcal{G}^l(a^u, a^l), -z]^T,$$

and

$$\xi(z,\ a^u,\ a^l) = \Big[z^T e - 1,\ \chi_2(z,\ a^u,\ a^l), \ldots, \chi_k(z,\ a^u,\ a^l)\Big]^T,$$

we have,

$$\Theta = \{(z,\ a^u,\ a^l):\ \Gamma(z,\ a^u,\ a^l) \leq 0,\ \xi(z,\ a^u,\ a^l) = 0\}.$$

From (3.1) and (3.3), we get

$$\left.\begin{array}{l} p\in\mathbb{R}_+^{p+q+k},\ q\in\mathbb{R}^k \\ 0 \in p\ \partial\Gamma(\hat{z},\ \hat{a}^u,\ \hat{a}^l) + q\ \partial\xi(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \\ \langle p,\ \Gamma(\hat{z},\ \hat{a}^u,\ \hat{a}^l)\rangle = 0 \end{array}\right\} \Longrightarrow [p=0,\ q=0].$$

Therefore, by [106, Theorem 6.14], we obtain

$$N((\hat{z},\ \hat{a}^u,\ \hat{a}^l),\Theta) \subseteq \left\{ \begin{pmatrix} -\lambda \ + \ \zeta e \ + \sum\limits_{r\in D^l\setminus\{1\}} \delta_r \nabla_z \chi_r(\hat{z},\ \hat{a}^u,\ \hat{a}^l) \\ \sum\limits_{i\in I} \alpha_i k_i^* \ + \ \sum\limits_{j\in J} \eta_j p_j^* \ + \sum\limits_{r\in D^l\setminus\{1\}} \delta_r(\hat{z}_r {a^u}^*_r - \hat{z}_1 {a^u}^*_1) \\ \sum\limits_{j\in J} \eta_j q_j^* \ + \sum\limits_{r\in D^l\setminus\{1\}} \delta_r(\hat{z}_r {a^l}^*_r - \hat{z}_1 {a^l}^*_1) \end{pmatrix} : \begin{array}{l} ({a^u}^*_s, {a^l}^*_s) \in \partial \mathcal{F}^l_s(\hat{a}^u,\ \hat{a}^l),\ k_i^* \in \partial \mathcal{G}^u_i(\hat{a}^u),\ (p_j^*,\ q_j^*) \in \partial \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) \\ \lambda_r \hat{z}_r = 0,\ r\in D^l,\ \lambda_r \geq 0, \\ \alpha_i \mathcal{G}^u_i(D^l \hat{x}_u) = 0,\ i\in I,\ \alpha_i \geq 0, \\ \eta_j \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l) = 0,\ j\in J,\ \eta_j \geq 0, \end{array} \right\} \tag{3.14}$$

incorporating (3.12), (3.13) and (3.14) we obtain the required result.

3.4 Example to validate Necessary Optimality Condition

The subsequent illustration serves to demonstrate the stated Theorem 3.2.

Example 3.1

$$(Q): \begin{cases} \mathbb{R}^2_+ - \min\limits_{a^u, a^l} \mathcal{F}^u(a^u,\ a^l) = (\mathcal{F}^u_1, \mathcal{F}^u_2)(a^u,\ a^l) = ((a^u-1)^2 + (a^l - \frac{1}{2})^2,\ |a^u - 1|) \\ \textit{subject to } \mathcal{G}^u_1(a^u) = a^{u2} - 4 \leq 0, \\ \qquad\qquad a^l \in \mathfrak{L}_{wef}(a^u), \end{cases}$$

where, for every $a^u \in \mathbb{R}$, $\mathfrak{L}_{wef}(a^u)$ is the collection of solutions of the below stated program

$$(Q_{a^u}): \begin{cases} \mathbb{R}^2_+ - \min\limits_{a^l} \mathcal{F}^l(a^u,\ a^l) = (\mathcal{F}^l_1,\ \mathcal{F}^l_2)(a^u,\ a^l) = (a^l + 2,\ 1 - a^l) \\ \textit{subject to } \mathcal{G}^l_1(a^u,\ a^l) = -a^l \leq 0, \\ \qquad\qquad \mathcal{G}^l_2(a^u,\ a^l) = a^l - 1 \leq 0, \\ \qquad\qquad (a^u,\ a^l) \in \mathbb{R}^2. \end{cases}$$

Therefore

$$(Q^1_{zx_u}): \begin{cases} \min\limits_{a^l} \mathcal{F}^l_1(a^u,\ a^l) = a^l + 2 \\ \textit{subject to } a^l \in K_1(z_1,\ z_2,\ a^u) \end{cases}$$

$$(Q^2_{zx_u}) : \begin{cases} \min\limits_{a^l} \mathcal{F}^l_2(a^u,\ a^l) = 1 - a^l \\ \textit{subject to } a^l \in K_2(z_1,\ z_2,\ a^u) \end{cases}$$

where

$$K_1(z_1,\ 1-z_1,\ a^u) : \begin{cases} [1-3z_1,\ 1] & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ [0,\ 1] & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$K_2(z_1,\ 1-z_1,\ a^u) : \begin{cases} [0,\ 1-3z_1] & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ \emptyset & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$\Psi_1(z_1,\ 1-z_1,\ a^u) : \begin{cases} 3(1-z_1) & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ 2 & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$\Psi_2(z_1,\ 1-z_1,\ a^u) : \begin{cases} 3z_1 & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ \infty & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$\rho_1(z_1,\ 1-z_1,\ a^u) : \begin{cases} a^l - 1 + 3z_1 & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ a^l & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$\rho_2(z_1,\ 1-z_1,\ a^u) : \begin{cases} 1 - a^l - 3z_1 & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ -\infty & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$\mathfrak{J}_1(z_1,\ 1-z_1,\ a^u) : \begin{cases} 1-3z_1 & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ 0 & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

$$\mathfrak{J}_2(z_1,\ 1-z_1,\ a^u) : \begin{cases} 1-3z_1 & \textit{if } 0 \leq z_1 \leq \frac{1}{3} \\ \emptyset & \textit{if } \frac{1}{3} < z_1 \leq 1 \end{cases}$$

and

$$\Theta = \Omega = \{(z,\ a^u,\ a^l) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} : z \in D^l,\ a^u \in X,\ a^l \in K(a^u),\ \chi(z, a^u, a^l) = 0\}$$

by simple calculation one can see that $\hat{w} = (\frac{1}{6}, \frac{5}{6}, 1, \frac{1}{2})$ *is a local optimal solution of* (Q^*)

$$(Q^*) : \begin{cases} \min\limits_{z,a^u,a^l} \ \phi(z,\ a^u,\ a^l) = \Delta_{-\mathrm{int}\mathbb{R}^2_+}((a^u-1)^2 + (a^l - \frac{1}{2})^2,\ |a^u - 1|) \\ \text{subject to} \ \ (z,\ a^u,\ a^l) \in \Omega. \end{cases}$$

On the one hand, we have

$$\left.\begin{array}{l} \alpha_1 \nabla \mathcal{G}^u_1(1) = 0 \\ \alpha_1 \mathcal{G}^u_1(1) = 0 \end{array}\right\} \Rightarrow \left\{ \ \alpha_1 = 0. \right.$$

The quantity (Q) *exhibits ul regularity at the point* $\hat{a}^u = 1$. *However,*

$$\left.\begin{array}{l} \eta_1 \mathcal{G}^l_1(1, \frac{1}{2}) = \eta_1(-\frac{1}{2}) = 0,\ \eta_1 \geq 0 \\ \eta_2 \mathcal{G}^l_2(1, \frac{1}{2}) = \eta_2(-\frac{1}{2}) = 0,\ \eta_2 \geq 0, \\ \sum\limits_{j=1}^{2} \eta_j \partial_{a^l} \mathcal{G}^l_j(1, \frac{1}{2}) + \delta_2 \partial_{a^l} \chi_2(\hat{w}) = 0,\ \delta_2 \geq 0. \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \eta_1 = \eta_2 = 0, \\ \delta_2 = 0. \end{array}\right.$$

Thus $(Q^s_{zx_u})$ *is ll regular at* $\hat{w}$. *Clearly* $\mathcal{N}(\hat{w},\ \Theta) = \{(0,0,0,0)\}$ *then* $bdN(\hat{w},\ \Theta) = \emptyset$. *For* $(\tau^*_1, \tau^*_2) \in \mathbb{R}^2_+ \setminus \{0\} \in$ *we have*

$$D^*\rho(\hat{w})(\tau^*_1, \tau^*_2) \cap -bdN(\hat{w}, \Theta) = \emptyset. \tag{3.15}$$

The set valued mappings $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *are inner semicompact at* $(\frac{1}{6},\ \frac{5}{6},\ 1,\ \frac{1}{2})$.
Then the opt. conds. are satisfied for

$$\begin{array}{l} (\gamma_1,\ \gamma_2,\ \alpha,\ \eta_1,\ \eta_2,\ \delta_2) = (1,\ 0,\ 0,\ 0,\ 0,\ \dfrac{1}{2}),\ (\nu_1,\ \nu_2) = (1,\ \dfrac{1}{2}), \\ (\eta^{1,t}_1,\ \eta^{1,t}_2,\ \eta^{2,t}_1,\ \eta^{2,t}_2) = (0,\ 0,\ 0,\ 0),\ (\delta^{1,t}_2, \delta^{2,t}_1) = (1,\ 1), \\ \sum\limits_{t=1}^{4} \beta^{1,t} = \sum\limits_{t=1}^{4} \beta^{2,t} = 1,\ {a^l}_{1,t} = {a^l}_{2,t} = \left\{\dfrac{1}{2}\right\},\ \forall t \in \{1,\ 2,\ 3,\ 4\}. \end{array}$$

Chapter 4

Optimality conditions and duality results for a robust bilevel programming problem [1]

4.1 Introduction

It has been seen in real-world problems that even relatively minor fluctuations of uncertain data can severely impair the feasibility and, hence, the significance of the nominal optimal solution as discussed in [16]. Therefore, an approach that generates "immunized against uncertainty" solutions is needed in applications. The only conventional approach of this kind is provided by stochastic programming, which substitutes the original constraints with their "chance versions" and assigns data fluctuation a probability distribution. This places a requirement on a candidate solution to satisfy the constraints with probability $\geq 1 - \epsilon, \ \epsilon \ll 1$ being a predetermined tolerance. However, there is no straightforward method to associate the data fluctuations with a probability distribution. To handle optimization problems with uncertain data, robust optimization can be seen as a supplement to stochastic programming. Here, the "uncertain-but-bounded" model of data fluctuation allows the uncertain data to flow through a specified uncertainty set. It requires that a candidate solution be robustly feasible - to satisfy the constraints regardless of how the data from this set are realized. One associates the original uncertain problem with its Robust Counterpart to obtain the robust optimal solution [29].

In this chapter, we have considered a BPP with uncertainty at the *ul* constraint. We consider here the case that the probability distribution which this data uncertainty follows is not known, but it is known that it belongs to an uncertainty set. Thus we have adopted the robust counterpart approach to deal with uncertainty. First, we transform the robust counterpart BPP into a sole level program via OVF reformulation. We have extended the ACQ to an extended non-smooth robust CQ. We have developed the *opt. conds.* in terms of sds. and CFs by using the concepts used by Kohli [76] and Gadhi [1] to deal with two levels and Chen et al. [29] to deal with the data uncertainty. Furthermore, MWD is introduced, and the relation between the two problems is established via weak and strong

[1] A part of this chapter has appeared as "*Optimality conditions and duality results for a robust bi-level programming problem*", RAIRO-Operations Research, 57 (2023) 525-539.

dlty. theorems. To the best of the author's knowledge, the *opt. conds.* for robust bilevel problems still need to be developed. Hence the results obtained here are new and will help researchers develop new theories in this exciting new field of robust BPPs.
The remainder of the chapter as pursues: Section 4.2 presents the robust BPPs and its reformulation to the single-level robust counterpart problem. Section 4.3 is devoted to the development of the robust CQ. In Section 4.4, we have developed the opt. cond. for the considered BPP under the appropriate assumptions.In Section 4.5 an application of $\mathcal{N}$. *opt. conds* is given. Section 4.6 is devoted to constructing MWD and establishing the weak and strong *dlty.* results.

4.2 Non-smooth Robust Bilevel Model

In this Section, we have considered an uncertain BPPs (Q). With the help of optimal value reformulation and the concept of robust counterpart, we transform (Q) to a single-level robust counterpart problem $(RBPP)$. Later in this Section, we have introduced the extended non-smooth RCQ. Let the problem (Q) be defined as:

$$(Q) \qquad \min_{a^u,\ a^l} \mathcal{F}^u(a^u, a^l)$$
$$\text{subject to } \mathcal{G}_i^u(a^u, \nu_i) \leq 0, \quad \forall i \in I,$$
$$a^l \in \Upsilon(a^u),$$

where for some sequentially compact topological space Ω_i, $\nu_i \in \Omega_i$ is an uncertain parameter. For each $a^u \in \mathbb{R}^{n_a}$ the parametric optimization problem (Q_{a^u}) has the set of optimal solutions $\Upsilon(a^u)$

$$(Q_{a^u}) \qquad \min_{a^l} \mathcal{F}^l(a^u, a^l)$$
$$\text{subject to } \mathcal{G}_j^l(a^u, a^l) \leq 0, \quad \forall j \in J,$$

where, $\mathcal{F}^u : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}$, is $l\mathcal{L}$ function, $\mathcal{G}_i^u : \mathbb{R}^{n_a} \times \Omega_i \to \mathbb{R},\ i \in I = \{1,\ 2, \ldots,\ n\}$ is a function which is convex in a^u, continuous in ν and is upper semicontinuous in (a^u, ν), $\mathcal{F}^l : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}$ and $\mathcal{G}_j^l : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R},\ j \in J = \{1,\ 2, \ldots,\ m\}$ are continuous convex functions and $\Upsilon(a^u) = \arg \min_{a^l}\{\mathcal{F}^l(a^u,\ a^l) :\ \mathcal{G}^l(a^u,\ a^l) \leq 0\}$; $n,\ n_1,\ n_2$ and m are whole numbers.
When the decision-maker lacks information on the probability distribution of the uncertain parameters, a well-known robust approach known as a robust counterpart is employed to analyze the problem.

The robust counterpart of (Q) is:

$$(RP') \qquad \min_{a^u,a^l} \mathcal{F}^u(a^u,a^l)$$
$$\text{subject to } \ \mathcal{G}_i^u(a^u,\nu_i) \le 0, \quad \forall \nu_i \in \Omega_i, \ \forall i \in I,$$
$$a^l \in \Upsilon(a^u),$$

where the uncertain constraints are applied to all alternative values of the parameters within their specified uncertainty sets $\Omega_i, \ i = 1, \ 2, \dots, \ n$. The problem (RP') can be viewed as the severest possible scenario of (Q). The robust counterpart is a model that resolves the uncertain worst-case scenario without using uncertain variables. Therefore, optimizing (Q) with (RP') is the robust technique (worst technique) for (Q). A feasible solution to the robust counterpart problem is the robust feasible solution to the uncertain problem (Q) which should, by definition, fulfill all realizations of the constraints from Ω_i, $i = 1, \ 2, \dots, \ n$ (uncertainty sets). An optimal solution of (RP') is a robust feasible solution with the best possible value of the objective.
Using OVF reformulation, we get

$$(RBPP) \qquad \min_{a^u,a^l} \mathcal{F}^u(a^u,a^l)$$
$$\text{subject to } \ \mathcal{G}_i^u(a^u,\nu_i) \le 0, \quad \forall \nu_i \in \Omega_i, \ \forall \ i \in I,$$
$$\mathcal{G}_j^l(a^u,a^l) \le 0, \qquad \forall j \in J,$$
$$\mathcal{F}^l(a^u,a^l) \le \varphi(a^u),$$

where,

$$\varphi(a^u) = \inf_{a^l}\{\mathcal{F}^l(a^u,a^l) : \ \mathcal{G}_j^l(a^u,a^l) \le 0, \quad \forall j \in J\}$$

is the OVF. Since $\mathcal{F}^l(., \ .)$ and $\mathcal{G}_j^l(., \ .)$, $j \in J$ are convex, therefore, $\varphi(.)$ is also a convex function.
Suppose

$$X = \{a^u \in \mathbb{R}^{n_a} : \ \mathcal{G}_i^u(a^u,\nu_i) \le 0, \quad \forall \nu_i \in \Omega_i, \ \forall i \in I\}$$

represents the ul constraints and

$$S(a^u) = \{a^l \in \mathbb{R}^{n_b} : \ \mathcal{G}_j^l(a^u,a^l) \le 0, \quad \forall j \in J\}$$

be the feasible set of ll-problem for a fixed a^u and

$$H = \{(a^u, \ a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} : \mathcal{F}^l(a^u,a^l) - \varphi(a^u) \le 0, \ \mathcal{G}_j^l(a^u,a^l) \le 0, \ j \in J,$$
$$\mathcal{G}_i^u(a^u,\nu_i) \le 0, \ \forall \ \nu_i \in \Omega_i, \ i \in I\}$$

be the feasible set of $(RBPP)$.

Remark 4.1 *If the second variable of the upper-level constraints become certain* $\forall i \in I$ *then the given problem (Q) reduces to the one studied by [76] and [1].*

Definition 4.1 *A robust feasible point* $(\hat{a}^u, \hat{a}^l) \in H$ *is called a robust local minimizer of* $(RBPP)$ *if there exist open balls* N_{a^u} *of* $\hat{a}^u$ *and* N_{a^l} *of* $\hat{a}^l$ *such that for each* $(a^u,\ a^l) \in H \cap (N_{a^u} \times N_{a^l})$ *one has* $\mathcal{F}^u(a^u, a^l) \geq \mathcal{F}^u(\hat{a}^u, \hat{a}^l)$.

On the lines of [139, Lemma 1.1], we have the following Lemma:

Lemma 4.1 *As long as (Q) admits a solution,* $(\hat{a}^u, \hat{a}^l)$ *is a solution to* (Q) *iff it is a robust local minimizer of* $(RBPP)$.

Lemma 4.2 [1] *Let* S *be a nonempty, convex and compact set and* K *be a convex cone. If*

$$\sup_{z \in S} \langle z,\ d \rangle \geq 0, \qquad \forall d \in K^-$$

then

$$0 \in S + cl\ K.$$

Let Ψ, $\Psi_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, be real-valued functions defined as follows:

$$\Psi_i(a^u) = \max_{\nu_i \in \Omega_i} \mathcal{G}_i^u(a^u, \nu_i)$$

and

$$\Psi(a^u) = \max_{i \in \{1, \ldots, n\}} \Psi_i(a^u).$$

Ψ_i is well defined, since $\mathcal{G}_i^u$ is upper semicontinuous and $\Omega_i \neq \emptyset$, convex and compact $\forall j \in J$.

Lemma 4.3 [29] *Let* I *represents a finite index set and* $h_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ *are* $l\mathcal{L}$ *functions. Set* $h(a^u) = \max_{i \in I} h_i(a^u)$. *Then* $\partial h(a^u) \subseteq conv\{\partial h_i(a^u) :\ i \in I(a^u)\}\ \forall\ a^u \in \mathbb{R}^n$, *where* $I(a^u) = \{i \in I :\ h_i(a^u) = h(a^u)\}$.

4.3 Extended Non-smooth Robust Constraint Qualification

For a nonlinear programming problem dealing with the smooth functions, a well-known CQ known as ACQ is used, which is weaker than most CQs and was introduced by [2]. However, since the single-level problem in our case is non-smooth and has an uncertain parameter, we should indeed extend the ACQ. So, the extended non-smooth robust CQ (RCQ) for all $\nu_i \in \Omega_i(\hat{a}^u)$ is defined as

$$\Big(\bigcup_{j\in J_m} conv\ \partial^* \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) \bigcup \bigcup_{i\in I_n} conv\ \partial\Psi(\hat{a}^u) \times \{0\} \\ \bigcup \partial\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial\varphi(\hat{a}^u) \times \{0\}\Big)^- \subseteq T(H,\ (\hat{a}^u, \hat{a}^l)),$$

where $I_n = \{i \in I :\ \Psi_i(\hat{a}^u) = 0\}$, $J_m = \{j \in J :\ \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) = 0\}$, $\Omega_i(\hat{a}^u) = \{\nu_i \in \Omega_i :\ \mathcal{G}^u_i(\hat{a}^u,\ \nu_i) = \Psi_i(\hat{a}^u)\}$ and $T(H,\ (\hat{a}^u, \hat{a}^l))$ represents the tangent cone to H at $(\hat{a}^u, \hat{a}^l)$ designed as

$$T(H,\ (\hat{a}^u, \hat{a}^l)) = \Big\{(l_1,\ l_2) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} :\ \exists t_n \downarrow 0 \text{ and } (d_{n_a},\ d_{n_b}) \to (l_1,\ l_2) \\ \text{such that } (\hat{a}^u, \hat{a}^l) + t_n(d_{n_a},\ d_{n_b}) \in H\Big\}.$$

4.4 Necessary Condition

The core of optimization theory is the concept of *opt. cond.*. The existence of *opt. conds.* enables the generation of efficient numerical approaches for the practical solution of a given mathematical program. $\mathcal{N}$. *opt. cond.* are beneficial because any local minimum must satisfy these conditions. As a result, one can only look for local (or global) minima among points that hold the $\mathcal{N}$. *opt. cond.*.
In this section, we present the $\mathcal{N}$. *opt. conds* for (Q) by using the optimistic approach. Before proceeding with the main results of this paper we need to define the *ll* regularity condition. The *ll* regularity of (Q_{a^u}) at $(\hat{a}^u, \hat{a}^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, is given as:

$$\left.\begin{array}{l} 0 \in \sum_{j\in J_m} \eta_j \partial_{a^l} \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l), \\ \eta_j \geq 0,\ \eta_j \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) = 0,\ j \in J_m \end{array}\right\} \Longrightarrow [\eta_j = 0,\ j \in J_m].$$

Theorem 4.1 (Necessary Condition) *Let $(\hat{a}^u, \hat{a}^l) \in H$ be a robust local minimizer of (Q). Assume that $\mathcal{F}^u$ is $l\mathcal{L}$ and admit bounded (USRCF) $\partial^*\mathcal{F}^u(\hat{a}^u, \hat{a}^l)$ at $(\hat{a}^u, \hat{a}^l)$, Ω_i is convex and that the function $\mathcal{G}^u_i(a^u,\ .)$ is concave on Ω_i, for each $a^u \in X$ and for each $i \in I$. Furthermore, we suppose that the extended non-smooth robust CQ holds at $(\hat{a}^u, \hat{a}^l)$. Suppose that the argminimum map Υ is inner semicompact at $\hat{a}^u$, that for each vector $a^l \in \Upsilon(\hat{a}^u)$, $(\hat{a}^u,\ a^l)$ is ll regular. Then, there exist $\hat{\nu}_i \in \Omega_i(\hat{a}^u)$, $\lambda_0 \geq 0$, $\mu_j \geq 0$, $j \in J_m$, $\tau_i \geq 0$, $i \in I_n$, $\lambda_j \geq 0$, $j \in J$ and also $x^{l*} \in \Upsilon(\hat{a}^u)$ along with the closedness of* *cone A, where*

$$A := \Big(\bigcup_{i\in I_n} conv\ \partial\Psi(\hat{a}^u)\times\{0\}\Big)\cup\Big(\bigcup_{j\in J_m} conv\ \partial^*\mathcal{G}^l_j(\hat{a}^u,\hat{a}^l)\Big) \cup\Big(\partial\mathcal{F}^l(\hat{a}^u,\hat{a}^l)-\partial\varphi(\hat{a}^u)\times\{0\}\Big),$$

such that the following conditions satisfy:

1. $0\in conv\ \partial^*\mathcal{F}^u(\hat{a}^u,\hat{a}^l) + \sum_{i\in I_n}\tau_i conv\ \partial_{a^u}\mathcal{G}^u_i(\hat{a}^u,\ \hat{\nu}_i) + \sum_{j\in J_m}\mu_j conv\ \partial^*\mathcal{G}^l_j(\hat{a}^u,\hat{a}^l) + \lambda_0\big(\partial\mathcal{F}^l(\hat{a}^u,\hat{a}^l) - \partial_{a^u}\mathcal{F}^l(\hat{a}^u,\ x^{l*})\times\{0\}\big) - \lambda_0\Big(\sum_{j\in J}\lambda_j\partial_{a^u}\mathcal{G}^l_j(\hat{a}^u,\ x^{l*})\times\{0\}\Big).$
2. $0\in\partial_{a^l}\mathcal{F}^l(\hat{a}^u,\ x^{l*}) + \sum_{j\in J}\lambda_j\partial_{a^l}\mathcal{G}^l_j(\hat{a}^u,\ x^{l*}).$
3. $\lambda_j\ge 0,\ 0=\lambda_j\mathcal{G}^l_j(\hat{a}^u,\ x^{l*}),\quad \forall j=1,2,\dots,m.$
4. $\mu_j\ge 0,\ 0=\mu_j\mathcal{G}^l_j(\hat{a}^u,\hat{a}^l),\quad \forall j=1,2,\dots,m.$
5. $\tau_i\ge 0,\ 0=\tau_i\mathcal{G}^u_i(\hat{a}^u,\ \hat{\nu}_i),\quad \forall i=1,2,\dots,n.$

Here, ∂, ∂_{a^u}, ∂_{a^l} symbolizes, respectively, the full and partial sds. of convex analysis.

Proof. Let $(l_1,\ l_2)\in T(H,\ (\hat{a}^u,\hat{a}^l))$, from the definition of tangent cone, $\exists t_n\downarrow 0$ and $(l_{n_a},\ l_{n_b})\to(l_1,\ l_2)$ such that $(\hat{a}^u,\hat{a}^l)+t_n(l_{n_a},\ l_{n_b})\in H\ \forall n$.
Since $(\hat{a}^u,\hat{a}^l)$ is robust local minimum of $\mathcal{F}^u$ over H, therefore, we have

$$\frac{\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_{n_a},\ l_{n_b}))-\mathcal{F}^u(\hat{a}^u,\hat{a}^l)}{t_n}\ge 0,\qquad \text{for sufficiently large } n. \tag{4.1}$$

Now,

$$\begin{aligned}&\frac{\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_{n_a},\ l_{n_b}))-\mathcal{F}^u(\hat{a}^u,\hat{a}^l)}{t_n}\\ &=\frac{\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_{n_a},\ l_{n_b}))-\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_1,\ l_2))}{t_n}\\ &\quad+\frac{\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_1,\ l_2))-\mathcal{F}^u(\hat{a}^u,\hat{a}^l)}{t_n}.\end{aligned} \tag{4.2}$$

Since $\mathcal{F}^u$ is $l\mathcal{L}$, thus,

$$\frac{\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_{n_a},\ l_{n_b}))-\mathcal{F}^u((\hat{a}^u,\hat{a}^l)+t_n(l_1,\ l_2))}{t_n}\to 0 \text{ as } n\to\infty.$$

Take on both sides of the equation (4.2) Limit supremum and from above and (4.1), we get

$$\limsup_{t_n \to 0^+} \frac{\mathcal{F}^u((\hat{a}^u, \hat{a}^l) + t_n(l_{n_a},\ l_{n_b})) - \mathcal{F}^u(\hat{a}^u, \hat{a}^l)}{t_n} = (\mathcal{F}^u)^+_d((\hat{a}^u, \hat{a}^l), (l_1,\ l_2)) \geq 0.$$

That is, we have

$$(\mathcal{F}^u)^+_d((\hat{a}^u, \hat{a}^l), (l_1,\ l_2)) \geq 0, \quad \forall (l_1,\ l_2) \in T(H,\ (\hat{a}^u, \hat{a}^l)).$$

Since $\partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l)$ is upper semiregular CF at $(\hat{a}^u, \hat{a}^l)$, we get

$$\sup_{\eta \in \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l)} \langle \eta,\ (l_1,\ l_2) \rangle \geq 0, \quad \forall (l_1,\ l_2) \in T(H,\ (\hat{a}^u, \hat{a}^l)). \tag{4.3}$$

Since extended non-smooth RCQ holds at $(\hat{a}^u, \hat{a}^l)$, we have

$$\sup_{\eta \in conv\ \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l)} \langle \eta,\ (l_1,\ l_2) \rangle \geq 0, \quad \forall (l_1,\ l_2) \in (cone\ A)^- \tag{4.4}$$

where $(cone\ A)^-$ is a negative polar cone of A and A is defined as

$$A = \Big(\bigcup_{i \in I_n} conv\ \partial \Psi(\hat{a}^u) \times \{0\} \Big) \cup \Big(\bigcup_{j \in J_m} conv\ \partial^* \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) \Big)$$
$$\cup \Big(\partial \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial \varphi(\hat{a}^u) \times \{0\} \Big).$$

Since $\partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l)$ is a closed set, $conv\ \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l)$ is a compact set [66, Theorem 1.4.3]. By Lemma 4.2, we get

$$(0,\ 0) \in conv\ \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l) + cl\ cone\ A.$$

Since $cone\ A$ is closed, we have

$$(0,\ 0) \in conv\ \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l) + cone\ A.$$

Therefore,

$$0 \in conv\ \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l) + \sum_{i \in I_n} cone\ \partial \Psi(\hat{a}^u) \times \{0\} + \sum_{j \in J_m} cone\ \partial^* \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l)$$
$$+ cone\ (\partial \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial \varphi(\hat{a}^u) \times \{0\}). \tag{4.5}$$

From Lemma 4.3, we obtain

$$\partial\Psi(\hat{a}^u) \subseteq conv\{\partial\Psi_i(\hat{a}^u) : \ i \in I_n\}, \tag{4.6}$$

by [83, Theorem 2.4 p. 290], we get

$$\partial\Psi_i(\hat{a}^u) = \{\xi_i : \ \exists \nu_i \in \Omega_i(\hat{a}^u) \text{ such that } \xi_i \in \partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \ \nu_i)\}, \tag{4.7}$$

for all $i \in I$, where $\Omega_i(\hat{a}^u) = \{\nu_i \in \Omega_i : \mathcal{G}_i^u(\hat{a}^u, \nu_i) = \Psi_i(\hat{a}^u)\} \neq \emptyset$ is convex and compact set. Thus, (4.6) and (4.7) imply that $\exists\ \hat{\nu} = (\hat{\nu}_1, \hat{\nu}_2, \ldots, \hat{\nu}_n) \in \Omega$ such that

$$\mathcal{G}_i^u(\hat{a}^u, \ \hat{\nu}_i) = \Psi_i(\hat{a}^u), \quad i \in I$$

and

$$\partial\Psi(\hat{a}^u) \subseteq conv \ \{\partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \ \hat{\nu}_i) : \ i \in I_n\}.$$

From above and (4.5), we obtain

$$\begin{aligned} 0 \in conv\ \partial^*\mathcal{F}^u(\hat{a}^u, \hat{a}^l) + \sum_{i\in I_n} cone\ conv\ \partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \ \hat{\nu}_i) \times \{0\} \\ + \sum_{j\in J_m} cone\ \partial^*\mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) + cone\ (\partial\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial\varphi(\hat{a}^u) \times \{0\}). \end{aligned}$$

Consequently, we can find scalars $\lambda_0 \geq 0, \mu_j \geq 0, j \in J_m$ and $\tau_i \geq 0, i \in I_n$ such that

$$\begin{aligned} 0 \in conv\ \partial^*\mathcal{F}^u(\hat{a}^u, \hat{a}^l) + \sum_{i\in I_n} \tau_i\ conv\ \partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \ \hat{\nu}_i) \times \{0\} \\ + \sum_{j\in J_m} \mu_j\ conv\ \partial^*\mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) + \lambda_0\ conv\ (\partial\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial\varphi(\hat{a}^u) \times \{0\}) \end{aligned}$$

and the complementary slackness conditions (4) and (5) holds. Since $\partial\mathcal{F}^l(\hat{a}^u, \hat{a}^l)$ and $\partial\varphi(\hat{a}^u)$ are convex, we obtain

$$\begin{aligned} 0 \in conv\ \partial^*\mathcal{F}^u(\hat{a}^u, \hat{a}^l) + \sum_{i\in I_n} \tau_i\ conv\ \partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \ \nu_i) \times \{0\} \\ + \sum_{j\in J_m} \mu_j\ conv\ \partial^*\mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) + \lambda_0\ (\partial\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial\varphi(\hat{a}^u) \times \{0\}). \end{aligned} \tag{4.8}$$

Moreover, with the help of the following relationship for convex, continuous function $\mathcal{F}^l(a^u, \ a^l)$ that holds, [97, Corollary 3.44]

$$\partial\mathcal{F}^l(a^u, \ a^l) \subset \partial_{a^u}\mathcal{F}^l(a^u, \ a^l) \times \partial_{a^l}\mathcal{F}^l(a^u, \ a^l). \tag{4.9}$$

By using (4.9) and [109, Theorem 8], the upper estimate of $\mathcal{C}.sd$ of the value function at $\hat{a}^u$ is as follows:

$$\partial\varphi(\hat{a}^u) := \Big[\bigcup_{a^l\in\Upsilon(\hat{a}^u)}\Big\{\bigcup_{(\lambda_1,\ldots,\lambda_m)\in\Lambda(\hat{a}^u,\ a^l)}\Big(\partial_{a^u}\mathcal{F}^l(\hat{a}^u,\ a^l)+\sum_{j\in J}\lambda_j\partial_{a^u}\mathcal{G}^l_j(\hat{a}^u,\ a^l)\Big)\Big\}\Big] \tag{4.10}$$

where $\Lambda(\hat{a}^u,\ a^l)$ is defined by

$$\Lambda(\hat{a}^u,\ a^l) := \Big\{(\lambda_1,\ldots,\lambda_m)\in\mathbb{R}^m : 0\in\partial_{a^l}\mathcal{F}^l(\hat{a}^u,\ a^l)+\sum_{j\in J}\lambda_j\partial_{a^l}\mathcal{G}^l_j(\hat{a}^u,\ a^l),$$
$$\lambda_j\geq 0,\ \lambda_j\mathcal{G}^l_j(\hat{a}^u,\ a^l)=0,\quad j\in J\Big\}. \tag{4.11}$$

Combining (4.8), (4.10) and (4.11), we have the $\mathcal{N}$. *opt. conds.* (1), (2) and (3).

4.5 Application of Necessary Condition

Example 4.1 *Let $(a^u,\ a^l)\in\mathbb{R}^2$ and $\nu\in\Omega=[0,\ 1]$. Clearly Ω is convex and compact. Consider the below stated robust BPP*

$$(Q)\qquad \min_{a^u,\ a^l}\ \mathcal{F}^u(a^u,a^l)=|a^u|+|a^l|+2a^u+a^l$$
$$\text{subject to}\ \ \mathcal{G}^u_1(a^u,\ \nu)=a^u-\nu\leq 0,$$
$$a^l\in\Upsilon(a^u),$$

where for each $a^u\in\mathbb{R}^{n_a}$, the parametric optimization problem (Q_{a^u}) has the set of optimal solutions $\Upsilon(a^u)$

$$(Q_{a^u})\qquad \min_{a^l}\ \mathcal{F}^l(a^u,a^l)=|a^u|+a^l$$
$$\text{subject to}\ \ \mathcal{G}^l_1(a^u,\ a^l)=-a^l\leq 0,$$
$$\mathcal{G}^l_2(a^u,\ a^l)=a^l-1\leq 0.$$

The OVF is

$$\varphi=\inf_{a^l}\{|a^u|+a^l:\ a^l\in[0,1]\}$$

and

$$\Psi=\max_{\nu\in[0,1]}\{a^u-\nu\}=a^u.$$

Collection of feasible points of reformulated problem $(RBPP)$ corresponding to Q is

$$H=\{(a^u,\ 0):a^u\leq 0\}.$$

Clearly $\hat{a} = (\hat{a^u},\ \hat{a^l}) = (0,\ 0) \in H$ *is the robust local minimizer of* (Q).

- *The set* $\partial^* \mathcal{F}^u(\hat{a}) = \{(1,\ 0), (1,\ 2), (3,\ 0), (3,\ 2)\}$ *is USRCF of* $\mathcal{F}^u$.

- *The function* $a^u - \nu$ *is concave on* Ω *for* a^u*=0.*

- *The non-smooth robust CQ holds at* $\hat{a}$*. Indeed, since*

$$\begin{aligned} C(\hat{a}) = \Big(&\underset{j \in J_m}{\cup} conv\ \partial^* \mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) \bigcup \underset{i \in I_n}{\cup} conv\ \partial \Psi(\hat{a}^u) \times \{0\} \\ &\bigcup \partial \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial \varphi(\hat{a}^u) \times \{0\} \Big) = \{(0,\ -1), (0,\ 1), (1,\ 0), (0,\ 1)\} \\ (C(\hat{a}))^- &\subseteq T(H,\ (0,\ 0)) = \{(a^u,\ 0) :\ a^u \leq 0\} \end{aligned}$$

- *The argminimum map* Υ *is inner semicompact at* $\hat{a^u} = 0$.

- (Q_{a^u}) *is ll regular at* $(0,\ 0)$ *since*

$$\left.\begin{aligned} &\eta_1(-1) + \eta_2(1) = 0 \\ &\eta_1,\ \eta_2 \geq 0,\ \eta_1(-a^l) + \eta_2(a^l - 1) = 0, \end{aligned}\right\} \Longrightarrow [\eta_1,\ \eta_2 = 0].$$

- *For* $(\lambda_0,\ \mu_1,\ \mu_2,\ \tau_1,\ \lambda_1,\ \lambda_2)$= $(1,\ 1,\ 0,\ 0,\ 1,\ 0)$, *the* $\mathcal{N}$. *opt. conds. (1)-(5) holds.*

4.6 Robust Bilevel Mond-Weir Dual

MWD and WD are the two widely used dual in literature. Due to the weaker assumptions used, MWD has an advantage over WD. Here we have formulated a robust bilevel MWD ($RBMWD$) corresponding to the considered robust BLPP. Moreover, we have developed the relationship between the solutions of these two problems in terms of weak and strong *dlty.* theorems.

$$
\begin{aligned}
(RBMWD) \qquad \max \quad & \mathcal{F}^u(u,\ v) \\
\text{subject to} \quad & 0 \in conv\ \partial^* \mathcal{F}^u(u,\ v) + \sum_{i \in I_n} \alpha_i\ conv\ \partial_{a^u} \mathcal{G}_i^u(u,\ \nu_i) \times \{0\} \\
& + \sum_{j \in J_m} \beta_j\ conv\ \partial^* \mathcal{G}_j^l(u,\ v)\ +\ \gamma \Big(\partial \mathcal{F}^l(u, v) \\
& - \partial \varphi(u) \times \{0\} \Big), \\
& \mathcal{F}^l(u, v) - \varphi(u) \geq 0, \\
& \alpha_i \mathcal{G}_i^u(u, \nu_i) \geq 0, \quad \nu_i \in \Omega_i, \\
& \beta_j \mathcal{G}_j^l(u,\ v) \geq 0, \\
& \gamma \geq 0, \quad \alpha_i \geq 0, \quad i \in I_n, \quad \beta_j \geq 0, \quad j \in J_m.
\end{aligned}
$$

Theorem 4.2 (Weak Duality) *Consider a feasible point* $(a^u,\ a^l)$ *of problem (Q), and let* $(u,\ v,\ \nu,\ \alpha,\ \beta,\ \gamma)$ *be a feasible point of problem* $(RBMWD)$*. Let* $\mathcal{F}^u(.,\ .)$ *be* ∂^**-pseudoconvex at* $(u,\ v)$ *and* $\mathcal{F}^l(.,\ .) - \varphi(.)$, $\mathcal{G}_i^u(.,\ \nu_i)$, $i \in I_n$ *and* $\mathcal{G}_j^l(.,\ .)$, $j \in J_m$ *are* ∂^**-quasiconvex at* $(u,\ v)$*, then*

$$
\mathcal{F}^u(a^u, a^l) \geq \mathcal{F}^u(u,\ v).
$$

Proof. By contrary, suppose that $\mathcal{F}^u(a^u, a^l) < \mathcal{F}^u(u,\ v)$. Since $\mathcal{F}^u$ is ∂^*-pseudoconvex at $(u,\ v)$, we have

$$
\langle \xi_1,\ (a^u - u,\ a^l - v) \rangle < 0, \qquad \forall \xi_1 \in \partial^* \mathcal{F}^u(u,\ v). \tag{4.12}
$$

Since $(u,\ v,\ \nu,\ \alpha,\ \beta,\ \gamma)$ is feasible for $(RBMWD)$, we obtain $\xi_1 \in$ conv $\partial^* \mathcal{F}^u(u,\ v)$, $\xi_2 \in$ conv $(\partial^* \mathcal{F}^l(u, v) - \partial^* \varphi(u) \times \{0\})$, $\eta_i \in$ conv $\partial_{a^u} \mathcal{G}_i^u(u,\ \nu_i)$ and $\mu_j \in$ conv $\partial^* \mathcal{G}_j^l(u,\ v)$ such that

$$
0 = \xi_1 + \gamma \xi_2 + \sum_{i \in I_n} \alpha_i \eta_i + \sum_{j \in J_m} \beta_j \mu_j. \tag{4.13}
$$

Since $(a^u,\ a^l)$ is feasible for (Q) and by Lemma 4.1 $(a^u,\ a^l)$ is feasible for $(RBPP)$. $(u,\ v,\ \nu,\ \alpha,\ \beta,\ \gamma)$ is feasible for $(RBMWD)$ thus we have $\mathcal{F}^l(a^u, a^l) - \varphi(a^u) \leq \mathcal{F}^l(u, v) - \varphi(u)$. From ∂^*-quasiconvexity of $\mathcal{F}^l(.,\ .) - \varphi(.)$, we get

$$
\langle \gamma \xi_2,\ (a^u - u,\ a^l - v) \rangle \leq 0, \quad \forall \xi_2 \in \partial^*(\mathcal{F}^l(u, v) - \varphi(u)) \text{ and } \gamma \geq 0. \tag{4.14}
$$

Similarly for $\alpha_i \geq 0$ and $\beta_j \geq 0$, $\alpha_i \mathcal{G}_i^u(a^u, \nu_i) \leq \alpha_i \mathcal{G}_i^u(u, \nu_i),\ i \in I_n$ and $\beta_j \mathcal{G}_j^l(a^u,\ a^l) \leq \beta_j \mathcal{G}_j^l(u, v),\ j \in J_m$. By ∂^*-quasiconvexity of $\mathcal{G}_i^u(.,\ \nu_i)$ and $\mathcal{G}_j^l(.,\ .)$, we get

$$\langle \sum_{i \in I_n} \alpha_i \eta_i,\ (a^u - u,\ 0) \rangle \leq 0, \tag{4.15}$$

$$\langle \sum_{j \in J_m} \beta_j \mu_j,\ (a^u - u,\ a^l - v) \rangle \leq 0. \tag{4.16}$$

From (4.12), (4.14) and (4.16), we have

$$\langle \xi_1 + \gamma \xi_2 + \sum_{j \in J_m} \beta_j \mu_j,\ (a^u - u,\ a^l - v) \rangle < 0, \tag{4.17}$$

from (4.15) and (4.17), we get the contradiction to (4.13).

Theorem 4.3 (Strong Duality) *Let $(\hat{a}^u, \hat{a}^l)$ be a robust minimizer of (Q) where the extended robust non-smooth CQ holds. Then, there exists $\hat{\nu} \in \Omega$, $(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma}) \in \mathbb{R}_+^{n+m+1}$ such as $(\hat{a}^u,\ \hat{a}^l,\ \hat{\nu},\ \hat{\alpha},\ \hat{\beta},\ \hat{\gamma})$ is feasible for $(RBMWD)$ and values of two obj. fns are equal. Moreover, if the hypotheses of Theorem 4.2 holds, then $(\hat{a}^u,\ \hat{a}^l,\ \hat{\nu},\ \hat{\alpha},\ \hat{\beta},\ \hat{\gamma})$ is a robust maximizer of $(RBMWD)$.*

Proof. Let$(\hat{a}^u, \hat{a}^l)$ be a robust minimizer of (Q) where the extended non-smooth robust CQ holds. Using Theorem 4.1 there exist $\hat{\nu} \in \Omega$ and $(\hat{\alpha},\ \hat{\beta},\ \hat{\gamma}) \in \mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+$ such that

$$\begin{aligned} 0 \in conv\ \partial^* \mathcal{F}^u(\hat{a}^u, \hat{a}^l) + \sum_{i \in I_n} \hat{\alpha}_i\ conv\ \partial_{a^u} \mathcal{G}_i^u(\hat{a}^u, \hat{\nu}) \times \{0\} \\ + \sum_{j \in J_m} \hat{\beta}_i\ conv\ \partial^* \mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) + \hat{\gamma}\ (\partial \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial \varphi(\hat{a}^u) \times \{0\}). \end{aligned}$$

By Lemma 4.1 $(\hat{a}^u, \hat{a}^l)$ is feasible for $(RBPP)$ and from

$$\varphi(a^u) = \inf_{a^l} \{\mathcal{F}^l(a^u, a^l) :\ \mathcal{G}_j^l(a^u, a^l) \leq 0, \quad \forall j \in J\}$$

we always have $\mathcal{F}^l(a^u, a^l) - \varphi(a^u) \geq 0,\ \forall (a^u,\ a^l)$, where $a^l \in S(a^u)$; thus the $(RBPP)$ inequality *cnstr* $\mathcal{F}^l(a^u, a^l) - \varphi(a^u) \leq 0,\ a^l \in S(a^u)$ and $a^u \in \mathbb{R}^{n_a}$ is equivalent to $\mathcal{F}^l(a^u, a^l) - \varphi(a^u) = 0$. Therefore, we have

$$\sum_{j \in J_m} \hat{\beta}_i \mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) = 0, \sum_{i \in I_n} \hat{\alpha}_i \mathcal{G}_i^u(\hat{a}^u, \hat{\nu}) = 0 \text{ and } \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \varphi(\hat{a}^u) = 0.$$

Hence $(\hat{a}^u,\ \hat{a}^l,\ \hat{\nu},\ \hat{\alpha},\ \hat{\beta},\ \hat{\gamma})$ is feasible for $(RBMWD)$ and the values of the two $obj.\ fns$ are equal.
Assume that $(\hat{a}^u,\ \hat{a}^l,\ \hat{\nu},\ \hat{\alpha},\ \hat{\beta},\ \hat{\gamma})$ is not a robust maximizer of $(RBMWD)$. Therefore there exist $(u,\ v,\ \nu,\ \alpha,\ \beta,\ \gamma)$ feasible for dual such that

$$\mathcal{F}^u(u,\ v) > \mathcal{F}^u(\hat{a}^u, \hat{a}^l).$$

But this contradicts the weak $dlty$. Theorem 4.2. Hence $(\hat{a}^u,\ \hat{a}^l,\ \hat{\nu},\ \hat{\alpha},\ \hat{\beta},\ \hat{\gamma})$ is a robust maximizer of $(RBMWD)$.

4.7 Multiobjective Robust Bilevel Model

A multi-objective robust bi-level model (MBP) has been considered in this Section. While all the functions at the ll are $l\mathcal{L}$, the ul has numerous $obj.$ and some data uncertainty associated with the $cnstr.fns$.

$$(MBP) \qquad \min_{a^u,\ a^l} \mathcal{F}^u(a^u, a^l) = \Big\{\mathcal{F}^u_1(a^u, a^l), \mathcal{F}^u_2(a^u, a^l), \dots, \mathcal{F}^u_q(a^u, a^l)\Big\}$$
$$\text{subject to } \ \mathcal{G}^u_i(a^u, \nu_i) \le 0, \quad \forall i \in I = \{1, 2, \dots, n\},$$
$$a^l \in \Lambda(a^u),$$

where for some sequentially compact topological space $\Omega_i,\ \nu_i \in \Omega_i$ is an uncertain parameter. For each $a^u \in \mathbb{R}^{n_a}$ the parametric optimization problem (MBP_{a^u}) has the set of optimal solutions $\Lambda(a^u)$

$$(MBP_{a^u}) \qquad \min_{\hat{a}^l} \mathcal{F}^l(a^u, a^l)$$
$$\text{subject to } \ \mathcal{G}^l_j(a^u, a^l) \le 0, \quad \forall j \in J = \{1, 2, \dots, m\}.$$

where, $\mathcal{F}^u_s : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}, s \in S = \{1, 2, \dots, q\}$, $\mathcal{F}^l : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R}$ and $\mathcal{G}^l_j : \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} \to \mathbb{R},\ j \in J = \{1,\ 2, \dots,\ m\}$ are $l\mathcal{L}$ functions and $\mathcal{G}^u_i : \mathbb{R}^{n_a} \times \Omega_i \to \mathbb{R},\ i \in I = \{1,\ 2, \dots,\ n\}$ is a function which is convex in a^u, continuous in ν and is upper semicontinuous in (a^u, ν), and $\Lambda(a^u) = \arg\min_{a^l}\{\mathcal{F}^l(a^u, a^l) :\ \mathcal{G}^l_j(a^u,\ a^l) \le 0, j \in J\}$; $(m, n, n_a, n_b, q) \ge 1$ are integers.
The robust counterpart of (MBP) is:

$$(MRP') \qquad \min_{x, a^l} \mathcal{F}^u(a^u, a^l) = \Big\{\mathcal{F}^u_1(a^u, a^l), \mathcal{F}^u_2(a^u, a^l), \dots, \mathcal{F}^u_q(a^u, a^l)\Big\}$$

$$\text{subject to } \ \mathcal{G}^u_i(a^u, \nu_i) \le 0, \quad \forall \nu_i \in \Omega_i,\ \forall i \in I,$$
$$a^l \in \Lambda(a^u),$$

where all possible parameter values fall inside the uncertain limitations, $\Omega_i,\ i = 1,\ 2, \ldots,\ n$, within the ranges of uncertainty that have been defined. By Lemma 4.1, $(\hat{a}^u, \hat{a}^l)$ is a solution of (MBP) if and only if its a robust local minimizer of (MRP')
The robust feasible set $H_{MRP'}$ of (MRP') is defined as

$$\begin{aligned} H_{MRP'} := \quad & \{(a^u, a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b} |\ \mathcal{G}_i^u(a^u, \nu_i) \leq 0, \forall \nu_i \in \Omega_i,\ \forall i \in I, \\ & \mathcal{G}_j^l(a^u, a^l) \leq 0,\ \forall j \in J, a^l \in \Lambda(a^u)\} \end{aligned}$$

Definition 4.2 *A pair $(\hat{a}^u, \hat{a}^l) \in H_{MRP'}$ is called a local robust weak eff. sol. of (MBP) if there exists neighbourhoods $\mathbb{N}_u$ of $\hat{a}^u$ and $\mathbb{N}_l$ of $\hat{a}^l$ such that*

$$\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\hat{a}^u, \hat{a}^l) \notin -\text{-int}\mathbb{R}^q_+, \qquad \forall (a^u, a^l) \in H_{MRP'} \cap (\mathbb{N}_u \times \mathbb{N}_l).$$

4.8 Optimality Conditions

Definition 4.3 *The nonsmooth generalized GCQ satisfies at $(\hat{a}^u, \hat{a}^l)$ if for each $\nu_i \in \Omega_i(\hat{a}^u)$ is defined as*

$$T(H_{MRP'}, (\hat{a}^u, \hat{a}^l))^* \subseteq cl(K)$$

where

$$\begin{aligned} K = \Big((\underset{i \in I_n}{\cup} conv\ \partial\Psi(\hat{a}^u) \times \{0\}) \bigcup (\underset{j \in J_m}{\cup} conv\ \partial^*\mathcal{G}_j^l(\hat{a}^u, \hat{a}^l)) \\ \bigcup (\partial\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial V(\hat{a}^u) \times \{0\}) \Big). \end{aligned}$$

Lemma 4.4 *Let $(\hat{a}^u, \hat{a}^l) \in H_{MRP'}$ be a local robust weak eff. sol. of (MBP). Then, $(\hat{a}^u, \hat{a}^l)$ is a local optimal point of robust counterpart BPP $(RCBPP)$ defined as:*

$$\begin{aligned} (RCBPP) \qquad & \min_{a^u, a^l} \quad \varphi(a^u, a^l) = \Delta_{-\text{int}\mathbb{R}^n_+}(\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\hat{a}^u, \hat{a}^l)) \\ & \textit{subject to} \quad \mathcal{G}_i^u(a^u, \nu_i) \leq 0, \quad \forall \nu_i \in \Omega_i, \forall i \in I, \\ & \qquad \mathcal{G}_j^l(a^u, a^l) \leq 0, \forall j \in J, \\ & \qquad \mathcal{F}^l(a^u, a^l) \leq V(a^u), \\ & \qquad (a^u, a^l) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}. \end{aligned}$$

Proof. As given $(\hat{a}^u, \hat{a}^l) \in H_{MRP'}$ is a local robust weak eff. sol. of (MBP) so there exists neighbourhoods $\mathbb{N}_u$ of $\hat{a}^u$ and $\mathbb{N}_l$ of $\hat{a}^l$ such that

$$\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\hat{a}^u, \hat{a}^l) \notin -\text{int}\mathbb{R}^n_+, \quad \forall (a^u, a^l) \in H_{MRP'} \cap (\mathbb{N}_u \times \mathbb{N}_l).$$

Then, by [82, Proposition 2.1],

$$\Delta_{-\text{int}\mathbb{R}^n_+}(\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\hat{a}^u, \hat{a}^l)) \geq 0, \forall (a^u, a^l) \in H_{MRP'} \cap (\mathbb{N}_u \times \mathbb{N}_l).$$

Therefore, $(\hat{a}^u, \hat{a}^l)$ provides a local solution to the following scalar problem with *cnstrs.*:

$$\begin{aligned} \min_{a^u, \hat{a}^l} \quad & \Delta_{-\text{int}\mathbb{R}^n_+}(\mathcal{F}^u(a^u, a^l) - \mathcal{F}^u(\hat{a}^u, \hat{a}^l)) \\ & (a^u, a^l) \in H_{MRP'}. \end{aligned}$$

By [139], $(\hat{a}^u, \hat{a}^l)$ is a local optimal point to $(RCBPP)$.

Theorem 4.4 *Let $\hat{u} = (\hat{a}^u, \hat{a}^l) \in H_{MRP'}$ be a robust local weak eff. sol. of (MBP). Suppose $\mathcal{F}^u s$, $s \in S$ admit bounded USRCF $\partial^*\mathcal{F}^u_s(\hat{a}^u, \hat{a}^l)$ at $(\hat{a}^u, \hat{a}^l)$, Ω_i is convex and $\mathcal{G}^u_i(a^u, .)$ is concave on Ω_i, for each $a^u \in X$, for each $i \in I$. $\mathcal{G}^l_j$, $j \in J$, admit UCF $\partial^*\mathcal{G}^l_j(\hat{a}^u, \hat{a}^l)$ at $(\hat{a}^u, \hat{a}^l)$. Let $\partial^*\mathcal{F}^u_k$, $k \in K$ are upper semicontinuous at $(\hat{a}^u, \hat{a}^l)$ and that the nonsmooth robust CQ satisfies at $(\hat{a}^u, \hat{a}^l)$. Assume that the argmin map Λ is inner semicontinuous at $(\hat{a}^u, \hat{a}^l)$ for each vector $a^l \in \Lambda(\hat{a}^u)$ (MBP_{a^u}) is ll regular at $(\hat{a}^u, a^l)$. Then there are $\lambda > 0$, $\gamma^* \geq 0, \gamma^* \neq 0_{\mathbb{R}^n}$, $\pi \geq 0$, $\xi \geq 0$ and $\mu \geq 0$ such that*

$$\begin{aligned} 0 \in conv\Big(&\partial^*_{a^u}(\gamma^* \circ \mathcal{F}^u)(\hat{a}^u, \hat{a}^l) + \sum_{i \in I(\hat{u})} \pi_i \partial^*_{a^u}\mathcal{G}^u_i(\hat{a}^u, \hat{\nu}_i) \\ &+ \sum_{j \in J(\hat{u})} \xi_j \partial^*_{a^u}\mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) + \tau(\partial_{a^u}\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial_{a^u}V(\hat{a}^u))\Big) \end{aligned} \tag{4.18}$$

$$\begin{aligned} 0 \in conv\Big(&\partial^*_{a^l}(\gamma^*.\mathcal{F}^u)(\hat{a}^u, \hat{a}^l) + \sum_{i \in I(\hat{u})} \pi_i \partial^*_{a^l}\mathcal{G}^u_i(\hat{a}, \hat{\nu}_i) \\ &+ \sum_{j \in J(\hat{u})} \xi_j \partial^*_{a^l}\mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) + \tau\partial_{a^l}\mathcal{F}^l(\hat{a}^u, \hat{a}^l)\Big) \end{aligned} \tag{4.19}$$

Proof. As $(\hat{a}^u, \hat{a}^l) \in H_{MRP'}$ is a robust local weak eff. sol. of (MBP) as per Lemma 4.4 it's a robust local optimal point of $RCBPP$. Then there exists a neighbourhood $\mathbb{U}$ of $(\hat{a}^u, \hat{a}^l)$ such that

$$\varphi(a^u, a^l) - \varphi(\hat{a}^u, \hat{a}^l) \notin -\text{int } \mathbb{R}^n_+ \quad (a^u, a^l) \in \mathbb{U} \cap H_{MRP'}.$$

Since $\Delta_{(-\text{int}\mathbb{R}^n_+)}$ and $\mathcal{F}^u$ are $l\mathcal{L}$, then there exists non negative integer κ such that φ is $l\mathcal{L}$

with Lipschitz constant $\kappa \geq 0$.
Let $l = (l_1, l_2) \in T(H_{MRP'}, (\hat{a}^u, \hat{a}^l))$ be arbitrarily selected. Therefore via the concept of tangent cone there exist $t_n \downarrow 0$ and $l_n \to l$ such that $(\hat{a}^u, \hat{a}^l) + t_n l_n \in H_{MRP'}$ for each n. For n sufficiently large $(\hat{a}^u, \hat{a}^l) + t_n l_n \in \mathbb{U}$. Now since $(\hat{a}^u, \hat{a}^l)$ is a robust local minimum of φ over $H_{MRP'}$, we have

$$\frac{\varphi((\hat{a}^u, \hat{a}^l) + t_n l_n) - \varphi(\hat{a}^u, \hat{a}^l)}{t_n} \geq 0.$$

Now,

$$\begin{aligned}\frac{\varphi((\hat{a}^u, \hat{a}^l) + t_n l_n) - \varphi(\hat{a}^u, \hat{a}^l)}{t_n} =& \frac{\varphi((\hat{a}^u, \hat{a}^l) + t_n l_n) - \varphi((\hat{a}^u, \hat{a}^l) + t_n l)}{t_n} \\ &+ \frac{\varphi((\hat{a}^u, \hat{a}^l) + t_n l) - \varphi(\hat{a}^u, \hat{a}^l)}{t_n}.\end{aligned}$$

Since φ is $l\mathcal{L}$ by taking limit supremum, we have

$$\varphi_d^+((\hat{a}^u, \hat{a}^l) + l) = \limsup_{t_n \to 0^+} \tfrac{\varphi((\hat{a}^u, \hat{a}^l) + t_n l_n) - \varphi(\hat{a}^u, \hat{a}^l)}{t_n} \geq 0.$$

Thus,

$$\varphi_d^+((\hat{a}^u, \hat{a}^l) + l) \geq 0, \quad \text{for all } l \in T(H_{MRP'}, (\hat{a}^u, \hat{a}^l))$$

Since φ is USRCF at $(\hat{a}^u, \hat{a}^l)$, we have

$$\sup_{\nu \in \partial^* \varphi(\hat{a}^u, \hat{a}^l)} \langle \nu, l \rangle \geq 0, \text{ for all } l \in T(H_{MRP'}, (\hat{a}^u, \hat{a}^l))$$

using [48, Theorem 3.5],

$$0 \in \overline{cl\ conv(\partial^* \varphi(\hat{a}^u, \hat{a}^l) + T(H_{MRP'}, (\hat{a}^u, \hat{a}^l))^-}.$$

Since the nonsmooth generalized GCQ retains at $(\hat{a}^u, \hat{a}^l)$, one obtains

$$0 \in cl[cl\ conv(\partial^* \varphi(\hat{a}^u, \hat{a}^l))] + cl\ cone K.$$

where

$$\begin{aligned}K = \Big((\bigcup_{i \in I_n} conv\ \partial \Psi(\hat{a}^u) \times \{0\}) \bigcup (\bigcup_{j \in J_m} conv\ \partial^* \mathcal{G}_j^l(\hat{a}^u, \hat{a}^l)) \\ \bigcup (\partial \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial V(\hat{a}^u) \times \{0\}) \Big).\end{aligned}$$

Since *cone* K is closed and via chain rule [71, Theorem 4.3], there exists $\gamma^* \in \partial^c \Delta_{(-\text{int}\mathbb{R}^n_+)}$

$$0 \in conv\partial^*(\gamma^* \circ \mathcal{F}^u)(\hat{a}^u, \hat{a}^l) + cone\Big((\underset{i \in I_n}{\cup} conv\ \partial\Psi(\hat{a}^u) \times \{0\}) \bigcup (\underset{j \in J_m}{\cup} conv\ \partial^* \mathcal{G}^l_j(\hat{a}^u,\ \hat{a}^l)) \bigcup (\partial \mathcal{F}^l(\hat{a}^u,\ \hat{a}^l) - \partial V(\hat{a}^u) \times \{0\}) \Big).$$

Since $\Delta_{(-\text{int}\mathbb{R}^n_+)}(\cdot)$ is a convex function and $\Delta_{(-\text{int}\mathbb{R}^n_+)}(0) = 0$, we have for all $\nu \in \mathbb{R}^n$

$$\Delta_{(-\text{int}\mathbb{R}^n_+)}(v) \geq \langle \gamma^*, v \rangle.$$

Hence, for all $v \in -\text{int}\mathbb{R}^n_+$

$$\langle \gamma^*, v \rangle \leq \Delta_{(-\text{int}\mathbb{R}^n_+)}(v) = -d(v, \mathbb{R}^n \setminus (-\text{int}\mathbb{R}^n_+)) \leq 0,$$

that is $\gamma^* \in \mathbb{R}^n_+$. From [82, Proposition 2.2], we have that $\gamma^* \neq 0$.
Consequently,

$$0 \in conv\ \partial^*(\gamma^* \circ \mathcal{F}^u)(\hat{a}^u, \hat{a}^l) + \sum_{i \in I_n} cone\ \partial\Psi(\hat{a}^u) + \sum_{j \in J_m} cone\ \partial^* \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) + cone\ (\partial^c \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial^c V(\hat{a}^u) \times \{0\})$$

Since $\mathcal{F}^l$ and V are $\mathcal{LC}$ near $(\hat{a}^u, \hat{a}^l)$ the function $\mathcal{F}^l - V$ is also $\mathcal{LC}$ near $(\hat{a}^u, \hat{a}^l)$. Consequently all these functions admit the $\mathcal{C}.sd$ $\partial^c f(\hat{a}^u, \hat{a}^l)$ and $\partial^c V(\hat{a}^u)$ are CFs at $(\hat{a}^u, \hat{a}^l)$ Then,

$$0 \in conv\ \partial^*(\gamma^* \circ \mathcal{F}^u)(\hat{a}^u, \hat{a}^l) + \sum_{i \in I_n} cone\ \partial\Psi(\hat{a}^u) + \sum_{j \in J_m} cone\ \partial^* \mathcal{G}^l_j(\hat{a}^u, \hat{a}^l) + cone\ (\partial^c \mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial^c V(\hat{a}^u) \times \{0\}) \tag{4.20}$$

From Lemma 4.3, we get

$$\partial\Psi(\hat{a}^u) \leq conv\ \{(\partial\Psi_i(\hat{a}^u)); \quad i \in I_n\} \tag{4.21}$$

By [83, Theorem 2.4], we obtain

$$\partial\Psi_i(\hat{a}^u) = \{\xi_i : \exists \nu_i \in \Omega_i(\hat{a}^u)\} \tag{4.22}$$

such that

$$\xi_i \in \partial_{a^u} G_i(\hat{a}, \nu_i)$$

for all $i \in I$, where $\Omega_i(\hat{a}^u) = \{\nu_i \in \Omega_i : \mathcal{G}_i^u(\hat{a}, \nu_i) = \Psi_i(\hat{a}^u)\} \neq \emptyset$ is convex and compact set. Thus (4.21) and (4.22) imply that there exists $\hat{\nu} = (\hat{\nu}_1, \hat{\nu}_2, \ldots, \hat{\nu}_n) \in \Omega$ such that

$$\mathcal{G}_i^u(\hat{a}^u, \hat{\nu}_i) = \Psi(\hat{a}^u),\ i \in I$$

and

$$\partial\Psi(\hat{a}^u) \subseteq conv\ \{\partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \hat{\nu}_i) : i \in I_n\}$$

from above and (4.20), we get

$$0 \in conv\ \partial^*(\gamma^* \circ \mathcal{F}^u)(\hat{a}^u, \hat{a}^l) + \sum_{i\in I_n} cone\ conv\partial_{a^u}\mathcal{G}_i^u(\hat{a}^u, \hat{\nu}_i) + \sum_{j\in J_m} cone\ \partial^*\mathcal{G}_j^l(\hat{a}^u, \hat{a}^l) + cone\ (\partial^c\mathcal{F}^l(\hat{a}^u, \hat{a}^l) - \partial^c V(\hat{a}^u) \times \{0\}).$$

Applying [109, Theorem 8], we get

$$\partial^c_{a^u} V(\hat{a}^u) \subseteq conv\ \{\bigcup_{a^l\in\Lambda(\hat{a}^u)} \bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\pi} \Big(\partial^c_{a^u}\mathcal{F}^l(\hat{a}^u, a^l) + \sum_{j\in J} \lambda_i \partial^c_{a^u}\mathcal{G}_j^l(\hat{a}^u, a^l)\Big)\}$$

where

$$\pi(\hat{a}^u, a^l) := \begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m : \lambda_j \geq 0,\ \lambda_j\mathcal{G}_j^l(\hat{a}^u, a^l) = 0,\ i \in I \\ 0 \in \partial^c_{a^l}\mathcal{F}^l(\hat{a}^u, \hat{a}^l) + \sum_{j\in J} \lambda_j \partial^c_{a^l}\mathcal{G}_j^l(\hat{a}^u, a^l) \end{cases}$$

Thus, there exist $a^{l*} \in \lambda(\hat{a}^u)$ and $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \in \pi(\hat{a}^u, a^{l*})$ satisfying (4.18) and (4.19).

Chapter 5

Multi-objective Fractional Bi-level Programming Problem [1]

5.1 Introduction

Models that effectively deal with fractional objectives, such as inventory/sales, output/employees, and profit/cost, are desirable for evaluating economic activities. Bi-level fractional programming is a class of the bi-level program that generalizes traditional fractional programming to address hierarchical systems with two decision levels. Bi-level fractional programming problems (BFPs) are used for representing a variety of decision-making scenarios. BFP have been explored by Calvete and Gale [24–26]. Wang et al. [129] utilize a multi-objective linear BFP technique to optimize the water consumption structure in a water-constrained environment. Recently, Kohli [77] proposed optimality criteria for the multi-objective BFP, in which the leader program is multi-objective fractional, and the follower's program is scalar. Laghdir [93] considered a multiobjective BFP and obtained *opt. conds*. in terms of sequences of Brøndsted-Rockafeller subdifferential. We relate a MWD to the multi-objective BFP discussed by Kohli [77] and obtain the *dlty*. results under the $\partial *$-pseudoconvexity and $\partial *$-quasiconvexity assumptions. In addition, we use an illustration to verify the weak *dlty*. theorem.
The layout of the Chapter looks like this: The case statement and the dual of the Mond-Weir type are discussed in Section 5.2. The relationship between the two models is studied in Section 5.3 via weak and strong *dlty*. theorems. Section 5.4 provides an illustrative example to validate our weak *dlty*. theorem.

5.2 Multi-objective Fractional Bi-level Programming Problem

Ratio optimization problem or fractional programming refers to optimizing the ratios within the constraints. Multi-objective fractional problems are those where one can simultaneously optimize multiple ratio objectives. A two-level, hierarchical system of

fractional programming problems is termed as a fractional BPP.
The multi-objective BFP [77]:

$$(MOFBLPP) \qquad \min \qquad \Big(\frac{\mathcal{F}^u_1(a^u,a^l)}{\mathcal{F}'^u_1(a^u,a^l)},\ldots,\frac{\mathcal{F}^u_p(a^u,a^l)}{\mathcal{F}'^u_p(a^u,a^l)}\Big)$$
$$\text{subject to} \qquad \mathcal{G}^u_i(a^u,a^l) \le 0, \quad i \in I,\ a^l \in \psi(a^u),$$

where, for each $a^u \in \mathbb{R}^{n_a}$, $\psi(a^u)$ is the collection of the optimal points of

$$\begin{aligned} \min \qquad & \mathcal{F}^l(a^u,a^l) \\ \text{subject to} \qquad & \mathcal{G}^l_j(a^u,a^l) \le 0, \quad j \in J, \end{aligned}$$

where $\mathcal{F}^u_k,\ \mathcal{F}'^u_k : \mathbb{R}^{n_a}\times\mathbb{R}^{n_b} \to \mathbb{R},\ k=1,\ldots,p,\ \mathcal{F}^l : \mathbb{R}^{n_a}\times\mathbb{R}^{n_b} \to \mathbb{R}, \mathcal{G}^u_i : \mathbb{R}^{n_a}\times\mathbb{R}^{n_b} \to \mathbb{R},\ i \in I = \{1,\ldots,m_1\}$ and $\mathcal{G}^l_j : \mathbb{R}^{n_a}\times\mathbb{R}^{n_b} \to \mathbb{R},\ j \in J = \{1,\ldots,m_2\};\ n_a \ge 1,\ n_b \ge 1,\ m_1 \ge 1$ and $m_2 \ge 1$ are integers. $\mathcal{F}^u_k(a^u,a^l) \ge 0,\ \mathcal{F}'^u_k(a^u,a^l) > 0$ for all $k = 1,\ldots,p$ and $(a^u,a^l) \in \mathbb{R}^{n_a}\times\mathbb{R}^{n_b},\ \mathcal{F}^l$ and $\mathcal{G}^l_j,\ j \in J$ are continuous, convex and

$$\psi(a^u) = \operatorname*{argmin}_{a^l}\{\mathcal{F}^l(a^u,a^l) : \quad \mathcal{G}^l_j(a^u,a^l) \le 0,\ j \in J\}.$$

By using OVF reformulation Kohli [77] reformulated $(MOFBLPP)$ as a single level multi-objective fractional program $(RMOFBLPP)$ defined as:

$$\begin{aligned} (RMOFBLPP) \qquad \min \qquad & \phi(a^u,a^l) = \Big(\frac{\mathcal{F}^u_1(a^u,a^l)}{\mathcal{F}'^u_1(a^u,a^l)},\ldots,\frac{\mathcal{F}^u_p(a^u,a^l)}{\mathcal{F}'^u_p(a^u,a^l)}\Big) \\ \text{subject to} \qquad & \mathcal{G}^u_i(a^u,a^l) \le 0, \quad i \in I, \\ & \mathcal{G}^l_j(a^u,a^l) \le 0, \quad j \in J, \\ & \mathcal{F}^l(a^u,a^l) - V(a^u) \le 0, \\ & (a^u,a^l) \in \mathbb{R}^{n_a}\times\mathbb{R}^{n_b}, \end{aligned}$$

where the OVF $V(a^u)$ is defined as

$$V(a^u) = \inf\{\mathcal{F}^l(a^u,a^l) :\ \mathcal{G}^l_j(a^u,a^l) \le 0,\ j \in J\}.$$

Let $E \subseteq \mathbb{R}^{n_a}\times\mathbb{R}^{n_b}$ defined as,

$$E = \{(a^u,a^l) \in \mathbb{R}^{n_a}\times\mathbb{R}^{n_b} :\ \mathcal{F}^l(a^u,a^l) - V(a^u) \le 0,\ \mathcal{G}^u_i(a^u,a^l) \le 0, i \in I,$$
$$\mathcal{G}^l_j(a^u,a^l) \le 0, j \in J\}$$

represent the collection of feasible points of $(RMOFBLPP)$.

We give the definitions of weak eff. and local weak eff. sol. as given in [77].

Definition 5.1 *A point $(a^{u^*}, a^{l^*}) \in E$ is said to be a weak eff. sol. of $(MOFBLPP)$ if there does not exist any $(a^u, a^l) \in E$ such that*

$$\frac{\mathcal{F}_k^u(a^u, a^l)}{\mathcal{F}_k'^u(a^u, a^l)} < \frac{\mathcal{F}_k^u(a^{u^*}, a^{l^*})}{\mathcal{F}_k'^u(a^{u^*}, a^{l^*})}, \quad k = 1, 2, \ldots, p.$$

Definition 5.2 *A feasible solution $(a^{u^*}, a^{l^*}) \in E$ is said to be a local weak eff. sol. of $(MOFBLPP)$ if there does exist n.b.d U_u of a^{u^*} and V_l of a^{l^*} such that any feasible solution $(a^u, a^l) \in (U_u \times V_l) \cap E$, the following holds*

$$\frac{\mathcal{F}_k^u(a^u, a^l)}{\mathcal{F}_k'^u(a^u, a^l)} < \frac{\mathcal{F}_k^u(a^{u^*}, a^{l^*})}{\mathcal{F}_k'^u(a^{u^*}, a^{l^*})}, \quad k = 1, 2, \ldots, p.$$

The collection of indexes of active constraints at point (a^{u^*}, a^{l^*}) are

$$I(a^{u^*}, a^{l^*}) = \{i \in I : \ \mathcal{G}_i^u(a^{u^*}, a^{l^*}) = 0\},$$
$$J(a^{u^*}, a^{l^*}) = \{j \in J : \ \mathcal{G}_j^l(a^{u^*}, a^{l^*}) = 0\}.$$

Let $(a^{u^*}, a^{l^*}) \in E$ be feasible for $(RMOFBLPP)$. Let $\mathcal{F}^l$, $\mathcal{G}_j^l$, $j \in J(a^{u^*}, a^{l^*})$, $\mathcal{G}_i^u$, $i \in I(a^{u^*}, a^{l^*})$ admit CFs $\partial^*\mathcal{F}^l(a^{u^*}, a^{l^*})$, $\partial^*\mathcal{G}_j^l(a^{u^*}, a^{l^*})$, $j \in J(a^{u^*}, a^{l^*})$ and $\partial^*\mathcal{G}_i^u(a^{u^*}, a^{l^*})$, $i \in I(a^{u^*}, a^{l^*})$, respectively at (a^{u^*}, a^{l^*}).
Problem $(RMOFBLPP)$ is known to satisfy ∂^*-$MOFBLPP\ CQ$ at (a^{u^*}, a^{l^*}) if for every $\tau \geq 0$, α_i, $i \in I(a^{u^*}, a^{l^*})$, β_j, $j \in J(a^{u^*}, a^{l^*})$ (not all zero) following holds

$$(0,0) \notin \text{cl}\Big\{ \quad \tau\text{conv}(\partial^*\mathcal{F}^l(a^{u^*}, a^{l^*}) - \partial^* V(a^{u^*}) \times \{0\}) + \sum_{i=I(a^{u^*}, a^{l^*})} \alpha_i^* \text{conv}\partial^*\mathcal{G}_i^u(a^{u^*}, a^{l^*}) + \sum_{j=J(a^{u^*}, a^{l^*})} \beta_j^* \text{conv}\partial^*\mathcal{G}_j^l(a^{u^*}, a^{l^*}) \Big\}.$$

Theorem 5.1 [77] *Let (a^{u^*}, a^{l^*}) be a local week eff. sol. of $(MOFBLPP)$. Suppose $\mathcal{F}_k^u$, $k = 1, 2, \ldots, p$ admit bounded CFs $\partial^*\mathcal{F}_k^u(a^{u^*}, a^{l^*})$, $k = 1, 2, \ldots, p$ at (a^{u^*}, a^{l^*}) which are upper regular and $\mathcal{F}_k'^u, k = 1, 2, \ldots, p$ admit bounded CFs $\partial^*\mathcal{F}_k'^u(a^{u^*}, a^{l^*})$, $k = 1, 2, \ldots, p$ at (a^{u^*}, a^{l^*}) which are upper regular if $\phi_k(a^{u^*}, a^{l^*}) > 0$ and lower regular if $\phi_k(a^{u^*}, a^{l^*}) < 0$. Let $\mathcal{F}^l$ admits a bounded CF $\partial^*\mathcal{F}^l(a^{u^*}, a^{l^*})$ at (a^{u^*}, a^{l^*}) which is upper regular and $\mathcal{G}_i^u$, $i \in I$, $\mathcal{G}_j^l, j \in J$ admit bounded CFs $\partial^*\mathcal{G}_i^u(a^{u^*}, a^{l^*})$, $i \in I$ and $\partial^*\mathcal{G}_j^l(a^{u^*}, a^{l^*})$, $j \in J$ respectively at (a^{u^*}, a^{l^*}). Let $\partial^*(\mathcal{F}_k^u(a^{u^*}, a^{l^*}) - \phi_k \mathcal{F}_k'^u(a^{u^*}, a^{l^*})), k =$*

$1, 2, \ldots, p$, $\partial^* \mathcal{F}^l, \partial^* \mathcal{G}_i^u$, $i \in I$ *and* $\partial^* \mathcal{G}_j^l$, $j \in J$ *are upper semicontinuous at* (a^{u^*}, a^{l^*}). *Also, suppose* $\mathcal{F}_k^u$, $\mathcal{F}_k'^u$, $k = 1, 2, \ldots, p$, $\mathcal{G}_i^u$, $i \in I$ *are continuous and convex. Let* ∂^*-*MOFBLPP CQ be satisfied at* (a^{u^*}, a^{l^*}). *Suppose for some* $k, k = 1, 2, \ldots, p, \partial^* \mathcal{F}_k^u(a^{u^*}, a^{l^*}) - \phi_k \mathcal{F}_k'^u(a^{u^*}, a^{l^*})$ *is an upper regular CF of* $\mathcal{F}_k^u - \phi \mathcal{F}_k'^u$ *at* (a^{u^*}, a^{l^*}) *and for some* $k_0 \neq k$, $\partial^* \mathcal{F}_{k_0}^u(a^{u^*}, a^{l^*}) - \phi_{x_{u0}}(a^{u^*}, a^{l^*}) \partial^* \mathcal{F}_{k_0}'^u(a^{u^*}, a^{l^*})$ *is a lower regular CF of* $\mathcal{F}_{k_0}^u - \phi_{k_0} \mathcal{F}_{k_0}'^u$ *at* (a^{u^*}, a^{l^*}). *Suppose that*

$$\begin{aligned}
\partial^* \Big(\sum_{k=1}^{p} \lambda_k (\mathcal{F}_k^u - \phi_k \mathcal{F}_k'^u) \Big)(a^{u^*}, a^{l^*}) = & \sum_{k=1}^{p} \partial^* (\lambda_k (\mathcal{F}_k^u - \phi \mathcal{F}_k'^u))(a^{u^*}, a^{l^*}) \\
= & \sum_{k=1}^{p} \lambda_k (\partial^* \mathcal{F}_k^u(a^{u^*}, a^{l^*}) - \phi(a^{u^*}, a^{l^*}) \partial^* \mathcal{F}_k'^u(a^{u^*}, a^{l^*})),
\end{aligned}$$

where $\lambda_k > 0$, $k = 1, 2, \ldots, p$ *and* $\sum_{k=1}^{p} \lambda_k = 1$. *Also, assume that* $\partial^*(\mathcal{F}^l(a^{u^*}, a^{l^*}) - V(a^{u^*})) = \partial^* \mathcal{F}^l(a^{u^*}, a^{l^*}) - \partial^* V(a^{u^*}) \times \{0\}$. *Then, there exist scalars* $\lambda_k' \geq 0$, *(not all zero),* $k = 1, 2, \ldots, p$, $\tau \geq 0, \alpha_i \geq 0, i \in I, \beta_j \geq 0, j \in J$ *with* $\sum_{k=1}^{p} \lambda_k + \tau + \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j = 1$ *such that*

$$\begin{aligned}
& 0 \in \sum_{k=1}^{p} \lambda_k^* conv(\partial^* \mathcal{F}_k^u(a^{u^*}, a^{l^*}) - \phi_k^* \partial^* \mathcal{F}_k'^u(a^{u^*}, a^{l^*})) + \sum_{i=1}^{m_1} \alpha_i^* conv \partial^* \mathcal{G}_i^u(a^{u^*}, a^{l^*}) \\
& \quad + \sum_{j=1}^{n_b} \beta_j^* conv \partial^* \mathcal{G}_j^l(a^{u^*}, a^{l^*}) + \tau conv(\partial^* \mathcal{F}^l(a^{u^*}, a^{l^*}) - \partial^* V(a^{u^*}) \times \{0\}), \\
& \alpha_i^* \mathcal{G}_i^u(a^{u^*}, a^{l^*}) = 0, \quad \beta_j^* \mathcal{G}_j^l(a^{u^*}, a^{l^*}) = 0.
\end{aligned}$$

5.2.1 Mond-Weir Dual

The importance of *dlty.* in mathematical programming arises from the fact that addressing a dual problem can sometimes be more straightforward than solving its corresponding primal problem.

Wolfe [134] established a dual problem in the context of a primal nonlinear programming problem under the assumption of convexity. Furthermore, Wolfe provided a proof for a collection of *dlty.* theorems. In their seminal work, Mond and Weir [95] proposed a novel framework to explore the concept of *dlty.*. Their model allowed for the relaxation of convexity restrictions on both the *obj.* and *cnstr.* fns. This was particularly significant as certain *dlty.* theorems may not be applicable to the Wolfe model when the functions involved are of a generalized convex nature.

We associate the following MWD with problem $(MOFBLPP)$:

$$(MWD) \qquad \max\ \phi(c_1,c_2) = \Big(\frac{\mathcal{F}_1^u(c_1,c_2)}{\mathcal{F}_1'^u(c_1,c_2)}, \ldots, \frac{\mathcal{F}_p^u(c_1,c_2)}{\mathcal{F}_p'^u(c_1,c_2)}\Big)$$

$$\text{subject to} \quad 0 \in \sum_{k=1}^{p} \lambda_k \text{conv}(\partial^* \mathcal{F}_k^u(c_1,c_2) - \phi_k \partial^* \mathcal{F}_k'^u(c_1,c_2))$$
$$+\sum_{i=1}^{m_1} \alpha_i \text{conv} \partial^* \mathcal{G}_i^u(c_1,c_2) + \sum_{j=1}^{n_b} \beta_j \text{conv} \partial^* \mathcal{G}_j^l(c_1,c_2)$$
$$+\tau \text{conv}(\partial^* \mathcal{F}^l(c_1,c_2) - \partial^* V(u) \times \{0\}),$$
$$\alpha_i \mathcal{G}_i^u(c_1,c_2) \geq 0, \quad \beta_j \mathcal{G}_j^l(c_1,c_2) \geq 0,$$
$$\mathcal{F}^l(c_1,c_2) - V(c_1) \geq 0,$$
$$\Big(\lambda_1, \ldots, \lambda_p, \alpha_1, \ldots, \alpha_{m_1}, \beta_1 \ldots, \beta_{m_2}, \tau\Big) \geq 0,$$
$$(\lambda_1, \ldots, \lambda_p) \neq (0, \ldots, 0),$$

where $\nu_k = \phi_k(c_1,c_2) = \dfrac{\mathcal{F}_k^u(c_1,c_2)}{\mathcal{F}_k'^u(c_1,c_2)}, \ k = 1,2,\ldots,p.$

5.3 Relation between $(MOFBLPP)$ and (MWD)

In this section, we have established the correlation between the primal and dual problems by examining the solutions of both problems.

Theorem 5.2 (Weak duality) *Let (a^u, a^l) be feasible for $(MOFBLPP)$ and $((c_1,c_2),\pi,\nu)$ be feasible for (MWD), where $\pi = (\lambda,\alpha,\beta,\tau)$. Suppose that $\partial^* \mathcal{F}_k^u(c_1,c_2),\ k \in K = \{1,2,\ldots,p\}$ is an upper regular CF of $\mathcal{F}_k^u(.,.),\ k \in K$ at (c_1,c_2) and $\partial^* \mathcal{F}_k'^u(.,.),\ k \in K$ is a lower regular CF of $\mathcal{F}_k'^u(.,.),\ k \in K$ at (c_1,c_2). If $\mathcal{F}_k^u(.,.) - \nu_k \mathcal{F}_k'^u(.,.),\ k \in K$ is ∂^*-pseudoconvex at (c_1,c_2), $\mathcal{F}^l(.,.) - V(.),\ \alpha_i \mathcal{G}_i^u(.,.),\ i \in I$ and $\beta_j \mathcal{G}_j^l(.,.),\ j \in J$ are ∂^*-quasiconvex at (c_1,c_2), then*

$$\phi(a^u,a^l) \nless \phi(c_1,c_2).$$

Proof. Since $\partial^* \mathcal{F}_k^u(c_1,c_2),\ k \in K$ is an upper regular CF of $\mathcal{F}_k^u(.,.),\ k \in K$ at (c_1,c_2) and $\partial^* \mathcal{F}_k'^u(.,.),\ k \in K$ is a lower regular CF of $\mathcal{F}_k'^u(.,.),\ k \in K$ at (c_1,c_2) via Remark 1.1 and Lemma 1.1, we have $\partial^* \mathcal{F}_k^u(c_1,c_2) - \nu_k \partial^* \mathcal{F}_k'^u(c_1,c_2),\ k \in K$ is a CF of $\mathcal{F}_k^u(.,.) - \nu_k \mathcal{F}_k'^u(.,.),\ k \in K$ at (c_1,c_2).
On the contrary, suppose that

$$\phi(a^u,a^l) < \phi(c_1,c_2).$$

Then,

$$\mathcal{F}^u_k(a^u, a^l) - \nu_k \mathcal{F}'^u_k(a^u, a^l) < 0, \quad k \in K, \tag{5.1}$$

where $\nu_k = \phi_k(c_1, c_2) = \dfrac{\mathcal{F}^u_k(c_1, c_2)}{\mathcal{F}'^u_k(c_1, c_2)}, \ k \in K.$

Since (u, v, π, ν) is feasible for (MWD), therefore, there exist

$$\begin{aligned}
&\mu_k \in \text{conv}(\partial^* \mathcal{F}^u_k(c_1, c_2) - \nu_k \partial^* \mathcal{F}'^u_k(c_1, c_2)), && k \in K,\\
&\sigma_i \in \text{conv } \partial^* \mathcal{G}^u_i(c_1, c_2), && i \in I,\\
&\rho_j \in \text{conv } \partial^* \mathcal{G}^l_j(c_1, c_2), && j \in J,\\
&\eta \in \text{conv } (\partial^* \mathcal{F}^l(c_1, c_2) - \partial^* V(c_1)),
\end{aligned}$$

such that

$$0 = \sum_{k=1}^{p} \lambda_k \mu_k + \sum_{i=1}^{m_1} \alpha_i \sigma_i + \sum_{j=1}^{n_b} \beta_j \rho_j + \tau \eta. \tag{5.2}$$

Using (5.1), the feasibility of (a^u, a^l) for $(MOFBLPP)$ and the feasibility of (u, v, π, ν) for (MWD), we get

$$\mathcal{F}^u_k(a^u, a^l) - \nu_k \mathcal{F}'^u_k(a^u, a^l) < 0 = \mathcal{F}^u_k(c_1, c_2) - \nu_k \mathcal{F}'^u_k(c_1, c_2), \quad k \in K, \tag{5.3}$$

and

$$\alpha_i \mathcal{G}^u_i(a^u, a^l) \leq 0 \leq \alpha_i \mathcal{G}^u_i(c_1, c_2), \qquad i \in I, \tag{5.4}$$

$$\beta_j \mathcal{G}^l_j(a^u, a^l) \leq 0 \leq \beta_j \mathcal{G}^l_j(c_1, c_2), \qquad j \in J, \tag{5.5}$$

$$\tau(\mathcal{F}^l(a^u, a^l) - V(a^u)) \leq 0 \leq \tau(\mathcal{F}^l(c_1, c_2) - V(c_1)). \tag{5.6}$$

Since $\mathcal{F}^u_k(.,.) - \nu_k \mathcal{F}^u_k(.,.), \ k \in K$ is ∂^*- pseudoconvex at (c_1, c_2), we have from (5.3)

$$\langle \mu_k, (a^u, a^l) - (c_1, c_2) \rangle < 0, \ \ \forall \mu_k \in \text{conv } \partial^*(\mathcal{F}^u_k(c_1, c_2) - \nu_k \mathcal{F}'^u_k(c_1, c_2)), \ k \in K,$$

as $\lambda \in \mathbb{R}^p_+ / \{0\}$,

$$\Big\langle \sum_{k=1}^{p} \lambda_k \mu_k, (a^u, a^l) - (c_1, c_2) \Big\rangle < 0, \ \ \forall \mu_k \in \text{conv}\partial^*(\mathcal{F}^u_k(c_1, c_2) - \nu_k \mathcal{F}'^u_k(c_1, c_2)). \tag{5.7}$$

By using ∂^*-quasiconvexity of $\alpha_i \mathcal{G}^u_i(.,.), \ i \in I$ and (5.4), we have

$$\langle \bar{\xi}_i, (a^u, a^l) - (c_1, c_2) \rangle \leq 0, \ \ \forall i \in I, \ \bar{\xi}_i \in \partial^*(\alpha_i \mathcal{G}^u_i)(c_1, c_2),$$

as $\bar{\xi}_i$ can be expressed as convex combination of finite members of $\partial^*(\alpha_i \mathcal{G}_i^u)(u,v)$. By Lemma 1.1, we have

$$\begin{aligned}\langle \xi_i, (a^u, a^l) - (c_1, c_2)\rangle \leq 0, \quad \forall i \in I,\ \xi_i \in \alpha_i \text{conv}\ \partial^* \mathcal{G}_i^u(c_1, c_2)\\ \Rightarrow \langle \alpha_i\sigma_i, (a^u, a^l) - (c_1, c_2)\rangle \leq 0, \quad \forall i \in I,\ \sigma_i \in \text{conv}\partial^* \mathcal{G}_i^u(c_1, c_2).\end{aligned} \tag{5.8}$$

Proceeding as above on using the ∂^*-quasiconvexity of $\beta_j \mathcal{G}_j^l(.,.)$, $j \in J$, $\mathcal{F}^l(.,.) - V(.)$ and (5.5), (5.6), we get

$$\begin{aligned}\langle \zeta_j, (a^u, a^l) - (c_1, c_2)\rangle \leq 0, \quad \forall j \in J,\ \zeta_j \in \partial^*(\beta_j \mathcal{G}_j^l)(c_1, c_2),\\ \langle \eta, (a^u, a^l) - (c_1, c_2)\rangle \leq 0, \quad \eta \in \text{conv}\ (\partial^* \mathcal{F}^l(c_1, c_2) - \partial^* V(u)),\end{aligned} \tag{5.9}$$

from (5.8) and (5.9), we obtain

$$\begin{aligned}\Big\langle \sum_{i=1}^{m_1} \alpha_i\sigma_i, (a^u, a^l) - (c_1, c_2)\Big\rangle \leq 0,\\ \Big\langle \sum_{j=1}^{n_b} \beta_j\rho_j, (a^u, a^l) - (c_1, c_2)\Big\rangle \leq 0,\\ \Big\langle \tau\eta, (a^u, a^l) - (c_1, c_2)\Big\rangle \leq 0,\end{aligned} \tag{5.10}$$

adding (5.7) and (5.10), we get

$$\Big\langle \sum_{k=1}^{p} \lambda_k\mu_k + \sum_{i=1}^{m_1} \alpha_i\sigma_i + \sum_{j=1}^{n_b} \beta_j\rho_j + \tau\eta, (a^u, a^l) - (c_1, c_2)\Big\rangle < 0,$$

which is a contradiction to (5.2).

Theorem 5.3 (Strong Duality) *Let (a^{u^*}, a^{l^*}) be a weak eff. sol. of $(MOFBLPP)$. Assume that the hypothesis of Theorem 5.1 hold. Then, there exist $(\pi, \nu) \in \mathbb{R}^{p+m_1+m_2+1} \times \mathbb{R}^p$ such that $(a^{u^*}, a^{l^*}, \pi^*)$ is feasible for the dual (MWD) and values of two obj. fns are equal. If, for each feasible (a^u, a^l, π, ν) of (MWD) hypothesis of Theorem 5.2 holds, then $(a^{u^*}, a^{l^*}, \pi^*, \nu^*)$ be an optimal solution of (MWD).*

Proof. Since (a^{u^*}, a^{l^*}) is a weak eff. sol. of $(MOFBLPP)$ and all the assumptions of Theorem 5.1 are satisfied, thus, there exist vectors $0 \neq \lambda^* \in \mathbb{R}^p_+, \alpha_i^* \in \mathbb{R}^{n_a}_+, \beta_j^* \in \mathbb{R}^{n_b}_+$ and

$\tau^* \in \mathbb{R}_+$ such that

$$0 \in \sum_{k=1}^{p} \lambda_k^* \text{conv}(\partial^* \mathcal{F}_k^u(a^{u^*}, a^{l^*}) - \nu_k^* \partial^* \mathcal{F}_k'^u(a^{u^*}, a^{l^*})) + \sum_{i=1}^{m_1} \alpha_i^* \text{conv} \partial^* \mathcal{G}_i^u(a^{u^*}, a^{l^*})$$
$$+ \sum_{j=1}^{n_b} \beta_j^* \text{conv} \partial^* \mathcal{G}_j^l(a^{u^*}, a^{l^*}) + \tau^* \text{conv}(\partial^* \mathcal{F}^l(a^{u^*}, a^{l^*}) - \partial^* V(a^{u^*}) \times \{0\}),$$
$$\alpha_i^* \mathcal{G}_i^u(a^{u^*}, a^{l^*}) = 0, \quad \beta_j^* \mathcal{G}_j^l(a^{u^*}, a^{l^*}) = 0.$$

Since $\mathcal{F}^l(a^{u^*}, a^{l^*}) - V(a^{u^*}) = 0$, implies that $(a^{u^*}, a^{l^*}, \pi^*, \nu^*)$ is a feasible for dual (MWD) where $\nu^* = \dfrac{\mathcal{F}_k^u(a^{u^*}, a^{l^*})}{\mathcal{F}_k'^u(a^{u^*}, a^{l^*})}$, $k = 1, 2, \ldots, p$. Let, if possible, $(a^{u^*}, a^{l^*}, \pi^*, \nu^*)$ be not a weak eff. sol. of (MWD). Then, there exists (u, v, π, ν) feasible for dual such that $\phi(u, v) > \phi(a^{u^*}, a^{l^*})$.

However, this is a contradiction to Theorem 5.2 as (a^{u^*}, a^{l^*}) is feasible for $(MFOBLPP)$ and (u, v, π, ν) is a feasible for (MWD). Hence, $(a^{u^*}, a^{l^*}, \pi^*, \nu^*)$ is a weak eff. sol. of (MWD).

5.4 Application of Weak Duality Theorem

Consider the problem

$$\begin{aligned} (MOFBLPP) \qquad \min \quad & \Big(\frac{\mathcal{F}_1^u(a^u, a^l)}{\mathcal{F}_1'^u(a^u, a^l)}, \frac{\mathcal{F}_2^u(a^u, a^l)}{\mathcal{F}_2'^u(a^u, a^l)}\Big) \\ \text{subject to} \quad & \mathcal{G}_i^u(a^u, a^l) \leq 0, \\ & a^l \in \psi(a^u), \end{aligned}$$

where, for each $a^u \in \mathbb{R}^{n_a}$, $\psi(a^u)$ is the collection of the optimal points of the following optimization program

$$\begin{aligned} \min \quad & \mathcal{F}^l(a^u, a^l) \\ \text{subject to} \quad & \mathcal{G}_j^l(a^u, a^l) \leq 0, \end{aligned}$$

where $\mathcal{F}_k^u,\ \mathcal{F}_k'^u,\ \mathcal{F}^l,\ \mathcal{G}_i^u,\ \mathcal{G}_j^l, : \mathbb{R}\times\mathbb{R}\to\mathbb{R},\ i=1,\ j=1,2,\ k=1,2$ are defined by:

$$\begin{aligned}
&\mathcal{F}_1^u(a^u,a^l)=a^u+a^l-1,\ \mathcal{F}_1'^u(a^u,a^l)=2a^u-5,\\
&\mathcal{F}_2^u(a^u,a^l)=a^{u2}+2a^l+3,\ \mathcal{F}_2'^u(a^u,a^l)=a^u+3,\\
&\mathcal{G}_1^u(a^u,a^l)=a^{u2}-4,\ \mathcal{F}^l(a^u,a^l)=a^u+a^l+10,\\
&\mathcal{G}_1^l(a^u,a^l)=-a^l,\ \mathcal{G}_2^l(a^u,a^l)=a^l-2.
\end{aligned}$$

The collection of feasible points of $(MOFBLPP)$ is $E=\{(a^u,0):a^u\in[-2,2]\}$.
Dual corresponding to it is:

$$\begin{aligned}
(MWD)\qquad \max\quad & \Big(\frac{\mathcal{F}_1^u(c_1,c_2)}{\mathcal{F}_1'^u(c_1,c_2)},\frac{\mathcal{F}_2^u(c_1,c_2)}{\mathcal{F}_2'^u(c_1,c_2)}\Big)\\
\text{subject to}\quad & 0\in\sum_{k=1}^{2}\lambda_k\text{conv}(\partial^*\mathcal{F}_k^u(c_1,c_2)-\nu_k\partial^*\mathcal{F}_k'^u(c_1,c_2))+\alpha_1\text{conv}\partial^*\mathcal{G}_1^u(c_1,c_2)\\
&\quad+\sum_{j=1}^{2}\beta_j\text{conv}\partial^*\mathcal{G}_j^l(c_1,c_2)+\tau\text{conv}(\partial^*\mathcal{F}^l(c_1,c_2)-\partial^*V(u)\times\{0\}),\\
&\alpha_1\mathcal{G}_1^u(c_1,c_2)\geq 0,\quad \beta_1\mathcal{G}_1^l(c_1,c_2)\geq 0,\quad \beta_2\mathcal{G}_2^l(c_1,c_2)\geq 0,\\
&\mathcal{F}^l(c_1,c_2)-V(c_1)\geq 0,
\end{aligned}$$

where $0\neq\lambda\in\mathbb{R}^2,\alpha_1\in\mathbb{R},\beta\in\mathbb{R}^2$, and $\tau\in\mathbb{R}$ and $\nu_k=\frac{\mathcal{F}_k^u(c_1,c_2)}{\mathcal{F}_k'^u(c_1,c_2)},\ k=1,2.$
$(c_1,c_2,\lambda_1,\lambda_2,\alpha_1,\beta_1,\beta_2,\tau)=(0,0,\frac{10}{40},\frac{6}{40},0,\frac{23}{40},0,\frac{1}{40})$ is feasible for dual (MWD).

$$\begin{aligned}
&\partial^*\mathcal{F}_1^u(0,0)=\{(1,1)\},\ \partial^*\mathcal{F}_1'^u(0,0)=\{(2,0)\},\\
&\partial^*\mathcal{F}_2^u(0,0)=\{(0,2)\},\ \partial^*\mathcal{F}_2'^u(0,0)=\{(1,0)\},\\
&\partial^*\mathcal{G}_1^u(0,0)=\{(0,0)\},\ \partial^*\mathcal{F}^l(0,0)=\{(1,1)\},\\
&\partial^*\mathcal{G}_1^l(0,0)=\{(0,-1)\},\ \partial^*\mathcal{G}_2^l(0,0)=\{(0,1)\},\\
&\partial^*V(0)=\{1\},\ \nu_1=1/5 \text{ and } \nu_2=1.
\end{aligned}$$

- $\mathcal{F}_1^u(.,.)-\nu_1\mathcal{F}_1'^u(.,.)$ and $\mathcal{F}_2^u(.,.)-\nu_2\mathcal{F}_2'^u(.,.)$ are ∂^*-pseudoconvex at $(0,0)$.
- $\mathcal{F}^l(.,.)-V(.),\ \alpha_1\mathcal{G}_1^u(.,.)$ and $\beta_j\mathcal{G}_j^l(.,.),\ j=1,2$ are ∂^*-quasiconvex at $(0,0)$.

We can see that for feasible point $(a^u,a^l)=(1,0)$ for $(MOFBLPP)$ and
$(c_1,c_2,\lambda_1,\lambda_2,\alpha_1,\beta_1,\beta_2,\tau)=(0,0,\frac{10}{40},\frac{6}{40},0,\frac{23}{40},0,\frac{1}{40})$ for (MWD)

$$\Big(\frac{\mathcal{F}_1^u(a^u,a^l)}{\mathcal{F}_1'^u(a^u,a^l)},\frac{\mathcal{F}_2^u(a^u,a^l)}{\mathcal{F}_2'^u(a^u,a^l)}\Big)=\Big(0,1\Big)\nless\Big(\frac{\mathcal{F}_1^u(c_1,c_2)}{\mathcal{F}_1'^u(c_1,c_2)},\frac{\mathcal{F}_2^u(c_1,c_2)}{\mathcal{F}_2'^u(c_1,c_2)}\Big)=(-2,1).$$

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