A STUDY

OF

ORDINARY, COLOURED AND SIGNED

GOLLNITZ – GORDON IDENTITIES

Thesis submitted in partial fulfillment of the requirement for the award of the degree of

Master of Science
in
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Submitted by

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UNDER

THE GUIDANCE OF

Dr. Meenakshi Rana

School of Mathematics and Computer Applications
Thapar University, Patiala
July, 2013

DEDICATED

TO

GOD AND MY PARENTS

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"A study of ordinary, coloured and signed Göllnitz-Gordon identities"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Meenakshi Rana.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

In this thesis, we studied combinatorial interpretation of generalized q - series given by Agarwal in 1986 (A.K. Agarwal, On a Generalized Partition Theorem, J. Indian Math. Soc. Vol. 50 (1986), pp.185-190) using ordinary partitions. In 2009 Agarwal and Rana (A.K. Agarwal and M. Rana, New Combinatorial Version of Göllnitz - Gordon identities, Utilitas Mathematica., Vol. 79(2009), pp.145-156) extended the interpretation given by Agarwal in 1986 using n - coloured partitions.In some particular cases these 2 - way combinatorial identities are extended to a 3 - way combinatorial identities which gives combinatorial interpretations of Göllnitz-Gordon identities using ordinary and n - colour partition discussed in Chapter 2. We have also explored the signed Göllnitz- Gordon identities which is due to Andrews .sills(G.E.Andrews, Euler's De Partition Numerorum,bull.Amer.Math.Soc. Vol 44 (2007),561-573) and bijection between ordinary and signed Göllnitz - Gordon identities due to Andrews V. Sills (Andrews V. Sills , On the Ordinary and Signed Göllnitz - Gordon Partitions , 2007.) discussed in Chapter 3. Chapter 1 is devoted to elementary study of partition Theory.

CONTENTS

CHAPTER 1

Introduction

The Theory of Partition is important branch of additive number theory. The concept of partition of non negative integers also belong to Combinatorics. Partitions first appeared in a letter written by Lebnitz in 1669 to John Bernoulli, asking him if he had investigated the number of ways in which given number can be expressed as sum of two or more integers.

The real development of partition theory started with Euler [7]. It was he who first discovered the important properties of the partition function. The theory has been further developed by many of other mathematicians which are Jacobi, Ramanujan, Sylvester, Gauss. In this thesis we study Gollnitz-Gordon identities using ordinary partition, coloured partition and signed partition.

1.1 Definition of Partition

Definition 1.1.1 A partition of a positive integer n is the finite non increasing sequence of positive integers whose sum is n. The number of partitions of n is denoted by $\mathrm{p}(n)$.

Example 1. The partitions of 7 are

$$7,$$

$$6+1,$$

$$5+2,\ 5+1+1,$$

$$4+3,\ 4+2+1,\ 4+1+1+1,$$

$$3+3+1,\ 3+2+2,\ 3+2+1+1,\ 3+1+1+1+1,$$

$$2+2+2+1,\ 2+2+1+1+1,\ 2+1+1+1+1+1,$$

$$1+1+1+1+1+1+1$$

so, $p(7) = 15$.

Remark 1. $p(n) = 0$ for $n < 0$ because we cannot write a negative number into sum of positive integers.

Also $p(0) = 1$.

Remark 2. We use an exponent to denote that part is repeated a certain number of times in partition.

Example 2.

$$\begin{aligned} p(2) = 2: &\quad 2 = (2),\ 1+1 = (1^2); \\ p(3) = 3: &\quad 3 = (3),\ 2+1 = (12),\ 1+1+1 = (1^3); \\ p(4) = 5: &\quad 4 = (4),\ 3+1 = (13),\ 2+2 = (2^2), \\ &\quad 2+1+1 = (1^2 2),\ 1+1+1+1 = (1^4). \end{aligned}$$

Definition 1.1.2 An ordered collection of partition is called a Composition. Thus $4+3$ and $3+4$ are two different compositions of 7. The number of compositions of n is denoted by $\mathrm{c}(n)$.

Remark 3. Number of compositions of $n = \mathrm{c}(n) = 2^{n-1}$.

Example 3. $p(4) = 5;\ c(4) = 2^{4-1} = 2^3 = 8.$

1.2 Generating function of Partition

Definition 1.2.1 The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n) = \frac{1}{(q;q)_\infty}$$

where $|q| < 1$ and $(q;q)_\infty$ is q - rising factorial defined by

$$(a;q)_n = \prod_{n=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{i+n})} = \prod_{i=0}^{n-1} (1-aq^i) \quad .$$

If n is a positive integer, then

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$

and when $a = q$

$$(q;q)_n = \prod_{i=1}^{n} (1-q^i).$$

.

Definition 1.2.2 Let p denote the set of all partitions and Let *p(S,n)* denote the number of partition of n that belong to a subset S of the set P of all partition.

In connection with above we consider partitions into distinct and odd parts as, Let D(n) denote the number of partitions of n into distinct parts, then generating function for this is

$$\sum_{n=0}^{\infty} D(n)q^n = \prod_{n=1}^{\infty}(1+q^n) = (-q;q)_\infty.$$

Example 4. We might consider D the set of all partitions with distinct parts

$p(D,7) = 5.$

The relevant partitions are 7, 6+1, 5+2, 4+3, 4+2+1.

Let O(n) denote the number of partitions of n into odd parts, then generating function for this is

$$\sum_{n=0}^{\infty} O(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})} = \frac{1}{(q;q^2)_\infty}.$$

Example 5. We might consider O the set of all partitions with distinct parts

$$p(O,7) = 5.$$

The relevant partitions are 7, 5+1+1, 3+3+1, 3+1+1+1, 1+1+1+1+1+1+1.

1.3 Graphical representation

Definition 1.3.1 The Ferrer's graph of a partition t_1, t_2,. . . ,t_i of n is a set of rows of equi-spaced dots aligned on the left where the *jth* row has t_j dots.

Example 6. The Ferrer's graph of the partition 6+4+3+1 of 14 as following,

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & & & & & \end{array}$$

by reading this graph horizontally, we see that first row has 6 dots, second row has 4, third row has 3, fourth row has 1.

Definition 1.3.2 The conjugate of a partition is obtained by interchanging ferrer's diagram's rows with its columns.

Example 7. The Ferrer's graph of partition 3+2+2 of 7 is

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \end{array}$$

The conjugate of above graph is

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & & \end{array}$$

is 3+3+1 of 7.

The conjugate of partition Π is denoted by Π^c.

Definition 1.3.3 A partition is said to be self conjugate if it is identical with its conjugate.

Example 8. Consider the partition $\Pi = 3 + 2 + 1$ of 6 then the Ferrer's graph is

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & & \end{array}$$

now if we read this graph vertically we get the same partition $\Pi^c = 3 + 2 + 1$ so this partition is self conjugate partition. In others words here $\Pi = \Pi^c$. So it is self conjugate partition.

1.4 n - Colour partition

Definition 1.4.1[2] An n-colour partition of a positive integer is a partition in which a part of size n, can come in n different colours denoted by subscripts $n_1, n_2, n_3 \ldots n_n$ and parts satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 \cdots$

Example 9. The n-colour partition of 3 are

$$3_1,\ 3_2,\ 3_3$$

$$2_1 + 1_1,\ 2_2 + 1_1$$

$$1_1 + 1_1 + 1_1$$

Definition 1.4.2 [2] Let $\Pi = (a_1)_{b1} + (a_2)_{b2} + \ldots + (a_r)_{br}$ be an n-color partition of v. We call $(a_i)_{a_i - b_i + 1}$ the conjugate of $(a_i) - b_i$. An n-color partition v obtained from Π by replacing each of its parts by its conjugate will be called the conjugate of Π and will be denoted by Π^c.

Example 10. If we consider $\Pi = 5_2 + 3_1$, an n-colour partition of 8,

then

$$\Pi^c = 5_{5-2+1} + 3_{3-1+1} = 5_4 + 3_3.$$

Definition 1.4.3 [2] The weighted difference of any pair of parts m_i, n_j is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

1.5 Some basic theorems on partition

Theorem 1.5.1 The number of partition of n into distinct parts is equal to the number of partitions of n into odd parts.

Proof. Given a partition of n into distinct parts replace each even parts by its two halves and repeat this process till even parts are left. Finally arrange the parts in non-increasing order. This will be a partition of n into odd parts.

Conversely, for a partition into odd parts go on adding two equal parts until there are no repetitions. Finally arrange the parts in non-increasing order. This gives rise to a partition into distinct parts.

Example 11. Consider the partition of 26 into distinct parts

$$\begin{aligned}
&26 = 10 + 8 + 5 + 3 \\
&\to 5 + 5 + 4 + 4 + 5 + 3 \\
&\to 5 + 5 + 2 + 2 + 2 + 2 + 5 + 3 \\
&\to 5 + 5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 5 + 3 \\
&\to 5 + 5 + 5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
\end{aligned}$$

Conversely

$$\begin{aligned}
26 &= 5 + 5 + 5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
&\to 10 + 5 + 3 + 2 + 2 + 2 + 2 \\
&\to 10 + 5 + 3 + 4 + 4
\end{aligned}$$

$\rightarrow 10 + 5 + 3 + 8$

$\rightarrow 10 + 8 + 5 + 3$

Theorem 1.5.2 The number of partitions of n into r parts is equal to the number of partitions of n in which largest part is r.

Proof. Consider a partition Π of n into r parts and draw its Ferrer's graph. Now read this Ferrer's graph vertically we get conjugate partition of Π which is a partition of n in which largest part is r.

. . . .
. . . . . .
. . . . .
and so on
. . . .
. . .

Consider a partition Π in which largest part is r. Draw its Ferrer's graph now read the graph vertically we get conjugate partition of Π which is a partition of n into r parts. So, we get one-one correspondence between the partitions of n into r parts and the partition of n in which largest part is r. Hence, we get the number of partitions of n into r parts is equal to the number of partitions of n in which largest part is r.

Example 12. Consider the partition Π = 4+3+3+2+1 of 13 then the Ferrer's graph is

. . . .
. . .
. . .
. .
.

read vertically, we get conjugate partition of Π

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and the new partition becomes 5+4+3+1 in this partition we see the number of parts is equal to the largest part in the original partition.

Theorem 1.5.3 The number of partitions of n in which each part appears two, three or five times equals the number of partitions of n into parts congruent to 2, 3, 4, 6, 9 or 10 modulo 12.

Proof. The Generating function of number of partitions n which each part appears two, three or five times is

$$\begin{aligned}
\prod_{n=0}^{\infty}(1+q^{2n}+q^{3n}+q^{5n}) &= \prod_{n=0}^{\infty}(1+q^{2n}+q^{3n}(1+q^{2n})) \\
&= \prod_{n=0}^{\infty}(1+q^{2n})(1+q^{3n}) \\
&= \prod_{n=0}^{\infty}\frac{(1-q^{4n})(1-q^{6n})}{(1-q^{2n})(1-q^{3n})} \\
&= \prod_{n=0}^{\infty}\frac{(1-q^{4n})(1-q^{6n})}{(1-q^{4n})(1-q^{4n+2})(1-q^{6n})(1-q^{6n+3})} \\
&= \prod_{n=0}^{\infty}\frac{1}{(1-q^{4n+2})(1-q^{6n+3})} \\
&= \prod_{n=0}^{\infty}\frac{1}{(1-q^{12n+2})(1-q^{12n+3})(1-q^{12n+6})(1-q^{12n+9})(1-q^{12n+10})}
\end{aligned}$$

is the Generating function of number of partitions of n into parts congruent to 2, 3, 6, 9 or 10 modulo 12.

CHAPTER 2

Combinatorial interpretation of Göllnitz - Gordon identities using ordinary and coloured partition.

2.1 Göllnitz - Gordon identities using ordinary partition

Theorem 2.1.1 The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear is equal to the number of partitions of n into parts which are congruent to 1, 4 or 7 (mod 8).

Theorem 2.1.2 The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3 is equal to the number of partitions of n into parts which are congruent to 3, 4 or 5(mod 8).

Theorem 2.1.3 The number of partitions of n into parts differing by at least 2 among which no two consecutive odd numbers appear and with each part being at least equal to 2 is equal to the number of partitions of n into parts which are congruent to 2, 3 or 7(mod 8).

Theorem 2.1.1 and **Theorem 2.1.2** are combinatorial interpretations of Göllnitz - Gordon identities which are due to Göllnitz [8] and were included in his (1960) unpublished honors baccalaureate thesis. However, essentially no one knew about the result until Gordon [10] (1965) independently rediscovered them and are commonly referred as Göllnitz - Gordon identities where **Theorem 2.1.3** is due to Göllnitz [9]. The analytic counterparts of the Göllnitz - Gordon partition identities and the Göllnitz identity are the q - series identities,

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \tag{2.1.1}$$

$$= 1+q+q^2+q^3+2q^4+2q^5+2q^6+3q^7+4q^8+5q^9+5q^{10}+\cdots$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty} \tag{2.1.2}$$
$$= 1+q^3+q^4+q^5+q^6+q^7+2q^8+2q^9+2q^{10}+\cdots$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)}}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^5;q^8)_\infty (q^6;q^8)_\infty} \tag{2.1.3}$$
$$= 1+q^2+q^3+q^4+q^5+2q^6+2q^7+2q^8+3q^9+4q^{10}+\cdots .$$

These analytic identities were published by Slater in 1952, Equation (2.1.1) is number 36 and Equation (2.1.2) is number 34 in slater paper [14]. The Equation (2.1.3) is Corollary 2.7, page 21, in [4]. However, it has recently been discovered by Sills et al [11] that two analytic identities equivalent to the analytic Göllnitz - Gordon identities were recorded by Ramanujan in his lost notebook, and thus that Ramanujan knew these identites more than 30 years before Slater rediscovered them (Andrews and Berndt (2008) page 37 [6]).

2.2 Main Theorem 1

In [1] Agarwal proved the following theorem:

Theorem 2.2.1 Given positive integer k, let $A_k(n)$ denote the number of partitions of n in which each part $\geq k$, minimal difference ≥ 2 between the parts, consecutive odd integers are not allowed if k is even and consecutive even integers are not allowed if k is odd. Then

$$\sum_{n=0}^{\infty} A_k(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+k-1)}. \tag{2.2.1}$$

Proof Let $A_k(m,n)$ denote the number of the partitions of the type enumerated by $A_k(n)$ with added restriction that there be exactly m parts. We shall first prove the identity

$$A_k(m,n) = A_k(m-1, n-k-2m+2) + A_k(m-1, n-k-4m+3) + A_k(m, n-2m). \tag{2.2.2}$$

We split the partitions enumerated by $A_k(m,n)$ into three classes:

(i) those that have least part equal to k

(ii)those that have least part equal to $k+1$

(iii)those that have least part greater than or equal to $k+2$

we now transform the partitions in class (i) by deleting the least part k and then subtracting 2 from all the remaining parts. This produces a partition of $n-k-2(m-1)$ into exactly $m-1$ parts, each of which $\geq k$; furthermore, since this transformation does not disturb the inequalities between the parts, we see that the transformed partition is of the type enumerated by

$$A_k(m-1, n-k-2m+2)$$

next we transform the partitions in class (ii) by deleting the least part $k+1$ and then subtracting 4 from all the remaining parts. This produces a partition of

$$n-(k+1)-4(m-1)=n-k-4m+3$$

into $m-1$ parts, each of which $\geq k$ (since originally the second smallest part was $\geq k+4$. Note that originally $k+2$ could not be the second smallest part because of the minimal difference between the parts ≥ 2. Also $k+3$ could not be the second smallest part as $k+1$ and $k+3$ are consecutive odd if k is even and consecutive even if k is odd.) furthermore, since the inequalities between the parts are not disturbed, we see that the transformed partition is of the type enumerated by

$$A_k(m-1,n-k-4m+3)$$

finally, we transform the partitions in class (iii) by subtracting 2 from each part. This produces a partition of $n-2m$ into m parts, each of which $\geq k$, as in the first two cases, here too, the inequalities between the parts are not disturbed, we see that the transformed partition is of the type enumerated by $A_k(m,n-2m)$. The above transformation establish a bijection between the partitions enumerated by $A_k(m,n)$ and those enumerated by

$$A_k(m,n)=A_k(m-1,n-k-2m+2)+A_k(m-1,n-k-4m+3)+A_k(m,n-2m)$$

let

$$f_k(z,q)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_k(m,n)z^mq^n$$

then identity (2.2.2) implies that

$$f_k(z,q)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(A_k(m-1,n-k-2m+2)+A_k(m-1,n-k-4m+3)+A_k(m,n-2m))z^mq^n$$

$$=zq^k\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_k(m-1,n-k-2m+2)(zq^2)^{m-1}q^{n-k-2m+2}$$

$$+zq^{k+1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_k(m-1,n-k-4m+3)(zq^4)^{m-1}q^{n-k-4m+3}$$

$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_k(m,n-2m)(zq^2)^mq^{n-2m}$$

$$=z_kq^kf_k(zq^2;q)+zq^{k+1}f_k(zq^4;q)+f_k(zq^2;q) \qquad (2.2.3)$$

setting

$$f_k(z,q)=\sum_{n=0}^{\infty}\alpha(n,k;q)z^n$$

and then comparing the coefficients of z^n on each side of (2.2.3), we see that

$$\alpha(n,k;q)=q^{2n-2+k}\alpha(n-1,k;q)+q^{4n-3+k}\alpha(n-1,k;q)+q^{2n}\alpha(n,k;q)$$

therefore

$$\alpha(n,k;q)=\frac{(1+q^{2n-1})q^{2n-2+k}}{1-q^{2n}}\alpha(n-1,k;q) \qquad (2.2.4)$$

iterating (2.2.4) $n-times$ and observing that

$$\alpha(0,k;q)=1$$

we get that

$$\alpha(n,k;q)=\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n(n+k-1)}$$

therefore

$$f_k(z,q)=\sum_{n=0}^{\infty}\alpha(n,k;q)z^n=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n(n+k-1)}z^n$$

now

$$\sum_{n=0}^{\infty}A_k(n)q^n=\sum_{n=0}^{\infty}[\sum_{m=0}^{\infty}A_k(m,n)]q^n$$

$$=f_k(1,q)$$

$$=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}$$

This completes the proof of the main theorem 1 .

2.2.1 Particular cases

If $k=1$, then Theorem 2.2.1 in view of Identity 2.1.1 becomes

Corollary 2.2.1(a)

$$A_1(n)=D_1(n) \qquad (2.2.1)_a$$

where $A_1(n)$ is the number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear.

where $D_1(n)$ is the number of partitions of n into parts which are congruent to 1, 4 or 7(mod 8).

Example 1. The following table gives the relevant partition of $A_1(n)$ and $D_1(n)$ when $n=1$ to 10.

n	$A_1(n)$	Relevant partition	$D_1(n)$	Relevant partition
1	1	1	1	1
2	1	2	1	1^2
3	1	3	1	1^3
4	2	4, 3+1	2	4, 1^4
5	2	5, 4+1	2	14^1, 1^5
6	2	6, 5+1	2	$1^2 4$, 1^6
7	3	7, 6+1, 5+2	3	7, $1^3 4$, 1^7
8	4	8, 7+1, 6+2, 5+3	4	17^1, 4^2, $1^4 4$, 1^8
9	5	9, 8+1, 7+2, 6+3, 5+3+1	5	9, $1^2 7$, 14^2, $1^5 4$, 1^9
10	5	10, 9+1, 8+2, 7+3, 6+3+1	5	$1^3 7$, $1^2 4^2$, $1^6 4$, $1^4 6$, 1^{10}

If $k = 2$, then Theorem 2.2.1 in view of Identity 2.1.3 becomes

Corollary 2.2.1(b)

$$A_2(n) = D_2(n) \qquad (2.2.1)_b$$

where $A_2(n)$ is the number of partitions of n into parts differing by at least 2 among which no two consecutive odd numbers appear and with each part being at least equal to 2.

where $D_2(n)$ is the number of partitions of n into parts which are congruent to 2, 3 or 7(mod 8).

Example 2. The following table gives the relevant partition of $A_2(n)$ and $D_2(n)$ when $n = 1$ to 10.

n	$A_2(n)$	Relevant partition	$D_2(n)$	Relevant partition
1	0	-	0	-
2	1	2	1	2
3	1	3	1	3
4	1	4	1	2^2
5	1	5	1	23
6	2	6, 4+2	2	3^2, 2^3
7	2	7, 5+2	2	7, $2^2 3$
8	2	8, 6+2	2	23^2, 2^4
9	3	9, 7+2, 6+3	3	27^1, 3^3, $2^3 3$
10	4	10, 8+2, 7+3, 6+4	4	10, 37^1, 2^5, $2^2 3^2$

If $k = 3$, then Theorem 2.2.1 in view of Identity 2.1.2 becomes

Corollary 2.2.1(c)

$$A_3(n) = D_3(n) \qquad (2.2.1)_c$$

where $A_3(n)$ is the number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3.

where $D_3(n)$ is the number of partitions of n into parts which are congruent to 3, 4 or 5(mod 8).

Example 3. The following table gives the relevant partition of $A_3(n)$ and $D_3(n)$ when $n = 1$ to 10.

n	$A_3(n)$	Relevant partition	$D_3(n)$	Relevant partition
1	0	-	0	-
2	0	-	0	-
3	1	3	1	3
4	1	4	1	4
5	1	5	1	5
6	1	6	1	3^2
7	1	7	1	34^1
8	2	8, 5+3	2	35^1, 4^2
9	2	9, 6+3	2	45^1, 3^3
10	2	10, 7+3	2	5^2, 3^24

2.3 Main theorem 2

In a attempt to unify the Göllnitz - Gordon partition functions appearing in Theorem 2.1.1, 2.1.2 and the Göllnitz partition function of Theorem 2.1.3, Agarwal in [1] defined a more generalized partition function $A_k(n)$ given in Theorem 2.2.1. Obviously, $A_1(n)$ and $A_3(n)$ are Göllnitz - Gordon functions of Theorem 2.1.1, 2.1.2 and $A_2(n)$ is Göllnitz function appearing in Theorem 2.1.3.

In [3] Agarwal and Rana interpret the right hand side of (2.2.1) as a generating function of a $n-$colour partition function. This result in an infinite family of 2 - way combinatorial identities. In some particular cases we get even 3 - way combinatorial identities. We discuss three such cases and obtain new combinatorial versions of each of the Theorem 2.1.1, 2.1.2 and 2.1.3. Rana and Agarwal further extended their result, given in [12]

Theorem 2.3.1 Given a positive integer k, let $B_k(n)$ denote the number of n-colour partitions of n such that parts are greater than or equal to k, parts used are of the type $(2l-1)_1$ and $(2l)_2$ if k is odd, $(2l-1)_2$ and $(2l)_1$ if k is even. The weighted difference between any two parts is non-negative and even. Then

$$A_k(n) = B_k(n) \tag{2.3.1}$$

for all n, where $A_k(n)$ is in last Theorem 2.2.1 .

Proof

We shall prove that

$$\sum_{n=0}^{\infty} B_k(n)q^n \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+k-1)} \tag{2.3.2}$$

Let $B_k(m,n)$ denote the number of the partitions of the type enumerated by $B_k(n)$ into exactly m parts. We shall first prove the identity,

$$B_k(m,n) = B_k(m-1, n-k-2m+2) + B_k(m-1, n-k-4m+3) + B_k(m, n-2m).$$

We give the proof of (2.3.1) for odd k as the proof for even k is similar.

We split the partitions enumerated by $B_k(m$,n) into three classes:

(i) those that have least part equal to k_1

(ii)those that have least part equal to $(k+1)_2$

(iii)those that have least part greater than or equal to $(k+2)_1$

We now transform the partitions in class (i) by deleting the least part k_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. This produces a partition of $n-k-2(m-1)$ into exactly $m-1$ parts, each of which $\geq k_1$(since originally the second smallest part was $\geq (k+2)_1$.
Obviously this transformation does not disturb the weighted difference condition between the parts and so the transformed partition is of the type enumerated by

$$B_k(m-1, n-k-2m+2)$$

next, we transform the partitions in class (ii) by deleting the least part $(k+1)_2$

and then subtracting 4 from all the remaining parts ignoring the subscripts. This produces a partition of

$$n-(k+1)-4(m-1)=n-k-4m+3$$

into $m-1$ parts, each of which $\geq k_1$ (since originally the second smallest part was $\geq (k+4)_1$. Note that originally $(k+2)_1$ and $(k+3)_2$ could not be the smallest part because of the weighted difference condition. Furthermore, since the weighted difference condition between the parts are not disturbed, we see that the transformed partition is of the type enumerated by

$$B_k(m-1,n-k-4m+3)$$

Finally, we transform the partitions in class (iii) by subtracting 2 from each part ignoring the subscripts. This produces a partition of $n-2m$ into m parts, each of which $\geq k_1$, since the weighted difference condition between the parts are not disturbed. We see that the transformed partition is of the type enumerated by $B_k(m,n-2m)$. The above transformation establish a bijection between the partitions enumerated by $B_k(m,n)$ and those enumerated by

$$B_k(m,n)=B_k(m-1,n-k-2m+2)+B_k(m-1,n-k-4m+3)+B_k(m,n-2m)$$

This proves the identity (2.3.2) for odd k.

Let

$$g_k(z,q)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}B_k(m,n)z^mq^n \tag{2.3.3}$$

using (2.3.2) in (2.3.3), we get the following q-functional equation

$$g_k(z,q) = z_k q^k g_k(zq^2;q) + zq^{k+1} g_k(zq^4;q) + g_k(zq^2;q). \tag{2.3.4}$$

setting

$$g_k(z,q) = \sum_{n=0}^{\infty} \beta_k(n;q) z^n$$

and then comparing the coefficients of z^n on each side of (2.3.4), we get

$$\beta_k(n;q) = q^{2n-2+k}\beta_k(n-1;q) + q^{4n-3+k}\beta(n-1;q) + q^{2n}\beta_k(n;q)$$

therefore

$$\beta_k(n;q) = \frac{(1+q^{2n-1})q^{2n-2+k}}{1-q^{2n}}\beta_k(n-1;q) \tag{2.3.5}$$

iterating (2.3.5) n - times and observing that

$$\beta_k(0;q) = 1$$

we see that

$$\beta(n,k;q) = \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+k-1)}$$

therefore

$$g_k(z,q) = \sum_{n=0}^{\infty} \beta_k(n;q) z^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n(n+k-1)} z^n.$$

More,

$$\sum_{n=0}^{\infty} B_k(n) q^v = \sum_{n=0}^{\infty}[\sum_{m=0}^{\infty} B_k(m,n)] q^n$$

$$= g_k(1,q)$$

$$= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}$$

This completes the proof of the main theorem 2 .

2.3.1 Particular cases

For $k = 1$, in view of (2.1.1) our Theorem 2.3.1 reduces to:

Corollary 2.3.1(a)

$$A_1(n) = B_1(n) = D_1(n) \qquad (2.3.1)_a$$

where $A_1(n)$ is as given in Theorem 2.2.1.

where $D_1(n)$ as given in corollary 2.2.1(a)

$(2.3.1)_a$ gives us following three identities in the usual sense,

$$A_1(n) = B_1(n) \qquad (2.4.1)_{a.1}$$

$$A_1(n) = D_1(n) \qquad (2.4.1)_{a.2}$$

$$B_1(n) = D_1(n) \qquad (2.4.1)_{a.3}$$

The case $(2.3.1)_{a.2}$ of $(2.3.1)_a$ is the first Göllnitz -Gordon [9] identity, that is, Theorem 2.1.1. The other two identities induced by $(2.3.1)_a$ are new combinatorial versions of it.

Example 4. The following table gives the relevant partition of $A_1(n)$, $B_1(n)$ and $D_1(n)$ when $n = 1$ to 10.

n	$A_1(n)$	Relevant partitions	$B_1(n)$	Relevant partitions	$D_1(n)$	Relevant partitions
1	1	1	1	1_1	1	1
2	1	2	1	2_2	1	1^2
3	1	3	1	3_1	1	1^3
4	2	4, 13	2	4_2, $3_1 1_1$	2	4, 1^4
5	2	5, 14	2	5_1, $4_2 1_1$	2	14^1, 1^5
6	2	6, 15	2	6_2,$5_1 1_1$	2	41^2, 1^6
7	3	7, 16, 25	3	7_1, $6_2 1_1$ $5_1 2_2$	3	7, 41^3, 1^7
8	4	8, 17, 26, 35	4	8_2, $7_1 1_1$, $6_2 2_2$, $5_1 3_1$	4	17^1, 4^2, 41^4, 1^8
9	5	9, 18, 27, 36, 135	5	9_1, $8_2 1_1$, $7_1 2_2$, $6_2 3_1$, $5_1 3_1 1_1$	5	9, 71^2, 14^2, 41^5, 1^9
10	5	10, 19, 28, 37, 136	5	10_2, $9_1 1_1$, $8_2 2_2$, $7_1 3_1$, $6_2 3_1 1_1$	5	71^3, 61^4, $4^2 1^2$, 41^6, 1^{10}

For $k = 2$, in view of (2.1.3) we get the following three way identity:

Corollary 2.3.1(b)

$$A_2(n) = B_2(n) = D_2(n) \qquad (2.3.1)_b$$

where $A_2(n)$ is as given in Theorem 2.2.1 .

where $D_2(n)$ as given in corollary 2.2.1(b)

$(2.3.1)_b$ gives us following three identities in the usual sense,

$$A_2(n) = B_2(n) \qquad (2.3.1)_{b.1}$$

$$A_2(n) = D_2(n) \qquad (2.3.1)_{b.2}$$

$$B_2(n) = D_2(n) \qquad (2.3.1)_{b.3}$$

The case $(2.3.1)_{b.2}$ of $(2.3.1)_b$ is the Göllnitz -Gordon [9] identity, that is, Theorem 2.1.3. The other two identities induced by $(2.3.1)_b$ are new combinatorial versions of it.

Example 5. The following table gives the relevant partition of $A_2(n)$, $B_2(n)$ and $D_2(n)$ when $n = 1$ to 10.

n	$A_2(n)$	Relevant partitions	$B_2(n)$	Relevant partitions	$D_2(n)$	Relevant partitions
1	0	-	0	-	0	-
2	1	2	1	2_1	1	2
3	1	3	1	3_2	1	3
4	1	4	1	4_1	1	2^2
5	1	5	1	5_2	1	23^1
6	2	6, 24	2	6_1, $4_1 2_1$	2	3^2, 2^3
7	2	7, 25	2	7_2, $5_2 2_1$	2	7, $2^2 3$
8	2	8, 26	2	8_1, $6_1 2_1$	2	23^2, 2^4
9	3	9, 27, 36	3	9_2, $7_2 2_1$, $6_1 3_2$	3	27^1, 3^3, $2^3 3$
10	4	10, 28, 37, 46	4	10_1, $8_1 2_1$, $7_2 3_2$, $6_1 4_1$	4	10, 37^1, $2^2 3^2$, 2^5

For $k = 3$, in view of (2.1.2) we get the following three way identity:

Corollary 2.3.1(c)

$$A_3(n) = B_3(n) = D_3(n) \qquad (2.3.1)_c$$

where $A_3(n)$ is as given in Theorem 2.2.1

where $D_3(n)$ as given in corollary 2.2.3

$(2.3.1)_c$ gives us following three identities in the usual sense,

$$A_3(n) = B_3(n) \qquad (2.3.1)_{c.1}$$

$$A_3(n) = D_3(n) \qquad (2.3.1)_{c.2}$$

$$B_3(n) = D_3(n) \qquad (2.3.1)_{c.3}$$

The case $(2.3.1)_{c.2}$ of $(2.3.1)_c$ is the second Göllnitz -Gordon [9] identity, that is, Theorem 2.2.1 . The other two identities induced by $(2.3.1)_c$ are new combinatorial versions of it.

Example 6. The following table gives the relevant partition of $A_3(n)$, $B_3(n)$ and $D_3(n)$ when $n = 1$ to 10.

n	$A_3(n)$	Relevant partitions	$B_3(n)$	Relevant partitions	$D_3(n)$	Relevant partitions
1	0	-	0	-	0	-
2	0	-	0	-	0	-
3	1	3	1	3_1	1	3
4	1	4	1	4_2	1	4
5	1	5	1	5_1	1	5
6	1	6	1	6_2	1	3^2
7	1	7	1	7_1	1	34^1
8	2	8, 35	2	8_2, $5_1 3_1$	2	35^1, 4^2
9	2	9, 36	2	9_1, $6_2 3_1$	2	45^1, 3^3
10	2	10, 37	2	10_2, $7_1 3_1$	2	5^2, 43^2

2.4 Conclusion

The main theorems of this chapter provide infinite 2-way combinatorial identity for each positive integer k using ordinary partition function and coloured partitions. For k = 1, 2, 3 we get a 3-way combinatorial identity, in particular Göllnitz-Gordon identities.

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CHAPTER 3

Signed Göllnitz - Gordon Identity

3.1 Signed partition

Definition 3.1.1[5] The Signed partition σ of an integer n as a pair of partitions (π, ν) where $n = \mid \pi \mid$ - $\mid \nu \mid$. The parts of π are positive parts of σ and the parts of ν are negative parts of σ.

Example 1. The signed partition of 5= 6+3+3+1-1-1-2-4.

The following theorem is due to Andrews[5].

Theorem 3.1.1 Let $C_1(n)$ denote the number of signed partitions of n where the negative parts are distinct, odd and smaller in magnitude than twice the number of positive parts and the positive parts are even and have magnitude atleast twice the number of positive parts. Let $D_1(n)$ be as defined in Corollary 2.2.1(a) .Then $C_1(n) = D_1(n)$ for all n.

Proof The result is variation on the first Göllnitz - Gordon identity

$$1 + \sum_{n=1,}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1, n\equiv 1,4 or 7 (mod 8)}^{\infty} \frac{1}{1-q^n}. \qquad (3.1.1)$$

Clearly the right hand side of 3.1.1 is generating function of $D_1(n)$ and left hand side can be written as

$$1 + \sum_{n=1}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}$$

$$= 1+\sum_{n=1}^{\infty} \frac{q^{\overbrace{2n+2n.....+2n}^{ntimes}}}{(1-q^2)(1-q^4).......(1-q^{2n})}(1+\frac{1}{q})(1+\frac{1}{q^3})...(1+\frac{1}{q^{2n-1}}) = \sum_{n\geqq 0} C_1(n)q^n.$$

Theorem 3.1.2 Let $C_2(n)$ denote the number of signed partitions of n where each positive part is even and $\geqslant (2n+2)$ and negative parts are odd, distinct and smaller than $2n$. Let $D_3(n)$ be as defined in corollary 2.2.3 . Then $C_2(n) = D_3(n)$ for all n.

Proof We have second Göllnitz-Gordon identity

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} = \prod_{n=1, n\equiv 3,4 or 5 (mod 8)}^{\infty} \frac{1}{1-q^n}. \tag{3.1.2}$$

The right hand side of 3.1.2 is generating function of $D_3(n)$ and left hand side can be written as

$$\frac{q^{n^2+2n}}{(q^2;q^2)_n} \prod_{k=1}^{n} (1+q^{(2k-1)})$$

$$= \frac{q^{n^2+2n} q^{n^2}}{(q^2;q^2)_n} \prod_{k=1}^{n} (1+q^{-(2k-1)})$$

$$= \frac{q^{2n^2+2n}}{(q^2;q^2)_n} \prod_{k=1}^{n} (1+q^{-(2k-1)})$$

$$= \sum_{n=1}^{\infty} \frac{q^{\overbrace{(2n+2)+(2n+2).....+(2n+2)}^{ntimes}}}{(1-q^2)(1-q^4).......(1-q^{2n})}(1+\frac{1}{q})(1+\frac{1}{q^3})...(1+\frac{1}{q^{2n-1}}) = \sum_{n\geqq 0} C_2(n)q^n.$$

The above two theorems provides the combinatorial interpretations of Göllnitz-Gordon identities using signed partition. In the next section we study the bijection between the ordinary Göllnitz-Gordon identities given in Chapter 2 (Corollary 2.2.1(a) and Corollary 2.2.3(c)) and signed Göllnitz-Gordon identities given in this section in Chapter 3.

3.2 A bijection between ordinary and signed Göllnitz - Gordon identities

Andrews V.sills [13] provided a bijection between the set of ordinary Göllnitz-Gordon partitions (those enumerated by $A_k(n)$ (k= 1,3) in Theorem 2.2.1) and Andrews "signed Göllnitz-Gordon partitions" enumerated by $C_k(n)$ (k= 1, 2) in Theorem 3.1.1 and Theorem 3.1.2 respectively.

Theorem 3.2.1 The map

$$g : G_{n,j} \to S_{n,j}$$

given by

$$(\gamma_1, \gamma_2, ..., \gamma_j) \to ((\pi_1, \pi_2, ..., \pi_j), (1^{f_1} 3^{f_3} ....(2j-1)^{f_{2j-1}}))$$

where

$G_{n,j}$ denote set of partitions $\gamma = (\gamma_1, \gamma_2, ...\gamma_j)$ of weight n and length j

and

$S_{n,j}$ denote set of signed partitions $\sigma = (\pi, \nu)$ of n ,

$$\pi_k = \gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2 \sum_{i=k+1}^{j} P(\gamma_i)$$

and

$$f_{2k-1} = P(\gamma_k)$$

is a bijection.

Proof. Suppose that $\gamma \in G_{n,j}$ and the image of γ under g is the signed partition $\sigma = (\pi, \nu)$.

Claim 1.

$$|\sigma| = |\pi| - |\nu| = n.$$

Proof of claim 1.

$$|\pi| - |\nu| = \sum_{k=1}^{j} (\gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2\sum_{i=k+1}^{j} P(\gamma_i)) - (\sum_{h=1}^{j} (2h-1)P(\gamma_k))$$

$$= \sum_{k=1}^{j} \gamma_k + 4\sum_{k=1}^{j} - 2\sum_{k=1}^{j} j - 2\sum_{k=1}^{j} 1 + \sum_{k=1}^{j} P(\gamma_k) + 2\sum_{k=1}^{j}\sum_{i=k+1}^{j} P(\gamma_i) - (\sum_{h=1}^{j}(2h-1)P(\gamma_k))$$

$$= (\sum_{k=1}^{j} \gamma_k) + \frac{4j(j+1)}{2} - 2j^2 - 2j + \sum_{k=1}^{j} P(\gamma_k) + 2\sum_{k=1}^{j}\sum_{i=k+1}^{j} P(\gamma_i) - (\sum_{h=1}^{j}(2h-1)P(\gamma_k))$$

$$= n - \sum_{h=1}^{j}(2h-2)P(\gamma_h) + 2\sum_{i=1}^{j}(h-1)P(\gamma_k) = n$$

Claim 2.

$$\pi_1 \geqq \pi_2 \geqq .... \geqq \pi_j.$$

Proof of claim 2. Fix k with $1 \leq k < j$.

$$\pi_k - \pi_{k+1} = \gamma_k + 4k - 2j - 2 + p(\gamma_k) + 2\sum_{i=k+1}^{j} p(\gamma_i)$$

$$- (\gamma_{k+1} + 4(k+1) - 2j - 2 + P(\gamma_{k+1}) + 2\sum_{i=k+2}^{j} P(\gamma_i))$$

$$= \gamma_k - \gamma_{k+1} + P(\gamma_k) + P(\gamma_{k+1}) - 4.$$

The minimum value of $\gamma_k - \gamma_{k+1}$ varies depending on the parities of γ_k and γ_{k+1}.

• If $\gamma_k \equiv \gamma_{k+1} \equiv 0(mod2)$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) + P(\gamma_{k+1}) - 4$$

$$\geq 4 + 0 + 0 - 4$$

$$= 0.$$

• If $\gamma_k \equiv 1(mod2) \quad and \quad \gamma_{k+1} \equiv 0(mod2)$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) + P(\gamma_{k+1}) - 4$$

$$\geq 3 + 1 + 0 - 4$$

$$= 0.$$

• If $\gamma_k \equiv 0(mod2) \quad and \quad \gamma_{k+1} \equiv 1(mod2)$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) + P(\gamma_{k+1}) - 4$$

$$\geq 3 + 0 + 1 - 4$$

$$= 0.$$

• If $\gamma_k \equiv \gamma_{k+1} \equiv 1(mod2)$, then

$$(\gamma_k - \gamma_{k+1}) + P(\gamma_k) + P(\gamma_{k+1}) - 4$$

$$\geq 2 + 1 + 1 - 4$$

$$= 0.$$

Claim 3. All of the π_k are at least 2j.

Proof of claim 3. By claim 2, it is sufficient to show that $\pi_j \geq 2j$.

If $\gamma_j = 1$, then

$$\pi_j = \gamma_j + 4j - 2j - 2 + P(\gamma_j) + 2\sum_{i=j+1}^{j} P(\gamma_i)$$

$$= \gamma_j + 2j - 2 + 1$$

$$\geq 1 + 2j - 2 + 1$$

=2j.

If $\gamma_j \geq 2$, and so

$$\pi_j = \gamma_j + 4j - 2j - 2 + P(\gamma_j) + 2\sum_{i=j+1}^{j} P(\gamma_i)$$

$$= \gamma_j + 2j - 2 + 1$$

$$\geq 2 + 2j - 2 + 0$$

=2j.

Claim 4. All parts of π are even.

Proof of claim 4.

$$\pi_k - \pi_{k+1} = \gamma_k + 4k - 2j - 2 + P(\gamma_k) + 2\sum_{i=k+1}^{j} P(\gamma_i)$$

$$\equiv \gamma_k + P(\gamma_k)(mod 2)$$

$$\equiv 0(mod 2)$$

.

Claim 5. All parts of ν are distinct,odd and atmost $2j - 1$.

Proof of claim 5. Claim 5 is clear from the definition of g together with the observation that $P(\gamma_i) \in$ 0,1 for any i.

Claim 6. The map g is invertible.

Proof of claim 6. Let

$$h : S_{n,j} \to G_{n,j}$$

be given by

$$((\pi_1, \pi_2, ...\pi_j), (1^{f_1}3^{f_3}....(2j-1)^{f_{2j-1}})) \to (\gamma_1, \gamma_2, ..., \gamma_3)$$

where

$$\gamma_k = \pi_k - 4k + 2j + 2 - f_{2k-1} - 2\sum_{i=k+1}^{j} f_{2i-1}.$$

for

$$1 \leq k \leq j.$$

We have $h(g(\gamma)) = \gamma$ for all $\gamma \in G_{n,j}$ and $g(h(\sigma)) = \sigma$ for all $\sigma \in S_{n,j}$. Thus h is inverse of g.

Hence, by the above claims g is a bijection.

3.3 conclusion

The purpose of this thesis is to study different combinatorial objects and to under their interdisplinary behaviour. Here we studied the combinatorial interpretations of Göllnitz - Gordon identities using ordinary, n- colour and signed partitions. We also explored the bijection between ordinary and signed Göllnitz - Gordon identities.

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