MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

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CERTIFICATE

*I hereby certify that the work which is being presented in the thesis entitled "**Multi-Index fixed charge bi-criterion transportation problem**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Mahesh Kumar Sharma.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

The Fixed charge bi-criterion transportation problem which is an extension of the bi-criterion transportation problem has been studied in the present work. In this type of problem, a fixed cost called the setup cost is incurred for every origin. In the bi-criterion transportation problem the cost of transportation is directly proportional to the number of units transported, but on account of quantity discounts, price breaks etc., the transportation may not be linear. Thus in real world situations, when a commodity is transported, a fixed cost is incurred in the objective function. The Multi-index fixed charge bi-criterion transportation problem has also been studied in which there are three indices. The problem is an extension of the Multi-index bi-criterion transportation problem, in which a fixed cost is incurred in the objective function.

The present thesis consists of three chapters. The first chapter is introductory in nature. In the second chapter the Fixed charge bi-criterion transportation problem given by Basu *et al.* (1994) is reviewed to find the cost-time trade-off pairs. Multi-index fixed charge bi-criterion transportation problem has been considered in chapter third which is an extension of the fixed charge bi-criterion transportation problem and may be thought of as a block in which the layers in all the directions form restricted transportation problem and an attempt has been made to find the cost-time trade-off pairs using an alternate approach of Ahuja and Arora (2001).

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CHAPTER-1

INTRODUCTION

Linear programming problems are similar in that they all require that a linear function of a set of variables is optimized while the variables themselves satisfy a number of linear constraints. All problems of this type can be solved by the use of Dantzig's simplex method in which the optimal vertex is located. A number of alternative methods have been constructed for special type of problems. The most useful have probably been those of the transportation type. Allocation method which is applied to a lot of very practical problems called Transportation problems in which the objective of the problem is to transport a commodity (single product) from more than one centre, called origins (or sources, or supply or capacity centres) to more than one places called destinations (sinks or demand or requirement centres) and the costs of transportation from each of the origins to each of the destinations being different and known. It is further assumed that

(i) The availability as well as requirements of the various centres are finite and contains the limited resources.

(ii) The cost of transportation is linear.

The problem is to transport the goods from various origins to different destinations in such a manner that the cost of transportation is minimum. The distinct feature of transportation problems is that sources and jobs must be expressed in terms of only one kind of unit.

1.1. CLASSICAL TRANSPORTATION PROBLEM

A certain class of linear programming problem known as transportation type problems, arises very frequently in practical applications. The classical transportation problem received its name because it arises naturally in the context of determining optimum shipping pattern. For example a product may be transported from factories to retail stores. The factories are the sources and the stores are the destinations. The amount of product that is available is known and the demands are also known. The problem is that different legs of the network joining the sources to the destinations have different costs associated with them. The aim is to find the minimum cost routing of products from the supply point to the destination. The general transportation problem can be formulated as: A product is available at each of m origins and it is required that given quantities of the product be shipped to each of n destinations. The minimum cost of shipping a unit of the product from any origin to any destination is known. The shipping schedule which minimizes the total cost of shipment is to be determined. The problem can be formulated as:

Minimize $$z = \sum_{i,j} c_{ij} x_{ij}$$

subject to $$\sum_{j=1}^{n} x_{ij} \le a_i \ , \quad a_i > 0, i = 1,2,...,m$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j \ , \quad b_j > 0, j = 1,2,...,n$$

$$x_{ij} \ge 0 \quad for\ all\ i\ and\ j$$

a_i is the quantity of the product available at origin i

b_j is the quantity of the product required at destination j

c_{ij} is the cost of shipping one unit from origin i to destination j

The aim is to minimize the objective function satisfying the above mentioned constraints. In the classical transportation problem of linear programming, the traditional objective is one of minimizing the total cost.

1.2. TIME MINIMIZING TRANSPORTATION PROBLEM

In a time minimizing transportation problem, the time of transporting goods from m origins to n destinations is minimized, satisfying certain conditions in respect of availabilities at sources and requirements at the destinations. The time minimizing transportation problem are of importance when it is required to transport perishable goods or during war days, it is required to transport food and ornaments in the shortest possible time and in so many other similar situations.

Thus, a time-minimizing transportation problem can be formulated as:

Minimize $$\left[\max_{(i,j)} \ t_{ij} \ / \ x_{ij} > 0 \right]$$

subject to $$\sum_{j=1}^{n} x_{ij} = a_i \ , \ (i = 1,2,...,m),$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ (j = 1,2,...,n).$$

Here t_{ij} is the time of transporting goods from the i^{th} origin, where the availability is a_i to the j^{th} destination, where the requirement is b_j. For any given feasible solution, $X = [x_{ij}]$ satisfying the above constraints, the time of transportation is the maximum of t_{ij}*'s* among the cells in which there are positive allocations, i.e., corresponding to the solution X, the time of transportation is

$$Z = \left[\underset{(i,j)}{Max} \; t_{ij} \;/\; x_{ij} > 0 \right]$$

The aim is to minimize this time of transportation.

This time of transportation remains independent of the amount of commodity sent so long as $x_{ij} > 0$. It is assumed that (i) the carriers have sufficient capacity to carry goods from an origin to destination in a single trip. (ii) they start simultaneously from there respective origins.

Thus, the basic difference between the cost minimizing transportation problem and the time minimizing problem is that whereas the cost of transportation changes with variations in the quantity of the commodity, the time involved remains unchanged, irrespective of the quantities of the commodity involved in the occupied cells in the time minimizing transportation problem. From a practical point of view, the cost minimizing transportation problem and the time minimizing transportation problem cannot be viewed as two independent problems, of one is interested in obtaining a solution which cost and time simultaneously. If the unit costs of transportation and the associated duration of transportation are given for each supply demand pair of points, then the cost-time trade-off solutions are of interest.

1.3. MULTI-OBJECTIVE TRANSPORTATION PROBLEM

Multi-objective optimization can be defined as:

" A vector of decision variables which satisfies constraints and optimizes a vector function whose elements represent the objective functions .Hence, the term "optimize" means finding such a solution which would give the values of all the objective functions acceptable to the designer."

Multi-objective problem is in which there are more than one objective satisfying a set of constraints. Kasana and Kumar (2003) formulated the multi-objective transportation problem as follows:

Consider *m* origins and *n* destinations and also the quantities available at each origin and the quantities to be transported to each destination. The total quantities required at the destinations may differ from the total quantities available at the origins. For such situations, the problem is balanced by introducing fictitious origin or destination; whichever is needed in order to get precisely the same quantities at the origins and the destinations. Specifically, a balanced transportation problem is considered as it amounts to no loss of generality.

Let x_{ij} be the quantity to be transported from origin i to destination j and for each fixed k: $k = 0,1,2,\ldots, p\text{-}1$, α^k_{ij}, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$; be the units of the parameter required for transporting one unit of the quantity from origin i to the destination j satisfying p objectives. The routing from the origin i to the destination j satisfying p objectives is to be determined.

The starting objective is known as primary and the others are classified as secondary.

The primary objective is to minimize

$$Z_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^0{}_{ij} x_{ij} \qquad (1.1)$$

and for k= 1,2,. . .,(p-1), also to minimize

$$x_k = \max\{\alpha^k{}_{ij} : x_{ij} \geq 0, i = 1,2, \ldots, m, j = 1,2, \ldots, n\} \qquad (1.2)$$

in order of the priorities to be assigned under the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \ , \ (i = 1,2,\ldots,m), \qquad (1.3)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ (j = 1,2,\ldots,n). \qquad (1.4)$$

The problem formulated above has p objective functions given by equations (1.1) and (1.2).

Keeping above in view Fixed Charge Bi-criterion Transportation Problem and Multi-Index FIxed Charge Bi-criterion Transportation Problem have been discussed in the succeeding chapters, the fixed charge bi-criterion transportation problem is an extension of the bi-criterion transportation problem in which a fixed cost is incurred for every origin. Multi-index fixed charge bi-criterion transportation problem is an extension of the fixed charge bi-criterion transportation problem which may be thought of as a block in which the layers in all directions form restricted transportation problem.

1.4. LITERATURE SURVEY

The Vogel's Approximation method (VAM) given by Rienfeld and Vogel (1958) is very often used for determining a low cost initial solution to the transportation problems. The method requires evaluation of a penalty cost for each row and column, which would have to be incurred if, instead of transporting over the cheapest route, we are forced to transport by the second cheapest route. The method was modified by Shimshak *et al. (*1981) which suggested that the least priority to the allocation of units in the dummy row or column can be assured by ignoring row or column differences involving the dummy row and column, and applying VAM as usual. The method was further modified by Goyal (1984) where the cost of transportation to or from a dummy should be assumed equal to the largest unit transportation cost before applying the VAM.

Bhatia *et al.* (1975) developed a technique for minimizing time in a transportation problem. The procedure involved finite number of iterations and is based on moving from one basic feasible solution to another till the last solution is arrived at. The algorithm given by them consists of determination of an initial basic feasible solution which can be found by the methods applicable in the case of the common cost minimizing transportation problem and finding an adjacent better basic feasible solution, the procedure is repeated until no better adjacent basic feasible solution can be found. This repeated procedure deals with the determination of a cell not in the basis which, when introduced will either reduce the time of transportation or reduce the allocation in at least one of the cells Є Q, where Q is the set of cell with positive allocations and corresponding time equal to the time of transportation. Garfinkel and Rao (1971) solved the problem by introducing a sufficiently large cost M on certain routes. Ramakrishnan (1977) developed another method of achieving a minimum time of transportation which is very different from other existing methods.

The transportation problem with two-objectives known as the bi-criterion transportation problem has been studied by many research workers. In this type of problem there are two objectives- one primary and the other secondary. The primary objective is to minimize the total cost of transportation problem and the secondary objective is to minimize the duration of transportation. Prakash (1981) reduced this problem to a goal-programming type problem discussed by Hughes and Grawiog (1973). To do this the set $\{t_{ij}: i=1,2,...,m; j=1,2,...,n\}$ is partitioned into subsets L_k $(k=1,2,...,q)$ in the following way. Each of the subsets L_k consists of t_{ij} having the same numerical value. L_1 consists of the t_{ij} having the greatest value, L_2 consists of the t_{ij} having the next lower greatest value and so on. Finally, L_q consists of the t_{ij} having the lowest value. After this, priority factors $M_0, M_1,...,M_q$ are assigned to

$$z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}, \qquad \sum_{L_1} x_{ij}, \ldots, \sum_{L_q} x_{ij},$$

respectively. Here $\sum_{L_k} x_{ij}$ is the sum of amounts x_{ij} transported from the origins to the destinations with the t_{ij} belonging to L_k $(k=1,2,...,q)$. The priority factors M_k have the relationship:

$$M_k \geq M_{k+1} \ (k= 0,1,...,q-1),$$

which implies that M_k is much larger than M_{k+1}. Now an alternate problem can be formulated whose optimal basic feasible solution would yield the desired solution. Prakash and Aggarwal (1992) dealt with the problem of establishing a route between two specified nodes through a network with two objectives, by reducing it to an assignment problem which is amenable to solution by the Hungarian method used for solving the standard assignment problem. Aneja and Nair (1979) presented an algorithm for determining the extreme points of the non-dominated set in the objective space instead of the decision space. The method consists of solving the same transportation problem repeatedly but with different objectives and each iteration gives either a new efficient extreme point or changes the direction of search in the objective space and the algorithm terminates when no more new efficient extreme point is available. For the purpose of implementation, at each iteration the efficient extreme point in the decision space corresponding to that revealed in the objective space is noted.

The fixed charge transportation problem has been investigated by many research workers. In this type of problem a fixed charge is associated with each route that can be opened, in addition to the variable transportation cost proportional to the amount of goods shipped. The problem was originally formulated by Dantzig and Hirsch (1954). Murty (1968) solved the fixed charge problem by ranking the extreme points. Gray (1971) presented an exact solution of this mixed integer programming problem by decomposing it into a master integer program and a series of transportation subprograms. To reduce the number of vertices that need to be examined, bounds are established on the maximum and minimum values of the total fixed cost, and feasibility conditions for the transportation problem are used extensively. Computational results show the method to be particularly suitable when fixed costs are large compared to variable costs. After that several procedures for solving the fixed charge transportation problem were developed. Bhatia *et al.* (1976) provided the cost-time trade-off pair in a transportation problem. Barr *et al.* (1981) presented a branch and bound algorithm for solving large scale fixed charge transportation problem where all cells do not exist. The algorithm exploits the absence of full problem density in several ways, thus yielding a procedure which is especially applicable in solving real world problems which are normally quite sparse. Cooper and Drebes (1967) provided two heuristic methods for moderate sized problems and the results indicate that the heuristic methods produce optimal solutions in well over 90 percent of the several hundred problems investigated and very close to optimal (a few percent) in the remaining cases. Gupta and Sharma (1983) presented branch and bound algorithm for solving linear fixed charge problems, the algorithm is an improvement over the algorithm described by Steinberg (1970) for calculating the lower bound on the various subsets of the feasible set, and helps in solving various structures of linear fixed charge problems. Sandrock (1988) presented a low technology algorithm for the solution of small fixed charge problems. Prasad *et al.* (1993) considered that the unit costs are piecewise linear non-increasing functions of time and it is assumed that all the transportations may take place concurrently. As such, the trade-offs they seek are between the total cost and the bottleneck time for completing all the transportations to all the demand points. They showed that a parametric method involving a finite sequence of parametric transportation problems revealed all the cost-time trade off solutions of the generalized trade-off problem. Also, a direct method is outlined for the case involving a finite set of discrete alternatives of unit cost-time pairs for each pair of supply-demand points. Basu *et al.* (1994) developed an algorithm for the optimum time-cost trade-off in a fixed charge bi-criterion transportation problem giving same priority to cost and time. Puri and Swarup (1974) established a method of solving the fixed charge problem by breaking it into two problems, one a simple linear programming problem and other a zero-one problem with no constraints of usual type. In this method different extreme points of the zero-one programming problem are enumerated and their corresponding x_j's defined in accordance with (1, 0) are tested for basic feasible solution of

AX=b, X≥0. Adlakha and Kowalski (2004) provided a simple algorithm for the source induced fixed-charge transportation problem. The source induced fixed charge transportation problem is a variation of regular fixed charge transportation problem in which a fixed cost is incurred for every supply point that is used in the solution, along with a variable cost that is proportional to the amount shipped. The problem is significantly different from the widely studied fixed charge transportation problem, where a fixed cost is incurred upon activation of a route. The introduction of the fixed costs in addition to the variable costs results in the objective function being a step function. The results of empirical tests of the effectiveness of the proposed algorithm are also presented. Balinski (1961) gave an approximate method of solving the fixed charge problem. Gottileb and Paulmann (1998) presented two genetic algorithms for solving fixed charge transportation problem. Both algorithms incorporate knowledge about the properties of optimal solutions, the algorithms mainly differ in the technique used to deal with the inherent constraints of fixed charge transportation problem, they compared both genetic algorithms on randomly generated instances.

Haley (1962) considered the Multi-Index Transportation Problem in which there are three indices and presented an algorithm to solve the problem, the method of solution is an extension of Modi-method. Multi-level fixed charge problems are mathematical optimization problems in which the separable portion of the objective function is the sum of piecewise continuous functions of a single variable. Haley (1963) gave the theorems justifying the method and an extension of the necessary conditions laid down by Schell. He also described the application of the technique to two special transportation problem and showed that the 'Three axial sums' problem of Schell can be written as 'Three planar sums'. Haley (1965) laid down a set of necessary conditions for a feasible solution to exist. He also proved that these conditions are sufficient. Morovek and Vlach (1967) presented the necessary conditions for the existence of the solution of the Multi-index transportation problem. Arnold and Soland (1969) described a branch and bound algorithm that finds a global solution to Multi-level fixed charge problem. The algorithm has the feature that a good feasible solution is generated at the start. Moreover, at each step of the algorithm an additional feasible solution is generated for comparison with the best solution found previously. Smith (1973) gave further necessary conditions for the existence of a solution to the Multi-index transportation problem.

1.5. PRESENT WORK

In the present thesis a fixed charge bi-criterion transportation problem studied by Basu *et al.* (1994) is revisited and a better solution has been found. The algorithm used in fixed charge bi-criterion transportation problem has also been applied to multi-index fixed charge bi-criterion transportation problem.

CHAPTER-2

FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

Bi-criterion transportation problem is an extension of the single objective transportation problem. In this type of problem there are two objectives: one of minimizing the total cost and the second is to minimize the total time of transportation.

Suppose there are m origins and n destinations, the quantities of a uniform product available at the origins and required at the destinations are given. The total quantity available at the sources is precisely the same as the total quantity required at the destinations and it is possible to transport to any destination from any origin. Moreover, the transportation starts simultaneously and the time of transportation from any origin to any destination does not depend on the amount of the product transported. Let

t_{ij} = the units of time of transportation of the product from origin i to destination j,

c_{ij} = the units of cost of transportation of one unit of the product from origin i to destination j,

a_i = the units of the product available at origin i,

b_j = the units of the product required at destination j,

x_{ij} = the number of units of the product transported from origin i to destination j.

It is required to determine the routing from the origins to the destination fulfilling two objectives in which the primary objective is to minimize the total cost of transportation and the secondary objective is to minimize the duration of transportation, subject to the below mentioned constraints. The mathematical formulation of this problem is as follows:

Minimize

$$z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{2.1}$$

and

$$z_2 = \max\{\, t_{ij} : x_{ij} > 0 \ (i = 1,2,...,m;\ j = 1,2,...,n)\} \tag{2.2}$$

with priorities being accorded to the two objectives in the order specified above and subject to the following constraints:

$$\sum_{j=1}^{n} x_{ij} = a_i \ ,\ \ a_i > 0, (i = 1,2,...,m), \tag{2.3}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ ,\ \ b_j > 0, (j = 1,2,...,n). \tag{2.4}$$

The problem formulated above is non linear in which the objective (2.1) is to minimize the total cost and the second objective is to minimize the total time.

2.1. FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

The fixed charge bi-criterion transportation problem is an extension of the bi-criterion transportation problem which is more complex to solve. In this type of problem a fixed cost called a setup cost is incurred for every origin. In the classical transportation problem the cost of transportation is directly proportional to the number of units transported, but on account of quantity discounts, price breaks etc. the transportation may not be linear. Thus in real world situations, when a commodity is transported, a fixed cost is incurred in the objective function. The fixed cost may represent the cost of renting a vehicle, landing fees in an airport, setup costs for machines in a manufacturing environment, a new facility costs money to be constructed etc. It also costs money to operate.

2.2. FORMULATION OF FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

The fixed charge bi-criterion transportation problem where the units of time and cost are taken in one standard scale was formulated by Basu *et al.* (1994) as follows:

(P_B)

Minimize $$\{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} F_i, \max\ [t_{ij} / x_{ij} > 0]\}$$

subject to $$\sum_{j=1}^{n} x_{ij} \le a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for \quad i = 1,2,\ldots, m; \quad j = 1,2,\ldots, n.$$

where

i =1, 2,. . . ,m, are the origins,

j = 1, 2,. . .,n, are the destinations,

x_{ij} = the amount transported from the i^{th} origin to the j^{th} destination,

c_{ij} = the variable cost per unit amount transported from i^{th} origin to the j^{th} destination,

t_{ij} = the time of transportation of the product from i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported, so long as $x_{ij} > 0$,

a_i = maximum capacity at origin i,

b_j = the demand at destination j,

F_i = the fixed cost associated with origin i.

The objective in fixed charge bi-criterion transportation problem is to minimize the total cost which includes both the variable cost and fixed cost and the total time of transportation satisfying the above constraints.

2.3. SOLUTION PROCEDURE

To solve the fixed charge bi-criterion transportation problem a procedure known as re-optimization procedure has been used in which cost is minimized without considering time and then time is minimized with respect to the minimum cost obtained, then after modifying cost with respect to the minimum time obtained in the last result, the cost is minimized and after that time is minimized with respect to the minimum cost of the last result. This process is repeated until an infeasible solution is obtained.

For re-optimization the above problem is separated into two problems (P) and (S), where

(P) Minimize the cost function $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} F_i$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for\ i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n.$$

and

(S) Minimize the time function $\max\limits_{\substack{1\le i\le m\\ 1\le j\le n}} [t_{ij} / x_{ij} > 0]$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \ge 0, \quad for \quad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n.$$

For formulation of F_i ($i = 1, 2,. . ., m$) it is assumed that F_i ($i = 1, 2,. . ., m$) has p-number of steps so that

$$F_i = \sum_{l=1}^{p} \delta_{il} F_{il}, \qquad i = 1,2,..., m,$$

where

$$\delta_{il} = 1, if \sum_{j=1}^{n} x_{ij} \geq A_{il},\ i = 1,2,..., m;\ l = 1,2,..., p$$
$$= 0, otherwise\ .$$

Here $$0 = A_{i1} < A_{i2} < . . . < A_{ip}.$$

$A_{i1}, A_{i2}, . . ., A_{ip}$ ($i = 1,2, . . . ,m$) are constants and F_{il} ($l = 1,2,. . .,p ; i = 1,2,. . .,m$) are fixed costs. Since fixed cost at each origin is considered, unbalanced transportation problem is to be taken into account. So, first the problems (P) and (S) are balanced using destinations. Then we have,

(P_1) Minimize $$z = \sum_{i=1}^{m} \sum_{j=1}^{n+1} c_{ij} x_{ij} + \sum_{i=1}^{m} F_i$$

subject to
$$\sum_{j=1}^{n+1} x_{ij} = a_i$$
$$\sum_{i=1}^{m} x_{ij} = b_j$$
$$x_{ij} \geq 0 \quad for \quad i = 1,2,..., m;\ \ j = 1,2,..., n, n+1$$

(S_1) Minimize $$T = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} [t_{ij} \ / \ x_{ij} > 0]$$

subject to
$$\sum_{j=1}^{n+1} x_{ij} = a_i$$
$$\sum_{i=1}^{m} x_{ij} = b_j$$
$$x_{ij} \geq 0, \quad for \quad i = 1,2,. .., m;\ \ j = 1,2,..., n, n+1.$$

In problems (P_1) and (S_1), the cost (variable cost and also fixed cost), and time associated with the dummy cells are all zero. At first iteration, let Z_1 be the minimum total cost of the problem (P_1) and T_1 be the optimal time of the problem (S_1) with respect to Z_1, then any schedule which is completed earlier than T_1 would cost more than Z_1. (Z_1, T_1) is called the cost-time trade-off pair at the first iteration.

Using re-optimization procedure, let after q^{th} iteration, the solution is infeasible. Therefore, we get the following complete set of time-cost trade-off pairs:

$$(Z_1, T_1), (Z_2, T_2), \ldots, (Z_q, T_q)$$

where

$$Z_1 < Z_2 < \ldots < Z_q \quad \text{and} \quad T_1 > T_2 > \ldots > T_q$$

Then, the minimum cost (Z_1) and minimum time (T_q) are identified among the above trade-off pairs. The pair (Z_1,T_q) with minimum cost and minimum time is termed the ideal solution. Except in trivial case the trade-off pair (Z_1, T_q) cannot be achieved in practice.

Let $$d_r = Z_r - Z_1$$

$$d_{q+r} = T_r - T_q$$

As same priority is given to cost and time, so d_r and d_{q+r} are minimized simultaneously among the trade-off pairs to obtain the optimal trade-off pair. So, to obtain the optimal solution, the following goal programming problem is to be solved:

(P') Minimize $\{d_r + d_{q+r}\}$

subject to $Z_r - Z_1 - d_r = 0$

$T_r - T_q - d_{q+r} = 0$

$d_r \geq 0,\ d_{q+r} \geq 0,\ r=1,2,\ldots,q.$

where d_r and d_{q+r} are the deviational variables. Let ($d_s + d_{q+s}$) be the solution of problem (P').

Now ($d_r + d_{q+r}$) is the distance $(D_1)_r$ from the trade-off pair (Z_r,T_r) to the ideal solution (Z_1,T_q).

So, we have,

$$(D_1)_{opt} = \text{Min } (D_1)_r,\ r=1,2,\ldots,q$$

$$= \text{Min } (d_r + d_{q+r}),$$

$$= (d_s + d_{q+s})$$

Hence, (Z_s,T_s) attains the optimal trade-off pair.

2.4. ALGORITHM

Basu *et al.* (1994) proposed the algorithm as:

Step 1: Take k=1, where k is the number of iterations in the algorithm.

Step 2: Find a basic feasible solution of the problem (P_1) with respect to the variable costs. Let B be the current basis.

Step 3: Calculate the fixed cost of the current basic feasible solution and denote this by F^1 (current) where

$$F^1 \text{ (current)} = \sum_{i=1}^{m} F_i$$

Step 4: Calculate (C_{ij}-u_i^1-v_j^1), for all $i, j \notin B$ and denote it by $(C_{ij})_1$, where u_i^1, v_j^1 are the dual variables for i=1,2,. . . ,m; j=1,2,. . .,n, n+1.

Step 5: Find $A_{ij}^1 = (C_{ij})_1 \times (E_{ij})_1$ for all $i, j \notin B$

where A_{ij}^1 is the change in cost which occurs for introducing a non-basic cell (i, j) with value $(E_{ij})_1$ into the basis by making reallocation.

Step 6: Compute F_{ij}^1 (Difference) = F_{ij}^1(NB) - F^1(Current)

where F_{ij}^1(NB) is the total fixed cost involved for introducing the variable x_{ij} with values $(E_{ij})_1$ for all $i, j \notin B$ into the current basis to form a new basis.

Step 7: Now add F_{ij}^1 (Difference) and A_{ij}^1 and denote it by Δ_{ij}^1 *i.e.* $\Delta_{ij}^1 = F_{ij}^1$ (Difference) + A_{ij}^1 for all $i, j \notin B$

Step 8: If all $\Delta_{ij}^1 \geq 0$, then go to Step 9; otherwise find min $\{\Delta_{ij}^1, \Delta_{ij}^1 < 0,\ i, j \notin B\}$, Then the variable x_{ij} associated with min (Δ_{ij}^1) will enter into the basis, where $i, j \notin B$. Go to Step 3.

Step 9: Let Z_1 be the optimal cost of P_1 and X_1 be the optimal solution corresponding to Z_1.

Step 10: Calculate T_1

$$T_1 = \max_{\substack{1 \le i \le m \\ 1 \le j \le n+1}} \left[t_{ij},\ x_{ij} > 0 \ \ according \ \ to \ \ X_1 \right]$$

Then the corresponding pair (Z_1, T_1) is called the cost-time trade-off pair at the first iteration.

Step 11: Define C_{ij}^k = M , if $t_{ij} \geq T_k$

$= C_{ij}$, if $t_{ij} < T_k$

where M is a sufficiently large positive number. Let (P_{K+1}) be the fixed charge bi-criterion transportation problem with variable cost C_{ij}^k.

Step 12: Find a basic feasible solution of the problem (P_{K+1}) with respect to the variable costs.

If the total variable cost $\geq$ M, then go to Step 21; otherwise go to Step 13.

Step 13: Find the fixed cost of the current basic feasible solution of problem (P_{K+1}) and denote this by F^{k+1}(current).

Step 14: Find $A_{ij}^{k+1} = (C_{ij})_{k+1} \times (E_{ij})_{k+1}$ for all $i, j \notin B$

where $(C_{ij})_{k+1} = C_{ij}^{k} - u_i^{k+1} - v_j^{k+1}$

Here u_i^{k+1}, v_j^{k+1} are dual variables for $i=1,2,...,m$; $j=1,2,...,n,n+1$. A_{ij}^{k+1} is the change in cost which occurs for introducing a non-basic cell (i,j) with value $(E_{ij})_{k+1}$ into the basis by making reallocation.

Step 15: Subtract F_{ij}^{k+1}(NB) *and* F^{k+1}(Current) and represent it by F_{ij}^{k+1} (Difference) i.e. F_{ij}^{k+1} (Difference) = F_{ij}^{k+1}(NB) – F^{k+1}(Current)

where F_{ij}^{k+1}(NB) is the total fixed cost involved for introducing the variable x_{ij} with values $(E_{ij})_{r+1}$ $(i, j \notin B)$ into the current basis to form a new basis.

Step 16: Find $\Delta_{ij}^{k+1} = F_{ij}^{k+1}$ (Difference) + A_{ij}^{k+1}, for all $i, j \notin B$.

Step 17: If all $\Delta_{ij}^{k+1} \geq 0$, then go to Step 18, otherwise find min $\{\Delta_{ij}^{k+1}, \Delta_{ij}^{k+1} < 0,\ i, j \notin B\}$

The variable x_{ij} associated with min (Δ_{ij}^{k+1}) will enter into the basis and then go to Step 13.

Step 18: Let Z_{k+1} be the optimal cost of problem (P_{k+1}) and X_{k+1} be the optimal solution corresponding to Z_{k+1}.

Step 19: Compute $T_{k+1} = \max\limits_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n+1}} [t_{ij}, x_{ij} > 0 \ \ according \ \ to \ \ X_{k+1}]$

Then the trade-off pair (Z_{k+1}, T_{k+1}) is called the cost-time trade-off pair at the $(k+1)^{\text{th}}$ iteration. Obviously, $Z_{k+1} > Z_k$, $T_{k+1} < T_k$.

Step 20: Set $k=k+1$, go to Step 11.

Step 21: Suppose after q^{th} iteration the solution is infeasible i.e. $Z_{q+1} \geq M$. Then identify the complete set of efficient trade-off pairs

$$(Z_1,T_1), (Z_2,T_2), \ldots, (Z_q,T_q)$$

$$\text{where } Z_1 < Z_2 < \ldots < Z_q \text{ and } T_1 > T_2 > \ldots > T_q$$

Step 22: Define the distance $(D_1)_r$ between the pair (Z_r,T_r) and the ideal solution (Z_1,T_q) as

$$(D_1)_r = (d_r + d_{q+r}),\ \ r=1,2,\ldots,q$$

Step 23: Calculate $(D_1)_{\text{opt}}= \text{Min}(D_1)_r$, $r=1,2,...,q$

$= (d_s + d_{q+s})$, say.

Then the trade-off pair (Z_s, T_s), offers the optimum solution.

2.5. NUMERICAL EXAMPLE

The above algorithm is explained by considering the following 3×3 fixed-charge bi-criterion transportation problem as considered by Basu *et al.* (1994) where the units of cost and time are taken in one standard scale.

$$\text{Minimize} \quad \left\{ \sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}x_{ij} + \sum_{i=1}^{3} F_i, \underset{\substack{1\le i\le 3\\ 1\le j\le 3}}{Max} \left[t_{ij} / x_{ij} > 0 \right] \right\}$$

$$\text{subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \; i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \; j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \quad i = 1,2,3; \; j = 1,2,3.$$

In order to solve the above problem, it is presented as two parts as shown below:

$$\text{Minimize the cost function} \quad \sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}x_{ij} + \sum_{i=1}^{3} F_i$$

$$\text{subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \; i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \; j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \quad i = 1,2,3; \; j = 1,2,3.$$

$$\text{and minimize the time function} \quad \max_{\substack{1\le i\le 3\\ 1\le j\le 3}} \left[t_{ij} / x_{ij} > 0 \right]$$

$$\text{subject to} \quad \sum_{j=1}^{3} x_{ij} \le a_i, \; i = 1,2,3,$$

$$\sum_{i=1}^{3} x_{ij} = b_j, \; j = 1,2,3,$$

$$x_{ij} \ge 0, \quad for \quad i = 1,2,3; \; j = 1,2,3.$$

Table 1 gives the values of variable cost C_{ij} (i=1, 2, 3; j=1, 2, 3) and Table 2 gives the values of time t_{ij} (i=1, 2, 3; j=1, 2, 3). The fixed costs are:

F_{11}=100, F_{12}=50, F_{13} =50

F_{21}=150, F_{22}=50, F_{23}=50

F_{31}=200, F_{32}=100, F_{33}=50

Table 1

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	a_i
1	5	9	9	19
2	4	6	2	10
3	2	1	1	11
b_j	5	8	15	

Table 2

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	a_i
1	15	8	2	19
2	10	13	11	10
3	6	9	17	11
b_j	5	8	15	

The total cost which is to be minimized is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}x_{ij} + \sum_{i=1}^{3} F_i$$

where $$F_i = \sum_{l=1}^{3} \delta_{il} F_{il} \ , \quad i = 1,2,3$$

where $$\delta_{i1} = 1, if \quad \sum_{j=1}^{3} x_{ij} \geq 0, \ i = 1,2,3,$$
$$= 0, otherwise \quad ;$$
$$\delta_{i2} = 1, if \quad \sum_{j=1}^{3} x_{ij} \geq 7, \ i = 1,2,3,$$
$$= 0, otherwise \ .$$

$$\delta_{i3} = 1, if \sum_{j=1}^{3} x_{ij} \geq 10, \; i = 1,2,3,$$
$$= 0, otherwise$$

Here F_i (i=1,2, 3) has three steps. On introducing a dummy destination $j=4$ with zero cost in Table 1 and Table 2, we get Table 3 and Table 4 respectively.

Table 3

Destination j→ Origin i↓	1	2	3	4	a_i
1	5	9	9	0	19
2	4	6	2	0	10
3	2	1	1	0	11
b_j	5	8	15	12	

Table 4

Destination j→ Origin i↓	1	2	3	4	a_i
1	15	8	2	0	19
2	10	13	11	0	10
3	6	9	17	0	11
b_j	5	8	15	12	

A basic feasible solution of problem (P_1) is given in Table 5.

The right hand side value of Table 5 gives the total fixed cost of the current solution.

Table 5

Destination j→ Origin i↓	1	2	3	4	F^1(current)
1	5 (5)	9 (2)	9	0 (12)	100
2	4	6 (6)	2 (4)	0	200
3	2	1	1 (11)	0	350
					650

Applying Step 4, the values ($C_{ij} - u_i - v_j$) are calculated, for all $i, j \notin B$ which are given in Table 6.

Table 6

ij	13	21	24	31	32	34
$C_{ij} - u_i - v_j$	4	2	3	1	-4	4

Applying Step 5, the values of A_{ij}^{1} are calculated, which are displayed in Table 7.

Table 7

Ij	13	21	24	31	32	34
A_{ij}^{1}	8	10	18	5	-24	24

Applying Step 6, the following results are obtained which are tabulated in Table 8.

Table 8

$ij \rightarrow$ $i \downarrow$	13	21	24	31	32	34
1	100	100	200	100	100	200
2	200	200	150	200	200	200
3	350	350	350	350	350	200
$F_{ij}(NB)$	650	650	700	650	650	600
$F_{ij}(Difference)$	0	0	50	0	0	-50

Applying Step 7, the values of Δ_{ij}^{1} are obtained, which are displayed in Table 9.

Table 9

ij	13	21	24	31	32	34
Δ_{ij}^{1}	8	10	68	5	-24	-26

In Table 9, it is observed that,

$$\text{Min}\ \{\Delta_{ij}^{1},\ \Delta_{ij}^{1} < 0,\ i, j \notin B\} = -26 \quad \text{at (3,4) cell .}$$

Therefore, the variable to enter the basis is x_{34} and the new solution is given in Table 10.

Table 10

Destination $j \rightarrow$ Origin $i \downarrow$	1	2	3	4	F^1(current)
1	5 (5)	9 (8)	9	0 (6)	200
2	4	6	2 (10)	0	200
3	2	1	1 (5)	0 (6)	200

600

Applying Step 4, the following results are obtained as shown in Table 11.

Table 11

ij	13	21	22	24	31	32
$C_{ij} - u_i - v_j$	8	-2	-4	-1	-3	-8

Applying Step 5, the values of A_{ij}^{1} are obtained, which are given in Table 12.

Table 12

ij	13	21	22	24	31	32
A_{ij}^{1}	40	-10	-24	-6	-15	-48

Applying Step 6, the following results are obtained which are tabulated in Table 13.

Table 13

$ij \rightarrow$ $i \downarrow$	13	21	22	24	31	32
1	200	150	100	200	150	100
2	200	200	200	150	200	200
3	0	300	350	350	300	350
$F_{ij}(NB)$	400	650	650	700	650	650
$F_{ij}(Difference)$	-200	50	50	100	50	50

Applying Step 7, the values of Δ_{ij}^{1} are calculated which are displayed in Table 14.

Table 14

ij	13	21	22	24	31	32
Δ_{ij}^{1}	-160	40	26	94	35	2

Here Δ_{ij}^{1} is negative at (1, 3) cell. Therefore, entering x_{13} into the basis, the following solution is obtained as shown in Table 15.

Table 15

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F^{1}(current)
1	5 (5)	9 (8)	9 (5)	0 (1)	200
2	4	6	2 (10)	0	200
3	2	1	1	0 (11)	0
					400

Applying Step 4, the following results are obtained as shown in Table 16.

Table 16

ij	21	22	24	31	32	33
$C_{ij} - u_i - v_j$	2	4	7	-3	-8	-8

Applying Step 5, the values of A_{ij}^{1} are calculated which are tabulated in Table 17.

Table 17

ij	21	22	24	31	32	33
A_{ij}^{1}	10	32	7	-15	-64	-40

Applying Step 6, the following results are obtained which are displayed in Table 18.

Table 18

$ij \rightarrow$ $i \downarrow$	21	22	24	31	32	33
1	200	200	200	200	150	200
2	200	200	200	200	200	200
3	0	0	0	200	300	200
$F_{ij}(NB)$	400	400	400	600	650	600
$F_{ij}(Difference)$	0	0	0	200	250	200

Applying Step 7, the following values of Δ_{ij}^1 are obtained which are tabulated in Table 19.

Table 19

ij	21	22	24	31	32	33
Δ_{ij}^1	10	32	7	185	186	160

Here all $\Delta_{ij}^1 \geq 0$ (for all $i, j \notin B$). Now applying Step 9 and Step 10, we get

$$Z_1 = 162 + 400 = 562$$

$$T_1 = 15$$

Hence, the first cost-time trade-off pair is (Z_1,T_1) = (562,15). Now, applying Step 11, the problem P_1 is modified to problem P_2 and the basic feasible solution of P_2 is given below in Table 20.

Table 20

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F^1(current)
1	M	9	9 (7)	0 (12)	100
2	4 (2)	6	2 (8)	0	200
3	2 (3)	1 (8)	M	0	350
					650

Applying Step 4, the values $(C_{ij} - u_i - v_j)$ are calculated, for all $i, j \notin B$ which are given in Table 21.

Table 21

ij	11	12	22	24	33	34
$C_{ij} - u_i - v_j$	M-11	-1	3	7	M	9

Applying Step 5, the values of A_{ij}^{1} are calculated, which are displayed in Table 22.

Table 22

ij	11	12	22	24	33	34
A_{ij}^{1}	2(M-11)	-2	6	56	3M	27

Applying Step 6, the following results are obtained which are tabulated in Table 23.

Table 23

$ij \rightarrow$ $i \downarrow$	11	12	22	24	33	34
1	100	100	100	200	100	150
2	200	200	200	150	200	200
3	350	350	350	350	350	300
$F_{ij}(NB)$	650	650	650	700	650	650
$F_{ij}(Difference)$	0	0	0	50	0	0

Applying Step 7, the values of Δ_{ij}^{1} are obtained, which are displayed in Table 24.

Table 24

ij	11	12	22	24	33	34
Δ_{ij}^{1}	2(M-11)	-2	6	106	3M	27

In Table 9, it is observed that

$$\text{Min}\ \{\Delta_{ij}^{1},\ \Delta_{ij}^{1} < 0,\ i, j \notin B\} = -2 \quad \text{at}\ \ (1,2)\ \text{cell}\ .$$

Therefore, the variable to enter the basis is x_{12} and the new solution is given in Table 25.

Table 25

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F^1(current)
1	M	9 (2)	9 (5)	0 (12)	100
2	4	6	2 (10)	0	200
3	2 (5)	1 (6)	M	0	350
					650

Applying Step 4, the following results are obtained as shown in Table 26.

Table 26

ij	11	21	22	24	33	34
$C_{ij} - u_i - v_j$	M-10	1	4	7	M-1	8

Applying Step 5, the values of A_{ij}^{1} are calculated which are given in Table 27.

Table 27

ij	11	21	22	24	33	34
A_{ij}^{1}	2(M-10)	2	8	70	5(M-1)	48

Applying Step 6, the following results are obtained which are tabulated in Table 28.

Table 28

$ij\rightarrow$ $i\downarrow$	11	21	22	24	33	34
1	100	100	100	200	100	200
2	200	200	200	0	200	200
3	350	350	350	350	350	200
$F_{ij}(NB)$	650	650	650	550	650	600
$F_{ij}(Difference)$	0	0	0	-100	0	-50

Applying Step 7, the values of Δ_{ij}^{1} are calculated which are displayed in Table 29.

Table 29

ij	11	21	22	24	33	34
Δ_{ij}^1	2(M-10)	2	8	-30	5(M-1)	-2

Here Δ_{ij}^1 is negative at (2,4) cell, Therefore, entering x_{24} into the basis, the following solution is obtained as shown in Table 30.

Table 30

Destination j→ Origin i↓	1	2	3	4	F^1(current)
1	M	9 (2)	9 (15)	0 (2)	200
2	4	6	2	0 (10)	0
3	2 (5)	1 (6)	M	0	350
					550

Applying Step 4, the following results are obtained as shown in Table 31.

Table 31

ij	11	21	22	23	33	34
$C_{ij} - u_i - v_j$	M-10	-6	-3	-7	M-1	8

Applying Step 5, the values of A_{ij}^1 are calculated which are tabulated in Table 32.

Table 32

ij	11	21	22	23	33	34
A_{ij}^1	2(M-10)	-12	-6	-70	6(M-1)	16

Applying Step 6, the following results are obtained which are displayed in Table 33.

Table 33

$ij \rightarrow$ $i \downarrow$	11	21	22	23	33	34
1	200	200	200	100	200	200
2	0	150	150	200	0	0
3	350	350	350	350	350	300
$F_{ij}(NB)$	550	700	700	650	550	500
$F_{ij}(Difference)$	0	150	150	100	0	-50

Applying Step 7, we get the following values of Δ_{ij}^{1} which are tabulated in Table 34.

Table 34

ij	11	21	22	23	33	34
Δ_{ij}^{1}	2(M-10)	138	144	30	6(M-1)	-34

Here Δ_{ij}^{1} is negative at (3,4) cell. Therefore, entering x_{34} into the basis the following solution is obtained as shown in Table 35.

Table 35

Destination $j \rightarrow$ Origin $i \downarrow$	1	2	3	4	F^{1}(current)
1	M	9 (4)	9 (15)	0	200
2	4	6	2	0 (10)	0
3	2 (5)	1 (4)	M	0 (2)	300
					500

Applying Step 4, the following results are obtained as shown in Table 36.

Table 36

ij	11	14	21	22	23	33
$C_{ij} - u_i - v_j$	M-10	-8	2	5	1	M-1

Applying Step 5, the values of A_{ij}^{1} are calculated which are tabulated in Table 37.

Table 37

ij	11	14	21	22	23	33
A_{ij}^{1}	4(M-10)	-16	10	20	4	4(M-1)

Applying Step 6, the following results are obtained which are displayed in Table 38.

Table 38

$ij \rightarrow$ $i \downarrow$	11	14	21	22	23	33
1	200	200	200	200	200	200
2	0	0	150	150	150	0
3	300	350	200	200	200	300
$F_{ij}(NB)$	500	550	550	550	550	500
$F_{ij}(Difference)$	0	50	50	50	50	0

Applying Step 7, the following values of Δ_{ij}^{1} are obtained which are tabulated in Table 39.

Table 39

ij	11	14	21	22	23	33
Δ_{ij}^{1}	4(M-10)	34	60	70	54	4(M-1)

Here all $\Delta_{ij}^{1} \geq 0$ (for all $i, j \notin B$). Now applying Step 9 and Step 10, we get

$$Z_2 = 185 + 500 = 685$$

$$T_2 = 9$$

Hence, the second cost-time trade-off pair is

$(Z_2, T_2) = (685, 9)$.

After second iteration, it is observed that the solution is infeasible. Hence, two cost-time trade-off pairs are obtained as follows:

(562,15), (685,9).

The result shows that the minimum cost is 562 which corresponds to the pair (562,15) and the minimum time is 9 which corresponds to the pair (685,9). So, the ideal solution corresponds to the pair (562, 9).

Table 40

Trade-off pairs	Ideal solution	Distance$(D_1)_r$ between Ideal solution and the trade-off pair	$(D_1)_{opt}$
(562,15)		6	
(685,9)	(562,9)	123	6

The distance of the trade-off pairs from the ideal solution is presented in Table 40.

2.6. CONCLUSION

The Fixed charge bi-criterion transportation problem considered by Basu *et al.* (1994) is reviewed and an improved cost-time trade-off pair has been obtained.

CHAPTER-3

MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

3.1. MULTI-INDEX BI-CRITERION TRANSPORTATION PROBLEM

An ordinary transportation problem can be written in the form of a two-dimensional table for i= 1,2,. . .,n; j=1,2,. . .,m, as shown in Fig.1. Each cell of this table represents one of the y_{ij}*'s*. When these y_{ij}*'s* are summed along the rows of the table they must equal b_i and when they are summed down the columns they must equal a_j.

$j\rightarrow$ $i\downarrow$	1	2	…	m	
1	y_{11}	y_{12}	…	y_{1m}	b_1
2	y_{21}	y_{22}	…	y_{2m}	b_2
:	…	…	…	:	
n	y_{n1}	y_{n2}	…	y_{nm}	b_n
	a_1	a_2	…	a_m	

Figure 1

An extension of the transportation type of problem was stated by Haley and may be thought of as a block in which the layers in all directions form restricted transportation problem. The solid problem can be set out as a three dimensional block for i=1, 2 ,. . . ,n; j=1,2,. . .,m; k=1, 2, . . . , p. Each cell of this block represents one of the x_{ijk}*'s* .When these are summed along the rows (for constant j and k) they equal A_{jk}. When they are summed along the columns (for constant k and i) they equal B_{ki}. When they are summed down the heights (for constant i and j) they equal E_{ij}. The arrangement of x_{ijk}*'s* and the boundary conditions are shown in Fig.2 and Fig.3.

The multi-index problem can be described as minimizing the cost and time of moving a set of p different commodities (k=1,2,…,p) from n origins ($i=1,2,…,n$) to m destinations (j=1,2,…,m).The equations then give rise to the conditions on the amount of the various type of combination that is available and required. Alternatively, the same set of restrictions arise when a single commodity has to be moved by different methods e.g. road, rail, sea, canal, air etc. Similarly the use of intermediate depots may require the use of a multi-index formulation.

A special type of problem where the method can be used is the capacitated transportation problem (each variable has an upper bound). Here both the cost and time have equal priorities. If the unit costs of transportation and the associated duration of transportation are given for each supply demand pair of points, then the cost-time trade-off solutions are of interest.

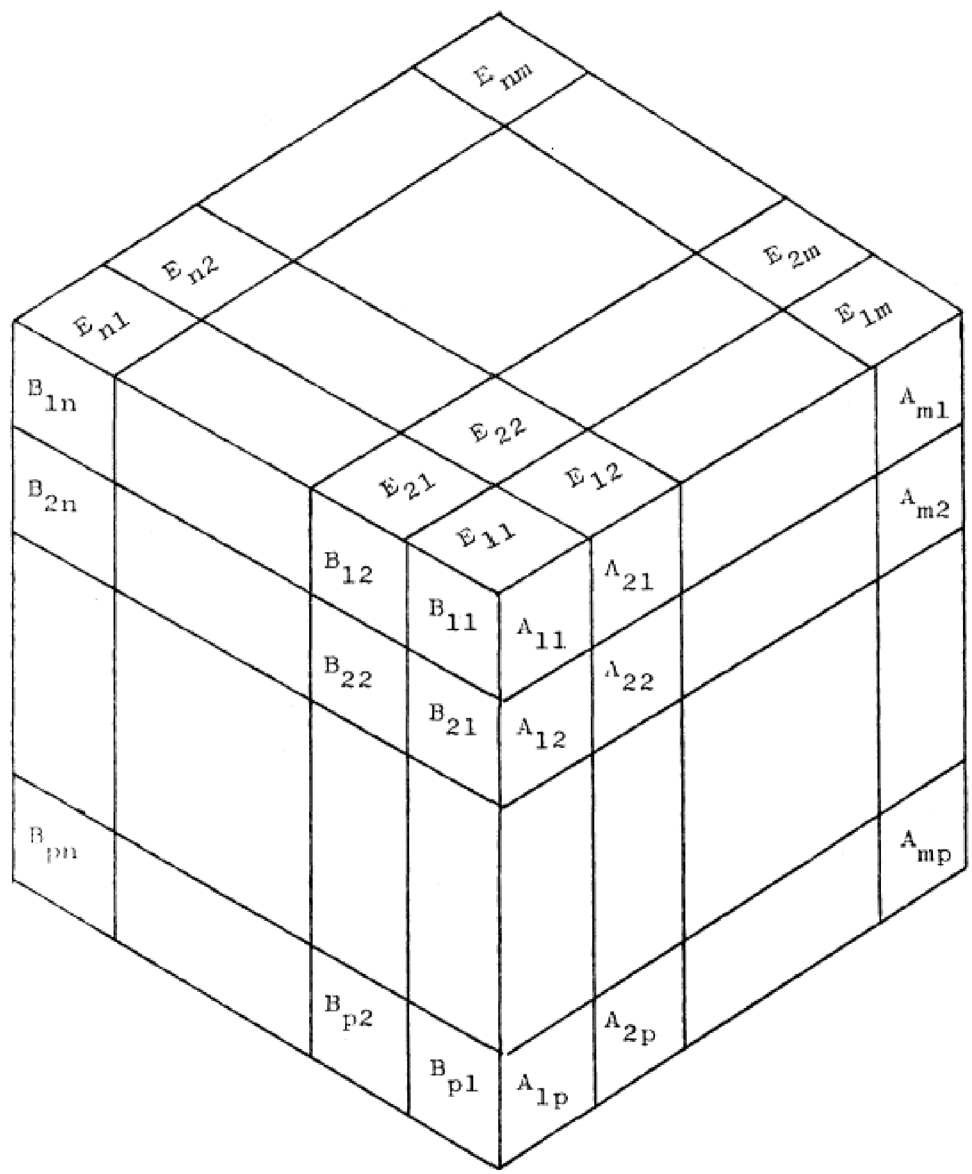

Figure 2

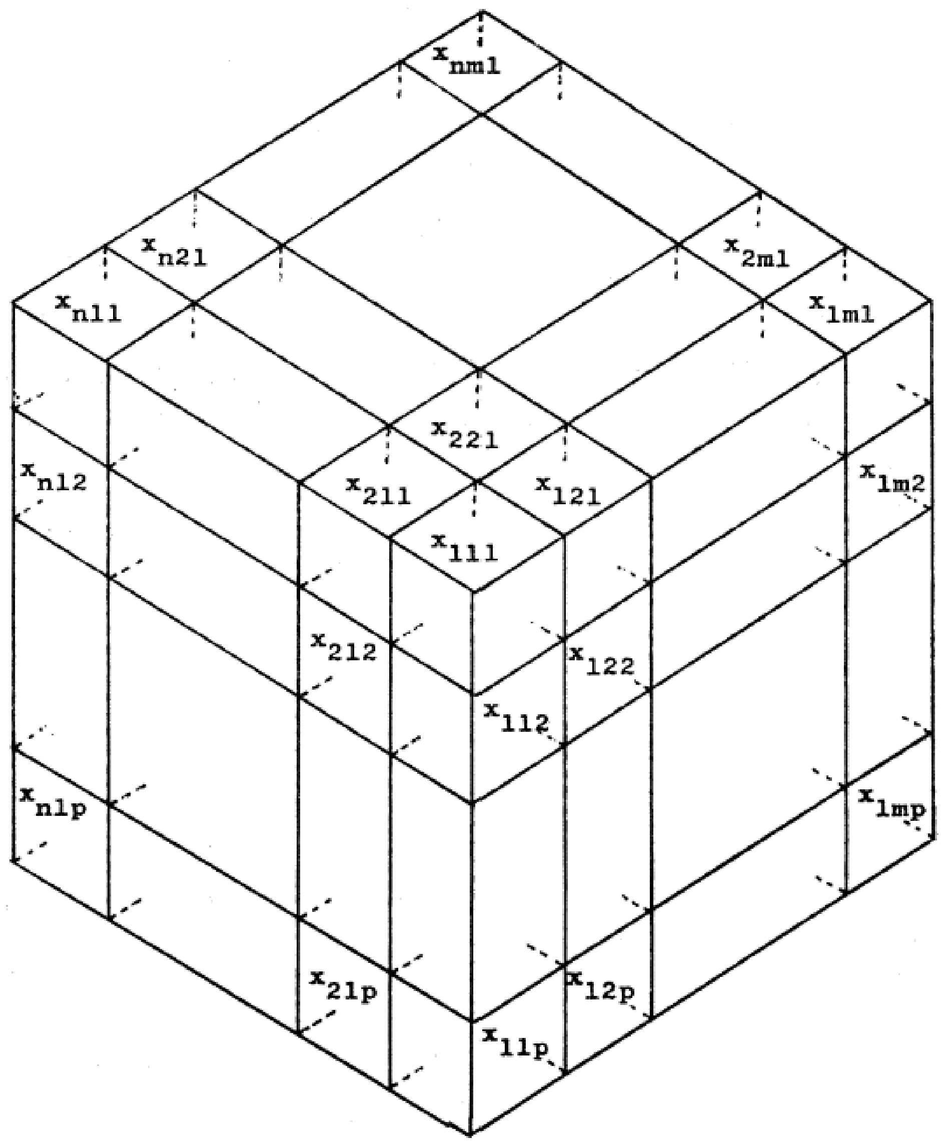

Figure 3

3.2. FORMULATION OF MULTI-INDEX BI-CRITERION TRANSPORTATION PROBLEM

The multi-index transportation problem in which there are *m* origins, *n* destinations and *p* type of commodities to be transported can be formulated as follows. In this problem there are two objectives one of minimizing the total cost and the other is to minimize the total time of transportation.

Minimize $z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk} x_{ijk}$

and $z_2 = \max\{t_{ijk} : x_{ijk} > 0 \ (i = 1,2,...,m;\ j = 1,2,...,n; k = 1,2,..., p)\}$

subject to

$$\sum_{i=1}^{m} x_{ijk} = A_{jk}$$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

$$x_{ijk} \geq 0, i = 1,2,...,m;\ j = 1,2,...,n; k = 1,2,..., p \ \ where$$

$$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki},\ \ \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij},\ \ \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p}\sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}$$

3.3. MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

Multi-index fixed charge bi-criterion transportation problem is an extension of the multi-index bi-criterion transportation problem in which a fixed cost is incurred in the objective function. The problem formulated by Ahuja and Arora (2001) is as follows:

(P) Minimize
$$\left\{ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} C_{ijk} x_{ijk} + \sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik\,i},\ \underset{\substack{1\leq i\leq m \\ 1\leq j\leq n \\ 1\leq k\leq p}}{Max} \left[t_{ijk} / x_{ijk} > 0 \right] \right\}$$

subject to

$$\sum_{i=1}^{m} x_{ijk} = A_{jk}$$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

$$x_{ijk} \geq 0, i = 1,2,...,m;\ j = 1,2,...,n; k = 1,2,..., p.$$

where $$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki}, \ \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij}, \ \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p}\sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}$$

Here $i = 1,2, \ldots, m$ are the origins,

$j = 1, 2, \ldots, n$ are the destinations,

$k = 1, 2, \ldots, p$ are the various types of commodities,

x_{ijk} is the amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination

c_{ijk} is the variable cost per unit amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination.

t_{ijk} is the time required to transport k^{th} type of commodity from the i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported , so long as $x_{ijk} > 0$.

F_{ik} is the fixed cost associated with origin i and commodity k.

where $$F_{ik} = \sum_{j=1}^{n} F_{ijk}\delta_{ijk}, \ i = 1,2,\ldots,m; \ k = 1,2,\ldots,p.$$

$$\delta_{ijk} = 1, \ if \quad x_{ijk} > 0$$
$$= 0, \ if \quad x_{ijk} = 0$$

A_{jk} is the total quantity of k^{th} type of commodity to be sent to the j^{th} destination,

B_{ki} is the total quantity of k^{th} type of commodity available at the i^{th} origin,

E_{ij} is the total quantity to be sent from i^{th} origin to the j^{th} destination.

In order to solve (P) it is separated into two problems (P') and (P'')

Problem (P') is:

Minimize $$z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}x_{ijk} + \sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik}$$

subject to the above constraints.

Problem (P'') is :

Minimize $$T = \max_{\substack{1\le i\le m\\ 1\le j\le n\\ 1\le k\le p}} [t_{ijk} \;/\; x_{ijk} > 0]$$

subject to the above constraints.

3.4. SOLUTION PROCEDURE

An initial basic feasible solution is calculated by using North-west corner rule, then the algorithm discussed in chapter- two is used to find the optimum cost-time trade -off pairs.

3.5. NUMERICAL EXAMPLE

Consider a 3×3×3 Multi-Index fixed charge bi-criterion transportation problem.

Minimize $$\left\{ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} C_{ijk}\, x_{ijk} + \sum_{i=1}^{3}\sum_{k=1}^{3} F_{ik\,i} ,\; \underset{\substack{1\le i\le 3\\ 1\le j\le 3\\ 1\le k\le 3}}{Max} \left[t_{ijk} \;/\; x_{ijk} > 0\right] \right\}$$

subject to $$\sum_{i=1}^{3} x_{ijk} = A_{jk}$$

$$\sum_{j=1}^{3} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{3} x_{ijk} = E_{ij}$$

$$x_{ijk} \ge 0, i = 1,2,3;\; j = 1,2,3;\; k = 1,2,3,$$

where $$\sum_{j=1}^{3} A_{jk} = \sum_{i=1}^{3} B_{ki},\;\; \sum_{k=1}^{3} B_{ki} = \sum_{j=1}^{3} E_{ij},\;\; \sum_{i=1}^{3} E_{ij} = \sum_{k=1}^{3} A_{jk}$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3} A_{jk} = \sum_{k=1}^{3}\sum_{i=1}^{3} B_{ki} = \sum_{i=1}^{3}\sum_{j=1}^{3} E_{ij}$$

Table 1 gives the values of the variable cost c_{ijk} (i =1,2,3; j=1,2,3; k =1,2,3) at the top left corner and the values of time t_{ijk} at the bottom right corner(i =1,2,3; j=1,2,3; k =1,2,3).The fixed costs are also given below.

Table 1

	j =1				j =2				j =3				B_{ki}		
i =1	8 3	7 5	6 4	E_{11}=10	5 8	6 6	10 8	E_{12} = 6	7 7	3 4	11 1	E_{13}=9	6	9	10
i =2	11 2	8 1	13 6	E_{21}=21	9 4	15 2	12 2	E_{22} = 9	13 6	7 1	8 1	E_{23} = 14	13	14	17
i =3	5 4	6 8	7 1	E_{31}=21	8 3	9 2	7 1	E_{32}=13	10 4	6 2	12 8	E_{33}=12	15	13	18
A_{jk}	15	17	20		8	11	9		11	8	16				

The fixed costs are

$F_{111} = 10$	$F_{121} = 30$	$F_{131} = 20$
$F_{112} = 20$	$F_{122} = 20$	$F_{132} = 20$
$F_{113} = 30$	$F_{123} = 20$	$F_{133} = 10$
$F_{211} = 10$	$F_{221} = 20$	$F_{231} = 20$
$F_{212} = 10$	$F_{222} = 10$	$F_{232} = 30$
$F_{213} = 40$	$F_{223} = 10$	$F_{233} = 10$
$F_{311} = 10$	$F_{321} = 40$	$F_{331} = 20$
$F_{312} = 20$	$F_{322} = 10$	$F_{332} = 30$
$F_{313} = 20$	$F_{323} = 10$	$F_{333} = 10$

The total cost which is to be minimized is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} c_{ijk} x_{ijk} + \sum_{i=1}^{3}\sum_{k=1}^{3} F_{ik}$$

where

$$F_{ik} = \sum_{l=1}^{3} \delta_{ijk} F_{ijk}, \quad i = 1,2,3\,; k = 1,2,3$$

where

$$\delta_{i1k} = 1, if \ \sum_{j=1}^{3} x_{ijk} \geq 0, \ i = 1,2,3; \ k = 1,2,3$$
$$= 0, otherwise;$$
$$\delta_{i2k} = 1, if \ \sum_{j=1}^{3} x_{ijk} \geq 7, \ i = 1,2,3; \ k = 1,2,3$$
$$= 0, otherwise;$$
$$\delta_{i3k} = 1, if \ \sum_{j=1}^{3} x_{ijk} \geq 10, \ i = 1,2,3; \ k = 1,2,3$$
$$= 0, otherwise.$$

Using the North-West Corner rule, an initial basic feasible solution is obtained as shown in Table 2, 3, 4.

Table 2

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3
1	8 (6)	5	7
2	11 (9)	9	13 (4)
3	5	8 (8)	10 (7)

k=1

Table 3

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3
1	7 (4)	6 (5)	3
2	8 (10)	15 (1)	7 (3)
3	6 (3)	9 (5)	6 (5)

k=2

Table 4

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3
1	6	10 (1)	11 (9)
2	13 (2)	12 (8)	8 (7)
3	7 (18)	7	12

k=3

The value of fixed cost, F (current) = 360

Applying Step 4, the values (C_{ijk} - u_{jk} - v_{ki} -w_{ij}) are calculated, for all $i, j, k \notin B$ which are given in Table 5.

Table 5

ijk	121	131	221	311	132	113	323	333
$C_{ijk} - u_{jk} - v_{ki} - w_{ij}$	0	0	-7	-2	0	-13	5	9

Applying Step 5, the values of A_{ijk}^{1} are obtained, which are displayed in Table 6.

Table 6

ijk	121	221	311	132	113	323	333
A_{ijk}^{1}	0	-7	-6	0	-13	25	45

Applying Step 6, the following results are obtained which are tabulated in Table 7.

Table 7

$ijk \rightarrow$ $ik \downarrow$	121	221	311	132	113	323	333
11	40	10	10	10	10	10	10
21	10	50	30	30	30	30	30
31	60	60	70	60	60	60	60
12	40	40	40	60	40	40	40
22	50	40	20	20	40	50	60
32	60	60	40	60	60	50	30
13	30	30	30	30	40	30	30
23	60	60	60	60	60	60	60
33	20	20	20	20	20	30	30
$F_{ij}(NB)$	370	370	320	350	360	360	350
$F_{ij}(Difference)$	10	10	-40	-10	0	0	-10

Applying Step 7, the values of Δ_{ijk}^{1} are calculated, which are displayed in Table 8.

Table 8

ijk	121	221	311	132	113	323	333
Δ_{ijk}^{1}	10	3	-46	-10	-13	25	35

In Table 8, it is observed that

$$\text{Min }\{\Delta_{ijk}^{1},\ \Delta_{ijk}^{1} < 0,\ i, j, k \notin B\} = -46 \text{ at } (3,1,1) \text{ cell.}$$

Therefore, the variable to enter the basis is x_{311} and the new solution is given in Table 9, 10, 11.

Table 9

Destination j→ Origin i↓	1	2	3
1	8 (6)	5	7
2	11 (6)	9	13 (7)
3	5 (3)	8 (8)	10 (4)

k=1

Table 10

Destination j→ Origin i↓	1	2	3
1	7 (4)	6 (5)	3
2	8 (13)	15 (1)	7 (0)
3	6	9 (5)	6 (8)

k=2

Table 11

Destination j→ Origin i↓	1	2	3
1	6	10 (1)	11 (9)
2	13 (2)	12 (8)	8 (7)
3	7 (18)	7	12

k=3

The value of fixed cost, F (current) = 320.

Applying Step 4,the following results are obtained as shown in Table 12.

Table 12

ijk	121	131	221	132	312	113	323	333
$C_{ijk} - u_{jk} - v_{ki} - w_{ij}$	0	0	-7	0	4	-13	1	5

Applying Step 5, the values of A_{ijk}^{1} are obtained, which are given in Table 13.

Table 13

ijk	221	113	333
A_{ijk}^{1}	-7	-13	20

Applying Step 6, the following results are obtained which are tabulated in Table 14.

Table 14

$ijk \rightarrow$ $ik \downarrow$	221	113	333
11	10	10	10
21	50	30	30
31	70	70	50
12	40	40	40
22	40	10	20
32	40	40	40
13	30	40	30
23	60	60	60
33	20	20	30
$F_{ij}(NB)$	360	320	310
$F_{ij}(Difference)$	40	0	-10

Applying Step 7, the values of Δ_{ijk}^{1} are calculated which are displayed in Table 15.

Table 15

ijk	221	113	333
Δ_{ijk}^{1}	33	-13	10

Here Δ_{ijk}^{1} is negative at (1, 1, 3) cell. Therefore, entering x_{113} into the basis, the following solution is obtained as shown in Table 16, 17, 18.

Table 16

Destination j→ Origin i↓	1	2	3
1	8 (6)	5	7
2	11 (6)	9	13 (7)
3	5 (3)	8 (8)	10 (4)

k=1

Table 17

Destination j→ Origin i↓	1	2	3
1	7 (3)	6 (6)	3
2	8 (14)	15	7 (0)
3	6	9 (5)	6 (8)

k=2

Table 18

Destination j→ Origin i↓	1	2	3
1	6 (1)	10 (0)	11 (9)
2	13 (1)	12 (9)	8 (7)
3	7 (18)	7	12

k=3

The value of fixed cost, F (current) = 320.

Applying Step 4, the following results are obtained as shown in Table 19.

Table 19

ijk	121	131	221	132	222	312	323	333
$C_{ijk} – u_{jk} - v_{ki} - w_{ij}$	0	-13	6	-13	13	2	-10	7

Applying Step 5, the values of A_{ijk}^{1} are obtained which are tabulated in Table 20.

Table 20

ijk	222	333
A_{ijk}^{1}	13	28

Applying Step 6, the following results are obtained which are displayed in Table 21.

Table 21

$ijk \rightarrow$ $ik \downarrow$	222	333
11	10	10
21	30	30
31	70	50
12	40	40
22	20	10
32	40	40
13	10	40
23	60	60
33	20	30
$F_{ij}(NB)$	320	310
$F_{ij}(Difference)$	0	-10

Applying Step 7, the following values of Δ_{ijk}^{1} are calculated which are tabulated in Table 22.

Table 22

ijk	222	333
Δ_{ijk}^{1}	13	18

Here all $\Delta_{ijk}^{1} \geq 0$ (for all $i, j, k \notin B$). Now applying Step 9 and Step 10, we get

$$Z_1 = 994 + 320 = 1314$$

$$T_1 = 6$$

Hence, the first cost-time trade-off pair is (Z_1, T_1) = (1314,6).

After second iteration, it is observed that the solution is infeasible. Hence, one cost-time trade-off pair is obtained as (1314, 6).

The result shows that the minimum cost is 1314 and the minimum time is 6 which corresponds to the pair (1314, 6). So, the ideal solution corresponds to the pair (1314, 6).

3.6. CONCLUSION

An alternative approach has been used to find the optimum cost-time trade off pairs for Multi-index fixed charge bi-criterion problem. A comparison with Ahuja and Arora (2001) shows that the total cost increases but the total time remains the same in cost-time trade-off pair.

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