# A STUDY ON FUZZY SHORTEST PATH PROBLEMS

***Thesis submitted in partial fulfillment of the requirement for***

***The award of the degree of***

***Masters of Science***

*In*

**Mathematics and Computing**

***Submitted by***

**Rajveer Kaur**

**Roll no. 301003018**

*Under the guidance of*

**Dr. Amit Kumar**

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**School of Mathematics and Computer Applications**

**Thapar University**

**Patiala-147004 (PUNJAB)**

**INDIA**

DEDICATED

TO

GOD, MY PARENTS AND MEHAR

# CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**A study on fuzzy shortest path problems**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Rajveer K

**Rajveer Kaur**

**Reg. no. 301003018**

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

10/07/12

**(Dr. Amit Kumar)**
*Supervisor*
Assistant Professor, SMCA
Thapar University, Patiala.

**Countersigned by:**

**Dr. S.S. Bhatia**
Professor & Head
Affairs
SMCA
Thapar University, Patiala.

**Dr. S.K. Mohapatra**
Dean of Academic

Thapar University
Patiala.

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Rajveer K.

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Rajveer Kaur

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# Chapter 1

# LITERATURE REVIEW

To find the shortest path from a source node to the other nodes is a fundamental matter in graph theory. In conventional shortest path problems, it is assumed that decision maker is certain about the parameters (distance, time etc.) between different nodes. But, in real life situations, there always exist uncertainty about the parameters between different nodes. In such cases, the parameters are represented by fuzzy numbers [27].

Klein [10] presented new models based on fuzzy shortest paths and given a general algorithm based on dynamic programming to solve the new models. Lin and Chern [13] considered the case that the arc lengths are fuzzy numbers and proposed an algorithm for finding the single most vital arc in a network. Okada and Gen [21] discussed the problem of finding the shortest paths from a fixed origin to a specified node in a network with arcs represented as intervals on real line. Li et al. [12] introduced the neural networks for solving fuzzy shortest path problems. Gent et al. [3] investigated the possibility of using genetic algorithms to solve shortest path problems. Shih and Lee [24] investigated multiple objective and multiple hierarchies minimum cost flow problems with fuzzy costs and fuzzy capacities in the arcs. Okada and Soper [22] concentrated on a shortest path problem on a network in which a fuzzy number, instead of a real number, is assigned to each arc length. Okada [20] concentrated on a shortest path problem on a network in which a fuzzy number, instead of a real number, is assigned to each arc length and introduced the concept of "degree of possibility" in which an arc is on the shortest path. Liu and Kao [15] investigated the network flow problems in that the arc lengths of the network are fuzzy numbers. Seda [23] dealed with the steiner tree problem on a graph in which a fuzzy number, instead of a real number, is assined to each edge.

Takahashi and Yamakami [25] discussed the shortest path problem with fuzzy parameters. He proposed a modified Okada's algorithm [20], using some properties observed by other authors. He also proposed a genetic algorithm to seek an approximated solution for large scale problems. Chuang and Kung [1] represented each arc length as a triangular fuzzy set and a new algorithm is proposed to deal with the fuzzy shortest path problems. Nayeem and Pal [19] considered a network with its arc lengths as imprecise number, instead of a real number, namely, interval number and triangular fuzzy number.

Ma and Chen [16] proposed an algorithm for the on-line fuzzy shortest path problems, based on the traditional shortest path problem in the domain of the operations research and the theory of the on-line algorithms. Kung and Chuang [11] proposed a new algorithm composed of fuzzy shortest path length procedure and similarity measure to deal with the fuzzy shortest path problem. Gupta and Pal [4] presented an algorithm for the shortest path problem when the connected arcs in a transportation network are represented as interval numbers.

Moazeni [18] discussed the shortest path problem from a specified node to every other node on a network in which a positive fuzzy quantity with finite support is assigned to each arc as its arc length. Chuang and Kung [2] pointed out that there are several methods reported to solve this kind of problem in the open literature. In these methods, they can obtain either the fuzzy shortest length or the shortest path and proposed an algorithm to obtain both. Ji et al. [6] considered the shortest path problem with fuzzy arc lengths. According to different decision criteria, the concepts of expected shortest path, a-shortest path and the shortest path in fuzzy environment are originally proposed, and three types of models are formulated. In order to solve these models, a hybrid intelligent algorithm integrating simulation and genetic algorithm is provided and some numerous examples are given to illustrate its effectiveness.

Hernandes et al. [5] proposed an iterative algorithm that assumes a generic ranking index for comparing the fuzzy numbers involved in the problem, in such a way that each time in which the decision-maker wants to solve a concrete problem (s)he can choose (or propose) the ranking index that best suits that problem. Yu & Wei [26] proposed a simple linear multiple objective programming to deal with the fuzzy shortest path problem. The proposed algorithm does not need to declare 0-1 variables to solve the fuzzy shortest path problem because it meets the requirements of the network linear programming constraints. Mahdavi et al. [17] proposed a dynamic programming approach to solve the fuzzy shortest chain problem using a suitable ranking method.

# Chapter 2

# SHORTEST PATH PROBLEMS WITH INTERVAL AND TRIANGULAR FUZZY ARC LENGTHS

Nayeem and Pal [19] proposed a new algorithm for solving shortest path problem on a network with imprecise edge weight. Kumar and Kaur [8] pointed out the shortcomings of the existing algorithm [19] and proposed a new algorithm to overcome these shortcomings. In this chapter the existing method [8] is presented.

### 2.1. Preliminaries

In this section, some basic definitions, addition of interval numbers and triangular fuzzy numbers and comparison methods of such numbers are presented. Also, the notations used throughout the paper are presented.

#### 2.1.1 Basic definitions

In this section, some basic definitions are presented [19].

**Definition 2.1.** An interval number is defined as $A = [a_L, a_R] = \{a : a_L \le a \le a_R\}$, where, $a_L$ and $a_R$ are the real numbers called the left end point and the right end point of the interval $A$.

Another way to represent an interval number in terms of midpoint and width is $A = \langle m(A), w(A) \rangle$, where $m(A) =$ midpoint of $A = \frac{a_R + a_L}{2}$ and $w(A) =$ half width of $A = \frac{a_R - a_L}{2}$.

**Definition 2.2.** Two interval numbers $A = \langle m_A, w_A \rangle$ and $B = \langle m_B, w_B \rangle$ are said to be non-dominating if

i. $m_A = m_B$ and

ii. $w_A = w_B$.

**Definition 2.3.** A triangular fuzzy number is represented by a triplet $\widetilde{A} = \langle m, \alpha, \beta \rangle$ with the membership function

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 - \dfrac{m-x}{\alpha} & \text{for } m - \alpha < x \le m \\ 1 - \dfrac{x-m}{\beta} & \text{for } m < x < m + \beta \\ 0 & \text{otherwise} \end{cases}$$

where, $m \in R$ and $\alpha, \beta > 0$.

**Definition 2.4.** Two triangular fuzzy numbers $\widetilde{A} = \langle a, \alpha, \beta \rangle$ and $\widetilde{B} = \langle b, \gamma, \delta \rangle$ are said to be non-dominating if

i. $a = b$ and

ii. $\alpha = \gamma$ or $\beta = \delta$ but, not both simultaneously.

### 2.1.2 Arithmetic operations

In this section, addition of interval fuzzy number and triangular fuzzy numbers are presented.

The addition of two interval numbers $A = [a_L, a_R]$ and $B = [b_L, b_R]$ is given by

$$A \oplus B = [a_L + b_L, a_R + b_R]$$

Alternately, in mean-width notations, if $A = \langle m_1, w_1 \rangle$ and $B = \langle m_2, w_2 \rangle$ Then,

$$A \oplus B = \langle m_1 + m_2, w_1 + w_2 \rangle$$

Let $\widetilde{A} = \langle m_1, \alpha_1, \beta_1 \rangle$ and $\widetilde{B} = \langle m_2, \alpha_2, \beta_2 \rangle$ be two triangular fu**zzy** numbers. Then,

$$\widetilde{A} \oplus \widetilde{B} = \langle m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 \rangle$$

### 2.2 Comparison of interval numbers and triangular fuzzy numbers

In this section the methods, used in the existing method [19] for comparing interval numbers and triangular fuzzy numbers, are presented

#### 2.2.1. Comparison of interval numbers

Nayeem and Pal [19] used acceptability index ($\mathcal{A}$-index) to the proposition '*A* is inferior to *B*' as $\mathcal{A}(A \prec B) = \dfrac{m_2 - m_1}{w_1 + w_2}$

In connection with this acceptability index, Nayeem and Pal [19] defined the total dominance and partial dominance of two interval numbers $A = \langle m_1, w_1 \rangle$ and $B = \langle m_2, w_2 \rangle$ one over another as follow:

i. If $\mathcal{A}(A \prec B) \geq 1$ then, *A* is said to be totally dominating over *B* in the sense of minimization and *B* is said to be totally dominating over *A* in the sense of maximization. We denote this by $A \prec B$, i.e., minimum $\{A, B\} = A$.

ii. If $0 < \mathcal{A}(A \prec B) < 1$ then, *A* is said to be partially dominating over *B* in the sense of minimization and *B* is said to be partially dominating over *A* in the sense of maximization. We denote this by $A \prec_P B$, i.e., minimum $\{A, B\} = A$.

iii. But, when, $\mathcal{A}(A \prec B) = 0$ i.e., $m_1 = m_2$ then we may not get an order relation from the above cases. Then, we may emphasize on the widths of the interval numbers *A* and *B*.

If $w_1 < w_2$ then the left end point of *A* is less than that of *B* and on finding a minimum distance, there is a chance that the distance may lie on A. But, at the same time, since the right end point of *A* is greater than that of *B*, if one prefers *A* to *B* in minimization then in worst case, he may be looser than one who prefers *B* to *A*. Thus, in such a situation an optimistic decision-maker would prefer *A* to *B* whereas a pessimistic decision-maker would do the converse.

### 2.2.2. Comparison of triangular fuzzy numbers

The acceptability index ($\mathcal{A}$-index) to the proposition '$\widetilde{A} = \langle a, \alpha, \beta \rangle$ is preferred to $\widetilde{B} = \langle b, \delta, \gamma \rangle$' is given by $\mathcal{A}(\widetilde{A} \prec \widetilde{B}) = \dfrac{b-a}{\beta+\gamma}$.

Using this $\mathcal{A}$-index Nayeem and Pal [17] defined the following ranking orders.

i. If $\mathcal{A}(\widetilde{A} \prec \widetilde{B}) \geq 1$ then, $\widetilde{A}$ is said to be totally dominating over $\widetilde{B}$ in case of minimization and the case is converse in case of maximization and this is denoted by $\widetilde{A} \prec \widetilde{B}$, i.e., minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$.

ii. If $0 < \mathcal{A}(\widetilde{A} \prec \widetilde{B}) < 1$ then, $\widetilde{A}$ is said to be partially dominating over $\widetilde{B}$ in the sense of minimization and $\widetilde{B}$ is said to be partially dominating over $\widetilde{A}$ in the sense of maximization. This is denoted by $\widetilde{A} \prec_P \widetilde{B}$, i.e., minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$.

## 2.3. Kumar and Kaur algorithm

In this section, the existing algorithm [8] for finding the fuzzy shortest path and fuzzy shortest distance of each node from source node is presented

The steps of the algorithm are summarized as follows:

**Step1**

Assume $\widetilde{e}_1 = \langle 0,0,0 \rangle$ (or $\langle 0,0 \rangle$ interval number) and label the source node (say node 1) as $[\langle 0,0,0 \rangle, -]$ (or $[\langle 0,0 \rangle, -]$).

**Step 2**

Find $\widetilde{e}_j = \text{minimum}\{\widetilde{e}_i \oplus \widetilde{e}_{ij} \,/\, i \in Np(j)\};\ j \neq 1,\ j = 2,3,\ldots,n.$

**Step 3**

If minimum occurs corresponding to unique value of $i$ i.e., $i = r$ then label node $j$ as $[\widetilde{e}_j, r]$. If minimum occurs corresponding to more than one values of $i$ then it represents that there are more than one fuzzy path between source node and node $j$ but fuzzy distance along all paths is $\widetilde{e}_j$, so choose any value of $i$.

**Step 4**

Let the destination node (node $n$) be labeled as $[\widetilde{e}_n, l]$, then the fuzzy shortest distance between source node and destination node is $\widetilde{e}_n$.

**Step 5**

Since, destination node is labeled as $[\widetilde{e}_n, l]$. So, to find the fuzzy shortest path between source node and destination node, check the label of node $l$. Let it be $[\widetilde{e}_l, p]$, now check the label of node $p$ and so on. Repeat the same procedure until node 1 is obtained.

**Step 6**

Now, the fuzzy shortest path can be obtained by combining all the nodes obtained by the Step 5.

### 2.3.1. Notation

In this section, the notation that will be used throughout the paper are presented.

$N = \{1,2,\ldots,n\}$ : The set of all nodes in a network.

$Np(j)$ : The set of all predecessor nodes of node $j$.

$e_i$ : The distance between node $i$ and first (source) node.

$e_{ij}$ : The distance between node $i$ and $j$.

$\widetilde{e}_i$ : The fuzzy distance between node $i$ and first (source) node.

$\widetilde{e}_{ij}$ : The fuzzy distance between node $i$ and $j$.

### 2.3.2. Illustrative example

To show the advantages of the Kumar and Kaur algorithm [8] over existing algorithm [19] the numerical examples presented in Nayeem and Pal [19] are solved by the Kumar and Kaur algorithm [8] and it is found that the results of Nayeem and Pal algorithm [19] and Kumar and Kaur algorithm [8] are same, while the existing algorithm [19] is very confusing to understand and to apply for finding the optimal solution compare to the existing algorithm [8].

**Example 2.1.** [19] The problem is to find the shortest path between source node (say node 1) and destination node (say node 6) on the network consists of 6 vertices $\{1,2,3,4,5,6\}$ and 11 edges $\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}, e_{35}, e_{36}, e_{45}, e_{46}, e_{56}\}$ the arc lengths of the network, shown in Figure 2.1 are all interval numbers and given by

$e_{12} = [10,12], e_{13} = [25,28],\ e_{14} = [19,20],\ e_{23} = [20,21],\ e_{24} = [30,35],\ e_{34} = [6.5,7.5],\ e_{35} = [38,40],\ e_{36} = [43,44],\ e_{45} = [35,40],\ e_{46} = [49,51],\ e_{56} = [12,13].$

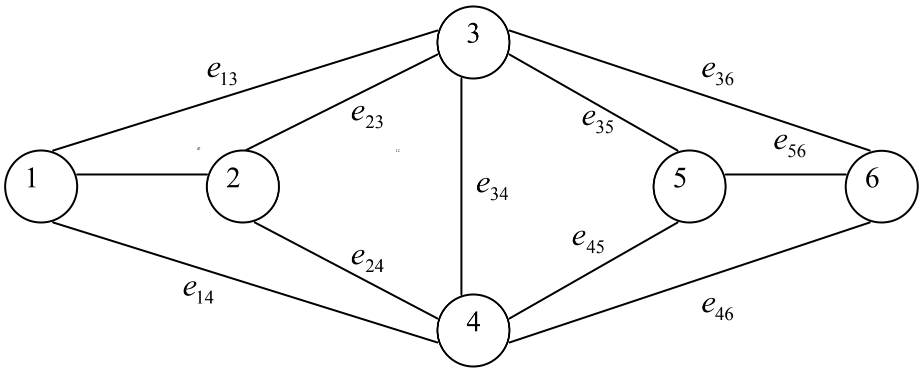


**Figure. 2.1** A network

**Solution** The mean-width notations of interval numbers as follow:

$e_{12} = \langle 11,1 \rangle,\ e_{13} = \langle 26.5,1.5 \rangle,\ e_{14} = \langle 19.5,0.5 \rangle,\ e_{23} = \langle 20.5,0.5 \rangle,\ e_{24} = \langle 32.5,2.5 \rangle,$
$e_{34} = \langle 7,0.5 \rangle,\ e_{35} = \langle 39,1 \rangle,\ e_{36} = \langle 43.5,0.5 \rangle,\ e_{45} = \langle 37.5,2.5 \rangle,\ e_{46} = \langle 50,1 \rangle,$
$e_{56} = \langle 12.5,0.5 \rangle.$

Since, node 6 is the destination node, so $n = 6$.

Assume $e_1 = \langle 0,0 \rangle$ and label the source node (say node 1) as $[\langle 0,0 \rangle, -]$, the values of $e_j$; $j = 2,3,4,5,6$ can be obtained as follows:

**Iteration 1**

Since, only node 1 is the predecessor node of node 2, so putting $i = 1$ and $j = 2$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_2$ is

$$e_2 = \text{minimum}\,\{e_1 \oplus e_{12}\} = \text{minimum}\,\{\langle 0,0 \rangle \oplus \langle 11,1 \rangle\} = \langle 11,1 \rangle$$

Since, minimum occurs corresponding to $i = 1$, so label node 2 as $[\langle 11,1 \rangle, 1]$.

**Iteration 2**

The predecessor nodes of the node 3 are node 1 and 2, so putting $i = 1, 2$ and $j = 3$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_3$ is

$$e_3 = \text{minimum}\{e_1 \oplus e_{13}, e_2 \oplus e_{23}\}$$

$$= \text{minimum}\{\langle 0,0\rangle \oplus \langle 26.5,1.5\rangle, \langle 11,1\rangle \oplus \langle 20.5,0.5\rangle\}$$

$$= \text{minimum}\{\langle 26.5,1.5\rangle, \langle 31.5,1.5\rangle\}$$

$$A(\langle 26.5,1.5\rangle \prec \langle 31.5,1.5\rangle) = \frac{31.5 - 26.5}{1.5 + 1.5} = 1.66 > 1.$$

Using Section 2.2.1, $\text{minimum}\{\langle 26.5,1.5\rangle, \langle 31.5,1.5\rangle\} = \langle 26.5,1.5\rangle$.

i.e., $e_3 = \langle 26.5,1.5\rangle$

Since, minimum occurs corresponding to $i = 1$, so label node 3 as $[\langle 26.5,1.5\rangle,1]$.

**Iteration 3**

The predecessor node of the node 4 is node 1, 2 and 3, so putting $i = 1, 2, 3$ and $j = 4$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_4$ is

$$e_4 = \text{minimum}\{e_1 \oplus e_{14}, e_2 \oplus e_{24}, e_3 \oplus e_{34}\}$$

$$= \text{minimum}\{\langle 0,0\rangle \oplus \langle 19.5,0.5\rangle, \langle 11,1\rangle \oplus \langle 32.5,2.5\rangle, \langle 26.5,1.5\rangle \oplus \langle 7,0.5\rangle\} = \langle 19.5,0.5\rangle$$

Since , minimum occurs corresponding to $i = 1$, so label node 4 as $[\langle 19.5,0.5\rangle,1]$.

**Iteration 4**

The predecessor nodes of the node 5 are node 3 and 4, so putting $i = 3, 4$ and $j = 5$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_5$ is

$$e_5 = \text{minimum}\{e_3 \oplus e_{35}, e_4 \oplus e_{45}\}$$

$$= \text{minimum}\{\langle 26.5,1.5\rangle \oplus \langle 39,1\rangle, \langle 19.5,0.5\rangle \oplus \langle 37.5,2.5\rangle\}$$

$$= \text{minimum}\{\langle 65.5,2.5\rangle, \langle 57,3\rangle\} = \langle 57,3\rangle$$

Since, minimum occurs corresponding to $i = 4$, so label node 5 as $[\langle 57,3\rangle,4]$.

**Iteration 5**

The predecessor nodes of the node 6 are node 3, 4 and 5, so putting $i = 3, 4, 5$ and $j = 6$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_6$ is

$$e_6 = \text{minimum}\{e_3 \oplus e_{36}, e_4 \oplus e_{46}, e_5 \oplus e_{56}\}$$

$$= \text{minimum}\{\langle 26.5,1.5\rangle \oplus \langle 43.5,0.5\rangle, \langle 19.5,0.5\rangle \oplus \langle 50,1\rangle, \langle 57,3\rangle \oplus \langle 12.5,0.5\rangle\}$$

$$= \text{minimum}\left\{\langle 70,2\rangle, \langle 69.5,1.5\rangle, \langle 69.5,3.5\rangle\right\}$$

$$e_6 = \langle 69.5,1.5\rangle \text{ or } \langle 69.5,3.5\rangle$$

Since, minimum occurs corresponding to $i = 4$, 5 so we can label node 6 as $[\langle 69.5,1.5\rangle,4]$ or $[\langle 69.5,3.5\rangle,5]$, if we label node 6 as $[\langle 69.5,1.5\rangle,4]$ then the corresponding shortest distance is 69.5 . Now the fuzzy shortest path between node 1 and node 6 can be obtained by using the following procedure:

Since, node 6 is labeled by $[\langle 69.5,1.5\rangle,4]$, which represents that we are coming from node 4. Node 4 is labeled by $[\langle 19.5,0.5\rangle,1]$, which represents that we are coming from node 1. Now the fuzzy shortest path between node 1 and node 6 is obtained by joining all the obtained nodes. Hence the fuzzy shortest path is $1 \to 4 \to 6$ and in the second case if we label node 6 as $[\langle 69.5,3.5\rangle,5]$ then the corresponding shortest distance is same i.e., 69.5 but the shortest path is $1 \to 4 \to 5 \to 6$.

**Example 2.2.** [19] Let us consider the same network, shown in Figure 2.1, with its arc lengths as triangular fuzzy numbers given by

$\tilde{e}_{12} = (10,11,12)$, $\tilde{e}_{13} = (25,27,28)$, $\tilde{e}_{14} = (19,20,22)$, $\tilde{e}_{23} = (20,21,21)$, $\tilde{e}_{24} = (30,34,35)$, $\tilde{e}_{34} = (6.5,7,8)$, $\tilde{e}_{35} = (30,30,32)$, $\tilde{e}_{36} = (43,44,45)$, $\tilde{e}_{45} = (39,40,40)$, $\tilde{e}_{46} = (49,50,52)$, $\tilde{e}_{56} = (9,9,10)$ and we are interested to find the fuzzy shortest path and fuzzy shortest path between the nodes 1 and 6.

**Solution** The triangular fuzzy numbers in the form of $\langle m, \alpha, \beta\rangle$, i.e., in terms of mean and the left-spreads and right-spreads are as follow:
$\tilde{e}_{12} = \langle 11,1,1\rangle$, $\tilde{e}_{13} = \langle 27,2,1\rangle$, $\tilde{e}_{14} = \langle 20,1,2\rangle$, $\tilde{e}_{23} = \langle 21,1,0\rangle$, $\tilde{e}_{24} = \langle 34,4,1\rangle$, $\tilde{e}_{34} = \langle 7,0.5,1\rangle$, $\tilde{e}_{35} = \langle 30,0,2\rangle$, $\tilde{e}_{36} = \langle 44,1,1\rangle$, $\tilde{e}_{45} = \langle 40,1,0\rangle$, $\tilde{e}_{46} = \langle 50,1,2\rangle$, $\tilde{e}_{56}\langle 9,0,1\rangle$.
Since node 6 is the destination node, so $n = 6$.
Assume $\tilde{e}_1 = \langle 0,0,0\rangle$ and label the source node (say node 1) as $[\langle 0,0,0\rangle,-]$, the values of $\tilde{e}_j$; $j = 2,3,4,5,6$ can be obtained as follows:

**Iteration 1**

Since, only node 1 is the predecessor node of node 2, so putting $i = 1$ and $j = 2$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $\widetilde{e}_2$ is

$$\widetilde{e}_2 = \text{minimum}\ \{\widetilde{e}_1 \oplus \widetilde{e}_{12}\} = \text{minimum}\ \{\langle 0,0,0\rangle \oplus \langle 11,1,1\rangle\} = \langle 11,1,1\rangle$$

Since, minimum occurs corresponding to $i = 1$, so label node 2 as $[\langle 11,1,1\rangle,1]$.

**Iteration 2**

The predecessor nodes of the node 3 are node 1 and 2, so putting $i = 1, 2$ and $j = 3$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $\widetilde{e}_3$ is

$$\widetilde{e}_3 = \text{minimum}\ \{\widetilde{e}_1 \oplus \widetilde{e}_{13}, \widetilde{e}_2 \oplus \widetilde{e}_{23}\}$$

$$= \text{minimum}\ \{\langle 0,0,0\rangle \oplus \langle 27,2,1\rangle, \langle 11,1,1\rangle \oplus \langle 21,1,0\rangle\}$$

$$= \text{minimum}\ \{\langle 27,2,1\rangle, \langle 32,2,1\rangle\}$$

Since, $\widetilde{A} = \langle a, \alpha, \beta\rangle = \langle 27,2,1\rangle$ and $\widetilde{B} = \langle b, \gamma, \delta\rangle = \langle 32,2,1\rangle$.

$$\mathcal{A}\ (\widetilde{A} \prec \widetilde{B}) = \frac{32-27}{1+2} = 1.66 > 1$$

So, using Section 2.2.1, $\text{minimum}\ \{\langle 27,2,1\rangle, \langle 32,2,1\rangle\} = \langle 27,2,1\rangle$.

i.e., $\widetilde{e}_3 = \langle 27,2,1\rangle$

Since, minimum occurs corresponding to $i = 1$, so label node 3 as $[\langle 27,2,1\rangle,1]$.

Similarly, $\widetilde{e}_4 = \langle 20,1,2\rangle$, label node 4 as $[\langle 20,1,2\rangle,1]$

$\widetilde{e}_5 = \langle 57,2,3\rangle$, label node 5 as $[\langle 57,2,3\rangle,3]$

$\widetilde{e}_6 = \langle 66,2,4\rangle$, label node 6 as $[\langle 66,2,4\rangle,5]$

Since, node 6 is the destination node of the given network, so the fuzzy shortest distance between node 1 and 6 is $\langle 66,2,4\rangle$ and the fuzzy shortest path is $1 \to 3 \to 5 \to 6$.

**Remark 2.1.** A node $i$ is said to be predecessor node of node $j$ if

(i) Node $i$ is directly connected to node $j$.

(ii) The direction of path, connecting node $i$ and $j$, is from $i$ to $j$.

**Remark 2.2.** If there is no uncertainty about any parameter then the proposed algorithm is also a new algorithm for finding the optimal solution for conventional shortest path problems.

## 2.4. Shortcomings of Nayeem and Pal algorithm

For solving the numerical examples the comparison methods presented in Nayeem and Pal [19] are used but there are the following shortcomings in these comparison methods:

(i) To show that the existing comparison method [19] can't be used for finding the fuzzy shortest path of real life problems the fuzzy shortest path and fuzzy shortest distance between node 1 and 4 of the network, shown in Figure 2.2, is obtained by the existing comparison method [19] and the obtained results are as follows:

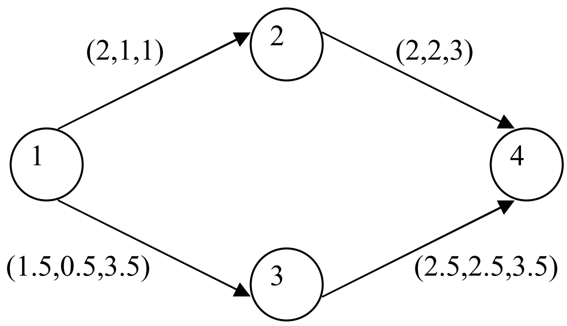


**Figure. 2.2.** A network

In the network shown in Figure. 2.2, there are two possible paths $1 \to 2 \to 4$ and $1 \to 3 \to 4$ between node 1 and 4. Using the existing comparison method [19] the distance between node 1 and node 4 along the first path i.e. $1 \to 2 \to 4$ is (4,3,4) while along the second path i.e. $1 \to 3 \to 4$ the distance between the node 1 and node 4 is (4,3,7). It is obvious from Definition 2.4 that the distances (4,3,4) and (4,3,7) are non-dominating and it is not possible to find the minimum between these distances so according to existing comparison method [19] the decision maker can choose either $1 \to 2 \to 4$ or $1 \to 3 \to 4$ i.e. using the existing comparison method it is not possible to choose the best from $1 \to 2 \to 4$ and $1 \to 3 \to 4$. But it is obvious from the values of the distances of paths that a decision maker will choose the path $1 \to 2 \to 4$. Since along this path the traveled distance will be between 1 unit and 8 unit and the maximum possibility is that it will be 4 unit while along the second path the traveled distance will be between 1 and 11 unit and the maximum possibility is that it will be 4 unit. Hence it can be concluded that existing comparison method Nayeem and Pal [19] should not be used to compare the fuzzy numbers for solving real life problems.

(ii) Nayeem and Pal [19] have pointed out that their method for comparison of different numbers is particular case of Okada and Soper [22] method but from the network, shown in Figure. 2.3, it is clear that the results are different using both existing method

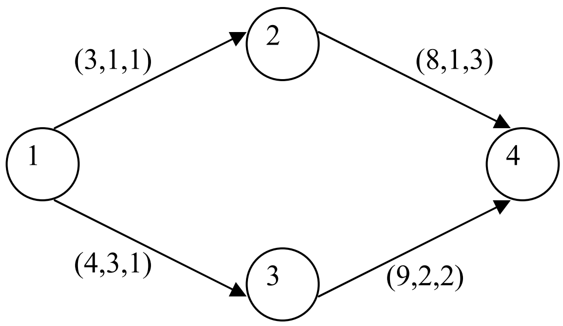


**Figure. 2.3.** A network

According to comparison method presented in Okada and Soper [22] the fuzzy shortest path and fuzzy shortest distance are $1 \rightarrow 3 \rightarrow 4$ and $(13,5,3)$ respectively, while using the comparison method presented in Nayeem and Pal [19] the fuzzy shortest path and fuzzy shortest distance are $1 \rightarrow 2 \rightarrow 4$ and $(11,2,4)$ respectively, i.e., according to Okada and Soper [22] the fuzzy shortest path is $1 \rightarrow 3 \rightarrow 4$ while according to Nayeem and Pal [19] the fuzzy shortest path is $1 \rightarrow 2 \rightarrow 4$.

(iii) Both the existing algorithms (Okada and Soper [22], Nayeem and Pal [19]) are very difficult and confusing to understand and to apply for a new decision maker, for finding the fuzzy optimal solution of shortest path problems occurring in real life problems.

(iv) In the real life problems, it is required to compare more than two fuzzy numbers (or interval numbers) simultaneously. But it is very difficult to compare a large number of fuzzy numbers simultaneously using the existing comparison method [19]. For example in 3rd and 5th iteration of Example 2.1 and Example 2.2, it is required to calculate the acceptability index of each pair i.e., it is not easy to find the minimum of three numbers.

To overcome the above shortcomings the existing comparison method [14] is used for solving the Examples 2.1 and Example 2.2.

### 2.5. Comparison of interval and triangular fuzzy numbers

Due to the shortcomings of the existing comparison method [19] it is better to use the following comparison method [14].

#### 2.5.1. Comparison of interval fuzzy numbers

Let $A=[a_L,a_R]$ and $B=[b_L,b_R]$ be two interval numbers then the symmetric triangular fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ corresponding to $A$ and $B$ are given by

$$\widetilde{A}=\left(\frac{a_L+a_R}{2},\frac{a_R-a_L}{2},\frac{a_R-a_L}{2}\right),\ \widetilde{B}=\left(\frac{b_L+b_R}{2},\frac{b_R-b_L}{2},\frac{b_R-b_L}{2}\right)\text{ and}$$

$$\Re(\widetilde{A})=\frac{a_L+a_R}{2},\ \Re(\widetilde{B})=\frac{b_L+b_R}{2}.$$

(i) $A\succ B$ if $\frac{a_L+a_R}{2}\succ\frac{b_L+b_R}{2}$

(ii) $A\prec B$ if $\frac{a_L+a_R}{2}\prec\frac{b_L+b_R}{2}$

(iii) $A\approx B$ if $\frac{a_L+a_R}{2}\approx\frac{b_L+b_R}{2}$

#### 2.5.2. Comparison of triangular fuzzy numbers

Let $\widetilde{A}=(m_1,\alpha_1,\beta_1)$ and $\widetilde{B}=(m_2,\alpha_2,\beta_2)$ be two triangular fuzzy number Then,

Where, $\Re(\widetilde{A})=m_1-\frac{1}{4}(\alpha_1-\beta_1)$ and $\Re(\widetilde{B})=m_2-\frac{1}{4}(\alpha_2-\beta_2)$.

(i) $\widetilde{A}\succ\widetilde{B}$ if $\Re(\widetilde{A})\succ\Re(\widetilde{B})$

(ii) $\widetilde{A}\prec\widetilde{B}$ if $\Re(\widetilde{A})\prec\Re(\widetilde{B})$

(iii) $\widetilde{A}\approx\widetilde{B}$ if $\Re(\widetilde{A})\approx\Re(\widetilde{B})$

## 2.6. Illustrative example

In Section 2.3, to solve the numerical examples the Kumar and Kaur algorithm [8] is used with existing comparison method [19] but due to shortcomings in the existing comparison method in this section the same numerical examples are solved using the Kumar and Kaur algorithm [8] with existing comparison method [14].

**Example 2.3.** Let the arc lengths of the network shown in Figure. 3.1 be all interval numbers and be given by

$e_{12} = [10,12],\ e_{13} = [25,28],\ e_{14} = [19,20],\ e_{23} = [20,21],\ e_{24} = [30,35],$

$e_{35} = [38,40],\ e_{36} = [43,44],\ e_{45} = [35,40],\ e_{46} = [49,51],\ e_{56} = [12,13]$

then we have to find out the shortest path between the vertices 1 and 6.

**Solution** Since, node 6 is the destination node, so $n = 6$ .Assume $e_1 = [0,0]$ and label the source node (say node 1) as $[[0,0],-]$, the values of $e_j$ ; $j = 2,3,4,5,6$ can be obtained as follows:

**Iteration 1**

Since, only node 1 is the predecessor node of node 2, so putting $i = 1$ and $j = 2$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_2$ is

$$e_2 = \text{minimum}\,\{e_1 \oplus e_{12}\} = \text{minimum}\,\{[0,0] \oplus [11,12]\} = [11,12]$$

Since, minimum occurs corresponding to $i = 1$, so label node 2 as $[[11,12],1]$.

**Iteration 2**

The predecessor nodes of the node 3 are node 1 and 2, so putting $i = 1, 2$ and $j = 3$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $e_3$ is

$$e_3 = \text{minimum}\,\{e_1 \oplus e_{13}, e_2 \oplus e_{23}\}$$
$$= \text{minimum}\,\{[0,0] \oplus [25,28], [10,12] \oplus [20,21]\}$$
$$= \text{minimum}\,\{[25,28],[30,33]\}$$

Since $\frac{25+28}{2} \prec \frac{30+33}{2}$.

So using Section 2.5.1, $\text{minimum}\,\{[25,28],[30,33]\} = [25,28]$.

i.e., $e_3 = [25,28]$

Since, minimum occurs corresponding to $i = 1$, so label node 3 as $\left[[25,28],1\right]$.

Similarly, $\widetilde{e}_4 = [19,20]$, label node 4 as $\left[[19,20],1\right]$

$\widetilde{e}_5 = [54,60]$, label node 5 as $\left[[54,60],4\right]$

$\widetilde{e}_6 = [68,71]$ or $[66,73]$, Since minimum occurs corresponding to $i = 4$, 5 so we can label node 6 as $\left[[68,71],4\right]$ or $\left[[66,73],5\right]$, if we label node 6 as $\left[[68,71],4\right]$ then the corresponding shortest distance is 69.5 and path is $1 \to 4 \to 6$ and in the second case if we label node 6 as $\left[[66,73],5\right]$ then the corresponding shortest distance is same i.e., 69.5 but the shortest path is $1 \to 4 \to 5 \to 6$.

**Example 2.4.** [19] Let us consider the same network with its arc lengths as triangular fuzzy numbers as shown in Example 2.2.

**Solution** Since node 6 is the destination node, so $n = 6$.

Assume $\widetilde{e}_1 = \langle 0,0,0 \rangle$ and label the source node (say node 1) as $[\langle 0,0,0 \rangle, -]$, the values of $\widetilde{e}_j$ ; $j = 2,3,4,5,6$ can be obtained as follows :

**Iteration 1**

Since, only node 1 is the predecessor node of node 2, so putting $i = 1$ and $j = 2$ in Step 2 of the Kumar and Kaur algorithm [8], the value of $\widetilde{e}_2$ is

$$\widetilde{e}_2 = \text{minimum } \{\widetilde{e}_1 \oplus \widetilde{e}_{12}\} = \text{minimum } \{\langle 0,0,0 \rangle \oplus \langle 11,1,1 \rangle\} = \langle 11,1,1 \rangle$$

Since, minimum occurs corresponding to $i = 1$, so label node 2 as $[\langle 11,1,1 \rangle, 1]$.

**Iteration 2**

The predecessor nodes of the node 3 are node 1 and 2, so putting $i = 1, 2$ and $j = 3$

In Step 2 of the Kumar and Kaur algorithm [8], the value of $\widetilde{e}_3$ is

$$\begin{aligned} \widetilde{e}_3 &= \text{minimum } \{\widetilde{e}_1 \oplus \widetilde{e}_{13}, \widetilde{e}_2 \oplus \widetilde{e}_{23}\} \\ &= \text{minimum } \{\langle 0,0,0 \rangle \oplus \langle 27,2,1 \rangle, \langle 11,1,1 \rangle \oplus \langle 21,1,0 \rangle\} \\ &= \text{minimum } \{\langle 27,2,1 \rangle, \langle 32,2,1 \rangle\} \end{aligned}$$

Since $\widetilde{A} = \langle 27,2,1 \rangle$ and $\widetilde{B} = \langle 32,2,1 \rangle$, using Section 2.5, $\Re(\widetilde{A}) = 26.75$ and $\Re(\widetilde{B}) = 31.75$.

Since $\Re(\widetilde{A}) \prec \Re(\widetilde{B})$, so minimum $\{\langle 27,2,1 \rangle, \langle 32,2,1 \rangle\} = \langle 27,2,1 \rangle$.

i.e., $\tilde{e}_3 = \langle 27,2,1 \rangle$

Since, minimum occurs corresponding to $i = 1$, so label node 3 as $[\langle 27,2,1 \rangle,1]$.

Similarly, $\tilde{e}_4 = \langle 20,1,2 \rangle$, label node 4 as $[\langle 20,1,2 \rangle,1]$

$\tilde{e}_5 = \langle 57,2,3 \rangle$, label node 5 as $[\langle 57,2,3 \rangle,3]$

$\tilde{e}_6 = \langle 66,2,4 \rangle$, label node 6 as $[\langle 66,2,4 \rangle,5]$

Since, node 6 is the destination node of the given network, so the fuzzy shortest distance between node 1 and 6 is $\langle 66,2,4 \rangle$ and the fuzzy shortest path is $1 \to 3 \to 5 \to 6$.

### 2.7. Advantages of Kumar and Kaur algorithm

In this section, it is shown that if the Kumar and Kaur algorithm [8] with existing comparison method [14] is used to solve the fuzzy shortest path problems then the shortcomings described in Section 2.4. are resolved

(i) Using the Kumar and Kaur algorithm [8] with existing comparison method [14] the fuzzy shortest path and fuzzy shortest distance between node 1 and 4, of the network shown in Figure. 3.2, are $1 \to 2 \to 4$ and (4,3,4) respectively.

(ii) The Kumar and Kaur algorithm [8] is very easy to understand and to apply for a new decision maker, for finding the fuzzy shortest path problems.

(iii) It is very easy to compare more then two fuzzy numbers (interval numbers) simultaneously.

### 2.8. Results and discussion

To compare the Kumar and Kaur algorithm [8] with existing algorithm [19] the numerical examples presented in Nayeem and Pal [19] are solved using the Kumar and Kaur algorithm [8] and the following results are obtained

.

(i) If the Kumar and Kaur algorithm [8] is applied with existing comparison method [19] then the obtained shortest path and shortest distance are same as obtained by the existing algorithm [19] but the existing algorithm is very confusing to understand and to apply for finding the optimal solution of shortest path problems for a new decision maker while the Kumar and Kaur algorithm [8] is very easy to understand and to apply for the same.

(ii) If the Kumar and Kaur algorithm [8] is applied with the existing comparison method [14] then it overcomes all the shortcomings, described in Section 3.4 and the shortest path and shortest distance are same as obtained by the existing algorithm.

## 2.9. Conclusions

On the basis of the presented study, it can be concluded that it is better to use Kumar and Kaur algorithm [8] as compared to existing algorithm [19] for solving fuzzy shortest path problems.

# Chapter 3

# SHORTEST PATH PROBLEMS WITH TRAPEZOIDAL FUZZY ARC LENGTHS

Kumar and Kaur [7] pointed out the shortcomings of the existing algorithm [15] and proposed a new algorithm to overcome these shortcomings. In this chapter, the existing method [7] is presented.

## 3.1. Preliminaries

In this section, some basic definitions, arithmetic operations and ranking function are presented.

### 3.1.1. Basic definitions

In this section, some basic definitions are presented [9].

**Definition 3.1.** The characteristic function $\mu_A$ of a crisp set $A \subseteq X$ assigns a value either $0$ or $1$ to each member in $X$. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set $X$ fall within a specified range $[0,1]$ i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned values indicate the membership grade of the element in the set $A$.

The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set.

**Definition 3.2.** A fuzzy number $\tilde{A} = (a,b,c)$ is said to be a triangular fuzzy number

if its membership function is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & a \le x \le b \\ 1 & x = b \\ \dfrac{(x-c)}{(b-c)}, & b \le x \le c \end{cases}$$

where $a, b, c \in R$

**Definition 3.3.** A fuzzy number $\widetilde{A} = (a, b, c, d)$ is called a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & a \le x \le b \\ 1 & b \le x \le c \\ \dfrac{(x-d)}{(c-d)}, & c \le x \le d \end{cases}$$

where, $a, b, c, d \in R$

### 3.1.2. Arithmetic operations

In this section, the arithmetic operations, triangular fuzzy numbers as well as proposed fuzzy numbers, are presented [9].

Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy number. Then,

(i) $\widetilde{A} \oplus \widetilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

(ii) $\widetilde{A} \ominus \widetilde{B} = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$

Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then,

(i) $\widetilde{A} \oplus \widetilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

(ii) $\widetilde{A} \ominus \widetilde{B} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$

### 3.1.3. Ranking function

A convenient method for comparing of fuzzy number is by use of ranking function [14].

A ranking function $\Re : F(R) \rightarrow R$, where F(R) is set of all fuzzy numbers defined on set of real numbers, maps each fuzzy number into a real number.

Let $\widetilde{A}$ and $\widetilde{B}$ be two triangular or trapezoidal fuzzy numbers. Then,

$$(i)\ \widetilde{A} \underset{\Re}{>} \widetilde{B} \quad \text{if } \Re(\widetilde{A}) > \Re(\widetilde{B})$$

$$(ii)\ \widetilde{A} \underset{\Re}{<} \widetilde{B} \quad \text{if } \Re(\widetilde{A}) < \Re(\widetilde{B})$$

$$(iii)\ \widetilde{A} \underset{\Re}{=} \widetilde{B} \quad \text{if } \Re(\widetilde{A}) = \Re(\widetilde{B})$$

For a triangular fuzzy number $\widetilde{A} = (a, b, c)$, Ranking function $\Re$ is given by $\Re(\widetilde{A}) = \frac{1}{4}(a + 2b + c)$ and for a trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$, ranking function $\Re$ is given by $\Re(\widetilde{A}) = \frac{1}{4}(a + b + c + d)$.

### 3.2. Shortcomings of the Liu and Kao alogrithm

In this section, shortcomings of the existing algorithm [15] pointed out by Kumar and Kaur [7] are presented.

i. For using the algorithm, proposed by Liu and Kao [15], a decision maker should have a good knowledge of linear programming formulation of fuzzy shortest path problems and the methods to solve the corresponding crisp linear programming problems.

ii. By using existing algorithm, to find the fuzzy shortest path and fuzzy shortest distance of each node from source node it is required to formulate and solve the problem several times which is very difficult and time consuming in case of a large network. For example: Let applying the existing algorithm on a network having seven nodes the fuzzy shortest path be $1 \rightarrow 2 \rightarrow 5 \rightarrow 7$ then to find the fuzzy shortest path and fuzzy shortest distance of nodes 3, 4 and 6 from source node (say 1) it is required to formulate and solve the problem again and again.

iii. It is very difficult to implement the existing algorithm to programming language.

### 3.3. Kumar and Kaur algorithm

In this section, the existing algorithm [7] for finding the fuzzy shortest path and fuzzy shortest distance of each node from source node is presented.

The steps of the algorithm are summarized as follow:

**Step 1**

Assume $\widetilde{d}_1 = (0,0,0,0)$ and label the source node (say node 1) as $[(0,0,0,0),-]$.

**Step 2**

Find $\widetilde{d}_j = \text{minimum}\{\widetilde{d}_i \oplus \widetilde{d}_{ij} / i \in Nd(j)\}; j \neq 1,\ j = 2,3,\ldots,n.$

**Step 3**

If minimum occurs corresponding to unique value of $i$ i.e., $i = r$ then label node $j$ as $[\widetilde{d}_j, r]$. If minimum occurs corresponding to more than one values of $i$ then it represents that there are more than one fuzzy path between source node and node $j$ but fuzzy distance along all paths is $\widetilde{d}_j$, so choose any value of $i$.

**Step 4**

Let the destination node (node $n$) be labeled as $[\widetilde{d}_n, l]$, then the fuzzy shortest distance between source node and destination node is $\widetilde{d}_n$.

**Step 5**

Since, destination node is labeled as $[\widetilde{d}_n, l]$. So, to find the fuzzy shortest path between source node and destination node, check the label of node $l$. Let it be $[\widetilde{d}_l, p]$, now check the label of node $p$ and so on. Repeat the same procedure until node 1 is obtained.

**Step 6**

Now, the fuzzy shortest path can be obtained by combining all the nodes obtained by the Step 5.

**Remark 3.1.** Let $\widetilde{A}_i; i = 1,2,\ldots,n$ be a set of fuzzy numbers. If $\Re(\widetilde{A}_k) < \Re(\widetilde{A}_i),$ for all $i$, then the fuzzy number $\widetilde{A}_k$ is the minimum of $\widetilde{A}_i; i = 1,2,\ldots,n$.

### 3.3.1. Notation

The notation that will be used throughout the chapter are as follow:

$N = \{1,2,\dots,n\}$ : The set of all nodes in a network.

$Nd(j)$ : The set of all predecessor nodes of node $j$.

$\widetilde{d}_i$ : The fuzzy distance between node $i$ and first (source) node.

$\widetilde{d}_{ij}$ : The fuzzy distance between node $i$ and $j$.

**Remark 3.2.** A node $i$ is said to be predecessor node of node $j$ if

(i) Node $i$ is directly connected to node $j$.

(ii) The direction of path, connecting node $i$ and $j$, is from $i$ to $j$.

### 3.3.2. Advantages of the Kumar and Kaur algorithm

By using the Kumar and Kaur algorithm [7] for finding the optimal solution for fuzzy shortest path problems, we have the following advantages:

i. We can find out fuzzy shortest distance and fuzzy shortest path of each node from the source node simultaneously i.e., it is not required to apply the proposed algorithm again and again for finding the fuzzy shortest distance and fuzzy shortest path for a particular node from source node.

ii. For using the Kumar and Kaur algorithm [7] a decision maker should have only the knowledge of ranking function and some arithmetic operations (addition and subtraction) of fuzzy numbers which is very easy to learn for a new decision maker.

iii. We do not use linear programming techniques.

iv. We do not use goal and parametric programming techniques.

v. The optimal solution is a fuzzy number.

vi. The proposed algorithm is very easy to understand and to apply

.

vii. There is no need of much knowledge of fuzzy linear programming, Zimmermann approach [28] and crisp linear programming.

viii. The proposed algorithms can be easily implemented into a programming language.

**3.3.3. Illustrative example**

Kumar and Kaur [7] solved the existing problem [15], presented in Example 2.1, to illustrate their proposed algorithm.

**Example 3.1.** [15] The problem is to find the shortest path between source node (say node 1) and destination node (say node 6) on the network with fuzzy distances shown in Figure 3.1

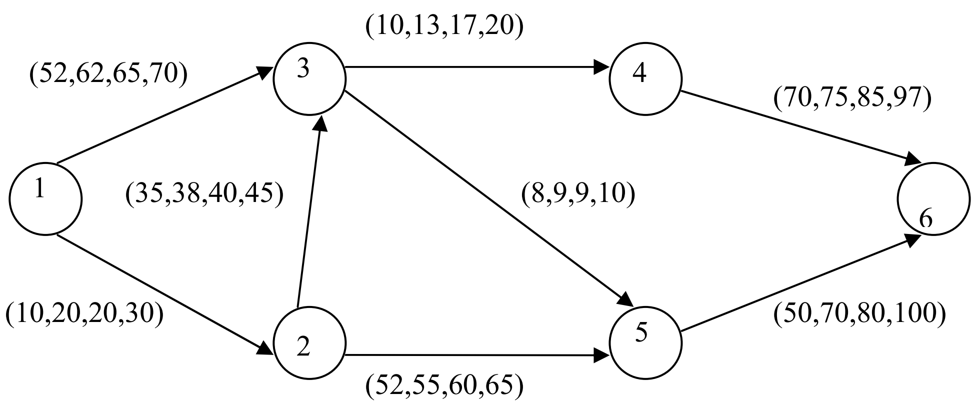


**Figure 3.1.** Network of the shortest path [15]

**Solution** Since, node 6 is the destination node, so $n = 6$.

Assume $\tilde{d}_1 = (0,0,0,0)$ and label the source node (say node 1) as $[(0,0,0,0), -]$, the values of $\tilde{d}_j$; $j = 2,3,4,5,6$ can be obtained as follows:

**Iteration 1**

Since, only node 1 is the predecessor node of node 2, so putting $i = 1$ and $j = 2$ in

Step 2 of the Kumar and Kaur algorithm [7], the value of $\tilde{d}_2$ is

$$\tilde{d}_2 = \text{minimum}\{\tilde{d}_1 \oplus \tilde{d}_{12}\} = \text{minimum}\{(0,0,0,0) \oplus (10,20,20,30)\} = (10,20,20,30)$$

Since minimum occurs corresponding to $i = 1$, so label node 2 as $[(10,20,20,30),1]$.

**Iteration 2**

The predecessor nodes of the node 3 are node 1 and 2, so putting $i = 1, 2$ and $j = 3$ in Step 2 of the Kumar and Kaur algorithm [7], the value of $\tilde{d}_3$ is

$$\begin{aligned} \tilde{d}_3 &= \text{minimum}\{\tilde{d}_1 \oplus \tilde{d}_{13},\ \tilde{d}_2 \oplus \tilde{d}_{23}\} \\ &= \text{minimum}\{(0,0,0,0) \oplus (52,62,65,70), (10,20,20,30) \oplus (35,38,40,45)\} \\ &= \text{minimum}\{(52,62,65,70), (45,58,60,75)\} \end{aligned}$$

$$\Re(52,62,65,70) = \frac{52+62+65+70}{4} = 62.25$$

$$\Re(45,58,60,75) = \frac{45+58+60+75}{4} = 59.5$$

Since, $\Re(45,58,60,75) < \Re(52,62,65,70)$

So, $\text{minimum}\{(52,62,65,70), (45,58,60,75)\} = (45,58,60,75)$

i.e $\tilde{d}_3 = (45,58,60,75)$

Since, minimum occurs corresponding to $i = 2$, so label node 3 as $[(45,58,60,75), 2]$.

**Iteration 3**

The predecessor node of the node 4 is node 3, so putting $i = 3$ and $j = 4$ in Step 2 of the Kumar and Kaur algorithm [7], the value of $\tilde{d}_4$ is

$$\tilde{d}_4 = \text{minimum}\{\tilde{d}_3 \oplus \tilde{d}_{34}\} = \text{minimum}\{(45,58,60,75) \oplus (10,13,17,20)\} = (55,71,77,95)$$

Since, minimum occurs corresponding to $i = 3$, so label node 4 as $[(55,71,77,95), 3]$.

**Iteration 4**

The predecessor nodes of the node 5 are node 2 and 3, so putting $i = 2, 3$ and $j = 5$ in Step 2 of the Kumar and Kaur algorithm [7], the value of $\tilde{d}_5$ is

$$\begin{aligned} \tilde{d}_5 &= \text{minimum}\{\tilde{d}_2 \oplus \tilde{d}_{25}, \tilde{d}_3 \oplus \tilde{d}_{35}\} \\ &= \text{minimum}\{(10,20,20,30) \oplus (52,55,60,65), (45,58,60,75) \oplus (8,9,9,10)\} \\ &= \text{minimum}\{(62,75,80,95), (53,67,69,85)\} \end{aligned}$$

$$\tilde{d}_5 = (53,67,69,85)$$

Since, minimum occurs corresponding to $i = 3$, so label node 5 as $[(53,67,69,85), 3]$.

**Iteration 5**

The predecessor nodes of the node 6 are node 4 and 5, so putting $i = 4, 5$ and $j = 6$ in Step 2 of the Kumar and Kaur algorithm [7], the value of $\tilde{d}_6$ is

$$\tilde{d}_6 = \text{minimum}\{\tilde{d}_4 \oplus \tilde{d}_{46}, \tilde{d}_5 \oplus \tilde{d}_{56}\}$$

$$= \text{minimum}\{(55,71,77,95) \oplus (70,75,85,97), (53,67,69,85) \oplus (50,70,80,100)\}$$

$$= \text{minimum}\{(125,146,162,192), (103,137,149,185)\}$$

$$\tilde{d}_6 = (103,137,149,185)$$

Since, minimum occurs corresponding to $i = 5$, so label node 6 as $[(103,137,149,185), 5]$.

Since, node 6 is the destination node of the given network, so the fuzzy shortest distance between node 1 and 6 is $(103, 137, 149, 185)$. Now the fuzzy shortest path between node 1 and node 6 can be obtained by using the following procedure:

Since, node 6 is labeled by $[(103, 137, 149, 185), 5]$, which represents that we are coming from node 5. Node 5 is labeled by $[(53, 67, 69, 85), 3]$, which represents that we are coming from node 3. Node 3 is labeled by $[(45, 58, 60, 75), 2]$, which represents that we are coming from node 2. Now node 2 is labeled by $[(10, 20, 20, 30), 1]$, which represents that we are coming from node 1. Now the fuzzy shortest path between node 1 and node 6 is obtained by joining all the obtained nodes. Hence the fuzzy shortest path is $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6$.

The fuzzy shortest distance and the fuzzy shortest path of all the nodes from node 1 is shown in the Table 3.1 and the labeling of each node is shown in Figure 3.2

**Table 3.1.** Tabular representation of different fuzzy shortest paths

| Node No. ($j$) | $\widetilde{d}_j$ | Fuzzy shortest path between $j^{th}$ and $1^{st}$ node |
|---|---|---|
| 2 | (10, 20, 20, 30) | $1 \to 2$ |
| 3 | (45, 58, 60, 75) | $1 \to 2 \to 3$ |
| 4 | (55, 71, 77, 95) | $1 \to 2 \to 3 \to 4$ |
| 5 | (53, 67, 69, 85) | $1 \to 2 \to 3 \to 5$ |
| 6 | (103, 137, 149, 185) | $1 \to 2 \to 3 \to 5 \to 6$ |

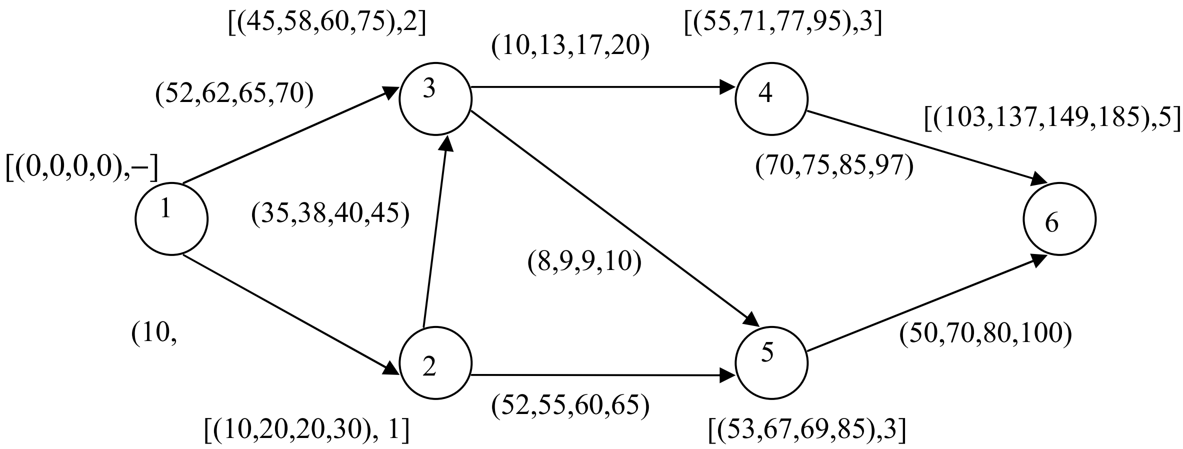


**Figure 3.2**. Network with shortest distance of each node from node 1

### 3.4. Results and discussion

In this section, the results obtained by using the existing algorithm [15] and the existing algorithm [7] are discussed.

i. Applying the existing algorithm [15] and existing algorithm [7] on the networks shown in Figure 3.1 the obtained fuzzy shortest path and fuzzy shortest distance between node 1 and node 6 are $1 \to 2 \to 3 \to 5 \to 6$ and $(103, 137, 149, 185)$ respectively. If a decision maker want to find the fuzzy shortest path and fuzzy shortest distance between node 1 and node 4, using the existing algorithm, then it is required to repeat the whole existing algorithm again while to find the fuzzy shortest path and fuzzy shortest distance between node 1 and node 4 by using the Kumar and Kaur algorithm [7], it is not required to repeat the whole procedure

ii. Since, in most of the real life networks the numbers of nodes are large, so it is very difficult and time consuming to repeat the whole procedure of existing algorithm again and again to find the fuzzy shortest path and fuzzy shortest distance of nodes from source node.

## 3.5. Conclusions

On the basis of presented study, it can be concluded that it is better to use the existing algorithm [7] as compared to existing algorithm [15] for solving fuzzy shortest path problems.

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