

A Study of New Topologies on \mathbb{R}

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CERTIFICATE

I hereby certify that the dissertation entitled, "A Study of New Topologies on R ", which is being submitted by Miss. Sakshi (Roll No. 301403015), in the partial fulfillment of the requirement for the award of the degree of Master of Science in the School of Mathematics, Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision and guidance of Dr. Jatinderdeep Kaur during the period from January 2016 to July 2016.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other University/ Institute for the award of any degree.


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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.



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ABSTRACT

The present dissertation entitled, “A Study of New Topologies on \mathbb{R} ” comprises certain investigations carried out by me at School of Mathematics, Thapar University, Patiala, under the supervision of **Dr. Jatinderdeep Kaur**, Assistant Professor, School of Mathematics, Thapar University, Patiala.

The aim of this work is to find new topologies with this new set of basis B_- and B_+ . The topologies generated from this new set of basis are different from standard, lower limit, upper limit and k -topologies. We compared well known topologies with new topologies on \mathbb{R} .

The whole work is divided into four chapters. Chapter 1 is introduction which includes brief account of definitions and their examples which will be required for the later chapters. In chapter 2, we have studied some main theorems which guarantees how topology can be generated from given base and vice-versa. The aim of chapter 3 is to study well known topologies on \mathbb{R} and their topological properties. In chapter 4, we introduced new topologies on real line and compared them with well known topologies on \mathbb{R} .

At the end of this dissertation, we have bibliography of research papers and books cited in the dissertation.

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Chapter 1

Introduction to Topology

1.1 Introduction

'Topology' is the area of mathematics which studies continuity and related concepts. Important fundamental notions like open sets, closed sets, limit points and continuity are introduced.

Originally coming from questions in analysis and differential geometry. L.Euler in 1736, the paper entitled, "**seven bridges of konigsberg**" [2] is regarded as one of important study in modern topology. In this paper, he pointed out that the choice of route inside each land mass is irrelevant. The only useful property of a route is the sequence of bridges crossed, he recognized that the key information was the number of bridges and the list of their end points which typically development of topology. It is difficult to fix a date for the starting of topology. The first time use of term *Topology* that we know of appeared in title of a book written by J.W. Listing in 1847. Further, many authors like **Cauchy, Riemann and Enrico Betti** studied various results in topology.

In present time topology is an important branch of pure mathematics. Topological ideas are present in almost all areas of today's mathematics. The subject of topology itself consists of several different branches such as point set topology, algebraic topology and differential topology, which have relatively little in common

Topology can be defined as “*the study of qualitative properties of certain objects that are invariant under a certain kind of transformation*”. The motivation insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. The square and the circle have many properties in common: they are both same dimensional objects and both separate the plane into two parts: inside and outside part.

Basic topology is concerned with boundaries . What is an open set? When is a set “in one piece”? When can we “put a set” inside a box”? What is a “boundary”? What is a ”neighborhood”? And so on.

1.2 Basic concepts and definitions

In this section, some basic definitions are presented. Here, we give a brief account of definitions and results which will be required for the later chapters. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience.

Definition 1.1 ([5],p.76) A *topology* on a set X is a collection T of subsets of X having the following properties:

- (i) ϕ and X are in T .
- (ii) The union of the elements of any subcollection of T is in T .
- (iii) The intersection of the elements of any finite subcollection of T is in T

If T is topology on X , then the pair (X, T) is called *Topological space*. The elements of T are called *open sets*. A subset $F \subseteq X$ is called *closed sets* if its complement is open.

Remark 1.2 Any finite intersection $U_1 \cap U_2 \dots \cap U_n$ of open sets is open. It is important to note that it is in general not true that an arbitrary(infinite) union of open

sets would be open.

Remark 1.3 Open and closed sets are not mutually exclusive. Infact, subsets that are both open and closed often exist.

Example 1.4 (Discrete Topological space)

Let X be arbitrary set, if we consider T as the collection of every subset of X , then (X, T) is called discrete topological space.

Example 1.5 (Indiscrete Topology)

Let X be a non-empty set, the collection consisting of X and ϕ only is also a topology on X , it is called the indiscrete topology.

These two examples are easy to understand. However, some more examples of important topological spaces are given below:

Example 1.6 Let $X = \mathbb{R}$, Consider the collection

$$T = \{U \subseteq \mathbb{R} \mid U \text{ is the union of open intervals}\}$$

where open interval is defined as $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ with $a, b \in \mathbb{R}$. This topology is called **Standard or Euclidean topology** on \mathbb{R} .

Example 1.7 (Co-Finite Topology)

Let X be a non-empty set, the collection of subsets of X consisting of the empty set together with those non-empty subsets of X whose complements are finite is called Co-finite Topology.

Example 1.8 (Co-Countable Topology)

Let X be a non-empty set, the collection of subsets of X consisting of the empty and all those non-empty subsets of X whose compliments are countable is called Co-countable Topology.

Definition 1.9 Suppose T_1 and T_2 are two topologies on the set X and if $T_1 \subseteq T_2$, then T_2 is *Finer Topology* than T_1 or T_1 is **coarser** than T_2 and if $T_1 \subset T_2$ then T_2 is **strictly finer** than T_1 . T_1 is comparable with T_2 if either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$.

Remark 1.10 Two topologies on X need not be comparable. For example Let $X = \{a, b, c\}$, and two topologies on X are $T_1 = \{\phi, \{a\}, X\}$ and $T_2 = \{\phi, \{b\}, X\}$. It can be easily noted that neither $T_1 \subseteq T_2$ nor $T_2 \subseteq T_1$, then T_1 and T_2 are not comparable.

Definition 1.11 ([5],p.78) If X is a set, a basis for a topology on X is a collection B of subsets of X such that:

- (i) For each $x \in X$, there is at least one basis element B containing x .
- (ii) If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis elements B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Example 1.12 If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X .

Example 1.13 The collection of all open intervals in the real line forms a base for the standard topology.

Definition 1.14 ([4],p.171) Let X be a topological space. A class B of open subsets of X is said to be an *open base* for X if every open set in X is a union of members of B .

Example 1.15 The class of open intervals form an open base for the topological space \mathbb{R} of real numbers with usual topology.

Definition 1.16 ([4],p.172) Let (X, T) be a topological space and $x \in X$. Then the class B_x of open sets containing points $x \in X$ is *local base* if for each open set G

containing x , there exists $G_x \in B_x$ such that $x \in G_x \subset G$.

Example 1.17 Let (\mathbb{R}, U) be a topological space and let $x \in \mathbb{R}$. Then the collection of $B_x = (x - \epsilon, x + \epsilon) : 0 < \epsilon \in \mathbb{R}$ forms a local base at x .

Definition 1.18 ([4],p.178) Let (X, T) be a topological space. The space X is said to be *First countable space* if for all $x \in X$, there exists a countable class B_x of open sets containing x such that every open set G containing x also contains a member of B_x .

Example 1.19 Every discrete topology (X, D) is first countable.

Definition 1.20([5],p.190) If a space X has a countable basis for its topology, then X is said to be *second countable space*.

Example 1.21 The topological space \mathbb{R} of real numbers with usual topology is second countable.

Definition 1.22 ([5],p.191) A subset A of a space X is said to be *dense* in X if $\bar{A} = X$, where $\bar{A} = A \cup A'$.

Example 1.23 The set Q of rational numbers is dense in \mathbb{R}

Definition 1.24 Let (X, T) be a topological space is said to be **separable** iff X contains a countable dense subset.

Example 1.25 The co-finite topological space \mathbb{R} of real numbers is separable.

Definition 1.26([4],p.209) Let X be a topological space and Y be a subset of X . A class $C = \{G_i : i \in I\}$ of open subsets of X is said to be *open cover* of Y if $Y \subseteq \cup G_i$.

Definition 1.27([4],p.209) A subclass of an open cover which is itself an open cover is called a *sub cover*.

Definition 1.28 ([5],p.192) A space for which every open cover contains a countable sub-covering is called a *Lindelof space*.

Example 1.29 The space (\mathbb{R}, U) is a Lindelof space.

Definition 1.30([4],p.273) A topological space (X, T) is said to be a T_0 -space if for every pair of distinct points x and y in X , there exists open set which contains one of these points but not the other.

Example 1.31 Every discrete topological space (X, D) is a T_0 – space.

Definition 1.32([4],p.273) A topological space (X, T) is said to be a T_1 - space if x and y are two distinct points of a topological space X , then there exists two open sets one containing x and not y and other containing y and not x .

Example 1.33 Every discrete topological space (X, D) is a T_1 -space.

Definition 1.34([4],p.282) A topological space (X, T) is said to be T_2 -space or *Haudroff space* iff for every pair of distinct points x, y of X , there exist neighbourhoods N of x and M of y such that $N \cap M = \phi$.

Example 1.35 Every discrete topological space (X, D) is a Hausdorff space.

Definition 1.36([4],p.302) A topological space (X, T) is said to be a *regular space* if F is a closed subset of X and x is a point of X not in F , then there exist two disjoint open sets G and H such that $x \in G$ and $F \subset H$.

Definition 1.37([4],p.302) A topological space (X, T) is said to be T_3 -space if it is regular and T_1 – space.

Example 1.38 Every discrete topological space (X, D) is a regular space.

Definition 1.39([4],p.319) A topological space (X, T) is said to be a *normal space* if F_1 and F_2 are disjoint closed subsets of X then there exists two disjoint subsets one containing F_1 and other containing F_2 .

Example 1.40 Let $X=\{a,b,c\}$ and $T=\{X,\phi,\{b\},\{a,c\}\}$. Then, (X, T) is normal space.

Definition 1.41([4],p.319) A topological space is said to be T_4 -*space* if it is normal and T_1 - *space*.

Now, I give a brief chapter wise resume of the results in this dissertation.

In chapter 2, we have studied some important results that deal with the construction of topology from basis. In reverse how to go from a topology to basis that generates it. Also, comparison of topology in terms of basis has been studied in this chapter .

In chapter 3, we have studied standard topology on real line and its properties. Also, some more well known topologies on \mathbb{R} and their properties have been studied.

In chapter 4, New topologies on real line have been introduced and compared them with standard topology and well known topologies on \mathbb{R} .

Chapter 2

Bases for Topology

2.1 Introduction

In mathematics, a base (or basis) B for a topological space X with topology T is a collection of open sets in T . Every open set in T can be written as union of elements of B . Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology. Bases are also useful to determine whether one topology is finer than another.

Definition 2.1([5],p.78) If X is a set, a *basis* for a topology on X is a collection B of subsets of X such that:

- (i) For each $x \in X$, there is at least one basis element B containing x .
- (ii) If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Remark 2.2 If \mathbb{B} satisfies these two conditions, then we define the *topology T generated by B* as follows:

A subset U of X is said to be open in X if for each $x \in U$, there is an element $B \in \mathbb{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of T .

Example 2.3 Let \mathbb{B} be the collection of all circular regions in the plane. Then \mathbb{B} satisfies both the conditions for a basis. The second condition is illustrated in fig

2.1 as shown below

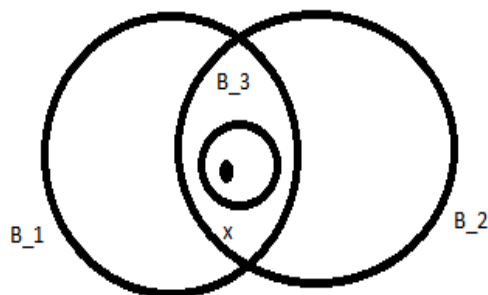


Figure 2.1:

In this chapter, we have studied how the topology can be generated from the given basis and vice-versa.

2.2 Main results

Theorem 2.4 *Let X be a set; let B be a basis for a topology T on X . Then T equals the collection of all unions of elements of B .*

Proof. By definition of basis, each element of B is itself an elements of T . Because T is a topology .This implies that $B \subset T$. Conversely, Let T be a topology on X and $U \in T$ and let x be any element of U . therefore, there exists $B_x \in B$, which contain x . such that, $x \in B_x \subset U$ Then $U = \bigcup_{x \in U} B_x$

So, U equals a union of elements of B . This shows that every open set U in X can be expressed as a union of basis elements. However, T is the collection of all unions of elements of basis.

□

Remark 2.5 Every open set U in X can be expressed as a union of basis element, However this set U is not a unique.

The following result describe how a basis can be generated from the the given topology:

Theorem 2.6 Let X be a topological space. Suppose that \mathcal{C} is a collection of open sets of X such that for each open set U of X and each x in U , there is an element C of \mathcal{C} such that $x \in C \subset U$. Then \mathcal{C} is a basis for the topology of X .

Proof. Given $x \in X$, by hypothesis, there exists $C \in \mathcal{C}$ such that $x \in C \subset X$.

Now, let $x \in C_1 \cap C_2$, where C_1 and C_2 are elements of \mathcal{C} .

Since C_1 and C_2 are open. So, $C_1 \cap C_2$ is also.

Therefore, by given hypothesis there exists C_3 in \mathcal{C} such that $x \in C_3 \subset C_1 \cap C_2$.

$\Rightarrow \mathcal{C}$ is the basis for the topology of X .

Let T_1 be the collection of open sets of X and let T_2 be the topology generated by \mathcal{C} . let $U \in T_1$ and $x \in U$, by given hypothesis, $C \in \mathcal{C}$ such that $x \in C \subset U \Rightarrow U$ is an element of T_2 .

Conversely, let $W \in T_2 \Rightarrow W$ is the union of elements of \mathcal{C}

by Theorem 2.4, each element of $\mathcal{C} \in T_1$ as T_1 is a topology. $W \in T_1 \Rightarrow T_1 = T_2$ □

Remark 2.7: As Topologies are given by bases, so it is useful to determine whether one topology is finer than another. This is given by the following theorem:

Theorem 2.8 Let \mathbf{B} and \mathbf{B}' be bases for the topologies T_1 and T_2 , respectively, on X . Then the following are equivalent:

- (i) T_2 is finer than T_1 .
- (ii) For each $x \in X$ and each basis element $B \in \mathbf{B}$ containing x , there is a basis element $B' \in \mathbf{B}'$ such that $x \in B' \subset B$.

Proof. Given an element U of T_1 , we wish to show that $U \in T_2$. Let $x \in U$. Since \mathbf{B} generates T_1 , there is an element $B \in \mathbf{B}$ such that $x \in B \subset U$, by given condition(ii) there exists $B' \in \mathbf{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in T_2$. Therefore, $T_1 \in T_2$ or T_2 is finer than T_1

Given $x \in X$ and $B \in \mathbf{B}$, with $x \in B$. Now, $B \in T_1$. From condition(i), T_2 is finer than T_1 . Therefore, $B \in T_2$. Since T_2 is generated by \mathbf{B}' , there is an element $B' \in \mathbf{B}'$ such that $x \in B' \subset B$. □

2.3 Examples to illustrate main results

Example 2.9 Let X be any non empty set and $B = \{X, \{a\}, \{c\}\}$ is a base. The topology that it generates: $T = \{\cup B : B \subseteq \mathbb{B}\}$ is a topology on X . In other words, the collection of all unions of members of B is a topology on X , the topology generated by the base is $\{\phi, \{a\}, \{c\}, \{a, c\}, X\}$.

Example 2.10 Let (X, T) is a topological space where $T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. which is the general procedure to follow in order to get a base for (X, T) ?

Let X be a non-empty set. A collection B of subsets of X is a base for some topology on X if it satisfies two conditions:

- (i) B covers X . That is, every point of X belongs to at least one members of B .
- (ii) If $B_1, B_2 \in B$ and $x \in B_1 \cap B_2$, then there is a B_3

The topology $T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ on the set $X = \{a, b\}$ has the base $B = \{\{a\}, \{b\}\}$

Note that a topology may have many different bases.

Example 2.11 Let $X = \{a, b, c\}$. Suppose $\mathbf{B} = \{\{a\}, X\}$ and $\mathbf{B}' = \{\{a\}, \{b\}, \{c\}\}$ be basis for topologies $T_1 = \{\phi, \{a\}, X\}$ and $T_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, X\}$ respectively. Clearly T_2 is finer than T_1 . Also for each $x \in X$ and each basis element $B \in \mathbf{B}$ containing x , there is a basis element $B' \in \mathbf{B}'$ such that $x \in B' \subset B$.

Chapter 3

Some known Topologies on Real Line \mathbb{R} and their Properties

In this chapter, we have defined some topological properties of real numbers \mathbb{R} and its subsets on the similar line.

Also, we discussed the properties of topologies on real line related to countability and separation axioms.

A countability axiom says when we can describe all open sets of a topological space, or all nhds. of a fixed point, using a countable basis. The separation axioms are about the use of topological space mean to distinguish disjoint sets and distinct points.

Definition 3.1 Let \mathbb{R} be the set of real numbers. Topology T consisting of ϕ and all subsets of \mathbb{R} which are union of open intervals of the form $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, then T is a topology on \mathbb{R} called *standard topology* or Usual Topology.

Definition 3.2 Let B' be the collection of half-open of the form $[a, b) = \{x | a \leq x < b\}$ where $a < b$, the topology generated by B' is called the *lower limit topology* on \mathbb{R} (generally it is denoted by \mathbb{R}_l).

Definition 3.3 Let B' be the collection of half-open of the form $(a, b] = \{x | a < x \leq b\}$ where $a < b$, the topology generated by B' is called the *upper limit topology* on

\mathbb{R} (generally it is denoted by \mathbb{R}_u).

Definition 3.4 Let K be the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let \mathbb{B}'' be the collection of all open intervals (a, b) , along with all sets of the form $(a, b) - K$. The topology generated by \mathbb{B}'' will be called the K -topology on \mathbb{R} and it is denoted by \mathbb{R}_k .

In this section, we study some properties of topologies define on real line .

3.1 Properties of standard topology

Property 3.1.1 The space (\mathbb{R}, U) is first countable.

Proof. Let $x \in \mathbb{R}$.

Consider the collection, $B_x = (x - 1/n, x + 1/n) : n \in \mathbb{N}$.

Let N be an arbitrary nhd. of x .

Then there exists $\epsilon > 0$ such that $x \in (x - \epsilon, x + \epsilon) \subseteq N$.

Let us choose a positive integer n_ϵ such that $1/n_\epsilon < \epsilon$, we have

$x \in (x - 1/n_\epsilon, x + 1/n_\epsilon) \subseteq (x - \epsilon, x + \epsilon) \subseteq N$.

Thus to each nhd. N of x there exists a member $(x - 1/n_\epsilon, x + 1/n_\epsilon)$ in B_x such that $(x - 1/n_\epsilon, x + 1/n_\epsilon) \subseteq N$

This shows that B_x is a local base at x .

Thus, every point of \mathbb{R} possesses a countable local base.

Hence (\mathbb{R}, U) is first countable.

□

Property 3.1.2 The space (\mathbb{R}, U) is second countable.

Proof. Let B be the family of all open intervals with rational end points.

Then, B is clearly a countable family of U - open sets.

Moreover, if G is a U - open set containing x , then we may find rational numbers a, b such that $x \in (a, b) \subseteq G$.

Thus for each U - open set G and each $x \in G$ there exists $(a, b) \in B$ such that $x \in (a, b) \subseteq G$.

Thus B is a countable base for U .

Hence (\mathbb{R}, U) is second countable.

□

Property 3.1.4 The space (\mathbb{R}, U) is a T_1 -space.

Proof. Let x and y be any two distinct real numbers.

Let $x < y$ and suppose that $y - x = k$.

Then there exists open sets $G, H \in U$ given by

$$G = (x - k/3, x + k/3)$$

and
$$H = (y - k/3, y + k/3)$$

such that $x \in G$ but y does not belong to G

also $y \in H$ but x does not belong to H .

Hence, (\mathbb{R}, U) is a T_1 -space.

□

Property 3.1.5 The space (\mathbb{R}, U) is Hausdorff space.

Proof. Let x and y be any two distinct points of \mathbb{R} so that $x \neq y$.

Let $|x - y| = k$. Then

$$(x - k/3, x + k/3) \text{ and } (y - k/3, y + k/3)$$

are disjoint U -nhs. of x and y respectively.

Hence (\mathbb{R}, U) is a Hausdorff space.

□

Property 3.1.6 The space (\mathbb{R}, U) is regular space and T_3 -space.

Proof. Let F be any closed subset of \mathbb{R} and let $x \notin F$.

Now, F is closed and $x \notin F$, so x is not a limit point of F .

Thus there exists an open interval $(x - \epsilon, x + \epsilon)$ containing x and containing no point of F i.e. $(x - \epsilon, x + \epsilon) \cap F = \phi$.

Let
$$H = (x - \epsilon/3, x + \epsilon/3)$$

and
$$G = \cup\{(y - \epsilon/3, y + \epsilon/3) : y \in F\}.$$

Then G and H are clearly open sets such that $F \subseteq G$ and $x \in H$.

Also, $G \cap H = \phi$,

For if $p \in G \cap H$, then

$p \in G \cap H \Rightarrow p \in G, p \in H.$

$\Rightarrow p \in (y - \epsilon/3, y + \epsilon/3)$ for some $y_0 \in F$ and $p \in (y - \epsilon/3, y + \epsilon/3)$

$\Rightarrow |p - y_0| < \epsilon/3$ and $|x - p| < \epsilon/3$

$\Rightarrow |x - y_0| = |x - p + p - y_0|$

$\leq |x - p| + |p - y_0|$

$< \epsilon/3 + \epsilon/3$

$< 2\epsilon/3$

$< \epsilon$

$\Rightarrow y_0 \in (x - \epsilon, x + \epsilon)$, where $y_0 \in F$.

This contradicts the fact that $(x - \epsilon, x + \epsilon)$ contains no limit point of F .

So, $G \cap H = \phi$.

Thus (\mathbb{R}, U) is a regular space.

The space (\mathbb{R}, U) is T_3 -space.

Since (\mathbb{R}, U) is a regular space and since (\mathbb{R}, U) is a T_1 -space, it follows that (\mathbb{R}, U) is a T_3 -space.

□

Property 3.1.7 The space (\mathbb{R}, U) is normal space and T_4 -space.

Proof. Let A and B be any two disjoint U -closed subsets of \mathbb{R} and let $a \in A$.

Then $a \notin B$ and so $a \in \mathbb{R} - B$.

Since B is U -closed, $\mathbb{R} - B$ must be an U -open set and so there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset (\mathbb{R} - B)$.

This implies that $(a - \delta, a + \delta)$ and B are disjoint.

Now, let $\bigcup\{(a - \delta/2, a + \delta/2) : a \in A\} = G$.

Similarly, if $b \in B$ there exist $\eta > 0$ such that $(b - \eta, b + \eta)$ and A are disjoint and

let $\bigcup\{(b - \eta/2, b + \eta/2) : b \in B\} = H$.

Thus G and H are U -open subsets of \mathbb{R} such that $A \subset G$ and $B \subset H$.

Now consider the intersection of G and H .

We claim that $G \cap H = \phi$.

Let, if possible, x be an element of $G \cap H$.

Then $x \in G \cap H \Rightarrow x \in G$ and $x \in H$.

$x \in G \Rightarrow x \in (a - \delta/2, a + \delta/2)$ for some $a \in A$.

$x \in H \Rightarrow x \in (b - \eta/2, b + \eta/2)$ for some $b \in B$.

Thus $|a - x| < \delta/2$ and

$|x - b| < \eta/2$ Therefore, $|a - b| = |a - x + x - b|$

$\leq |a - x| + |x - b|$

$\leq \delta/2 + \eta/2$.

Now if we choose δ such that $\delta < \eta$ then $|a - b| < \eta$ and so $a \in (b - \eta, b + \eta)$ which is contradiction of the fact that $(b - \eta, b + \eta)$ and A are disjoint.

So, $G \cap H = \phi$.

Thus (\mathbb{R}, U) is a normal space.

The space (\mathbb{R}, U) is T_4 -space.

Since (\mathbb{R}, U) is a normal space and since (R, U) is a T_1 -space, it follows that (R, U) is a T_4 -space.

□

Next, in this section we have studied the properties of lower limit topology on \mathbb{R} .

3.2 Properties of lower limit topology

Property 3.2.1 The space \mathbb{R}_l is first countable.

Proof. Let $x \in \mathbb{R}$.

Consider the collection, $B_x = \{[x, x + 1/n) : n \in \mathbb{N}\}$.

Let N be an arbitrary nhd. of x .

Then there exist $\epsilon > 0$ such that $x \in [x, x + \epsilon) \subseteq N$.

Let us choose a positive integer n_ϵ such that $1/n_\epsilon < \epsilon$, we have

$x \in [x, x + 1/n_\epsilon) \subseteq [x, x + \epsilon) \subseteq N$.

Thus to each nhd. N of x there exist a member $[x, x + 1/n_\epsilon)$ in B_x such that $[x, x + 1/n_\epsilon) \subseteq N$

This shows that B_x is a local base at x .

Moreover, B_x is clearly countable .

Thus, every point of \mathbb{R} possesses a countable local base.

Hence the \mathbb{R}_l is first countable.

□

Property 3.2.2 The space \mathbb{R}_l is not a second countable space.

Proof. Let B be a basis for \mathbb{R}_l .

Choose for each x , an element B_x of B such that $x \in B_x \subset [x, x + 1)$.

If $x \neq y$, then $B_x \neq B_y$,

Since $B_x = \inf B_x$

and $B_y = \inf B_y$.

Therefore, B must be uncountable .

Hence, \mathbb{R}_l is not a second countable space.

□

Property 3.2.3 The space \mathbb{R}_l is a T_1 -space.

Proof. Let x and y be any two distinct real numbers.

Let $x < y$ and suppose that $y - x = k$.

Then there exists open sets $G, H \in U$ given by

$$G = [x, x + k/3)$$

and $H = [y, y + k/3)$

such that $x \in G$ but $y \notin G$

also $y \in H$ but $x \notin H$.

Hence, \mathbb{R}_l is a T_1 -space.

□

Property 3.2.4 The space \mathbb{R}_l is Hausdroff space.

Proof. Let x and y be any two distinct points of \mathbb{R}_l so that $x < y$.

$[x, y)$ and $[y, y + 1)$

are disjoint $U - nhds.$ of x and y respectively.

Hence \mathbb{R}_l is a Hausdorff space.

□

Property 3.2.5 The space \mathbb{R}_l is a regular space.

Proof. Let F be any closed subset of \mathbb{R}_l and let $x \notin F$.

Now, F is closed and $x \notin F$, so x is not a limit point of F .

Thus there exists an open interval $[x, x + \epsilon)$ containing x and containing no point of F i.e. $[x, x + \epsilon) \cap F = \phi$.

Let $H = [x, x + \epsilon/3)$

and $G = \cup\{[y, y + \epsilon/3) : y \in F\}$.

Then G and H are clearly open sets such that $F \subseteq G$ and $x \in H$.

Also, $G \cap H = \phi$,

For if $p \in G \cap H$, then

$p \in G \cap H \Rightarrow p \in G, p \in H$.

$\Rightarrow p \in [y, y + \epsilon/3)$ for some $y_0 \in F$ and $p \in [x, x + \epsilon/3)$

$\Rightarrow |p - y_0| < \epsilon/3$ and $|x - p| < \epsilon/3$

$\Rightarrow |x - y_0| = |x - p + p - y_0|$

$\leq |x - p| + |p - y_0|$

$< \epsilon/3 + \epsilon/3$

$< 2\epsilon/3$

$< \epsilon$

$\Rightarrow y_0 \in [x, x + \epsilon)$, where $y_0 \in F$.

This contradiction the facts that $[x, x + \epsilon)$ contains no limit point of F .

So, $G \cap H = \phi$.

Thus \mathbb{R}_l is a regular space.

The space \mathbb{R}_l is T_3 -space.

Since \mathbb{R}_l is a regular space and since \mathbb{R}_l is a T_1 -space, it follows that R_l is a T_3 -space.

□

Property 3.2.6 The space \mathbb{R}_l is a normal space.

Proof. Let A and B are disjoint closed in \mathbb{R}_l , where A and B are one-point sets.

Because one-point set are closed in \mathbb{R}_l , since the topology \mathbb{R}_l is finer than that of \mathbb{R} .

For each point a of A choose a basis element $[a, x_a)$ not intersecting B

For each point b of B choose a basis element $[b, x_b)$ not intersecting A .

$$G = \bigcup [a, x_a) \quad \text{and} \quad H = \bigcup [b, x_b)$$

are disjoint open sets about A and B , respectively. So, $G \cap H = \phi$.

Thus \mathbb{R}_l is a normal space.

The space \mathbb{R}_l is T_4 -space.

Since \mathbb{R}_l is a normal space and since \mathbb{R}_l is a T_1 -space, it follows that \mathbb{R}_l is a T_4 -space.

□

Remark 3.5 The result of all the properties of upper limit topology is on the same lines of lower limit topologies.

Remark 3.6 The lower limit topology is not second countable space.

Remark 3.7 Every second countable space is separable.

Chapter 4

New Topologies on \mathbb{R} and Their Properties

4.1 Introduction

Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology, and because many topologies are mostly defined in term of a base which generates them.

Example: The collection of all open intervals in the real line forms a base for a topology on the real line because, the intersection of two open intervals is itself an open interval or empty. Infact, this is a base for standard topology on real numbers.

Remark: However, base is not unique many bases, even of different sizes, may generate the same topology for example, the open intervals with rational end points are a base for the standard real topology and the open intervals with irrational end points are also a base for standard topology. But these two sets are completely disjoint and both properly contained in the base of all open intervals.

4.2 New topologies on \mathbb{R}

Now, we introduce new basis on \mathbb{R} as follows:

Definition 4.1 Let S denote the set of all numbers of the form $1/n$ for $n \in Z_+$, let B_- be the collection of all half open intervals $[a, b)$ along with all sets of the form

$[a, b) - \{a + S\}$. The topology generated by B_- will be called *lower limit S-topology* and denoted by \mathbb{R}_l^+ .

If \mathbb{R}_l^+ is generated by unions of all intervals contained in $B_- = \{[a, b) \cup [a, b) - \{a + S\} \cup |a \leq b\}$ then B_- is the base for lower limit S-topology. It must satisfied certain conditions to verify that B_- is indeed a base.

Clearly, the first condition is obvious

$$\mathbb{R} = \bigcup_{a, b \in \mathbb{R}} [a, b) \cup [a, b) - \{a + S\}$$

Second condition :If there exists $x \in B_1 \cap B_2$, we have to prove there exists B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

Proof. Let B_1 and B_2 be elements of B_- , and suppose that $x \in B_1 \cap B_2$. We need to find B_3 such that $x \in B_3 \subset B_1 \cap B_2$. If neither of the sets B_1 and B_2 has the form $[a, b) - \{a + S\}$, they are half open intervals, say $B_1 = [a, b)$ $B_2 = [c, d)$. We can take $p = \max\{a, c\}$ and $q = \min\{b, d\}$ and obtain $x \in [p, q) \subset B_1 \cap B_2$. If $B_1 = [a, b) - \{a + S\}$. We have $x \in (([a, b) - \{a + S\}) \cap ([c, d) - \{c + S\}))$ for $[a, b)$ and $[c, d)$. Then taking p, q as before we obtain $x \in [p, q) - p + S \subset B_1 \cap B_2$. Thus the collection B is basis. □

Definition 4.2 Let S denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, let B_+ be the collection of all half open intervals $(a, b]$ along with all sets of the form $(a, b] - \{b - S\}$. The topology generated by B_+ will be called *upper limit S-topology* and denoted by \mathbb{R}_u^+ .

If \mathbb{R}_u^+ is generated by unions of all intervals contained in $B_+ = \{(a, b] \cup (a, b] - \{b - S\} \cup |a \leq b\}$ then B_+ is the base for upper limit S-topology. It must satisfied certain conditions to verify that B_+ is indeed a base.

Clearly, the first condition is obvious

$$\mathbb{R} = \bigcup_{a, b \in \mathbb{R}} (a, b] \cup (a, b] - \{b - S\}$$

Second condition :If there exists $x \in B_1 \cap B_2$, we have to prove there exists B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

Proof. Let B_1 and B_2 be elements of B_+ , and suppose that $x \in B_1 \cap B_2$. We need to find B_3 such that $x \in B_3 \subset B_1 \cap B_2$. If neither of the sets B_1 and B_2 has the form $(a, b] - \{b - S\}$, they are half open intervals, say $B_1 = (a, b]$ $B_2 = (c, d]$. We can take $p = \max\{a, c\}$ and $q = \min\{b, d\}$ and obtain $x \in (p, q] \subset B_1 \cap B_2$. If $B_1 = (a, b] - \{b - S\}$. We have $x \in ((a, b] - \{b - S\}) \cap ((c, d] - \{d - S\})$ for $(a, b]$ and $(c, d]$. Then taking p, q as before we obtain $x \in [p, q] - p + S \subset B_1 \cap B_2$. Thus the collection B_+ is basis. □

These are the following theorems to illustrate the comparison:

4.3 Comparison of well known topologies on \mathbb{R}

In this section, we have studied the comparison of well known topologies on \mathbb{R} .

Theorem 4.3.1 *The k -topology is strictly finer than the standard topology.*

Proof. Suppose that $U \subset \mathbb{R}$ is open in standard topology on \mathbb{R} . We show that U is also open in \mathbb{R}_k . Suppose that $x \in U$. We can find (a, b) such that $x \in (a, b) \subset U$ by the definition of the standard topology on \mathbb{R} . But $(a, b) \in B_k$ by definition. So that U is open in \mathbb{R}_k . Now the set $(-1, 1) - k$ is open in \mathbb{R}_k , because it is basis element. However, we can see that $(-1, 1) - k$ is not in \mathbb{R} equipped with the standard topology for $0 \in (-1, 1) - k$ and yet any open interval $(-\epsilon, +\epsilon)$ about 0 must intersect k the set $k = \{1/n, n \in \mathbb{N}\}$, for example by taking n such that $1/n < \epsilon$ so the k -topology is finer than the standard topology on \mathbb{R} . □

Theorem 4.3.2 *The topologies of \mathbb{R}_l is strictly finer than the standard topologies on \mathbb{R} .*

Proof. Let T, T' be the topologies of \mathbb{R}, \mathbb{R}_l respectively. Given a basis element (a, b) for T and a point x of (a, b) , the basis element $[x, b)$ for T' contains x and lies in (a, b) . On the other hand, given the basis element $[x, d)$ for T' , there is no open interval (a, b) that contains x and lies in $[x, d)$. Thus T' is strictly finer than T . □

Theorem 4.3.3 *The topologies of \mathbb{R}_u is strictly finer than the standard topologies on \mathbb{R} .*

Proof. Let T, T' be the topologies of \mathbb{R}, \mathbb{R}_u respectively. Given a basis element (a, b) for T and a point x of (a, b) , the basis element $(a, x]$ for T' contains x and lies in (a, b) . On the other hand, given the basis element $(c, x]$ for T' , there is no open interval (a, b) that contains x and lies in $(c, x]$. Thus T' is strictly finer than T . \square

Theorem 4.3.4 *The topologies lower limit and upper limit are not comparable.*

Proof. The topologies T_1 and T_2 on a set X are incomparable iff there exists $A \subset X$ such that $A \in T_1, A \notin T_2$ (which shows that $T_1 \subseteq T_2$ does not hold) and there also exists some $B \subset X$ such that $B \in T_2, B \notin T_1$, which similarly disproves the other inclusion.

For the lower limit topology and upper limit topology on the reals we can indeed take $A = [0, 1)$, which is (basic) open in one but not open in the other (as there is not set of the form $(a, b]$ that contains 0 and is contained in A), and $B = (0, 1]$, with a similar argument regarding instead of 0.

\square

Theorem 4.3.5 *The topologies lower limit and k -topology are not comparable.*

Proof. We claim that $(-1, 1) - k \in \mathbb{R}_k$ but $(-1, 1) - k \notin \mathbb{R}_l$. The fact that $(-1, 1) - k \in \mathbb{R}_k$ follows from the definition of the basis.

Now, suppose there was a basis element $[a, b) \in \mathbb{R}_l$ such that $0 \in [a, b) \subseteq (-1, 1) - k$. Clearly, $b > 0$ otherwise $0 \notin [a, b)$. But, by the definition of archimedean principle there exists some $n \in \mathbb{N}$ such that $1/n < b$ thus $1/n \in [a, b)$ but this contradicts that $[a, b) \subseteq (-1, 1) - k$. It follows that non such $[a, b)$ exists.

We now claim that $[0, 1) \in \mathbb{R}_l$ but $[0, 1) \notin \mathbb{R}_k$. To see this suppose that $(a, b) - k$ is a basic element for \mathbb{R}_k . Such that $0 \in (a, b) - k \subseteq [0, 1)$. Now, clearly $a < 0$ otherwise $0 \notin (a, b) - k$. But, by the definition of archimedean principle there exists some $n \in \mathbb{N}$ such that $a < 1/n < 0$ thus $-1/n \in (a, b)$ and thus $-1/n \in (a, b) - k$ but this contradicts that $(a, b) - k \subseteq [0, 1)$. It follows that non such $(a, b) - k$ exists.

\square

Theorem 4.3.6 *The topology upper limit and k -topology are not comparable*

Proof. To see this we note that $(0, 1]$ is open in the upper limit topology, but any set of the $(a, b) - k$ must have $b > 1$ and so it will entrap points not $(0, 1]$ that not removed by k . Also, using a very argument one can show that $(-1, 1) - k$ is not open in upper limit topology. □

4.4 Comparison of new topologies with well known topologies on \mathbb{R}

In this section, we have presented the comparison of topologies on \mathbb{R} with newly defined topologies on \mathbb{R} .

Theorem 4.4.1 *The lower S-topology is strictly finer than the lower limit topology.*

Proof. Suppose that $U \subset \mathbb{R}_l$ is open in lower limit topology on \mathbb{R} . We show that U is also open in \mathbb{R}_l^+ . Suppose that $x \in U$. We can find $[a, b)$ such that $x \in [a, b) \subset U$ by the definition of the lower limit topology on \mathbb{R} . But $[a, b) \in B_-$ by definition. So that U is open in \mathbb{R}_l^+ . Now the set $[a, b) - a + S$ is open in \mathbb{R}_l^+ , because it is basis element. However, we can see that $[a, b) - \{a + S\}$ is not in R_l equipped with the lower limit topology for $a \in [a, b) - \{a + S\}$ and yet any open interval $(-\epsilon, +\epsilon)$ about a must intersect $\{a + S\}$ the set $S = \{1/n, n \in \mathbb{N}\}$, for example by taking n such that $1/n < \epsilon$ so the lower S-topology is finer than the lower limit topology on \mathbb{R} . □

Theorem 4.4.2 *The upper S-topology is strictly finer than the upper limit topology.*

Proof. Suppose that $U \subset \mathbb{R}_u$ is open in upper limit topology on \mathbb{R} . We show that U is also open in \mathbb{R}_u^+ . Suppose that $x \in U$. We can find $(a, b]$ such that $x \in (a, b] \subset U$ by the definition of the upper limit topology on R . But $(a, b] \in B_+$ by definition. So that U is open in \mathbb{R}_u^+ . Now the set $(a, b] - \{b - S\}$ is open in \mathbb{R}_u^+ , because it is basis element. However, we can see that $(a, b] - \{b - S\}$ is not in \mathbb{R}_u equipped with the lower limit topology for $b \in (a, b] - \{b - S\}$ and yet any open interval $(-\epsilon, +\epsilon)$ about b must intersect $\{b - S\}$ the set $S = \{1/n, n \in \mathbb{N}\}$, for example by taking n

such that $1/n < \epsilon$ so the upper S-topology is finer than the upper limit topology on \mathbb{R} .

□

4.5 Conclusion

In this chapter, we have concluded that newly defined topologies on \mathbb{R} are strictly finer than the standard, lower limit and upper limit topologies. But \mathbb{R}_l^+ and \mathbb{R}_u^+ are not comparable with k-topology.

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