# FUZZY VARIABLE

# LINEAR PROGRAMMING PROBLEMS

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**Submitted by**

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**INDIA**

**DEDICATED**

**TO**

**MY PARENTS, GOD AND MEHAR**

# CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"Fuzzy variable linear programming problems "** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

During the last four decades, a large number of mathematical tools have been developed in Operational Research. For the linear programming problems in the crisp scenario, the aim is to maximize or minimize a linear objective function subject to linear constraints. But in many practical situations, the decision maker may not be in a position to specify the objective and/or constraint functions precisely but rather can specify them in a "fuzzy sense". In such situations , it is desirable to use some fuzzy variable linear programming type of modeling.

This thesis is devoted to solve fuzzy variable linear programming problems. The main topics are fuzzy simplex method , fuzzy dual simplex method and sensitivity analysis for fuzzy variable linear programming problems.

The chapter-wise summary of the thesis is as follows:

**Chapter 1** is introductory in nature. This chapter includes basic definitions, arithmetic operations and concepts used throughout the work. A brief review of the work done in the area of fuzzy variable linear programming problem is also presented.

In **Chapter 2 ,** fuzzy simplex method for solving fuzzy variable linear programming problem , is presented .

In **Chapter 3,** some duality results and fuzzy dual simplex method for solving fuzzy variable linear programming problem , is presented . To illustrate the presented method, a numerical example is solved.

In **Chapter 4,** a method to deal with sensitivity analysis for fuzzy variable linear programming problems is presented . To illustrate this method some numerical examples are solved.

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# CHAPTER 1

# INTRODUCTION AND LITERATURE REVIEW

## 1.1 Introduction

The notion of fuzzy set is widely spread to various fields. The application to mathematical programming has relatively long history. This is the fruit of the continuous efforts of the researchers in that topic. The fuzzy mathematical programming was developed for treating uncertainties in the setting of optimization problems. It treats decision making problem under fuzzy goals and constraints. The fuzzy goals and constraints represent the flexibility of the target values of objective functions and the elasticity of constraints.

The basic definitions used throughout the work are as follows [30]:

**Definition 1.1**

A crisp set or a classical set $A$ is defined as a collection of distinct and distinguishable objects. The objects are called elements of $A$ . A crisp set $A$ , defined on the universal set $X$ , can also be represented by $A = \{(x, \mu_A(x)); x \in X\}$ where $\mu_A : X \to \{0, 1\}$ is called characteristic function defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \text{ ;} \\ 0, & \text{otherwise .} \end{cases}$$

**Definition 1.2**

The characteristic function $\mu_A$ of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in $X$. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set $X$ fall within a specific range $[0,1]$ i.e., $\mu_{\tilde{A}} : X \rightarrow [0,1]$. The assigned values indicate the membership grade of the element in the set $A$.

The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set. $\mu_{\tilde{A}}(x)$ is the degree of membership of $x$ in $\tilde{A}$ .

**Definition 1.3**

Let $\widetilde{A}$ be a fuzzy set and $\alpha$ be real number in the interval $[0\ ,\ 1]$ . The crisp set $A_\alpha$ defined by $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ is called $\alpha$ - cut of $\tilde{A}$ .

**Definition 1.4**

A fuzzy set $\tilde{A}$ , defined on the universal set of real numbers $R$ , is said to be a fuzzy number if its membership function has the following characteristics:

1) $\mu_{\tilde{A}} : R \rightarrow [0,1]$ is continuous.
2) $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, c] \cup [d, \infty)$.
3) Is strictly increasing on $[c,a]$ and strictly decreasing on $[b,d]$ .
4) $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a,b]$ .

**Definition 1.5**

The support of a fuzzy set $\tilde{A}$ is the crisp subset of $X$ and is represented as :

$$Supp(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) > 0\}.$$

## 1.2 Arithmetic operations on fuzzy numbers

Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers then

$$\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta)$$

$$\tilde{a} - \tilde{b} = (a^L - b^U, a^U - b^L, \alpha + \theta, \beta + \gamma)$$

$$x \geq 0,\ x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta)$$

$$x \leq 0,\ x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha)$$

## 1.3 Ranking function

A convenient method for comparing of the fuzzy numbers is by use of ranking function . A ranking function defined as $\Re : F(R) \to R$ maps each fuzzy number into a real line. If $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ are two trapezoidal fuzzy numbers then ,

$$\tilde{a} \succeq \tilde{b} \text{ iff } \Re(\tilde{a}) \geqslant \Re(\tilde{b})$$

$$\tilde{a} \succ \tilde{b} \text{ iff } \Re(\tilde{a}) > \Re(\tilde{b})$$

$$\tilde{a} \approx \tilde{b} \text{ iff } \Re(\tilde{a}) = \Re(\tilde{b})$$

where $\Re(\tilde{a}) = a^L + a^U + \frac{1}{2}(\beta - \alpha)$ and $\Re(\tilde{b}) = b^L + b^U + \frac{1}{2}(\theta - \gamma)$ .

**Lemma 1.1** Let $\Re$ be any linear ranking function . Then

i)$\tilde{a} \succeq \tilde{b}$ iff $\tilde{a} - \tilde{b} \succeq 0$ iff $-\tilde{b} \succeq -\tilde{a}$

ii) If $\tilde{a} \succeq \tilde{b}$ and $\tilde{c} \succeq \tilde{d}$ , then $\tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{d}$.

## 1.4 Literature review

The concept of fuzzy mathematical programming on a general level was first proposed by Tanaka et al. [54] in the framework of the fuzzy decision of Bellman and Zadeh [6]. The first formulation of fuzzy linear programming (FLP) is proposed by Zimmermann [67]. Afterwards, many authors have considered various kinds of FLP problems and have proposed several approaches for solving these problems [15-19,35-39,44,47].

In addition, some authors have used the concept of comparison of fuzzy numbers to solve fuzzy linear programming problems. In fact, most convenient methods are based on the concept of comparison of fuzzy numbers by using linear ranking functions [18,21,35-39].

Of course , linear ranking functions have been proposed by researchers to suit their requirements of the problem under consideration and conceivably there are no generally accepted criteria for application of ranking functions. Nevertheless, usually in such situations authors define a crisp model which is equivalent to an FLP problem and then use optimal solution of the model as the optimal solution of the FLP problem. Based on this idea Maleki et al. [39] proposed a new method for solving the fuzzy variable linear programming (FVLP) problem. Ebrahimnejad et al. [18] gave another efficient method namely primaldual simplex algorithm to obtain a fuzzy solution of FVLP problems. Also, Nasseri and Ebrahimnejad [44] applied a fuzzy primal simplex method [37] to solve flexible linear programming problems directly without solving any auxiliary problem. Moreover, Ebrahimnejad and Nasseri [15] used the complementary slackness to solve FVLP problems without the need of a simplex table.

In published works on fuzzy linear programming there are only a few papers dealing with stability or sensitivity analysis in fuzzy mathematical programming. Sensitivity analysis in FLP problems (with crisp parameters and soft constraints) was first considered by Hamacher et al. [26], where a functional relationship between changes of parameters of the right-hand side and those of the optimal value of the primal objective function was derived for almost all conceivable cases. Sensitivity analysis for fuzzy linear fractional programming problems was studied by Dutta et al. [13]. Tanaka et al. [55] have discussed the value of information in an FLP problem (with symmetrical triangular fuzzy numbers) via sensitivity analysis. Fuller [22] investigated the stability of the solution of FLP problems with respect to changes of centers of fuzzy parameters. He showed that the solution to these problems is stable under variations in the membership function of the fuzzy coefficients.

Ebrahimnejad [14] generalized the concept of sensitivity analysis on the parameters of the crisp linear programming [2] to the fuzzy number linear programming and showed that the fuzzy primal simplex algorithm stated in [37] and the fuzzy dual simplex algorithm presented in [43] is useful for post optimality analysis on linear programming problems with fuzzy numbers.

# CHAPTER 2

# FUZZY SIMPLEX METHOD FOR SOLVING FUZZY VARIABLE LINEAR PROGRAMMING PROBLEMS

In this chapter , the fuzzy simplex method for solving fuzzy variable linear programming problems is presented .

## 2.1 Fuzzy variable linear programming problems

A fuzzy variable linear programming problem is defined as follows:

$$\begin{aligned}
&min \quad \widetilde{z} \approx c\widetilde{x} \\
&subject\ to \\
&\qquad A\widetilde{x} \approx \widetilde{b}, \\
&\qquad \widetilde{x} \succeq \widetilde{0},
\end{aligned}$$

$$Or$$

$$\begin{aligned}
&max\ \widetilde{z} \approx c\widetilde{x} \\
&subject\ to \\
&\qquad A\widetilde{x} \approx \widetilde{b}, \\
&\qquad \widetilde{x} \succeq \widetilde{0},
\end{aligned} \tag{2.1}$$

where , $\widetilde{b} \in (F(R))^m$, $\widetilde{x} \in (F(R))^n, A \in R^{m\times n}, c^T \in R^n$ .

**Definition 2.1**

A fuzzy vector $\widetilde{x} \in (F(R))^n$ is a fuzzy feasible solution to (2.1) iff $\widetilde{x}$ satisfies the constraints of the problem.

**Definition 2.2**

A fuzzy feasible solution $\widetilde{x_*}$ is a fuzzy optimal solution for (2.1), if for all fuzzy feasible solution $\widetilde{x}$ for (2.1), $c\widetilde{x_*} \succeq c\widetilde{x}$.

**2.2 Fuzzy basic feasible solution**

Consider the system $A\widetilde{x} \approx \widetilde{b}$ and $\widetilde{x} \succeq 0$. Let $A = [a_{ij}]_{m\times n}$. Assume $rank(A) = m$. Partition $A$ as $[B \;\; N]$ where $B$ , $m \times m$ is nonsingular . It is obvious that $rank(B) = m$ . Let $y_j$ be the solution to $By = a_j$. It is apparent that the basic solution.

$$\widetilde{x}_B \approx (\widetilde{x}_{B1}, ..., \widetilde{x}_{Bm})^T \approx B^{-1}\widetilde{b}\, ,\; \widetilde{x}_N \approx 0 \tag{2.2}$$

is a solution of $A\widetilde{x} \approx \widetilde{b}$ . We call $\widetilde{x}$ , accordingly partitioned as $(\widetilde{x}_B^T \;\; \widetilde{x}_N^T)^T$ , a fuzzy basic solution corresponding to the basis $B$. If $\widetilde{b}_B \succeq 0$ , then the fuzzy basic solution is feasible and the corresponding fuzzy objective value is $\widetilde{z} \approx c_B\widetilde{x}_B$ , where , $c_B = (c_{B_1}, ..., c_{B_m})$. Now , corresponding to every fuzzy nonbasic variable $\widetilde{x}_j, 1 \leqslant j \leqslant n, j \neq B_i$, and $i = 1, ..., m$, define

$$z_j = c_B y_j = c_B B^{-1} a_j$$

If $\widetilde{x}_B \succ 0$, then $\widetilde{x}$ is called a nondegenerate fuzzy basic feasible solution and if at least one component of $\widetilde{x}_B$ is zero, then $\widetilde{x}$ is called a degenerate fuzzy basic feasible solution.

**Theorem 2.1**

If a fuzzy variable linear programming problem is nondegenerate then a fuzzy basic feasible solution $\widetilde{x}_B \approx B^{-1}\widetilde{b}$ , $\widetilde{x}_N \approx 0$ is optimal if and only if $z_j = c_B B^{-1} a_j \leqslant c_j$, for minimization ( $z_j \geqslant c_j$, for maximization) for all $1 \leq j \leq n$ .

**Proof:** Suppose $\widetilde{x_*} \approx (\widetilde{x}_B^T \quad \widetilde{x}_N^T)^T$ is a fuzzy basic feasible solution to (2.1) (minimization) , where $\widetilde{x}_B = B^{-1}\widetilde{b}, \widetilde{x}_N = \widetilde{0}$ . Then the corresponding fuzzy objective value is

$$\widetilde{z_*} \approx c\widetilde{x_*} \approx c_B\widetilde{x}_B \approx c_B B^{-1}\widetilde{b}.$$

On the other hand , for any fuzzy basic feasible solution $\widetilde{x}$ to (2.1)

$$\widetilde{b} \approx A\widetilde{x} \approx B\widetilde{x}_B + N\widetilde{x}_N$$

Hence , this can be rewritten as follows :

$$\widetilde{x}_B \approx B^{-1}\widetilde{b} - B^{-1}N\widetilde{x}_N$$

Then, for any basic feasible solution of (2.1) ,

$$\widetilde{z} \approx c\widetilde{x} \approx c_B\widetilde{x}_B + c_N\widetilde{x}_N \approx c_B B^{-1}\widetilde{b} - (c_B B^{-1}N - c_N)\widetilde{x}_N \ .$$

$$\approx c_B B^{-1}\widetilde{b} - \sum_{j=1}^{n}(c_B B^{-1}a_j - c_j)\widetilde{x}_j \approx c_B B^{-1}\widetilde{b} - \sum_{j=1}^{n}(z_j - c_j)\widetilde{x}_j.$$

Hence ,

$$\widetilde{z} \approx \widetilde{z}_* - \sum_{j \neq B_i}(z_j - c_j)\widetilde{x}_j.$$

Now, if for all $j$ , $1 \leqslant j \leqslant n$ $z_j \leqslant c_j$ , then from feasibility of $\widetilde{x}$ , $(z_j - c_j)\widetilde{x}_j \preceq \widetilde{0}$, and then $\sum_{j \neq B_i}(z_j - c_j)\widetilde{x}_j \preceq \widetilde{0}$ . Therefore it follows that $\widetilde{z} \succeq \widetilde{z}_*$ and so $\widetilde{x}_*$ is optimal. For "*only if*" part , let $\widetilde{x}_*$ be a fuzzy optimal basic feasible solution to (2.1) . For $j = B_i$ , $1 \leqslant i \leqslant m$, we know that $z_j - c_j = 0$. From above , it is obvious that if for any nonbasic variable $\widetilde{x}_j$ we have $z_j > c_j$ , then we can enter $\widetilde{x}_j$ into the basis and obtain $\widetilde{z}_* \succ \widetilde{z}$ (because the problem is nondegenerate and $\widetilde{x}_j \succ \widetilde{0}$ in the new basis). This is a contradiction to $\widetilde{z}_*$ being optimal . Hence , $z_j \leqslant c_j, 1 \leqslant j \leqslant n$.

### 2.3 Optimal solution of fuzzy variable linear programming problems

Consider the fuzzy variable linear programming problem as is defined in (2.1).

$$max\ \widetilde{z} \approx c_B\widetilde{x}_B + c_N\widetilde{x}_N$$

$$subject\ to$$

$$B\widetilde{x}_B + N\widetilde{x}_N \approx \widetilde{b},$$

$$\widetilde{x}_B, \widetilde{x}_N \succeq 0,$$

Then , it is possible to write $\widetilde{x}_B \approx B^{-1}\widetilde{b} - B^{-1}N\widetilde{x}_N$ and $\widetilde{z} \approx c_B(B^{-1}\widetilde{b} - B^{-1}N\widetilde{x}_N) + c_N\widetilde{x}_N$. Also, it can be written as $\widetilde{x}_B + B^{-1}N\widetilde{x}_N \approx B^{-1}\widetilde{b}$ , and also for objective function $\widetilde{z} + (c_BB^{-1}N - c_N)\widetilde{x}_N \approx c_BB^{-1}\widetilde{b}$.

Currently $\widetilde{x}_N \approx 0$ so $\widetilde{x}_B \approx B^{-1}\widetilde{b}$ , and $\widetilde{z} \approx c_BB^{-1}\widetilde{b}$ . And the above fuzzy variable linear programming problem can be rewritten in the following table format:

**Table 2.1**

**Fuzzy simplex table**

| | $\widetilde{z}$ | $\widetilde{x}_B$ | $\widetilde{x}_N$ | R.H.S. |
|---|---|---|---|---|
| $\widetilde{z}$ | 1 | 0 | $c_BB^{-1}N - c_N$ | $c_BB^{-1}\widetilde{b}$ |
| $\widetilde{x}_B$ | 0 | 1 | $B^{-1}N$ | $B^{-1}\widetilde{b}$ |

The Table (2.1) gives all the information that is needed to proceed with the simplex method . The cost row in the above table is $(\gamma_j)_{j\neq B_i} = (c_BB^{-1}a_j - c_j)_{j\neq B_i} = (z_j - c_j)_{j\neq B_i}$. According to the optimality condition for these problems if $\gamma_j \geq 0$ for all $j \neq B_i$ then the solution is optimal . On the other hand , if $\gamma_l < 0$ , for a $\ell \neq B_i$

then $\widetilde{x}_{B_r}$ can be exchanged with $\widetilde{x}_l$ . Then compute the vector $y_l = B^{-1}a_l$ . If $y_l \leq 0$ , then $\widetilde{x}_l$ can be increased indefinitely and then the optimal objective is unbounded. On the other hand , if $y_l$ has at least one positive component , then the increase in will be blocked by one of the current basic variables, which drops to zero.

### 2.3.1 Pivoting

If $\widetilde{x}_l$ enters the basis and $\widetilde{x}_{B_r}$ leaves the basis then pivoting on $y_{rl}$ can be stated as follows:

1) Divide row $r$ by $y_{rl}$.

2) For $i = 1, ..., m$ and $i \neq r$ update the $i^{th}$ row by adding to it $-y_{il}$ times the new $r^{th}$ row.

3) Update row zero by adding to it $\gamma_l$ times the new $r^{th}$ row.

**Theorem 2.2**

If in a fuzzy simplex table, an $l$ exists such that $z_l - c_l < 0$ and there exists a basic index $i$ such that $y_{il} > 0$ , then a pivoting row $r$ can be found so that pivoting on $y_{rl}$ will yield a fuzzy feasible table with a corresponding nondecreasing objective value.

**Proof:** A criterion is needed for choosing a fuzzy basic variable to leave the basis so that the new simplex table will remain feasible and the new objective value is nondecreasing. Assume column $l$ is the pivot column. Also, suppose that $\widetilde{x} \approx (\widetilde{x}_B^T \ \widetilde{x}_N^T)^T$ is a fuzzy basic feasible solution to the fuzzy variable linear programming problem , where $\widetilde{x}_B \approx B^{-1}\widetilde{b}$ and $\widetilde{x}_N \approx 0$. Then , the corresponding fuzzy objective value is $\widetilde{z} \approx c_B B^{-1}\widetilde{b} \approx c_B \widetilde{y}_0$.

On the other hand , for any fuzzy basic feasible solution to the fuzzy variable linear programming problem

$$\widetilde{x}_B + \sum_{j \neq B_i} y_j \widetilde{x}_j \approx \widetilde{y}_0$$

where , $y_j = B^{-1} a_j$.

So, if $\widetilde{x}_l$ enters into the basis it can be written as

$$\widetilde{x}_B \approx \widetilde{y}_0 - y_l \widetilde{x}_l$$

Since , we want $\widetilde{x}_B$ be feasible , hence $\widetilde{y}_{i0} - y_{il}\widetilde{x}_l \succeq 0$, for all $i = 1, ..., m$.

If $y_{il} \leqslant 0$ , then it is obvious that the above condition will hold.

Hence , for all $y_{il} > 0$ , it must be needed that

$$\widetilde{x}_l \preceq \frac{\widetilde{y}_{i0}}{y_{il}}$$

To satisfy the definition of linear ranking function it is sufficient to let

$$\frac{\widetilde{y}_{r0}}{y_{rl}} \approx min\{\frac{\widetilde{y}_{i0}}{y_{il}} | y_{il} > 0\}$$

Also , for any fuzzy basic feasible solution to the fuzzy variable linear programming problem

$$\widetilde{z} \approx c_B \widetilde{y}_0 - \sum_{j \neq B_i} (z_j - c_j) \widetilde{x}_j$$

So, if $\widetilde{x}_l$ is allowed to enter into the basis , then

$$\widetilde{z} \approx c_B \widetilde{y}_0 - (z_l - c_l)\widetilde{x}_l$$

Now , the new objective value is nondecreasing , since

$$\widetilde{z} \approx c_B \widetilde{y}_0 - (z_l - c_l)\widetilde{x}_l \succeq c_B \widetilde{y}_0$$

Using the fact that $(z_l - c_l)\widetilde{x}_l \preceq 0$.

**Theorem 2.3**

If for any fuzzy basic feasible solution to the fuzzy variable linear programming problem there is some column not in basis for which $z_l - c_l < 0$ and $y_{il} \leqslant 0$, $i = 1, ..., m$, then the fuzzy variable linear programming problem has an unbounded solution.

**Proof:** Suppose that $\widetilde{x}_B$ is a fuzzy basic solution to the fuzzy variable linear programming problem , so

$$\widetilde{x}_{B_i} + \sum_{j \neq B_i} y_{ij} \widetilde{x}_j \approx \widetilde{y}_{i0}, i = 1, ..., m, j = 1, ..., n,$$

$$or$$

$$\widetilde{x}_{B_i} \approx \widetilde{y}_{i0} - \sum_{j \neq B_i} y_{ij} \widetilde{x}_j, i = 1, ..., m, j = 1, ..., n,$$

Now , if $\widetilde{x}_l$ enter into the basis , then $\widetilde{x}_l \succ 0$ and $\widetilde{x}_j \approx 0$ , for all $j \neq B_i \cup l$. Since $y_{il} \leqslant 0, i = 1, ..., m$, hence

$$\widetilde{y}_{i0} - y_{il} \widetilde{x}_l \succeq 0$$

Therefore , the current fuzzy basic solution will remain feasible. Now , the value of $\hat{z}$ for the above fuzzy feasible solution is as following:

$$\begin{aligned}
\hat{z} &\approx c_B \widetilde{x}_B + c_N \widetilde{x}_N \\
&\approx \sum_{i=1}^{m} c_{B_i} (\widetilde{y}_{i0} - y_{il} \widetilde{x}_l) + c_l \widetilde{x}_l \\
&\approx \sum_{i=1}^{m} c_{B_i} \widetilde{y}_{i0} - (\sum_{i=1}^{m} c_{B_i} y_{il} - c_l) \widetilde{x}_l \\
&\approx c_B \widetilde{y}_0 - (c_B \widetilde{y}_l - c_l) \widetilde{x}_l \\
&\approx \widetilde{z} - (z_l - c_l) \widetilde{x}_l.
\end{aligned}$$

So,

$$\hat{z} \approx \widetilde{z} - (z_l - c_l)\widetilde{x}_l. \tag{2.3}$$

Hence, $\widetilde{x}_l$ can be entered into the basis with arbitrarily large fuzzy value. Then , from (2.3) , there is unbounded solution.

### 2.3.2 Fuzzy simplex method

Suppose that a basic feasible solution with basis $B$ is given. Then:

**Step 1:** The basic feasible solution is given by $\widetilde{x}_B \approx B^{-1}\widetilde{b} \approx \widetilde{y}_0$ and $\widetilde{x}_N \approx 0$. The fuzzy objective $\widetilde{z} \approx c_B B^{-1}\widetilde{b} \approx c_B\widetilde{y}_0$.

**Step 2:** Calculate $w = c_B B^{-1}$ and $y_0 = \Re(\widetilde{y}_0)$. For each nonbasic variable , calculate $\gamma_j = z_j - c_j = c_B B^{-1}a_j - cj = wa_j - c_j$. Let $\gamma_l = min_j\{\gamma_j\}$. If $\gamma_l \geqslant 0$, then stop ; the current solution is optimal. Otherwise go to Step 3.

**Step 3:** Calculate $y_l = B^{-1}a_l$. If $y_l \leqslant 0$, then stop ; the optimal solution is unbounded. Otherwise determine the index of the variable $\widetilde{x}_{B_r}$ leaving the basis as follows :

$$\frac{y_{r0}}{y_{rl}} = min\{\frac{y_{i0}}{y_{il}} \;\mid\; y_{il} > 0\}$$

Update $\widetilde{y}_{i0}$ by replacing $\widetilde{y}_{i0} - \frac{\widetilde{y}_{r0}}{y_{rl}}y_{il}$ for $i \neq r$ and $\widetilde{y}_{r0}$ by replacing $\frac{\widetilde{y}_{r0}}{y_{rl}}$ . Also , update $\widetilde{z}$ by replacing $\widetilde{z} - \frac{\widetilde{y}_{r0}}{y_{rl}}(z_l - c_l)$ . Then , update $B$ by replacing $a_{B_r}$ with $a_l$ and go to Step 2.

### 2.3.3 Illustrative example

For an illustration of the fuzzy simplex method a fuzzy variable linear programming problem is solved by use of fuzzy simplex method.

**Example 2.1**

$$max\ \widetilde{z} \approx 3\widetilde{x_1} + 4\widetilde{x_2}$$

$$subject\ to$$

$$3\widetilde{x_1} + \widetilde{x_2} \preceq (2,4,1,3)$$

$$2\widetilde{x_1} - 3\widetilde{x_2} \preceq (3,5,2,1)$$

$$\widetilde{x_1}, \widetilde{x_2} \succeq 0$$

**Solution:** The chosen problem can be written as

$$3\widetilde{x_1} + \widetilde{x_2} + \widetilde{x_3} \approx (2,4,1,3)$$

$$2\widetilde{x_1} - 3\widetilde{x_2} + \widetilde{x_4} \approx (3,5,2,1)$$

$$\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{x_4} \succeq 0$$

The first table is as follows:

| Basis | $\widetilde{x_1}$ | $\widetilde{x_2}$ | $\widetilde{x_3}$ | $\widetilde{x_4}$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\widetilde{z}$ | $-3$ | $-4$ | 0 | 0 | $\widetilde{0}$ | 0 |
| $\widetilde{x_3}$ | 3 | 1 | 1 | 0 | (2,4,1,3 ) | 7 |
| $\widetilde{x_4}$ | 2 | $-3$ | 0 | 1 | (3,5,2,1) | 7.5 |

Now , the obtained values are $\gamma_1 = z_1 - c_1 = -3 < 0$ and $\gamma_2 = z_2 - c_2 = -4 < 0$. Then, $\gamma_2 < \gamma_1$. Hence , related fuzzy nonbasic variable to $\gamma_2$, that is $\widetilde{x_2}$ is an entering variable. Therefore , according to the minimum ratio test given in Step 3 of fuzzy simplex algorithm , $\widetilde{x_3}$ is a leaving variable. Now , after pivoting the new table is:

| Basis | $\widetilde{x}_1$ | $\widetilde{x}_2$ | $\widetilde{x}_3$ | $\widetilde{x}_4$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\widetilde{z}$ | 9 | 0 | 4 | 0 | (8,16,4,12) | 28 |
| $\widetilde{x}_2$ | 3 | 1 | 1 | 0 | (2,4,1,3 ) | 7 |
| $\widetilde{x}_4$ | 11 | 0 | 3 | 1 | (9,17,5,10) | 28.5 |

According to the above table , for fuzzy nonbasic variables $\widetilde{x}_1, \widetilde{x}_3$ , $\gamma_1 = 9 > 0, \gamma_3 = 4 > 0$. Hence , using the optimality condition for the fuzzy variable linear programming problems , the optimal fuzzy solution is obtained i.e. $\widetilde{x}_1{}^* \approx (0,0,0,0), \widetilde{x}_2{}^* \approx (2,4,1,3), \widetilde{x}_3{}^* \approx (0,0,0,0), \widetilde{x}_4{}^* \approx (9,17,5,10)$ and $\widetilde{z} \approx (8,16,4,12)$ with $\Re(\widetilde{z}) = 28$.

## 2.4 Conclusion

In this chapter , the fuzzy simplex method for solving fuzzy variable linear programming problems is presented and the presented method is illustrated by solving a fuzzy variable linear programming problem .

# CHAPTER 3

# DUALITY RESULTS AND FUZZY DUAL SIMPLEX METHOD FOR FUZZY VARIABLE LINEAR PROGRAMMING PROBLEMS

In this chapter the dual problem of the linear programming problem with fuzzy variables and some duality results are presented. Also , the fuzzy dual simplex method for solving fuzzy variable linear programming problems is presented. The presented method is illustrated by solving a fuzzy variable linear programming problem.

## 3.1 Formulation of the dual problem

Consider the following fuzzy variable linear programming problem below:

$$\begin{aligned}
&min\ \tilde{z} \approx c\tilde{x} \\
&subject\ to \\
&\quad A\tilde{x} \preceq \tilde{b}, \\
&\quad \tilde{x} \succeq \tilde{0}, \\
&\qquad or \\
&min\ \tilde{z} \approx \sum_{j=1}^{n} c_j \tilde{x}_j, \\
&subject\ to \\
&\quad \sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq \tilde{b}_i, \quad i = 1, ..., m, \\
&\quad \tilde{x}_j \succeq \tilde{0}, \qquad j = 1, 2, ..., n,
\end{aligned} \tag{3.1}$$

where $\tilde{x} = (\tilde{x_1}, ..., \tilde{x_n})^T \in (F(R))^n, \tilde{b} = (\tilde{b_1}, ..., \tilde{b_m})^T \in (F(R))^m, A = [a_{ij}]_{m \times n}, c = (c_1, ..., c_n)^T \in R^n, \tilde{b_i} = (b_i^L, b_i^U, b_i^\alpha, b_i^\beta) \in F((R)), \tilde{x_j} = (x_j^L, x_j^U, x_j^\alpha, x_j^\beta) \in F((R))$.

It is obvious that problem (3.1) is equivalent to problem (3.2) given below:

$$\begin{aligned}
&min\ z = \Re(c\tilde{x}) \\
&subject\ to \\
&\qquad \Re(A\tilde{x}) = \Re(\tilde{b}), \\
&\qquad \Re(\tilde{x}) \geqslant 0,
\end{aligned} \tag{3.2}$$

where , $\Re(\tilde{x}) = (\Re(\tilde{x_1}), ..., \Re(\tilde{x_n}))^T \in R^n, \Re(\tilde{b}) = (\Re(\tilde{b_1}), ..., \Re(\tilde{b_m}))^T \in R^m$.

In the above notations , $\Re(\tilde{x_j}) = c_L x_j^L + c_U x_j^U + c_\alpha x_j^\alpha + c_\beta x_j^\beta, \quad \Re(\tilde{b_i}) = c_L b_i^L + c_U b_i^U + c_\alpha b_i^\alpha + c_\beta b_i^\beta$, where , $c_L, c_U, c_\alpha, c_\beta$ are constants , at least one of which is nonzero.

**Theorem 3.1**

Problems (3.1) and (3.2) are equivalent to the following problem:

$$\begin{aligned}
&min\ z = c\Re(\tilde{x}) \\
&subject\ to \\
&\qquad A\Re(\tilde{x}) \leqslant \Re(\tilde{b}), \\
&\qquad \Re(\tilde{x}) \geqslant 0.
\end{aligned} \tag{3.3}$$

**Proof.** Since , $\Re$ is a linear ranking function , so ,

$$\begin{aligned}
\Re(c\tilde{x}) = \Re(\sum_{j=1}^{n} c_j \tilde{x_j}) &= \sum_{j=1}^{n} \Re(c_j \tilde{x_j}) \\
&= \sum_{j=1}^{n} c_j \Re(\tilde{x_j}) = c\Re(\tilde{x}),
\end{aligned}$$

where , $\Re(\tilde{x}) = (\Re(\tilde{x_1}), ..., \Re(\tilde{x_n}))^T \in R^n$.

On the other hand ,

$$\Re(\sum_{j=1}^{n} a_{ij}\tilde{x_j}) = \sum_{j=1}^{n}\Re(a_{ij}\tilde{x_j}) = \sum_{j=1}^{n} a_{ij}\Re(\tilde{x_j}).$$

So , if we denote the $i^{th}$ row of matrix $A$ by $\bar{a}_i$, we have

$$\begin{aligned}
\Re(A\tilde{x}) &= (\Re(\bar{a_1}\tilde{x}), ..., \Re(\bar{a_m}\tilde{x}))^T \\
&= (\Re(\sum_{j=1}^{n} a_{1j}\tilde{x_j}), ..., \Re(\sum_{j=1}^{n} a_{mj}\tilde{x_j}))^T \\
&= (\sum_{j=1}^{n} a_{1j}\Re(\tilde{x_j}), ..., \sum_{j=1}^{n} a_{mj}\Re(\tilde{x_j}))^T \\
&= (\bar{a_1}\Re(\tilde{x}), ..., \bar{a_m}\Re(\tilde{x})^T = A\Re(\tilde{x}).
\end{aligned}$$

For any feasible $\tilde{x} \in (F(R))^n$ for (3.1)

$$\sum_{j=1}^{n} a_{ij}\Re(\tilde{x_j}) \leqslant \Re(\tilde{b_i}), \qquad i = 1, 2, ..., m,$$

$$\Re(\tilde{x_j}) \geqslant 0, \qquad\qquad j = 1, 2, ..., n,$$

or , using a linear ranking function ,

$$\sum_{j=1}^{n} a_{ij}(c_L x_j^L + c_U x_j^U + c_\alpha x_j^\alpha + c_\beta x_j^\beta) \leqslant c_L b_i^L + c_U b_i^U + c_\alpha b_i^\alpha + c_\beta b_i^\beta \qquad i = 1, 2, ..., m,$$

$$c_L x_j^L + c_U x_j^U + c_\alpha x_j^\alpha + c_\beta x_j^\beta \leqslant 0, \quad j = 1, 2, ..., n. \tag{3.4}$$

Now , for any optimal solution $\tilde{x^*}$ for (3.1)

$$c\tilde{x^*} \preceq c\tilde{x}$$

for any feasible solution $\tilde{x}$ for (3.1) , or,

$$\Re(c\tilde{x^*}) = c\Re(\tilde{x^*}) \leqslant \Re(c\tilde{x}) = c\Re(\tilde{x}),$$

*or*

$$\sum_{j=1}^{n} c_j \Re(\tilde{x_j^*}) \leqslant \sum_{j=1}^{n} c_j \Re(\tilde{x_j^*}),$$

or, using linear ranking function ,

$$\sum_{j=1}^{n} c_j (c_L x_j^{*L} + c_U x_j^{*U} + c_\alpha x_j^{*\alpha} + c_\beta x_j^{*\beta})$$

$$\leqslant \sum_{j=1}^{n} c_j (c_L x_j^{L} + c_U x_j^{U} + c_\alpha x_j^{\alpha} + c_\beta x_j^{\beta}) \tag{3.5}$$

Therefore , $\tilde{x^*} = (\tilde{x_1^*}, ..., \tilde{x_n^*})^T$, where $\tilde{x_j^*} = (x_j^{*L}, x_j^{*U}, x_j^{*\alpha}, x_j^{*\beta})^T \in F(R)$, is optimal for (3.1) , if and only if $x^* = (x_1^{*L}, x_1^{*U}, x_1^{*\alpha}, x_1^{*\beta}, ..., x_n^{*L}, x_n^{*U}, x_n^{*\alpha}, x_n^{*\beta})^T \in R^{4n}$ is an optimal solution for (3.2).

**Theorem 3.2**

The dual of (3.1) is defined to be:

$$\begin{aligned} &max\ \tilde{u} \approx y\tilde{b} \\ &subject\ to \\ &\qquad yA \leqslant c, \\ &\qquad y \leqslant 0, \end{aligned} \tag{3.6}$$

Now, shown that the dual of (3.1) is indeed (3.6) , using the usual definition of the dual in linear programming . Using Theorem (3.1) and applying a linear ranking function , (3.1) can be written as

$$min\ z = \sum_{j=1}^{n} c_j (c_L x_j^{L} + c_U x_j^{U} + c_\alpha x_j^{\alpha} + c_\beta x_j^{\beta}),$$

*subject to*

$$\sum_{j=1}^{n} a_{ij} (c_L x_j^{L} + c_U x_j^{U} + c_\alpha x_j^{\alpha} + c_\beta x_j^{\beta}) \leqslant \Re(\tilde{b_i}), \qquad i = 1, 2, ..., m,$$

$$c_L x_j^{L} + c_U x_j^{U} + c_\alpha x_j^{\alpha} + c_\beta x_j^{\beta} \geqslant 0, \qquad j = 1, 2, ..., n \tag{3.7}$$

The dual of (3.7) is

$$max\ u = \sum_{i=1}^{m} y_i(c_L b_i^L + c_U b_i^U + c_\alpha b_i^\alpha + c_\beta b_i^\beta) = \sum_{i=1}^{m} y_i \Re(\tilde{b_i}),$$

*subject to*

$$\sum_{i=1}^{m} a_{ij} c_L y_i + c_L y_{m+1} = c_L c_j,$$

$$\sum_{i=1}^{m} a_{ij} c_U y_i + c_U y_{m+1} = c_U c_j,$$

$$\sum_{i=1}^{m} a_{ij} c_\alpha y_i + c_\alpha y_{m+1} = c_\alpha c_j,$$

$$\sum_{i=1}^{m} a_{ij} c_\beta y_i + c_\beta y_{m+1} = c_\beta c_j, \qquad j = 1, 2, ..., n$$

$$y_1, ..., y_m \leqslant 0, \quad y_{m+1} \geqslant 0. \tag{3.8}$$

If any of $c_L, c_U, c_\alpha, c_\beta$ is equal to zero then the corresponding constraint is not present , and since at least one of the values $c_L, c_U, c_\alpha, c_\beta$ is nonzero , the corresponding constraints reduce to $\sum_{i=1}^{m} a_{ij} y_i + y_{m+1} = c_j$ , for $j = 1, 2, ..., n$. Then (3.8) can be written as

$$max\ u = \sum_{i=1}^{m} y_i \Re(\tilde{b_i}),$$

*subject to*

$$\sum_{i=1}^{m} y_i a_{ij} \leqslant c_j, \qquad j = 1, 2, ..., n,$$

$$y_i \leqslant 0, \qquad i = 1, 2, ..., m \tag{3.9}$$

or, using matrix notations,

$$max\ u = y\Re(\tilde{b_i}),$$

*subject to*

$$yA \leqslant c,$$

$$y \leqslant 0, \tag{3.10}$$

Since problems (3.6) and (3.10) are equivalent , then problem (3.6) is the dual of (3.1) in the usual definition of the dual in linear programming.

**Lemma 3.1**

The dual of the problem (3.6) is the problem (3.1).

**Proof.** Problems (3.6) and (3.1) are , respectively , equivalent to problems (3.9) and (3.7) . Knowing from linear programming that the dual of (3.9) is (3.7) , it follows that the dual of (3.6) is (3.1).

**Theorem 3.3**

If $\tilde{x_0}$ and $w_0$ are feasible solutions to fuzzy variable linear programming problem and its dual , then $c\tilde{x_0} \succeq w_0\tilde{b}$.

**Proof.** Multiplying $A\tilde{x_0} \succeq \tilde{b}$ on the left by $w_0 \leqslant 0$ and $w_0 A \leqslant c$ on the right by $\tilde{x_0} \succeq \tilde{0}$ . Thus , $c\tilde{x_0} \succeq w_0 A\tilde{x_0} \succeq w_0\tilde{b}$.

**Corollary 3.1**

If $\tilde{x_0}$ and $w_0$ are feasible solutions to fuzzy variable linear programming problem and its dual and $c\tilde{x_0} \approx w_0\tilde{b}$ , then $\tilde{x_0}$ and $w_0$ are optimal solutions to their respective problems.

**Proof.** Theorem 3.3 indicates that $c\tilde{x} \succeq w_0\tilde{b} \approx c\tilde{x_0}$ , for each feasible solution $\widetilde{x}$ of fuzzy variable linear programming problem . Thus , $\tilde{x_0}$ is an optimal solution to the fuzzy variable linear programming problem.

Also , $w\tilde{b} \preceq c\tilde{x_0} \approx w_0\tilde{b}$ , for each feasible solution $w$ of dual fuzzy linear programming problem. Thus , $w_0$ is an optimal solution to the dual fuzzy linear programming problem

**Theorem 3.4 (Strong duality)**

If either a fuzzy variable linear programming problem or its dual has an optimal solution , then both problems have optimal solutions and the two optimal value of ranking functions for the fuzzy objective values are equal. (If fact , if $\tilde{x^*}$ with $\tilde{x^*_B} \approx B^{-1}\tilde{b}$ , and $\tilde{x^*_N} \approx \tilde{0}$ is optimal solution of the primal problem then the crisp vector $w^* = c_B B^{-1}$, where $B$ is the optimal basis , is optimal solution of the dual problem.).

**Proof.** First , assume that the fuzzy variable linear programming problem has a fuzzy optimal solution , and $rank(A) = m$ . Let $\tilde{y} \succeq \tilde{0}$ be the fuzzy slack variable for the constraints $A\tilde{x} \preceq \tilde{b}$ . The new equivalent problem to the fuzzy variable linear programming problem is

$$
\begin{aligned}
& min\ \tilde{z} \approx c\tilde{x} + 0\tilde{y} \\
& subject\ to \\
& \quad A\tilde{x} + \tilde{y} \approx \tilde{b}, \\
& \quad \tilde{x} \succeq \tilde{0}, \\
& \quad \tilde{y} \succeq \tilde{0}.
\end{aligned} \tag{3.11}
$$

Assume $B$ is the optimal basis matrix and $\tilde{x_*} \approx (\tilde{x_B}^T \quad \tilde{0}^T)^T \approx (\tilde{b^T} B^{-T} \quad \tilde{0}^T)^T$ is the fuzzy basic optimal solution corresponding to the fuzzy variable linear programming problem . From Theorem 2.1 of Chapter 2

$$c_B B^{-1} a_j - c_j \leqslant 0, \qquad j = 1, 2, ..., n, n+1, ..., n+m,$$

or equivalently ,

$$c_B B^{-1} a_j \leqslant c_j, \qquad j = 1, 2, ..., n,$$

$$c_B B^{-1} e_i \leqslant 0, \qquad i = 1, 2, ..., m.$$

Hence ,

$$c_B B^{-1} A \leqslant c$$

$$c_B B^{-1} \leqslant 0.$$

Now, let $w_* = c_B B^{-1}$. Using the above inequalities ,

$$w_* A \leqslant c$$

$$w_* \leqslant 0$$

Thus , $w_*$ is a feasible solution to the dual

$$w_* \tilde{b} \approx c_B B^{-1} \tilde{b} \approx c_B \tilde{x_B} \approx c \tilde{x_*}$$

and hence

$$w_* \tilde{b} \approx c \tilde{x_*}$$

Therefore , the result follows immediately from Theorem 3.3.

**Remark 3.1**

We emphasize that the basic solution to the fuzzy variable linear programming problem is a fuzzy vector that is uniquely determined by the optimal basis $B$. The solution to the dual problem is a unique crisp solution, however , depending solely on the basis matrix $B$ , with the objective value of the dual having the same rank as the primal . We realize that the duality result obtained here are independent of the choice of the linear ranking function . It is clear that we can use any other linear ranking function , and although the solution obtained may be different but the duality result are still valid for the new solution . In the , the dual simplex algorithm that we present later will find the fuzzy solution of the primal as well as crisp solution of the dual problem along with the induced basis corresponding to the ranking function being used (we point out that the optimal basis corresponding to a linear ranking function does not change if the ranking function is multiplied by a positive constant). As for the types of the fuzzy data in the model and the assumption of fuzziness in the variables , the choice and compatibility of the ranking function for the ranking function for the linear programming model should be the decision maker's main concern. For trapezoidal fuzzy numbers and variables , the linear ranking function defined in Chapter 1 is deemed to be appropriate.

For an illustration of the above theorem consider the following fuzzy variable linear programming problem and its dual.

**Example 3.1**

$$min\ \tilde{z} \approx 6\tilde{x_1} + 10\tilde{x_2}$$

$$subject\ to$$

$$3\tilde{x_1} + 4\tilde{x_2} \succeq (5, 8, 2, 5),$$

$$3\tilde{x_1} + 4\tilde{x_2} \succeq (6, 10, 2, 6),$$

$$\tilde{x_1}, \tilde{x_2} \succeq \tilde{0}.$$

$$max\ \tilde{u} \approx (5, 8, 2, 5)w_1 + (6, 10, 2, 6)w_2$$

$$subject\ to$$

$$2w_1 + 3w_2 \leqslant 6,$$

$$5w_1 + 4w_2 \leqslant 10,$$

$$w_1, w_2 \geqslant 0.$$

We see that the fuzzy basic optimal solution for the fuzzy variable linear programming problem is $\tilde{x_1} \approx (\frac{-5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7})$ , $\tilde{x_2} \approx (\frac{-2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7})$ with $\Re(\tilde{x_1}) = \frac{15}{14}$ , $\Re(\tilde{x_2}) = \frac{32}{7}$ and also the optimal objective value is $\tilde{z} \approx (\frac{-62}{7}, \frac{300}{7}, \frac{360}{7}, \frac{418}{7})$ with $\Re(\tilde{z}) = 38.1428$ , with the optimal basis matrix

$$\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}$$

Hence , using $w = c_B B^{-1}$ , $w = (\frac{6}{7}, \quad \frac{10}{7})$, which is equal to the optimal solution of the dual . On the other hand , $\Re(\tilde{u}) = 38.1428$ . Therefore , both problems have optimal solutions and the two optimal value of ranking functions for the fuzzy objective values are equal.

**Theorem 3.5**

Let $\tilde{x_*}$ and $w_*$ be any feasible solutions to fuzzy variable linear programming problem and its corresponding Dual fuzzy variable linear programming problem . Then $\tilde{x_*}$ and $w_*$ are respectively optimal if and only if

$$(w_* A - c)\tilde{x_*} \approx \tilde{0}, \ \ w_*(\tilde{b} - A\tilde{x_*}) \approx \tilde{0}. \tag{3.12}$$

**Proof.** Suppose that $\tilde{x_*}$ and $w_*$ are feasible solutions to fuzzy variable linear programming problems and Dual fuzzy variable linear programming problems , respectively . Therefore,

$$A\tilde{x_*} \preceq \tilde{b} \tag{3.13}$$

and

$$w_*A \leqslant c. \tag{3.14}$$

Multiplying $A\tilde{x_*} \preceq \tilde{b}$ on the left by $w_* \leqslant 0$ yields

$$w_*A\tilde{x_*} \preceq w_*\tilde{b}. \tag{3.15}$$

Multiplying $w_*A \leqslant c$ on the right side by $\tilde{x_*} \succeq \tilde{0}$ yields

$$w_*A\tilde{x_*} \preceq c\tilde{x_*}. \tag{3.16}$$

Therefore ,

$$w_*\tilde{b} \preceq w_*A\tilde{x_*} \preceq c\tilde{x_*}. \tag{3.17}$$

On the other hand , since $\tilde{x_*}$ and $w_*$ are optimal solutions to the primal and dual problems , respectively , then by Theorem 3.1 , $w_*\tilde{b} \approx c\tilde{x_*}$ and (3.17) is written as

$$w_*\tilde{b} \approx w_*A\tilde{x_*} \approx c\tilde{x_*}. \tag{3.18}$$

From (3.18)

$$(w_*A-c)\tilde{x_*} \approx \tilde{0}, w_*(\tilde{b}-A\tilde{x_*}) \approx \tilde{0}. \tag{3.19}$$

The converse of the theorem follows from the fact that $(w_*A - c)\tilde{x}* \approx \tilde{0}$ and $w_*(\tilde{b} - A\tilde{x_*}) \approx \tilde{0}$ imply that $c\tilde{x_*} \approx w_*\tilde{b}$ . Therefore , optimality of $\tilde{x_*}$ and $w_*$ follows from Corollary 3.1.

### 3.2 Primal optimality and dual feasibility

Consider the following fuzzy variable linear programming problem ,

$$\begin{aligned} &min\ \tilde{z} \approx c\tilde{x} \\ &subject\ to \\ &\quad A\tilde{x} \preceq \tilde{b}, \\ &\quad \tilde{x} \succeq \tilde{0}, \end{aligned} \tag{3.20}$$

where , $\tilde{x} = (\tilde{x_1}, ..., \tilde{x_n})^T, \tilde{b} = (\tilde{b_1}, ..., \tilde{b_m})^T, A = [a_{ij}]_{m\times n}, c = (c_1, ..., c_n), , \tilde{b_i}, \tilde{x_j} \in F((R)), c_j, a_{ij} \in R \ \ for \ \ all \ \ i, j$ .

Now, we may rewrite (3.20) as follows:

$$\begin{aligned}
&min\ \tilde{z} \approx c\tilde{x} + 0\tilde{y} \\
&subject\ to \\
&\quad A\tilde{x} + \tilde{y} \approx \tilde{b}, \\
&\quad \tilde{x} \succeq \tilde{0}, \\
&\quad\ \ \tilde{y} \succeq \tilde{0}.
\end{aligned} \tag{3.21}$$

where , $\tilde{y} = (\tilde{y_1}, ..., \tilde{y_m})^T$.

Define $\bar{x} \in (F(R))^{n+m}$ and $\bar{c} \in R^{n+m}$ as

$$x_j = \begin{cases} x_j, & j = 1, 2, ..., n; \\ y_{j-n}, & j = n+1, n+2, ..., n+m. \end{cases}$$

$$c_j = \begin{cases} c_j, & j = 1, 2, ..., n; \\ 0, & j = n+1, n+2, ..., n+m. \end{cases} \tag{3.22}$$

Suppose that a basic solution for (3.21) is given by $\bar{x_B} \approx B^{-1}\tilde{b}$ , with the basis matrix $B$ . Now , let $z_j = \bar{c_B} B^{-1} \bar{a_j}, \tilde{y_0} \approx B^{-1}\tilde{b}$, where $\bar{c_B} = (\bar{c}_{B_1}, ..., \bar{c}_{B_m})$, and $\bar{a_j}$ is the $j^{th}$ column of the coefficient matrix $[A \quad I]$ . Consider Table 3.1 , where $(\bar{x_B})_r$ is the $r_{th}$ fuzzy basic variable , $y_j = B^{-1}\bar{a}_j$, and $\Re(\tilde{y_{r0}})$ is the real number corresponding to the fuzzy number $\tilde{y_{r0}}$ obtained by a linear ranking function . Suppose that for $j = 1, 2, ..., n+m$ , we have

$$z_j - \bar{c_j} \leqslant 0 \tag{3.23}$$

Define $w = \bar{c_B}B^{-1}$, where $w = (w_1, ..., w_m)$. For $j = 1, 2, ..., n$, we have

$$y_{0j} = z_j - \bar{c_j} = \bar{c_B}B^{-1}a_j - c_j = wa_j - c_j,$$

Therefore , for $j = 1, ..., n$, from $z_j - c_j \leqslant 0$. Then ,

$$wA \leqslant c \tag{3.24}$$

**Table 3.1**

**A dual feasible simplex table**

| Basis | $\bar{x_1}$ | ... | $\bar{x_l}$ | ... | $\bar{x}_{n+m}$ | R.H.S. | $\Re(R.H.S)$ |
|---|---|---|---|---|---|---|---|
| $\tilde{z}$ | $z_1 - \bar{c_1}$ | ... | $z_l - \bar{c_l}$ | ... | $z_{n+m} - \bar{c}_{n+m}$ | $\bar{c}_B\tilde{y_0}$ | $\Re(\bar{c_B}\tilde{y_0})$ |
| $(\bar{x}_B)_1$ | $y_{11}$ | ... | $y_{1l}$ | ... | $y_{1,n+m}$ | $\tilde{y_{10}}$ | $\Re(\tilde{y_{10}})$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $(\bar{x}_B)_r$ | $y_{r1}$ | ... | $y_{rl}$ | ... | $y_{r,n+m}$ | $\tilde{y_{r0}}$ | $\Re(\tilde{y_{r0}})$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| $(\bar{x}_B)_m$ | $y_{m1}$ | ... | $y_{ml}$ | ... | $y_{m,n+m}$ | $\tilde{y_{m0}}$ | $\Re(\tilde{y_{m0}})$ |

On the other hand , using (3.23) ,we have

$$0 \geqslant z_{n+i} - c_{n+i} = c_B B^{-1}e_i - 0 = we_i = w_i, \quad i = 1, 2, ..., m,$$

and hence ,

$$w \leqslant 0, \tag{3.25}$$

yielding the dual feasibility . If $\Re(\tilde{y_{r0}}) \geqslant 0$, for all $r = 1, 2, ..., m$, then a fuzzy feasible solution for the fuzzy variable linear programming problem is at hand . Moreover , we will have ,

$$\bar{c}\bar{x} \approx \bar{c_B}\tilde{y_0} \approx \bar{c_B}B^{-1}\tilde{b} \approx w\tilde{b},$$

and thus , by Corollary 3.1 , establishing the optimality of $\bar{x}$ and $w$ for the fuzzy variable linear programming problems and its dual. Therefore , we have the following result.

**Theorem 3.6**

The optimality criteria $z_j - \bar{c_j} \leqslant 0$ for all $j$ , for the fuzzy variable linear programming problem is equivalent to the feasibility condition for the Dual fuzzy variable linear programming problem. If , in addition , $\bar{x}$ corresponding to a basis is primal feasible then $\bar{x}$ is optimal for the fuzzy variable linear programming problem and $w = \tilde{c_B}B^{-1}$ is optimal to the fuzzy variable linear programming problem.

**Proof.** Now, let us assume that the Dual fuzzy variable linear programming problem is feasible and $\bar{x}$ , corresponding to a basis $B$ , is dual feasible but primal infeasible (and hence not optimal). That is , we have ,

$$y_{oj} = z_j - \bar{c_j} \leqslant 0, \quad j = 1, ........, n + m$$

and there exists at least one $r$ such that $\tilde{y_{r0}} \prec \tilde{0}$ . We can immediately deduce that the primal must be finite. Thus, the fuzzy variable linear programming problem can be either infeasible (in which case , the dual fuzzy variable linear programming problem is unbounded) , or it has an optimal solution . In what follows we will show how to work on row $r$ of the simplex table corresponding to B,as the pivoting row, and either (1) detect infeasibility of the fuzzy variable linear programming problem (or unboundedness of the Dual fuzzy variable linear programming problem) , or (2) find a column $l$, as the pivoting column , to pivot on $y_{rl}$ and obtain a new dual feasible table with a nondecreasing primal objective value. We explain these cases as Lemma 3.2 and 3.3 below.

**Lemma 3.2**

If in a dual feasible simplex table an $r$ exists such that $\tilde{y_{r0}} \prec \tilde{0}$ and $y_{rj} \geqslant 0$ for all $j$ , then the fuzzy variable linear programming problem is infeasible.

**Proof.** Suppose that a dual simplex table is feasible , and an $r$ exists such that $\tilde{y_{r0}} \prec \tilde{0}$ and $y_{ij} \geqslant 0$ , for all $j$ . Corresponding to the $rth$ row of the table , we have $\sum_j y_{rj}\bar{x_j} \approx \tilde{y_{r0}}$ . Since $y_{rj} \geqslant 0$ for all $j$ and $\bar{x_j}$ is required to be nonnegative , then $\sum_j y_{rj}\bar{x_j} \succeq \tilde{0}$ for any fuzzy basic feasible solution. However , $\tilde{y_{r0}} \prec \tilde{0}$ . This shows that fuzzy variable linear programming problem is infeasible.

**Lemma 3.3**

If in a dual feasible simplex table , an $r$ exists such that $\tilde{y_{r0}} \prec \tilde{0}$ and there exists a nonbasic index $j$ such that $y_{rj} < 0$ , then a pivoting column $l$ can be found so that pivoting on $y_{rl}$ will yield a dual feasible table with a corresponding nondecreasing objective value.

**Proof.** A new criterion is needed for choosing a nonbasic fuzzy variable to enter the basis so that the new simplex table will remain dual feasible and the new objective value is nondecreasing . Assuming column $l$ is the pivot column . Pivoting on the pivot $y_{rl}$ will result in the new $0^{th}$ row as follows:

$$\hat{y_{0j}} = y_{0j} - \frac{y_{0l}}{y_{rl}} y_{rj}, \quad j = 1, 2, ..., n+m, \;\; j \neq B_i. \tag{3.26}$$

For the new table to be dual feasible it is needed that

$$\hat{y_{0j}} \leqslant 0, \quad j \neq B_i, \tag{3.27}$$

which , using (3.26) , results in

$$\frac{y_{0l}}{y_{rl}} \leqslant \frac{y_{0j}}{y_{rj}}, \quad \forall j \neq B_i, \tag{3.28}$$

To satisfy (3.28) , it is sufficient to let

$$\frac{y_{0l}}{y_{rl}} = min\left\{\frac{y_{0j}}{y_{rj}}|y_{rj} < 0\right\} \quad , j \neq B_i \tag{3.29}$$

Note that the new objective value is nondecreasing , since

$$\hat{y_{00}} = \Re\left(\tilde{y_{00}} - \frac{y_{0l}}{y_{rl}}\tilde{y_{r0}}\right) = \Re\left(\tilde{y_{00}}\right) - \Re\left(\frac{y_{0l}}{y_{rl}}\tilde{y_{r0}}\right) \geqslant \Re\left(\tilde{y_{00}}\right),$$

using the fact that

$$\Re\left(\frac{y_{0l}}{y_{rl}}\tilde{y_{r0}}\right) \leqslant 0.$$

### 3.3 Fuzzy dual simplex method

(1) Given a basis $B$ for the fuzzy variable linear programming problem such that $y_{0j} = z_j - \bar{c_j} \leqslant 0$ for all $j$. Compute the simplex table.

(2) If $\tilde{y_0} \succeq \tilde{0}$ then Stop (the current solution is optimal) else select the pivot row $r$ with $\tilde{y_{r0}} \prec \tilde{0}$ (that is , $r$ so that $\Re(\tilde{y_{r0}}) < 0$ ).

(3) If $y_{rj} \geqslant 0$ for all $j$ then Stop (the fuzzy variable linear programming problem is infeasible ) else select the pivot column $l$ by the following minimum ratio test:

$$\frac{y_{0l}}{y_{rl}} = min\left\{\frac{y_{0j}}{y_{rj}}|y_{rj} < 0\right\} \quad , j \neq B_i$$

(4) Pivot on $y_{rl}$ and go to (2).

**Remark 3.2**

One suggestion for choice of $r$ in Step (2) may be so that

$$\Re(\tilde{y_{r0}}) = min_{1 \leqslant i \leqslant m}\{\Re(\tilde{y_{i0}})\}$$

**Remark 3.3**

Pivoting on $y_{rl}$ in Step (4) is the usual Gaussian elimination process that, using $y_{rl}$ , converts column $l$ to the unit vector $e_r$ yielding the new simplex table corresponding to the new basis (the one that is obtained by replacing column $r$ of $B$ with $a_l$ ).

### 3.4. A numerical example

**Example 3.2**

$$min\ \tilde{z} \approx 6\tilde{x_1} + 10\tilde{x_2}$$

$$subject\ to$$

$$2\tilde{x_1} + 5\tilde{x_2} \succeq (5, 8, 2, 5),$$

$$3\tilde{x_1} + 4\tilde{x_2} \succeq (6, 10, 2, 6),$$

$$\tilde{x_1}, \tilde{x_2} \succeq \tilde{0}.$$

Its first dual feasible simplex table as follows :

$$-2\tilde{x_1} - 5\tilde{x_2} + \tilde{x_3} \approx (-8, -5, 5, 2),$$

$$-3\tilde{x_1} - 4\tilde{x_2} + \tilde{x_4} \approx (-10, -6, 6, 2),$$

$$\tilde{x_1}, ..., \tilde{x_4} \succeq \tilde{0}.$$

or equivalently ,

| Basis | $\tilde{x_1}$ | $\tilde{x_2}$ | $\tilde{x_3}$ | $\tilde{x_4}$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\tilde{z}$ | $-6$ | $-10$ | 0 | 0 | (0,0,0,0) | 0 |
| $\tilde{x_3}$ | $-2$ | $-5$ | 1 | 0 | $(-8,-5,5,2)$ | $-7.25$ |
| $\tilde{x_4}$ | $-3$ | $-4$ | 0 | 1 | $(-10,-6,6,2)$ | $-9$ |

$\tilde{x_4}$ is a leaving variable and $\tilde{x_1}$ is an entering variable. The new table is

| Basis | $\tilde{x_1}$ | $\tilde{x_2}$ | $\tilde{x_3}$ | $\tilde{x_4}$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\tilde{z}$ | 0 | $-2$ | 0 | $-2$ | (12,20,4,12) | 18 |
| $\tilde{x_3}$ | 0 | $-7/3$ | 1 | $-2/3$ | $(-4,5/3,19/3,6)$ | $-15/12$ |
| $\tilde{x_1}$ | 1 | $4/3$ | 0 | $-1/3$ | (2,10/3,2/3,2) | 3 |

$\tilde{x_3}$ is a leaving variable and $\tilde{x_2}$ is an entering variable. The next table is

| Basis | $\tilde{x_1}$ | $\tilde{x_2}$ | $\tilde{x_3}$ | $\tilde{x_4}$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\tilde{z}$ | 0 | 0 | $-6/7$ | $-10/7$ | $(-62/7,300/7,360/7,418/7)$ | 267/7 |
| $\tilde{x_2}$ | 0 | 1 | $-3/7$ | $2/7$ | $(-5/7,12/7,18/7,19/7)$ | 15/14 |
| $\tilde{x_1}$ | 1 | 0 | $4/7$ | $-5/7$ | $(-2/7,30/7,30/7,38/7)$ | 32/7 |

Therefore, the optimal solution of the fuzzy variable linear programming problem obtained by the dual method is $\tilde{x_1} \approx \left(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7}\right)$ , $\tilde{x_2} \approx \left(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7}\right)$ and $\tilde{z} \approx \left(\frac{-62}{7},\frac{300}{7},\frac{360}{7},\frac{418}{7}\right)$.

Now if we consider the optimal solution , showing four decimal places for the fractional parts , we have

$$\tilde{x_1} \approx (-0.2857, 4.2857, 4.2857, 5.4286),$$

$$\tilde{x_2} \approx (-0.7143, 1.7143, 2.5714, 2.7143),$$

$$\tilde{z} \approx (-8.8571, 42.8571, 51.4286, 59.7143),$$

$$and \quad \Re(\tilde{z}) = 38.1428$$

### 3.5 Conclusion

In this chapter , the dual of linear programming problem with trapezoidal fuzzy variables and some duality results based on certain linear ranking functions , for the fuzzy primal and fuzzy dual problems are presented . The duality results using certain general linear ranking functions on trapezoidal fuzzy numbers are presented which are natural extensions of the results for linear programming problems with crisp data . Using these results and the feasible primal simplex table , fuzzy dual simplex method for solving the primal and the dual problems directly is presented.

# CHAPTER 4

# SENSITIVITY ANALYSIS FOR FUZZY VARIABLE LINEAR PROGRAMMING PROBLEMS

In this chapter , a method to deal with sensitivity analysis of fuzzy variable linear programming problem , is presented. The presented method is illustrated with the help of some numerical examples.

## 4.1 Sensitivity analysis

It is important to be able to find the new optimal solution of the fuzzy variable linear programming problem as other estimates of some of the fuzzy data become available, without the expensive task of resolving the problem from scratch. It is important to update the current solution in a way that takes care of these factors. It is desirable to examine the effect of relaxing some of the constraints on the value of the optimal fuzzy objective without having to resolve the problem. These topics constitute sensitivity analysis.

Consider the following fuzzy variable linear programming problem,

$$
\begin{aligned}
&\mathit{min} \quad \widetilde{z} \approx c\widetilde{x} \\
&\mathit{subject\ to} \\
&\qquad A\widetilde{x} \preceq \widetilde{b}, \\
&\qquad \widetilde{x} \succeq \widetilde{0},
\end{aligned}
\tag{4.1}
$$

where , $\widetilde{b} \in (F(R))^m, \widetilde{x} \in (F(R))^n, A \in R^{m\times n}, c^T \in R^n$ .

Suppose that we have an optimal fuzzy solution for (4.1) by $\widetilde{x}_B \approx B^{-1}\widetilde{b}$, with basis matrix $B$. We shall describe how to make use of the optimality conditions in order to find the new optimal fuzzy solution, if some of the fuzzy variable linear programming problem data change. The following variations in the fuzzy variable linear programming problem will be considered.

1) Change in the fuzzy right hand side vector $\widetilde{b}$,

2) Change in the constraint matrix $A$ ,

3) Addition of a new constraint ,

4) Change in the cost vector c ,

5) Addition of a new activity (trapezoidal fuzzy variable) .

### 4.1.1 Change in the fuzzy right hand-side vector $\widetilde{b}$,

Suppose that fuzzy right-hand-side vector $\widetilde{b}$ is replaced by $\widetilde{d}$ , where $\widetilde{d} = (\widetilde{d}_1, ..., \widetilde{d}_m)^T$ , and $\widetilde{d}_i \in F(R)$. Then the new fuzzy variable linear programming problem is as follows :

$$\begin{aligned}
&min \quad \widetilde{z} \approx c\widetilde{x} \\
&subject\ to \\
&\qquad A\widetilde{x} \approx \widetilde{d}, \\
&\qquad \widetilde{x} \succeq \widetilde{0},
\end{aligned} \tag{4.2}$$

Therefore , $B^{-1}\widetilde{b}$ will replaced by $B^{-1}\widetilde{d}$. Since , $z_j - c_j \leq 0$ for all for all nonbasic fuzzy variables, the only possible violation of optimality is that the new fuzzy vector $B^{-1}\widetilde{d}$ may have some negative entries. If $B^{-1}\widetilde{d} \succeq \widetilde{0}$ , then the same basis remains optimal, and the values of the basic fuzzy variables are $B^{-1}\widetilde{d}$ and the objective has fuzzy value $c_B B^{-1}\widetilde{d}$. Otherwise the dual method is used to find the new optimal fuzzy solution by restoring feasibility.

Consider the following linear programming problem with fuzzy variables.

**Example 4.1**

$$min \qquad \widetilde{z} \approx 6\widetilde{x_1} + 10\widetilde{x_2}$$

$$subject\ to$$

$$2\widetilde{x_1} + 5\widetilde{x_2} \succeq (5, 8, 2, 5)$$

$$3\widetilde{x_1} + 4\widetilde{x_2} \succeq (6, 10, 2, 6)$$

$$\widetilde{x_1}, \widetilde{x_2} \succeq \widetilde{0},$$

The optimal solution of above fuzzy variable linear programming problem is $\widetilde{x_1}^* \approx (0,0,0,0)$ , $\widetilde{x_2}^* \approx (2,4,3,1)$ and $\widetilde{z} = (-16, -8, 12, 4)$ with $\Re(\widetilde{z}) = 28$.

Now , suppose that the right hand side is replaced by $\widetilde{d} \approx \begin{bmatrix} (1,4,1,3) \\ (1,2,1,2) \end{bmatrix}$

The inverse of the current basis matrix is $B^{-1} = \begin{pmatrix} -3/7 & 2/7 \\ 4/7 & -5/7 \end{pmatrix}$ and hence $B^{-1}\widetilde{d} \approx \begin{bmatrix} (-10/7, 1/7, 11/7, 1) \\ (-6/7, 11/7, 14/7, 17/7) \end{bmatrix}$ . Then , $\widetilde{d} \approx B^{-1}\widetilde{d} \not\succeq \widetilde{0}$ with $\Re(\widetilde{x}_2) = -11/7$ , $\Re(\widetilde{x}_1) = 13/14$ , and so the new fuzzy solution is not feasible . Then , we can use from the dual method for finding the new optimal solution by restoring feasibility or determine infeasibility of the new fuzzy variable linear programming problem shown in Table 4.1.

**Table 4.1**

| Basis | $\widetilde{x}_1$ | $\widetilde{x}_2$ | $\widetilde{x}_3$ | $\widetilde{x}_4$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\widetilde{z}$ | 0 | 0 | $-6/7$ | $-10/7$ | $(-196/7, 16/7, 214/7, 152/7)$ | $-142/14$ |
| $\widetilde{x}_2$ | 0 | 1 | $-3/7$ | $-2/7$ | $(-10/7, 1/7, 11/7, 1)$ | $-11/7$ |
| $\widetilde{x}_1$ | 1 | 0 | $4/7$ | $-5/7$ | $(-6/7, 11/7, 14/7, 17/7)$ | $13/14$ |

Since , $\Re(-10/7, 1/7, 11/7, 1) = -11/7 < 0$, therefore $\widetilde{x}_2$ is leaving variable and $\widetilde{x}_3$ is entering variable (Table 4.2) .

**Table 4.2**

| Basis | $\widetilde{x}_1$ | $\widetilde{x}_2$ | $\widetilde{x}_3$ | $\widetilde{x}_4$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\widetilde{z}$ | 0 | $-2$ | 0 | $-6/7$ | $(-132/49, 846/49, 708/49, 1026/49)$ | $873/49$ |
| $\widetilde{x}_3$ | 0 | $-7/3$ | 1 | $2/3$ | $(5/3, 16/3, 5/3, 13/3)$ | $25/3$ |
| $\widetilde{x}_1$ | 1 | $4/3$ | 0 | $-23/21$ | $(-22/49, 141/49, 118/49, 171/49)$ | $291/98$ |

Now since , $\Re(\widetilde{x}_1) = 291/98 > 0$ and $\Re(\widetilde{x}_3) = 25/3 > 0$ . Therefore , the optimal fuzzy solution of the problem is $\widetilde{x}_3 \approx (5/3, 16/3, 5/3, 13/3)$ and $\widetilde{x}_1 \approx (-22/49, 141/49, 118/49, 171/49)$. Then , $\widetilde{z} \approx (-132/49, 846/49, 708/49, 1026/49)$ with $\Re(\widetilde{z}) = 873/49$

### 4.1.2 Change in the constraint matrix $A$

Here the effect of change some of the entries of the constraint matrix $A$ is discussed . We only consider changes involving nonbasic column and omit other case. Therefore, suppose that in the fuzzy variable linear programming problem problem , is defined in Example 4.1, the nonbasic column $a_k$ is modified to $\widetilde{a}_k$ .

Then the new updating column is $\bar{y}_k = B^{-1}\bar{a}_k$ and $\bar{z}_k - c_k = c_B B^{-1}\bar{a}_k - c_k = c_B\bar{y}_k - c_k$.

Hence if $\bar{z}_k - c_k \leq 0$, then the old solution is optimal ; otherwise the fuzzy primal simplex method is continued , after column $k$ of the tableau is updated , by introducing the fuzzy nonbasic variable $\widetilde{x}_k$ .

**Example 4.2**

$$min \qquad \widetilde{z} \approx 6\widetilde{x}_1 + 10\widetilde{x}_2$$

$$subject\ to$$

$$2\widetilde{x}_1 + 5\widetilde{x}_2 \succeq (5, 8, 2, 5)$$

$$3\widetilde{x}_1 + 4\widetilde{x}_2 \succeq (6, 10, 2, 6)$$

$$\widetilde{x}_1, \widetilde{x}_2 \succeq \widetilde{0},$$

The optimal solution of above fuzzy variable linear programming problem is $\widetilde{x}_1{}^* \approx (-2/7, 30/7, 30/7, 38/7)$ , $\widetilde{x}_2{}^* \approx (-5/7, 12/7, 18/7, 19/7)$ and $\widetilde{z} \approx (-62/7, 300/7, 360/7, 418/7)$ with $\Re(\widetilde{z}) = 38.1428$.

Suppose that the column $a_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in above fuzzy variable linear programming problem , is replaced by $\bar{a}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Hence , $\bar{z}_2 - c_2 = c_B B^{-1} \bar{a}_2 - c_2 = \dfrac{-230}{7} < 0$.

Therefore , the current fuzzy solution is still optimal with respect to the new fuzzy variable linear programming problem.

### 4.1.3 Adding a new constraint

Consider the fuzzy variable linear programming problem , as defined in (4.1). Suppose that we have an optimal fuzzy basic solution by $\widetilde{x}_B = \widetilde{y}_0 = B^{-1}\widetilde{b}$, with basis matrix $B$ . Also , suppose that a new constraint is added to the problem. Then, the new fuzzy variable linear programming problem is given as following:

$$
\begin{aligned}
& min \quad \widetilde{z} \approx c\widetilde{x} \\
& subject\ to \\
& \qquad A\widetilde{x} \preceq \widetilde{b}\ ,\ \ p\widetilde{x} \preceq \widetilde{b}_0 \\
& \qquad \widetilde{x} \succeq \widetilde{0},
\end{aligned}
\tag{4.3}
$$

where , $\widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n)^T, \widetilde{x} = (\widetilde{b}_1, ..., \widetilde{b}_n)^T, c = (c_1, ..., c_n), A = [a_{ij}]_{m\times n}$ , $\widetilde{b}_i$ , $\widetilde{b}_0$ , $\widetilde{x}_j \in F(R)$ , $c_j$ , $a_{ij} \in R$ , $p^T \in R^n$, for all $i, j$.

Then with add up fuzzy basic variable $\widetilde{x}_{k+1}$, the new constraints can be rewritten as follows: $\quad p\widetilde{x} + \widetilde{x}_{k+1} \approx \widetilde{b}_0 \quad or$

$$p_B\widetilde{x}_B + p_N\widetilde{x}_N + \widetilde{x}_{k+1} \approx \widetilde{b}_0.$$

where $p_B$ and $p_N$ are the multiplication vectors of fuzzy basic and nonbasic variable in the new constraint , respectively. Then , the new basis matrix is $\bar{B} = \begin{pmatrix} B & 0 \\ p_B & 1 \end{pmatrix}$ and its order is $(m+1)^2$ . So ,

$$\bar{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -p_B B^{-1} & 1 \end{pmatrix}$$

Now we consider $\bar{c}_B$ as the multiplications vector of fuzzy basic variables in objective and $\bar{a}_j$ as the multiplications vector of $j^t h$ fuzzy variables in constraints , then we have :

$$(z_j - c_j)_{new} = \bar{c}_B \bar{B}^{-1} \bar{a}_j - c_j =$$
$$\begin{pmatrix} c_B & 0 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ -p_B B^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_j \\ p_j \end{pmatrix} - c_j =$$
$$c_B B^{-1} a_j - c_j = z_j - c_j.$$

**Remark 4.1**

Adding a new constraint to the fuzzy variable linear programming problem is ineffective on the optimality condition.

Now we show that the adding a new constraint can be effective on the feasibility condition.

**Example 4.3**

Consider the fuzzy variable linear programming problem given in Example 4.2. Suppose that we add a new constraint by $\widetilde{x}_1 + 2\widetilde{x}_2 \succeq \widetilde{b}_0$, where $\widetilde{b}_0 = (2, 3, 1, 2)$. According to Remark 4.1 we should investigate the feasibility condition for the new constraint by the current optimal fuzzy solution. By use of the final tableau, the optimal fuzzy solution is as follows:

$$\widetilde{x}_1 = (-2/7, 30/7, 30/7, 38/7), \widetilde{x}_1 = (-5/7, 12/7, 18/7, 19/7).$$

Then, the left side of the new constraint is $\widetilde{x}_1 + 2\widetilde{x}_2 \approx (-12/7, 54/7, 66/7, 76/7)$, and $\Re(\widetilde{x}_1 + 2\widetilde{x}_2) = 47/7$. .In the other hand , we have $\Re(\widetilde{b}_0) = 11/2$. Hence, the optimal fuzzy solution of the problem is given in Example 4.2 for the new constraint is feasible , too. Therefore , the current optimal fuzzy solution is still optimal.

**Example 4.4**

Consider the fuzzy variable linear programming problem given in Example 4.2. Suppose that we add a new constraint by $\widetilde{x}_1 + 2\widetilde{x}_2 \succeq \widetilde{b}_0$, where $\widetilde{b}_0 \approx (2, 3, 1, 2)$. Note that the current optimal fuzzy solution does not satisfy in the new constraint. Then , with add up the fuzzy variable $\widetilde{x}_5$ as basic variable of the added constraint we obtain $\widetilde{x}_5 \approx (-40/7, 33/7, 83/7, 80/7)$ with $\Re(\widetilde{x}_5) = -17/14$. So , $\widetilde{x}_5 \prec \widetilde{0}$ . Hence , Table 4.3 with the constraint can be considered .

**Table 4.3**

| Basis | $\bar{x}_1$ | $\bar{x}_2$ | $\bar{x}_3$ | $\bar{x}_4$ | R.H.S. | $\bar{R}$ |
|---|---|---|---|---|---|---|
| $\bar{z}$ | 0 | 0 | $-6/7$ | $-10/7$ | $(-62/7, 300/7, 360/7, 418/7)$ | 267/7 |
| $\bar{x}_2$ | 0 | 1 | $-3/7$ | $2/7$ | $(-5/7, 12/7, 18/7, 19/7)$ | 15/14 |
| $\bar{x}_1$ | 1 | 0 | $4/7$ | $-5/7$ | $(-2/7, 30/7, 30/7, 38/7)$ | 32/7 |
| $\bar{x}_5$ | 0 | 0 | $2/7$ | $9/7$ | $(-40/7, 33/7, 83/7, 80/7)$ | $-17/14$ |

Now , by use of dual method , the basic fuzzy variable $\widetilde{x}_5$ is eligible to leave the basis, but we are not able to determine any variable to enter the basis. Therefore , we follow that the new fuzzy variable linear programming problem is not feasible.

### 4.1.4 Change in the cost vector c

Here, a particular situation that could affect the optimality of the current fuzzy solution , is considered . Suppose that a fuzzy optimal basic feasible solution is given and cost coefficient of one (or more) of the fuzzy variables is changed from $c_k$ to $\bar{c}_k$ . These changes only affect the optimality of the fuzzy solution , that is dual feasibility may be lost. In general there are two cases :

1) The new cost row satisfies the optimality condition and the fuzzy solution remains unchanged.

2) The optimality condition is not satisfied, in which case the fuzzy primal simplex method is used to recover optimality.

In the first case suppose that the cost coefficient of a fuzzy nonbasic variable is changed , that is $c_k \rightarrow \bar{c}_k$ , where$k \neq B_i, (i = 1, ..., m)$. In this case $c_B$ is not affected , and hence $z_j = c_B B^{-1} a_j$ is not changed for any $j$. Thus $z_k - c_k$ is replaced by $z_k - \bar{c}_k$. If $z_k - \bar{c}_k \leq 0$, the current fuzzy solution is still optimal with respect to the new fuzzy variable linear programming problem , and if $z_k - \bar{c}_k$ is positive , then fuzzy variable $\widetilde{x}_k$ must be introduced into basis and the fuzzy primal simplex method is continued as usual.

In the second case suppose that the cost coefficient of a fuzzy basic variable is changed. In this case $z_j = c_B B^{-1} a_j$ is changed for any $j$ , since $c_B$ is replaced by $\bar{c}_B$ .Let the new value of $z_j$ be $\bar{z}_j k = B_i$ , and $e_k^T = (0, ..., 0, 1, 0, ..., 0)$ is the $kth$ unit vector . Then $\bar{z}_j - c_j = \bar{c}_B B^{-1} a_j - c_j$.

It is obvious that for $j = k$ , $\bar{z}_j - \bar{c}_j = 0$, since

$$\bar{z}_j - \bar{c}_j = \bar{c}_B B^{-1} a_j - \bar{c}_j = \bar{c}_B e_k - \bar{c}_k = \bar{c}_{B_t} - \bar{c}_k = \bar{c}_k - \bar{c}_k = 0.$$

Therefore , it is enough that we update the cost row only for $j \neq B_i$ and evaluate the optimality condition for new fuzzy variable linear programming problem . It can be obtained by

$$\widetilde{z}_{new} \approx \bar{c}_B B^{-1}\widetilde{b} \approx \bar{c}_B \bar{y}_0 \approx$$

$$c_B\bar{y}_0 + (\bar{c}_{B_t} - c_{B_t})\widetilde{y}_{y0} \approx \widetilde{z} + (\bar{c}_{B_t} - c_{B_t})\widetilde{y}_{t0}.$$

**Example 4.5**

Consider the fuzzy variable linear programming problem given in Example 4.2. Suppose that the cost vector $c = (6\ ,\ 10)$ is replaced by $\bar{c} = (7\ ,\ 9)$. Then $c_B$ is changed to $\bar{c} = (7\ ,\ 9)$, since $\widetilde{x}_1, \widetilde{x}_2$ are fuzzy basic variables. Then the coefficient of the fuzzy nonbasic variables in zero row are as follows (Table 4.4):

$$\bar{z}_3 - c_3 = \bar{c}_B B^{-1} a_3 - c_3 = \frac{15}{7},$$

$$\bar{z}_4 - c_4 = \bar{c}_B B^{-1} a_4 - c_4 = \frac{-59}{7}.$$

**Table 4.4**

| Basis | $\widetilde{x}_1$ | $\widetilde{x}_2$ | $\widetilde{x}_3$ | $\widetilde{x}_4$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\widetilde{z}$ | 0 | 0 | $15/7$ | $-59/7$ | $(-62/7, 300/7, 360/7, 418/7)$ | $267/7$ |
| $\widetilde{x}_2$ | 0 | 1 | $-3/7$ | $-2/7$ | $(-5/7, 12/7, 18/7, 19/7)$ | $15/14$ |
| $\widetilde{x}_1$ | 1 | 0 | $4/7$ | $-5/7$ | $(-2/7, 30/7, 30/7, 38/7)$ | $32/7$ |

Since , $\bar{z}_3 - c_3 > 0$ , $\widetilde{x}_3$ is an entering variable , and $\widetilde{x}_1$ is a leaving variable. Therefore , by use of the fuzzy primal simplex method in Table 4.5 as follows

**Table 4.5**

| Basis | $\widetilde{x}_1$ | $\widetilde{x}_2$ | $\widetilde{x}_3$ | $\widetilde{x}_4$ | R.H.S. | $\Re(R.H.S.)$ |
|---|---|---|---|---|---|---|
| $\widetilde{z}$ | $-15/4$ | 0 | 0 | $-161/28$ | $(-117/14, 621/14, 729/14, 837/14)$ | $279/7$ |
| $\widetilde{x}_2$ | $3/4$ | 1 | 0 | $-23/28$ | $(-13/14, 69/14, 81/14, 95/14)$ | $75/17$ |
| $\widetilde{x}_3$ | $7/4$ | 0 | 1 | $-5/4$ | $(-1/2, 15/2, 15/2, 19/2)$ | 9 |

Since , $\bar{z}_1 - c_1 = \dfrac{-15}{4} < 0$ , $\bar{z}_4 - c_4 = \dfrac{-161}{28}$, therefore , fuzzy optimal solution is obtained.

### 4.1.5 Adding a new activity

Suppose a new activity $\widetilde{x}_{n+1}$ with cost coefficient $c_{n+1}$ and consumption column $a_{n+1}$ is considered for possible production . Now , it is necessary to calculate $\bar{z}_{n+1} - c_{n+1}$ and in the next example optimality conditions are discussed.

**Example 4.6**

Consider the fuzzy variable linear programming problem given in Example 4.2. Suppose that a new activity $\widetilde{x}_5 \succeq \widetilde{0}$ with $c_5 = -2$ and $a_5 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is introduced , the new problem will be as follows

$$min \qquad \widetilde{z} \approx 6\widetilde{x}_1 + 10\widetilde{x}_2 - 2\widetilde{x}_5$$

*subject to*

$$2\widetilde{x}_1 + 5\widetilde{x}_2 - 2\widetilde{x}_5 \succeq (5, 8, 2, 5)$$

$$3\widetilde{x_1} + 4\widetilde{x_2} + \widetilde{x_5} \succeq (6, 10, 2, 6)$$

$$\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_5} \succeq \widetilde{0},$$

Then , $z_5 = \bar{c}_B B^{-1} a_5 = \begin{pmatrix} 4/7 \\ \\ -13/7 \end{pmatrix}$, $\bar{z}_5 - c_5 = \dfrac{-92}{7} < 0.$

Hence , the current fuzzy solution is still optimal.

## 4.2 Conclusion

In this chapter , a method to deal with sensitivity analysis of fuzzy variable linear programming problem , is presented and the presented method is illustrated with the help of some numerical examples.

# BIBLIOGRAPHY

[1] Bazaraa , M.S., Jarvis , J.J. , Sherali , H.D., *Linear Programming and Network Flows* , Second ed. , John Wiley , New York , 1990.

[2] Bazaraa , M.S., Jarvis , J.J. , Sherali , H.D., *Linear Programming and Network Flows* , third ed. , John Wiley , New York , 2005.

[3] Bector , C.R. , Chandra , S, *On duality in linear programming under fuzzy environment* , Fuzzy Sets and Systems , Vol. 125, 2002 , pp. 317-325.

[4] Bector , C.R. , Chandra , S , *Fuzzy Mathematical Programming and Fuzzy Matrix Games* , Studies in Fuzziness and Soft Computing , Vol. 169, Springer , Berlin , 2005.

[5] Bector , C.R. , Chandra , S, Vijay , V., *Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs* , Fuzzy Sets and Systems , Vol. 146, 2004 , pp. 253-269.

[6] Bellman , R.E., Zadeh , L.A. , *Decision making in a fuzzy environment*, Management Science , Vol. 17, 1970, pp. 141-164.

[7] Bezdek , J.C. , *Fuzzy models - What are they , and Why ?*, IEEE Transactions on Fuzzy Systems , Vol. 1 , 1993 , pp. 1-9.

[8] Buckley , J., Feuring , T., *Evolutionary algorithm solution to fuzzy problems : fuzzy linear programming* , Fuzzy Sets and Systems , Vol. 109, 2000 , pp. 35-53.

[9] Cabrerizo , F. J., Alonso , S., Herrera-Viedma , E. , *A consensus model for group decision making problems with unbalanced fuzzy linguistic information* , Inter-

national Journal of Information Technology and Decision Making , Vol. 8 , 2009 , pp. 109-131.

[10] Campos , L., Verdegay , J.L., *Linear programming problems and ranking of fuzzy numbers* , Fuzzy Sets and Systems , Vol. 32, 1989 , pp. 1-11.

[11] Chanas , S., *The use of parametric programming in fuzzy linear programming* , Fuzzy Sets and Systems , Vol. 11, 1983 , pp. 243-251.

[12] Delgado , M., Verdegay , J. L. , Vila, M. A., *A general model for fuzzy linear programming* , Fuzzy Sets and Systems , Vol. 29 ,1989 , pp. 21-29.

[13] Dutta , D., Rao , J. R. , Tiwari, R. N. , *Sensitivity analysis in fuzzy linear fractional programming problem* , Fuzzy Sets and Systems , Vol. 48 , 1992 , pp. 211-216.

[14] Ebrahimnejad , A. , *Sensitivity analysis in fuzzy number linear programming problems* , Mathematical and Computer Modelling , Vol. 53 , pp. 1878-1888.

[15] Ebrahimnejad , A. , Nasseri , S. H. , *Using complementary slackness property to solve linear programming with fuzzy parameters* , Fuzzy Information and Engineering , Vol. 3 , 2009 , pp. 233-245.

[16] Ebrahimnejad , A. , Nasseri , S. H. , *A dual simplex method for bounded linear programmes with fuzzy numbers* , International Journal of Mathematics in Operational Research , Vol. 2 , 2010 , pp. 762-779.

[17] Ebrahimnejad , A. , Nasseri , S. H. , Hosseinzadeh Lotfi , F . , *Bounded linear programs with trapezoidal fuzzy numbers* , International Journal of Uncertainty ,

Fuzziness and Knowledge - Based Systems , Vol. 18 , 2010 , pp. 269-286.

[18] Ebrahimnejad , A. , Nasseri , S. H. , Hosseinzadeh Lotfi , F., and Soltanifar , M. , *A primal- dual method for linear programming problems with fuzzy variables* , European Journal of Industrial Engineering , Vol. 4 , 2010 , pp. 189-209.

[19] Ebrahimnejad , A. , Nasseri , S. H., Mansourzadeh , S.M. , *Bounded primal simplex algorithm for bounded linear programming with fuzzy cost coefficients* , International Journal of Operations Research and Information Systems , Vol. 2 , 2011 , pp. 96-120.

[20] Fang , S.C., Hu, C.F., *Linear programming with fuzzy coefficients in constraints* , Computers and Mathematics with Applications , Vol. 37, 1999, pp. 63-76

[21] Fortemps , P., Roubens , M., *Ranking and defuzzification methods based on area compensation* , Fuzzy Sets and Systems , Vol. 82, 1996, pp. 319-330.

[22] Fuller , R., *On stability in fuzzy linear programming problems* , Fuzzy Sets and Systems , Vol. 30 , 1989 , pp. 339-344.

[23] Ganesan , K., Veeramani , P., *Fuzzy linear programming with trapezoidal fuzzy numbers* , Annals of Operations Research , Vol. 143 , 2006, pp. 305-315.

[24] Gao , Y. , Zhang , G. , Lu , J. , *A fuzzy multi-objective bilevel decision support system* , International Journal of Information Technology and Decision Making , Vol. 8 , 2009 , pp. 93-108.

[25] Gupta , P. , Bhatia , D. , *Sensitivity analysis in fuzzy multiobjective linear fractional programming problem* , Fuzzy Sets and Systems , Vol. 122 , 2001 , pp.

229-236.

[26] Hamacher , H. , Leberling , H. , Zimmermann , H. J., *Sensitivity analysis in fuzzy linear programming* , Fuzzy Sets and Systems , Vol. 1 , 1978 , pp. 269-281.

[27] Hong , D. H. , *An additive decomposition of fuzzy numbers* , Kybernetika , Vol. 39 , 2003 , pp. 289-294.

[28] Inuiguchi , M., Ramik, J., Tanino, T., Vlach, M., *Satisficing solutions and duality in interval and fuzzy linear programming* , Fuzzy Sets and Systems , Vol. 135, 2003 , pp. 151-177.

[29] Katagiri , H. , Sakawa , M. , Kato , K. , Nishizaki , I . , *A fuzzy random multiobjective* 01 *programming based on the expectation optimization model using possibility and necessity measures* , Mathematical and Computer Modelling , Vol. 40 , 2004 , pp. 411-421.

[30] Klir , G.J., Yuan, B., Fuzzy Sets and Fuzzy Logic: Theory and Applications , Prentice -Hall, Englewood Cliffs , New Jersey , 1995.

[31] Lai ,Y.J., Hwang , C.L., *Fuzzy Mathematical Programming Methods and Applications* , Springer , Berlin , 1992.

[32] Lai ,Y.J., Hwang , C.L., *A new approach to some possibilistic linear programming problem* , Fuzzy Sets and Systems , Vol. 49, 1992 , pp. 121-133.

[33] Liu, Y.J., Shi , Y., Liu , Y.H., *Duality of MC2 linear programming : a constructive approach* , Journal of Mathematical Analysis and Applications , Vol. 194, 1995, pp. 389-413.

[34] Mahadavi -Amiri, N., Nasseri , S.H., *Duality in fuzzy variable linear programming* , 4th World Enformatika Conference , WEC'05 , June 24- 26, 2005, Istanbul , Turkey.

[35] Mahadavi -Amiri, N., Nasseri , S.H., *Duality in fuzzy number linear programming problem by use of a certain linear ranking function* , Applied Mathematics and Computation , Vol. 180, 2006, pp. 206-216 .

[36] Mahdavi - Amiri , N., Nasseri , S.H., *Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables* , Fuzzy Sets and Systems , Vol. 158 , 2007 , pp. 1961-1978.

[37] Mahdavi-Amiri , N. , Nasseri , S. H. , Yazdani , A., *Fuzzy primal simplex algorithms for solving fuzzy linear programming problems* , Iranian Journal of Operations Research , Vol. 1 , 2009 , pp. 68-84.

[38] Maleki , H.R. , *Ranking functions and their applications to fuzzy linear programming* , Far East Journal of Mathematical Sciences , Vol. 4 , 2002 , pp. 283-301.

[39] Maleki , H.R., Tata , M., Mashinchi , M., *Linear programming with fuzzy variables*, Fuzzy Sets and Systems , Vol. 109, 2000 , pp. 21-33.

[40] Mishmast Nehi , H., Maleki , H.R., Mashinchi , M., *Multiobjective linear programming with fuzzy variables* , Far East Journal of Mathematical Sciences , Vol. 5, 2002, pp. 155-172.

[41] Mishmast Nehi , H. , Maleki , H.R., Mashinchi , M. , *Solving fuzzy number*

*linear programming problem by Lexicographic ranking function* , Italian Journal of Pure and Applied Mathematics , Vol. 15, 2004 , pp. 9-20.

[42] Nasseri , S. H. , Ardil , E . , *Simplex method for fuzzy variable linear programming problems* , Proceedings of World Academy of Science , Engineering and Technology , Vol. 8 , 2005 , pp. 198-202.

[43] Nasseri , S. H. , Ebrahimnejad , A. , *A fuzzy dual simplex method for fuzzy number linear programming problem* , Advances in Fuzzy Sets and Systems , Vol. 5 , 2010 , pp. 81-95.

[44] Nasseri , S. H. , Ebrahimnejad , A. , *A fuzzy primal simplex algorithm and its application for solving flexible linear programming problems* , European Journal of Industrial Engineering , Vol. 4 , 2010 , pp. 372-389.

[45] Nasseri , S. H. , Ebrahimnejad , A. , *Sensitivity analysis on linear programming problems with trapezoidal fuzzy variables* , International Journal of Operations Research and Information Systems , Vol. 2 , 2011 , pp. 22-39.

[46] Nasseri , S. H. , Mahdavi-Amiri , N. , *Some duality results on linear programming problems with symmetric fuzzy numbers* , Fuzzy Information and Engineering , Vol. 1 , 2009 , pp. 59-66.

[47] Nasseri , S.H. , Ebrahimnejad , A. , Mizuno , S. , *Duality in fuzzy linear programming with symmetric trapezoidal numbers* , Applications and Applied Mathematics , Vol. 5 , 2010 , pp. 1467-1482.

[48] Pahlavani , A. , *A hybrid algorithm of improved case-based reasoning and*

*multi-attribute decision making in fuzzy environment for investment loan evaluation* , International Journal of Information and Decision Sciences , Vol. 2 , 2010 , pp. 17-49.

[49] Ramik , J. , *Duality in fuzzy linear programming based on fuzzy relations , Proc. of* $10^{th}$ *IFSA World Congress* , June 29- July 2, 2003, Istanbul , Turkey.

[50] Ramik , J. , *Duality in fuzzy linear programming : some new concepts and results* , Fuzzy Optimization and Decision Making , Vol. 4, 2005 , pp. 25-39.

[51] Rodder , W., Zimmermann , H.J., *Duality in fuzzy linear programming* , in: A.V. Fiacco , K.O. Kortanek (Eds.), Proceedings of International Symposium on External Methods and Systems Analysis , Berlin , Heidelberg , New York, 1980 , University of Texas at Austin , 1977 , pp. 415-427.

[52] Rommelfanger , H., Hanuscheck , R., and Wolf , J.,*Linear programming with fuzzy objective*, Fuzzy Sets and Systems , Vol. 29, 1989, pp. 31-48.

[53] Roubens , M., *Inequality constraints between fuzzy numbers and their use in mathematical programming* , in: R. Slowinski , J. Teghem (Eds.), Stochastic Versus Fuzzy Approaches To Multiobjective Mathematical Programming Under Uncertainty , Kluwer Academic Publishers , Dordrecht , 1991, pp. 321-330.

[54] Tanaka, H., Asai, K., Okuda ,T., *On fuzzy mathematical programming* , Journal of Cybernetics , Vol. 3, 1984 , pp. 37-46.

[55] Tanaka , H. , Ichihashi , H. , and Asai , K. , *A value of information in fuzzy linear programming problems via sensitivity analysis* , Fuzzy Sets and Systems , Vol.

18 , 1986 , pp. 119-186.

[56] Thomaidis , N. S. , Nikitakos , N. , Dounias , G. D. , *The evaluation of information technology projects: a fuzzy multicriteria decision-making approach* , International Journal of Information Technology and Decision Making , Vol. 5 , 2006 , pp. 89-122.

[57] Vasant , P., Nagarajan , R. , Yaacab , S., *Decision making in industrial production planning using fuzzy linear programming* , IMA , Journal of Management Mathematics , Vol. 15, 2004, pp. 53-55.

[58] Verdegay , J.L., *A dual approach to solve the fuzzy linear programming problems*, Fuzzy Sets and Systems , Vol. 14, 1984, pp. 131-141.

[59] Wang , X., Kerre , E., *Reasonable properties for the ordering of fuzzy quantities* , Fuzzy Sets and Systems , Vol. 118, 2001, pp. 375-405.

[60] Wang , M. , Liang , L. , *Fuzzy context dependent data envelopment analysis* , International Journal of Data Analysis Techniques and Strategies , Vol. 1 , 2009 , pp. 211-233.

[61] Wu , H.C., *Duality theory in fuzzy linear programming problems with fuzzy coefficients* , Fuzzy Optimization and Decision Making, Vol. 2, 2003, pp. 61-73.

[62] Wu , H.C., *Duality theorems in fuzzy mathematical programming problems based on the concept of necessity* , Fuzzy Sets and Systems , Vol. 139, 2003, pp. 363-377.

[63] Yager , R.R., *A procedure for ordering fuzzy subsets of the unit interval* , Information Sciences , Vol. 24, 1981, pp. 143-161.

[64] Yager , R. R. , Filev , D. P. , Essentials of fuzzy modeling and control , New York , NY: John Wiley and Sons , 1994.

[65] Zadeh , L.A. , Fuzzy sets and systems , Presented at the Symposium on Systems Theory, Polytechnic Institute of Brooklyn , April 20-22 , 1965 , (republished in : International Journal of General Systems , Vol. 17 , 1990 , pp. 129-138).

[66] Zimmermann , H.J. , *Description and optimization of fuzzy systems* , International Journal of General Systems , Vol. 2 , 1976 , pp. 209-215.

[67] Zimmermann , H.J., *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems , Vol. 1, 1978 , pp. 45-55.

[68] Zimmermann , H.J., *Applications of fuzzy sets theory to mathematical programming* , Information Sciences , Vol. 36, 1985, pp. 29-58.