SOME ALGORITHMS FOR SOLVING SCHEDULING PROBLEMS IN FUZZY ENVIRONMENT USING CPM

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CERTIFICATE

This is to certify that the thesis entitled, **"Some Algorithms for Solving Scheduling Problems in Fuzzy Environment using CPM"**, submitted by Mrs. Parmpreet Kaur in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidates own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

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TO

MY DAUGHTER

“MEHAR”

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Patiala (Parmpreet Kaur)

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Abstract

In this thesis, the shortcomings and limitations of the existing methods for finding the optimal fuzzy project completion time as well as for finding minimum fuzzy crashing cost (additional fuzzy cost) corresponding to specific fuzzy project completion time are pointed out. Also, new methods are proposed to overcome the limitations as well as to resolve the shortcomings of the existing methods.

The chapter wise summary of the thesis is as follows:

Chapter 1

In this chapter, a brief review of the work done in the area of fuzzy critical path problems and fuzzy project crashing problems are presented.

Chapter 2

Liang and Han [84] proposed a method to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a fuzzy number. In this chapter, shortcomings of this existing method occurring due to existing subtraction operation are pointed out. To overcome these shortcomings a new subtraction operation, named as Mehar's subtraction (Mehar is my lovely daughter), is proposed and it is shown that by using Mehar's subtraction all

the shortcomings of the existing method [84] are resolved.

Chapter 3

Liu [93] proposed a method based on fuzzy linear programming approach to find optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number. In this chapter, it is shown that on applying this method more than one fuzzy numbers, representing the optimal fuzzy project completion time, are obtained which contradicts the uniqueness property of optimal fuzzy project completion time. Also, it is shown that the same shortcoming is occurring in the modified Liang and Han method as well as in the existing Liang and Han method [84]. To overcome these shortcomings, a new method, named as Mehar's method based on Kaufmann and Gupta ranking approach, is proposed by modifying the existing method [93] and some modifications are also suggested in the modified Liang and Han method.

Chapter 4

In the previous chapter, it is shown that by using the proposed Mehar's method based on Kaufmann and Gupta ranking approach and modified Liang and Han method based on Kaufmann and Gupta ranking approach all the shortcomings of Liu method [93] and the modified Liang and Han method are resolved. Since, Kaufmann and Gupta ranking approach [70] is applicable only for finding the maximum and minimum of triangular fuzzy numbers so the methods, proposed in previous chapter, can not be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number. In this chapter, it is shown that it is not genuine to use the existing ranking approaches for finding the maximum and minimum of trapezoidal fuzzy numbers and a new ranking approach, by extending Kaufmann and Gupta ranking approach, is proposed for finding the maximum and minimum of trapezoidal fuzzy numbers. Also, on the basis of extended Kaufmann and Gupta ranking approach new methods, by modifying the methods proposed in previous chapter, are proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

Chapter 5

In this chapter, limitations of the methods, proposed in previous chapter, are pointed out and to overcome these limitations a ranking approach by modifying Farhadinia ranking approach [55] as well as new methods based on modified Farhadinia ranking approach are proposed.

Chapter 6

In this chapter, it is shown that it is not genuine to apply the modified Farhadinia ranking approach for finding the maximum and minimum of LR flat fuzzy numbers and a new ranking approach, named as Mehar's ranking approach, is proposed for finding the maximum and minimum of LR flat fuzzy numbers. Also, on the basis of proposed Mehar's ranking approach a new method, named as Mehar's method based on proposed Mehar's ranking approach, is proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. Also, the modified Liang and Han method based on modified Farhadinia ranking approach is modified on the basis of proposed Mehar's ranking approach.

Chapter 7

In many situations, there is need to complete the project in a fuzzy time which is less than the optimal initial fuzzy project completion time. To handle such situations, Chen and Tsai [30] proposed a method for finding the minimum fuzzy crashing cost (additional fuzzy cost) for completing the project within specific fuzzy time which is less than the optimal initial fuzzy project completion time. In this chapter, the shortcomings of this method are pointed out and to overcome these shortcomings a new method, named as JMD (JAI MATA (MEHAR) DI) method, is proposed. The advantages of the proposed method over the existing method [30] is discussed. Also, a new representation of LR flat fuzzy numbers, named as JMD representation of LR flat fuzzy numbers, is proposed and shown that it is better to represent the parameters of any fuzzy linear programming problem by JMD LR flat fuzzy numbers as compared to the existing representation of LR flat fuzzy numbers.

Chapter 8

Finally, in this chapter future scope is suggested.

List of Research Papers

1. A. Kumar, **P. Kaur**, A new method for fuzzy critical path analysis in project networks with a new representation of triangular fuzzy numbers, Applications and Applied Mathematics: An International Journal, 5 (2010) 1442−1466.

2. A. Kumar, **P. Kaur**, A new approach for fuzzy critical path analysis, International Journal of Mathematics in Operational Research, 3 (2011) 341−357.

3. A. Kumar, **P. Kaur**, Exact optimal solution of critical path problems, Applications and Applied Mathematics: An International Journal, 6 (2011) 1992−2008.

4. A. Kumar, **P. Kaur**, A new method for solving fuzzy critical path problems based on fuzzy linear programming formulation, The Journal of Fuzzy Mathematics, 20 (2012) 103−117.

5. A. Kumar, **P. Kaur**, J. Kaur, Fuzzy optimal solution of fully fuzzy project crashing problems with new representation of LR flat fuzzy numbers, Lecture Notes in Computer Science, 6743 (2011) 171−174.

6. **P. Kaur**, A. Kumar, A new ranking approach and its application for solving fuzzy critical path problems, South African Journal of Industrial Engineering (Accepted).

7. **P. Kaur**, A. Kumar, A modified ranking approach for solving fuzzy critical path problems with *LR* flat fuzzy numbers, Control and Cybernetics (Accepted).

8. A. Kumar, **P. Kaur**, Modification in Chen and Tsai's method for solving time-cost trade-off problems of project networks in fuzzy environment, Neural Computing and Applications (Communicated).

9. A. Kumar, **P. Kaur**, A new approach for finding the exact fuzzy optimal solution of fuzzy critical path problems, International Journal of Computational Intelligence Systems (Communicated).

Papers Presented in Conferences

1. A. Kumar, **P. Kaur**, A new algorithm for solving fuzzy critical path problems, 8^{th} Trinnial Conference of Association of Asian Pacific Operational Research Societies (APORS), Jaipur, India, Dec 6-9 2009.

2. A. Kumar, **P. Kaur**, Mehar's method for solving fuzzy critical path problems with generalized *LR* fuzzy parameters, International Congress on Productivity, Quality, Reliability, Optimization and Modelling, Delhi, India, Feb 7-8 2011.

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Chapter 1

Introduction

A project is a collection of activities to accomplish a specific objective. Large and complex projects, such as constructing a building, implementing plant layout decisions, introducing a new product, designing a computer system and planning a military invasion etc. are common in all areas of industry and government. Project scheduling deals with the determination of activity times in a way which minimizes the project completion cost. Project management involves project scheduling, planning, evaluating, and controlling.

In todays highly competitive business environment, project managements ability to schedule activities and monitor progress within strict cost, time, and performance guidelines is becoming increasingly important to obtain competitive priorities such as on time delivery and customization. A project is deemed complete if work along all paths (a path through a project network is one of the routes from the starting node to the ending node) is complete. After the durations and precedence relations of activities have been determined, project management techniques are used to calculate the completion time of the project. Since the beginning of 1960s critical path method (CPM) [73] have become widely recognized as valuable tools for the planning and scheduling of projects. This method provides an excellent way of calculating the shortest completion time and the critical activities for a project.

In many situations, the project manager must complete the project in a time, which is less than the critical path, under budget constraint. In general, the duration of any activity of a project can be controlled by the allocation of more resources to the activity. Therefore, project crashing is a method for shortening the project duration by reducing the time of one or more of the critical activities to less than its normal activity time.

In the real world, the project is executed in an environment the uncertainty being one the principal features of it. One of these uncertainties in project planning process is estimating the activities duration. The duration of activities is usually estimated by the experts and considering their judgment and expertise. The experts use terms like almost, a little more, about, more or less etc. These terms clearly show some kind of uncertainty. Some of the methods exist for using this uncertainty in approximating the duration of the activities, the most important and employed of which being the probable methods like project evaluation and review technique as well as graphical evaluation and review technique. These methods use probability distributions such as normal distribution and beta distribution for estimating the duration of the project activities. To use the probability distributions, the repeatable random samples are needed, which is not possible properly due to the unique activities of the project and their little antecedent. Moreover, when using a probability distribution, the scheduling variables depend upon the distribution treatment, causing restriction of the project scheduling.

One basic solution for solving such problems is using the fuzzy theory [174]. The fuzzy theory established a new attitude towards different sciences including the project scheduling. This is a way towards realizing the project scheduling models through considering the uncertainty in decision making parameters and using the experts mental models.

The main advantages of methodologies based on fuzzy theory are that they do not require prior predictable regularities or posterior frequency distributions and they can deal with imprecise input information containing feelings and emotions quantified based on the decision-makers subjective judgment. The comparison of the fuzzy approach and stochastic approach to project scheduling can be found in Shipley et al. [141].

1.1 Literature review

In this section, a brief review of the work done in the area of fuzzy critical path problems and fuzzy project crashing problems are presented.

Gazdik [59] proposed a technique called FNET based on a combination of fuzzy sets and the theory of graphs and used it for finding critical path as well as for calculating fuzzy project completion time. McCahon and Lee [105] proposed a method for determining fuzzy project completion time and the degree of criticality of each path for such project network problems in which time of each activity is represented by a triangular fuzzy number. Also, McCahon and Lee used possibility theory to determine the possibilities of the project completion given the fuzzy project completion time. Kaufmann and Gupta [70] proposed a method for finding the critical path and optimal fuzzy project completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number.

Rommelfanger [132] pointed out that in the literature extended subtraction is used for calculating latest starting times due to which the latest starting times turn more and more fuzzy while getting closure to the end of the calculation process and some times the fuzzy latest starting times of the first tasks even reduce fuzzy intervals with negative elements. To overcome this shortcoming of the existing methods, Rommelfanger proposed a method for determining latest starting and finishing times of such project network problems in which time of each activity is represented by an LR flat fuzzy number.

Nasution [123] proposed an extension of FNET [59] and introduced an interactive fuzzy subtraction in the backward calculations to show that fuzzy numbers could be exploited further in the network. Chang et al. [21] proposed a method for calculating the fuzzy project completion time and the degree of criticality for each path of such project network problems in which time of each activity is represented by a triangular fuzzy number. Mon et al. [111] adopted the concept of α-cut, the decision-makers index of optimism, and fuzzy program evaluation and review technique to analyze the time cost trade-off problem with activity times being fuzzy triangular distribution or fuzzy normal distribution. Yao and Lin [171] proposed a method for comparing triangular fuzzy numbers and used it to obtain critical paths of such project network problems in which time of each activity is represented by a triangular fuzzy number.

Chanas and Zielinski [18] proposed two methods for calculating the path degree of criticality of such project network problems in which all the activity times are represented by different type of LR flat fuzzy numbers. Chen and Chang [25] proposed a fuzzy project evaluation and review technique algorithm for finding multiple possible critical paths of such fuzzy project network problems in which time of each activity is represented by a trapezoidal fuzzy number. Chanas et al. [16] offer some methods that can be used for determining the possible criticality of paths but can not be used to solve the problem of the degree of the necessary criticality of activities. Lin [87] proposed an approach to develop a fuzzy critical path method based on statistical data for solving fuzzy critical path problems.

Lin and Yao [91] proposed an approach for implementing a fuzzy critical path method for activity networks based on statistical confidence-interval estimates and a signed-distance ranking for $(1-\alpha)$ fuzzy number levels. Dubois et al. [42] pointed out that the existing methods can be used for computing the possible values of the earliest starting times by means of a forward recursion procedure but can not be used for computing possible values of the latest starting times and proposed a rigorous treatment of this problem in the frame work of possibility theory. Liu [93] proposed a method based on mathematical programming formulation for finding the optimal solutions of fuzzy critical path problems as well as fuzzy project crashing problems. Liang and Han [84] proposed an algorithm for finding the critical path and the optimal fuzzy project completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

Zielinski [181] pointed out that the existing methods can be used only for computing the intervals of possible values of the latest starting times and float of activities of such series parallel networks in which time of each activity is represented by a fuzzy number. But, the existing methods can not be used for non-series parallel networks and proposed polynomial algorithms to overcome this limitation of the existing methods. Chao-guang et al. [22] proposed a method based on genetic algorithms for solving such time-cost trade-off problems in which each parameter is represented by a triangular fuzzy number. Han et al. [63] used the existing method [84] to tackle the problem in fuzzy airports ground operation decision analysis. Liang [81] presented an interactive fuzzy linear programming approach for minimizing total costs with reference to direct, indirect and penalty costs of project management decision problems in fuzzy environment.

Chen [24] proposed an approach based on the extension principle and linear programming formulation for finding the membership function of fuzzy total duration time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. Soltani and Haji [143] proposed a method for computing earliest times, latest times and slack times of such project scheduling problems in which time of each activity is represented by a trapezoidal fuzzy number. Chen and Huang [29] proposed a method for computing the fuzzy bounds for starting time, finishing time and critical degree of each activity as well as critical degree of each critical path of such project network problems in which time of each activity is represented by a triangular fuzzy number.

Chen and Hsueh [28] pointed out that it is difficult to use the existing approach [24] for solving such critical path problems in which activity times are represented by different type of LR flat fuzzy numbers and proposed a simple approach for the same. Sharafi et al. [140] presented a method for calculating earliest and latest start and finish time as well as slack time of such fuzzy project scheduling problems in which time of each activity is represented by a triangular fuzzy number. Liberatore [85] proposed a method for determining fuzzy set of critical path lengths and fuzzy activity criticality of such project network problems in which time of each activity is represented by a triangular fuzzy number. Lin [88] proposed an approach based on statistical confidence-interval estimates and a distance ranking method for solving fuzzy critical path and fuzzy time-cost trade-off problems. Lin [90] proposed an approach for solving fuzzy project crashing in project management based on statistical confidence interval estimates.

Sireesha and Shankar [142] proposed a method for computing total float time of each activity and critical path of such fuzzy project network problems in which time of each activity is represented by a triangular fuzzy number. Shankar et at. [138] pointed out the limitations of the existing method [25] and proposed an analytical method based on a new defuzzification formulas for finding critical path of such fuzzy project network problems in which time of each activity is represented by a trapezoidal fuzzy number. Shankar et al. [139] proposed an analytical method based on metric distance ranking approach [26] for finding critical path of such fuzzy project network problems in which time of each activity is represented by a trapezoidal fuzzy number. Shahsavari pour et al. [135] proposed an algorithm based on fuzzy linear programming model for determining the critical path of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

Zareei et al. [177] proposed a method based on linear programming models to calculate fuzzy earliest and fuzzy latest events time of such project scheduling problems in which time of each activity is represented by an $L-R$ fuzzy number and derived the membership functions of earliest and latest times of events by calculating lower and upper bounds of α-cuts of earliest and latest times. Chen and Tsai [30] pointed out that if the parameters in a project network are represented by fuzzy numbers then the obtained minimum fuzzy crashing cost and optimal fuzzy activity times should also be fuzzy numbers and proposed a method which is significantly different from other existing methods to find the minimum fuzzy crashing cost and optimal fuzzy activity times.

Madhuri et al. [97] proposed a method based on linear programming models to calculate fuzzy earliest and fuzzy latest events time of such project scheduling problems in which time of each activity is represented by an $L-L$ fuzzy number and derived the membership functions of earliest and latest times of events by calculating lower and upper bounds of α-cuts of earliest and latest fuzzy times. Rao and Shankar [128] proposed a method based on lexicographic ordering for finding critical path of such fuzzy project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

After reviewing the literature, it can be concluded that there are several shortcomings and limitations in the existing methods for finding the optimal fuzzy project completion time as well as for finding minimum fuzzy crashing cost (additional fuzzy cost) corresponding to specific fuzzy project completion time. In this thesis, new methods are proposed to overcome the limitations as well as to resolve the shortcomings of the existing methods.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Liang and Han [84] proposed a method to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a fuzzy number. In this chapter, shortcomings of this existing method occurring due to existing subtraction operation are pointed out. To overcome these shortcomings a new subtraction operation, named as Mehar's subtraction (Mehar is my lovely daughter), is proposed and it is shown that by using Mehar's subtraction all the shortcomings of the existing method [84] are resolved.

Chapter 3

Liu [93] proposed a method based on fuzzy linear programming approach to find optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number. In this chapter, it is shown that on applying this method more than one fuzzy numbers, representing the optimal fuzzy project completion time, are obtained which contradicts the uniqueness property of optimal fuzzy project completion time. Also, it is shown that the same shortcoming is occurring in the modified Liang and Han method as well as in the existing Liang and Han method [84]. To overcome these shortcomings, a new method, named as Mehar's method based on Kaufmann and Gupta ranking approach, is proposed by modifying the existing method [93] and some modifications are also suggested in the modified Liang and Han method.

Chapter 4

In the previous chapter, it is shown that by using the proposed Mehar's method based on Kaufmann and Gupta ranking approach and modified Liang and Han method based on Kaufmann and Gupta ranking approach all the shortcomings of Liu method [93] and the modified Liang and Han method are resolved. Since, Kaufmann and Gupta ranking approach [70] is applicable only for finding the maximum and minimum of triangular fuzzy numbers so the methods, proposed in previous chapter, can not be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number. In this chapter, it is shown that it is not genuine to use the existing ranking approaches for finding the maximum and minimum of trapezoidal fuzzy numbers and a new ranking approach, by extending Kaufmann and Gupta ranking approach, is proposed for finding the maximum and minimum of trapezoidal fuzzy numbers. Also, on the basis of extended Kaufmann and Gupta ranking approach new methods, by modifying the methods proposed in previous chapter, are proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

Chapter 5

In this chapter, limitations of the methods, proposed in previous chapter, are pointed out and to overcome these limitations a ranking approach by modifying Farhadinia ranking approach [55] as well as new methods based on modified Farhadinia ranking approach are proposed.

Chapter 6

In this chapter, it is shown that it is not genuine to apply the modified Farhadinia ranking approach for finding the maximum and minimum of LR flat fuzzy numbers and a new ranking approach, named as Mehar's ranking approach, is proposed for finding the maximum and minimum of LR flat fuzzy numbers. Also, on the basis of proposed Mehar's ranking approach a new method, named as Mehar's method based on proposed Mehar's ranking approach, is proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. Also, the modified Liang and Han method based on modified Farhadinia ranking approach is modified on the basis of proposed Mehar's ranking approach.

Chapter 7

In many situations, there is need to complete the project in a fuzzy time which is less than the optimal initial fuzzy project completion time. To handle such situations, Chen and Tsai [30] proposed a method for finding the minimum fuzzy crashing cost (additional fuzzy cost) for completing the project within specific fuzzy time which is less than the optimal initial fuzzy project completion time. In this chapter, the shortcomings of this method are pointed out and to overcome these shortcomings a new method, named as JMD (JAI MATA (MEHAR) DI) method, is proposed. The advantages of the proposed method over the existing method [30] is discussed. Also, a new representation of LR flat fuzzy numbers, named as JMD representation of LR flat fuzzy numbers, is proposed and shown that it is better to represent the parameters of any fuzzy linear programming problem by JMD LR flat fuzzy numbers as compared to the existing representation of LR flat fuzzy numbers.

Chapter 8

Finally, in this chapter future scope is suggested.

Chapter 2

Modification In Liang And Han Method On The Basis Of Proposed Mehar's Subtraction

Liang and Han [84] proposed a method to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a fuzzy number. In this chapter, shortcomings of this existing method occurring due to existing subtraction operation are pointed out. To overcome these shortcomings a new subtraction operation, named as Mehar's subtraction (Mehar is my lovely daughter), is proposed and it is shown that by using Mehar's subtraction all the shortcomings of the existing method [84] are resolved.

2.1 Preliminaries

In this section, some basic concepts of fuzzy set theory and fuzzy critical path method are presented.

2.1.1 Fuzzy set theory

In this section, some basic concepts of fuzzy set theory are presented [70].

The contents of this chapter are published in the *International Journal of Mathematics in Operational Research* 3 (2011) 341-357.

2.1.1.1 Basic definitions

In this section, some basic definitions of fuzzy set theory are presented.

Definition 2.1 Let X be a classical set of objects. Then, the set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$, where $\mu_{\widetilde{A}} : X \to [0,1]$, is called a fuzzy set in X. The evaluation function $\mu_{\widetilde{A}}(x)$ is called the membership function.

Definition 2.2 Let $\widetilde{A}$ be a fuzzy set in X and $\lambda \in [0,1]$ be a real number. Then, a classical set $A^{\lambda} = \{x \in X : \mu_{\widetilde{A}}(x) \geq \lambda\}$ is called a λ-level set or λ-cut of $\widetilde{A}$.

Definition 2.3 A fuzzy set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$ is called a normalized fuzzy set if and only if $\underset{x \in X}{\text{Supremum}}\{\mu_{\widetilde{A}}(x)\} = 1$.

Definition 2.4 A fuzzy set $\widetilde{A}$ is called a convex fuzzy set if and only if $\mu_{\widetilde{A}}(\alpha x_1 + (1-\alpha)x_2) \geq \text{minimum}\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\} \ \forall \ x_1, x_2 \in X, \ \alpha \in [0,1]$.

Definition 2.5 A convex normalized fuzzy set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in \mathbb{R}\}$ on the real line $\mathbb{R}$ is called a fuzzy number if and only if $\mu_{\widetilde{A}}(x)$ is piecewise continuous in $\mathbb{R}$.

Definition 2.6 A fuzzy number $\widetilde{A}$ is said to be a non-negative fuzzy number if and only if $\mu_{\widetilde{A}}(x) = 0 \ \forall \ x < 0$.

Definition 2.7 A fuzzy number $\widetilde{A}$ defined on the universal set of real numbers $\mathbb{R}$, denoted as $\widetilde{A} = (a, b, c)$, is said to be a triangular fuzzy number if its membership function, $\mu_{\widetilde{A}}(x)$, is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)} & , \quad a \leq x < b \\ 1 & , \quad x = b \\ \frac{(x-c)}{(b-c)} & , \quad b < x \leq c \\ 0 & , \quad \text{otherwise} \end{cases}$$

Definition 2.8 Let $\widetilde{A} = (a, b, c)$ be a triangular fuzzy number. Then, its λ-cut A^{λ} is defined as follows:

$$A^{\lambda} = [a + (b-a)\lambda, c - (c-b)\lambda], \ 0 \leq \lambda \leq 1$$

Definition 2.9 A triangular fuzzy number $\widetilde{A} = (a, b, c)$ is said to be a non-negative triangular fuzzy number if and only if $a \geq 0$.

Definition 2.10 A triangular fuzzy number $\widetilde{A} = (a, b, c)$ is said to be a zero triangular fuzzy number if and only if $a = 0, b = 0$ and $c = 0$.

Definition 2.11 Two triangular fuzzy numbers $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ are said to be equal i.e., $\widetilde{A_1} = \widetilde{A_2}$ if and only if $a_1 = a_2, b_1 = b_2$ and $c_1 = c_2$.

2.1.1.2 Arithmetic operations

In this section, some arithmetic operations between two triangular fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are presented.

(i) Let $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then,

$$\widetilde{A_1} \oplus \widetilde{A_2} = (a_1 + a_2, b_1 + b_2, c_1 + c_2).$$

(ii) Let $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then,

$$\widetilde{A_1} \ominus \widetilde{A_2} = (a_1 - c_2, b_1 - b_2, c_1 - a_2).$$

(iii) Let $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ be two non-negative triangular fuzzy numbers. Then, $\widetilde{A_1} \otimes \widetilde{A_2} \simeq (a_1a_2, b_1b_2, c_1c_2)$.

(iv) Let $\widetilde{A} = (a, b, c)$ be any triangular fuzzy number. Then,

$$\gamma\widetilde{A} = \begin{cases} (\gamma a, \gamma b, \gamma c) & \gamma \geq 0 \\ (\gamma c, \gamma b, \gamma a) & \gamma \leq 0 \end{cases}$$

2.1.2 Some basic definitions related to fuzzy critical path method

In this section, some basic definitions related to fuzzy critical path method are presented.

Definition 2.12 [145] Any individual operation, which utilizes resources and has an end and a beginning, is called activity. It is usually classified into the following categories:

(i) ***Predecessor activity:*** An activity which must be completed before one or more other activities start is known as predecessor activity.

(ii) ***Successor activity:*** An activity which start immediately after one or more of other activities are completed is known as successor activity.

(iii) ***Dummy activity:*** An activity which does not consume either any resource or time but merely depicts the technological dependence is known as dummy activity.

Definition 2.13 [145] An event represent the beginning or end of the activity and as such it consumes no time. It has no time duration and does not consume any resources. It is also known as node. An event is not complete until all the activities flowing into it are completed.

Definition 2.14 [84] Fuzzy earliest starting time of node i $(\widetilde{FES_i})$ is the fuzzy earliest possible time for the node i from which the activity start.

Definition 2.15 [84] Fuzzy earliest starting time of activity (i, j) $(\widetilde{FES_{ij}})$ is the fuzzy earliest starting time of the tail end event(node) i.e., $\widetilde{FES_{ij}} = \widetilde{FES_i}$.

Definition 2.16 [84] Fuzzy earliest finishing time of activity (i, j) $(\widetilde{FEF_{ij}})$ is equal to the sum of fuzzy earliest starting time of the activity (i, j) and the fuzzy normal duration time of the activity (i, j) i.e., $\widetilde{FEF_{ij}} = \widetilde{FES_{ij}} \oplus \widetilde{FNT_{ij}} = \widetilde{FES_i} \oplus \widetilde{FNT_{ij}}$.

Definition 2.17 [84] Fuzzy earliest finishing time of node j $(\widetilde{FEF_j})$ is equal to the maximum of fuzzy earliest finishing time of all the activities ending into that node i.e., $\widetilde{FEF_j} = \text{maximum}\{\widetilde{FES_i} \oplus \widetilde{FNT_{ij}} \mid i \in NP(j), j \neq 1, j \in N\}$

where,

N : The set of all nodes in a project network

$NP(j)$: The set of all nodes connected to all predecessor activities of node j, i.e.,

$$NP(j) = \{i \quad | \ (i,j) \in F(j),\ i \in N\}$$

(i,j) : The activity between nodes i and j

$F(j)$: The set of all predecessor activities of node j

Definition 2.18 [84] Fuzzy latest finishing time of node j ($\widetilde{FLF_j}$) is equal to the fuzzy earliest finishing time of node j i.e., $\widetilde{FLF_j} = \widetilde{FEF_j}$

Definition 2.19 [84] Fuzzy latest finishing time of the activity (i,j) ($\widetilde{FLF_{ij}}$) is equal to the fuzzy latest finishing time of node j i.e., $\widetilde{FLF_{ij}} = \widetilde{FLF_j}$.

Definition 2.20 [84] Fuzzy latest starting time of the activity (i,j) ($\widetilde{FLS_{ij}}$) is obtained by subtracting the fuzzy normal time of the activity (i,j) from the fuzzy latest finishing time of the activity (i,j) i.e., $\widetilde{FLS_{ij}} = \widetilde{FLF_{ij}} \ominus \widetilde{FNT_{ij}} = \widetilde{FLF_j} \ominus \widetilde{FNT_{ij}}$.

Definition 2.21 [84] Fuzzy latest starting time of node i ($\widetilde{FLS_i}$) is the minimum of fuzzy latest starting time of all the activities originating from that node i.e.,
$\widetilde{FLS_i} = \text{minimum}\{\widetilde{FLF_k} \ominus \widetilde{FNT_{jk}} \mid k \in NS(j), j \neq n, j \in N\}$,

where,

$NS(j)$: The set of all nodes connected to all successor activities of node j,

i.e., $NS(j) = \{k \quad | \ (j,k) \in S(j),\ k \in N\}$,

$S(j)$: The set of all successor activities of node j,

n : the destination node.

Definition 2.22 [84] Fuzzy slack time is the amount of fuzzy time that an activity can be delayed past its earliest start or earliest finish without delaying the total fuzzy completion time of the project.

2.2 Method for finding the maximum and minimum of triangular fuzzy numbers used in Liang and Han method

Liang and Han [84] proposed a method to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a fuzzy number. In this section, the method for finding the maximum and minimum of triangular fuzzy numbers used in Liang and Han method [84] is presented.

Liang and Han [84] used the following method for finding maximum and minimum of n triangular fuzzy numbers $\widetilde{A_i} = (a_i, b_i, c_i)$, $i = 1, 2, ..., n$:

Step 1 Choose any value of β where, $0 \leq \beta \leq 1$ and calculate ranking value of A_i ($\Re(A_i)$) by using the following expression:

$\Re(\widetilde{A_i}) = \beta[(c_i - x_1)/(x_2 - x_1 - b_i + c_i)] + (1 - \beta)[1 - (x_2 - a_i)/(x_2 - x_1 + b_i - a_i)]$

where, $x_1 = \underset{1 \leq i \leq n}{\text{minimum}}\{a_i\}$ and $x_2 = \underset{1 \leq i \leq n}{\text{maximum}}\{c_i\}$.

Step 2 Find $\underset{1 \leq i \leq n}{\text{maximum}}\{\Re(\widetilde{A_i})\}$ and check that maximum occurs corresponding to unique value of i or not.

Case (2a) If maximum occurs corresponding to unique value of i, say $i = t$ then $\underset{1 \leq i \leq n}{\text{maximum}}\{\widetilde{A_i}\} = \widetilde{A_t}$

Case (2b) If maximum occurs for p values of i, say $i = 1, 2, ..., p$ where, $2 \leq p \leq n$ then go to Step 3.

Step 3 Find $\underset{1 \leq i \leq p}{\text{maximum}}\{m(\widetilde{A_i})\} = \underset{1 \leq i \leq p}{\text{maximum}}\{b_i\}$ and check that maximum occurs corresponding to unique value of i or not.

Case (3a) If maximum occurs corresponding to unique value of i, say $i = \theta$ then $\underset{1 \leq i \leq p}{\text{maximum}}\{\widetilde{A_i}\} = \widetilde{A_\theta}$

Case (3b) If maximum occurs for l values of i, say $i = 1, 2, ..., l$ where, $2 \leq l \leq p$

then all the triangular fuzzy numbers $\widetilde{A_1}$, $\widetilde{A_2}$, ..., $\widetilde{A_l}$ can be treated as maximum values.

Remark 2.1: The value of β can be referred as the decision maker's risk attitude index. If $\beta < 0.5$, it implies that the decision maker is a risk averter. If $\beta = 0.5$, it implies that the risk attitude of decision maker is neutral. If $\beta > 0.5$, it implies that the decision maker is risk lover.

2.2.1 Illustrative examples

In this section, to illustrate the method, presented in Section 2.2, some numerical examples are solved by assuming $\beta = 0.5$.

Example 2.1 Let (14, 25, 39) and (12, 18, 28) be two triangular fuzzy numbers. Then, find maximum$\{(14, 25, 39), (12, 18, 28)\}$.

Solution: Using the method, presented in Section 2.2, maximum$\{(14, 25, 39), (12, 18, 28)\}$ can be obtained as follows:

Step 1 Since from Step 1 of the method, presented in Section 2.2, values of x_1 and x_2 are 12 and 39 respectively.

$$\text{So, } \Re(14, 25, 39) = 0.5[\tfrac{39-12}{39-12-25+39}] + (1-0.5)[1 - \tfrac{39-14}{39-12+25-14}] = 0.50032$$

$$\Re(12, 18, 28) = 0.5[\tfrac{28-12}{39-12-18+28}] + (1-0.5)[1 - \tfrac{39-12}{39-12+18-12}] = 0.307$$

Step 2 Since, maximum$\{\Re(14, 25, 39), \Re(12, 18, 28)\} = \Re(14, 25, 39)$

So, maximum$\{(14, 25, 39), (12, 18, 28)\} = (14, 25, 39)$

Example 2.2 Let (0, 15, 32) and $(-11, 18, 30)$ be two triangular fuzzy numbers. Then, find minimum$\{(0, 15, 32), (-11, 18, 30)\}$.

Solution: Using the method, presented in Section 2.2, minimum$\{(0, 15, 32), (-11, 18, 30)\}$ can be obtained as follows:

Step 1 Since from Step 1 of the method, presented in Section 2.2, values of x_1 and x_2 are -11 and 32 respectively.

So, $\Re(0,15,32) = 0.5[\frac{32-(-11)}{32-(-11)-15+32}]+(1-0.5)[1-\frac{32-0}{32-(-11)+15-0}] = 0.58$

$\Re(-11,18,30) = 0.5[\frac{30-(-11)}{32-(-11)-18+30}]+(1-0.5)[1-\frac{32-(-11)}{39-(-11)+18-(-11)}] = 0.46$

Step 2 Since, minimum$\{\Re(0,15,32),\Re(-11,18,30)\} = \Re(-11,18,30)$

So, minimum$\{(0,15,32),(-11,18,30)\} = (-11,18,30)$

Example 2.3 Let (1, 4, 7) and $(2,4,6)$ be two triangular fuzzy numbers. Then, find maximum$\{(1,4,7),(2,4,6)\}$.

Solution: Using the method, presented in Section 2.2, maximum$\{(1,4,7),(2,4,6)\}$ can be obtained as follows:

Step 1 Since from Step 1 of the method, presented in Section 2.2, values of x_1 and x_2 are 1 and 7 respectively.

So, $\Re(1,4,7) = 0.5[\frac{7-1}{7-1-4+7}]+(1-0.5)[1-\frac{7-1}{7-1+4-1}] = 0.5$

$\Re(2,4,6) = 0.5[\frac{6-1}{7-1-4+6}]+(1-0.5)[1-\frac{7-2}{7-1+4-2}] = 0.5$

Step 2 Since, $\Re(1,4,7) = \Re(2,4,6)$ so go to Step 3

Step 3 Using Step 3 of the method, presented in Section 2.2, values of $m(1,4,7)$ and $m(2,4,6)$ are 4 and 4 respectively.

Since, $m(1,4,7) = m(2,4,6)$ so both (1, 4, 7) and (2, 4, 6) can be treated as maximum.

2.3 Liang and Han method

Liang and Han [84] proposed the following method for calculating the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number:

Step 1 Assume the fuzzy earliest starting time of the starting node $\widetilde{FES_1}$ as (0, 0, 0) and calculate the fuzzy earliest starting time of node j ($\widetilde{FES_j}$) by using the expression:

$$\widetilde{FES_j} = \text{maximum}\{\widetilde{FES_i} \oplus \widetilde{FNT_{ij}} \mid i \in NP(j),\, j \neq 1,\, j \in N\}$$

Step 2 Assume the fuzzy latest finishing time of the last node $\widetilde{FLF_n}$ equal to the fuzzy earliest starting time of the last node $\widetilde{FES_n}$ and calculate the fuzzy latest finishing time of node j ($\widetilde{FLF_j}$) by using the expression:

$$\widetilde{FLF_j} = \text{minimum}\{\widetilde{FLF_k} \ominus \widetilde{FNT_{jk}} \mid k \in NS(j),\, j \neq n,\, j \in N\}$$

Step 3 Calculate the fuzzy total slack time of the activity (i, j) ($\widetilde{FTS_{ij}}$) by using the expression:

$$\widetilde{FTS_{ij}} = \widetilde{FLF_j} \ominus (\widetilde{FES_i} \oplus \widetilde{FNT_{ij}});\ i, j \in N$$

Step 4 Find minimum$\{\sum_{(i,j)\in S_k} \widetilde{FTS_{ij}}, \forall\ k = 1, 2, ..., p\}$,

where,

p: Number of possible paths between starting and last node.

S_k: Set of all the activities of the k^{th} path

If the minimum occurs corresponding to $k = c$ then the path P_c will be the critical path.

Step 5 Calculate the optimal fuzzy completion time $\widetilde{FCT}$ of the project by using the expression:

$$\widetilde{FCT} = \sum_{(i,j)\in S_c} \widetilde{FNT_{ij}}$$

2.3.1 Illustrative example

In this section, the existing method [84] is illustrated by solving a numerical example.

Example 2.4 Find the optimal fuzzy completion time of the project network, shown in Figure 2.1, in which the fuzzy normal time $(\widetilde{FNT_{ij}})$ of the activity (i, j) is represented by the following triangular fuzzy numbers:
$\widetilde{FNT_{12}} = (5,\ 8,\ 14)$, $\widetilde{FNT_{13}} = (2,\ 4,\ 7)$, $\widetilde{FNT_{23}} = (3,\ 7,\ 10)$, $\widetilde{FNT_{24}} = (7,\ 10,\ 14)$, $\widetilde{FNT_{34}} = (6,\ 10,\ 15)$

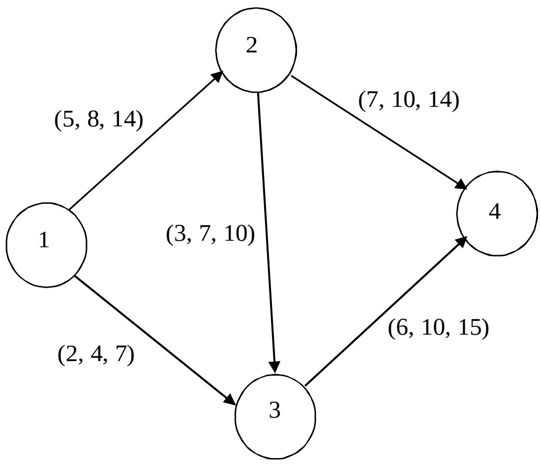

Figure 2.1 Fuzzy activity times in the project network

Solution: Using the method, presented in Section 2.3, the optimal fuzzy completion time of the project network, shown in Figure 2.1, can be obtained as follows:

Step 1 Using Step 1 of Liang and Han method [84], presented in Section 2.3, by assuming $\widetilde{FES_1} = (0,\ 0,\ 0)$ values of $\widetilde{FES_j}$, $j = 2, 3, 4$ can be obtained as follows:

$$\widetilde{FES_2} = \widetilde{FES_1} \oplus \widetilde{FNT_{12}} = (0,\ 0,\ 0) \oplus (5,\ 8,\ 14) = (5,\ 8,\ 14)$$

$$\widetilde{FES_3} = \text{maximum } \{\widetilde{FES_1} \oplus \widetilde{FNT_{13}},\ \widetilde{FES_2} \oplus \widetilde{FNT_{23}}\}$$

$$= \text{maximum}\{(0, 0, 0) \oplus (2, 4, 7), (5, 8, 14) \oplus (3, 7, 10)\}$$

$$= \text{maximum}\{(2, 4, 7), (8, 15, 24)\} = (8, 15, 24)$$

$$\widetilde{FES_4} = \text{maximum } \{\widetilde{FES_2} \oplus \widetilde{FNT_{24}},\ \widetilde{FES_3} \oplus \widetilde{FNT_{34}}\}$$

$$= \text{maximum}\{(5, 8, 14) \oplus (7, 10, 14), (8, 15, 24) \oplus (6, 10, 15)\}$$

$$= \text{maximum}\{(12,18,28),(14,25,39)\} = (14,25,39)$$

Step 2 Using Step 2 of Liang and Han method, presented in Section 2.3, by assuming $\widetilde{FLF_4} = \widetilde{FES_4} = (14,\ 25,\ 39)$ values of $\widetilde{FLF_j}$, $j = 3,2,1$ can be obtained as follows:

$$\widetilde{FLF_3} = \widetilde{FLF_4} \ominus \widetilde{FNT_{34}} = (14,\ 25,\ 39) \ominus (6,\ 10,\ 15) = (-1,15,33)$$

$$\widetilde{FLF_2} = \text{minimum}\{\widetilde{FLF_4} \ominus \widetilde{FNT_{24}},\ \widetilde{FLF_3} \ominus \widetilde{FNT_{23}}\}$$

$$= \text{minimum}\{(14,25,39) \ominus (7,10,14),(-1,15,33) \ominus (3,7,10)\}$$

$$= \text{minimum}\{(0,15,32),(-11,8,30)\} = (-11,8,30)$$

$$\widetilde{FLF_1} = \text{minimum}\{\widetilde{FLF_3} \ominus \widetilde{FNT_{13}}, \widetilde{FLF_2} \ominus \widetilde{FNT_{12}}\}$$

$$= \text{minimum}\{(-1,15,33) \ominus (2,4,7),(-11,8,30) \ominus (5,8,14)\}$$

$$= \text{minimum}\{(-8,11,31),(-25,0,25)\}$$

i.e., $\widetilde{FLF_1} = (-25,0,25)$

Step 3 Using Step 3 of Liang and Han method, presented in Section 2.3, values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$$\widetilde{FTS_{12}} = \widetilde{FLF_2} \ominus (\widetilde{FES_1} \oplus \widetilde{FNT_{12}})$$

$$= (-11,8,30) \ominus ((0,\ 0,\ 0) \oplus (5,\ 8,\ 14))$$

$$= (-11,8,30) \ominus (5,\ 8,\ 14)$$

$$= (-25,0,25)$$

$$\widetilde{FTS_{13}} = \widetilde{FLF_3} \ominus (\widetilde{FES_1} \oplus \widetilde{FNT_{13}})$$

$$= (-1,15,33) \ominus ((0,\ 0,\ 0) \oplus (2,\ 4,\ 7))$$

$$= (-1,15,33) \ominus (2,\ 4,\ 7)$$

$$= (-8,11,31)$$

$$\widetilde{FTS_{23}} = \widetilde{FLF_3} \ominus (\widetilde{FES_2} \oplus \widetilde{FNT_{23}})$$

$$= (-1, 15, 33) \ominus ((5,\ 8,\ 14) \oplus (3,\ 7,\ 10))$$

$$= (-1, 15, 33) \ominus (8,\ 15,\ 24)$$

$$= (-25, 0, 25)$$

$$\widetilde{FTS_{24}} = \widetilde{FLF_4} \ominus (\widetilde{FES_2} \oplus \widetilde{FNT_{24}})$$

$$= (14,\ 25,\ 39) \ominus ((5,\ 8,\ 14) \oplus (7,\ 10,\ 14))$$

$$= (14,\ 25,\ 39) \ominus (12,\ 18,\ 28)$$

$$= (-14, 7, 27)$$

$$\widetilde{FTS_{34}} = \widetilde{FLF_4} \ominus (\widetilde{FES_3} \oplus \widetilde{FNT_{34}})$$

$$= (14,\ 25,\ 39) \ominus ((8,\ 15,\ 24) \oplus (6,\ 10,\ 15))$$

$$= (14,\ 25,\ 39) \ominus (14,\ 25,\ 39)$$

$$= (-25, 0, 25)$$

Step 4 Using Step 4 of Liang and Han method, presented in Section 2.3,

$$\text{minimum}\{\widetilde{FTS_{12}} \oplus \widetilde{FTS_{24}}, \widetilde{FTS_{12}} \oplus \widetilde{FTS_{23}} \oplus \widetilde{FTS_{34}}, \widetilde{FTS_{13}} \oplus \widetilde{FTS_{34}}\}$$

$$= \text{minimum}\{(-25, 0, 25) \oplus (-14, 7, 27), (-25, 0, 25) \oplus (-25, 0, 25) \oplus (-25, 0, 25),$$

$$(-8, 11, 31) \oplus (-25, 0, 25)\}$$

$$= \text{minimum}\{(-39, 7, 52), (-75, 0, 75), (-33, 11, 56)\} = (-75, 0, 75)$$

Since, minimum is occurring corresponding to the path $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ so it is the critical path for the chosen project network.

Step 5 Using Step 5 of Liang and Han method, presented in Section 2.3, the optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FNT_{12}} \oplus \widetilde{FNT_{23}} \oplus \widetilde{FNT_{34}}$$

$$= (5,\ 8,\ 14) \oplus (3,\ 7,\ 10) \oplus (6,\ 10,\ 15)$$

$$= (14,\ 25,\ 39)$$

2.4 Shortcoming of Liang and Han method occurring due to existing subtraction

It is obvious from Step 2 and Step 3 of Liang and Han method, presented in Section 2.3, that there is need to subtract two non-negative triangular fuzzy numbers for calculating the values of $\widetilde{FLF_j}$ and $\widetilde{FTS_{ij}}$. Since, $\widetilde{FLF_j}$ and $\widetilde{FTS_{ij}}$ are representing the time and negative time has no physical meaning. So, $\widetilde{FLF_j}$ and $\widetilde{FTS_{ij}}$ should be non-negative triangular fuzzy numbers. However, it is obvious from the numerical example that all the obtained values of $\widetilde{FLF_j}$ and $\widetilde{FTS_{ij}}$ are not non-negative triangular fuzzy numbers e.g., $\widetilde{FLF_3}$ = (14, 25, 39) $\ominus$ (6, 10, 15) = (−1, 15, 33) is a triangular fuzzy number with a negative part. It depicts that time may be negative. But, the negative time has no physical meaning and it is not feasible since it is not defined in the project scheduling. To avoid this problem, if the Minkowski's subtraction [70, pp. 223]
$\widetilde{A_1} \ominus \widetilde{A_2} = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$ where, $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ are triangular fuzzy numbers, is used.

Then, the result may be a triangular fuzzy number with negative part or may not be a triangular fuzzy number.

Example 2.5 Let $\widetilde{A} = (10, 15, 30)$ and $\widetilde{B} = (12, 13, 15)$. Then, $\widetilde{A} \ominus \widetilde{B} = \widetilde{A'} =$ (−2, 2, 15) is a triangular fuzzy number with negative part.

Example 2.6 Let $\widetilde{C} = (10, 20, 22)$ and $\widetilde{D} = (5, 10, 15)$. Then, $\widetilde{C} \ominus \widetilde{D} = \widetilde{C'} =$ (5, 10, 7) is not a triangular fuzzy number because it violates the convex conditions $(5 \leq 10 \leq 7)$.

Hence, it is also not genuine to use the Minkowski's subtraction.

So, there is need to define subtraction of non-negative triangular fuzzy numbers in such a way that $\widetilde{A'}$ and $\widetilde{C'}$ are non-negative triangular fuzzy numbers.

2.5 Proposed Mehar's subtraction

In this section, to resolve the shortcomings of existing subtraction, pointed out in Section 2.4, a new subtraction operation, $\ominus_M$, named as Mehar's subtraction, of non-negative triangular fuzzy numbers is proposed.

Let $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ be two non-negative triangular fuzzy numbers. Then,

$\widetilde{A_1} \ominus_M \widetilde{A_2} = (a, b, c)$.

$$\text{where, } \begin{cases} a = \text{maximum}\{0, (a_1 - a_2)\} \\ b = a + \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} \\ c = b + \text{maximum } \{0, (c_1 - b_1) - (c_2 - b_2)\} \end{cases}$$

2.5.1 Advantages of proposed Mehar's subtraction

In this section, advantages of proposed Mehar's subtraction over existing subtraction [70] are discussed.

2.5.1.1 Non-negativity property of proposed Mehar's subtraction

In this section, it is proved that if $\widetilde{A_1}$ and $\widetilde{A_2}$ are two non-negative triangular fuzzy numbers then $\widetilde{A_1} \ominus_M \widetilde{A_2}$ will always be a non-negative triangular fuzzy number.

Proposition 2.1 Let $\widetilde{A_1} = (a_1, b_1, c_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2)$ be two non-negative triangular fuzzy numbers. Then, $\widetilde{A_1} \ominus_M \widetilde{A_2} = (a, b, c)$

$$\text{where, } \begin{cases} a = \text{maximum}\{0, (a_1 - a_2)\} \\ b = a + \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} \\ c = b + \text{maximum } \{0, (c_1 - b_1) - (c_2 - b_2)\} \end{cases}$$

is always a non-negative triangular fuzzy number i.e.,

(i) $a \geq 0$ (ii) $b - a \geq 0$ (iii) $c - b \geq 0$

Proof: (i) Since, a_1 and a_2 are two independent real numbers so, $a_1 > a_2$ or $a_1 = a_2$ or $a_1 < a_2$.

Case (i) If $a_1 > a_2$ i.e., $a_1 - a_2 > 0$ then,

$$a = \text{maximum}\{0, (a_1 - a_2)\} = a_1 - a_2 > 0$$

Case (ii) If $a_1 = a_2$ i.e., $a_1 - a_2 = 0$ then,

$$a = \text{maximum}\{0, (a_1 - a_2)\} = 0$$

Case (iii) If $a_1 < a_2$ i.e., $a_1 - a_2 < 0$ then,

$$a = \text{maximum}\{0, (a_1 - a_2)\} = 0$$

Hence, $a \geq 0$

(ii) Since, $b_1 - a_1$ and $b_2 - a_2$ are two independent real numbers so, $(b_1 - a_1) - (b_2 - a_2) > 0$ or $(b_1 - a_1) - (b_2 - a_2) = 0$ or $(b_1 - a_1) - (b_2 - a_2) < 0$

Case (i) If $(b_1 - a_1) > (b_2 - a_2)$ i.e., $(b_1 - a_1) - (b_2 - a_2) > 0$ then,

$$b - a = \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} = (b_1 - a_1) - (b_2 - a_2) > 0$$

Case (ii) If $(b_1 - a_1) = (b_2 - a_2)$ i.e., $(b_1 - a_1) - (b_2 - a_2) = 0$ then,

$$b - a = \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} = 0$$

Case (iii) If $(b_1 - a_1) < (b_2 - a_2)$ i.e., $(b_1 - a_1) - (b_2 - a_2) < 0$ then,

$$b - a = \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} = 0$$

Hence, $b - a \geq 0$

(iii) Since, $c_1 - b_1$ and $c_2 - b_2$ are two independent real numbers so, $(c_1 - b_1) -$

$(c_2 - b_2) > 0$ or $(c_1 - b_1) - (c_2 - b_2) = 0$ or $(c_1 - b_1) - (c_2 - b_2) < 0$

Case (i) If $(c_1 - b_1) > (c_2 - b_2)$ i.e., $(c_1 - b_1) - (c_2 - b_2) > 0$ then,

$$c - b = \text{maximum}\{0, (c_1 - b_1) - (c_2 - b_2)\} = (c_1 - b_1) - (c_2 - b_2) > 0$$

Case (ii) If $(c_1 - b_1) = (c_2 - b_2)$ i.e., $(c_1 - b_1) - (c_2 - b_2) = 0$ then,

$$c - b = \text{maximum}\{0, (c_1 - b_1) - (c_2 - b_2)\} = 0$$

Case (iii) If $(c_1 - b_1) < (c_2 - b_2)$ i.e., $(c_1 - b_1) - (c_2 - b_2) < 0$ then,

$$c - b = \text{maximum}\{0, (c_1 - b_1) - (c_2 - b_2)\} = 0$$

Hence, $c - b \geq 0$.

2.5.1.2 Cancelation property of proposed Mehar's subtraction

In crisp environment, the cancelation properties $(A + B) - B = A$ and $B + (A - B) = A$ are always satisfied but the same is not true in fuzzy environment i.e., neither $(\widetilde{A} \oplus \widetilde{B}) \ominus \widetilde{B} = \widetilde{A}$ nor $\widetilde{B} \oplus (\widetilde{A} \ominus \widetilde{B}) = \widetilde{A}$ e.g., Let $\widetilde{A} = (15, 30, 40)$ and $\widetilde{B} = (10, 12, 13)$ be two triangular fuzzy numbers. Then,

$$\begin{aligned}(\widetilde{A} \oplus \widetilde{B}) \ominus \widetilde{B} &= ((15, 30, 40) \oplus (10, 12, 13)) \ominus (10, 12, 13)\\ &= (25, 42, 53) \ominus (10, 12, 13)\\ &= (12, 30, 43) \neq (15, 30, 40)\end{aligned}$$

Also, $\widetilde{B} \oplus (\widetilde{A} \ominus \widetilde{B}) = (10, 12, 13) \oplus ((15, 30, 40) \ominus (10, 12, 13))$

$$\begin{aligned}&= (10, 12, 13) \oplus (2, 18, 30)\\ &= (12, 30, 43) \neq (15, 30, 40)\end{aligned}$$

In this section, it is proved that for proposed Mehar's subtraction this property will always be satisfied.

Proposition 2.2 Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two non-negative triangular fuzzy numbers. Then, $(\widetilde{A} \oplus \widetilde{B}) \ominus_M \widetilde{B} = \widetilde{A}$ and $\widetilde{B} \oplus (\widetilde{A} \ominus_M \widetilde{B}) = \widetilde{A}$.

Proof: $(\widetilde{A} \oplus \widetilde{B}) \ominus_M \widetilde{B}$

$= ((a_1, b_1, c_1) \oplus (a_2, b_2, c_2)) \ominus_M (a_2, b_2, c_2)$

$= (a_1 + a_2, b_1 + b_2, c_1 + c_2) \ominus_M (a_2, b_2, c_2)$

$= (a_1 + a_2 - a_2,\ (a_1 + a_2 - a_2 + \text{maximum}\{0, (b_1 + b_2 - a_1 - a_2) - (b_2 - a_2)\},$

$(a_1 + a_2 - a_2 + \text{maximum}\{0, (b_1 + b_2 - a_1 - a_2) - (b_2 - a_2)\} + \text{maximum}\{0, (c_1$

$+ c_2 - b_1 - b_2) - (c_2 - b_2)\})$

$= (a_1,\ a_1 + \text{maximum}\{0, b_1 - a_1\},\ a_1 + \text{maximum}\{0, b_1 - a_1\} + \text{maximum}\{0, c_1 -$

$b_1\})$

$= (a_1,\ a_1 + b_1 - a_1,\ a_1 + b_1 - a_1 + c_1 - b_1) \qquad (\because b_1 - a_1, c_1 - b_1 \geq 0)$

$= (a_1, b_1, c_1) = \widetilde{A}$

Hence, $(\widetilde{A} \oplus \widetilde{B}) \ominus_M \widetilde{B} = \widetilde{A}$.

Similarly, it can be proved that $\widetilde{B} \oplus (\widetilde{A} \ominus_M \widetilde{B}) = \widetilde{A}$.

2.5.2 Illustrative examples

In this section, to illustrate proposed Mehar's subtraction and to show the advantages of proposed Mehar's subtraction over existing subtraction, the values of $\widetilde{A'}$ and $\widetilde{C'}$, calculated in Example 2.5 and Example 2.6, are calculated using proposed Mehar's subtraction instead of the existing subtraction. Also, it is shown that by using proposed Mehar's subtraction, the shortcoming occurring in the values of $\widetilde{A'}$ and $\widetilde{C'}$, pointed in Section 2.4, is resolved.

Example 2.7 Let $\widetilde{A} = (10, 15, 30)$, $\widetilde{B} = (12, 13, 15)$ and $\widetilde{A} \ominus_M \widetilde{B} = \widetilde{A'} = (a, b, c)$. Then, the values of a, b and c can be calculated as follows:

$a = \text{maximum}\{0, (10 - 12)\} = 0$

$b = 0 + \text{maximum}\{0, (15 - 10) - (13 - 12)\} = 0 + \text{maximum } \{0, (5 - 1)\}$

$= 0 + 4 = 4$

$c = 4 + \text{maximum } \{0, (30 - 15) - (15 - 13)\} = 4 + \text{maximum } \{0, (15 - 2)\}$

$= 4 + 13 = 17$

Hence, (10, 15, 30) $\ominus_M$ (12, 13, 15) = (0, 4, 17) is a non-negative triangular fuzzy number.

Example 2.8 Let $\widetilde{C}$ = (10, 20, 22), $\widetilde{D}$ = (5, 10, 15) and $\widetilde{C} \ominus_M \widetilde{D} = \widetilde{C'} = (a, b, c)$. Then, the values of a, b and c can be calculated as follows:

$a = \text{maximum}\{0, (10 - 5)\} = 5$

$b = 5 + \text{maximum}\{0, (20 - 10) - (10 - 5)\} = 5 + \text{maximum } \{0, (10 - 5)\}$

$= 5 + 5 = 10$

$c = 10 + \text{maximum } \{0, (22 - 20) - (15 - 10)\} = 10 + \text{maximum } \{0, (2 - 5)\}$

$= 10 + 0 = 10$

Hence, (10, 20, 22) $\ominus_M$ (5, 10, 15) = (5, 10, 10) is a non-negative triangular fuzzy number.

It can be easily seen that if the values of $\widetilde{A'}$ and $\widetilde{C'}$ are calculated using the proposed Mehar's subtraction instead of existing subtraction then there does not exist any negative part in the obtained values of $\widetilde{A'}$ and $\widetilde{C'}$. On the basis of the obtained results, it can be concluded that it is better to use the proposed Mehar's subtraction instead of existing subtraction.

2.6 Modified Liang and Han method

The shortcomings of Liang and Han method [84], pointed in Section 2.4, can be resolved by the following modifications in Step 2 and Step 3 of the existing method [84]:

Step 2 Calculate $\widetilde{FLF_j}$ by using the expression:

$\widetilde{FLF_j}$ = minimum$\{\widetilde{FLF_k} \ominus_M \widetilde{FNT_{jk}}$ / $k \in NS(j)$, $j \neq n$, $j \in N\}$ instead of using the expression:

$\widetilde{FLF_j}$ = minimum$\{\widetilde{FLF_k} \ominus \widetilde{FNT_{jk}}$ / $k \in NS(j)$, $j \neq n$, $j \in N\}$.

Step 3 Calculate the fuzzy total slack time of the activity (i, j) $(\widetilde{FTS_{ij}})$ by using the expression:

$\widetilde{FTS_{ij}} = \widetilde{FLF_j} \ominus_M (\widetilde{FES_i} \oplus \widetilde{FNT_{ij}})$; $i, j \in N$ instead of using the expression:

$\widetilde{FTS_{ij}} = \widetilde{FLF_j} \ominus (\widetilde{FES_i} \oplus \widetilde{FNT_{ij}})$; $i, j \in N$.

2.7 Advantages of modified Liang and Han method

Since, in the modified Liang and Han method proposed Mehar's subtraction is used and Mehar's subtraction is proposed in such a manner that if Mehar's subtraction is used for subtracting two non-negative triangular fuzzy numbers then the obtained number will always be a non-negative triangular fuzzy number. So, on applying the modified Liang and Han method all the shortcomings of existing Liang and Han method [84], occurring due to using existing subtraction, are resolved.

To show the advantages of modified Liang and Han method over the existing Liang and Han method the project network problem, chosen in Example 2.4, is solved by using the modified Liang and Han method. Also, it is shown that by using the modified Liang and Han method the shortcoming, pointed out in Section

2.4, is resolved.

The results of the numerical example, obtained by using the existing Liang and Han method and the modified Liang and Han method, are shown in Table 2.1

Table 2.1: Values of $\widetilde{FES_j}$, $\widetilde{FLF_j}$ and $\widetilde{FCT}$ obtained by using the existing Liang and Han method and modified Liang and Han method

Methods	Results of Example 2.4			
	$\widetilde{FES_j}$	$\widetilde{FLF_j}$	Critical path	$\widetilde{FCT}$
Existing Liang and Han method [84]	$\widetilde{FES_1} = (0, 0, 0)$ $\widetilde{FES_2} = (5, 8, 14)$ $\widetilde{FES_3} = (8, 15, 24)$ $\widetilde{FES_4} = (14, 25, 39)$	$\widetilde{FLF_1} = (-25, 0, 25)$ $\widetilde{FLF_2} = (-11, 8, 30)$ $\widetilde{FLF_3} = (-1, 15, 33)$ $\widetilde{FLF_4} = (14, 25, 39)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)
Modified Liang and Han method	$\widetilde{FES_1} = (0, 0, 0)$ $\widetilde{FES_2} = (5, 8, 14)$ $\widetilde{FES_3} = (8, 15, 24)$ $\widetilde{FES_4} = (14, 25, 39)$	$\widetilde{FLF_1} = (0, 0, 0)$ $\widetilde{FLF_2} = (5, 8, 14)$ $\widetilde{FLF_3} = (8, 15, 24)$ $\widetilde{FLF_4} = (14, 25, 39)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)

It is obvious from the values, shown in Table 2.1, that the optimal fuzzy completion time ($\widetilde{FCT}$) obtained by using the existing Liang and Han method and the modified Liang and Han method are same. Also, it is obvious that in the values of $\widetilde{FLF_1}$, $\widetilde{FLF_2}$ and $\widetilde{FLF_3}$ obtained by using the existing Liang and Han method there exist negative part due to which the values of $\widetilde{FLF_1}$, $\widetilde{FLF_2}$ and $\widetilde{FLF_3}$ has no physical meaning. However, in the values of $\widetilde{FLF_1}$, $\widetilde{FLF_2}$ and $\widetilde{FLF_3}$ obtained by using the modified Liang and Han method there does not exist any negative part.

2.8 Conclusions

On the basis of presented study, it can be concluded that it is better to use modified Liang and Han method instead of using the existing Liang and Han method to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number.

Chapter 3

New Methods Based On Kaufmann And Gupta Ranking Approach For Finding Unique Optimal Fuzzy Project Completion Time

Liu [93] proposed a method based on fuzzy linear programming approach to find optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number. In this chapter, it is shown that on applying this method more than one fuzzy numbers, representing the optimal fuzzy project completion time, are obtained which contradicts the uniqueness property of optimal fuzzy project completion time. Also, it is shown that the same shortcoming is occurring in the modified Liang and Han method as well as in the existing Liang and Han method [84]. To overcome these shortcomings, a new method, named as Mehar's method based on Kaufmann and Gupta ranking approach, is proposed by modifying the existing method [93] and some modifications are also suggested in the modified Liang and Han method.

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3.1 Liu method

Liu [93] proposed the following method to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number.

Step 1 Formulate the chosen problem into the fuzzy linear programming problem ($P_{3.1}$):

$$\left.\begin{aligned} &\text{Maximize} \sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x_{ij}) \\ &\text{subject to} \\ &\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i=1 \\ -1 & , i=n \\ 0 & , i\in N-\{1,n\} \end{cases} \\ &x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A. \end{aligned}\right\} \quad (P_{3.1})$$

where,

A: set of all activities (i,j),

$\widetilde{t}_{ij}$: the fuzzy time duration of the activity (i,j),

N: the set of nodes,

n: the destination node,

1: the source node,

x_{ij}: the decision variable denoting the amount of flow in the activity (i,j).

Step 2 Suppose the fuzzy linear programming problem ($P_{3.1}$) have h feasible solutions and $\{x_{ij}^w\}$ is the w^{th} feasible solution then the aim is to find the feasible solution with the largest objective value i.e., the aim is to find $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^w)\right\}$.

Although, till now there is no unique way to compare fuzzy numbers but Liu [93] have used the concept that if $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum_{(i,j)\in A}(\Re(\widetilde{t}_{ij})\ x_{ij}^w)\right\}$ is $\sum_{(i,j)\in A}(\Re(\widetilde{t}_{ij})\ x_{ij}^\eta)$

then $\underset{1 \le w \le h}{\text{maximum}} \left\{ \sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x^{w}_{ij}) \right\}$ will also be $\sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x^{\eta}_{ij})$, where, $\Re(\widetilde{t}_{ij}) = \Re(a_{ij}, b_{ij}, c_{ij})$ $= \frac{a_{ij}+2b_{ij}+c_{ij}}{4}$ represents the Yager ranking index [160] of a triangular fuzzy number $\widetilde{t}_{ij} = (a_{ij}, b_{ij}, c_{ij})$.

In other words, Liu [93] have assumed that the optimal solution of the fuzzy linear programming problem $(P_{3.1})$ can be obtained by solving the crisp linear programming problem $(P_{3.2})$:

$$\text{Maximize} \sum_{(i,j)\in A} (\Re(\widetilde{t}_{ij})\ x_{ij})$$

subject to

$$\left.\begin{aligned} &\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & ,\ i = 1 \\ -1 & ,\ i = n \\ 0 & ,\ i \in N - \{1, n\} \end{cases} \\ &\qquad\qquad x_{ij}\ \ge\ 0\ \forall\ (i,j)\ \in A. \end{aligned}\right\} \qquad (P_{3.2})$$

Step 3 Solve the crisp linear programming problem $(P_{3.2})$ to find the optimal solution $\{x_{ij}\}$.

Step 4 Use the optimal solution $\{x_{ij}\}$, obtained in Step 3, to find the critical path and also put the obtained optimal values of x_{ij} in $\sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x_{ij})$ to find the optimal fuzzy completion time of the project.

3.2 Shortcomings of Liu method

If there exist more than one critical paths for a project network problem then the optimal fuzzy completion time of the project should be same corresponding to all the critical paths.

In this section, a fuzzy project network problem, chosen in Example 3.1, is solved by using Liu method [93] and shown that by using this method more than one fuzzy numbers, representing the optimal fuzzy completion time of the same project, are obtained which contradicts the uniqueness property of optimal fuzzy project completion time.

Example 3.1 Find the optimal fuzzy completion time of the project, shown by Figure 3.1, in which the fuzzy time duration $(\widetilde{t}_{ij})$ of the activity (i, j) is represented by the following triangular fuzzy numbers:

$\widetilde{t}_{12} = (2, 4, 6)$, $\widetilde{t}_{13} = (9, 13, 17)$, $\widetilde{t}_{23} = (7, 9, 11)$, $\widetilde{t}_{24} = (12, 19, 26)$, $\widetilde{t}_{34} = (6, 10, 14)$

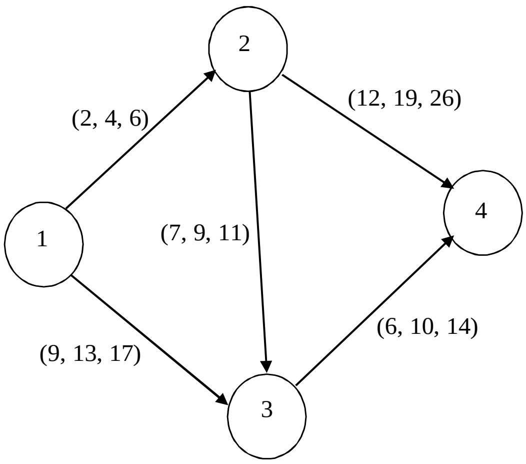

Figure 3.1 Structure of the project network

3.2.1 Optimal fuzzy completion time of the chosen problem

Using the existing method [93] the optimal fuzzy completion time of the project network problem, chosen in Example 3.1, can be obtained as follows:

Step 1 Using Step 1 of the Liu method [93] presented in Section 3.1 the problem, chosen in Example 3.1, can be formulated as follows:

Maximize ((2, 4, 6) $x_{12} \oplus$ (9, 13, 17) $x_{13} \oplus$ (7, 9, 11) $x_{23} \oplus$ (12, 19, 26) $x_{24} \oplus$ (6, 10, 14) x_{34})

subject to

$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$

Step 2 Using Step 2 of Liu method [93] the fuzzy linear programming problem, obtained in Step 1, can be written as:

Maximize $(\Re(2,4,6)\ x_{12} + \Re(9,13,17)\ x_{13} + \Re(7,9,11)\ x_{23} + \Re(12,19,26)\ x_{24} + \Re(6,10,14)\ x_{34})$

subject to

$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$

i.e., Maximize $(4\ x_{12} + 13\ x_{13} + 9\ x_{23} + 19\ x_{24} + 10\ x_{34})$

subject to

$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$

Step 3 On solving crisp linear programming problem, obtained in Step 2, the following three optimal solutions are obtained:

(i) $x_{12} = x_{24} = 1$ and $x_{13} = x_{23} = x_{34} = 0$

(ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{13} = x_{24} = 0$

(iii) $x_{13} = x_{34} = 1$ and $x_{12} = x_{23} = x_{24} = 0$

Step 4 Using the optimal values of x_{ij}, obtained from Step 3, the following three critical paths are obtained :

(i) $1 \Rightarrow 2 \Rightarrow 4$

(ii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$

(iii) $1 \Rightarrow 3 \Rightarrow 4$

Putting the obtained optimal values of x_{ij}, obtained from Step 3, in ((2, 4,

6) $x_{12} \oplus$ (9, 13, 17) $x_{13} \oplus$ (7, 9, 11) $x_{23} \oplus$ (12, 19, 26) $x_{24} \oplus$ (6, 10, 14) x_{34}), the optimal fuzzy project completion times corresponding to paths $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ are (14, 23, 32), (15, 23, 31) and (15, 23, 31) respectively.

3.2.2 Results and discussion

The critical paths and the optimal fuzzy project completion time for the problem, chosen in Example 3.1, obtained by using the Liu method [93] are shown in Table 3.1

Table 3.1 Results obtained by using Liu method

Example	Critical path	Optimal fuzzy project completion time
3.1	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)
	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(15, 23, 31)
	$1 \Rightarrow 3 \Rightarrow 4$	(15, 23, 31)

If there exist more than one critical paths for a project network problem then the optimal fuzzy completion time of the project should be same corresponding to all the critical paths. However, it can be seen from the obtained solution, presented in Table 3.1, that the optimal fuzzy project completion time corresponding to path $1 \Rightarrow 2 \Rightarrow 4$ is different from the optimal fuzzy project completion time corresponding to paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ i.e., $(14, 23, 32) \neq (15, 23, 31)$, only $\Re(14, 23, 32) = \Re(15, 23, 31) = 23$.

Since $(14, 23, 32)$ and $(15, 23, 31)$ are two different fuzzy numbers so their physical interpretation will also be different i.e., for the optimal fuzzy completion time of a same project two different interpretations will be required which is not acceptable for real life problems.

3.3 Shortcomings of modified Liang and Han method

In this section, the project network problem, chosen in Example 3.1, is solved by modified Liang and Han method and shown that the same shortcoming, pointed out in Section 3.2, is also occurring in the modified Liang and Han method.

3.3.1 Optimal fuzzy completion time of the chosen problem

Using the modified Liang and Han method, the optimal fuzzy completion time of the project, chosen in Example 3.1, can be obtained as follows:

Step 1

Using Step 1 of the modified Liang and Han method by assuming $\widetilde{FES_1} = (0, 0, 0)$ values of $\widetilde{FES_j}$, $j = 2, 3, 4$ can be obtained as follows:

$\widetilde{FES_2} = \widetilde{FES_1} \oplus \widetilde{FNT_{12}} = (0, 0, 0) \oplus (2, 4, 6) = (2, 4, 6)$

$\widetilde{FES_3} = \text{maximum}\{\widetilde{FES_1} \oplus \widetilde{FNT_{13}}, \widetilde{FES_2} \oplus \widetilde{FNT_{23}}\}$

$= \text{maximum}\{(0, 0, 0) \oplus (9, 13, 17), (2, 4, 6) \oplus (7, 9, 11)\}$

$= \text{maximum}\{(9, 13, 17), (9, 13, 17)\} = (9, 13, 17)$

$\widetilde{FES_4} = \text{maximum}\{\widetilde{FES_2} \oplus \widetilde{FNT_{24}}, \widetilde{FES_3} \oplus \widetilde{FNT_{34}}\}$

$= \text{maximum}\{(2, 4, 6) \oplus (12, 19, 26), (9, 13, 17) \oplus (6, 10, 14)\}$

$= \text{maximum}\{(14, 23, 32), (15, 23, 31)\}$

Since, $\Re(14, 23, 32) = \Re(15, 23, 31)$ and $m(14, 23, 32) = m(15, 23, 31)$

So, by using the existing approach used in existing method [84], discussed in Section 2.2, both (14, 23, 32) and (15, 23, 31) can be assumed as maximum i.e., further calculations can be done by the following two ways.

Case 1: Treating $\widetilde{FES_4} = (14, 23, 32)$ as maximum.

Case 2: Treating $\widetilde{FES_4} = (15, 23, 31)$ as maximum.

Case 1: Remaining steps by assuming $\widetilde{FES_4}$ = (14, 23, 32) as maximum

Step 2 Using Step 2 of the modified Liang and Han method by assuming $\widetilde{FLF_4} =$ (14, 23, 32) values of $\widetilde{FLF_j}$, $j = 3, 2, 1$ can be obtained as follows:

$$\widetilde{FLF_3} = \widetilde{FLF_4} \ominus_M \widetilde{FNT_{34}} = (14,\ 23,\ 32) \ominus_M (6,\ 10,\ 14) = (8, 13, 18)$$

$$\widetilde{FLF_2} = \text{minimum}\{\widetilde{FLF_4} \ominus_M \widetilde{FNT_{24}},\ \widetilde{FLF_3} \ominus_M \widetilde{FNT_{23}}\}$$

$$= \text{minimum}\{(14, 23, 32) \ominus_M (12, 19, 26), (8, 13, 18) \ominus_M (7, 9, 11)\}$$

$$= \text{minimum}\{(2, 4, 6), (1, 4, 7)\}$$

Since, $\Re(2, 4, 6) = \Re(1, 4, 7)$ and $m(2, 4, 6) = m(1, 4, 7)$

So, by using the existing approach used in existing method [84], discussed in Section 2.2, both (2, 4, 6) and (1, 4, 7) can be assumed as minimum i.e., further calculations can be done by the following two ways.

Case (1a): Treating $\widetilde{FLF_2} = (2, 4, 6)$ as minimum.

Case (1b): Treating $\widetilde{FLF_2} = (1, 4, 7)$ as minimum.

Case (1a): Remaining steps by assuming $\widetilde{FLF_2}$ = (2, 4, 6) as minimum

Assuming $\widetilde{FLF_2} = (2, 4, 6)$ value of $\widetilde{FLF_1}$ can be obtained as follows:

$$\widetilde{FLF_1} = \text{minimum}\{\widetilde{FLF_2} \ominus_M \widetilde{FNT_{12}},\ \widetilde{FLF_3} \ominus_M \widetilde{FNT_{13}}\}$$

$$= \text{minimum}\{(2, 4, 6) \ominus_M (2, 4, 6), (8, 13, 18) \ominus_M (9, 13, 17)\}$$

$$= \text{minimum}\{(0, 0, 0), (0, 1, 2)\}$$

$$= (0,\ 0,\ 0)$$

Step 3 Using Step 3 of the modified Liang and Han method values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$$\widetilde{FTS}_{12} = \widetilde{FLF}_2 \ominus_M (\widetilde{FES}_1 \oplus \widetilde{FNT}_{12})$$
$$= (2, 4, 6) \ominus_M ((0,\ 0,\ 0) \oplus (2,\ 4,\ 6))$$
$$= (2, 4, 6) \ominus_M (2,\ 4,\ 6)$$
$$= (0, 0, 0)$$
$$\widetilde{FTS}_{13} = \widetilde{FLF}_3 \ominus_M (\widetilde{FES}_1 \oplus \widetilde{FNT}_{13})$$
$$= (8, 13, 18) \ominus_M ((0,\ 0,\ 0) \oplus (9,\ 13,\ 17))$$
$$= (0, 1, 2)$$
$$\widetilde{FTS}_{23} = \widetilde{FLF}_3 \ominus_M (\widetilde{FES}_2 \oplus \widetilde{FNT}_{23})$$
$$= (8, 13, 18) \ominus_M ((2,\ 4,\ 6) \oplus (7,\ 9,\ 11))$$
$$= (0, 1, 2)$$
$$\widetilde{FTS}_{24} = \widetilde{FLF}_4 \ominus_M (\widetilde{FES}_2 \oplus \widetilde{FNT}_{24})$$
$$= (14,\ 23,\ 32) \ominus_M ((2,\ 4,\ 6) \oplus (12,\ 19,\ 26))$$
$$= (0, 0, 0)$$
$$\widetilde{FTS}_{34} = \widetilde{FLF}_4 \ominus_M (\widetilde{FES}_3 \oplus \widetilde{FNT}_{34})$$
$$= (14,\ 23,\ 32) \ominus_M ((9,\ 13,\ 17) \oplus (6,\ 10,\ 14))$$
$$= (0, 1, 2)$$

Step 4 Using Step 4 of modified Liang and Han method,

$$\text{minimum}\{\widetilde{FTS}_{12} \oplus \widetilde{FTS}_{24}, \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{23} \oplus \widetilde{FTS}_{34}, \widetilde{FTS}_{13} \oplus \widetilde{FTS}_{34}\}$$
$$= \text{minimum}\{(0, 0, 0) \oplus (0, 0, 0), (0, 0, 0) \oplus (0, 1, 2) \oplus (0, 1, 2), (0, 1, 2) \oplus (0, 1, 2)\}$$
$$= \text{minimum}\{(0, 0, 0), (0, 2, 4), (0, 2, 4)\} = (0,\ 0,\ 0)$$

Since, minimum is occurring corresponding to the path $1 \Rightarrow 2 \Rightarrow 4$ so it is the critical path for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method, optimal fuzzy completion time of the project is

$\widetilde{FCT} = \widetilde{FNT}_{12} \oplus \widetilde{FNT}_{24} = (2,\ 4,\ 6) \oplus (12,\ 19,\ 26) = (14,\ 23,\ 32)$

Therefore, on choosing $\widetilde{FLF}_2 = (2,\ 4,\ 6)$, the obtained critical path is $1 \Rightarrow 2 \Rightarrow 4$ and the obtained optimal fuzzy project completion time is (14, 23, 32)

Case (1b): Remaining steps by assuming $\widetilde{FLF}_2$ = (1, 4, 7) as minimum

Assuming $\widetilde{FLF}_2 = (1,\ 4,\ 7)$ value of $\widetilde{FLF}_1$ can be obtained as follows:

$$\widetilde{FLF}_1 = \text{minimum}\{\widetilde{FLF}_2 \ominus_M \widetilde{FNT}_{12},\ \widetilde{FLF}_3 \ominus_M \widetilde{FNT}_{13}\}$$

$$= \text{minimum}\{(1,4,7) \ominus_M (2,4,6), (8,13,18) \ominus_M (9,13,17)\}$$

$$= \text{minimum}\{(0,1,2), (0,1,2)\}$$

$$\widetilde{FLF}_1 = (0,\ 1,\ 2)$$

Step 3 Using Step 3 of the modified Liang and Han method values of $\widetilde{FTS}_{ij}$ can be obtained as follows:

$$\widetilde{FTS}_{12} = \widetilde{FLF}_2 \ominus_M (\widetilde{FES}_1 \oplus \widetilde{FNT}_{12})$$

$$= (1,4,7) \ominus_M ((0,\ 0,\ 0) \oplus (2,\ 4,\ 6))$$

$$= (1,4,7) \ominus_M (2,\ 4,\ 6)$$

$$= (0,1,2)$$

$$\widetilde{FTS}_{13} = \widetilde{FLF}_3 \ominus_M (\widetilde{FES}_1 \oplus \widetilde{FNT}_{13})$$

$$= (8,13,18) \ominus_M ((0,\ 0,\ 0) \oplus (9,\ 13,\ 17))$$

$$= (8,13,18) \ominus_M (9,\ 13,\ 17)$$

$$= (0,1,2)$$

$$\widetilde{FTS}_{23} = \widetilde{FLF}_3 \ominus_M (\widetilde{FES}_2 \oplus \widetilde{FNT}_{23})$$

$$= (8,13,18) \ominus_M ((2,\ 4,\ 6) \oplus (7,\ 9,\ 11))$$

$$= (8,13,18) \ominus_M (9,\ 13,\ 17)$$

$$= (0,1,2)$$

$$\widetilde{FTS_{24}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{24}})$$
$$= (14,\ 23,\ 32) \ominus_M ((2,\ 4,\ 6) \oplus (12,\ 19,\ 26))$$
$$= (14,\ 23,\ 32) \ominus_M (14,\ 23,\ 32))$$
$$= (0, 0, 0)$$
$$\widetilde{FTS_{34}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_3} \oplus \widetilde{FNT_{34}})$$
$$= (14,\ 23,\ 32) \ominus_M ((9,\ 13,\ 17) \oplus (6,\ 10,\ 14))$$
$$= (14,\ 23,\ 32) \ominus_M (15,\ 23,\ 31))$$
$$= (0, 1, 2)$$

Step 4 Using Step 4 of the modified Liang and Han method,

$$\text{minimum}\{\widetilde{FTS_{12}} \oplus \widetilde{FTS_{24}}, \widetilde{FTS_{12}} \oplus \widetilde{FTS_{23}} \oplus \widetilde{FTS_{34}}, \widetilde{FTS_{13}} \oplus \widetilde{FTS_{34}}\}$$
$$= \text{minimum}\{(0,1,2) \oplus (0,0,0), (0,1,2) \oplus (0,1,2) \oplus (0,1,2), (0,1,2) \oplus (0,1,2)\}$$
$$= \text{minimum}\{(0,1,2), (0,3,6), (0,2,4)\} = (0,\ 1,\ 2)$$

Since, the minimum is occurring corresponding to the path $1 \Rightarrow 2 \Rightarrow 4$ so it is the critical path for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method the optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FNT_{12}} \oplus \widetilde{FNT_{24}} = (2,\ 4,\ 6) \oplus (12,\ 19,\ 26) = (14,\ 23,\ 32)$$

Therefore, on choosing $\widetilde{FES_4} = (14,\ 23,\ 32)$, the critical path is $1 \Rightarrow 2 \Rightarrow 4$ and the optimal fuzzy project completion time is (14, 23, 32).

Case 2: Remaining steps by assuming $\widetilde{FES_4}$ = (15, 23, 31) as maximum

Step 2 Using Step 2 of the modified Liang and Han method by assuming $\widetilde{FLF_4} =$ (15, 23, 31) values of $\widetilde{FLF_j}$, $j = 3, 2, 1$ can be obtained as follows:

$$\widetilde{FLF_3} = \widetilde{FLF_4} \ominus_M \widetilde{FNT_{34}} = (15,\ 23,\ 31) \ominus_M (6,\ 10,\ 14) = (9, 13, 17)$$

$$\widetilde{FLF_2} = \text{minimum}\{\widetilde{FLF_4} \ominus_M \widetilde{FNT_{24}}, \widetilde{FLF_3} \ominus_M \widetilde{FNT_{23}}\}$$

$$= \text{minimum}\{(15, 23, 31) \ominus_M (12, 19, 26), (9, 13, 17) \ominus_M (7, 9, 11)\}$$

$$= \text{minimum}\{(3, 4, 5), (2, 4, 6)\}$$

Since, $\Re(2, 4, 6) = \Re(3, 4, 5)$ and $m(3, 4, 5) = m(2, 4, 6)$ so, by using the existing approach used in existing method [84] discussed in Section 2.2, both (2, 4, 6) and (3, 4, 5) can be assumed as minimum i.e., further calculations can be done by the following two ways:

Case (2a): Treating $\widetilde{FLF_2}$ = (2, 4, 6) as minimum.

Case (2b): Treating $\widetilde{FLF_2}$ = (3, 4, 5) as minimum.

Case (2a): Remaining steps by assuming $\widetilde{FLF_2}$ = (2, 4, 6) as minimum

Assuming $\widetilde{FLF_2}$ = (2, 4, 6) value of $\widetilde{FLF_1}$ can be obtained as follows:

$$\widetilde{FLF_1} = \text{minimum}\{\widetilde{FLF_2} \ominus_M \widetilde{FNT_{12}}, \widetilde{FLF_3} \ominus_M \widetilde{FNT_{13}}\}$$

$$= \text{minimum}\{(2, 4, 6) \ominus_M (2, 4, 6), (9, 13, 17) \ominus_M (9, 13, 17)\}$$

$$= \text{minimum}\{(0, 0, 0), (0, 0, 0)\}$$

$$= (0,\ 0,\ 0)$$

Step 3 Using Step 3 of the modified Liang and Han method values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$$\widetilde{FTS_{12}} = \widetilde{FLF_2} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{12}})$$

$$= (2, 4, 6) \ominus_M ((0,\ 0,\ 0) \oplus (2,\ 4,\ 6))$$

$$= (2, 4, 6) \ominus_M (2,\ 4,\ 6)$$

$$= (0, 0, 0)$$

$$\widetilde{FTS_{13}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{13}})$$

$$= (9, 13, 17) \ominus_M ((0,\ 0,\ 0) \oplus (9,\ 13,\ 17))$$

$$= (0, 0, 0)$$

$$\widetilde{FTS_{23}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{23}})$$

$$= (9, 13, 17) \ominus_M ((2,\ 4,\ 6) \oplus (7,\ 9,\ 11))$$

$$= (0, 0, 0)$$

$$\widetilde{FTS_{24}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{24}})$$

$$= (15,\ 23,\ 31) \ominus_M ((2,\ 4,\ 6) \oplus (12,\ 19,\ 26))$$

$$= (15,\ 23,\ 31) \ominus_M (14,\ 23,\ 32)$$

$$= (1, 1, 1)$$

$$\widetilde{FTS_{34}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_3} \oplus \widetilde{FNT_{34}})$$

$$= (15,\ 23,\ 31) \ominus_M ((9,\ 13,\ 17) \oplus (6,\ 10,\ 14))$$

$$= (0, 0, 0)$$

Step 4 Using Step 4 of the modified Liang and Han method,

$$\text{minimum}\{\widetilde{FTS_{12}} \oplus \widetilde{FTS_{24}}, \widetilde{FTS_{12}} \oplus \widetilde{FTS_{23}} \oplus \widetilde{FTS_{34}}, \widetilde{FTS_{13}} \oplus \widetilde{FTS_{34}}\}$$

$$= \text{minimum}\{(0, 0, 0) \oplus (1, 1, 1), (0, 0, 0) \oplus (0, 0, 0) \oplus (0, 0, 0), (0, 0, 0) \oplus (0, 0, 0)\}$$

$$= \text{minimum}\{(1, 1, 1), (0, 0, 0), (0, 0, 0)\} = (0,\ 0,\ 0)$$

Since, the minimum is occurring corresponding to the paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ so, these are the critical paths for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method the optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FTS_{12}} \oplus \widetilde{FTS_{23}} \oplus \widetilde{FTS_{34}} \text{ and } \widetilde{FNT_{13}} \oplus \widetilde{FNT_{34}}$$

$$= (2,\ 4,\ 6) \oplus (7,\ 9,\ 11) \oplus (6,\ 10,\ 14) \text{ and } (9,\ 13,\ 17) \oplus (6,\ 10,\ 14)$$

$$= (15,\ 23,\ 31)$$

Therefore, on choosing $\widetilde{FLF_2} = (2,\ 4,\ 6)$, the obtained critical paths are $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 2 \Rightarrow 4$ and the obtained optimal fuzzy project completion time is

(15, 23, 31).

Case (2b): Remaining steps by assuming $\widetilde{FLF_2}$ = (3, 4, 5) as minimum

Assuming $\widetilde{FLF_2}$ = (3, 4, 5) value of $\widetilde{FLF_1}$ can be obtained as follows:

$$\widetilde{FLF_1} = \text{minimum}\{\widetilde{FLF_2} \ominus_M \widetilde{FNT_{12}}, \widetilde{FLF_3} \ominus_M \widetilde{FNT_{13}}\}$$

$$= \text{minimum}\{(3,4,5) \ominus_M (2,4,6), (9,13,17) \ominus_M (9,13,17)\}$$

$$= \text{minimum}\{(1,1,1), (0,0,0)\}$$

$$\widetilde{FLF_1} = (0,\ 0,\ 0)$$

Step 3 Using Step 3 of the modified Liang and Han method values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$$\widetilde{FTS_{12}} = \widetilde{FLF_2} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{12}})$$

$$= (3,4,5) \ominus_M ((0,\ 0,\ 0) \oplus (2,\ 4,\ 6))$$

$$= (3,4,5) \ominus_M (2,\ 4,\ 6)$$

$$= (1,1,1)$$

$$\widetilde{FTS_{13}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{13}})$$

$$= (9,13,17) \ominus_M ((0,\ 0,\ 0) \oplus (9,\ 13,\ 17))$$

$$= (0,0,0)$$

$$\widetilde{FTS_{23}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{23}})$$

$$= (9,13,17) \ominus_M ((2,\ 4,\ 6) \oplus (7,\ 9,\ 11))$$

$$= (0,0,0)$$

$$\widetilde{FTS_{24}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{24}})$$

$$= (15,\ 23,\ 31) \ominus_M ((2,\ 4,\ 6) \oplus (12,\ 19,\ 26))$$

$$= (15,\ 23,\ 31) \ominus_M (14,\ 23,\ 32)$$

$$= (1,1,1)$$

$$\widetilde{FTS}_{34} = \widetilde{FLF}_4 \ominus_M (\widetilde{FES}_3 \oplus \widetilde{FNT}_{34})$$

$$= (15,\ 23,\ 31) \ominus_M ((9,\ 13,\ 17) \oplus (6,\ 10,\ 14))$$

$$= (0, 0, 0)$$

Step 4 Using Step 4 of the modified Liang and Han method,

$$\text{minimum}\{\widetilde{FTS}_{12} \oplus \widetilde{FTS}_{24}, \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{23} \oplus \widetilde{FTS}_{34}, \widetilde{FTS}_{13} \oplus \widetilde{FTS}_{34}\}$$

$$= \text{minimum}\{(1, 1, 1) \oplus (1, 1, 1), (1, 1, 1) \oplus (0, 0, 0) \oplus (0, 0, 0), (0, 0, 0) \oplus (0, 0, 0)\}$$

$$= \text{minimum}\{(2, 2, 2), (1, 1, 1), (0, 0, 0)\} = (0,\ 0,\ 0)$$

Since, the minimum is occurring corresponding to the path $1 \Rightarrow 3 \Rightarrow 4$ so this is the critical path for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method the obtained optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FNT}_{13} \oplus \widetilde{FNT}_{34}$$

$$= (9,\ 13,\ 17) \oplus (6,\ 10,\ 14) = (15,\ 23,\ 31)$$

Therefore, on choosing $\widetilde{FLF}_2 = (3,\ 4,\ 5)$, the obtained critical path is $1 \Rightarrow 3 \Rightarrow 4$ and the obtained optimal fuzzy project completion time is (15, 23, 31).

3.3.2 Results and discussion

The critical paths and the optimal fuzzy project completion time for the problem, chosen in Example 3.1, obtained by using the modified Liang and Han method are shown in Table 3.2

Table 3.2 Results obtained by using modified Liang and Han method

Example 3.1		
Critical Path	$1 \Rightarrow 2 \Rightarrow 4$	$1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$
Optimal fuzzy project completion time	$\widetilde{FCT} = (14,\ 23,\ 32)$	$\widetilde{FCT} = (15,\ 23,\ 31)$

Since, the results, shown in Table 3.2, obtained by using the modified Liang and Han method, is same as the results, shown in Table 3.1, obtained by using Liu method [93]. So, the same shortcomings, pointed out in Section 3.2, occurring in the results obtained by using the existing method [93] is also occurring in the results obtained by using the modified Liang and Han method.

Remark 3.1: Although, the shortcomings, pointed out in Section 3.2, will also occur in the existing Liang and Han method [84] but in Chapter 2 it is shown that the modified Liang and Han method is better than the existing Liang and Han method so in this chapter the shortcomings of existing Liang and Han method [84] are not discussed.

3.4 Origin of shortcomings

The shortcomings, pointed out in Section 3.2 and Section 3.3, are occurring due to the following reasons:

If A_1 and A_2 are two real numbers such that $A_1 \neq A_2$ then maximum$\{A_1, A_2\}$ (or minimum$\{A_1, A_2\}$) will be either A_1 or A_2. On the same direction, if $\widetilde{A_1}$ and $\widetilde{A_2}$ are two fuzzy numbers such that $\widetilde{A_1} \neq \widetilde{A_2}$ then maximum$\{\widetilde{A_1}, \widetilde{A_2}\}$ (or minimum$\{\widetilde{A_1}, \widetilde{A_2}\}$) should be either $\widetilde{A_1}$ or $\widetilde{A_2}$. However, for the maximum$\{\widetilde{A_1}, \widetilde{A_2}\}$ (or minimum$\{\widetilde{A_1}, \widetilde{A_2}\}$) obtained by using the approaches, used in Liang and Han method [84] and Liu method [93], this property is not satisfying e.g., Since, $\widetilde{A_1} = (14,$ $23, 32)$ and $\widetilde{A_2} = (15, 23, 31)$ are two triangular fuzzy numbers such that $\widetilde{A_1} \neq \widetilde{A_2}$ so maximum$\{\widetilde{A_1}, \widetilde{A_2}\}$ (or minimum$\{\widetilde{A_1}, \widetilde{A_2}\}$) should be either $\widetilde{A_1}$ or $\widetilde{A_2}$. However, using the method, presented in Section 2.2, used in Liang and Han method [84], maximum$\{\widetilde{A_1}, \widetilde{A_2}\}$ = minimum$\{\widetilde{A_1}, \widetilde{A_2}\}$ = $\widetilde{A_1}$ and $\widetilde{A_2}$. Similarly, since, $\widetilde{A_1} = (1,$

4, 7) and $\widetilde{A_2} = (2,\ 4,\ 6)$ are two triangular fuzzy numbers such that $\widetilde{A_1} \neq \widetilde{A_2}$ so maximum$\{\widetilde{A_1}, \widetilde{A_2}\}$ (or minimum$\{\widetilde{A_1}, \widetilde{A_2}\}$) should be either $\widetilde{A_1}$ or $\widetilde{A_2}$. However, using existing ranking approach [160], used in Liu method [93] maximum$\{\widetilde{A_1}, \widetilde{A_2}\} =$ minimum$\{\widetilde{A_1}, \widetilde{A_2}\} = \widetilde{A_1}$ and $\widetilde{A_2}$.

3.5 Existing ranking approaches for finding maximum and minimum of triangular fuzzy numbers

In Section 3.4, it is pointed out that the shortcomings in Liu method as well as in the modified Liang and Han method are occurring due to the existing ranking approaches, used in the existing methods [84, 93], for finding maximum and minimum of fuzzy numbers. So, one may try to resolve these shortcomings by using some other existing ranking approaches for finding maximum and minimum of triangular fuzzy numbers instead of the existing ranking approaches used in existing methods [84,93]. Although, there are several existing ranking approaches for finding maximum and minimum of triangular fuzzy numbers. However, in this section, some existing ranking approaches [2, 11, 27, 31, 33, 58, 70, 79, 92, 113, 151, 160, 172], mostly used in the literature for finding maximum and minimum of triangular fuzzy numbers, are presented.

3.5.1 Kaufmann and Gupta ranking approach

In this section, the existing ranking approach [70] for finding maximum and minimum of two triangular fuzzy numbers is presented.

Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find $\Re(\widetilde{A}) = \frac{a_1+2b_1+c_1}{4}$ and $\Re(\widetilde{B}) = \frac{a_2+2b_2+c_2}{4}$

Case (i) If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then go to Step 2.

Step 2 Find $mode(\widetilde{A}) = b_1$ and $mode(\widetilde{B}) = b_2$

Case (i) If $mode(\widetilde{A}) > mode(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $mode(\widetilde{A}) < mode(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $mode(\widetilde{A}) = mode(\widetilde{B})$ then go to Step 3.

Step 3 Find $divergence(\widetilde{A}) = c_1 - a_1$ and $divergence(\widetilde{B}) = c_2 - a_2$

Case (i) If $divergence(\widetilde{A}) > divergence(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $divergence(\widetilde{A}) < divergence(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $divergence(\widetilde{A}) = divergence(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$

3.5.2 Wang and Lee ranking approach

In this section, the existing ranking approach [151] for finding maximum and minimum of triangular fuzzy numbers is presented.

Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find $x_{\widetilde{A}} = \frac{\int_{a_1}^{b_1} x(\frac{x-a_1}{b_1-a_1})dx + \int_{b_1}^{c_1} x(\frac{x-c_1}{b_1-c_1})dx}{\int_{a_1}^{b_1} (\frac{x-a_1}{b_1-a_1})dx + \int_{b_1}^{c_1} (\frac{x-c_1}{b_1-c_1})dx}$ and $x_{\widetilde{B}} = \frac{\int_{a_2}^{b_2} x(\frac{x-a_2}{b_2-a_2})dx + \int_{b_2}^{c_2} x(\frac{x-c_2}{b_2-c_2})dx}{\int_{a_2}^{b_2} (\frac{x-a_2}{b_2-a_2})dx + \int_{b_2}^{c_2} (\frac{x-c_2}{b_2-c_2})dx}$.

Case (i) If $x_{\widetilde{A}} > x_{\widetilde{B}}$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $x_{\widetilde{A}} < x_{\widetilde{B}}$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $x_{\widetilde{A}} = x_{\widetilde{B}}$ then go to Step 2.

Step 2 Find $y_{\widetilde{A}} = \frac{\int_0^1 y(a_1+(b_1-a_1)y)dy + \int_0^1 (y(c_1+(b_1-c_1)y)dy}{\int_0^1 (a_1+(b_1-a_1)y)dy + \int_0^1 (c_1+(b_1-c_1)y)dy}$ and

$$y_{\widetilde{B}} = \frac{\int_0^1 y(a_2+(b_2-a_2)y)dy + \int_0^1 (y(c_2+(b_2-c_2)y)dy}{\int_0^1 (a_2+(b_2-a_2)y)dy + \int_0^1 (c_2+(b_2-c_2)y)dy}.$$

Case (i) If $y_{\widetilde{A}} > y_{\widetilde{B}}$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $y_{\widetilde{A}} < y_{\widetilde{B}}$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $y_{\widetilde{A}} = y_{\widetilde{B}}$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \text{minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and } \widetilde{B}.$$

3.5.3 Kumar et al. ranking approach

In this section, the existing ranking approach [79] for finding maximum and minimum of triangular fuzzy numbers is presented.

Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find RM($\widetilde{A}$) $= Rank(\widetilde{A}) = \frac{a_1+2b_1+c_1}{4}$ and RM($\widetilde{B}$) $= Rank(\widetilde{B}) = \frac{a_2+2b_2+c_2}{4}$

Case (i) If RM($\widetilde{A}$) $>$ RM($\widetilde{B}$) then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If RM($\widetilde{A}$) $<$ RM($\widetilde{B}$) then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If RM($\widetilde{A}$) $=$ RM($\widetilde{B}$) then go to Step 2.

Step 2 Find RM($\widetilde{A}$) $= mode(\widetilde{A}) = b_1$ and RM($\widetilde{B}$) $= mode(\widetilde{B}) = b_2$

Case (i) If RM($\widetilde{A}$) $>$ RM($\widetilde{B}$) then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If RM($\widetilde{A}$) $<$ RM($\widetilde{B}$) then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If RM($\widetilde{A}$) $=$ RM($\widetilde{B}$) then $\widetilde{A} \approx \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} =$ minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and $\widetilde{B}$.

3.5.4 Other existing ranking approaches

In this section, some other existing ranking approaches [2, 11, 27, 31, 33, 58, 92, 113, 160, 172] for finding maximum and minimum of triangular fuzzy numbers are presented.

Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then, use the formulae, shown in Table 3.3, to calculate $\Re(\widetilde{A})$ and $\Re(\widetilde{B})$ and check that $\Re(\widetilde{A}) > \Re(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$ or $\Re(\widetilde{A}) < \Re(\widetilde{B})$.

Case (i): If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and } \text{minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii): If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and } \text{minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii): If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \text{minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and } \widetilde{B}.$$

Table 3.3 Ranking formulae used in some existing ranking approaches

Ranking approaches	Ranking formulae for a triangular fuzzy number $\widetilde{A} = (a, b, c)$
Yager [160]	$\Re(\widetilde{A}) = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x(\frac{x-c}{b-c})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c (\frac{x-c}{b-c})dx}$
Murakami et al. [113]	$\Re(\widetilde{A}) = x_{\widetilde{A}}$ or $y_{\widetilde{A}}$ where, $x_{\widetilde{A}} = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x(\frac{x-c}{b-c})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c (\frac{x-c}{b-c})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(a+(b-a)y))dy + \int_0^1 (y(c+(b-c)y))dy}{\int_0^1 (a+(b-a)y)dy + \int_0^1 (c+(b-c)y)dy}$
Liou and Wang [92]	$\Re(\widetilde{A}) = \frac{a+2b+c}{4}$
Chen et al. [27]	$\Re(\widetilde{A}) = \frac{a+2b+c}{4}$
Cheng [31]	$\Re(\widetilde{A}) = \sqrt{x_{\widetilde{A}}^2 + y_{\widetilde{A}}^2}$ where, $x_{\widetilde{A}} = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x(\frac{x-c}{b-c})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c (\frac{x-c}{b-c})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(a+(b-a)y))dy + \int_0^1 (y(c+(b-c)y))dy}{\int_0^1 (a+(b-a)y)dy + \int_0^1 (c+(b-c)y)dy}$
Yao and Wu [172]	$\Re(\widetilde{A}) = b + \frac{1}{4}(c - 2b + a)$
Chu and Tsao [33]	$\Re(\widetilde{A}) = x_{\widetilde{A}} . y_{\widetilde{A}}$ where, $x_{\widetilde{A}} = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x(\frac{x-c}{b-c})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c (\frac{x-c}{b-c})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 y(a+(b-a)y)dy + \int_0^1 y(c+(b-c)y)dy}{\int_0^1 (a+(b-a)y)dy + \int_0^1 (c+(b-c)y)dy}$
Asady and Zendehnam [11]	$\Re(\widetilde{A}) = \frac{a+2b+c}{4}$
Garcia and Lamata [58]	$\Re(\widetilde{A}) = \frac{a+4b+c}{6}$
Abbasbandy and Hajjari [2]	$\Re(\widetilde{A}) = b + \frac{1}{12}(c - 2b + a)$

3.6 Kaufmann and Gupta ranking approach vs other existing ranking approaches

In this section, to show the advantage of Kaufmann and Gupta ranking approach [70] over other existing ranking approaches [2,11,27,31,33,58,79,92,113,151, 160,172] the maximum and minimum of triangular fuzzy numbers $\widetilde{A} = (14, 23, 32)$

and $\widetilde{B} = (15,\ 23,\ 31)$ obtained by using Kaufmann and Gupta ranking approach [70] and other existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 92, 113, 151, 160, 172] are shown in Table 3.4

Table 3.4 Maximum and minimum of triangular fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained by using existing ranking approaches

Ranking approaches	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A},\widetilde{B}\}$	minimum$\{\widetilde{A},\widetilde{B}\}$
Yager [160]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Murakami et al. [113]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Liou and Wang [92]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Chen et al. [27]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Cheng [31]	$\Re(\widetilde{A}) = 23.005$	$\Re(\widetilde{B}) = 23.005$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Yao and Wu [172]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Chu and Tsao [33]	$\Re(\widetilde{A}) = 11.5$	$\Re(\widetilde{B}) = 11.5$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Asady and Zendehnam [11]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Garcia and Lamata [58]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Wang and Lee [151]	$y_{\widetilde{A}} = 0.5$	$y_{\widetilde{B}} = 0.5$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Abbasbandy and Hajjari [2]	$\Re(\widetilde{A}) = 23$	$\Re(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Kumar et al. [79]	$\mathrm{RM}(\widetilde{A}) = 23$	$\mathrm{RM}(\widetilde{B}) = 23$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Kaufmann and Gupta [70]	$\Re(\widetilde{A}) = 23$ $mode(\widetilde{A}) = 23$ $divergence(\widetilde{A}) = 18$	$\Re(\widetilde{B}) = 23$ $mode(\widetilde{B}) = 23$ $divergence(\widetilde{B}) = 16$	$\widetilde{A}$	$\widetilde{B}$

It is obvious from the Definition 2.11 that $\widetilde{A} \neq \widetilde{B}$ i.e., maximum$\{\widetilde{A},\widetilde{B}\}$ and minimum$\{\widetilde{A},\widetilde{B}\}$ should be either $\widetilde{A}$ or $\widetilde{B}$. However, on the basis of results, shown in Table 3.4, obtained by using the existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 92, 113, 151, 160, 172] except Kaufmann and Gupta ranking approach [70], it can be concluded that maximum$\{\widetilde{A},\widetilde{B}\}$ and minimum$\{\widetilde{A},\widetilde{B}\} = \widetilde{A}$ and $\widetilde{B}$ i.e., if instead of existing ranking approaches used in existing methods [84, 93] any other existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 92, 113, 151, 160, 172] except Kaufmann and Gupta ranking approach [70] will be used for finding maximum and minimum of triangular fuzzy numbers in the existing method [93] and in the modified Liang and Han method then more than one fuzzy numbers, representing the optimal fuzzy project completion time, may be obtained.

3.6.1 Validity of Kaufmann and Gupta ranking approach

It is obvious that if $\widetilde{A}$ and $\widetilde{B}$ are two triangular fuzzy numbers such that $\Re(\widetilde{A}) > \Re(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) > mode(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) = mode(\widetilde{B})$, $divergence(\widetilde{A}) > divergence(\widetilde{B})$ then using the existing ranking approach, presented in Section 3.5.1, maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$.

In this section, it is proved that if $\widetilde{A}$ and $\widetilde{B}$ are two such triangular fuzzy numbers for which all the three conditions of Kaufmann and Gupta ranking approach [70] are satisfied then $\widetilde{A} = \widetilde{B}$.

Proposition 3.1 [70] Let $\widetilde{A} = (a_1, b_1, c_1)$ and $\widetilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers such that

(i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$ (ii) $mode(\widetilde{A}) = mode(\widetilde{B})$ (iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$.

Then, $\widetilde{A} = \widetilde{B}$ i.e., $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

Proof: (i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$

$$\Rightarrow \frac{a_1+2b_1+c_1}{4} = \frac{a_2+2b_2+c_2}{4}$$

$$\Rightarrow a_1+2b_1+c_1 = a_2+2b_2+c_2 \tag{3.1}$$

(ii) $mode(\widetilde{A}) = mode(\widetilde{B})$

$$\Rightarrow b_1 = b_2 \tag{3.2}$$

(iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$

$$\Rightarrow c_1 - a_1 = c_2 - a_2 \tag{3.3}$$

Solving (3.1), (3.2) and (3.3)

$$a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2$$

i.e., $\widetilde{A} = \widetilde{B}$

3.7 Proposed methods based on Kaufmann and Gupta ranking approach

As discussed in Section 3.2 and Section 3.3, that the shortcomings in Liu method [93] and in the modified Liang and Han method are occurring due to using the existing ranking approaches, discussed in Section 3.5 and in Section 2.2 of Chapter 2, which are used for finding maximum and minimum of triangular fuzzy numbers. Also, in Section 3.6, it is proved that if $\widetilde{A}$ and $\widetilde{B}$ are two triangular fuzzy numbers then maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$ obtained by using Kaufmann and Gupta ranking approach, will always be either $\widetilde{A}$ or $\widetilde{B}$ but maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$ obtained by using existing ranking approaches [2,11,27,31,33, 58,79,92,113,151,160,172] may be $\widetilde{A}$ as well as $\widetilde{B}$.

In this section, to overcome the shortcomings of Liu method [93] and the modified Liang and Han method, pointed out in Section 3.2 and Section 3.3, some modifications in the modified Liang and Han method on the basis of Kaufmann and Gupta ranking approach are suggested and also on the basis of Kaufmann and Gupta ranking approach, a new method, named as Mehar's method based on Kaufmann and Gupta ranking approach, is proposed by modifying Liu method [93].

3.7.1 Modified Liang and Han method based on Kaufmann and Gupta ranking approach

If in the modified Liang and Han method instead of the existing approach, discussed in Section 2.2, the existing Kaufmann and Gupta ranking approach, discussed in Section 3.5.1, is used for finding the maximum and minimum of triangular fuzzy numbers, then always a unique triangular fuzzy number, representing the optimal fuzzy project completion time, will be obtained i.e., the shortcoming of the modified Liang and Han method will be resolved.

3.7.2 Proposed Mehar's method based on Kaufmann and Gupta ranking approach

In this section, to overcome the shortcomings, discussed in Section 3.2, a new method, named as Mehar's method based on Kaufmann and Gupta ranking approach, is proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number.

The steps of proposed method are as follows:

Step 1 Find the critical path and the optimal fuzzy completion time of the chosen problem by using the existing method discussed in Section 3.1.

Case (i): If a unique critical path and hence a unique fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained optimal fuzzy project completion time is the optimal fuzzy completion time of the project and the obtained critical path is the only critical path of the project.

Case (ii): If more than one critical paths are obtained then go to Step 2.

Step 2 Check that the fuzzy numbers, representing the optimal fuzzy project completion time, corresponding to all the critical paths are same or not.

Case (i): If a unique triangular fuzzy number, representing the optimal fuzzy project completion time, is obtained then the critical paths obtained in Step 1 corresponding to which the obtained fuzzy number is obtained are critical paths of the project and the obtained triangular fuzzy number will represent the optimal fuzzy completion time of the project.

Case (ii): If more than one triangular fuzzy numbers, representing the optimal fuzzy project completion time, are obtained then go to Step 3.

Step 3 Let using previous steps p different triangular fuzzy numbers $\widetilde{T}_1$, $\widetilde{T}_2$, ..., $\widetilde{T}_p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}_1 \neq \widetilde{T}_2 \neq ... \neq \widetilde{T}_p$ but $\Re(\widetilde{T}_1) = \Re(\widetilde{T}_2) = ... = \Re(\widetilde{T}_p) = u_1$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\text{Maximize} \sum_{(i,j)\in A} (mode(a_{ij}, b_{ij}, c_{ij})\ x_{ij})$$

subject to

$$\left.\begin{array}{l} \sum_{(i,j)\in A} (\Re(a_{ij}, b_{ij}, c_{ij})\ x_{ij}) = u_1, \\ \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ x_{ij} \geq 0 \ \forall\ (i,j) \in A. \end{array}\right\} \qquad (P_{3.3})$$

Case (i): If by putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A} (a_{ij}, b_{ij}, c_{ij})\ x_{ij})$ a unique triangular fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained triangular fuzzy number will represent the optimal fuzzy completion time of the project and all the critical paths can be obtained by using the obtained optimal values of x_{ij}.

Case (ii): If more than one triangular fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 4.

Step 4 Let using previous steps l triangular fuzzy numbers $\widetilde{T}'_1$, $\widetilde{T}'_2$, ..., $\widetilde{T}'_l$, where, $l \leq p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}'_1 \neq \widetilde{T}'_2 \neq ... \neq \widetilde{T}'_l$ but $\Re(\widetilde{T}'_1) = \Re(\widetilde{T}'_2) = ... = \Re(\widetilde{T}'_l) = u_1$ and $mode(\widetilde{T}'_1) = mode(\widetilde{T}'_2) = ... = mode(\widetilde{T}'_l) = u_2$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$
\begin{array}{l}
\text{Maximize} \sum_{(i,j)\in A} (divergence(a_{ij}, b_{ij}, c_{ij})\ x_{ij}) \\
\text{subject to} \\
\left.
\begin{array}{c}
\sum_{(i,j)\in A} (\Re(a_{ij}, b_{ij}, c_{ij})\ x_{ij}) = u_1, \\
\sum_{(i,j)\in A} (mode(a_{ij}, b_{ij}, c_{ij})\ x_{ij}) = u_2, \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\
x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A.
\end{array}
\right\} \qquad (P_{3.4})
\end{array}
$$

Now, putting the obtained optimal values of x_{ij} in ($\sum_{(i,j)\in A} (a_{ij}, b_{ij}, c_{ij})\ x_{ij}$) a unique triangular fuzzy number, representing the optimal fuzzy project completion time, will be obtained and the critical paths can be obtained by using the obtained optimal values of x_{ij}.

3.8 Advantages of proposed methods based on Kaufmann and Gupta ranking approach

The main advantage of the modified Liang and Han method based on Kaufmann and Gupta ranking approach and the proposed Mehar's method based on Kaufmann and Gupta ranking approach over the modified Liang and Han method and Liu method [93] is that on applying the modified Liang and Han method and Liu method [93] more than one triangular fuzzy numbers, representing the optimal fuzzy completion time of the same project, may be obtained due to which there will be different interpretations for the optimal fuzzy project completion time of a same project which is not genuine. While, by using the modified Liang and Han method based on Kaufmann and Gupta ranking approach and proposed Mehar's method based on

Kaufmann and Gupta ranking approach always a unique triangular fuzzy number, representing the optimal fuzzy project completion time, is obtained. So, there will be a unique interpretation for the optimal fuzzy completion time of project.

To show the advantages of the modified Liang and Han method based on Kaufmann and Gupta ranking approach and proposed Mehar's method based on Kaufmann and Gupta ranking approach over the modified Liang and Han method and Liu method [93], the fuzzy project network problem, chosen in Example 3.1, is solved by using the modified Liang and Han method based on Kaufmann and Gupta ranking approach as well as proposed Mehar's method based on Kaufmann and Gupta ranking approach and it is shown that a unique fuzzy number, representing the optimal fuzzy project completion time, is obtained.

3.8.1 Optimal fuzzy completion time of the chosen problem by using proposed Mehar's method based on Kaufmann and Gupta ranking approach

Using the proposed Mehar's method based on Kaufmann and Gupta ranking approach the optimal fuzzy completion time of the project, chosen in Example 3.1, can be obtained as follows:

Step 1 It is obvious from the results of Example 3.1, shown in Table 3.1, that the critical paths for the chosen problem are $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$. Also, the optimal fuzzy project completion time corresponding to the paths $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ are (14, 23, 32), (15, 23, 31) and (15, 23, 31) respectively.

Since, more than one critical paths are obtained i.e., Case (ii) of Step 1 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach is satisfied, so go to Step 2 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach.

Step 2 Since, fuzzy numbers (14, 23, 32) and (15 , 23, 31), representing the optimal fuzzy completion time of the project, are different i.e., Case (ii) of Step 2 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach is satisfied, so go to Step 3 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach.

Step 3 Since, $u_1 = \Re(14, 23, 32) = \Re(15, 23, 31) = 23$. So, using Step 3 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(mode(2,4,6)\ x_{12} + mode(9,13,17)\ x_{13} + mode(7,9,11)\ x_{23} + mode(12,19,26)\ x_{24} + mode(6,10,14)\ x_{34})$

subject to

$\Re(2,4,6)\ x_{12} + \Re(9,13,17)\ x_{13} + \Re(7,9,11)\ x_{23} + \Re(12,19,26)\ x_{24} + \Re(6,10,14)\ x_{34} = 23,$

$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$

On solving the crisp linear programming problem the following three optimal solutions are obtained:

(i) $x_{12} = x_{24} = 1$ and $x_{13} = x_{23} = x_{34} = 0$

(ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{13} = x_{24} = 0$

(iii) $x_{13} = x_{34} = 1$ and $x_{12} = x_{23} = x_{24} = 0$

Putting the obtained optimal values of x_{ij} in ((2, 4, 6) $x_{12} \oplus$ (9, 13, 17) $x_{13} \oplus$

(7, 9, 11) x_{23} ⊕ (12, 19, 26) x_{24} ⊕ (6, 10, 14) x_{34}), the obtained optimal fuzzy project completion times are (14, 23, 32), (15, 23, 31) and (15, 23, 31) respectively. Since (14, 23, 32) ≠ (15, 23, 31) i.e., Case (ii) of Step 3 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach is satisfied, so go to Step 4 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach.

Step 4 Since, $u_2 = mode(14, 23, 32) = mode(15, 23, 31) = 23$. So, using Step 4 of the proposed Mehar's method based on Kaufmann and Gupta ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(divergence(2, 4, 6)\ x_{12} + divergence(9, 13, 17)\ x_{13} + divergence(7, 9, 11)$
$x_{23} + divergence(12, 19, 26)\ x_{24} + divergence(6, 10, 14)\ x_{34})$

subject to

$\Re(2, 4, 6)\ x_{12} + \Re(9, 13, 17)\ x_{13} + \Re(7, 9, 11)\ x_{23} + \Re(12, 19, 26)\ x_{24} + \Re(6, 10, 14)\ x_{34}$
$= 23,$

$mode(2, 4, 6)\ x_{12} + mode(9, 13, 17)\ x_{13} + mode(7, 9, 11)\ x_{23} + mode(12, 19, 26)\ x_{24}$
$+\ mode(6, 10, 14)\ x_{34} = 23,$

$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12},\ x_{13},\ x_{23},\ x_{24},\ x_{34} \geq 0$

On solving the crisp linear programming problem the obtained optimal solution is $x_{12} = x_{24} = 1$ and $x_{13} = x_{23} = x_{34} = 0$.

Putting the obtained optimal values of x_{ij} in ((2, 4, 6) x_{12} ⊕ (9, 13, 17) x_{13} ⊕ (7, 9, 11) x_{23} ⊕ (12, 19, 26) x_{24} ⊕ (6, 10, 14) x_{34}) a unique triangular fuzzy number (14, 23, 32), representing the optimal fuzzy project completion time, is obtained and using the same values of x_{ij} the obtained critical path is $1 \Rightarrow 2 \Rightarrow 4$.

3.8.2 Optimal fuzzy completion time of the chosen problem by using the modified Liang and Han method based on Kaufmann and Gupta ranking approach

Using the modified Liang and Han method based on Kaufmann and Gupta ranking approach the optimal fuzzy completion time of the project, chosen in Example 3.1, can be obtained as follows:

Step 1 Using Step 1 of the modified Liang and Han method based on Kaufmann and Gupta ranking approach by assuming $\widetilde{FES_1} = (0,\ 0,\ 0)$ the values of $\widetilde{FES_j}$, $j = 2, 3, 4$ can be obtained as follows:

$$\widetilde{FES_2} = \widetilde{FES_1} \oplus \widetilde{FNT_{12}} = (0,\ 0,\ 0) \oplus (2,\ 4,\ 6) = (2,\ 4,\ 6)$$

$$\widetilde{FES_3} = \text{maximum}\{\widetilde{FES_1} \oplus \widetilde{FNT_{13}},\ \widetilde{FES_2} \oplus\ \widetilde{FNT_{23}}\}$$

$$= \text{maximum}\{(0, 0, 0) \oplus (9, 13, 17), (2, 4, 6) \oplus (7, 9, 11)\}$$

$$= \text{maximum}\{(9, 13, 17), (9, 13, 17)\} = (9, 13, 17)$$

$$\widetilde{FES_4} = \text{maximum}\{\widetilde{FES_2} \oplus \widetilde{FNT_{24}},\ \widetilde{FES_3} \oplus\ \widetilde{FNT_{34}}\}$$

$$= \text{maximum}\{(2, 4, 6) \oplus (12, 19, 26), (9, 13, 17) \oplus (6, 10, 14)\}$$

$$= \text{maximum}\{(14, 23, 32), (15, 23, 31)\}$$

$$= (14,\ 23,\ 32) \qquad \Big(\because divergence(14, 23, 32) > divergence(15, 23, 31)\Big)$$

Step 2 Using Step 2 of the modified Liang and Han method based on Kaufmann and Gupta ranking approach by assuming $\widetilde{FLF_4} = (14,\ 23,\ 32)$ the values of $\widetilde{FLF_j}$, $j = 3, 2, 1$ can be obtained as follows:

$$\widetilde{FLF_3} = \widetilde{FLF_4} \ominus_M \widetilde{FNT_{34}} = (14,\ 23,\ 32) \ominus_M (6,\ 10,\ 14) = (8, 13, 18)$$

$$\widetilde{FLF_2} = \text{minimum}\{\widetilde{FLF_4} \ominus_M \widetilde{FNT_{24}},\ \widetilde{FLF_3} \ominus_M \widetilde{FNT_{23}}\}$$

$$= \text{minimum}\{(14, 23, 32) \ominus_M (12, 19, 26), (8, 13, 18) \ominus_M (7, 9, 11)\}$$

$$= \text{minimum}\{(2, 4, 6), (1, 4, 7)\}$$

$$= (2,\ 4,\ 6) \qquad \Big(\because divergence(2, 4, 6) < divergence(1, 4, 7)\Big)$$

$$\widetilde{FLF_1} = \text{minimum}\{\widetilde{FLF_2} \ominus_M \widetilde{FNT_{12}},\ \widetilde{FLF_3} \ominus_M \widetilde{FNT_{13}}\}$$

$$= \text{minimum}\{(2, 4, 6) \ominus_M (2, 4, 6), (8, 13, 18) \ominus_M (9, 13, 17)\}$$

$$= \text{minimum}\{(0, 0, 0), (0, 1, 2)\}$$

$$= (0,\ 0,\ 0) \qquad \Big(\because \Re(0, 0, 0) < \Re(0, 1, 2)\Big)$$

Step 3 Using Step 3 of the modified Liang and Han method based on Kaufmann and Gupta ranking approach the values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$$\widetilde{FTS_{12}} = \widetilde{FLF_2} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{12}})$$

$$= (2, 4, 6) \ominus_M ((0,\ 0,\ 0) \oplus (2,\ 4,\ 6))$$

$$= (0, 0, 0)$$

$$\widetilde{FTS_{13}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{13}})$$

$$= (8, 13, 18) \ominus_M ((0,\ 0,\ 0) \oplus (9,\ 13,\ 17))$$

$$= (8, 13, 18) \ominus_M (9,\ 13,\ 17)$$

$$= (0, 1, 2)$$

$$\widetilde{FTS_{23}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{23}})$$

$$= (8, 13, 18) \ominus_M ((2,\ 4,\ 6) \oplus (7,\ 9,\ 11))$$

$$= (0, 1, 2)$$

$$\widetilde{FTS_{24}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{24}})$$

$$= (14,\ 23,\ 32) \ominus_M ((2,\ 4,\ 6) \oplus (12,\ 19,\ 26))$$

$$= (0, 0, 0)$$

$$\widetilde{FTS}_{34} = \widetilde{FLF}_4 \ominus_M (\widetilde{FES}_3 \oplus \widetilde{FNT}_{34})$$

$$= (14,\ 23,\ 32) \ominus_M ((9,\ 13,\ 17) \oplus (6,\ 10,\ 14))$$

$$= (0, 1, 2)$$

Step 4 Using Step 4 of the modified Liang and Han method based on Kaufmann and Gupta ranking approach

$$\text{minimum}\{\widetilde{FTS}_{12} \oplus \widetilde{FTS}_{24}, \widetilde{FTS}_{12} \oplus \widetilde{FTS}_{23} \oplus \widetilde{FTS}_{34}, \widetilde{FTS}_{13} \oplus \widetilde{FTS}_{34}\}$$

$$= \text{minimum}\{(0, 0, 0) \oplus (0, 0, 0), (0, 0, 0) \oplus (0, 1, 2) \oplus (0, 1, 2), (0, 1, 2) \oplus (0, 1, 2)\}$$

$$= \text{minimum}\{(0, 0, 0), (0, 2, 4), (0, 2, 4)\}$$

$$= (0,\ 0,\ 0) \qquad \Big(\because \Re(0, 0, 0) < \Re(0, 1, 2)\Big)$$

Since, minimum is occurring corresponding to the path $1 \Rightarrow 2 \Rightarrow 4$ so it is the critical path for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method based on Kaufmann and Gupta ranking approach the optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FNT}_{12} \oplus \widetilde{FNT}_{24}$$

$$= (2,\ 4,\ 6) \oplus (12,\ 19,\ 26) = (14,\ 23,\ 32)$$

3.8.3 Comparative study

The results of the problem, chosen in Example 3.1, obtained by using the modified Liang and Han method, Liu method [93], modified Liang and Han method based on Kaufmann and Gupta ranking approach and proposed Mehar's method based on Kaufmann and Gupta ranking approach are shown in Table 3.5

Table 3.5 Results of the chosen problem obtained using existing, modified and proposed methods

Example	Modified Liang and Han method		Liu method [93]		Modified Liang and Han method based on Kaufmann and Gupta ranking approach		Proposed Mehar's method based Kaufmann and Gupta ranking approach	
	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time
3.1	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 3 \Rightarrow 4$	(15, 23, 31), (14, 23, 32), (15, 23, 31)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 3 \Rightarrow 4$	(15, 23, 31), (14, 23, 32), (15, 23, 31)	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)

It is obvious from the results, shown in Table 3.5, that on solving the chosen problem by using the modified Liang and Han method proposed in Chapter 2 and Liu method [93] more than one fuzzy numbers, representing the optimal fuzzy completion time of the same project, are obtained which is not appropriate. While, on solving the same problem by using the modified Liang and Han method based on Kaufmann and Gupta ranking approach and proposed Mehar's method based on Kaufmann and Gupta ranking approach a unique fuzzy number, representing the optimal fuzzy project completion time, is obtained.

3.9 Conclusion

On the basis of presented study, it can be concluded that it is better to use modified Liang and Han method based on Kaufmann and Gupta ranking approach and the proposed Mehar's method based on Kaufmann and Gupta ranking approach as compared to modified Liang and Han method and Liu method [93] to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number.

Chapter 4

New Methods Based On Proposed Extension Of Kaufmann And Gupta Ranking Approach For Finding Unique Optimal Fuzzy Project Completion Time

In the previous chapter, it is shown that by using the proposed Mehar's method based on Kaufmann and Gupta ranking approach and modified Liang and Han method based on Kaufmann and Gupta ranking approach all the shortcomings of Liu method [93] and the modified Liang and Han method are resolved. Since, Kaufmann and Gupta ranking approach [70] is applicable only for finding the maximum and minimum of triangular fuzzy numbers so the methods, proposed in previous chapter, can not be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number. In this chapter, it is shown that it is not genuine to use the existing ranking approaches for finding the maximum and minimum of trapezoidal fuzzy numbers and a new ranking approach, by extending Kaufmann and Gupta ranking approach, is proposed for finding the maximum and minimum

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of trapezoidal fuzzy numbers. Also, on the basis of extended Kaufmann and Gupta ranking approach new methods, by modifying the methods proposed in previous chapter, are proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

4.1 Preliminaries

In this section, some basic definitions and arithmetic operations between two trapezoidal fuzzy numbers are presented [70].

4.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 4.1 A fuzzy number $\widetilde{A}$ defined on the universal set of real numbers $\mathbb{R}$, denoted as $\widetilde{A} = (a, b, c, d)$, is said to be a trapezoidal fuzzy number if its membership function, $\mu_{\widetilde{A}}(x)$, is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)} & , \quad a \leq x < b \\ 1 & , \quad b \leq x \leq c \\ \frac{(x-d)}{(c-d)} & , \quad c < x \leq d \\ 0 & , \quad \text{otherwise} \end{cases}$$

Definition 4.2 Let $\widetilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then, its λ-cut A^{λ} is defined as follows:

$$A^{\lambda} = [a + (b - a)\lambda, d - (d - c)\lambda], \ 0 \leq \lambda \leq 1$$

Definition 4.3 A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be a non-negative trapezoidal fuzzy number if and only if $a \geq 0$.

Definition 4.4 A trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ is said to be a zero trapezoidal fuzzy number if and only if $a = 0, b = 0, c = 0$ and $d = 0$.

Definition 4.5 Two trapezoidal fuzzy numbers $\widetilde{A_1} = (a_1, b_1, c_1, d_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2, d_2)$ are said to be equal i.e., $\widetilde{A_1} = \widetilde{A_2}$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2$ and $d_1 = d_2$.

4.1.2 Arithmetic operations

In this section, some arithmetic operations between two trapezoidal fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are presented.

(i) Let $\widetilde{A_1} = (a_1, b_1, c_1, d_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, $\widetilde{A_1} \oplus \widetilde{A_2} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$.

(ii) Let $\widetilde{A_1} = (a_1, b_1, c_1, d_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2, d_2)$ be two non-negative trapezoidal fuzzy numbers. Then, $\widetilde{A_1} \otimes \widetilde{A_2} \simeq (a_1a_2, a_1d_2, a_2d_1, d_1d_2)$

(iii) Let $\widetilde{A} = (a, b, c, d)$ be any trapezoidal fuzzy number. Then,

$$\gamma\widetilde{A} = \begin{cases} (\gamma a, \gamma b, \gamma c, \gamma d) & \gamma \geq 0 \\ (\gamma d, \gamma c, \gamma b, \gamma a) & \gamma \leq 0 \end{cases}$$

Remark 4.1 If $b = c$ then a trapezoidal fuzzy number (a, b, c, d) is said to be triangular fuzzy number and is denoted as (a, b, b, d) or (a, c, c, d) or (a, b, d) or (a, c, d).

4.1.3 Proposed Mehar's subtraction for trapezoidal fuzzy numbers

In this section, by modifying Mehar's subtraction for triangular fuzzy numbers, proposed in Chapter 2, Mehar's subtraction for trapezoidal fuzzy numbers is proposed.

Let $\widetilde{A_1} = (a_1, b_1, c_1, d_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2, d_2)$ be two non-negative trapezoidal fuzzy numbers. Then,

$\widetilde{A_1} \ominus_M \widetilde{A_2} = (a, b, c, d)$

$$\text{where, } \begin{cases} a = \text{maximum}\{0, (a_1 - a_2)\} \\ b = a + \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} \\ c = b + \text{maximum}\{0, (c_1 - b_1) - (c_2 - b_2)\} \\ d = c + \text{maximum}\{0, (d_1 - c_1) - (d_2 - c_2)\} \end{cases}$$

Proposition 4.1 Let $\widetilde{A_1} = (a_1, b_1, c_1, d_1)$ and $\widetilde{A_2} = (a_2, b_2, c_2, d_2)$ be two non-negative trapezoidal fuzzy numbers. Then, $\widetilde{A_1} \ominus_M \widetilde{A_2} = (a, b, c, d)$

$$\text{where, } \begin{cases} a = \text{maximum}\{0, (a_1 - a_2)\} \\ b = a + \text{maximum}\{0, (b_1 - a_1) - (b_2 - a_2)\} \\ c = b + \text{maximum}\{0, (c_1 - b_1) - (c_2 - b_2)\} \\ d = c + \text{maximum}\{0, (d_1 - c_1) - (d_2 - c_2)\} \end{cases}$$

is always a non-negative trapezoidal fuzzy number i.e.,

(i) $a \geq 0$ (ii) $b - a \geq 0$ (iii) $c - b \geq 0$ (iv) $d - c \geq 0$.

Proof: Proof is same as in Chapter 2.

4.2 Existing ranking approaches for finding the maximum and minimum of trapezoidal fuzzy numbers

Since, the limitations of the methods, proposed in previous chapter, is occurring due to the existing Kaufmann and Gupta ranking approach, so to overcome the limitations of the methods, proposed in previous chapter, one may try to modify the methods, proposed in previous chapter, with the help of existing ranking approaches for finding the maximum and minimum of trapezoidal fuzzy numbers. Although, there are several existing ranking approaches for finding the maximum and minimum of trapezoidal fuzzy numbers. However, in this section, some existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 84, 92, 113, 151, 160, 172], mostly used in the literature for finding the maximum and minimum of trapezoidal fuzzy numbers, are presented.

4.2.1 Existing ranking approach used by Liang and Han for finding the maximum and minimum of trapezoidal fuzzy numbers

In this section, the existing ranking approach, used by Liang and Han [84] for finding the maximum and minimum of n trapezoidal fuzzy numbers $\widetilde{A_i} = (a_i, b_i, c_i, d_i)$; $i = 1, 2, ..., n$, is presented.

Step 1 Choose any value of β where, $0 \leq \beta \leq 1$ and calculate

$\Re(\widetilde{A_i}) = \beta[(d_i - x_1)/(x_2 - x_1 - c_i + d_i)] + (1 - \beta)[1 - (x_2 - a_i)/(x_2 - x_1 + b_i - a_i)]$

where, $x_1 = \underset{1 \leq i \leq n}{\text{minimum}}\{a_i\}$ and $x_2 = \underset{1 \leq i \leq n}{\text{maximum}}\{d_i\}$.

Step 2 Find $\underset{1 \leq i \leq n}{\text{maximum}}\{\Re(\widetilde{A_i})\}$ and check that maximum occurs corresponding to unique value i or not.

Case (2a) If maximum occurs corresponding to unique value of i, say $i = t$ then $\underset{1 \leq i \leq n}{\text{maximum}}\{\widetilde{A_i}\} = \widetilde{A_t}$

Case(2b) If maximum occurs for p values of i, say $i = 1, 2, ..., p$ where, $2 \leq p \leq n$ then go to Step 3.

Step 3 Find $\underset{1 \leq i \leq p}{\text{maximum}}\{m(\widetilde{A_i})\} = \underset{1 \leq i \leq p}{\text{maximum}}\{b_i + c_i\}$ and check that maximum occurs corresponding to unique value of i or not.

Case (3a) If maximum occurs corresponding to unique value of i, say $i = \theta$ then $\underset{1 \leq i \leq p}{\text{maximum}}\{\widetilde{A_i}\} = \widetilde{A_\theta}$

Case (3b) If maximum occurs for l values of i, say $i = 1, 2, ..., l$ where, $2 \leq l \leq p$ then all the trapezoidal fuzzy numbers $\widetilde{A_1}$, $\widetilde{A_2}$, ..., $\widetilde{A_l}$ can be treated as maximum values.

4.2.2 Wang and Lee ranking approach

In this section, the existing ranking approach [151] for finding the maximum and minimum of trapezoidal fuzzy numbers is presented.

Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find $x_{\widetilde{A}} = \frac{\int_{a_1}^{b_1} x(\frac{x-a_1}{b_1-a_1})dx + \int_{b_1}^{c_1} x dx + \int_{c_1}^{d_1} x(\frac{x-d_1}{c_1-d_1})dx}{\int_{a_1}^{b_1} (\frac{x-a_1}{b_1-a_1})dx + \int_{b_1}^{c_1} dx + \int_{c_1}^{d_1} (\frac{x-d_1}{c_1-d_1})dx}$ and

$$x_{\widetilde{B}} = \frac{\int_{a_2}^{b_2} x(\frac{x-a_2}{b_2-a_2})dx + \int_{b_2}^{c_2} x dx + \int_{c_2}^{d_2} x(\frac{x-d_2}{c_2-d_2})dx}{\int_{a_2}^{b_2} (\frac{x-a_2}{b_2-a_2})dx + \int_{b_2}^{c_2} dx + \int_{c_2}^{d_2} (\frac{x-d_2}{c_2-d_2})dx}.$$

Case (i) If $x_{\widetilde{A}} > x_{\widetilde{B}}$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $x_{\widetilde{A}} < x_{\widetilde{B}}$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $x_{\widetilde{A}} = x_{\widetilde{B}}$ then go to Step 2.

Step 2 Find $y_{\widetilde{A}} = \frac{\int_0^1 (y(a_1+(b_1-a_1)y))dy + \int_0^1 (y(d_1+(c_1-d_1)y))dy}{\int_0^1 (a_1+(b_1-a_1)y)dy + \int_0^1 (d_1+(c_1-d_1)y)dy}$ and

$$y_{\widetilde{B}} = \frac{\int_0^1 (y(a_2+(b_2-a_2)y))dy + \int_0^1 (y(d_2+(c_2-d_2)y))dy}{\int_0^1 (a_2+(b_2-a_2)y)dy + \int_0^1 (d_2+(c_2-d_2)y)dy}.$$

Case (i) If $y_{\widetilde{A}} > y_{\widetilde{B}}$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $y_{\widetilde{A}} < y_{\widetilde{B}}$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $y_{\widetilde{A}} = y_{\widetilde{B}}$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \text{minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and } \widetilde{B}.$$

4.2.3 Kumar et al. ranking approach

In this section, the existing ranking approach [79] for finding the maximum and minimum of trapezoidal fuzzy numbers is presented.

Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find RM$(\widetilde{A}) = Rank(\widetilde{A}) = \frac{a_1+b_1+c_1+d_1}{4}$ and RM$(\widetilde{B}) = Rank(\widetilde{B}) = \frac{a_2+b_2+c_2+d_2}{4}$

Case (i) If RM$(\widetilde{A}) >$ RM$(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If RM$(\widetilde{A}) <$ RM$(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If RM$(\widetilde{A}) =$ RM$(\widetilde{B})$ then go to Step 2.

Step 2 Find RM$(\widetilde{A}) = mode(\widetilde{A}) = \frac{b_1+c_1}{2}$ and RM$(\widetilde{B}) = mode(\widetilde{B}) = \frac{b_2+c_2}{2}$

Case (i) If RM$(\widetilde{A}) >$ RM$(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If RM$(\widetilde{A}) <$ RM$(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If RM$(\widetilde{A}) =$ RM$(\widetilde{B})$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \text{minimum } \{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and } \widetilde{B}.$$

4.2.4 Other existing ranking approaches

In this section, some other existing ranking approaches [2, 11, 27, 31, 33, 58, 92, 113, 160, 172] for finding the maximum and minimum of trapezoidal fuzzy numbers are presented.

Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, use the formulae, shown in Table 4.1, to calculate $\Re(\widetilde{A})$ and $\Re(\widetilde{B})$ and check that $\Re(\widetilde{A}) > \Re(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$ or $\Re(\widetilde{A}) < \Re(\widetilde{B})$.

Case (i): If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii): If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii): If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} =$ minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and $\widetilde{B}$.

Table 4.1 Ranking formulae used in some existing ranking approaches

Ranking approaches	Ranking formulae for a trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$
Yager [160]	$\Re(\widetilde{A}) = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x dx + \int_c^d x(\frac{x-d}{c-d})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c dx + \int_c^d (\frac{x-d}{c-d})dx}$
Murakami et al. [113]	$\Re(\widetilde{A}) = x_{\widetilde{A}}$ or $y_{\widetilde{A}}$ where, $x_{\widetilde{A}} = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x dx + \int_c^d x(\frac{x-d}{c-d})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c dx + \int_c^d (\frac{x-d}{c-d})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(a+(b-a)y))dy + \int_0^1 (y(d+(c-d)y))dy}{\int_0^1 (a+(b-a)y)dy + \int_0^1 (d+(c-d)y)dy}$
Liou and Wang [92]	$\Re(\widetilde{A}) = \frac{a+b+c+d}{4}$
Chen et al. [27]	$\Re(\widetilde{A}) = \frac{a+b+c+d}{4}$
Cheng [31]	$\Re(\widetilde{A}) = \sqrt{x_{\widetilde{A}}^2 + y_{\widetilde{A}}^2}$ where, $x_{\widetilde{A}} = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x dx + \int_c^d x(\frac{x-d}{c-d})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c dx + \int_c^d (\frac{x-d}{c-d})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(a+(b-a)y))dy + \int_0^1 (y(d+(c-d)y))dy}{\int_0^1 (a+(b-a)y)dy + \int_0^1 (d+(c-d)y)dy}$
Yao and Wu [172]	$\Re(\widetilde{A}) = \frac{b+c}{2} + \frac{1}{4}(d-c-b+a)$
Chu and Tsao [33]	$\Re(\widetilde{A}) = x_{\widetilde{A}} . y_{\widetilde{A}}$ where, $x_{\widetilde{A}} = \frac{\int_a^b x(\frac{x-a}{b-a})dx + \int_b^c x dx + \int_c^d x(\frac{x-d}{c-d})dx}{\int_a^b (\frac{x-a}{b-a})dx + \int_b^c dx + \int_c^d (\frac{x-d}{c-d})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 y(a+(b-a)y)dy + \int_0^1 y(d+(c-d)y)dy}{\int_0^1 (a+(b-a)y)dy + \int_0^1 (d+(c-d)y)dy}$
Asady and Zendehnam [11]	$\Re(\widetilde{A}) = \frac{a+b+c+d}{4}$
Garcia and Lamata [58]	$\Re(\widetilde{A}) = \frac{a+2b+2c+d}{6}$
Abbasbandy and Hajjari [2]	$\Re(\widetilde{A}) = \frac{b+c}{2} + \frac{1}{12}(d-c-b+a)$

4.3 Shortcomings of existing ranking approaches

To show the shortcomings of existing ranking approaches [2, 11, 27, 31, 33, 58, 70, 79, 84, 92, 113, 151, 160, 172] the maximum and minimum of trapezoidal fuzzy numbers $\widetilde{A} = (8,\ 12,\ 16,\ 20)$ and $\widetilde{B} = (7,\ 13,\ 15,\ 21)$, obtained by using the existing ranking approaches [2, 11, 27, 31, 33, 58, 70, 79, 84, 92, 113, 151, 160, 172], are shown in Table 4.2

Table 4.2 Maximum and minimum of trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained by using existing ranking approaches

Ranking approaches	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A}, \widetilde{B}\}$	minimum$\{\widetilde{A}, \widetilde{B}\}$
Yager [160]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Murakami et al. [113]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Liou and Wang [92]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Chen et al. [27]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Cheng [31]	$\Re(\widetilde{A}) = 14.008$	$\Re(\widetilde{B}) = 14.008$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Yao and Wu [172]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Chu and Tsao [33]	$\Re(\widetilde{A}) = 7$	$\Re(\widetilde{B}) = 7$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Liang and Han [84]	$\Re(\widetilde{A}) = 0.5$	$\Re(\widetilde{B}) = 0.5$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Asady and Zendehnam [11]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Garcia and Lamata [58]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Wang and Lee [151]	$y_{\widetilde{A}} = 0.5$	$y_{\widetilde{B}} = 0.5$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Abbasbandy and Hajjari [2]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Kumar et al. [79]	$\mathrm{RM}(\widetilde{A}) = 14$	$\mathrm{RM}(\widetilde{A}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Kaufmann and Gupta [70]	Not applicable	Not applicable	Not applicable	Not applicable

It is obvious from Definition 4.5 that $\widetilde{A} \neq \widetilde{B}$ i.e., maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$ should be either $\widetilde{A}$ or $\widetilde{B}$. However, on the basis of results, shown in Table 4.2, obtained by using the existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 84, 92, 113, 151, 160, 172] it can be concluded that maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and $\widetilde{B}$ i.e., if modified Liang and Han method based on Kaufmann and Gupta ranking approach and proposed Mehar's method based on Kaufmann and

Gupta ranking approach will be modified on the basis of any of the existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 84, 92, 113, 151, 160, 172] then more than one trapezoidal fuzzy numbers, representing the optimal fuzzy project completion time, may be obtained. Hence, it is not genuine to modify the modified Liang and Han method based on Kaufmann and Gupta ranking approach and proposed Mehar's method based on Kaufmann and Gupta ranking approach on the basis of any of the existing ranking approaches [2, 11, 27, 31, 33, 58, 79, 84, 92, 113, 151, 160, 172].

Remark 4.1: In Section 4.3 the shortcomings of some ranking approaches are pointed out. The same shortcomings is also occurring in the remaining existing ranking approaches which are not discussed in this chapter.

4.4 Proposed extension of Kaufmann and Gupta ranking approach

In this section, to overcome the shortcomings of existing ranking approaches [2,11,27,31,33,58,79,84,92,113,151,160,172] and limitation of Kaufmann and Gupta ranking approach [70], a new ranking approach by extending Kaufmann and Gupta ranking approach is proposed for finding the maximum and minimum of trapezoidal fuzzy numbers.

Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$.

Step 1 Find $\Re(\widetilde{A}) = \frac{a_1+b_1+c_1+d_1}{4}$ and $\Re(\widetilde{B}) = \frac{a_2+b_2+c_2+d_2}{4}$

Case (i) If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then go to Step 2.

Step 2 Find $mode(\widetilde{A}) = \frac{b_1+c_1}{2}$ and $mode(\widetilde{B}) = \frac{b_2+c_2}{2}$

Case (i) If $mode(\widetilde{A}) > mode(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $mode(\widetilde{A}) < mode(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $mode(\widetilde{A}) = mode(\widetilde{B})$ then go to Step 3.

Step 3 Find $divergence(\widetilde{A}) = \frac{d_1+c_1-a_1-b_1}{2}$ and $divergence(\widetilde{B}) = \frac{d_2+c_2-a_2-b_2}{2}$

Case (i) If $divergence(\widetilde{A}) > divergence(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $divergence(\widetilde{A}) < divergence(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $divergence(\widetilde{A}) = divergence(\widetilde{B})$ then go to Step 4.

Step 4 Find $Left\ spread(\widetilde{A}) = \frac{b_1-a_1}{2}$ and $Left\ spread(\widetilde{B}) = \frac{b_2-a_2}{2}$

Case (i) If $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $Left\ spread(\widetilde{A}) < Left\ spread(\widetilde{B})$ then maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$

Remark 4.2: In Step 4 of proposed extension of Kaufmann and Gupta ranking approach instead of $Left\ spread(\widetilde{A}) = \frac{b_1-a_1}{2}$ and $Left\ spread(\widetilde{B}) = \frac{b_2-a_2}{2}$ the $Right\ spread(\widetilde{A}) = \frac{d_1-c_1}{2}$ and $Right\ spread(\widetilde{B}) = \frac{d_2-c_2}{2}$ respectively can also be used.

4.4.1 Validity of the proposed extension of Kaufmann and Gupta ranking approach

It is obvious that if $\widetilde{A}$ and $\widetilde{B}$ are two trapezoidal fuzzy numbers such that $\Re(\widetilde{A}) > \Re(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) > mode(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) = mode(\widetilde{B})$, $divergence(\widetilde{A}) > divergence(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) = mode(\widetilde{B})$, $divergence(\widetilde{A}) = divergence(\widetilde{B})$, $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$ then using the proposed extension of Kaufmann and Gupta ranking approach, presented in Section 4.4, maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$.

In this section, it is proved that if $\widetilde{A}$ and $\widetilde{B}$ are two such trapezoidal fuzzy numbers for which all the four conditions of proposed extension of Kaufmann and Gupta ranking approach are satisfied then $\widetilde{A} = \widetilde{B}$. Also, it is proved that if $Left\ spread(\widetilde{A}) \gtreqless Left\ spread(\widetilde{B})$ then $Right\ spread(\widetilde{A}) \gtreqless Right\ spread(\widetilde{B})$.

Proposition 4.1 Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers such that

(i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$ (ii) $mode(\widetilde{A}) = mode(\widetilde{B})$ (iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$ (iv) $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$. Then, $\widetilde{A} = \widetilde{B}$ i.e., $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$

Proof: (i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$

$$\Rightarrow \frac{a_1+b_1+c_1+d_1}{4} = \frac{a_2+b_2+c_2+d_2}{4}$$

$$\Rightarrow a_1+b_1+c_1+d_1 = a_2+b_2+c_2+d_2 \tag{4.1}$$

(ii) $mode(\widetilde{A}) = mode(\widetilde{B})$

$$\Rightarrow \frac{b_1+c_1}{2} = \frac{b_2+c_2}{2}$$

$$\Rightarrow b_1+c_1 = b_2+c_2 \tag{4.2}$$

(iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$

$\Rightarrow \frac{c_1+d_1-a_1-b_1}{2} = \frac{c_2+d_2-a_2-b_2}{2}$

$$\Rightarrow c_1 + d_1 - a_1 - b_1 = c_2 + d_2 - a_2 - b_2 \quad (4.3)$$

(iv) $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$

$\Rightarrow \frac{b_1-a_1}{2} = \frac{b_2-a_2}{2}$

$$\Rightarrow b_1 - a_1 = b_2 - a_2 \quad (4.4)$$

Solving (4.1), (4.2), (4.3) and (4.4)

$a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2, \quad d_1 = d_2$

i.e., $\widetilde{A} = \widetilde{B}$

Proposition 4.2 Let $\widetilde{A} = (a_1, b_1, c_1, d_1)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers such that

(i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$ (ii) $mode(\widetilde{A}) = mode(\widetilde{B})$ (iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$.

Then,

(a) $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$ iff $Right\ spread(\widetilde{A}) > Right\ spread(\widetilde{B})$

(b) $Left\ spread(\widetilde{A}) < Left\ spread(\widetilde{B})$ iff $Right\ spread(\widetilde{A}) < Right\ spread(\widetilde{B})$

(c) $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$ iff $Right\ spread(\widetilde{A}) = Right\ spread(\widetilde{B})$

Proof: Since, $\Re(\widetilde{A}) = \Re(\widetilde{B})$

$mode(\widetilde{A}) = mode(\widetilde{B})$

$divergence(\widetilde{A}) = divergence(\widetilde{B})$

so from Proposition 4.1,

$$a_1+b_1+c_1+d_1 = a_2+b_2+c_2+d_2 \quad (4.5)$$

$$b_1+c_1 = b_2+c_2 \tag{4.6}$$

$$c_1+d_1-a_1-b_1 = c_2+d_2-a_2-b_2 \tag{4.7}$$

Subtracting (4.6) from (4.5)

$$a_1+d_1 = a_2+d_2 \tag{4.8}$$

Subtracting (4.6) from (4.8)

$$(d_1 - c_1) - (b_1 - a_1) = (d_2 - c_2) - (b_2 - a_2)$$

$$\Rightarrow (d_1-c_1)-(d_2-c_2) = (b_1-a_1)-(b_2-a_2) \tag{4.9}$$

(a) $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$

$$\Leftrightarrow \tfrac{b_1-a_1}{2} > \tfrac{b_2-a_2}{2}$$

$$\Leftrightarrow (b_1-a_1)-(b_2-a_2) > 0 \tag{4.10}$$

$$\Leftrightarrow (d_1 - c_1) - (d_2 - c_2) > 0 \qquad \text{(Using equation 4.9)}$$

$$\Leftrightarrow d_1 - c_1 > d_2 - c_2$$

$$\Leftrightarrow \tfrac{d_1-c_1}{2} > \tfrac{d_2-c_2}{2}$$

$$\Leftrightarrow Right\ spread(\widetilde{A}) > Right\ spread(\widetilde{B})$$

Similarly, (b) and (c) can be easily proved.

4.4.2 Advantages of the proposed extension of Kaufmann and Gupta ranking approach

The main advantages of the proposed extension of Kaufmann and Gupta ranking approach over Kaufmann and Gupta ranking approach [70] is that Kaufmann and Gupta ranking approach can be used for finding the maximum and minimum of triangular fuzzy numbers but can not be used for finding the maximum and minimum of trapezoidal fuzzy numbers while the proposed extension of Kaufmann and Gupta ranking approach can be used for finding the maximum and minimum of triangular fuzzy numbers as well as trapezoidal fuzzy numbers.

To show the advantages of proposed extension of Kaufmann and Gupta ranking approach the maximum and minimum of trapezoidal fuzzy numbers $\widetilde{A}=(8, 12, 16, 20)$ and $\widetilde{B}=(7, 13, 15, 21)$, chosen in Section 4.3, obtained by using the proposed extension of Kaufmann and Gupta ranking approach is shown in Table 4.3

Table 4.3 Maximum and minimum of trapezoidal fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained by using proposed extension of Kaufmann and Gupta ranking approach

Ranking approach	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A},\widetilde{B}\}$	minimum$\{\widetilde{A},\widetilde{B}\}$
Proposed extension of Kaufmann and Gupta ranking approach	$\Re(\widetilde{A})=14$ $mode(\widetilde{A})=14$ $divergence(\widetilde{A})=8$ $Left\ spread(\widetilde{A})=2$	$\Re(\widetilde{B})=14$ $mode(\widetilde{B})=14$ $divergence(\widetilde{B})=8$ $Left\ spread(\widetilde{B})=3$	$\widetilde{B}$	$\widetilde{A}$

4.5 Proposed methods based on proposed extension of Kaufmann and Gupta ranking approach

Due to the limitations of Kaufmann and Gupta ranking approach [70] the methods, proposed in previous chapter, can be used only to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number but can not be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

In this section, to overcome the limitations of the methods, proposed in previous chapter, new methods on the basis of proposed extension of Kaufmann and Gupta ranking approach are proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

4.5.1 Modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach

If in the modified Liang and Han method based on Kaufmann and Gupta ranking approach, proposed in Chapter 3, instead of using existing Kaufmann and Gupta ranking approach [70] the proposed extension of Kaufmann and Gupta ranking approach will be used then always a unique trapezoidal fuzzy number representing the optimal fuzzy project completion time will be obtained.

4.5.2 Proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach

In this section, to overcome the limitations of proposed Mehar's method based on Kaufmann and Gupta ranking approach a new method, named as Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach, is proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number.

The steps of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach are as follows:

Step 1 Find the critical path and optimal fuzzy project completion time of the chosen problem by using the existing method [93] discussed in Section 3.1.

Case (i): If a unique critical path and hence a unique fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained optimal fuzzy project completion time is the optimal fuzzy completion time of the project and the obtained critical path is the only critical path of the project.

Case (ii): If more than one critical paths are obtained then go to Step 2.

Step 2 Check that the fuzzy numbers, representing the optimal fuzzy project completion time, corresponding to all the critical paths are same or not.

Case (i): If a unique trapezoidal fuzzy number, representing the optimal fuzzy project completion time, is obtained then all the critical paths, obtained in Step 1, corresponding to which the obtained trapezoidal fuzzy number is obtained, are critical paths of the project and the obtained trapezoidal fuzzy number will represent the optimal fuzzy completion time of the project.

Case (ii): If more than one trapezoidal fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 3.

Step 3 Let using previous steps p different trapezoidal fuzzy numbers $\widetilde{T}_1$, $\widetilde{T}_2$, ..., $\widetilde{T}_p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}_1 \neq \widetilde{T}_2 \neq ... \neq \widetilde{T}_p$ but $\Re(\widetilde{T}_1) = \Re(\widetilde{T}_2) = ... = \Re(\widetilde{T}_p) = u_1$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\text{Maximize} \sum_{(i,j)\in A} (mode(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij})$$

subject to

$$\left.\begin{array}{c} \sum\limits_{(i,j)\in A} (\Re(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) = u_1, \\ \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A. \end{array}\right\} \qquad (P_{4.1})$$

Case (i): If by putting the obtained optimal values of x_{ij} in $(\sum\limits_{(i,j)\in A} ((a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}))$ a unique trapezoidal fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained trapezoidal fuzzy number will represent the optimal fuzzy completion time of the project and all the critical paths can be obtained by using the obtained optimal values of x_{ij}.

Case (ii): If more than one trapezoidal fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 4.

Step 4 Let using previous steps l trapezoidal fuzzy numbers $\widetilde{T}'_1$, $\widetilde{T}'_2$, ..., $\widetilde{T}'_l$, where, $l \leq p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}'_1 \neq \widetilde{T}'_2 \neq ... \neq \widetilde{T}'_l$ but $\Re(\widetilde{T}'_1) = \Re(\widetilde{T}'_2) = ... = \Re(\widetilde{T}'_l) = u_1$ and $mode(\widetilde{T}'_1) = mode(\widetilde{T}'_2) = ... = mode(\widetilde{T}'_l) = u_2$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\text{Maximize} \sum_{(i,j)\in A} (divergence(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij})$$

subject to

$$\left.\begin{array}{l} \sum_{(i,j)\in A} (\Re(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) = u_1, \\ \sum_{(i,j)\in A} (mode(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) = u_2, \\ \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ x_{ij} \geq 0\ \forall\ (i,j) \in A. \end{array}\right\} \quad (P_{4.2})$$

Case (i): If by putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A} ((a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}))$ a unique trapezoidal fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained trapezoidal fuzzy number will represent the optimal fuzzy completion time of the project and all the critical paths can be obtained by using the obtained optimal values of x_{ij}.

Case (ii): If more than one trapezoidal fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 5.

Step 5 Let using previous steps k trapezoidal fuzzy numbers $\widetilde{T_1''}$, $\widetilde{T_2''}$, ..., $\widetilde{T_k''}$, where $k \leq l$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T_1''} \neq \widetilde{T_2''} \neq ... \neq \widetilde{T_k''}$ but $\Re(\widetilde{T_1''}) = \Re(\widetilde{T_2''}) = ... = \Re(\widetilde{T_k''}) = u_1$, $mode(\widetilde{T_1''}) = mode(\widetilde{T_2''}) = ... = mode(\widetilde{T_k''}) = u_2$ and $divergence(\widetilde{T_1''}) = divergence(\widetilde{T_2''}) = ... = divergence(\widetilde{T_k''}) = u_3$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\begin{aligned}
&\text{Maximize } \sum_{(i,j)\in A} (Left\ spread(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) \\
&\text{subject to} \\
&\left.\begin{aligned}
&\sum_{(i,j)\in A} (\Re(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) = u_1, \\
&\sum_{(i,j)\in A} (mode(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) = u_2, \\
&\sum_{(i,j)\in A} (divergence(a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}) = u_3, \\
&\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\
&x_{ij} \geq 0 \ \forall\ (i,j) \in A.
\end{aligned}\right\} \quad (P_{4.3})
\end{aligned}$$

Now, putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A} ((a_{ij}, b_{ij}, c_{ij}, d_{ij})\ x_{ij}))$ a unique trapezoidal fuzzy number, representing the optimal fuzzy project completion time, will be obtained and the critical paths can be obtained by using the obtained optimal values of x_{ij}.

4.6 Illustrative example

In this section, to illustrate the methods, proposed in Section 4.5, the problem, chosen in Example 4.1, is solved by both the proposed methods.

Example 4.1 Find the optimal fuzzy completion time of the project, shown in Figure 4.1, in which the fuzzy time duration $(\widetilde{t}_{ij})$ of the activity (i, j) is represented by the following trapezoidal fuzzy numbers:

$\widetilde{t}_{12} = (2, 3, 5, 7)$, $\widetilde{t}_{23} = (2, 4, 5, 8)$, $\widetilde{t}_{24} = (5, 8, 12, 14)$, $\widetilde{t}_{34} = (2, 5, 6, 7)$

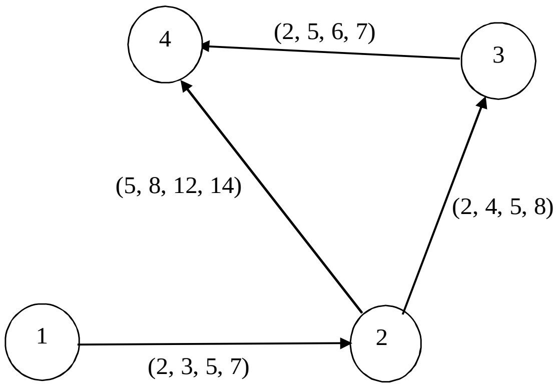

Figure 4.1 Project network of the illustrated example

4.6.1 Optimal fuzzy completion time of the chosen problem by using proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach

Using the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach the optimal fuzzy completion time of the project, chosen in Example 4.1, can be obtained as follows:

Step 1 On solving Example 4.1 by using existing method [93] the obtained critical paths are $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. Also, the optimal fuzzy project completion time corresponding to the paths $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are (7, 11, 17, 21) and (6, 12, 16, 22) respectively.

Since, more than one critical paths are obtained i.e., Case (ii) of Step 1 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach is satisfied, so go to Step 2 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach.

Step 2 Since, trapezoidal fuzzy numbers (7, 11, 17, 21) and (6, 12, 16, 22), representing the optimal fuzzy completion time of the project, are different i.e., Case (ii) of Step 2 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach is satisfied, so go to Step 3 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach.

Step 3 Since, $u_1 = \Re(7,11,17,21) = \Re(6,12,16,22) = 14$. So, using Step 3 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(mode(2,3,5,7)\; x_{12} + mode(2,4,5,8)\; x_{23} + mode(5,8,12,14)\; x_{24} + mode(2,5,6,7)\; x_{34})$

subject to

$\Re(2,3,5,7)\; x_{12} + \Re(2,4,5,8)\; x_{23} + \Re(5,8,12,14)\; x_{24} + \Re(2,5,6,7)\; x_{34} = 14,$

$x_{12} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12},\, x_{23},\, x_{24},\, x_{34} \geq 0$

On solving the crisp linear programming problem the following two optimal solutions are obtained:

(i) $x_{12} = x_{24} = 1$ and $x_{23} = x_{34} = 0$

(ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{24} = 0$

Putting the obtained optimal values of x_{ij} in ((2, 3, 5, 7) $x_{12} \oplus$ (2, 4, 5, 8) x_{23} $\oplus$ (5, 8, 12, 14) $x_{24} \oplus$ (2, 5, 6, 7) x_{34}), the obtained optimal fuzzy project completion times are (7, 11, 17, 21) and (6, 12, 16, 22) respectively. Since (7, 11, 17, 21) $\neq$

(6, 12, 16, 22) i.e., Case (ii) of Step 3 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach is satisfied, so go to Step 4 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach.

Step 4 Since, $u_2 = mode(7, 11, 17, 21) = mode(6, 12, 16, 22) = 14$. So, using Step 4 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(divergence(2, 3, 5, 7)\ x_{12} + divergence(2, 4, 5, 8)\ x_{23} + divergence(5, 8, 12,$
$14)\ x_{24} + divergence(2, 5, 6, 7)\ x_{34})$

subject to

$\Re(2, 3, 5, 7)\ x_{12} + \Re(2, 4, 5, 8)\ x_{23} + \Re(5, 8, 12, 14)\ x_{24} + \Re(2, 5, 6, 7)\ x_{34} = 14,$

$mode(2, 3, 5, 7)\ x_{12} + mode(2, 4, 5, 8)\ x_{23} + mode(5, 8, 12, 14)\ x_{24} + mode(2, 5, 6, 7)\ x_{34}$
$= 14,$

$x_{12} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12},\ x_{23},\ x_{24},\ x_{34} \geq 0.$

On solving the crisp linear programming problem the following two optimal solutions are obtained:

(i) $x_{12} = x_{24} = 1$ and $x_{23} = x_{34} = 0$

(ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{24} = 0$

Putting the obtained optimal values of x_{ij} in ((2, 3, 5, 7) $x_{12} \oplus$ (2, 4, 5, 8) $x_{23} \oplus$ (5, 8, 12, 14) $x_{24} \oplus$ (2, 5, 6, 7) x_{34}), the obtained optimal fuzzy project completion times are (7, 11, 17, 21) and (6, 12, 16, 22). Since (7, 11, 17, 21) $\neq$

(6, 12, 16, 22) i.e., Case (ii) of Step 4 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach is satisfied, so go to Step 5 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach.

Step 5 Since, $u_2 = divergence(7, 11, 17, 21) = divergence(6, 12, 16, 22) = 10$. So, using Step 4 of the proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(Left\ spread(2, 3, 5, 7)\ x_{12} + Left\ spread(2, 4, 5, 8)\ x_{23} + Left\ spread(5, 8, 12, 14)\ x_{24} + Left\ spread(2, 5, 6, 7)\ x_{34})$

subject to

$\Re(2, 3, 5, 7)\ x_{12} + \Re(2, 4, 5, 8)\ x_{23} + \Re(5, 8, 12, 14)\ x_{24} + \Re(2, 5, 6, 7)\ x_{34} = 14,$

$mode(2, 3, 5, 7)\ x_{12} + mode(2, 4, 5, 8)\ x_{23} + mode(5, 8, 12, 14)\ x_{24} + mode(2, 5, 6, 7)\ x_{34} = 14,$

$divergence(2, 3, 5, 7)\ x_{12} + divergence(2, 4, 5, 8)\ x_{23} + divergence(5, 8, 12, 14)\ x_{24} + divergence(2, 5, 6, 7)\ x_{34} = 10,$

$x_{12} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12}, x_{23}, x_{24}, x_{34} \geq 0$

On solving the crisp linear programming problem the obtained optimal solution is $x_{12} = x_{23} = x_{34} = 1$ and $x_{24} = 0$. Putting the obtained optimal values of x_{ij} in $((2, 3, 5, 7)\ x_{12} \oplus (2, 4, 5, 8)\ x_{23} \oplus (5, 8, 12, 14)\ x_{24} \oplus (2, 5, 6, 7)\ x_{34})$ a unique trapezoidal fuzzy number (6, 12, 16, 22), representing the optimal fuzzy project completion time, is obtained and using the same values of x_{ij} the obtained critical path is $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

4.6.2 Optimal fuzzy completion time of the chosen problem by using the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach

Using the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach the optimal fuzzy completion time of the project, chosen in Example 4.1, can be obtained as follows:

Step 1 Using Step 1 of the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach by assuming $\widetilde{FES_1} = (0,\ 0,\ 0,\ 0)$ the values of $\widetilde{FES_j}$, $j = 2, 3, 4$ can be obtained as follows:

$$\widetilde{FES_2} = \widetilde{FES_1} \oplus \widetilde{FNT_{12}} = (0,\ 0,\ 0,\ 0) \oplus (2,\ 3,\ 5,\ 7) = (2,\ 3,\ 5,\ 7)$$

$$\widetilde{FES_3} = \widetilde{FES_2} \oplus \widetilde{FNT_{23}} = (2, 3, 5, 7) \oplus (2, 4, 5, 8) = (4, 7, 10, 15)$$

$$\widetilde{FES_4} = \text{maximum}\{\widetilde{FES_2} \oplus \widetilde{FNT_{24}},\ \widetilde{FES_3} \oplus\ \widetilde{FNT_{34}}\}$$

$$= \text{maximum}\{(2, 3, 5, 7) \oplus (5, 8, 12, 14), (4, 7, 10, 15) \oplus (2, 5, 6, 7)\}$$

$$= \text{maximum}\{(7, 11, 17, 21), (6, 12, 16, 22)\}$$

$$= (6, 12, 16, 22)\ \Big(\because Left\ spread(7, 11, 17, 21) < Left\ spread(6, 12, 16, 22)\Big)$$

Step 2 Using Step 2 of the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach by assuming $\widetilde{FLF_4} = (6,\ 12,\ 16,\ 22)$ the values of $\widetilde{FLF_j}$, $j = 3, 2, 1$ can be obtained as follows:

$$\widetilde{FLF_3} = \widetilde{FLF_4} \ominus_M \widetilde{FNT_{34}} = (6,\ 12,\ 16,\ 22) \ominus_M (2,\ 5,\ 6,\ 7) = (4, 7, 10, 15)$$

$$\widetilde{FLF_2} = \text{minimum}\{\widetilde{FLF_4} \ominus_M \widetilde{FNT_{24}},\ \widetilde{FLF_3} \ominus_M \widetilde{FNT_{23}}\}$$

$$= \text{minimum}\{(6, 12, 16, 22) \ominus_M (5, 8, 12, 14), (4, 7, 10, 15) \ominus_M (2, 4, 5, 8)\}$$

$= \text{minimum}\{(1,4,4,8),(2,3,5,7)\}$

$= (2,\ 3,\ 5,\ 7) \quad \Big(\because \mathit{Left\ spread}(1,4,4,8) > \mathit{Left\ spread}(2,3,5,7)\Big)$

$\widetilde{FLF_1} = \widetilde{FLF_2} \ominus_M \widetilde{FNT_{12}}$

$= (2,\ 3,\ 5,\ 7) \ominus_M (2,\ 3,\ 5,\ 7) = (0,\ 0,\ 0,\ 0)$

Step 3 Using Step 3 of the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach the values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$\widetilde{FTS_{12}} = \widetilde{FLF_2} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{12}})$

$= (2,3,5,7) \ominus_M ((0,\ 0,\ 0,\ 0) \oplus (2,\ 3,\ 5,\ 7))$

$= (0,0,0,0)$

$\widetilde{FTS_{23}} = \widetilde{FLF_3} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{23}})$

$= (4,7,10,15) \ominus_M ((2,\ 3,\ 5,\ 7) \oplus (2,\ 4,\ 5,\ 8))$

$= (0,0,0,0)$

$\widetilde{FTS_{24}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{24}})$

$= (6,\ 12,\ 16,\ 22) \ominus_M ((2,\ 3,\ 5,\ 7) \oplus (5,\ 8,\ 12,\ 14))$

$= (6,\ 12,\ 16,\ 22) \ominus_M (7,\ 11,\ 17,\ 21)$

$= (0,2,2,4)$

$\widetilde{FTS_{34}} = \widetilde{FLF_4} \ominus_M (\widetilde{FES_3} \oplus \widetilde{FNT_{34}})$

$= (6,\ 12,\ 16,\ 22) \ominus_M ((4,\ 7,\ 10,\ 15) \oplus (2,\ 5,\ 6,\ 7))$

$= (0,0,0,0)$

Step 4 Using Step 4 of the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach,

$\text{minimum}\{\widetilde{FTS_{12}} \oplus \widetilde{FTS_{24}}, \widetilde{FTS_{12}} \oplus \widetilde{FTS_{23}} \oplus \widetilde{FTS_{34}}\}$

$= \text{minimum}\{(0,0,0,0) \oplus (0,2,2,4),\ (0,0,0,0) \oplus (0,0,0,0) \oplus (0,0,0,0)\}$

$= \text{minimum}\{(0,2,2,4),(0,0,0,0)\}$

$= (0,\ 0,\ 0,\ 0) \qquad \Big(\because \Re(0,0,0,0) < \Re(0,2,2,4)\Big)$

Since, the minimum is occurring corresponding to the path $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ so it is the critical path for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach, the optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FNT}_{12} \oplus \widetilde{FNT}_{23} \oplus \widetilde{FNT}_{34}$$

$$= (2,\ 3,\ 5,\ 7) \oplus (2,\ 4,\ 5,\ 8) \oplus (2,\ 5,\ 6,\ 7) = (6,\ 12,\ 16,\ 22)$$

4.7 Advantages of proposed methods

The main advantages of the methods, proposed in this chapter, over the methods, proposed in previous chapter, is that the methods, proposed in previous chapter, can be used only to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a triangular fuzzy number but can not be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by a trapezoidal fuzzy number. While, the methods proposed in this chapter can be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is either represented by a triangular fuzzy number or a trapezoidal fuzzy number.

To show the advantages of the methods, proposed in this chapter, over the methods, proposed in previous chapter, the results of the problem, chosen in Example 2.4, 3.1 and 4.1, obtained by using the methods proposed in this chapter and the methods proposed in previous chapter are shown in Table 4.4

Table 4.4 Results of the chosen problems obtained by using the methods proposed in previous chapter and in this chapter

Example	Proposed Mehar's method based on Kaufmann and Gupta ranking approach		Modified Liang and Han method based on Kaufmann and Gupta ranking approach		Proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach		Modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach	
	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time
2.4	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)
3.1	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)
4.1	Not applicable	Not applicable	Not applicable	Not applicable	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(6, 12, 16, 22)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(6, 12, 16, 22)

The results, presented in Table 4.4, can be explained as follows:

1. Since, proposed Mehar's method based on Kaufmann and Gupta ranking approach and modified Liang and Han method based on Kaufmann and Gupta ranking approach can be used only for solving such project network problems in which time of each activity is represented by a triangular fuzzy number. So, the problems, chosen in Example 2.4 and Example 3.1, in which time of each activity is represented by a triangular fuzzy number can be solved by using the methods proposed in previous chapter. But, the problem, chosen in Example 4.1, in which time of each activity is represented by a trapezoidal fuzzy number can not be solved by using the methods proposed in previous chapter.

2. Since, the methods, proposed in this chapter, can be used for solving such project network problems in which time of each activity is either represented by a triangular fuzzy number or by a trapezoidal fuzzy number. So, the problems, chosen in Example 2.4, Example 3.1 and Example 4.1, in which time of each activity is either represented by a triangular fuzzy number or by a trapezoidal fuzzy number can be solved by using the methods proposed in

this chapter.

4.8 Conclusion

On the basis of presented study, it can be concluded that it is better to use the methods proposed in this chapter as compared to the methods proposed in previous chapter for finding the optimal fuzzy completion time of such project network problems in which the time of each activity is either represented by a triangular fuzzy number or a trapezoidal fuzzy number.

Chapter 5

New Methods Based On Modified Farhadinia Ranking Approach For Finding Unique Optimal Fuzzy Project Completion Time

In this chapter, limitations of the methods, proposed in previous chapter, are pointed out and to overcome these limitations new methods are proposed by modifying the methods proposed in previous chapter.

5.1 Preliminaries

In the literature [43, 182], it is pointed out that the computational efforts required to solve a fuzzy linear programming problem can be reduced, if the decision maker express his data using LR flat fuzzy numbers. All kinds of crisp numbers, triangular fuzzy numbers and trapezoidal fuzzy numbers are LR flat fuzzy numbers. So, LR flat fuzzy numbers are frequently used to increase the computational efficiency without limiting the generality beyond the acceptable limits and facilities the ease of acquisition of data to solve real life problems.

In this section, some basic definitions and arithmetic operations between two

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LR flat fuzzy numbers are presented.

5.1.1 Basic definitions

In this section, some basic definitions are presented.

Definition 5.1 [43] A function $L : [0, \infty) \to [0, 1]$ (or $R : [0, \infty) \to [0, 1]$) is said to be reference function of fuzzy number if and only if

(i) $L(0) = 1$ (or $R(0) = 1$)

(ii) L (or R) is non-increasing on $[0, \infty)$.

Definition 5.2 [43] A fuzzy number $\widetilde{A}$ defined on universal set of real numbers $\mathbb{R}$, denoted as $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{\underline{a}-x}{a^L}) & , \quad x \leq \underline{a} \\ R(\frac{x-\overline{a}}{a^R}) & , \quad x \geq \overline{a} \\ 1 & , \quad \underline{a} \leq x \leq \overline{a} \end{cases}$$

where, a^L and a^R are non-negative real numbers.

The cases $a^L = 0$ and/or $a^R = 0$ are admissible. It is assumed that $L(\frac{\underline{a}-x}{0}) = 0$ and/or $R(\frac{x-\overline{a}}{0}) = 0$. Thus, any interval $[\underline{a}, \overline{a}]$ and any real number a are also LR flat fuzzy numbers and can be written as $(\underline{a}, \overline{a}, 0, 0)_{LR}$ and $(a, a, 0, 0)$ respectively.

Definition 5.3 [43] Let $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ be an LR flat fuzzy number and α be a real number in the interval $[0, 1]$. Then, the crisp set $A_\alpha = \{x \in X : \mu_{\widetilde{A}}(x) \geq \alpha\} = [\underline{a} - a^L L^{-1}(\alpha), \overline{a} + a^R R^{-1}(\alpha)]$, is said to be an α-cut of $\widetilde{A}$.

Definition 5.4 [43] An LR flat fuzzy number $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a zero LR flat fuzzy number if and only if $\underline{a} = 0, \overline{a} = 0, a^L = 0$ and $a^R = 0$.

Definition 5.5 [43] Two LR flat fuzzy numbers $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ are said to be equal i.e., $\widetilde{A_1} = \widetilde{A_2}$ if and only if $\underline{a_1} = \underline{a_2}, \overline{a_1} =$

$\overline{a_2}, a_1^L = a_2^L$ and $a_1^R = a_2^R$.

Definition 5.6 [37] An LR flat fuzzy number $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a non-negative LR flat fuzzy number if and only if $\underline{a} - a^L \geq 0$.

Remark 5.1 If $\underline{a} = \overline{a}$ then an LR flat fuzzy number $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be an LR fuzzy number and is denoted as $(\underline{a}, \underline{a}, a^L, a^R)_{LR}$ or $(\overline{a}, \overline{a}, a^L, a^R)_{LR}$ or $(\underline{a}, a^L, a^R)_{LR}$ or $(\overline{a}, a^L, a^R)_{LR}$.

Remark 5.2 If $\underline{a} = \overline{a}$ and $L(x) = R(x) = \text{maximum}\{0, 1-x\}$ then an LR flat fuzzy number $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a triangular fuzzy number and is denoted as (a, b, c) where, $a = \underline{a} - a^L(\text{or } \overline{a} - a^L), b = \underline{a}(\text{or } \overline{a}), c = \underline{a} + a^R(\text{or } \overline{a} + a^R)$.

Remark 5.3 If $\underline{a} \neq \overline{a}$ and $L(x) = R(x) = \text{maximum}\{0, 1-x\}$ then an LR flat fuzzy number $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ is said to be a trapezoidal fuzzy number and is denoted as (a, b, c, d) where, $a = \underline{a} - a^L, b = \underline{a}, c = \overline{a}, d = \overline{a} + a^R$.

5.1.2 Arithmetic operations

In this section, some arithmetic operations between two LR flat fuzzy numbers are presented [43].

Let $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers. Then,

(i) $\widetilde{A_1} \oplus \widetilde{A_2} = (\underline{a_1} + \underline{a_2}, \overline{a_1} + \overline{a_2}, a_1^L + a_2^L, a_1^R + a_2^R)_{LR}$

(ii) $\lambda\widetilde{A_1} = \begin{cases} (\lambda\underline{a_1}, \lambda\overline{a_1}, \lambda a_1^L, \lambda a_1^R)_{LR} & \lambda \geq 0 \\ (\lambda\overline{a_1}, \lambda\underline{a_1}, -\lambda a_1^R, -\lambda a_1^L)_{RL} & \lambda \leq 0 \end{cases}$

Let $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two non-negative LR flat fuzzy numbers. Then,

(iii) $\widetilde{A_1} \otimes \widetilde{A_2} \simeq (\underline{a_1}\ \underline{a_2}, \overline{a_1}\ \overline{a_2}, \underline{a_1}a_2^L + \underline{a_2}a_1^L - a_1^L a_2^L, \overline{a_1}a_2^R + \overline{a_2}a_1^R + a_1^R a_2^R)_{LR}$.

5.2 Limitations of the methods proposed in previous chapter

Since, the extension of Kaufmann and Gupta ranking approach, proposed in previous chapter, can be used for finding the maximum and minimum of triangular and trapezoidal fuzzy numbers but can not be used for finding the maximum and minimum of LR flat fuzzy numbers. So, the methods, proposed in previous chapter, can not be used for finding the unique optimal fuzzy project completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number e.g., the problems, chosen in Example 5.1 and Example 5.2, in which time of each activity is represented by an LR flat fuzzy number can not be solved using the methods proposed in previous chapter.

Example 5.1 Find the optimal fuzzy completion time of the project, shown in Figure 5.1, in which the fuzzy time duration $(\widetilde{t}_{ij})$ of the activity (i, j) is represented by the following LR flat fuzzy numbers with $L(x) = R(x) = e^{-|x|}$:

$\widetilde{t}_{12} = (5, 6, 2, 1)_{LR}$, $\widetilde{t}_{23} = (7, 8, 1, 1)_{LR}$, $\widetilde{t}_{24} = (13, 14, 2, 5)_{LR}$, $\widetilde{t}_{34} = (6, 9, 1, 1)_{LR}$

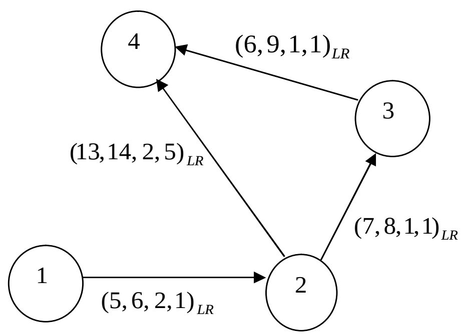

Figure 5.1 Structure of the project network

Example 5.2 A research and development department divides the whole work for developing a new power supply for a console television set into different activities. The activities and its time duration, represented by LR flat fuzzy numbers with

$L(x) = \max\{0, 1 - x^2\}$ and $R(x) = e^{-|x|}$, are shown in Table 5.1 and the network, connecting the different activities, is shown in Figure 5.2. Find the optimal fuzzy completion time of the project.

Table 5.1 Fuzzy time for each activity

Job	Activity	Description	Duration ($\widetilde{D_{ij}}$) (days)
A	(1, 2)	Determine output voltages	$(2, 2.3, 0.6, 0.8)_{LR}$
B	(2, 3)	Determine whether to use solid state rectifier	$(5, 6, 1, 1.2)_{LR}$
C	(3, 4)	Choose rectifiers	$(7, 7.8, 1, 1.1)_{LR}$
D	(3, 5)	Choose filters	$(4, 4.5, 0.7, 2.7)_{LR}$
E	(4, 6)	Choose transformers	$(2, 4.2, 0.7, 1.8)_{LR}$
F	(5, 6)	Choose chassis	$(5, 5.5, 1, 2.2)_{LR}$
G	(4, 7)	Choose rectifier mounting	$(5, 9.6, 1.1, 1.3)_{LR}$
H	(6, 7)	Layout chassis	$(3, 3.4, 0.4, 1.5)_{LR}$
I	(7, 8)	Build and test	$(6, 6.3, 0.3, 0.6)_{LR}$

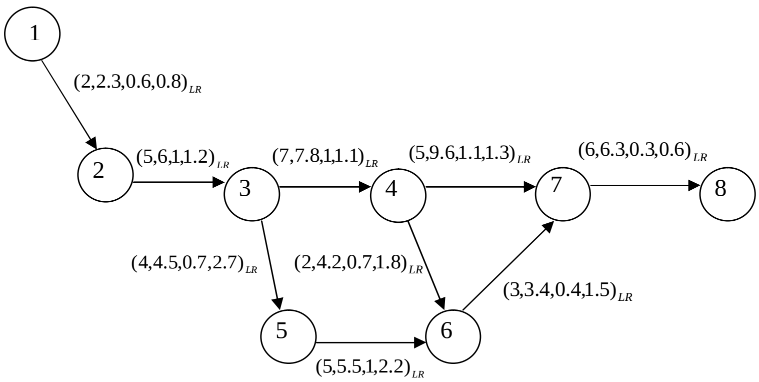

Figure 5.2 Fuzzy activity times in the project network

5.3 Existing ranking approaches for finding the maximum and minimum of LR flat fuzzy numbers

Since, the limitations of the methods, proposed in previous chapter, is occurring due to the limitations of proposed extension of Kaufmann and Gupta ranking approach, so to overcome the limitations of the methods, proposed in previous chapter, one may try to modify the methods, proposed in previous chapter, with the help of existing ranking approaches for finding the maximum and minimum of LR flat fuzzy numbers. Although, there are several existing ranking approaches for finding the maximum and minimum of LR flat fuzzy numbers. However, in this section, some existing ranking approaches [2,11,31,33,55,58,79,92,113,151,160,172], mostly used in the literature for finding the maximum and minimum of LR flat fuzzy numbers, are presented.

5.3.1 Wang and Lee ranking approach

In this section, the existing ranking approach [151] for comparing LR flat fuzzy numbers is presented.

Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers then use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find $x_{\widetilde{A}} = \frac{\int_{\underline{a_1}-a_1^L}^{\underline{a_1}} xL(\frac{\underline{a_1}-x}{a_1^L})dx + \int_{\underline{a_1}}^{\overline{a_1}} xdx + \int_{\overline{a_1}}^{\overline{a_1}+a_1^R} xR(\frac{x-\overline{a_1}}{a_1^R})dx}{\int_{\underline{a_1}-a_1^L}^{\underline{a_1}} L(\frac{\underline{a_1}-x}{a_1^L})dx + \int_{\underline{a_1}}^{\overline{a_1}} dx + \int_{\overline{a_1}}^{\overline{a_1}+a_1^R} R(\frac{x-\overline{a_1}}{a_1^R})dx}$ and

$$x_{\widetilde{B}} = \frac{\int_{\underline{a_2}-a_2^L}^{\underline{a_2}} xL(\frac{\underline{a_2}-x}{a_2^L})dx + \int_{\underline{a_2}}^{\overline{a_2}} xdx + \int_{\overline{a_2}}^{\overline{a_2}+a_2^R} xR(\frac{x-\overline{a_2}}{a_2^R})dx}{\int_{\underline{a_2}-a_2^L}^{\underline{a_2}} L(\frac{\underline{a_2}-x}{a_2^L})dx + \int_{\underline{a_2}}^{\overline{a_2}} dx + \int_{\overline{a_2}}^{\overline{a_2}+a_2^R} R(\frac{x-\overline{a_2}}{a_2^R})dx}.$$

Case (i) If $x_{\widetilde{A}} > x_{\widetilde{B}}$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $x_{\widetilde{A}} < x_{\widetilde{B}}$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $x_{\widetilde{A}} = x_{\widetilde{B}}$ then go to Step 2.

Step 2 Find $y_{\widetilde{A}} = \frac{\int_0^1 (y(\underline{a_1}-a_1^L L^{-1}(y)))dy + \int_0^1 (y(\overline{a_1}+a_1^R R^{-1}(y)))dy}{\int_0^1 (\underline{a_1}-a_1^L L^{-1}(y))dy + \int_0^1 (\overline{a_1}+a_1^R R^{-1}(y))dy}$ and

$$y_{\widetilde{B}} = \frac{\int_0^1 (y(\underline{a_2}-a_2^L L^{-1}(y)))dy + \int_0^1 (y(\overline{a_2}+a_2^R R^{-1}(y)))dy}{\int_0^1 (\underline{a_2}-a_2^L L^{-1}(y))dy + \int_0^1 (\overline{a_2}+a_2^R R^{-1}(y))dy}.$$

Case (i) If $y_{\widetilde{A}} > y_{\widetilde{B}}$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $y_{\widetilde{A}} < y_{\widetilde{B}}$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A},\widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A},\widetilde{B}\} = \widetilde{A}$$

Case (iii) If $y_{\widetilde{A}} = y_{\widetilde{B}}$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A},\widetilde{B}\} = \text{minimum}\{\widetilde{A},\widetilde{B}\} = \widetilde{A} \text{ and } \widetilde{B}.$$

5.3.2 Farhadinia ranking approach

In this section, the existing ranking approach [55] for finding the maximum and minimum of LR flat fuzzy numbers is presented.

Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A},\widetilde{B}\}$ and minimum$\{\widetilde{A},\widetilde{B}\}$:

Step 1 Find $C(\widetilde{A})= \inf\{x \in \text{Supp}(\widetilde{A}) : \mu_{\widetilde{A}}(x) = 1\} = \underline{a_1}$ and

$$C(\widetilde{B}) = \inf\{x \in \text{Supp}(\widetilde{B}) : \mu_{\widetilde{B}}(x) = 1\} = \underline{a_2}$$

Case (i) If $C(\widetilde{A}) > C(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A},\widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A},\widetilde{B}\} = \widetilde{B}$$

Case (ii) If $C(\widetilde{A}) < C(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A},\widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A},\widetilde{B}\} = \widetilde{A}$$

Case (iii) If $C(\widetilde{A}) = C(\widetilde{B})$ then go to Step 2.

Step 2 Find $L(\widetilde{A}) = \inf \text{Supp}(\widetilde{A}) = \underline{a_1} - a_1^L$ and

$$L(\widetilde{B}) = \inf \text{Supp}(\widetilde{B}) = \underline{a_2} - a_2^L$$

Case (i) If $L(\widetilde{A}) > L(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A},\widetilde{B}\} = \widetilde{A} \text{ and minimum } \{\widetilde{A},\widetilde{B}\} = \widetilde{B}$$

Case (ii) If $L(\widetilde{A}) < L(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A},\widetilde{B}\} = \widetilde{B} \text{ and minimum } \{\widetilde{A},\widetilde{B}\} = \widetilde{A}$$

Case (iii) If $L(\widetilde{A}) = L(\widetilde{B})$ then go to Step 3.

Step 3 Find $W(\widetilde{A}) = |\text{Supp}(\widetilde{A})| = \overline{a_1} - \underline{a_1} + a_1^L + a_1^R$

and $W(\widetilde{B}) = |\text{Supp}(\widetilde{B})| = \overline{a_2} - \underline{a_2} + a_2^L + a_2^R$

Case (i) If $W(\widetilde{A}) > W(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $W(\widetilde{A}) < W(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $W(\widetilde{A}) = W(\widetilde{B})$ then go to Step 4.

Step 4 Find $S(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = \overline{a_1} - \underline{a_1} + a_1^L \int_0^1 L^{-1}(\alpha)d\alpha + a_1^R \int_0^1 R^{-1}(\alpha)d\alpha$

and $S(\widetilde{B}) = \int \mu_{\widetilde{B}}(x)dx = \overline{a_2} - \underline{a_2} + a_2^L \int_0^1 L^{-1}(\alpha)d\alpha + a_2^R \int_0^1 R^{-1}(\alpha)d\alpha$

Case (i) If $S(\widetilde{A}) > S(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $S(\widetilde{A}) < S(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $S(\widetilde{A}) = S(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$.

Remark 5.4: For an LR flat fuzzy number $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$

$S(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = \int_{\underline{a}-a^L}^{\underline{a}} L(\frac{\underline{a}-x}{a^L})dx + \int_{\underline{a}}^{\overline{a}} dx + \int_{\overline{a}}^{\overline{a}+a^R} R(\frac{x-\overline{a}}{a^R})dx = \overline{a} - \underline{a}$

$+ a^L \int_0^1 L^{-1}(\alpha)d\alpha + a^R \int_0^1 R^{-1}(\alpha)d\alpha$

5.3.3 Kumar et al. ranking approach

In this section, the existing ranking approach [79] for finding the maximum and minimum of LR flat fuzzy numbers is presented.

Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find RM$(\widetilde{A}) = Rank(\widetilde{A}) = \frac{1}{2}\,[\int_0^1 \underline{a_1} d\alpha - \int_0^1 a_1^L L^{-1}(\alpha) d\alpha + \int_0^1 \overline{a_1} d\alpha + \int_0^1 a_1^R R^{-1}(\alpha) d\alpha]$

and RM$(\widetilde{B}) = Rank(\widetilde{B}) = \frac{1}{2}\,[\int_0^1 \underline{a_2} d\alpha - \int_0^1 a_2^L L^{-1}(\alpha) d\alpha + \int_0^1 \overline{a_2} d\alpha + \int_0^1 a_2^R R^{-1}(\alpha) d\alpha]$

Case (i) If RM$(\widetilde{A}) >$ RM$(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If RM$(\widetilde{A}) <$ RM$(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If RM$(\widetilde{A}) =$ RM$(\widetilde{B})$ then go to Step 2.

Step 2 Find RM$(\widetilde{A}) = mode(\widetilde{A}) = \frac{1}{2}[\int_0^1 \underline{a_1} d\alpha + \int_0^1 \overline{a_1} d\alpha]$

and RM$(\widetilde{B}) = mode(\widetilde{B}) = \frac{1}{2}[\int_0^1 \underline{a_2} d\alpha + \int_0^1 \overline{a_2} d\alpha]$

Case (i) If RM$(\widetilde{A}) >$ RM$(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If RM$(\widetilde{A}) <$ RM$(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If RM$(\widetilde{A}) =$ RM$(\widetilde{B})$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

maximum$\{\widetilde{A}, \widetilde{B}\} =$ minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and $\widetilde{B}$.

5.3.4 Other existing ranking approaches for finding the maximum and minimum of LR flat fuzzy numbers

In this section, some other existing ranking approaches [2, 11, 31, 33, 58, 92, 113, 160, 172] for finding the maximum and minimum of LR flat fuzzy numbers are presented.

Let $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\widetilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ be two LR flat fuzzy numbers.

Then, use the formulae, shown in Table 5.2, to calculate $\Re(\widetilde{A})$ and $\Re(\widetilde{B})$ and check that $\Re(\widetilde{A}) > \Re(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$ or $\Re(\widetilde{A}) < \Re(\widetilde{B})$.

Case (i): If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii): If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii): If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then $\widetilde{A} \approx \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \text{minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and } \widetilde{B}.$$

Table 5.2 Ranking formulae used in some existing ranking approaches

Ranking approaches	Ranking formulae for an LR flat fuzzy number $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$
Yager [160]	$\Re(\widetilde{A}) = \frac{\int_{\underline{a}-a^L}^{\underline{a}} xL(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} xdx+\int_{\overline{a}}^{\overline{a}+a^R} xR(\frac{x-\overline{a}}{a^R})dx}{\int_{\underline{a}-a^L}^{\underline{a}} L(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} dx+\int_{\overline{a}}^{\overline{a}+a^R} R(\frac{x-n}{a^R})dx}$
Murakami et al. [113]	$\Re(\widetilde{A}) = x_{\widetilde{A}}$ or $y_{\widetilde{A}}$ where, $x_{\widetilde{A}} = \frac{\int_{\underline{a}-a^L}^{\underline{a}} xL(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} xdx+\int_{\overline{a}}^{\overline{a}+a^R} xR(\frac{x-\overline{a}}{a^R})dx}{\int_{\underline{a}-a^L}^{\underline{a}} L(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} dx+\int_{\overline{a}}^{\overline{a}+a^R} R(\frac{x-\overline{a}}{a^R})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(\underline{a}-a^L L^{-1}(y)))dy+\int_0^1 (y(\overline{a}+a^R R^{-1}(y)))dy}{\int_0^1 (\underline{a}-a^L L^{-1}(y))dy+\int_0^1 (\overline{a}+a^R R^{-1}(y))dy}$
Liou and Wang [92]	$\Re(\widetilde{A}) = \frac{1}{2}[\int_0^1 (\underline{a} - a^L L^{-1}(y))dy + \int_0^1 (\overline{a} + a^R R^{-1}(y))dy]$
Cheng [31]	$\Re(\widetilde{A}) = \sqrt{x_{\widetilde{A}}^2 + y_{\widetilde{A}}^2}$ where, $x_{\widetilde{A}} = \frac{\int_{\underline{a}-a^L}^{\underline{a}} xL(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} xdx+\int_{\overline{a}}^{\overline{a}+a^R} xR(\frac{x-\overline{a}}{a^R})dx}{\int_{\underline{a}-a^L}^{\underline{a}} L(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} dx+\int_{\overline{a}}^{\overline{a}+a^R} R(\frac{x-\overline{a}}{a^R})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(\underline{a}-a^L L^{-1}(y)))dy+\int_0^1 (y(\overline{a}+a^R R^{-1}(y)))dy}{\int_0^1 (\underline{a}-a^L L^{-1}(y))dy+\int_0^1 (\overline{a}+a^R R^{-1}(y))dy}$
Yao and Wu [172]	$\Re(\widetilde{A}) = \frac{1}{2}\int_0^1 [D_L(y) + D_R(y)]dy$, where, $D_L(y) = \underline{a} - a^L L^{-1}(y)$ and $D_R(y) = \overline{a} + a^R R^{-1}(y)$
Chu and Tsao [33]	$\Re(\widetilde{A}) = x_{\widetilde{A}}.y_{\widetilde{A}}$ where, $x_{\widetilde{A}} = \frac{\int_{\underline{a}-a^L}^{\underline{a}} xL(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} xdx+\int_{\overline{a}}^{\overline{a}+a^R} xR(\frac{x-\overline{a}}{a^R})dx}{\int_{\underline{a}-a^L}^{\underline{a}} L(\frac{\underline{a}-x}{a^L})dx+\int_{\underline{a}}^{\overline{a}} dx+\int_{\overline{a}}^{\overline{a}+a^R} R(\frac{x-\overline{a}}{a^R})dx}$, $y_{\widetilde{A}} = \frac{\int_0^1 (y(\underline{a}-a^L L^{-1}(y)))dy+\int_0^1 (y(\overline{a}+a^R R^{-1}(y)))dy}{\int_0^1 (\underline{a}-a^L L^{-1}(y))dy+\int_0^1 (\overline{a}+a^R R^{-1}(y))dy}$
Asady and Zendehnam [11]	$\Re(\widetilde{A}) = \frac{1}{2}\int_0^1 (\underline{A}(r) + \overline{A}(r))dr$, where, $\underline{A}(r) = \underline{a} - a^L L^{-1}(r)$ and $\overline{A}(r) = \overline{a} + a^R R^{-1}(r)$
Garcia and Lamata [58]	$\Re(\widetilde{A}) = \frac{1}{3}[\int_0^1 (\underline{a} - a^L L^{-1}(y))dy + \int_0^1 (\overline{a} + a^R R^{-1}(y))dy] + \frac{1}{6}[\int_0^1 (\underline{a} + \overline{a})dy]$
Abbasbandy and Hajjari [2]	$\Re(\widetilde{A}) = \frac{1}{2}[\int_0^1 ((\underline{a} - a^L L^{-1}(y)) + (\overline{a} + a^R R^{-1}(y)) + \underline{a} + \overline{a})dy]$

5.4 Shortcomings of existing ranking approaches

To show the shortcomings of existing ranking approaches [2, 11, 31, 33, 55, 58, 79, 92, 113, 151, 160, 172] the maximum and minimum of LR flat fuzzy numbers $\widetilde{A} =$

$(12, 16, 4, 4)_{LR}$ and $\widetilde{B} = (13, 15, 6, 6)_{LR}$ with $L(x) = R(x) = \max\{0, 1-x^2\}$, obtained by using the existing ranking approaches, are shown in Table 5.3

Table 5.3 Maximum and minimum of LR flat fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained by using existing ranking approaches

Ranking approaches	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A}, \widetilde{B}\}$	minimum$\{\widetilde{A}, \widetilde{B}\}$
Yager [160]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Murakami et al. [113]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Liou and Wang [92]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Cheng [31]	$\Re(\widetilde{A}) = 14.008$	$\Re(\widetilde{B}) = 14.008$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Yao and Wu [172]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Chu and Tsao [33]	$\Re(\widetilde{A}) = 7$	$\Re(\widetilde{B}) = 7$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Asady and Zendehnam [11]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Garcia and Lamata [58]	$\Re(\widetilde{A}) = 14$	$\Re(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Wang and Lee [151]	$y_{\widetilde{A}} = 0.5$	$y_{\widetilde{B}} = 0.5$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Abbasbandy and Hajjari [2]	$\Re(\widetilde{A}) = 28$	$\Re(\widetilde{B}) = 28$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Kumar et al. [79]	RM$(\widetilde{A}) = 14$	RM$(\widetilde{B}) = 14$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$
Proposed extension of Kaufmann and Gupta ranking approach	Not applicable	Not applicable	Not applicable	Not applicable
Farhadinia [55]	$C(\widetilde{A})$ =12	$C(\widetilde{B}) = 13$	$\widetilde{B}$	$\widetilde{A}$

It is obvious from Definition 5.5 that $\widetilde{A} \neq \widetilde{B}$ i.e., maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$ should be either $\widetilde{A}$ or $\widetilde{B}$. However, on the basis of results, shown in Table 5.3, obtained by using the existing ranking approaches [2, 11, 31, 33, 58, 79, 92, 113, 151, 160, 172] it can be concluded that maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$ $= \widetilde{A}$ and $\widetilde{B}$. Also, it is obvious from the results, shown in Table 5.3, that by using Farhadinia ranking approach a unique LR flat fuzzy number, representing maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$, is obtained. But, on the basis of this result it can not be concluded that it is not genuine to use Farhadinia ranking approach for finding the maximum and minimum of LR flat fuzzy numbers. To show the shortcomings of Farhadinia ranking approach [55] the maximum and minimum of

LR flat fuzzy numbers $\widetilde{A} = (10, 12, 3, 5)_{LR}$ and $\widetilde{B} = (10, 13, 3, 4)_{LR}$ with $L(x) = \max\{0, 1 - x^2\}$ and $R(x) = e^{-|x|}$ obtained by using Farhadinia ranking approach is shown in Table 5.4

Table 5.4 Maximum and minimum of LR flat fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained using Fardahinia ranking approach

Ranking approach	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A}, \widetilde{B}\}$	minimum$\{\widetilde{A}, \widetilde{B}\}$
Farhadinia ranking approach [55]	$C(\widetilde{A}) = 10$ $L(\widetilde{A}) = 7$ $W(\widetilde{A}) = 10$ $S(\widetilde{A}) = 9$	$C(\widetilde{B}) = 10$ $L(\widetilde{B}) = 7$ $W(\widetilde{B}) = 10$ $S(\widetilde{B}) = 9$	$\widetilde{A}$ and $\widetilde{B}$	$\widetilde{A}$ and $\widetilde{B}$

On the basis of results, shown in Table 5.3 and Table 5.4, obtained by using the existing ranking approaches [2, 11, 31, 33, 55, 58, 79, 92, 113, 151, 160, 172] it can be concluded that maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and $\widetilde{B}$ i.e., if modified Liang and Han based on proposed extension of Kaufmann and Gupta ranking approach and proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach will be modified on the basis of any of the existing ranking approaches [2, 11, 31, 33, 55, 58, 79, 92, 113, 151, 160, 172] then more than one LR flat fuzzy numbers, representing the optimal fuzzy project completion time, may be obtained. Hence, it is not genuine to modify the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach and proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach on the basis of the existing ranking approaches [2, 11, 31, 33, 55, 58, 79, 92, 113, 151, 160, 172].

Remark 5.5: In Section 5.4 the shortcomings of some ranking approaches are pointed out. The same shortcomings is also occurring in the remaining existing ranking approaches which are not discussed in this chapter.

5.5 Modified Farhadinia ranking approach

In this section, to overcome the shortcomings of existing ranking approaches [2, 11, 31, 33, 55, 58, 79, 92, 113, 151, 160, 172], a new ranking approach, named as modified Farhadinia ranking approach, by modifying the parameters of Farhadinia ranking approach [55], is proposed for finding the maximum and minimum of LR flat fuzzy numbers.

Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers. Then, use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find $C^M(\widetilde{A}) = \inf\{x \in \text{Supp}(\widetilde{A}) : \mu_{\widetilde{A}}(x) = 1\} = \underline{a_1}$ and

$$C^M(\widetilde{B}) = \inf\{x \in \text{Supp}(\widetilde{B}) : \mu_{\widetilde{B}}(x) = 1\} = \underline{a_2}$$

Case (i) If $C^M(\widetilde{A}) > C^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $C^M(\widetilde{A}) < C^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $C^M(\widetilde{A}) = C^M(\widetilde{B})$ then go to Step 2.

Step 2 Find $L^M(\widetilde{A}) = \underline{a_1} - a_1^L \int_0^1 L^{-1}(\alpha)d\alpha$ and

$$L^M(\widetilde{B}) = \underline{a_2} - a_2^L \int_0^1 L^{-1}(\alpha)d\alpha$$

Case (i) If $L^M(\widetilde{A}) > L^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $L^M(\widetilde{A}) < L^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $L^M(\widetilde{A}) = L^M(\widetilde{B})$ then go to Step 3.

Step 3 Find $W^M(\widetilde{A}) = \overline{a_1} - \underline{a_1} + 2a_1^L \int_0^1 L^{-1}(\alpha)d\alpha + 2a_1^R \int_0^1 R^{-1}(\alpha)d\alpha$

and $W^M(\widetilde{B}) = \overline{a_2} - \underline{a_2} + 2a_2^L \int_0^1 L^{-1}(\alpha)d\alpha + 2a_2^R \int_0^1 R^{-1}(\alpha)d\alpha$

Case (i) If $W^M(\widetilde{A}) > W^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $W^M(\widetilde{A}) < W^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $W^M(\widetilde{A}) = W^M(\widetilde{B})$ then go to Step 4.

Step 4 Find $S^M(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = \overline{a_1} - \underline{a_1} + a_1^L \int_0^1 L^{-1}(\alpha)d\alpha + a_1^R \int_0^1 R^{-1}(\alpha)d\alpha$

and $S^M(\widetilde{B}) = \int \mu_{\widetilde{B}}(x)dx = \overline{a_2} - \underline{a_2} + a_2^L \int_0^1 L^{-1}(\alpha)d\alpha + a_2^R \int_0^1 R^{-1}(\alpha)d\alpha$

Case (i) If $S^M(\widetilde{A}) > S^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $S^M(\widetilde{A}) < S^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $S^M(\widetilde{A}) = S^M(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$.

5.5.1 Validity of the modified Farhadinia ranking approach

It is obvious that if $\widetilde{A}$ and $\widetilde{B}$ are two LR flat fuzzy numbers such that $C^M(\widetilde{A}) > C^M(\widetilde{B})$ or $C^M(\widetilde{A}) = C^M(\widetilde{B})$, $L^M(\widetilde{A}) > L^M(\widetilde{B})$ or $C^M(\widetilde{A}) = C^M(\widetilde{B})$, $L^M(\widetilde{A}) = L^M(\widetilde{B})$, $W^M(\widetilde{A}) > W^M(\widetilde{B})$ or $C^M(\widetilde{A}) = C^M(\widetilde{B})$, $L^M(\widetilde{A}) = L^M(\widetilde{B})$, $W^M(\widetilde{A}) = W^M(\widetilde{B})$, $S^M(\widetilde{A}) > S^M(\widetilde{B})$ then using the modified Farhadinia ranking approach, presented in Section 5.5, maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$.

In this section, it is proved that if $\widetilde{A}$ and $\widetilde{B}$ are two such LR flat fuzzy numbers for which all the four conditions of modified Farhadinia ranking approach are satisfied then $\widetilde{A} = \widetilde{B}$.

Proposition 5.1 Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers such that

(i) $C^M(\widetilde{A}) = C^M(\widetilde{B})$ (ii) $L^M(\widetilde{A}) = L^M(\widetilde{B})$ (iii) $W^M(\widetilde{A}) = W^M(\widetilde{B})$

(iv) $S^M(\widetilde{A}) = S^M(\widetilde{B})$

Then, $\widetilde{A} = \widetilde{B}$.

Proof: (i) $C^M(\widetilde{A}) = C^M(\widetilde{B})$

$$\Rightarrow \underline{a_1} = \underline{a_2} \tag{5.1}$$

(ii) $L^M(\widetilde{A}) = L^M(\widetilde{B})$

$$\Rightarrow \underline{a_1} - a_1^L \int_0^1 L^{-1}(\alpha)d\alpha = \underline{a_2} - a_2^L \int_0^1 L^{-1}(\alpha)d\alpha \tag{5.2}$$

(iii) $W^M(\widetilde{A}) = W^M(\widetilde{B})$

$$\Rightarrow \overline{a_1} - \underline{a_1} + 2a_1^L \int_0^1 L^{-1}(\alpha)d\alpha + 2a_1^R \int_0^1 R^{-1}(\alpha)d\alpha = \overline{a_2} - \underline{a_2} + 2a_2^L \int_0^1 L^{-1}(\alpha)d\alpha + 2a_2^R \int_0^1 R^{-1}(\alpha)d\alpha \tag{5.3}$$

(iv) $S^M(\widetilde{A}) = S^M(\widetilde{B})$

$$\Rightarrow \overline{a_1} - \underline{a_1} + a_1^L \int_0^1 L^{-1}(\alpha)d\alpha + a_1^R \int_0^1 R^{-1}(\alpha)d\alpha = \overline{a_2} - \underline{a_2} + a_2^L \int_0^1 L^{-1}(\alpha)d\alpha + a_2^R \int_0^1 R^{-1}(\alpha)d\alpha \tag{5.4}$$

On solving (5.1), (5.2), (5.3) and (5.4)

$\underline{a_1} = \underline{a_2}$, $\overline{a_1} = \overline{a_2}$, $a_1^L = a_2^L$ and $a_1^R = a_2^R$

i.e., $\widetilde{A} = \widetilde{B}$

Remark 5.5: For an LR flat fuzzy number $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$

$S^M(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)\ dx = \int_{\underline{a}-a^L}^{\underline{a}} L(\frac{\underline{a}-x}{a^L})\ dx\ +\ \int_{\underline{a}}^{\overline{a}}\ dx + \int_{\overline{a}}^{\overline{a}+a^R} R(\frac{x-\overline{a}}{a^R})\ dx\ =\ \overline{a}\ -\ \underline{a}$

$+\ a^L \int_0^1 L^{-1}(\alpha)\ d\alpha + a^R \int_0^1 R^{-1}(\alpha)\ d\alpha$

5.5.2 Advantages of the modified Farhadinia ranking approach

The main advantages of the modified Farhadinia ranking approach over the approach, proposed in previous chapter, is that the approach, proposed in previous chapter, can be used for finding the maximum and minimum of triangular and trapezoidal fuzzy numbers but can not be used for finding the maximum and minimum of LR flat fuzzy numbers. While, the modified Farhadinia ranking approach can be used for comparing triangular, trapezoidal as well as LR flat fuzzy numbers.

To show the advantages of modified Farhadinia ranking approach the maximum and minimum of LR flat fuzzy numbers $\widetilde{A} = (10, 12, 3, 5)_{LR}$ and $\widetilde{B} = (10, 13, 3, 4)_{LR}$ with $L(x) = \max\{0, 1 - x^2\}$ and $R(x) = e^{-|x|}$, chosen in Section 5.4, obtained by using the modified Farhadinia ranking approach is shown in Table 5.5

Table 5.5 Maximum and minimum of LR flat fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained by using modified Farhadinia ranking approach

Ranking approach	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A}, \widetilde{B}\}$	minimum$\{\widetilde{A}, \widetilde{B}\}$
Modified Farhadinia ranking approach	$C^M(\widetilde{A}) = 10$ $L^M(\widetilde{A}) = 8$ $W^M(\widetilde{A}) = 16$	$C^M(\widetilde{B}) = 10$ $L^M(\widetilde{B}) = 8$ $W^M(\widetilde{B}) = 15$	$\widetilde{A}$	$\widetilde{B}$

5.6 Proposed Mehar's subtraction for LR flat fuzzy numbers

Since, in the modified Liang and Han method there is need to subtract fuzzy numbers and in the literature [43] it is pointed out that an LR flat fuzzy number $\widetilde{B}$ can be subtracted from an LR flat fuzzy number $\widetilde{A}$ if one of the following conditions is satisfied.

(i) $\widetilde{A}$ is an LR flat fuzzy number and $\widetilde{B}$ is a RL flat fuzzy number.

(ii) For both the fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, $L(x) = R(x)$.

So, Mehar's subtraction, proposed in Chapter 2, for LR flat fuzzy numbers can be modified in the following way:

(i) Let $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ be a non-negative LR flat fuzzy number and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{RL}$ be a non-negative RL flat fuzzy number. Then,

$\widetilde{A_1} \ominus_M \widetilde{A_2} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$

$$\text{where, } \begin{cases} \underline{a} = \text{maximum}\{0, (\underline{a_1} - a_1^L) - (\underline{a_2} - a_2^L)\} + \text{ maximum}\{0, (a_1^L - a_2^L)\} \\ \overline{a} = \underline{a} + \text{ maximum}\{0, (\overline{a_1} - \underline{a_1}) - (\overline{a_2} - \underline{a_2})\} \\ a^L = \text{maximum } \{0, (a_1^L - a_2^L)\} \\ a^R = \text{maximum } \{0, (a_1^R - a_2^R)\} \end{cases}$$

(ii) Let $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two non-negative LR flat fuzzy numbers such that $L(x) = R(x)$. Then,

$\widetilde{A_1} \ominus_M \widetilde{A_2} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$

$$\text{where, } \begin{cases} \underline{a} = \text{maximum}\{0, (\underline{a_1} - a_1^L) - (\underline{a_2} - a_2^L)\} + \text{ maximum}\{0, (a_1^L - a_2^L)\} \\ \overline{a} = \underline{a} + \text{ maximum}\{0, (\overline{a_1} - \underline{a_1}) - (\overline{a_2} - \underline{a_2})\} \\ a^L = \text{maximum } \{0, (a_1^L - a_2^L)\} \\ a^R = \text{maximum } \{0, (a_1^R - a_2^R)\} \end{cases}$$

Proposition 5.2 Let $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two non-negative LR flat fuzzy numbers such that $L(x) = R(x)$. Then,

$\widetilde{A_1} \ominus_M \widetilde{A_2} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$

$$\text{where, } \begin{cases} \underline{a} = \text{maximum}\{0, (\underline{a_1} - a_1^L) - (\underline{a_2} - a_2^L)\} + \text{ maximum}\{0, (a_1^L - a_2^L)\} \\ \overline{a} = \underline{a} + \text{ maximum}\{0, (\overline{a_1} - \underline{a_1}) - (\overline{a_2} - \underline{a_2})\} \\ a^L = \text{maximum } \{0, (a_1^L - a_2^L)\} \\ a^R = \text{maximum } \{0, (a_1^R - a_2^R)\} \end{cases}$$

is always a non-negative LR flat fuzzy number i.e.,

(i) $\underline{a} - a^L \geq 0$ (ii) $\overline{a_1} - \underline{a} \geq 0$ (iii) $a^L \geq 0$ (iv) $a^R \geq 0$.

Proof: Proof is same as in Chapter 2.

5.7 Proposed methods based on modified Farhadinia ranking approach

In this section, to overcome the limitations of the methods, pointed out in Section 5.2, new methods are proposed on the basis of modified Farhadinia ranking approach to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number.

5.7.1 Modified Liang and Han method based on modified Farhadinia ranking approach

If in the modified Liang and Han method instead of proposed extension of Kaufmann and Gupta ranking approach, modified Farhadinia ranking approach and instead of Mehar's subtraction for trapezoidal fuzzy numbers, proposed Mehar's subtraction for LR flat fuzzy numbers are used then the limitation of the modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach, pointed out in Section 5.2, will be resolved.

5.7.2 Proposed Mehar's method based on modified Farhadinia ranking approach

In this section, to overcome the limitations of proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach, a new method, named as Mehar's method based on modified Farhadinia ranking approach, is proposed.

The steps of the proposed Mehar's method based on modified Farhadinia ranking approach are as follows:

Step 1 Formulate the chosen fuzzy project network problem into the fuzzy linear programming problem ($P_{5.1}$):

$$\text{Maximize} \sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x_{ij})$$

subject to

$$\left.\begin{array}{l} \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ \qquad x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A. \end{array}\right\} \qquad (P_{5.1})$$

where, A: set of all activities (i, j),

$\widetilde{t}_{ij}$: the fuzzy time duration of the activity (i, j).

N: the set of nodes,

n: the destination node,

1: the source node,

x_{ij}: the decision variable denoting the amount of flow in the activity (i, j).

Step 2 Suppose the fuzzy linear programming problem ($P_{5.1}$) have h feasible solutions and $\{x_{ij}^w\}$ is the w^{th} feasible solution then the aim is to find the feasible solution with the largest objective value i.e., the goal is to find $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum\limits_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^w)\right\}$.

Since, in the modified Farhadinia ranking approach it is assumed that if $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum\limits_{(i,j)\in A}(C^M(\widetilde{t}_{ij})\ x_{ij}^w)\right\}$ is $\sum\limits_{(i,j)\in A}(C^M(\widetilde{t}_{ij})\ x_{ij}^\eta)$ then $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum\limits_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^w)\right\}$ will also be $\sum\limits_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^\eta)$, where, $C^M(\widetilde{t}_{ij}) = C^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR} = \underline{a_{ij}}$ represents the first modified Farhadinia ranking index of an LR flat fuzzy number $\widetilde{t}_{ij}$ $= (\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}$.

So, by using the modified Farhadinia ranking approach the optimal solution of the fuzzy linear programming problem $(P_{5.1})$ can be obtained by solving the crisp linear programming problem $(P_{5.2})$:

$$\left.\begin{array}{l}
\text{Maximize} \sum_{(i,j)\in A} (C^M(\widetilde{t}_{ij})\ x_{ij}) \\
\text{subject to} \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i=1 \\ -1 & , i=n \\ 0 & , i\in N-\{1,n\} \end{cases} \\
\qquad x_{ij} \geq 0 \ \forall\ (i,j) \in A.
\end{array}\right\} \qquad (P_{5.2})$$

Step 3 Solve the crisp linear programming problem $(P_{5.2})$ to find the optimal solution $\{x_{ij}\}$.

Step 4 Use the optimal solution $\{x_{ij}\}$, obtained in Step 3, to find the critical path and also put the obtained optimal values of x_{ij} in $\sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x_{ij})$ to find the optimal fuzzy completion time of the project.

Case (i): If a unique critical path and hence a unique fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained optimal fuzzy project completion time is the optimal fuzzy completion time of the project and the obtained critical path is the only critical path of the project.

Case (ii): If more than one critical paths are obtained then go to Step 5.

Step 5 Check that the fuzzy numbers, representing the optimal fuzzy project completion time, corresponding to all the critical paths are same or not.

Case (i): If a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, is obtained then all the critical paths, obtained in Step 4, corresponding to which the obtained fuzzy number is obtained, are critical paths of the project and the obtained LR flat fuzzy number will represent the optimal fuzzy completion time of the project.

Case (ii): If more than one LR flat fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 6.

Step 6 Let using the previous steps p different LR flat fuzzy numbers $\widetilde{T}_1$, $\widetilde{T}_2$, ..., $\widetilde{T}_p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}_1 \neq \widetilde{T}_2 \neq ... \neq \widetilde{T}_p$ but $C^M(\widetilde{T}_1) = C^M(\widetilde{T}_2) = ... = C^M(\widetilde{T}_l) = u_1$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\text{Maximize} \sum_{(i,j)\in A} (L^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij})$$

subject to

$$\left.\begin{aligned} &\sum_{(i,j)\in A} (C^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_1, \\ &\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ &x_{ij} \geq 0\ \forall\ (i,j) \in A. \end{aligned}\right\} \qquad (P_{5.3})$$

Case (i): If by putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A} ((\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}))$ a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained LR flat fuzzy number will represent the optimal fuzzy completion time of the project.

Case (ii): If more than one LR flat fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 7.

Step 7 Let using the previous steps l LR flat fuzzy numbers $\widetilde{T}'_1$, $\widetilde{T}'_2$, ..., $\widetilde{T}'_l$, where $l \leq p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}'_1 \neq \widetilde{T}'_2 \neq ... \neq \widetilde{T}'_l$ but $C^M(\widetilde{T}'_1) = C^M(\widetilde{T}'_2) = ... = C^M(\widetilde{T}'_l) = u_1$ and $L^M(\widetilde{T}'_1) = L^M(\widetilde{T}'_2)$

$= ... = L^M(\widetilde{T_l'}) = u_2$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$
\begin{aligned}
&\text{Maximize } \sum_{(i,j)\in A} (W^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij})\\
&\text{subject to}\\
&\left.\begin{array}{l}
\sum_{(i,j)\in A} (C^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_1,\\
\sum_{(i,j)\in A} (L^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_2,\\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases}\\
x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A.
\end{array}\right\} \quad (P_{5.4})
\end{aligned}
$$

Case (i): If by putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A}((\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}))$ a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained LR flat fuzzy number will represent the optimal fuzzy completion time of the project.

Case (ii): If more than one LR flat fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 8.

Step 8 Let using the previous steps k LR flat fuzzy numbers $\widetilde{T_1''}$, $\widetilde{T_2''}$, ..., $\widetilde{T_k''}$, where $k \leq l$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T_1''} \neq \widetilde{T_2''} \neq ... \neq \widetilde{T_k''}$ but $C^M(\widetilde{T_1''}) = C^M(\widetilde{T_2''}) = ... = C^M(\widetilde{T_k''}) = u_1$, $L^M(\widetilde{T_1''}) = L^M(\widetilde{T_2''}) = ... = L^M(\widetilde{T_k''}) = u_2$ and $W^M(\widetilde{T_1''}) = W^M(\widetilde{T_2''}) = ... = W^M(\widetilde{T_k''}) = u_3$ (say). Then, find the optimal solution of the following crisp linear programming problem:

Maximize $\sum_{(i,j)\in A} (S^M(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij})$

subject to

$$\left.\begin{array}{l}
\sum\limits_{(i,j)\in A}(C^M(\underline{a_{ij}},\overline{a_{ij}},a^L_{ij},a^R_{ij})_{LR}\ x_{ij})=u_1,\\
\sum\limits_{(i,j)\in A}(L^M(\underline{a_{ij}},\overline{a_{ij}},a^L_{ij},a^R_{ij})_{LR}\ x_{ij})=u_2,\\
\sum\limits_{(i,j)\in A}(W^M(\underline{a_{ij}},\overline{a_{ij}},a^L_{ij},a^R_{ij})_{LR}\ x_{ij})=u_3,\\
\sum\limits_{j:(i,j)\in A}x_{ij}-\sum\limits_{j:(j,i)\in A}x_{ji}=\begin{cases}1 & ,\ i=1\\ -1 & ,\ i=n\\ 0 & ,\ i\in N-\{1,n\}\end{cases}\\
x_{ij}\ \geq\ 0\ \forall\ (i,j)\ \in A.
\end{array}\right\}\qquad (P_{5.5})$$

Now, putting the obtained optimal values of x_{ij} in $(\sum\limits_{(i,j)\in A}((\underline{a_{ij}},\overline{a_{ij}},a^L_{ij},a^R_{ij})\ x_{ij}))$ a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, will be obtained and the critical paths can be obtained by using the obtained optimal values of x_{ij}.

5.8 Advantages of proposed Mehar's method based on modified Farhadinia ranking approach over modified Liang and Han method based on modified Farhadinia ranking approach

Since, in the modified Liang and Han method based on modified Farhadinia ranking approach proposed Mehar's subtraction for LR flat fuzzy numbers is used so due to its existence conditions, discussed in Section 5.6, the modified Liang and Han method based on modified Farhadinia ranking approach can be used only to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by such an LR flat fuzzy number for which the condition $L(x) = R(x)$ is satisfied. But, the same method can not be used for finding the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by such an LR flat fuzzy number for which the condition $L(x) = R(x)$ is not satisfied.

While, the proposed Mehar's method based on modified Farhadinia ranking approach can be used to find the unique optimal fuzzy completion time of all such project network problems in which time of each activity is either represented by such an LR flat fuzzy number for which the condition $L(x) = R(x)$ is satisfied or such an LR flat fuzzy number for which the condition $L(x) \neq R(x)$ is satisfied e.g., the unique optimal fuzzy completion time of the project network problem, chosen in Example 5.1, can be obtained by using the modified Liang and Han method based on modified Farhadinia ranking approach. But, the unique optimal fuzzy completion time of the project network problem, chosen in Example 5.2, can not be obtained by using the same method. However, the unique optimal fuzzy completion time of both the project network problems, chosen in Example 5.1 and Example 5.2, can be obtained by using the proposed Mehar's method based on modified Farhadinia ranking approach.

To illustrate the proposed Mehar's method based on modified Farhadinia ranking approach and modified Liang and Han method based on modified Farhadinia ranking approach the project network problems, chosen in Example 5.1 and Example 5.2, are solved by both the methods.

5.9 Optimal fuzzy completion time of the first chosen problem

In this section, to illustrate the proposed Mehar's method based on modified Farhadinia ranking approach and modified Liang and Han method based on modified Farhadinia ranking approach the project network problem, chosen in Example 5.1, is solved by both the methods.

5.9.1 Optimal fuzzy completion time of the first chosen problem by using the proposed Mehar's method based on modified Farhadinia ranking approach

Using the proposed Mehar's method based on modified Farhadinia ranking approach the optimal fuzzy completion time of the project, chosen in Example 5.1, can be obtained as follows:

Step 1 Using Step 1 of the proposed Mehar's method based on modified Farhadinia ranking approach the problem, chosen in Example 5.1, can be formulated as follows:

Maximize $((5,6,2,1)_{LR}\, x_{12} \oplus (7,8,1,1)_{LR}\, x_{23} \oplus (13,14,2,5)_{LR}\, x_{24} \oplus (6,9,1,1)_{LR}\, x_{34})$

subject to

$x_{12} = 1, \qquad x_{12} - x_{24} - x_{23} = 0, \qquad x_{23} - x_{34} = 0, \qquad x_{24} + x_{34} = 1,$

$x_{12},\, x_{23},\, x_{24},\, x_{34} \geq 0$

Step 2 Using Step 2 of the proposed Mehar's method based on modified Farhadinia ranking approach, the fuzzy linear programming problem, obtained in Step 1, can be converted into the following crisp linear programming problem:

Maximize $(C^M(5,6,2,1)_{LR}\ x_{12} + C^M(7,8,1,1)_{LR}\ x_{23} + C^M(13,14,2,5)_{LR}\ x_{24} +$
$C^M(6,9,1,1)_{LR}\ x_{34})$

subject to

$x_{12} = 1, \qquad x_{12} - x_{24} - x_{23} = 0, \qquad x_{23} - x_{34} = 0, \qquad x_{24} + x_{34} = 1,$

$x_{12},\, x_{23},\, x_{24},\, x_{34} \geq 0$

i.e., Maximize $(5\ x_{12} + 7\ x_{23} + 13\ x_{24} + 6\ x_{34})$

subject to

$x_{12} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$

$x_{12}, x_{23}, x_{24}, x_{34} \geq 0$

Step 3 On solving crisp linear programming problem, obtained in Step 2, the following two optimal solutions are obtained:

(i) $x_{12} = x_{24} = 1$ and $x_{23} = x_{34} = 0$

(ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{24} = 0$

Step 4 Using the optimal values of x_{ij}, obtained from Step 3, the following two critical paths are obtained :

(i) $1 \Rightarrow 2 \Rightarrow 4$

(ii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$

Putting the optimal values of x_{ij}, obtained from Step 3, in $((5, 6, 2, 1)_{LR}\, x_{12} \oplus (7, 8, 1, 1)_{LR}\, x_{23} \oplus (13, 14, 2, 5)_{LR}\, x_{24} \oplus (6, 9, 1, 1)_{LR}\, x_{34})$, the optimal fuzzy project completion times corresponding to the paths $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are $(18, 20, 4, 6)_{LR}$ and $(18, 23, 4, 3)_{LR}$ respectively. Since, more than one critical paths are obtained i.e., Case (ii) of Step 4 of the proposed Mehar's method based on modified Farhadinia ranking approach is satisfied, so go to Step 5 of the proposed Mehar's method based on modified Farhadinia ranking approach.

Step 5 It is obvious from results, the optimal fuzzy completion time of the project corresponding to the critical paths $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are $(18, 20, 4, 6)_{LR}$ and $(18, 23, 4, 3)_{LR}$ respectively. Since, the LR flat fuzzy numbers $(18, 20, 4, 6)_{LR}$ and $(18, 23, 4, 3)_{LR}$, representing the optimal fuzzy completion time of the project, are different i.e., Case (ii) of Step 5 of the proposed Mehar's method based on modified Farhadinia ranking approach is satisfied, so go to Step 6 of the proposed Mehar's method based on modified Farhadinia ranking approach.

Step 6 Since, $u_1 = C^M(18,20,4,6)_{LR} = C^M(18,23,4,3)_{LR} = 18$. So, using Step 6 of the proposed Mehar's method based on modified Farhadinia ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(L^M(5,6,2,1)_{LR}\ x_{12} + L^M(7,8,1,1)_{LR}\ x_{23} + L^M(13,14,2,5)_{LR}\ x_{24}$

$+ L^M(6,9,1,1)_{LR}\ x_{34})$

subject to

$C^M(5,6,2,1)_{LR}\, x_{12} + C^M(7,8,1,1)_{LR}\, x_{23} + C^M(13,14,2,5)_{LR}\, x_{24} + C^M(6,9,1,1)_{LR}$

$x_{34} = 18$,

$x_{12} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1,$

$x_{12},\ x_{23},\ x_{24},\ x_{34} \geq 0$

On solving the crisp linear programming problem the following two optimal solutions are obtained:

(i) $x_{12} = x_{24} = 1$ and $x_{23} = x_{34} = 0$

(ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{24} = 0$

Putting the obtained optimal values of x_{ij} in $((5,6,2,1)_{LR}\ x_{12} \oplus (7,8,1,1)_{LR}\ x_{23} \oplus (13,14,2,5)_{LR}\ x_{24} \oplus (6,9,1,1)_{LR}\ x_{34})$, the obtained optimal fuzzy project completion times are $(18,20,4,6)_{LR}$ and $(18,23,4,3)_{LR}$ respectively. Since $(18,20,4,6)_{LR} \neq (18,23,4,3)_{LR}$ i.e., Case (ii) of Step 6 of the proposed Mehar's method based on modified Farhadinia ranking approach is satisfied, so go to Step 7 of the proposed Mehar's method based on modified Farhadinia ranking approach.

Step 7 Since, $u_2 = L^M(18,20,4,6)_{LR} = L^M(18,23,4,3)_{LR} = 14$. So, using Step 7 of the proposed Mehar's method based on modified Farhadinia ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(W^M(5,6,2,1)_{LR}\ x_{12} + W^M(7,8,1,1)_{LR}\ x_{23} + W^M(13,14,2,5)_{LR}\ x_{24} + W^M(6,9,1,1)_{LR}\ x_{34})$

subject to

$L^M(5,6,2,1)_{LR}\, x_{12} + L^M(7,8,1,1)_{LR}\, x_{23} + L^M(13,14,2,5)_{LR}\, x_{24} + L^M(6,9,1,1)_{LR}\, x_{34} = 14,$

$C^M(5,6,2,1)_{LR}\, x_{12} + C^M(7,8,1,1)_{LR}\, x_{23} + C^M(13,14,2,5)_{LR}\, x_{24} + C^M(6,9,1,1)_{LR}\, x_{34} = 18,$

$x_{12} = 1, \qquad x_{12} - x_{24} - x_{23} = 0, \qquad x_{23} - x_{34} = 0, \qquad x_{24} + x_{34} = 1,$

$x_{12},\ x_{23},\ x_{24},\ x_{34} \geq 0$

On solving the crisp linear programming problem the following optimal solution is obtained:

$x_{12} = x_{24} = 1$ and $x_{23} = x_{34} = 0$.

Putting the obtained optimal values of x_{ij} in $((5,6,2,1)_{LR}\, x_{12} \oplus (7,8,1,1)_{LR}\, x_{23} \oplus (13,14,2,5)_{LR}\, x_{24} \oplus (6,9,1,1)_{LR}\, x_{34})$ a unique LR flat fuzzy number $(18,20,4,6)_{LR}$, representing the optimal fuzzy project completion time, is obtained and using the same values of x_{ij} the obtained critical path is $1 \Rightarrow 2 \Rightarrow 4$.

5.9.2 Optimal fuzzy completion time of the first chosen problem by using the modified Liang and Han method based on modified Farhadinia ranking approach

Using the modified Liang and Han method based on modified Farhadinia ranking approach the optimal fuzzy completion time of the project, chosen in Example 5.1, can be obtained as follows:

Step 1 Using Step 1 of the modified Liang and Han method based on modified Farhadinia ranking approach by assuming $\widetilde{FES_1} = (0,0,0,0)_{LR}$ the values of $\widetilde{FES_j}$, $j = 2,3,4$ can be obtained as follows:

$$\widetilde{FES_2} = \widetilde{FES_1} \oplus \widetilde{FNT_{12}} = (0,0,0,0)_{LR} \oplus (5,6,2,1)_{LR} = (5,6,2,1)_{LR}$$

$$\begin{aligned}\widetilde{FES_3} &= \widetilde{FES_2} \oplus \widetilde{FNT_{23}} \\ &= (5,6,2,1)_{LR} \oplus (7,8,1,1)_{LR} \\ &= (12,14,3,2)_{LR}\end{aligned}$$

$$\begin{aligned}\widetilde{FES_4} &= \text{maximum } \{\widetilde{FES_2} \oplus \widetilde{FNT_{24}},\ \widetilde{FES_3} \oplus\ \widetilde{FNT_{34}}\} \\ &= \text{maximum}\{(5,6,2,1)_{LR} \oplus (13,14,2,5)_{LR}, (12,14,3,2)_{LR} \oplus (6,9,1,1)_{LR}\} \\ &= \text{maximum}\{(18,20,4,6)_{LR}, (18,23,4,3)_{LR}\} \\ &= (18,20,4,6)_{LR} \qquad \left(\because W^M(18,20,4,6)_{LR} > W^M(18,23,4,3)_{LR}\right)\end{aligned}$$

Step 2 Using Step 2 of the modified Liang and Han method based on modified Farhadinia ranking approach by assuming $\widetilde{FLF_4} = (18,20,4,6)_{LR}$ the values of $\widetilde{FLF_j}$, $j = 3,2,1$ can be obtained as follows:

$$\widetilde{FLF_3} = \widetilde{FLF_4} \ominus_M \widetilde{FNT_{34}} = (18,20,4,6)_{LR} \ominus_M (6,9,1,1)_{LR} = (12,12,3,5)_{LR}$$

$$\begin{aligned}\widetilde{FLF_2} &= \text{minimum}\{\widetilde{FLF_4} \ominus_M \widetilde{FNT_{24}},\ \widetilde{FLF_3} \ominus_M \widetilde{FNT_{23}}\} \\ &= \text{minimum}\{(18,20,4,6)_{LR} \ominus_M (13,14,2,5)_{LR}, (12,12,3,5)_{LR} \ominus_M (7,8,1,1)_{LR}\} \\ &= \text{minimum}\{(5,6,2,1)_{LR}, (5,5,2,4)_{LR}\} \\ &= (5,6,2,1)_{LR} \qquad \left(\because W^M(5,6,2,1)_{LR} < W^M(5,5,2,4)_{LR}\right)\end{aligned}$$

$$\begin{aligned}\widetilde{FLF_1} &= \widetilde{FLF_2} \ominus_M \widetilde{FNT_{12}} \\ &= (5,6,2,1)_{LR} \ominus_M (5,6,2,1)_{LR} = (0,0,0,0)_{LR}\end{aligned}$$

Step 3 Using Step 3 of the modified Liang and Han method based on modified Farhadinia ranking approach, the values of $\widetilde{FTS_{ij}}$ can be obtained as follows:

$$\begin{aligned}
\widetilde{FTS_{12}} &= \widetilde{FLF_2} \ominus_M (\widetilde{FES_1} \oplus \widetilde{FNT_{12}}) \\
&= (5,6,2,1)_{LR} \ominus_M ((0,0,0,0)_{LR} \oplus (5,6,2,1)_{LR}) \\
&= (0,0,0,0)_{LR} \\
\widetilde{FTS_{23}} &= \widetilde{FLF_3} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{23}}) \\
&= (12,12,3,5)_{LR} \ominus_M ((5,6,2,1)_{LR} \oplus (7,8,1,1)_{LR}) \\
&= (12,12,3,5)_{LR} \ominus_M (12,14,3,2)_{LR} \\
&= (0,0,0,3)_{LR} \\
\widetilde{FTS_{24}} &= \widetilde{FLF_4} \ominus_M (\widetilde{FES_2} \oplus \widetilde{FNT_{24}}) \\
&= (18,20,4,6)_{LR} \ominus_M ((5,6,2,1)_{LR} \oplus (13,14,2,5)_{LR}) \\
&= (0,0,0,0)_{LR} \\
\widetilde{FTS_{34}} &= \widetilde{FLF_4} \ominus_M (\widetilde{FES_3} \oplus \widetilde{FNT_{34}}) \\
&= (18,20,4,6)_{LR} \ominus_M ((12,14,3,2)_{LR} \oplus (6,9,1,1)_{LR}) \\
&= (0,0,0,3)_{LR}
\end{aligned}$$

Step 4 Using Step 4 of the modified Liang and Han method based on modified Farhadinia ranking approach,

$$\begin{aligned}
&\text{minimum}\{\widetilde{FTS_{12}} \oplus \widetilde{FTS_{24}},\ \widetilde{FTS_{12}} \oplus \widetilde{FTS_{23}} \oplus \widetilde{FTS_{34}}\} \\
&= \text{minimum}\{(0,0,0,0)_{LR} \oplus (0,0,0,0)_{LR},\ (0,0,0,0)_{LR} \oplus (0,0,0,3)_{LR} \oplus (0,0,0,3)_{LR}\} \\
&= \text{minimum}\{(0,0,0,0)_{LR}, (0,0,0,6)_{LR}\} \\
&= (0,0,0,0)_{LR} \qquad \left(\because W^M(0,0,0,0)_{LR} < W^M(0,0,0,6)_{LR}\right)
\end{aligned}$$

Since, the minimum is occurring corresponding to the path $1 \Rightarrow 2 \Rightarrow 4$ so it is the critical path for the chosen project network.

Step 5 Using Step 5 of the modified Liang and Han method based on modified

Farhadinia ranking approach, the optimal fuzzy completion time of the project is

$$\widetilde{FCT} = \widetilde{FNT}_{12} \oplus \widetilde{FNT}_{24}$$

$$= (5, 6, 2, 1)_{LR} \oplus (13, 14, 2, 5)_{LR} = (18, 20, 4, 6)_{LR}$$

5.10 Optimal fuzzy completion time of the second chosen problem

In this section, to show the advantages of proposed Mehar's method based on modified Farhadinia ranking approach the project network problem, chosen in Example 5.2, is solved by both the methods.

5.10.1 Optimal fuzzy completion time of the second chosen problem by using the proposed Mehar's method based on modified Farhadinia ranking approach

Using the proposed Mehar's method based on modified Farhadinia ranking approach the optimal fuzzy completion time of the project, chosen in Example 5.2, can be obtained as follows:

Step 1 Using Step 1 of the proposed Mehar's method based on modified Farhadinia ranking approach the problem, chosen in Example 5.2, can be formulated as follows:

Maximize $((2, 2.3, 0.6, 0.8)_{LR}\, x_{12} \oplus (5, 6, 1, 1.2)_{LR}\, x_{23} \oplus (7, 7.8, 1, 1.1)_{LR}\, x_{34} \oplus (4, 4.5, 0.7, 2.7)_{LR}\, x_{35} \oplus (5, 5.5, 1, 2.2)_{LR}\, x_{56} \oplus (5, 9.6, 1.1, 1.3)_{LR}\, x_{47} \oplus (2, 4.2, 0.7, 1.8)_{LR}\, x_{46} \oplus (3, 3.4, 0.4, 1.5)_{LR}\, x_{67} \oplus (6, 6.3, 0.3, 0.6)_{LR}\, x_{78})$

subject to

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{35} - x_{34} = 0, \quad x_{34} - x_{47} - x_{46} = 0,$

$x_{35} - x_{56} = 0, \quad x_{46} + x_{56} - x_{67} = 0, \quad x_{47} + x_{67} - x_{78} = 0, \quad x_{78} = 1,$

$x_{12},\, x_{23},\, x_{34},\, x_{35},\, x_{46},\, x_{47},\, x_{56},\, x_{67},\, x_{78} \geq 0$

Step 2 Using Step 2 of the proposed Mehar's method based on modified Farhadinia ranking approach the fuzzy linear programming problem, obtained in Step 1, can be converted into the following crisp linear programming problem:

Maximize $(C^M(2, 2.3, 0.6, 0.8)_{LR}\, x_{12} + C^M(5, 6, 1, 1.2)_{LR}\, x_{23} + C^M(7, 7.8, 1, 1.1)_{LR}\, x_{34}$ $+\, C^M(4, 4.5, 0.7, 2.7)_{LR}\, x_{35} + C^M(5, 5.5, 1, 2.2)_{LR}\, x_{56} + C^M(5, 9.6, 1.1, 1.3)_{LR}\, x_{47} +$ $C^M(2, 4.2, 0.7, 1.8)_{LR}\, x_{46} + C^M(3, 3.4, 0.4, 1.5)_{LR}\, x_{67} + C^M(6, 6.3, 0.3, 0.6)_{LR}\, x_{78})$

subject to

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{35} - x_{34} = 0, \quad x_{34} - x_{47} - x_{46} = 0,$

$x_{35} - x_{56} = 0, \quad x_{46} + x_{56} - x_{67} = 0, \quad x_{47} + x_{67} - x_{78} = 0, \quad x_{78} = 1,$

$x_{12}, x_{23}, x_{34}, x_{35}, x_{46}, x_{47}, x_{56}, x_{67}, x_{78} \geq 0$

i.e., Maximize $(2\, x_{12} + 5\, x_{23} + 7\, x_{34} + 4\, x_{35} + 5\, x_{56} + 5\, x_{47} + 2\, x_{46} + 3\, x_{67} + 6\, x_{78})$

subject to

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{35} - x_{34} = 0, \quad x_{34} - x_{47} - x_{46} = 0,$

$x_{35} - x_{56} = 0, \quad x_{46} + x_{56} - x_{67} = 0, \quad x_{47} + x_{67} - x_{78} = 0, \quad x_{78} = 1,$

$x_{12}, x_{23}, x_{34}, x_{35}, x_{46}, x_{47}, x_{56}, x_{67}, x_{78} \geq 0$

Step 3 On solving crisp linear programming problem, obtained in Step 2, the following three optimal solutions are obtained:

(i) $x_{12} = x_{23} = x_{34} = x_{47} = x_{78} = 1$ and $x_{35} = x_{56} = x_{46} = x_{67} = 0$

(ii) $x_{12} = x_{23} = x_{35} = x_{56} = x_{67} = x_{78} = 1$ and $x_{34} = x_{46} = x_{47} = 0$

(iii) $x_{12} = x_{23} = x_{34} = x_{46} = x_{67} = x_{78} = 1$ and $x_{35} = x_{56} = x_{47} = 0$

Step 4 Using the optimal values of x_{ij}, obtained from Step 3, the following three critical paths are obtained :

(i) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 7 \Rightarrow 8$

(ii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$

(iii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$

Putting the optimal values of x_{ij}, obtained from Step 3, in $((2, 2.3, 0.6, 0.8)_{LR}$ $x_{12} \oplus (5, 6, 1, 1.2)_{LR}\, x_{23} \oplus (7, 7.8, 1, 1.1)_{LR}\, x_{34} \oplus (4, 4.5, 0.7, 2.7)_{LR}\, x_{35} \oplus (5, 5.5, 1,$ $2.2)_{LR}\, x_{56} \oplus (5, 9.6, 1.1, 1.3)_{LR}\, x_{47} \oplus (2, 4.2, 0.7, 1.8)_{LR}\, x_{46} \oplus (3, 3.4, 0.4, 1.5)_{LR}\, x_{67}$ $\oplus\, (6, 6.3, 0.3,$

$0.6)_{LR}\, x_{78})$, the optimal fuzzy project completion times corresponding to paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 7 \Rightarrow 8$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$ and $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$ are $(25, 32, 4, 5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(25, 30, 4, 7)_{LR}$ respectively.

Since, more than one critical paths are obtained i.e., Case (ii) of Step 4 of the proposed Mehar's method based on modified Farhadinia ranking approach is satisfied, so go to Step 5 of the proposed Mehar's method based on modified Farhadinia ranking approach.

Step 5 Since, the LR flat fuzzy numbers $(25, 32, 4, 5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(25, 30, 4, 7)_{LR}$, representing the optimal fuzzy completion time of the project, are different i.e., Case (ii) of Step 5 of the proposed Mehar's method based on modified Farhadinia ranking approach is satisfied, so go to Step 6 of the proposed Mehar's method based on modified Farhadinia ranking approach.

Step 6 Since, $u_1 = C^M(25, 32, 4, 5)_{LR} = C^M(25, 28, 4, 9)_{LR} = C^M(25, 30, 4, 7)_{LR} = 25$. So, using Step 6 of the proposed Mehar's method based on modified Farhadinia ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(L^M(2, 2.3, 0.6, 0.8)_{LR}\, x_{12} + L^M(5, 6, 1, 1.2)_{LR}\, x_{23} + L^M(7, 7.8, 1, 1.1)_{LR}\, x_{34}$ $+\, L^M(4, 4.5, 0.7, 2.7)_{LR}\, x_{35} + L^M(5, 5.5, 1, 2.2)_{LR}\, x_{56} + L^M(5, 9.6, 1.1, 1.3)_{LR}\, x_{47} +$ $L^M(2, 4.2, 0.7, 1.8)_{LR}\, x_{46} + L^M(3, 3.4, 0.4, 1.5)_{LR}\, x_{67} + L^M(6, 6.3, 0.3, 0.6)_{LR}\, x_{78})$

subject to

$C^M(2, 2.3, 0.6, 0.8)_{LR}\, x_{12} + C^M(5, 6, 1, 1.2)_{LR}\, x_{23} + C^M(7, 7.8, 1, 1.1)_{LR}\, x_{34} + C^M(4, 4.5,$
$0.7, 2.7)_{LR}\, x_{35} + C^M(5, 5.5, 1, 2.2)_{LR}\, x_{56} + C^M(5, 9.6, 1.1, 1.3)_{LR}\, x_{47} + C^M(2, 4.2, 0.7,$
$1.8)_{LR}\, x_{46} + C^M(3, 3.4, 0.4, 1.5)_{LR}\, x_{67} + C^M(6, 6.3, 0.3, 0.6)_{LR}\, x_{78} = 25,$

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{35} - x_{34} = 0, \quad x_{34} - x_{47} - x_{46} = 0,$

$x_{35} - x_{56} = 0, \quad x_{46} + x_{56} - x_{67} = 0, \quad x_{47} + x_{67} - x_{78} = 0, \quad x_{78} = 1,$

$x_{12}, x_{23}, x_{34}, x_{35}, x_{46}, x_{47}, x_{56}, x_{67}, x_{78} \geq 0$

On solving the crisp linear programming problem the following three optimal solutions are obtained:

(i) $x_{12} = x_{23} = x_{34} = x_{47} = x_{78} = 1$ and $x_{35} = x_{56} = x_{46} = x_{67} = 0$

(ii) $x_{12} = x_{23} = x_{35} = x_{56} = x_{67} = x_{78} = 1$ and $x_{34} = x_{46} = x_{47} = 0$

(iii) $x_{12} = x_{23} = x_{34} = x_{46} = x_{67} = x_{78} = 1$ and $x_{35} = x_{56} = x_{47} = 0$

Putting the obtained optimal values of x_{ij} in $((2, 2.3, 0.6, 0.8)_{LR}\, x_{12} \oplus (5, 6, 1, 1.2)_{LR}\, x_{23} \oplus (7, 7.8, 1, 1.1)_{LR}\, x_{34} \oplus (4, 4.5, 0.7, 2.7)_{LR}\, x_{35} \oplus (5, 5.5, 1, 2.2)_{LR}\, x_{56} \oplus (5, 9.6, 1.1, 1.3)_{LR}\, x_{47} \oplus (2, 4.2, 0.7, 1.8)_{LR}\, x_{46} \oplus (3, 3.4, 0.4, 1.5)_{LR}\, x_{67} \oplus (6, 6.3, 0.3, 0.6)_{LR}\, x_{78})$, the obtained optimal fuzzy project completion times are $(25, 32, 4, 5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(25, 30, 4, 7)_{LR}$. Since, $(25, 32, 4, 5)_{LR} \neq (25, 28, 4, 9)_{LR} \neq (25, 30, 4, 7)_{LR}$ i.e., Case (ii) of Step 6 of the proposed Mehar's method based on modified Farhadinia ranking approach is satisfied, so go to Step 7 of the proposed Mehar's method based on modified Farhadinia ranking approach.

Step 7 Since, $u_2 = L^M(25, 32, 4, 5)_{LR} = L^M(25, 28, 4, 9)_{LR} = L^M(25, 30, 4, 7)_{LR} = \frac{67}{3}$. So, using Step 7 of the proposed Mehar's method based on modified Farhadinia ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(W^M(2,2.3,0.6,0.8)_{LR}\, x_{12} + W^M(5,6,1,1.2)_{LR}\, x_{23} + W^M(7,7.8,1,1.1)_{LR}$ $x_{34} + W^M(4,4.5,0.7,2.7)_{LR}\, x_{35} + W^M(5,5.5,1,2.2)_{LR}\, x_{56} + W^M(5,9.6,1.1,1.3)_{LR}\, x_{47}$ $+ W^M(2,4.2,0.7,1.8)_{LR}\, x_{46} + W^M(3,3.4,0.4,1.5)_{LR}\, x_{67} + W^M(6,6.3,0.3,0.6)_{LR}\, x_{78})$

subject to

$C^M(2,2.3,0.6,0.8)_{LR}\, x_{12} + C^M(5,6,1,1.2)_{LR}\, x_{23} + C^M(7,7.8,1,1.1)_{LR}\, x_{34} + C^M(4,4.5,$ $0.7,2.7)_{LR}\ x_{35} + C^M(5,5.5,1,2.2)_{LR}\, x_{56} + C^M(5,9.6,1.1,1.3)_{LR}\, x_{47} + C^M(2,4.2,0.7,$ $1.8)_{LR}\ x_{46} + C^M(3,3.4,0.4,1.5)_{LR}\ x_{67} + C^M(6,6.3,0.3,0.6)_{LR}\ x_{78} = 25,$

$L^M(2,2.3,0.6,0.8)_{LR}\, x_{12} + L^M(5,6,1,1.2)_{LR}\, x_{23} + L^M(7,7.8,1,1.1)_{LR}\, x_{34} + L^M(4,4.5,$ $0.7,2.7)_{LR}\, x_{35} + L^M(5,5.5,1,2.2)_{LR}\, x_{56} + L^M(5,9.6,1.1,1.3)_{LR}\, x_{47} + L^M(2,4.2,0.7,$ $1.8)_{LR}\ x_{46} + L^M(3,3.4,0.4,1.5)_{LR}\ x_{67} + L^M(6,6.3,0.3,0.6)_{LR}\ x_{78} = \frac{67}{3},$

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{35} - x_{34} = 0, \quad x_{34} - x_{47} - x_{46} = 0,$

$x_{35} - x_{56} = 0, \quad x_{46} + x_{56} - x_{67} = 0, \quad x_{47} + x_{67} - x_{78} = 0, \quad x_{78} = 1,$

$x_{12},\ x_{23},\ x_{34},\ x_{35},\ x_{46},\ x_{47},\ x_{56},\ x_{67},\ x_{78} \geq 0$

On solving the crisp linear programming problem the following optimal solution is obtained:

$x_{12} = x_{23} = x_{35} = x_{56} = x_{67} = x_{78} = 1$ and $x_{34} = x_{46} = x_{47} = 0$.

Putting the obtained optimal values of x_{ij} in $((2,2.3,0.6,0.8)_{LR}\ x_{12} \oplus (5,6,1,$ $1.2)_{LR}\ x_{23} \oplus (7,7.8,1,1.1)_{LR}\ x_{34} \oplus (4,4.5,0.7,2.7)_{LR}\ x_{35} \oplus (5,5.5,1,2.2)_{LR}\ x_{56} \oplus$ $(5,9.6,1.1,1.3)_{LR}\ x_{47} \oplus (2,4.2,0.7,1.8)_{LR}\ x_{46} \oplus (3,3.4,0.4,1.5)_{LR}\ x_{67} \oplus (6,6.3,0.3,$ $0.6)_{LR}\ x_{78})$, a unique LR flat fuzzy number $(25,28,4,9)_{LR}$, representing the optimal fuzzy project completion time, is obtained and using the same values of x_{ij} the obtained critical path is $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$.

5.10.2 Optimal fuzzy completion time of the second chosen problem by using the modified Liang and Han method based on modified Farhadinia ranking approach

Using the modified Liang and Han method based on modified Farhadinia ranking approach the optimal fuzzy completion time of the project, chosen in Example 5.2, can be obtained as follows:

Step 1 Using Step 1 of the modified Liang and Han method based on modified Farhadinia ranking approach by assuming $\widetilde{FES_1} = (0,0,0,0)_{LR}$ the values of $\widetilde{FES_j}$, $j = 2,3,4,5,6,7,8$ can be obtained as follows:

$$\widetilde{FES_2} = \widetilde{FES_1} \oplus \widetilde{FNT_{12}} = (0,0,0,0)_{LR} \oplus (2,2.3,0.6,0.8)_{LR} = (2,2.3,0.6,0.8)_{LR}$$

$$\begin{aligned}\widetilde{FES_3} &= \widetilde{FES_2} \oplus \widetilde{FNT_{23}} \\ &= (2,2.3,0.6,0.8)_{LR} \oplus (5,6,1,1.2)_{LR} \\ &= (7,8.3,1.6,2)_{LR}\end{aligned}$$

$$\begin{aligned}\widetilde{FES_4} &= \widetilde{FES_3} \oplus \widetilde{FNT_{34}} \\ &= (7,8.3,1.6,2)_{LR} \oplus (7,7.8,1,1.1)_{LR} \\ &= (14,16.1,2.6,3.1)_{LR}\end{aligned}$$

$$\begin{aligned}\widetilde{FES_5} &= \widetilde{FES_3} \oplus \widetilde{FNT_{35}} \\ &= (7,8.3,1.6,2)_{LR} \oplus (4,4.5,0.7,2.7)_{LR} \\ &= (11,12.8,2.3,4.7)_{LR}\end{aligned}$$

$$\begin{aligned}\widetilde{FES_6} &= \text{maximum } \{\widetilde{FES_4} \oplus \widetilde{FNT_{46}},\ \widetilde{FES_5} \oplus \ \widetilde{FNT_{56}}\} \\ &= \text{maximum}\{(14,16.1,2.6,3.1)_{LR} \oplus (2,4.2,0.7,1.8)_{LR}, (11,12.8,2.3,4.7)_{LR} \\ &\qquad \oplus (5,5.5,1,2.2)_{LR}\}\end{aligned}$$

$$= \text{maximum}\{(16, 20.3, 3.3, 4.9)_{LR}, (16, 18.3, 3.3, 6.9)_{LR}\}$$

$$= (16, 18.3, 3.3, 6.9)_{LR} \left(\because W^M(16, 20.3, 3.3, 4.9)_{LR} < W^M(16, 18.3, 3.3, 6.9)_{LR} \right)$$

$$\widetilde{FES}_7 = \text{maximum } \{\widetilde{FES}_4 \oplus \widetilde{FNT}_{47}, \ \widetilde{FES}_6 \oplus \ \widetilde{FNT}_{67}\}$$

$$= \text{maximum}\{(14, 16.1, 2.6, 3.1)_{LR} \oplus (5, 9.6, 1.1, 1.3)_{LR}, (16, 18.3, 3.3, 6.9)_{LR} \oplus (3, 3.4, 0.4, 1.5)_{LR}\}$$

$$= \text{maximum}\{(19, 25.7, 3.7, 4.4)_{LR}, (19, 21.7, 3.7, 8.4)_{LR}\}$$

$$= (19, 21.7, 3.7, 8.4)_{LR} \left(\because W^M(19, 25.7, 3.7, 4.4)_{LR} < W^M(19, 21.7, 3.7, 8.4)_{LR} \right)$$

$$\widetilde{FES}_8 = \widetilde{FES}_7 \oplus \widetilde{FNT}_{78}$$

$$= (19, 21.7, 3.7, 8.4)_{LR} \oplus (6, 6.3, 0.3, 0.6)_{LR}$$

$$= (25, 28, 4, 9)_{LR}$$

Step 2 Since, proposed Mehar's subtraction is not defined for such LR flat fuzzy numbers for which $L(x) \neq R(x)$. So, this problem can not be solved further.

5.11 Comparative study

The results of fuzzy project network problems, chosen in Example 2.4, 3.1, 4.1, 5.1 and 5.2, obtained by using the methods, proposed in previous chapters and the methods proposed in this chapter, are shown in Table 5.6

Table 5.6 Results of the chosen problems obtained using the proposed methods

Example	Proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach		Modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach		Modified Liang and Han method based on modified Farhadinia ranking approach		Proposed Mehar's method based on modified Farhadinia ranking approach	
	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time
2.4	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(14, 25, 39)
3.1	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)	$1 \Rightarrow 2 \Rightarrow 4$	(14, 23, 32)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(15, 23, 31)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(15, 23, 31)
4.1	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(6, 12, 16, 22)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(6, 12, 16, 22)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(6, 12, 16, 22)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	(6, 12, 16, 22)
5.1	Not applicable	Not applicable	Not applicable	Not applicable	$1 \Rightarrow 2 \Rightarrow 4$	$(18, 20, 4, 6)_{LR}$	$1 \Rightarrow 2 \Rightarrow 4$	$(18, 20, 4, 6)_{LR}$
5.2	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8$	$(25, 28, 4, 9)_{LR}$

The results, presented in Table 5.6, can be explained as follows:

1. The proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach and modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach can be used only to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by either a triangular fuzzy number or a trapezoidal fuzzy number. Since, in the problems, chosen in Example 2.4, Example 3.1 and Example 4.1, time of each activity activity is represented by either a triangular fuzzy number or a trapezoidal fuzzy number so the optimal fuzzy project completion time of all these problems can be obtained by these methods. However, in the problems, chosen in Example 5.1 and Example 5.2, time of each activity is represented by an LR flat fuzzy number so the optimal fuzzy project completion time of these problems can not be obtained by these methods.

2. The modified Liang and Han method based on modified Farhadinia ranking approach can be used to find the optimal fuzzy project completion time of such project network problems in which time of each activity is represented by such an LR flat fuzzy number for which $L(x) = R(x)$. Since, in the problems chosen in Example 2.4, Example 3.1, Example 4.1 and Example 5.1, time of each activity is represented by such an LR flat fuzzy number for which $L(x) = R(x)$ so the optimal fuzzy project completion time of all these problems can be obtained by the modified Liang and Han method based on modified Farhadinia ranking approach. However, in the problem, chosen in Example 5.2, time of each activity is represented by such an LR flat fuzzy

number for which $L(x) \neq R(x)$ so the optimal fuzzy project completion time of this problem can not be obtained by this method.

3. The proposed Mehar's method based on modified Farhadinia ranking approach can be used to find the optimal fuzzy project completion time of such project network problems in which time of each activity is represented by such an LR flat fuzzy number for which $L(x) = R(x)$ or such an LR flat fuzzy number for which $L(x) \neq R(x)$ so the optimal fuzzy project completion time of all the problems, chosen in Example 2.4, 3.1, 4.1, 5.1, 5.2, can be obtained by this method.

5.12 Conclusion

On the basis of presented study, it can be concluded that it is better to use the methods proposed in this chapter as compared to the methods proposed in previous chapter for finding the optimal fuzzy completion time of such project network problems in which time of each activity is either represented by a triangular fuzzy number or a trapezoidal fuzzy number or an LR flat fuzzy number. Also, it can be concluded that it is better to use the proposed Mehar's method based on modified Farhadinia ranking approach as compared to the modified Liang and Han method based on modified Farhadinia ranking approach to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number.

Chapter 6

New Methods Based On Proposed Mehar's Ranking Approach For Finding Unique Optimal Fuzzy Project Completion Time

In this chapter, it is shown that it is not genuine to apply the modified Farhadinia ranking approach for finding the maximum and minimum of LR flat fuzzy numbers and a new ranking approach, named as Mehar's ranking approach, is proposed for finding the maximum and minimum of LR flat fuzzy numbers. Also, on the basis of proposed Mehar's ranking approach a new method, named as Mehar's method based on proposed Mehar's ranking approach, is proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. Also, the modified Liang and Han method based on modified Farhadinia ranking approach is modified on the basis of proposed Mehar's ranking approach.

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6.1 Shortcomings of modified Farhadinia ranking approach

In Step 1 of the modified Farhadinia ranking approach it is assumed that if $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ are two LR flat fuzzy numbers such that $C^M(\widetilde{A}) > C^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e., the maximum and minimum of LR flat fuzzy numbers, obtained by using modified Farhadinia ranking approach, is independent from $L(x)$ and $R(x)$ which is not genuine. e.g., since for LR flat fuzzy numbers $\widetilde{A} = (13, 14, 2, 3)_{LR}$ and $\widetilde{B} = (10, 12, 3, 4)_{LR}$ the condition $C^M(\widetilde{A}) > C^M(\widetilde{B})$ is satisfying so according to modified Farhadinia ranking approach, proposed in previous chapter, $\widetilde{A} \succ \widetilde{B}$ for all values of $L(x)$ and $R(x)$ i.e, maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$. Since, maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ obtained by using modified Farhadinia ranking approach is independent from $L(x)$ and $R(x)$ so it is not genuine to use modified Farhadinia ranking approach for finding the maximum and minimum of LR flat fuzzy numbers.

6.2 Proposed Mehar's ranking approach

In this section, to overcome the shortcomings of modified Farhadinia ranking approach, a new ranking approach, named as Mehar's ranking approach, is proposed for finding maximum and minimum of LR flat fuzzy numbers.

Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers then use the following steps to find maximum$\{\widetilde{A}, \widetilde{B}\}$ and minimum$\{\widetilde{A}, \widetilde{B}\}$:

Step 1 Find $\Re(\widetilde{A}) = \frac{1}{2}\,[\int_0^1 \underline{a_1} d\alpha - \int_0^1 a_1^L L^{-1}(\alpha) d\alpha + \int_0^1 \overline{a_1} d\alpha + \int_0^1 a_1^R R^{-1}(\alpha) d\alpha]$ and

$$\Re(\widetilde{B}) = \frac{1}{2}\,[\int_0^1 \underline{a_2} d\alpha - \int_0^1 a_2^L L^{-1}(\alpha) d\alpha + \int_0^1 \overline{a_2} d\alpha + \int_0^1 a_2^R R^{-1}(\alpha) d\alpha]$$

Case (i) If $\Re(\widetilde{A}) > \Re(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $\Re(\widetilde{A}) < \Re(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $\Re(\widetilde{A}) = \Re(\widetilde{B})$ then go to Step 2.

Step 2 Find $mode(\widetilde{A}) = \frac{1}{2}\,[\int_0^1 \underline{a_1}d\alpha + \int_0^1 \overline{a_1}d\alpha]$ and $mode(\widetilde{B}) = \frac{1}{2}\,[\int_0^1 \underline{a_2}d\alpha + \int_0^1 \overline{a_2}d\alpha]$

Case (i) If $mode(\widetilde{A}) > mode(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $mode(\widetilde{A}) < mode(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $mode(\widetilde{A}) = mode(\widetilde{B})$ then go to Step 3.

Step 3 Find $divergence(\widetilde{A}) = \int_0^1 \overline{a_1}d\alpha + \int_0^1 a_1^R R^{-1}(\alpha)d\alpha - \int_0^1 \underline{a_1}d\alpha + \int_0^1 a_1^L L^{-1}(\alpha)d\alpha$

and $divergence(\widetilde{B}) = \int_0^1 \overline{a_2}d\alpha + \int_0^1 a_2^R R^{-1}(\alpha)d\alpha - \int_0^1 \underline{a_2}d\alpha + \int_0^1 a_2^L L^{-1}(\alpha)d\alpha$

Case (i) If $divergence(\widetilde{A}) > divergence(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $divergence(\widetilde{A}) < divergence(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $divergence(\widetilde{A}) = divergence(\widetilde{B})$ then go to Step 4.

Step 4 Find $Left\ spread(\widetilde{A}) = \int_0^1 a_1^L L^{-1}(\alpha)d\alpha$ and $Left\ spread(\widetilde{B}) = \int_0^1 a_2^L L^{-1}(\alpha)d\alpha$

Case (i) If $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$$

Case (ii) If $Left\ spread(\widetilde{A}) < Left\ spread(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum}\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$

Remark 6.1: In Step 4 of proposed Mehar's ranking approach instead of $Left\ spread(\widetilde{A}) = \int_0^1 a_1^L L^{-1}(\alpha)d\alpha$ and $Left\ spread(\widetilde{B}) = \int_0^1 a_2^L L^{-1}(\alpha)d\alpha$ the $Right\ spread(\widetilde{A}) = a_1^R \int_0^1 R^{-1}(\alpha)d\alpha$ and $Right\ spread(\widetilde{B}) = a_2^R \int_0^1 R^{-1}(\alpha)d\alpha$ respectively can also be used.

6.2.1 Validity of proposed Mehar's ranking approach

It is obvious that if $\widetilde{A}$ and $\widetilde{B}$ are two LR flat fuzzy numbers such that $\Re(\widetilde{A}) > \Re(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) > mode(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) = mode(\widetilde{B})$, $divergence(\widetilde{A}) > divergence(\widetilde{B})$ or $\Re(\widetilde{A}) = \Re(\widetilde{B})$, $mode(\widetilde{A}) = mode(\widetilde{B})$, $divergence(\widetilde{A}) = divergence(\widetilde{B})$, $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$ then using the proposed Mehar's ranking approach, presented in Section 6.2, maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$.

In this section, it is proved that if $\widetilde{A}$ and $\widetilde{B}$ are two such LR flat fuzzy numbers for which all the four conditions of proposed Mehar's ranking approach are satisfied then $\widetilde{A} = \widetilde{B}$ and also it is proved that if $Left\ spread(\widetilde{A}) \gtreqless Left\ spread(\widetilde{B})$ then $Right\ spread(\widetilde{A}) \gtreqless Right\ spread(\widetilde{B})$.

Proposition 6.1 Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers such that

(i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$ (ii) $mode(\widetilde{A}) = mode(\widetilde{B})$ (iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$

(iv) $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$

Then, $\widetilde{A} = \widetilde{B}$.

Proof. $\Re(\widetilde{A}) = \Re(\widetilde{B})$

$$\Rightarrow \frac{1}{2}\ [\int_0^1 \underline{a_1}d\alpha - \int_0^1 a_1^L L^{-1}(\alpha)d\alpha + \int_0^1 \overline{a_1}d\alpha + \int_0^1 a_1^R R^{-1}(\alpha)d\alpha] = \frac{1}{2}\ [\int_0^1 \underline{a_2}d\alpha$$

$- \int_0^1 a_2^L L^{-1}(\alpha)d\alpha + \int_0^1 \overline{a_2}d\alpha + \int_0^1 a_2^R R^{-1}(\alpha)d\alpha]$

$$\Rightarrow \underline{a_1} - \int_0^1 a_1^L L^{-1}(\alpha)d\alpha + \overline{a_1} + \int_0^1 a_1^R R^{-1}(\alpha)d\alpha = \underline{a_2} - \int_0^1 a_2^L L^{-1}(\alpha)d\alpha + \overline{a_2} + \int_0^1 a_2^R R^{-1}(\alpha)d\alpha \quad (6.1)$$

(ii) $mode(\widetilde{A}) = mode(\widetilde{B})$

$\Rightarrow \frac{1}{2}\ [\int_0^1 \underline{a_1}d\alpha + \int_0^1 \overline{a_1}d\alpha] = \frac{1}{2}\ [\int_0^1 \underline{a_2}d\alpha + \int_0^1 \overline{a_2}d\alpha]$

$$\Rightarrow \underline{a_1} + \overline{a_1} = \underline{a_2} + \overline{a_2} \quad (6.2)$$

(iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$

$$\Rightarrow \int_0^1 \overline{a_1}d\alpha + \int_0^1 a_1^R R^{-1}(\alpha)d\alpha - \int_0^1 \underline{a_1}d\alpha + \int_0^1 a_1^L L^{-1}(\alpha)d\alpha = \int_0^1 \overline{a_2}d\alpha + \int_0^1 a_2^R R^{-1}(\alpha)d\alpha - \int_0^1 \underline{a_2}d\alpha + \int_0^1 a_2^L L^{-1}(\alpha)d\alpha \quad (6.3)$$

(iv) $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$

$\Rightarrow \int_0^1 a_1^L L^{-1}(\alpha)d\alpha = \int_0^1 a_2^L L^{-1}(\alpha)d\alpha$

$$\Rightarrow a_1^L = a_2^L \quad (6.4)$$

On solving (6.1), (6.2) and (6.4)

$$a_1^R = a_2^R \quad (6.5)$$

On solving (6.3), (6.4) and (6.5)

$$\overline{a_1} - \underline{a_1} = \overline{a_2} - \underline{a_2} \quad (6.6)$$

On solving (6.2) and (6.6)

$\underline{a_1} = \underline{a_2}$, $\overline{a_1} = \overline{a_2}$

i.e., $\underline{a_1} = \underline{a_2}$, $\overline{a_1} = \overline{a_2}$, $a_1^L = a_2^L$ and $a_1^R = a_2^R$ so, $\widetilde{A} = \widetilde{B}$.

Proposition 6.2 Let $\widetilde{A} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{B} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ be two LR flat fuzzy numbers such that

(i) $\Re(\widetilde{A}) = \Re(\widetilde{B})$ (ii) $mode(\widetilde{A}) = mode(\widetilde{B})$ (iii) $divergence(\widetilde{A}) = divergence(\widetilde{B})$

Then,

(a) $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$ iff $Right\ spread(\widetilde{A}) > Right\ spread(\widetilde{B})$

(b) $Left\ spread(\widetilde{A}) < Left\ spread(\widetilde{B})$ iff $Right\ spread(\widetilde{A}) < Right\ spread(\widetilde{B})$

(c) $Left\ spread(\widetilde{A}) = Left\ spread(\widetilde{B})$ iff $Right\ spread(\widetilde{A}) = Right\ spread(\widetilde{B})$

Proof: Since, $\Re(\widetilde{A}) = \Re(\widetilde{B})$

$$mode(\widetilde{A}) = mode(\widetilde{B})$$

$$divergence(\widetilde{A}) = divergence(\widetilde{B})$$

so from Proposition 6.1,

$$\underline{a_1} - \int_0^1 a_1^L L^{-1}(\alpha)d\alpha + \overline{a_1} + \int_0^1 a_1^R R^{-1}(\alpha)d\alpha = \underline{a_2} - \int_0^1 a_2^L L^{-1}(\alpha)d\alpha + \overline{a_2} + \int_0^1 a_2^R R^{-1}(\alpha)d\alpha \quad (6.7)$$

$$\underline{a_1} + \overline{a_1} = \underline{a_2} + \overline{a_2} \quad (6.8)$$

$$\overline{a_1} + \int_0^1 a_1^R R^{-1}(\alpha)d\alpha - \underline{a_1} + \int_0^1 a_1^L L^{-1}(\alpha)d\alpha = \overline{a_2} + \int_0^1 a_2^R R^{-1}(\alpha)d\alpha - \underline{a_2} + \int_0^1 a_2^L L^{-1}(\alpha)d\alpha \quad (6.9)$$

Subtracting (6.8) from (6.7)

$$(a_1^R - a_2^R) \int_0^1 R^{-1}(\alpha)d\alpha = (a_1^L - a_2^L) \int_0^1 L^{-1}(\alpha)d\alpha \quad (6.10)$$

(a) $Left\ spread(\widetilde{A}) > Left\ spread(\widetilde{B})$

$$\Leftrightarrow \int_0^1 a_1^L L^{-1}(\alpha)d\alpha > \int_0^1 a_2^L L^{-1}(\alpha)d\alpha$$

$$\Leftrightarrow (a_1^L - a_2^L) \int_0^1 L^{-1}(\alpha)d\alpha > 0 \quad (6.11)$$

$$\Leftrightarrow (a_1^R - a_2^R) \int_0^1 R^{-1}(\alpha)d\alpha > 0 \quad \text{(Using equation 6.10)}$$

$$\Leftrightarrow a_1^R \int_0^1 R^{-1}(\alpha)d\alpha > a_2^R \int_0^1 R^{-1}(\alpha)d\alpha$$

$$\Leftrightarrow Right\ spread(\widetilde{A}) > Right\ spread(\widetilde{B})$$

Similarly, (b) and (c) can be easily proved.

6.3 Advantages of proposed Mehar's ranking approach

Although, both the modified Farhadinia ranking approach and proposed Mehar's ranking approach can be used to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. But, as discussed in Section 6.1 that the maximum and minimum of LR flat fuzzy numbers obtained by using the modified Farhadinia ranking approach is independent from $L(x)$ and $R(x)$ while the maximum and minimum of LR flat fuzzy numbers obtained by using the proposed Mehar's ranking approach is depending upon $L(x)$ and $R(x)$. So, it is better to use proposed Mehar's ranking approach for finding the maximum and minimum of LR flat fuzzy numbers

To show the advantages of proposed Mehar's ranking approach over modified Farhadinia ranking approach the maximum and minimum of LR flat fuzzy numbers $\widetilde{A} = (30, 37, 2, 3)_{LR}$ and $\widetilde{B} = (25, 42, 4, 4.5)_{LR}$ with different values of $L(x)$ and $R(x)$, obtained by using modified Farhadinia ranking approach and proposed Mehar's ranking approach, are shown in Table 6.1

Table 6.1 Maximum and minimum of $\widetilde{A}$ and $\widetilde{B}$ obtained by using modified Farhadinia ranking approach and proposed Mehar's ranking approach

$L(x)$	$R(x)$	Modified Farhadinia ranking approach			Proposed Mehar's ranking approach		
		Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A}, \widetilde{B}\}$	Ranking value of $\widetilde{A}$	Ranking value of $\widetilde{B}$	maximum$\{\widetilde{A}, \widetilde{B}\}$
$\max\{0, 1-x^2\}$	$\max\{0, 1-x^4\}$	$C^M(\widetilde{A}) = 30$	$C^M(\widetilde{B}) = 25$	$\widetilde{A}$	$\Re(\widetilde{A}) = 34.033$	$\Re(\widetilde{B}) = 33.96$	$\widetilde{A}$
$\max\{0, 1-x^2\}$	$\max\{0, 1-x\}$	$C^M(\widetilde{A}) = 30$	$C^M(\widetilde{B}) = 25$	$\widetilde{A}$	$\Re(\widetilde{A}) = 33.58$	$\Re(\widetilde{B}) = 33.29$	$\widetilde{A}$
$\max\{0, 1-x\}$	$\max\{0, 1-x^2\}$	$C^M(\widetilde{A}) = 30$	$C^M(\widetilde{B}) = 25$	$\widetilde{A}$	$\Re(\widetilde{A}) = 34$ $mode(\widetilde{A}) = 33.5$ $divergence(\widetilde{A}) = 10$	$\Re(\widetilde{B}) = 34$ $mode(\widetilde{B}) = 33.5$ $divergence(\widetilde{B}) = 22$	$\widetilde{B}$
$\max\{0, 1-x^4\}$	$\max\{0, 1-x\}$	$C^M(\widetilde{A}) = 30$	$C^M(\widetilde{B}) = 25$	$\widetilde{A}$	$\Re(\widetilde{A}) = 33.45$	$\Re(\widetilde{B}) = 33.025$	$\widetilde{A}$
$\max\{0, 1-x\}$	$e^{-\|x\|}$	$C^M(\widetilde{A}) = 30$	$C^M(\widetilde{B}) = 25$	$\widetilde{A}$	$\Re(\widetilde{A}) = 34.5$	$\Re(\widetilde{B}) = 34.75$	$\widetilde{B}$

It is obvious from the results, shown in Table 6.1, that the maximum and minimum of LR flat fuzzy numbers obtained by using the modified Farhadinia ranking approach is depending only upon the values of C^M and independent from $L(x)$ and $R(x)$. While, the maximum and minimum of LR flat fuzzy numbers obtained by using the proposed Mehar's ranking approach is depending upon $L(x)$ and $R(x)$.

6.4 Proposed methods based on proposed Mehar's ranking approach

Due to the shortcomings of modified Farhadinia ranking approach, it is not genuine to apply the methods, proposed in Chapter 5, for finding the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. In this section, to overcome the shortcomings of proposed methods based on modified Farhadinia ranking approach new methods on the basis of proposed Mehar's ranking approach are proposed to find the unique optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number.

6.4.1 Modified Liang and Han method based on proposed Mehar's ranking approach

If in the modified Liang and Han method instead of modified Farhadinia ranking approach proposed Mehar's ranking approach is used then the shortcomings of modified Liang and Han method based on modified Farhadinia ranking approach will be resolved.

6.4.2 Proposed Mehar's method based on proposed Mehar's ranking approach

In this section, to resolve the shortcomings of proposed Mehar's method based on modified Farhadinia ranking approach, a new method, based on proposed Mehar's ranking approach, is proposed.

The steps of proposed Mehar's method based on proposed Mehar's ranking approach are as follows:

Step 1 Formulate the chosen fuzzy project network problem into the fuzzy linear programming problem $(P_{6.1})$:

$$\left.\begin{array}{l} \text{Maximize} \sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x_{ij}) \\ \text{subject to} \\ \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A. \end{array}\right\} \quad (P_{6.1})$$

where, A: set of all activities (i, j),

$\widetilde{t}_{ij}$: the fuzzy time duration of the activity (i, j),

N: the set of nodes,

n: the destination node,

1: the source node,

x_{ij}: the decision variable denoting the amount of flow in the activity (i, j).

Step 2 Suppose the fuzzy linear programming problem $(P_{6.1})$ have h feasible solutions and $\{x_{ij}^w\}$ is the w^{th} feasible solution then the aim is to find the feasible solution with the largest objective value i.e., the goal is to find $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^w)\right\}$.

In proposed Mehar's ranking approach it is assumed that if $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum_{(i,j)\in A}(\Re(\widetilde{t}_{ij})\ x_{ij}^w)\right\}$ is $\sum_{(i,j)\in A}(\Re(\widetilde{t}_{ij})\ x_{ij}^\eta)$ then $\underset{1\leq w\leq h}{\text{maximum}}\left\{\sum_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^w)\right\}$ will also be $\sum_{(i,j)\in A}(\widetilde{t}_{ij}\ x_{ij}^\eta)$, where, $\Re(\widetilde{a_{ij}}) = \Re(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR} = \frac{1}{2}\ [\int_0^1 \underline{a_{ij}} d\alpha - \int_0^1 a_{ij}^L L^{-1}(\alpha) d\alpha + \int_0^1 \overline{a_{ij}} d\alpha + \int_0^1 a_{ij}^R R^{-1}(\alpha) d\alpha]$ i.e., by using proposed Mehar's ranking approach the optimal solution of the fuzzy linear programming problem $(P_{6.1})$

can be obtained by solving the crisp linear programming problem ($P_{6.2}$):

Maximize $\sum_{(i,j)\in A} (\Re(\widetilde{t}_{ij})\ x_{ij})$

subject to

$$\left.\begin{array}{l} \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\ \\ \qquad\qquad x_{ij} \ \geq \ 0 \ \forall \ (i,j) \ \in A. \end{array}\right\} \qquad (P_{6.2})$$

Step 3 Solve the crisp linear programming problem ($P_{6.2}$) to find the optimal solution $\{x_{ij}\}$.

Step 4 Use the optimal solution $\{x_{ij}\}$, obtained in Step 3, to find the critical path and also put the obtained optimal values of x_{ij} in $\sum_{(i,j)\in A} (\widetilde{t}_{ij}\ x_{ij})$ to find the optimal fuzzy completion time of the project.

Case (i): If a unique critical path and hence a unique fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained fuzzy project completion time is the optimal fuzzy completion time of the project and the obtained critical path is the only critical path of the project.

Case (ii): If more than one critical paths are obtained then go to Step 5.

Step 5 Check that the LR flat fuzzy numbers, representing the optimal fuzzy project completion time, corresponding to all the critical paths are same or not.

Case (i): If a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, is obtained then all the critical paths, obtained in Step 4, corresponding to which the obtained LR flat fuzzy number is obtained, are critical paths of the project and the obtained LR flat fuzzy number will represent the optimal fuzzy completion time of the project.

Case (ii): If more than one LR flat fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 6.

Step 6 Let using the previous steps p different LR flat fuzzy numbers $\widetilde{T}_1$, $\widetilde{T}_2$, ..., $\widetilde{T}_p$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}_1 \neq \widetilde{T}_2 \neq ... \neq \widetilde{T}_p$ but $\Re(\widetilde{T}_1) = \Re(\widetilde{T}_2) = ... = \Re(\widetilde{T}_l) = u_1$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\left.\begin{array}{l}
\text{Maximize} \sum\limits_{(i,j)\in A} (mode(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) \\
\text{subject to} \\
\sum\limits_{(i,j)\in A} (\Re(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_1, \\
\sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\
x_{ij} \geq 0\ \forall\ (i,j) \in A.
\end{array}\right\} \quad (P_{6.3})$$

Case (i): If by putting the obtained optimal values of x_{ij} in ($\sum\limits_{(i,j)\in A}((\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}$ x_{ij})) a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained LR flat fuzzy number will represent the optimal fuzzy completion time of the project and all the critical paths can be obtained by using the obtained optimal values of x_{ij}.

Case (ii): If more than one LR flat fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 7.

Step 7 Let using the previous steps l LR flat fuzzy numbers $\widetilde{T}'_1$, $\widetilde{T}'_2$, ..., $\widetilde{T}'_l$, where, $l \leq p$, representing the fuzzy project completion time, be obtained i.e., $\widetilde{T}'_1 \neq \widetilde{T}'_2 \neq$

$... \neq \widetilde{T}'_l$ but $\Re(\widetilde{T}'_1) = \Re(\widetilde{T}'_2) = ... = \Re(\widetilde{T}'_l) = u_1$ and $mode(\widetilde{T}'_1) = mode(\widetilde{T}'_2) = ... = mode(\widetilde{T}'_l) = u_2$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\text{Maximize} \sum_{(i,j)\in A} (divergence(\underline{a_{ij}}, \overline{a_{ij}}, a^L_{ij}, a^R_{ij})_{LR}\ x_{ij})$$

subject to

$$\left.\begin{array}{l}
\sum_{(i,j)\in A} (\Re(\underline{a_{ij}}, \overline{a_{ij}}, a^L_{ij}, a^R_{ij})_{LR}\ x_{ij}) = u_1, \\
\sum_{(i,j)\in A} (mode(\underline{a_{ij}}, \overline{a_{ij}}, a^L_{ij}, a^R_{ij})_{LR}\ x_{ij}) = u_2, \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\
x_{ij} \geq 0\ \forall\ (i,j) \in A.
\end{array}\right\} \quad (P_{6.4})$$

Case (i): If by putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A} ((\underline{a_{ij}}, \overline{a_{ij}}, a^L_{ij}, a^R_{ij})_{LR}\ x_{ij}))$ a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, is obtained then the obtained LR flat fuzzy number will represent the optimal fuzzy completion time of the project and all the critical paths can be obtained by using the obtained optimal values of x_{ij}.

Case (ii): If more than one LR flat fuzzy numbers, representing the optimal fuzzy completion time of the project, are obtained then go to Step 8.

Step 8 Let using the previous steps k LR flat fuzzy numbers $\widetilde{T}''_1, \widetilde{T}''_2, ..., \widetilde{T}''_k$, where $k \leq l$, representing the optimal fuzzy project completion time, be obtained i.e., $\widetilde{T}''_1 \neq \widetilde{T}''_2 \neq ... \neq \widetilde{T}''_k$ but $\Re(\widetilde{T}''_1) = \Re(\widetilde{T}''_2) = ... = \Re(\widetilde{T}''_k) = u_1$, $mode(\widetilde{T}''_1) = mode(\widetilde{T}''_2) = ... = mode(\widetilde{T}''_k) = u_2$ and $divergence(\widetilde{T}''_1) = divergence(\widetilde{T}''_2) = ... = divergence(\widetilde{T}''_k) = u_3$ (say). Then, find the optimal solution of the following crisp linear programming problem:

$$\text{Maximize} \sum_{(i,j)\in A} (Left\ spread(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij})$$

subject to

$$\left.\begin{array}{l}
\sum_{(i,j)\in A} (\Re(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_1, \\
\sum_{(i,j)\in A} (mode(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_2, \\
\sum_{(i,j)\in A} (divergence(\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}) = u_3, \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1 & , i = 1 \\ -1 & , i = n \\ 0 & , i \in N - \{1, n\} \end{cases} \\
x_{ij} \ \geq\ 0\ \forall\ (i,j)\ \in A.
\end{array}\right\} \quad (P_{6.5})$$

Now, putting the obtained optimal values of x_{ij} in $(\sum_{(i,j)\in A}((\underline{a_{ij}}, \overline{a_{ij}}, a_{ij}^L, a_{ij}^R)_{LR}\ x_{ij}))$ a unique LR flat fuzzy number, representing the optimal fuzzy project completion time, will be obtained and the critical paths can be obtained by using the obtained optimal values of x_{ij}.

6.5 Illustrative example

To illustrate the proposed Mehar's method based on proposed Mehar's ranking approach and modified Liang and Han method based on proposed Mehar's ranking approach the project network problem, chosen in Example 6.1, is solved by both the methods.

Example 6.1 A contractor divides the whole work for constructing a house into different activities. The activities and its time duration, represented by LR flat numbers with $L(x) = \text{maximum}\{0, 1 - x\}$ and $R(x) = \text{maximum}\{0, 1 - x^2\}$, are shown in Table 6.2 and the network, connecting the different activities, is shown in

Figure 6.1. Find the optimal fuzzy completion time to construct the house.

Table 6.2 Fuzzy normal time for each activity

Activity	Description	Fuzzy time duration ($\widetilde{t}_{ij}$) (days)
(1, 2)	Build foundation	$(7, 7.1, 0.2, 2)_{LR}$
(2, 3)	Build walls and ceilings	$(10, 10.2, 0.5, 2.5)_{LR}$
(3, 5)	Build roofs	$(4, 7.7, 1, 2.5)_{LR}$
(3, 6)	Do electrical wiring	$(9, 9.7, 5.3, 6)_{LR}$
(3, 4)	Put in windows	$(2, 3, 1.5, 2)_{LR}$
(4, 5)	Put on siding	$(3, 3.7, 1.5, 2)_{LR}$
(5, 6)	Paint house	$(3, 4, 0.3, 0.5)_{LR}$

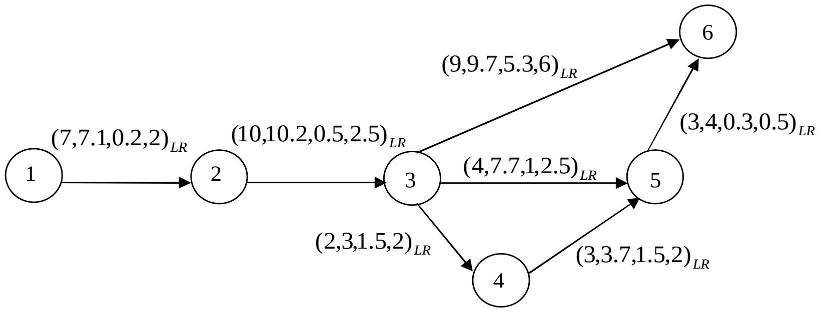

Figure 6.1 Project network of the illustrated example

6.6 Optimal fuzzy completion time of the chosen problem

In this section, the project network problem, chosen in Example 6.1, is solved by both the methods.

6.6.1 Optimal fuzzy completion time of the chosen problem by using proposed Mehar's method based on proposed Mehar's ranking approach

Using the proposed Mehar's method based on proposed Mehar's ranking approach the optimal fuzzy completion time of the project, chosen in Example 6.1, can be obtained as follows:

Step 1 Using Step 1 of proposed Mehar's method based on proposed Mehar's ranking approach the project network problem, chosen in Example 6.1, can be formulated as follows:

Maximize $((7, 7.1, 0.2, 2)_{LR}\ x_{12} \oplus (10, 10.2, 0.5, 2.5)_{LR}\ x_{23} \oplus (2, 3, 1.5, 2)_{LR}\ x_{34} \oplus (4, 7.7, 1, 2.5)_{LR}\ x_{35} \oplus (9, 9.7, 5.3, 6)_{LR}\ x_{36} \oplus (3, 3.7, 1.5, 2)_{LR}\ x_{45} \oplus (3, 4, 0.3, 0.5)_{LR}\ x_{56})$

subject to

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{34} - x_{35} - x_{36} = 0, \quad x_{34} - x_{45} = 0,$

$x_{35} + x_{45} - x_{56} = 0,\ x_{36} + x_{56} = 1,$

$x_{12},\ x_{23},\ x_{34},\ x_{35},\ x_{36},\ x_{45},\ x_{56} \geq 0$

Step 2 Using Step 2 of proposed Mehar's method based on proposed Mehar's ranking approach, the fuzzy linear programming problem obtained in Step 1 can be converted into the following crisp linear programming problem:

Maximize $(\Re(7, 7.1, 0.2, 2)_{LR}\ x_{12} + \Re(10, 10.2, 0.5, 2.5)_{LR}\ x_{23} + \Re(2, 3, 1.5, 2)_{LR}\ x_{34} + \Re(4, 7.7, 1, 2.5)_{LR}\ x_{35} + \Re(9, 9.7, 5.3, 6)_{LR}\ x_{36} + \Re(3, 3.7, 1.5, 2)_{LR}\ x_{45} + \Re(3, 4, 0.3, 0.5)_{LR}\ x_{56})$

subject to

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{34} - x_{35} - x_{36} = 0, \quad x_{34} - x_{45} = 0,$

$x_{35} + x_{45} - x_{56} = 0,\ x_{36} + x_{56} = 1,$

$x_{12},\ x_{23},\ x_{34},\ x_{35},\ x_{36},\ x_{45},\ x_{56} \geq 0$

i.e., Maximize $(\frac{23}{3}\ x_{12} + \frac{1297}{120}\ x_{23} + \frac{67}{24}\ x_{34} + \frac{193}{30}\ x_{35} + \frac{401}{40}\ x_{36} + \frac{437}{120}\ x_{45} + \frac{431}{120}\ x_{56})$

subject to

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{34} - x_{35} - x_{36} = 0, \quad x_{34} - x_{45} = 0,$

$x_{35} + x_{45} - x_{56} = 0$, $x_{36} + x_{56} = 1$,

$x_{12}, x_{23}, x_{34}, x_{35}, x_{36}, x_{45}, x_{56} \geq 0$

Step 3 On solving crisp linear programming problem, obtained in Step 2, the following three optimal solutions are obtained:

(i) $x_{12} = x_{23} = x_{36} = 1$ and $x_{34} = x_{35} = x_{45} = x_{56} = 0$

(ii) $x_{12} = x_{23} = x_{34} = x_{45} = x_{56} = 1$ and $x_{35} = x_{36} = 0$

(iii) $x_{12} = x_{23} = x_{35} = x_{56} = 1$ and $x_{34} = x_{36} = x_{45} = 0$

Step 4 Using the optimal values of x_{ij}, obtained from Step 3, the following three critical paths are obtained :

(i) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 6$

(ii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6$

(ii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6$

Putting the optimal values of x_{ij}, obtained from Step 3, in $((7, 7.1, 0.2, 2)_{LR}\, x_{12} \oplus (10, 10.2, 0.5, 2.5)_{LR}\, x_{23} \oplus (2, 3, 1.5, 2)_{LR}\, x_{34} \oplus (4, 7.7, 1, 2.5)_{LR}\, x_{35} \oplus (9, 9.7, 5.3, 6)_{LR}\, x_{36} \oplus (3, 3.7, 1.5, 2)_{LR}\, x_{45} \oplus (3, 4, 0.3, 0.5)_{LR}\, x_{56})$, the optimal fuzzy project completion times corresponding to paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 6$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6$ and $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6$ are $(26, 27, 6, 10.5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(24, 29, 2, 7.5)_{LR}$ respectively.

Since, more than one critical paths are obtained i.e., Case (ii) of Step 4 of proposed Mehar's method based on proposed Mehar's ranking approach is satisfied, so go to Step 5 of proposed Mehar's method based on proposed Mehar's ranking approach.

Step 5 Since, the fuzzy numbers $(26, 27, 6, 10.5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(24, 29, 2, 7.5)_{LR}$, representing the optimal fuzzy completion time of the project, are different i.e., Case

(ii) of Step 5 of proposed Mehar's method based on proposed Mehar's ranking approach is satisfied, so go to Step 6 of proposed Mehar's method based on proposed Mehar's ranking approach.

Step 6 Since, $u_1 = \Re(26, 27, 6, 10.5)_{LR} = \Re(25, 28, 4, 9)_{LR} = \Re(24, 29, 2, 7.5)_{LR} = \frac{57}{2}$. So, using Step 6 of the proposed Mehar's method based on proposed Mehar's ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(mode(7, 7.1, 0.2, 2)_{LR}\, x_{12} + mode(10, 10.2, 0.5, 2.5)_{LR}\, x_{23} + mode(2, 3, 1.5, 2)_{LR}\, x_{34} + mode(4, 7.7, 1, 2.5)_{LR}\, x_{35} + mode(9, 9.7, 5.3, 6)_{LR}\, x_{36} + mode(3, 3.7, 1.5, 2)_{LR}\, x_{45} + mode(3, 4, 0.3, 0.5)_{LR}\, x_{56})$

subject to

$\Re(7, 7.1, 0.2, 2)_{LR}\, x_{12} + \Re(10, 10.2, 0.5, 2.5)_{LR}\, x_{23} + \Re(2, 3, 1.5, 2)_{LR}\, x_{34} + \Re(4, 7.7, 1, 2.5)_{LR}\, x_{35} + \Re(9, 9.7, 5.3, 6)_{LR}\, x_{36} + \Re(3, 3.7, 1.5, 2)_{LR}\, x_{45} + \Re(3, 4, 0.3, 0.5)_{LR}\, x_{56} = \frac{57}{2}$,

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{34} - x_{35} - x_{36} = 0, \quad x_{34} - x_{45} = 0,$

$x_{35} + x_{45} - x_{56} = 0,\ x_{36} + x_{56} = 1,$

$x_{12},\ x_{23},\ x_{34},\ x_{35},\ x_{36},\ x_{45},\ x_{56} \geq 0$

On solving the crisp linear programming the following three optimal solutions are obtained:

(i) $x_{12} = x_{23} = x_{36} = 1$ and $x_{34} = x_{35} = x_{45} = x_{56} = 0$

(ii) $x_{12} = x_{23} = x_{34} = x_{45} = x_{56} = 1$ and $x_{35} = x_{36} = 0$

(iii) $x_{12} = x_{23} = x_{35} = x_{56} = 1$ and $x_{34} = x_{36} = x_{45} = 0$

Putting the obtained optimal values of x_{ij} in $((7, 7.1, 0.2, 2)_{LR}\, x_{12} \oplus (10, 10.2, 0.5, 2.5)_{LR}\, x_{23} \oplus (2, 3, 1.5, 2)_{LR}\, x_{34} \oplus (4, 7.7, 1, 2.5)_{LR}\, x_{35} \oplus (9, 9.7, 5.3, 6)_{LR}\, x_{36}$

$\oplus$ $(3, 3.7, 1.5, 2)_{LR}$ x_{45} $\oplus$ $(3, 4, 0.3, 0.5)_{LR}$ x_{56}), the obtained optimal fuzzy project completion times are $(26, 27, 6, 10.5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(24, 29, 2, 7.5)_{LR}$. Since $(26, 27, 6, 10.5)_{LR} \neq (25, 28, 4, 9)_{LR} \neq (24, 29, 2, 7.5)_{LR}$ i.e., Case (ii) of Step 6 of proposed Mehar's method based on proposed Mehar's ranking approach is satisfied, so go to Step 7 of proposed Mehar's method based on proposed Mehar's ranking approach.

Step 7 Since, $u_2 = mode(26, 27, 6, 10.5)_{LR} = mode(25, 28, 4, 9)_{LR} = mode(24, 29, 2, 7.5)_{LR} = \frac{53}{2}$. So, using Step 7 of the proposed Mehar's method based on proposed Mehar's ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(divergence(7, 7.1, 0.2, 2)_{LR}\ x_{12} + divergence(10, 10.2, 0.5, 2.5)_{LR}\ x_{23} + divergence(2, 3, 1.5, 2)_{LR}\ x_{34} + divergence(4, 7.7, 1, 2.5)_{LR}\ x_{35} + divergence(9, 9.7, 5.3, 6)_{LR}\ x_{36} + divergence(3, 3.7, 1.5, 2)_{LR}\ x_{45} + divergence(3, 4, 0.3, 0.5)_{LR}\ x_{56})$

subject to

$\Re(7, 7.1, 0.2, 2)_{LR}\, x_{12} + \Re(10, 10.2, 0.5, 2.5)_{LR}\, x_{23} + \Re(2, 3, 1.5, 2)_{LR}\, x_{34} + \Re(4, 7.7, 1, 2.5)_{LR} x_{35} + \Re(9, 9.7, 5.3, 6)_{LR}\ x_{36} + \Re(3, 3.7, 1.5, 2)_{LR}\ x_{45} + \Re(3, 4, 0.3, 0.5)_{LR}\ x_{56} = \frac{57}{2}$,

$mode(7, 7.1, 0.2, 2)_{LR}\ x_{12} + mode(10, 10.2, 0.5, 2.5)_{LR}\ x_{23} + mode(2, 3, 1.5, 2)_{LR}\ x_{34} + mode(4, 7.7, 1, 2.5)_{LR}\ x_{35} + mode(9, 9.7, 5.3, 6)_{LR}\ x_{36} + mode(3, 3.7, 1.5, 2)_{LR}\ x_{45} + mode(3, 4, 0.3, 0.5)_{LR}\ x_{56} = \frac{53}{2}$,

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{34} - x_{35} - x_{36} = 0, \quad x_{34} - x_{45} = 0,$

$x_{35} + x_{45} - x_{56} = 0,\ x_{36} + x_{56} = 1,$

$x_{12},\ x_{23},\ x_{34},\ x_{35},\ x_{36},\ x_{45},\ x_{56} \geq 0$

On solving the crisp linear programming the following three optimal solutions are obtained:

(i) $x_{12} = x_{23} = x_{36} = 1$ and $x_{34} = x_{35} = x_{45} = x_{56} = 0$

(ii) $x_{12} = x_{23} = x_{34} = x_{45} = x_{56} = 1$ and $x_{35} = x_{36} = 0$

(iii) $x_{12} = x_{23} = x_{35} = x_{56} = 1$ and $x_{34} = x_{36} = x_{45} = 0$

Putting the obtained optimal values of x_{ij} in $((7, 7.1, 0.2, 2)_{LR}\, x_{12} \oplus (10, 10.2, 0.5, 2.5)_{LR}\, x_{23} \oplus (2, 3, 1.5, 2)_{LR}\, x_{34} \oplus (4, 7.7, 1, 2.5)_{LR}\, x_{35} \oplus (9, 9.7, 5.3, 6)_{LR}\, x_{36} \oplus (3, 3.7, 1.5, 2)_{LR}\, x_{45} \oplus (3, 4, 0.3, 0.5)_{LR}\, x_{56})$, the obtained optimal fuzzy project completion times are $(26, 27, 6, 10.5)_{LR}$, $(25, 28, 4, 9)_{LR}$ and $(24, 29, 2, 7.5)_{LR}$. Since $(26, 27, 6, 10.5)_{LR} \neq (25, 28, 4, 9)_{LR} \neq (24, 29, 2, 7.5)_{LR}$ i.e., Case (ii) of Step 7 of proposed Mehar's method based on proposed Mehar's ranking approach is satisfied, so go to Step 8 of proposed Mehar's method based on proposed Mehar's ranking approach.

Step 8 Since, $u_3 = divergence(26, 27, 6, 10.5)_{LR} = divergence(25, 28, 4, 9)_{LR} = divergence(24, 29, 2, 7.5)_{LR} = 11$. So, using Step 8 of the proposed Mehar's method based on proposed Mehar's ranking approach there is need to find the optimal solution of the following crisp linear programming problem:

Maximize $(Left\ spread(7, 7.1, 0.2, 2)_{LR}\, x_{12} + Left\ spread(10, 10.2, 0.5, 2.5)_{LR}\, x_{23} + Left\ spread(2, 3, 1.5, 2)_{LR}\, x_{34} + Left\ spread(4, 7.7, 1, 2.5)_{LR}\, x_{35} + Left\ spread(9, 9.7, 5.3, 6)_{LR}\, x_{36} + Left\ spread(3, 3.7, 1.5, 2)_{LR}\, x_{45} + Left\ spread(3, 4, 0.3, 0.5)_{LR}\, x_{56})$

subject to

$\Re(7, 7.1, 0.2, 2)_{LR}\, x_{12} + \Re(10, 10.2, 0.5, 2.5)_{LR}\, x_{23} + \Re(2, 3, 1.5, 2)_{LR}\, x_{34} + \Re(4, 7.7, 1, 2.5)_{LR}\, x_{35} + \Re(9, 9.7, 5.3, 6)_{LR}\, x_{36} + \Re(3, 3.7, 1.5, 2)_{LR}\, x_{45} + \Re(3, 4, 0.3, 0.5)_{LR}\, x_{56} = \frac{57}{2}$,

$mode(7, 7.1, 0.2, 2)_{LR}\, x_{12} + mode(10, 10.2, 0.5, 2.5)_{LR}\, x_{23} + mode(2, 3, 1.5, 2)_{LR}\, x_{34}$

$+ \, mode(4, 7.7, 1, 2.5)_{LR} \, x_{35} + mode(9, 9.7, 5.3, 6)_{LR} \, x_{36} + mode(3, 3.7, 1.5, 2)_{LR} \, x_{45}$

$+ \, mode(3, 4, 0.3, 0.5)_{LR} \, x_{56} = \frac{53}{2},$

$divergence(7, 7.1, 0.2, 2)_{LR} \, x_{12} + divergence(10, 10.2, 0.5, 2.5)_{LR} \, x_{23} + divergence(2, 3, 1.5, 2)_{LR} \, x_{34} + divergence(4, 7.7, 1, 2.5)_{LR} \, x_{35} + divergence(9, 9.7, 5.3, 6)_{LR} \, x_{36} + divergence(3, 3.7, 1.5, 2)_{LR} \, x_{45} + divergence(3, 4, 0.3, 0.5)_{LR} \, x_{56} = 11,$

$x_{12} = 1, \quad x_{12} - x_{23} = 0, \quad x_{23} - x_{34} - x_{35} - x_{36} = 0, \quad x_{34} - x_{45} = 0,$

$x_{35} + x_{45} - x_{56} = 0, \; x_{36} + x_{56} = 1,$

$x_{12}, \, x_{23}, \, x_{34}, \, x_{35}, \, x_{36}, \, x_{45}, \, x_{56} \geq 0$

On solving the crisp linear programming the following optimal solution is obtained:

$x_{12} = x_{23} = x_{36} = 1$ and $x_{34} = x_{35} = x_{45} = x_{56} = 0.$

Putting the obtained optimal values of x_{ij} in $((5, 13, 2, 2)_{LR} \, x_{12} \oplus (10, 10.2, 0.5, 2.5)_{LR} \, x_{23} \oplus (2, 3, 1.5, 2)_{LR} \, x_{34} \oplus (4, 7.7, 1, 2.5)_{LR} \, x_{35} \oplus (9, 9.7, 5.3, 6)_{LR} \, x_{36} \oplus (3, 3.7, 1.5, 2)_{LR} \, x_{45} \oplus (3, 4, 0.3, 0.5)_{LR} \, x_{56})$ a unique LR flat fuzzy number $(26, 27, 6, 10.5)_{LR}$, representing the optimal fuzzy project completion time, is obtained and using the same values of x_{ij} the obtained critical path is $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 6$.

6.6.2 Optimal fuzzy completion time of the chosen problem by using the modified Liang and Han method based on proposed Mehar's ranking approach

In the previous chapter it is pointed out that proposed Mehar's subtraction is not defined for such LR flat fuzzy numbers for which $L(x) \neq R(x)$. Since, in the chosen project network problem the time of each activity is represented by such an LR flat fuzzy number for which $L(x) \neq R(x)$, so this problem can not be solved by using the modified Liang and Han method based on proposed Mehar's ranking approach.

6.7 Comparative study

The results of fuzzy project network problems, chosen in Example 2.4, Example 3.1, Example 4.1, Example 5.1, Example 5.2 and Example 6.1, obtained by using the methods, proposed in previous chapters and the methods proposed in this chapter, are shown in Table 6.3

Table 6.3 Results of chosen problems obtained using the methods proposed in Chapter 4 and methods proposed in this chapter

Example	Proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach		Modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach		Modified Liang and Han method based on proposed Mehar's ranking approach		Proposed Mehar's method based on proposed Mehar's ranking approach	
	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time	Critical path	Optimal fuzzy project completion time
2.4	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(14, 25, 39)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(14, 25, 39)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(14, 25, 39)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(14, 25, 39)$
3.1	$1 \Rightarrow 2 \Rightarrow 4$	$(14, 23, 32)$	$1 \Rightarrow 2 \Rightarrow 4$	$(14, 23, 32)$	$1 \Rightarrow 2 \Rightarrow 4$	$(14, 23, 32)$	$1 \Rightarrow 2 \Rightarrow 4$	$(14, 23, 32)$
4.1	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(6, 12, 16, 22)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(6, 12, 16, 22)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(6, 12, 16, 22)$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(6, 12, 16, 22)$
5.1	Not applicable	Not applicable	Not applicable	Not applicable	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(18, 23, 4, 3)_{LR}$	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(18, 23, 4, 3)_{LR}$
5.2	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 7 \Rightarrow 8$	$(25, 32, 4, 5)_{LR}$
6.1	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	Not applicable	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 6$	$(26, 27, 6, 10.5)_{LR}$

The results, presented in Table 6.3, can be explained as follows:

1. The proposed Mehar's method based on proposed extension of Kaufmann and Gupta ranking approach and modified Liang and Han method based on proposed extension of Kaufmann and Gupta ranking approach can be used only to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by either a triangular fuzzy number or a trapezoidal fuzzy number. Since, in the problems, chosen in Example 2.4, Example 3.1 and Example 4.1, time of each activity is represented by either a triangular fuzzy number or a trapezoidal fuzzy number so the optimal fuzzy project completion time of all these problems can be obtained by these

methods. However, in the problems, chosen in Example 5.1, Example 5.2 and Example 6.1, time of each activity is represented by an LR flat fuzzy number so the optimal fuzzy project completion time of these problems can not be obtained by these methods.

2. The modified Liang and Han method based on proposed Mehar's ranking approach can be used to find the optimal fuzzy project completion time of such project network problems in which time of each activity is represented by such an LR flat fuzzy number for which $L(x) = R(x)$. Since, in the problems chosen in Example 2.4, Example 3.1, Example 4.1 and Example 5.1, time of each activity is represented by such an LR flat fuzzy number for which $L(x) = R(x)$ so the optimal fuzzy project completion time of all these problems can be obtained by this method. However, in the problems, chosen in Example 5.2 and Example 6.1, time of each activity is represented by such an LR flat fuzzy number for which $L(x) \neq R(x)$ so the optimal fuzzy project completion time of these problems can not be obtained by this method.

3. The proposed Mehar's method based on proposed Mehar's ranking approach can be used to find the optimal fuzzy project completion time of such project network problems in which time of each activity is represented by such an LR flat fuzzy number for which $L(x) = R(x)$ or such an LR flat fuzzy number for which $L(x) \neq R(x)$ so the optimal fuzzy project completion time of all the problems, chosen in Example 2.4, Example 3.1, Example 4.1, Example 5.1, Example 5.2 and Example 6.1, can be obtained by this method.

6.8 Conclusions

On the basis of the presented study, it can be concluded that it is better to use the methods proposed in this chapter as compared to the existing methods and the methods proposed in previous chapters to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number. Also, it can be concluded that it is better to use the proposed Mehar's method based on proposed Mehar's ranking approach as compared to the modified Liang and Han method based on proposed Mehar's ranking approach to find the optimal fuzzy completion time of such project network problems in which time of each activity is represented by an LR flat fuzzy number.

Chapter 7

A New Method For Finding The Unique Optimal Fuzzy Crashing Cost Corresponding To Specific Fuzzy Project Completion Time

In many situations, there is need to complete the project in a fuzzy time which is less than the optimal initial fuzzy project completion time. To handle such situations, Chen and Tsai [30] proposed a method for finding the minimum fuzzy crashing cost (additional fuzzy cost) for completing the project within specific fuzzy time which is less than the optimal initial fuzzy project completion time. In this chapter, the shortcomings of this method are pointed out and to overcome these shortcomings a new method, named as JMD (JAI MATA (MEHAR) DI) method, is proposed. The advantages of the proposed method over the existing method [30] is discussed. Also, a new representation of LR flat fuzzy numbers, named as JMD representation of LR flat fuzzy numbers, is proposed and shown that it is better to represent the parameters of any fuzzy linear programming problem by JMD LR flat fuzzy numbers as compared to the existing representation of LR flat fuzzy numbers.

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7.1 Chen and Tsai method

Chen and Tsai [30] proposed the following method for finding the minimum fuzzy crashing cost ($\widetilde{TC}$), for completing the project within specific fuzzy time ($\widetilde{PCT}$) which is less than the optimal initial fuzzy project completion time ($\widetilde{NT_n}$):

Step 1 Check that $(CT_{ij})^U_{\alpha=0} \leq (NT_{ij})^L_{\alpha=0}$ $\forall$ $(i,j) \in A$ and $\sum\limits_{(i,j)\in A}(CT^{CP}_{ij})^U_{\alpha=0} \leq (PCT)^L_{\alpha=0}$.

where, A is the set of activities (i,j),

$(CT_{ij})^U_{\alpha=0}$ is the upper bound of the α-cut of fuzzy crash time $\widetilde{CT_{ij}}$ of the activity (i,j) corresponding to $\alpha = 0$,

$(NT_{ij})^L_{\alpha=0}$ is the lower bound of the α-cut of fuzzy normal time $\widetilde{NT_{ij}}$ of the activity (i,j) corresponding to $\alpha = 0$,

$(CT^{CP}_{ij})^U_{\alpha=0}$ is the upper bound of the α-cut of fuzzy crash time of the activity (i,j) on the critical path corresponding to $\alpha = 0$,

$(PCT)^L_{\alpha=0}$ is the lower bound of the α-cut of specified fuzzy project completion time $\widetilde{PCT}$ corresponding to $\alpha = 0$.

Case (i) If $(CT_{ij})^U_{\alpha=0} \leq (NT_{ij})^L_{\alpha=0}$ $\forall$ $(i,j) \in A$ and $\sum\limits_{(i,j)\in A}(CT^{CP}_{ij})^U_{\alpha=0} \leq (PCT)^L_{\alpha=0}$ then the chosen problem is feasible and go to Step 2.

Case (ii) If there exist atleast one activity (i,j) such that $(CT_{ij})^U_{\alpha=0} > (NT_{ij})^L_{\alpha=0}$ or $\sum\limits_{(i,j)\in A}(CT^{CP}_{ij})^U_{\alpha=0} > (PCT)^L_{\alpha=0}$ then the chosen problem is infeasible.

Step 2 Formulate the chosen problem into fuzzy linear programming problem ($P_{7.1}$):

Minimize $\sum\limits_{(i,j)\in A}(\widetilde{C_{ij}} \otimes (\widetilde{NT_{ij}} - x_{ij}))$

subject to

$$x_{ij} \geq \widetilde{CT}_{ij},$$

$$x_{ij} \leq \widetilde{NT}_{ij}, \qquad (P_{7.1})$$

$$T_n \leq \widetilde{PCT},$$

$$T_i + x_{ij} - T_j \leq 0,$$

$$T_j, x_{ij} \geq 0,\ i = 1, 2, ..., n,\ j = 1, 2, ..., n,\ (i, j) \in A$$

where,

T_i: Earliest time for the node i,

T_j: Earliest time for the node j,

$\widetilde{CT}_{ij}$: Fuzzy crash time of the activity (i, j),

$\widetilde{NT}_{ij}$: Fuzzy normal time of the activity (i, j),

$\widetilde{C}_{ij}$: Incremental direct cost per unit decrease in the activity time,

$\widetilde{PCT}$: Specified fuzzy project completion time,

x_{ij}: Time for the activity (i, j).

Step 3 Since the term $\sum_{(i,j)\in A} (\widetilde{C}_{ij} \otimes \widetilde{NT}_{ij})$ is a non variable constant, so the minimization of objective function of the problem $(P_{7.1})$ is equivalent to Maximize $\sum_{(i,j)\in A} (\widetilde{C}_{ij}\ x_{ij})$ i.e., the fuzzy optimal value of the fuzzy linear programming problem $(P_{7.1})$ can be obtained by subtracting the fuzzy optimal value of fuzzy linear programming problem $(P_{7.2})$ from $\sum_{(i,j)\in A} (\widetilde{C}_{ij} \otimes \widetilde{NT}_{ij})$

Maximize $\sum_{(i,j)\in A} (\widetilde{C}_{ij}\ x_{ij})$

subject to $\qquad (P_{7.2})$

Constraints of $(P_{7.1})$

Step 4 Solve the problem $(P_{7.3})$ to find the lower bound of the α-cut of optimal values of x_{ij} and $\sum_{(i,j)\in A} (\widetilde{C_{ij}}\ x_{ij})$ corresponding to some values of $\alpha \in [0,\ 1]$.

$$\underset{\substack{(C_{ij})^L_\alpha \le c_{ij} \le (C_{ij})^U_\alpha \\ (NT_{ij})^L_\alpha \le nt_{ij} \le (NT_{ij})^U_\alpha \\ (CT_{ij})^L_\alpha \le ct_{ij} \le (CT_{ij})^U_\alpha \\ (PCT)^L_\alpha \le pct \le (PCT)^U_\alpha \\ ct_{ij} \le nt_{ij} \\ \sum_{(i,j)\in A} ct^{CP}_{ij} \le pct\ \forall (i,j)\in A}}{\text{Minimize}} \begin{cases} \text{Maximize} \quad \sum_{(i,j)\in A} (c_{ij} x_{ij}) \\ \text{subject to} \\ x_{ij} \ge ct_{ij}, \\ x_{ij} \le nt_{ij}, \\ T_i + x_{ij} - T_j \le 0, \\ T_n \le pct, \\ T_j, x_{ij} \ge 0, i = 1, 2, ..., n,\ j = 1, 2, ..., n, (i,j) \in A \end{cases} \quad (P_{7.3})$$

Step 5 Solve the problem $(P_{7.4})$ to find the upper bound of the α-cut of optimal values of x_{ij} and $\sum_{(i,j)\in A} (\widetilde{C_{ij}}\ x_{ij})$ corresponding to same values of $\alpha \in [0,\ 1]$ for which the lower bounds are calculated in Step 4.

$$\underset{\substack{(C_{ij})^L_\alpha \le c_{ij} \le (C_{ij})^U_\alpha \\ (NT_{ij})^L_\alpha \le nt_{ij} \le (NT_{ij})^U_\alpha \\ (CT_{ij})^L_\alpha \le ct_{ij} \le (CT_{ij})^U_\alpha \\ (PCT)^L_\alpha \le pct \le (PCT)^U_\alpha \\ ct_{ij} \le nt_{ij} \\ \sum_{(i,j)\in A} ct^{CP}_{ij} \le pct\ \forall (i,j)\in A}}{\text{Maximize}} \begin{cases} \text{Maximize} \quad \sum_{(i,j)\in A} (c_{ij} x_{ij}) \\ \text{subject to} \\ x_{ij} \ge ct_{ij}, \\ x_{ij} \le nt_{ij}, \\ T_i + x_{ij} - T_j \le 0, \\ T_n \le pct, \\ T_j, x_{ij} \ge 0, i = 1, 2, ..., n,\ j = 1, 2, ..., n, (i,j) \in A \end{cases} \quad (P_{7.4})$$

Step 6 Put the lower and upper bounds of the α-cut of fuzzy optimal value of $\sum_{(i,j)\in A} (\widetilde{C_{ij}}\ x_{ij})$, say Z^L_α and Z^U_α, obtained from Step 4 and Step 5, in $(TC)^L_\alpha = \sum_{(i,j)\in A} ((C_{ij})^L_\alpha\ (NT_{ij})^L_\alpha) - Z^L_\alpha$ and $(TC)^U_\alpha = \sum_{(i,j)\in A} ((C_{ij})^U_\alpha\ (NT_{ij})^U_\alpha) - Z^U_\alpha$ respectively to find the lower and upper bounds of the α-cut of minimum fuzzy crashing cost $\widetilde{TC}$.

Step 7 Use the values of lower and upper bounds of the α-cut of fuzzy optimal values of x_{ij}, obtained from Step 4 and Step 5, to obtain the optimal fuzzy time $\widetilde{x_{ij}}$ of the activity (i, j).

Step 8 Use the values of $(TC)^L_\alpha$ and $(TC)^U_\alpha$, obtained from Step 6, to obtain the minimum fuzzy crashing cost $\widetilde{TC}$.

7.2 Shortcomings of Chen and Tsai method

It is not genuine to apply the existing method [30] due to the following reasons:

1. To point out one of the shortcomings of the existing method [30], the graphical representation of α-cut of an LR fuzzy number is shown in Figure 7.1 and the results of an existing problem [30, Example 1, pp. 393] are shown in Table 7.1.

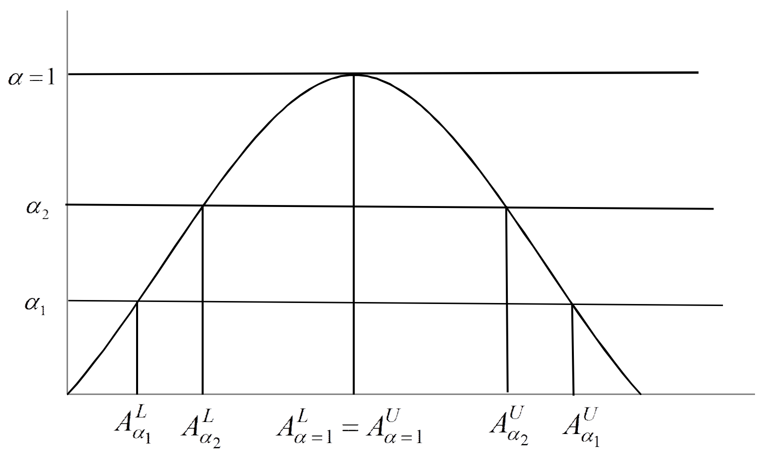

Figure 7.1 Lower and upper bounds of α - cut of an *LR* fuzzy number $\widetilde{A}$

Table 7.1 The lower and upper bounds of the optimal fuzzy activity time of the existing problem [30]

α	x^*_{ij}	x^*_{12}	x^*_{13}	x^*_{25}	x^*_{24}	x^*_{34}	x^*_{45}	x^*_{56}
1.0	Upper	6	12	14	6	2	6	8
	Lower	6	12	14	6	2	6	8
0.9	Upper	5.8	12.1	14.6	6.2	2.5	5.8	7.8
	Lower	6	11.8	14	5.9	2.1	6.1	8
0.8	Upper	5.6	12.2	15.2	6.4	3	5.6	7.6
	Lower	6	11.6	14	5.8	2.2	6.2	8
0.7	Upper	5.4	12.3	15.8	6.6	3.5	5.4	7.4
	Lower	6	11.4	14	5.7	2.3	6.3	8
0.6	Upper	5.2	12.4	16.4	6.8	4	5.2	7.2
	Lower	6	11.2	14	5.6	2.4	6.4	8
0.5	Upper	5	12.5	17	7	4.5	5	7
	Lower	6	11	14	5.5	2.5	6.5	8
0.4	Upper	4.8	12.6	17.6	7.2	4.6	5.2	6.8
	Lower	6	10.8	14	5.4	2.6	6.6	8
0.3	Upper	4.6	12.7	18.2	7.4	4.7	5.4	6.6
	Lower	6	10.6	14	5.3	2.7	6.7	8
0.2	Upper	4.4	12.8	18.8	7.6	4.8	5.6	6.4
	Lower	6	10.4	14	5.2	2.8	6.8	8
0.1	Upper	4.7	12.9	18.9	7.8	4.9	5.8	6.2
	Lower	6	10.2	14	5.1	2.9	6.9	8
0	Upper	5	13	19	8	5	6	6
	Lower	6	10	14	5	3	7	8

It is obvious from the graphical representation of the α-cuts of an LR fuzzy number $\widetilde{A}$, shown in Figure 7.1, that if $A^L_{\alpha_1}$, $A^U_{\alpha_1}$ and $A^L_{\alpha_2}$, $A^U_{\alpha_2}$ are the lower and upper bounds of the α-cut of an LR fuzzy number $\widetilde{A}$ corresponding to two different values of α, say α_1 and α_2 respectively then the following conditions should always be satisfied:

(i) $A^L_{\alpha_1} \leq A^U_{\alpha_1}\ \forall\ \alpha_1 \in [0,\ 1]$

(ii) $A^L_{\alpha_2} \leq A^U_{\alpha_2}\ \forall\ \alpha_2 \in [0,\ 1]$

(iii) $A^L_{\alpha_1} \leq A^L_{\alpha_2}$ and $A^U_{\alpha_1} \geq A^U_{\alpha_2}\ \forall\ 0 \leq \alpha_1 < \alpha_2 \leq 1$

(iv) $A^L_{\alpha} \leq A^L_{\alpha=1} = A^U_{\alpha=1} \leq A^U_{\alpha}\ \forall\ \alpha \in [0, 1]$

However, it can be easily seen from the existing results, shown in Table 7.1, that

(i) For the lower and upper bounds of an α-cut of x_{12}, x_{45} and x_{56}, the condition $A_\alpha^L \leq A_\alpha^U \ \forall \ \alpha = 0, 0.1, 0.2, ..., 0.9$ is not satisfying.

(ii) For the lower bounds of α-cut of x_{34} and x_{45}, the condition $A_{\alpha_1}^L \leq A_{\alpha_2}^L \ \forall \ 0 \leq \alpha_1 < \alpha_2 \leq 1$ is not satisfying.

(iii) For the upper bounds of α-cut of x_{56}, the condition $A_{\alpha_1}^U \geq A_{\alpha_2}^U \ \forall \ 0 \leq \alpha_1 < \alpha_2 \leq 1$ is not satisfying.

(iv) For the lower bounds of α-cut of x_{34} and x_{45}, the condition $A_\alpha^L \leq A_{\alpha=1}^L \ \forall \ \alpha \in [0, 1]$ and for the upper bounds of α-cut of x_{45} and x_{56}, the condition $A_{\alpha=1}^U \leq A_\alpha^U \ \forall \ \alpha \in [0, 1]$ is not satisfying.

So, by using the lower and upper bounds of α-cut of x_{12}, x_{34}, x_{45} and x_{56}, obtained by using the existing method [30], LR fuzzy numbers, representing the optimal fuzzy activity times x_{12}, x_{34}, x_{45} and x_{56}, can not be obtained.

2. If A is any real number then the values of lower bound and upper bound of the α-cut of A will also be A for all values of $\alpha \in [0, 1]$. But, it can be easily seen from the existing results, shown in Table 7.1, that the values of lower and upper bound of the α-cut of x_{ij} are not same i.e., x_{ij} should be represented by a fuzzy number. Similarly, the parameters T_j should also be represented by fuzzy numbers. While, it can be easily seen that in the existing formulation ($P_{7.1}$), used by Chen and Tsai [30] for finding the minimum fuzzy crashing cost for completing the project within specific fuzzy time, these parameters are represented by crisp numbers which is not genuine.

3. Chen and Tsai [30] have pointed out that since in fuzzy linear programming problem ($P_{7.1}$) the quantity $\sum\limits_{(i,j)\in A} (\widetilde{C_{ij}} \otimes \widetilde{NT_{ij}})$ is a constant so the fuzzy

optimal value of fuzzy linear programming problem ($P_{7.1}$) can be obtained by subtracting the fuzzy optimal value of the fuzzy linear programming problem ($P_{7.2}$) i.e., $\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes x_{ij})$ from $\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes \widetilde{NT_{ij}})$ i.e., Chen and Tsai [30] have assumed that $\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes (\widetilde{NT_{ij}} \ominus x_{ij}))$ can be written as $\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes \widetilde{NT_{ij}}) \ominus \sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes x_{ij})$. Since, $\widetilde{x_{ij}}$ is a fuzzy number and in the literature [43] it is pointed out that for fuzzy numbers $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ the distributive property $\widetilde{A} \otimes (\widetilde{B} \ominus \widetilde{C}) = \widetilde{A} \otimes \widetilde{B} \ominus \widetilde{A} \otimes \widetilde{C}$ is not necessarily satisfied. So, $\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes (\widetilde{NT_{ij}} \ominus x_{ij})) \neq \sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes \widetilde{NT_{ij}}) \ominus \sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes x_{ij})$ i.e., it is not genuine to use existing fuzzy linear programming problem ($P_{7.1}$) in its present form for finding the minimum fuzzy crashing cost for completing the project within specific fuzzy time.

7.3 Drawbacks of using other existing methods

Although, Chen and Tsai [30] have used the method based on Zadeh's extension principle [182] for solving fuzzy linear programming problem ($P_{7.1}$) but there are several other methods in the literature for solving fuzzy linear programming problems. So, due to the shortcomings of the existing method [30], pointed out in Section 7.2, one may try to use other existing methods for solving fuzzy linear programming problem ($P_{7.1}$).

To find the fuzzy optimal solution of fuzzy linear programming problem by using other existing methods, firstly it is converted into the crisp linear programming problem and then the optimal solution of obtained crisp linear programming problem is used to find the fuzzy optimal solution of fuzzy linear programming problem. To convert any fuzzy linear programming problem into the crisp linear programming problem there is need to convert fuzzy inequality constraints into crisp inequality constraints and also there is need to find the minimum (or maximum) of fuzzy numbers.

In this section, the methods, used in the existing methods [7, 44–49, 53, 98–103, 114–122], for converting the fuzzy inequality constraints into crisp inequality constraints and to find the minimum (maximum) of fuzzy numbers are presented. Also, drawbacks of using these methods as well as Mehar's ranking approach, proposed in previous chapter for finding the minimum and maximum of fuzzy numbers, are pointed out.

7.3.1 Existing methods for converting fuzzy inequality constraints into crisp inequality constraints

In this section, the methods, used in the existing methods [7,44–49,53,98–103, 114–122], for converting fuzzy inequality constraints into crisp inequality constraints are presented.

7.3.1.1 Fan et al. method

Fan et al. [53] have used the following method to find LR flat fuzzy number $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$.

Step 1 Assuming $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$, $(\underline{c}, \overline{c}, c^L, c^R)_{LR}$ and $(\underline{x}, \overline{x}, x^L, x^R)_{LR}$ as non-negative LR flat fuzzy numbers and using arithmetic operations, defined in Section 5.1.2, the fuzzy inequality constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$ can be converted into fuzzy inequality constraint $(\underline{a}\,\underline{x} + \underline{b}, \overline{a}\,\overline{x} + \overline{b}, \underline{a}x^L + \underline{x}a^L - a^L x^L + b^L, \overline{a}x^R + \overline{x}a^R + a^R x^R + b^R)_{LR}$

$\preceq (\underline{c}\ \underline{x} + \underline{d}, \overline{c}\ \overline{x} + \overline{d}, \underline{c}x^L + \underline{x}c^L - c^L x^L + d^L, \overline{c}x^R + \overline{x}c^R + c^R x^R + d^R)_{LR}$.

Step 2 Convert the fuzzy inequality constraint $(\underline{a}\ \underline{x} + \underline{b}, \overline{a}\ \overline{x} + \overline{b}, \underline{a}x^L + \underline{x}a^L - a^L x^L + b^L, \overline{a}x^R + \overline{x}a^R + a^R x^R + b^R)_{LR} \preceq (\underline{c}\ \underline{x} + \underline{d}, \overline{c}\ \overline{x} + \overline{d}, \underline{c}x^L + \underline{x}c^L - c^L x^L + d^L, \overline{c}x^R + \overline{x}c^R + c^R x^R + d^R)_{LR}$ into the following four crisp inequality constraints:

$\underline{a}\ \underline{x} + \underline{b} \leq \underline{c}\ \underline{x} + \underline{d}$, $\overline{a}\ \overline{x} + \overline{b} \leq \overline{c}\ \overline{x} + \overline{d}$, $\underline{a}x^L + \underline{x}a^L - a^L x^L + b^L \leq \underline{c}x^L + \underline{x}c^L - c^L x^L + d^L$, $\overline{a}x^R + \overline{x}a^R + a^R x^R + b^R \leq \overline{c}x^R + \overline{x}c^R + c^R x^R + d^R$.

Step 3 Find the values of $\underline{x}, \overline{x}, x^L$ and x^R satisfying the crisp inequality constraints $\underline{a}\ \underline{x} + \underline{b} \leq \underline{c}\ \underline{x} + \underline{d}$, $\overline{a}\ \overline{x} + \overline{b} \leq \overline{c}\ \overline{x} + \overline{d}$, $\underline{a}x^L + \underline{x}a^L - a^L x^L + b^L \leq \underline{c}x^L + \underline{x}c^L - c^L x^L + d^L$, $\overline{a}x^R + \overline{x}a^R + a^R x^R + b^R \leq \overline{c}x^R + \overline{x}c^R + c^R x^R + d^R$ as well as restrictions of non-negative LR flat fuzzy numbers $\underline{x} - x^L \geq 0$, $\overline{x} - \underline{x} \geq 0$, $x^L \geq 0$, $x^R \geq 0$.

Step 4 Use the values of $\underline{x}, \overline{x}, x^L$ and x^R, obtained from Step 3, to find the non-negative LR flat fuzzy number $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$.

7.3.1.2 Other existing methods

In all the existing methods [7, 44–49, 98–103, 114–122] the following method is used to find an LR flat fuzzy number $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$.

Step 1 Assuming $(\underline{a}, \overline{a}, a^L, a^R)_{LR}, (\underline{c}, \overline{c}, c^L, c^R)_{LR}$ and $(\underline{x}, \overline{x}, x^L, x^R)_{LR}$ as non-negative LR flat fuzzy numbers and using arithmetic operations, defined in Section 5.1.2, the fuzzy inequality constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$ can be converted into fuzzy inequality constraint $(\underline{a}\ \underline{x} + \underline{b}, \overline{a}\ \overline{x} + \overline{b}, \underline{a}x^L + \underline{x}a^L - a^L x^L + b^L, \overline{a}x^R + \overline{x}a^R + a^R x^R + b^R)_{LR}$

$\preceq (\underline{c}\ \underline{x} + \underline{d}, \overline{c}\ \overline{x} + \overline{d}, \underline{c}x^L + \underline{x}c^L - c^Lx^L + d^L, \overline{c}x^R + \overline{x}c^R + c^Rx^R + d^R)_{LR}$.

Step 2 Convert the fuzzy inequality constraint $(\underline{a}\ \underline{x} + \underline{b}, \overline{a}\ \overline{x} + \overline{b}, \underline{a}x^L + \underline{x}a^L - a^Lx^L + b^L, \overline{a}x^R + \overline{x}a^R + a^Rx^R + b^R)_{LR} \preceq (\underline{c}\ \underline{x} + \underline{d}, \overline{c}\ \overline{x} + \overline{d}, \underline{c}x^L + \underline{x}c^L - c^Lx^L + d^L, \overline{c}x^R + \overline{x}c^R + c^Rx^R + d^R)_{LR}$ into the following crisp inequality constraint:

$\frac{1}{2}[\underline{a}\ \underline{x} + \underline{b} + \overline{a}\ \overline{x} + \overline{b}] + \frac{1}{4}[(\overline{a}x^R + \overline{x}a^R + a^Rx^R + b^R) - (\underline{a}x^L + \underline{x}a^L - a^Lx^L + b^L)] \leq \frac{1}{2}[\underline{c}\ \underline{x} + \underline{d} + \overline{c}\ \overline{x} + \overline{d}] + \frac{1}{4}[(\overline{c}x^R + \overline{x}c^R + c^Rx^R + d^R) - (\underline{c}x^L + \underline{x}c^L - c^Lx^L + d^L)]$

Step 3 Find the values of $\underline{x}, \overline{x}, x^L$ and x^R satisfying the crisp inequality constraint $\frac{1}{2}[\underline{a}\ \underline{x} + \underline{b} + \overline{a}\ \overline{x} + \overline{b}] + \frac{1}{4}[(\overline{a}x^R + \overline{x}a^R + a^Rx^R + b^R) - (\underline{a}x^L + \underline{x}a^L - a^Lx^L + b^L)] \leq \frac{1}{2}[\underline{c}\ \underline{x} + \underline{d} + \overline{c}\ \overline{x} + \overline{d}] + \frac{1}{4}[(\overline{c}x^R + \overline{x}c^R + c^Rx^R + d^R) - (\underline{c}x^L + \underline{x}c^L - c^Lx^L + d^L)]$ as well as restrictions of non-negative LR flat fuzzy numbers $\underline{x} - x^L \geq 0$, $\overline{x} - \underline{x} \geq 0$, $x^L \geq 0$, $x^R \geq 0$.

Step 4 Use the values of $\underline{x}, \overline{x}, x^L$ and x^R, obtained from Step 3, to find the non-negative LR flat fuzzy number $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$.

7.3.2 Existing methods for finding the minimum and maximum of fuzzy numbers

In this section, the methods, used in the existing methods [7,44–49,53,98–103, 114–122], for finding the minimum and maximum of fuzzy numbers are presented.

7.3.2.1 Fan et al. method

Fan et al. [53] have used the following method to find the minimum (or maximum) of n LR flat fuzzy numbers $(\underline{a_i}, \overline{a_i}, a_i^L, a_i^R)_{LR}$; $i = 1, 2, ..., n$.

Step 1 Find $\Re(\underline{a_i}, \overline{a_i}, a_i^L, a_i^R)_{LR} = \frac{1}{3}[\int_0^1 \frac{1}{2}(\underline{a_i} + \overline{a_i})d\lambda] + \frac{2}{3}[\frac{1}{2}\int_0^1(\underline{a_i} - a_i^L L^{-1}(\lambda))d\lambda + \frac{1}{2}\int_0^1(\overline{a_i} + a_i^R R^{-1}(\lambda))d\lambda]$.

Step 2 Find $\underset{1\leq i\leq n}{\text{minimum}}\{\Re(\underline{a_i}, \overline{a_i}, a_i^L, a_i^R)_{LR}\}$.

If $\underset{1\le i\le n}{\text{minimum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \Re\big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$ then

$\underset{1\le i\le n}{\text{minimum}}\{(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$.

Similarly, find $\underset{1\le i\le n}{\text{maximum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\}$.

If $\underset{1\le i\le n}{\text{maximum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \Re\big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$ then

$\underset{1\le i\le n}{\text{maximum}}\{(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$.

7.3.2.2 Other existing methods

In all existing methods [7,44–49,98–103,114–122], the following method is used to find the minimum (or maximum) of n LR flat fuzzy numbers $(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}$; $i=1,2,...,n$.

Step 1 Find $\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR} = \frac{1}{2}[\int_0^1(\underline{a_i} - a_i^L L^{-1}(\lambda))d\lambda + \int_0^1(\overline{a_i} + a_i^R R^{-1}(\lambda))d\lambda]$.

Step 2 Find $\underset{1\le i\le n}{\text{minimum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\}$.

If $\underset{1\le i\le n}{\text{minimum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \Re\big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$ then

$\underset{1\le i\le n}{\text{minimum}}\{(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$.

Similarly, find $\underset{1\le i\le n}{\text{maximum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\}$.

If $\underset{1\le i\le n}{\text{maximum}}\{\Re(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \Re\big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$ then

$\underset{1\le i\le n}{\text{maximum}}\{(\underline{a_i},\overline{a_i},a_i^L,a_i^R)_{LR}\} = \big((\underline{a})^\theta,(\overline{a})^\theta,(a^L)^\theta,(a^R)^\theta\big)_{LR}$.

7.3.3 Drawbacks of using existing methods for converting the fuzzy inequality constraints into crisp inequality constraints

For all values of x which satisfies the crisp constraint $ax+b \le cx+d$, there always exist a non-negative real number s such that $ax+b+s = cx+d$. On the same direction for all values of $\widetilde{x} = (\underline{x},\overline{x},x^L,x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$ there should exist a non-negative LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$. However, the values of $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$, obtained by using the methods, presented in Section 7.3.1, satisfies the fuzzy inequality constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$, but it is not always possible to find a non-negative LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$.

To show this drawback of the existing methods, one non-negative value of $\widetilde{x}$ which will satisfy the fuzzy inequality constraint $(2, 3, 1, 1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5, 8, 1, 5)_{LR} \preceq (4, 6, 2, 2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1, 2, 1, 1)_{LR}$, where $L(x) = R(x) = \max\{0, 1 - x\}$, is obtained by the methods, presented in Section 7.3.1, and it is shown that there does not exist any non-negative LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(2, 3, 1, 1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5, 8, 1, 5)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (4, 6, 2, 2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1, 2, 1, 1)_{LR}$.

7.3.3.1 Fuzzy solution by using Fan et al. method

Using Fan et al. method, discussed in Section 7.3.1.1, the non-negative values of $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(2, 3, 1, 1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5, 8, 1, 5)_{LR} \preceq (4, 6, 2, 2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1, 2, 1, 1)_{LR}$ can be obtained as follows:

Step 1 Using the arithmetic operations, defined in Section 5.1.2, the fuzzy inequality constraint $(2, 3, 1, 1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5, 8, 1, 5)_{LR} \preceq (4, 6, 2, 2)_{LR} \otimes (\underline{x}, \overline{x},$

$x^L, x^R)_{LR} \oplus (1,2,1,1)_{LR}$ can be converted into the fuzzy inequality constraint $(2\underline{x}+5, 3\overline{x}+8, \underline{x}+x^L+1, \overline{x}+4x^R+5)_{LR} \preceq (4\underline{x}+1, 6\overline{x}+2, 2\underline{x}+2x^L+1, 2\overline{x}+8x^R+1)_{LR}$

Step 2 Using Step 2 of the method, presented in Section 7.3.1.1, the fuzzy inequality constraint, obtained in Step 1, can be converted into the following four crisp inequality constraints:

$2\underline{x}+5 \leq 4\underline{x}+1,\ 3\overline{x}+8 \leq 6\overline{x}+2,\ \underline{x}+x^L+1 \leq 2\underline{x}+2x^L+1,\ \overline{x}+4x^R+5 \leq 2\overline{x}+8x^R+1$

i.e., $\underline{x} \geq 2$, $\overline{x} \geq 2$, $\underline{x}+x^L \geq 0$, $\overline{x}+4x^R \geq 4$

Step 3 There will be infinite values of $\underline{x}, \overline{x}, x^L$ and x^R which will satisfy the crisp inequality constraints $\underline{x} \geq 2$, $\overline{x} \geq 2$, $\underline{x}+x^L \geq 0$, $\overline{x}+4x^R \geq 4$ as well as restrictions of non-negative LR flat fuzzy numbers $\underline{x}-x^L \geq 0$, $\overline{x}-\underline{x} \geq 0$, $x^L \geq 0$, $x^R \geq 0$. One of these values are 3, 4, 1 and 2 respectively.

Step 4 Using the values of $\underline{x}, \overline{x}, x^L$ and x^R, obtained from Step 3, the non-negative value of $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(2,3,1,1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5,8,1,5)_{LR} \preceq (4,6,2,2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1,2,1,1)_{LR}$ is $(3,4,1,2)_{LR}$.

7.3.3.2 Fuzzy solution by using other existing methods

Using the method, used in the other existing methods, discussed in Section 7.3.1.2, the non-negative values of $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(2,3,1,1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5,8,1,5)_{LR} \preceq (4,6,2,2)_{LR}$ $\otimes$ $(\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1,2,1,1)_{LR}$ can be obtained as follows:

Step 1 Using the arithmetic operations, defined in Section 5.1.2, the fuzzy inequality constraint $(2,3,1,1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5,8,1,5)_{LR} \preceq (4,6,2,2)_{LR} \otimes (\underline{x}, \overline{x}, x^L,$ $x^R)_{LR} \oplus (1,2,1,1)_{LR}$ can be converted into the fuzzy inequality constraint $(2\underline{x}+$

$5, 3\overline{x}+8, \underline{x}+x^L+1, \overline{x}+4x^R+5)_{LR} \preceq (4\underline{x}+1, 6\overline{x}+2, 2\underline{x}+2x^L+1, 2\overline{x}+8x^R+1)_{LR}$

Step 2 Using Step 2 of the method, presented in Section 7.3.1.2, the fuzzy inequality constraint, obtained in Step 1, can be converted into the following crisp inequality constraint:

$3\underline{x}+7\overline{x}-x^L+4x^R+30 \le 6\underline{x}+14\overline{x}-2x^L+8x^R+6$

i.e., $3\underline{x}+7\overline{x}-x^L+4x^R \ge 24$

Step 3 There will be infinite values of $\underline{x}, \overline{x}, x^L$ and x^R which will satisfy the crisp inequality constraint $3\underline{x}+7\overline{x}-x^L+4x^R \ge 24$ as well as restrictions of non-negative LR flat fuzzy numbers $\underline{x}-x^L \ge 0$, $\overline{x}-\underline{x} \ge 0$, $x^L \ge 0$, $x^R \ge 0$. One of these values are 3, 4, 1 and 2 respectively.

Step 4 Using the values of $\underline{x}, \overline{x}, x^L$ and x^R, obtained from Step 3, the non-negative value of $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ which will satisfy the fuzzy inequality constraint $(2, 3, 1, 1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5, 8, 1, 5)_{LR} \preceq (4, 6, 2, 2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1, 2, 1, 1)_{LR}$ is $(3, 4, 1, 2)_{LR}$.

7.3.3.3 Drawback of the obtained fuzzy solution

On putting the value of $\widetilde{x} = (3, 4, 1, 2)_{LR}$, obtained from Step 4 of Section 7.3.3.1, in the fuzzy inequality constraint $(2, 3, 1, 1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (5, 8, 1, 5)_{LR} \preceq (4, 6, 2, 2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1, 2, 1, 1)_{LR}$ it is converted into $(11, 20, 5, 17)_{LR} \preceq (13, 26, 9, 25)_{LR}$. Since, $11 \le 13$, $20 \le 26$, $5 \le 9$ and $17 \le 25$ so according to Fan et al. method [53], $(11, 20, 5, 17)_{LR}$ is less than $(13, 26, 9, 25)_{LR}$. However, it is not possible to find any non-negative LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(11, 20, 5, 17)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (13, 26, 9, 25)_{LR}$.

Similarly, on putting the value of $\widetilde{x} = (3, 4, 1, 2)_{LR}$, obtained from the Step 4 of Section 7.3.3.2, in the fuzzy inequality constraint $(2,3,1,1)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR}$ $\oplus\ (5,8,1,5)_{LR} \preceq (4,6,2,2)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (1,2,1,1)_{LR}$, it is converted into $(11,20,5,17)_{LR} \preceq (13,26,9,25)_{LR}$. Since, $\Re(11,20,5,17)_{LR} = 17.5 \leq \Re(13,26,9,25)_{LR} = 22.1667$ so according to the method, used in the existing methods [7, 44–49, 98–103, 114–122], $(11,20,5,17)_{LR}$ is less than $(13,26,9,25)_{LR}$. However, it is not possible to find any non-negative LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(11,20,5,17)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (13,26,9,25)_{LR}$.

On the basis of these results, it can be concluded that it is not genuine to use the methods, used in the existing methods, for converting the fuzzy inequality constraints into crisp inequality constraints.

7.3.4 Drawbacks of using the existing methods as well as proposed Mehar's ranking approach for finding the minimum (or maximum) of fuzzy numbers

If a_t represents the minimum (or maximum) of n real numbers $a_1, a_2, ..., a_n$ then for a_t the following conditions are always satisfied.

(i) a_t is a unique real number.

(ii) There exist n non-negative real numbers s_i, $i = 1, 2, ..., n$ such that

$$a_t + s_i = a_i \quad (\text{or } a_t = a_i + s_i), i = 1, 2, ..., n.$$

On the same direction, if $\widetilde{a}_t$ represents the minimum (or maximum) of n fuzzy numbers $\widetilde{a}_1, \widetilde{a}_2, ..., \widetilde{a}_n$ then for $\widetilde{a}_t$ the following conditions should always be satisfied.

(i) $\widetilde{a}_t$ should be a unique fuzzy number.

(ii) There should exist n non-negative fuzzy numbers $\widetilde{s}_i$, $i = 1, 2, ..., n$ such that $\widetilde{a}_t \oplus \widetilde{s}_i = \widetilde{a}_i \quad (\text{or } \widetilde{a}_t = \widetilde{a}_i \oplus \widetilde{s}_i), i = 1, 2, ..., n.$

However, for the minimum (or maximum) of fuzzy numbers, obtained by using the methods discussed in Section 7.3.2, these conditions are not necessarily satisfied. Although, the minimum (or maximum) of fuzzy numbers, obtained by using Mehar's ranking approach proposed in Chapter 6, is a unique fuzzy number but the second condition is not necessarily satisfied.

To show this drawback of existing methods [7, 44–49, 53, 98–103, 114–122] the minimum (or maximum) of fuzzy numbers $\widetilde{a_1} = (3,4,2,3)_{LR}$, $\widetilde{a_2} = (2,5,1,2)_{LR}$, $\widetilde{a_3} = (6,8,3,5)_{LR}$ and $\widetilde{a_4} = (5,9,1,3)_{LR}$ where $L(x) = R(x) = \max\{0, 1-x\}$, are obtained by the methods, presented in Section 7.3.2, and it is shown that the minimum of these numbers, obtained by using the existing methods [7,44–49,53,98–103,114–122], is neither a unique fuzzy number nor it is possible to find any non-negative fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$, $\widetilde{s_3}$ and $\widetilde{s_4}$ which will satisfy the condition minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_1} = \widetilde{a_1}$, minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_2} = \widetilde{a_2}$, minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_3} = \widetilde{a_3}$ and minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_4} = \widetilde{a_4}$.

Similarly, it is shown that the maximum of these numbers, obtained by using the existing methods [7,44–49,53,98–103,114–122], is neither a unique fuzzy number nor it is possible to find any non-negative LR flat fuzzy numbers $\widetilde{s_1'}$, $\widetilde{s_2'}$, $\widetilde{s_3'}$ and $\widetilde{s_4'}$ which will satisfy the condition maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_1} \oplus \widetilde{s_1'}$, maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_2} \oplus \widetilde{s_2'}$, maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_3} \oplus \widetilde{s_3'}$ and maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_4} \oplus \widetilde{s_4'}$.

Also, to show the drawback of Mehar's ranking approach, proposed in Chapter 6, it is shown that although the minimum and maximum of these numbers, obtained by using proposed Mehar's ranking approach, is a unique LR flat fuzzy number but it is not possible to find any non-negative LR flat fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$, $\widetilde{s_3}$, $\widetilde{s_4}$ and

$\widetilde{s_1'}$, $\widetilde{s_2'}$, $\widetilde{s_3'}$, $\widetilde{s_4'}$ which will satisfy the following condition:

minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_1} = \widetilde{a_1}$, minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_2} = \widetilde{a_2}$, minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_3} = \widetilde{a_3}$, minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} \oplus \widetilde{s_4} = \widetilde{a_4}$ and maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_1} \oplus \widetilde{s_1'}$, maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_2} \oplus \widetilde{s_2'}$, maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_3} \oplus \widetilde{s_3'}$, maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\} = \widetilde{a_4} \oplus \widetilde{s_4'}$.

7.3.4.1 Minimum (or maximum) of fuzzy numbers by using Fan et al. method

Using Fan et al. method, discussed in Section 7.3.2.1, the minimum (or maximum) of LR flat fuzzy numbers $\widetilde{a_1} = (3, 4, 2, 3)_{LR}$, $\widetilde{a_2} = (2, 5, 1, 2)_{LR}$, $\widetilde{a_3} = (6, 8, 3, 5)_{LR}$ and $\widetilde{a_4} = (5, 9, 1, 3)_{LR}$ with $L(x) = R(x) = \max\{0, 1 - x\}$ can be obtained as follows:

Step 1 $\Re(\widetilde{a_1}) = \frac{22}{6}$, $\Re(\widetilde{a_2}) = \frac{22}{6}$, $\Re(\widetilde{a_3}) = \frac{22}{3}$ and $\Re(\widetilde{a_4}) = \frac{22}{3}$.

Step 2 minimum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\} = \Re(\widetilde{a_1}) = \Re(\widetilde{a_2}) = \frac{22}{6}$ and

maximum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\} = \Re(\widetilde{a_3}) = \Re(\widetilde{a_4}) = \frac{22}{3}$

Since, minimum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\}$ are $\Re(\widetilde{a_1})$ and $\Re(\widetilde{a_2})$ so minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\}$ are $\widetilde{a_1}$ and $\widetilde{a_2}$.

Similarly, since maximum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\}$ are $\Re(\widetilde{a_3})$ and $\Re(\widetilde{a_4})$ so maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\}$ are $\widetilde{a_3}$ and $\widetilde{a_4}$.

7.3.4.2 Minimum (or maximum) of fuzzy numbers by using other existing methods

Using the method, discussed in Section 7.3.2.2, the minimum (or maximum) of LR flat fuzzy numbers $\widetilde{a_1} = (3, 4, 2, 3)_{LR}$, $\widetilde{a_2} = (2, 5, 1, 2)_{LR}$, $\widetilde{a_3} = (6, 8, 3, 5)_{LR}$ and $\widetilde{a_4} = (5, 9, 1, 3)_{LR}$ with $L(x) = R(x) = \max\{0, 1 - x\}$ can be obtained as follows:

Step 1 $\Re(\widetilde{a_1}) = \frac{15}{4}$, $\Re(\widetilde{a_2}) = \frac{15}{4}$, $\Re(\widetilde{a_3}) = \frac{15}{2}$ and $\Re(\widetilde{a_4}) = \frac{15}{2}$.

Step 2 minimum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\} = \Re(\widetilde{a_1}) = \Re(\widetilde{a_2}) = \frac{15}{4}$ and

maximum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\} = \Re(\widetilde{a_3}) = \Re(\widetilde{a_4}) = \frac{15}{2}$

Since, minimum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\}$ are $\Re(\widetilde{a_1})$ and $\Re(\widetilde{a_2})$ so minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\}$ are $\widetilde{a_1}$ and $\widetilde{a_2}$.

Similarly, maximum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\}$ are $\Re(\widetilde{a_3})$ and $\Re(\widetilde{a_4})$ so maximum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\}$ are $\widetilde{a_3}$ and $\widetilde{a_4}$.

7.3.4.3 Minimum (or maximum) of fuzzy numbers by using proposed Mehar's ranking approach

Using Mehar's ranking approach, proposed in previous chapter, the minimum (or maximum) of LR flat fuzzy numbers $\widetilde{a_1} = (3, 4, 2, 3)_{LR}$, $\widetilde{a_2} = (2, 5, 1, 2)_{LR}$, $\widetilde{a_3} = (6, 8, 3, 5)_{LR}$ and $\widetilde{a_4} = (5, 9, 1, 3)_{LR}$ with $L(x) = R(x) = \max\{0, 1 - x\}$ can be obtained as follows:

Step 1 $\Re(\widetilde{a_1}) = \frac{15}{4}$, $\Re(\widetilde{a_2}) = \frac{15}{4}$, $\Re(\widetilde{a_3}) = \frac{15}{2}$ and $\Re(\widetilde{a_4}) = \frac{15}{2}$.

Step 2 Since, minimum$\{\Re(\widetilde{a_1}), \Re(\widetilde{a_2}), \Re(\widetilde{a_3}), \Re(\widetilde{a_4})\} = \frac{15}{4}$ which is corresponding to two LR flat fuzzy numbers $\widetilde{a_1}$ and $\widetilde{a_2}$ so go to Step 3.

Step 3 $mode(\widetilde{a_1}) = \frac{7}{2}$, $mode(\widetilde{a_2}) = \frac{7}{2}$.

Step 4 Since, minimum$\{mode(\widetilde{a_1}), mode(\widetilde{a_2})\} = \frac{7}{2}$ which is corresponding to two LR flat fuzzy numbers $\widetilde{a_1}$ and $\widetilde{a_2}$ so go to Step 5.

Step 5 $divergence(\widetilde{a_1}) = \frac{7}{2}$, $divergence(\widetilde{a_2}) = \frac{9}{2}$

Step 6 Since, minimum$\{divergence(\widetilde{a_1}), divergence(\widetilde{a_2})\} = \frac{7}{2}$ which is corresponding to a unique LR flat fuzzy number $\widetilde{a_1}$ so minimum$\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{a_4}\}$ is $\widetilde{a_1} = (3, 4, 2, 3)_{LR}$.

Similarly, maximum of LR flat fuzzy numbers $\widetilde{a_1} = (3, 4, 2, 3)_{LR}$, $\widetilde{a_2} = (2, 5, 1, 2)_{LR}$, $\widetilde{a_3} = (6, 8, 3, 5)_{LR}$ and $\widetilde{a_4} = (5, 9, 1, 3)_{LR}$ with $L(x) = R(x) = \max\{0, 1-x\}$, obtained by using proposed Mehar's ranking approach, is $\widetilde{a_3} = (6, 8, 3, 5)_{LR}$.

7.3.4.4 Drawbacks of the obtained minimum (or maximum) of fuzzy numbers

It is obvious from the results, obtained in Section 7.3.4.1 and Section 7.3.4.2 that by using the methods used in existing methods [7, 44–49, 53, 98–103, 114–122] for finding the minimum and maximum of fuzzy numbers, two distinct LR flat fuzzy numbers $\widetilde{a_1}$ and $\widetilde{a_2}$, representing the minimum and maximum, are obtained and if out of them $\widetilde{a_1}$ is assumed as minimum then it is not possible to find non-negative LR flat fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$ and $\widetilde{s_3}$ such that $\widetilde{a_1} \oplus \widetilde{s_1} = \widetilde{a_2}$, $\widetilde{a_1} \oplus \widetilde{s_2} = \widetilde{a_3}$ and $\widetilde{a_1} \oplus \widetilde{s_3} = \widetilde{a_4}$. Similarly, if $\widetilde{a_2}$ is assumed as minimum then it is not possible to find non-negative LR flat fuzzy numbers $\widetilde{s_1^{'}}$, $\widetilde{s_2^{'}}$ and $\widetilde{s_3^{'}}$ such that $\widetilde{a_2} \oplus \widetilde{s_1^{'}} = \widetilde{a_1}$, $\widetilde{a_2} \oplus \widetilde{s_2^{'}} = \widetilde{a_3}$ and $\widetilde{a_2} \oplus \widetilde{s_3^{'}} = \widetilde{a_4}$.

The same shortcomings is also occurring in the maximum of fuzzy numbers obtained by using the existing methods [7, 44–49, 53, 98–103, 114–122].

Also, it is obvious from the results, obtained in Section 7.3.4.3, that although by using Mehar's ranking approach, proposed in Chapter 6, a unique LR flat fuzzy number $(3, 4, 2, 3)_{LR}$, representing the minimum, is obtained but it is not possible to find non-negative LR flat fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$ and $\widetilde{s_3}$ such that $(3, 4, 2, 3)_{LR} \oplus \widetilde{s_1} = (2, 5, 1, 2)_{LR}$, $(3, 4, 2, 3)_{LR} \oplus \widetilde{s_2} = (6, 8, 3, 5)_{LR}$ and $(3, 4, 2, 3)_{LR} \oplus \widetilde{s_3} = (5, 9, 1, 3)_{LR}$. Similarly, it is obvious from the results, obtained in Section 7.3.4.3, that although by using Mehar's ranking approach, proposed in Chapter 6, a unique LR flat fuzzy number $(6, 8, 3, 5)_{LR}$, representing the maximum, is obtained but it is not possible to find non-negative LR flat fuzzy numbers $\widetilde{s_1^{'}}$, $\widetilde{s_2^{'}}$ and $\widetilde{s_3^{'}}$ such that $(3, 4, 2, 3)_{LR} \oplus \widetilde{s_1^{'}} = (6, 8, 3, 5)_{LR}$, $(2, 5, 1, 2)_{LR} \oplus \widetilde{s_2^{'}} = (6, 8, 3, 5)_{LR}$ and $(5, 9, 1, 3)_{LR} \oplus \widetilde{s_3^{'}} = (6, 8, 3, 5)_{LR}$.

On the basis of these results, it can be concluded that it is not genuine to use the methods, used in the existing methods [7, 44–49, 53, 98–103, 114–122] as well as Mehar's ranking approach proposed in Chapter 6, for finding the minimum and maximum of LR flat fuzzy numbers.

7.4 Proposed methods

Since, all the shortcomings, pointed out in the first point of Section 7.2, Section 7.3.3 and Section 7.3.4 are occurring due to the shortcomings in the methods used for converting the fuzzy inequality constraints into crisp inequality constraints and also due to the shortcomings in the methods used for finding the minimum and maximum of fuzzy numbers. So, in this section, a method for converting the fuzzy inequality constraints into crisp inequality constraints and a method for finding the minimum and maximum of fuzzy numbers are proposed. Also, on the basis of these methods, a new method is proposed for finding the unique minimum fuzzy crashing cost for completing the project within a specific fuzzy time.

7.4.1 Proposed JMD method for converting fuzzy inequality constraints into crisp inequality constraints

In this section, a new method for converting fuzzy inequality constraints into crisp inequality constraints, named as JMD method for converting fuzzy inequality constraints into crisp inequality constraints, is proposed.

If $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\widetilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ are two LR flat fuzzy numbers such that $\underline{a} - a^L \leq \underline{b} - b^L$, $a^L \leq b^L$, $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$, $a^R \leq b^R$, then there will always exist a unique non-negative LR flat fuzzy number $\widetilde{C} = (\underline{c}, \overline{c}, c^L, c^R)_{LR}$ such that $\widetilde{A} \oplus \widetilde{C} = \widetilde{B}$. Similarly, If $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\widetilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ are two LR flat fuzzy numbers such that $\underline{a}-a^L \geq \underline{b}-b^L$, $a^L \geq b^L$, $\overline{a}-\underline{a} \geq \overline{b}-\underline{b}$, $a^R \geq b^R$, then there will always exist a unique non-negative LR flat fuzzy number $\widetilde{C} = (\underline{c}, \overline{c}, c^L, c^R)_{LR}$ such that $\widetilde{A} = \widetilde{B} \oplus \widetilde{C}$.

So, the fuzzy inequality constraints $\widetilde{A} \preceq \widetilde{B}$ and $\widetilde{A} \succeq \widetilde{B}$ should be replaced by the crisp inequality constraints $\underline{a} - a^L \leq \underline{b} - b^L$, $a^L \leq b^L$, $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$, $a^R \leq b^R$ and $\underline{a} - a^L \geq \underline{b} - b^L$, $a^L \geq b^L$, $\overline{a} - \underline{a} \geq \overline{b} - \underline{b}$, $a^R \geq b^R$ respectively.

Proposition 7.1 If $\widetilde{A} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$ and $\widetilde{B} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$ are two LR flat fuzzy numbers such that $\underline{a} - a^L \leq \underline{b} - b^L$, $a^L \leq b^L$, $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$, $a^R \leq b^R$, then prove that there will always exist a unique non-negative LR flat fuzzy number $\widetilde{C} = (\underline{c}, \overline{c}, c^L, c^R)_{LR}$ such that $\widetilde{A} \oplus \widetilde{C} = \widetilde{B}$.

Proof: Since, $\underline{a} - a^L \leq \underline{b} - b^L \quad (7.1)$

$$a^L \leq b^L \tag{7.2}$$

$$\overline{a} - \underline{a} \leq \overline{b} - \underline{b} \tag{7.3}$$

$$a^R \leq b^R \tag{7.4}$$

So, there will always exist unique non-negative real numbers s_1, s_2, s_3 and s_4 such that

$$\underline{a} - a^L + s_1 = \underline{b} - b^L$$

$$a^L + s_2 = b^L$$

$$\overline{a} - \underline{a} + s_3 = \overline{b} - \underline{b}$$

$$a^R + s_4 = b^R$$

which implies that

$$s_1 = \underline{b} - b^L - \underline{a} + a^L$$

$$s_2 = b^L - a^L$$

$$s_3 = \overline{b} - \underline{b} - \overline{a} + \underline{a}$$

$s_4 = b^R - a^R$

Now, it is proved that the number $\widetilde{C} = (\underline{c}, \overline{c}, c^L, c^R)_{LR} = (s_1+s_2, s_1+s_2+s_3, s_2, s_4)_{LR}$, constructed by using the obtained values of s_1, s_2, s_3 and s_4, is the unique non-negative LR flat fuzzy number such that $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \oplus (\underline{c}, \overline{c}, c^L, c^R)_{LR} = (\underline{b}, \overline{b}, b^L, b^R)_{LR}$.

(1) Since, $(\underline{c}, \overline{c}, c^L, c^R)_{LR}$ satisfies all the conditions of a non-negative LR flat fuzzy number i.e.,

(i) $\underline{c} - c^L = (s_1 + s_2) - s_2 = s_1 = \underline{b} - b^L - \underline{a} + a^L \geq 0$ (from (7.1))

(ii) $\overline{c} - \underline{c} = (s_1 + s_2 + s_3) - (s_1 + s_2) = s_3 = \overline{b} - \underline{b} - \overline{a} + \underline{a} \geq 0$ (from (7.3))

(iii) $c^L = s_2 = b^L - a^L \geq 0$ (from (7.2))

(iv) $c^R = s_4 = b^R - a^R \geq 0$ (from (7.4))

So, $(\underline{c}, \overline{c}, c^L, c^R)_{LR}$ is a non-negative LR flat fuzzy number.

(2) Since, s_1, s_2, s_3 and s_4 are unique real numbers so, $s_1 + s_2, s_1 + s_2 + s_3$ will be unique real numbers and hence LR flat fuzzy number $(\underline{c}, \overline{c}, c^L, c^R)_{LR} = (s_1 + s_2, s_1 + s_2 + s_3, s_2, s_4)_{LR}$, obtained from these real numbers, will also be a unique LR flat fuzzy number.

(3) $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \oplus (\underline{c}, \overline{c}, c^L, c^R)_{LR}$

$= (\underline{a}, \overline{a}, a^L, a^R)_{LR} \oplus (s_1 + s_2, s_1 + s_2 + s_3, s_2, s_4)_{LR}$

$= (\underline{a}, \overline{a}, a^L, a^R)_{LR} \oplus (\underline{b} - b^L - \underline{a} + a^L + b^L - a^L, \underline{b} - b^L - \underline{a} + a^L + b^L - a^L + \overline{b} - \underline{b} - \overline{a} + \underline{a}, b^L - a^L, b^R - a^R)_{LR}$

$= (\underline{a}, \overline{a}, a^L, a^R)_{LR} \oplus (\underline{b} - \underline{a}, \overline{b} - \overline{a}, b^L - a^L, b^R - a^R)_{LR}$

$= (\underline{b}, \overline{b}, b^L, b^R)_{LR}$.

7.4.2 Proposed JMD method for finding the minimum and maximum of LR flat fuzzy numbers

In this section, a new method for finding the minimum and maximum of fuzzy numbers, named as JMD method for finding the minimum and maximum of fuzzy numbers, is proposed.

If the minimum of LR flat fuzzy numbers $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ is defined by

$$\text{minimum}\{\widetilde{A}, \widetilde{B}\} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$$

$$\text{where, } \begin{cases} \underline{a} = \text{minimum}\{\underline{a_1} - a_1^L, \underline{a_2} - a_2^L\} + \text{minimum}\{a_1^L, a_2^L\} \\ \overline{a} = \text{minimum}\{\underline{a_1} - a_1^L, \underline{a_2} - a_2^L\} + \text{minimum}\{a_1^L, a_2^L\} + \\ \quad \text{minimum}\{\overline{a_1} - \underline{a_1}, \overline{a_2} - \underline{a_2}\} \\ a^L = \text{minimum}\{a_1^L, a_2^L\} \\ a^R = \text{minimum}\{a_1^R, a_2^R\} \end{cases}$$

Then, one of the following cases will be satisfied:

Case (i) minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and there will exist a unique non-negative LR flat fuzzy number $\widetilde{C}$ such that $\widetilde{A} \oplus \widetilde{C} = \widetilde{B}$

Case (ii) minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and there will exist a unique non-negative LR flat fuzzy number $\widetilde{C}$ such that $\widetilde{B} \oplus \widetilde{C} = \widetilde{A}$

Case (iii) minimum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{C}$ and there will exist two unique non-negative LR flat fuzzy numbers $\widetilde{C_1}$ and $\widetilde{C_2}$ such that $\widetilde{C} \oplus \widetilde{C_1} = \widetilde{A}$ and $\widetilde{C} \oplus \widetilde{C_2} = \widetilde{B}$.

Similarly, if the maximum of LR flat fuzzy numbers $\widetilde{A_1} = (\underline{a_1}, \overline{a_1}, a_1^L, a_1^R)_{LR}$ and $\widetilde{A_2} = (\underline{a_2}, \overline{a_2}, a_2^L, a_2^R)_{LR}$ is defined by

$$\text{maximum}\{\widetilde{A}, \widetilde{B}\} = (\underline{a}, \overline{a}, a^L, a^R)_{LR}$$

$$\text{where,}\begin{cases}\underline{a} = \text{ maximum}\{\underline{a_1} - a_1^L, \underline{a_2} - a_2^L\} \ + \text{ maximum}\{a_1^L, a_2^L\} \\ \overline{a} = \text{ maximum}\{\underline{a_1} - a_1^L, \underline{a_2} - a_2^L\} \ + \text{ maximum}\{a_1^L, a_2^L\} + \\ \qquad \text{maximum}\{\overline{a_1} - \underline{a_1}, \overline{a_2} - \underline{a_2}\} \\ a^L = \text{maximum}\{a_1^L, a_2^L\} \\ a^R = \text{ maximum}\{a_1^R, a_2^R\}\end{cases}$$

Then, one of the following cases will be satisfied:

Case (i) maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and there will exist a unique non-negative LR flat fuzzy number $\widetilde{C}$ such that $\widetilde{B} \oplus \widetilde{C} = \widetilde{A}$

Case (ii) maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and there will exist a unique non-negative LR flat fuzzy number $\widetilde{C}$ such that $\widetilde{A} \oplus \widetilde{C} = \widetilde{B}$

Case (iii) maximum$\{\widetilde{A}, \widetilde{B}\} = \widetilde{C}$ and there will exist two unique non-negative LR flat fuzzy numbers $\widetilde{C_1}$ and $\widetilde{C_2}$ such that $\widetilde{C} = \widetilde{A} \oplus \widetilde{C_1}$ and $\widetilde{C} = \widetilde{B} \oplus \widetilde{C_2}$.

Proposition 7.2 Let $\underset{1\le i\le n}{\text{minimum}}\{\widetilde{A_i}\} = \underset{1\le i\le n}{\text{minimum}}\{(\underline{a_i}, \overline{a_i}, {a_i}^L, {a_i}^R)_{LR}\} = \widetilde{A_t} = (\underline{a_t}, \overline{a_t}, {a_t}^L, {a_t}^R)_{LR}$ where,

$$\underline{a_t} = \underset{1\le i\le n}{\text{minimum}}\{\underline{a_i} - {a_i}^L\} + \underset{1\le i\le n}{\text{minimum}}\{{a_i}^L\},$$

$$\overline{a_t} = \underset{1\le i\le n}{\text{minimum}}\{\underline{a_i} - {a_i}^L\} + \underset{1\le i\le n}{\text{minimum}}\{{a_i}^L\} + \underset{1\le i\le n}{\text{minimum}}\{\overline{a_i} - \underline{a_i}\},$$

$${a_t}^L = \underset{1\le i\le n}{\text{minimum}}\{{a_i}^L\},$$

$${a_t}^R = \underset{1\le i\le n}{\text{minimum}}\{{a_i}^R\}$$

then prove that there will always exist n unique non-negative LR flat fuzzy numbers $\widetilde{C_i} = (\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR};\ i = 1, 2, ..., n$ such that $\widetilde{A_i} = \widetilde{A_t} \oplus \widetilde{C_i};\ i = 1, 2, ..., n$.

Proof Since, $\underline{a_t} = \underset{1\le i\le n}{\text{minimum}}\{\underline{a_i} - {a_i}^L\} + \underset{1\le i\le n}{\text{minimum}}\{{a_i}^L\}$,

$$\overline{a_t} = \underset{1\le i\le n}{\text{minimum}}\{\underline{a_i} - {a_i}^L\} + \underset{1\le i\le n}{\text{minimum}}\{{a_i}^L\} + \underset{1\le i\le n}{\text{minimum}}\{\overline{a_i} - \underline{a_i}\},$$

$${a_t}^L = \underset{1\le i\le n}{\text{minimum}}\{{a_i}^L\},$$

$${a_t}^R = \underset{1\le i\le n}{\text{minimum}}\{{a_i}^R\}$$

So,

$$\underline{a_t} = \underset{1 \le i \le n}{\text{minimum}}\{\underline{a_i} - {a_i}^L\} + {a_t}^L,$$

$$\overline{a_t} = \underline{a_t} + \underset{1 \le i \le n}{\text{minimum}}\{\overline{a_i} - \underline{a_i}\},$$

$${a_t}^L = \underset{1 \le i \le n}{\text{minimum}}\{{a_i}^L\},$$

$${a_t}^R = \underset{1 \le i \le n}{\text{minimum}}\{{a_i}^R\}$$

which implies that

$$\underline{a_t} - {a_t}^L = \underset{1 \le i \le n}{\text{minimum}}\{\underline{a_i} - {a_i}^L\},$$

$$\overline{a_t} - \underline{a_t} = \underset{1 \le i \le n}{\text{minimum}}\{\overline{a_i} - \underline{a_i}\},$$

$${a_t}^L = \underset{1 \le i \le n}{\text{minimum}}\{{a_i}^L\},$$

$${a_t}^R = \underset{1 \le i \le n}{\text{minimum}}\{{a_i}^R\}$$

which implies that

$$\underline{a_t} - {a_t}^L \le \underline{a_i} - {a_i}^L, \tag{7.5}$$

$$\overline{a_t} - \underline{a_t} \le \overline{a_i} - \underline{a_i}, \tag{7.6}$$

$${a_t}^L \le {a_i}^L, \tag{7.7}$$

$${a_t}^R \le {a_i}^R; \forall\ i = 1, 2, ..., n \tag{7.8}$$

There will always exist $4n$ unique non-negative real numbers s_1^i, s_2^i, s_3^i and s_4^i such that

$$\underline{a_t} - {a_t}^L + s_1^i = \underline{a_i} - {a_i}^L,$$

$$\overline{a_t} - \underline{a_t} + s_2^i = \overline{a_i} - \underline{a_i},$$

$${a_t}^L + s_3^i = {a_i}^L,$$

$${a_t}^R + s_4^i = {a_i}^R; \forall\ i = 1, 2, ..., n$$

which implies that

$$s_1^i = \underline{a_i} - {a_i}^L - \underline{a_t} + {a_t}^L,$$

$$s_2^i = \overline{a_i} - \underline{a_i} - \overline{a_t} + \underline{a_t},$$

$s_3^i = {a_i}^L - {a_t}^L,$

$s_4^i = {a_i}^R - {a_t}^R; \forall\ i = 1, 2, ..., n$

Now, it is proved that the i^{th} LR flat fuzzy number $\widetilde{C}_i = (\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR} = (s_1^i + s_3^i, s_1^i + s_2^i + s_3^i, s_3^i, s_4^i)_{LR}$, constructed by using the obtained values of s_1^i, s_2^i, s_3^i and s_4^i, is the unique non-negative LR flat fuzzy number such that $(\underline{a_i}, \overline{a_i}, {a_i}^L, {a_i}^R)_{LR} = (\underline{a_t}, \overline{a_t}, {a_t}^L, {a_t}^R)_{LR} \oplus (\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR}$

(1) Since, $(\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR}$ satisfies all the conditions of non-negative LR flat fuzzy number i.e.,

(i) $\underline{c_i} - c_i^L = (s_1^i + s_3^i) - s_3^i = s_1^i = \underline{a_i} - {a_i}^L - \underline{a_t} + {a_t}^L \geq 0$ (from (7.5))

(ii) $\overline{c_i} - \underline{c_i} = (s_1^i + s_2^i + s_3^i) - (s_1^i + s_3^i) = s_2^i = \overline{a_i} - \underline{a_i} - \overline{a_t} + \underline{a_t} \geq 0$ (from (7.6))

(iii) $c_i^L = s_3^i = {a_i}^L - {a_t}^L \geq 0$ (from (7.7))

(iv) $c_i^R = s_4^i = {a_i}^R - {a_t}^R \geq 0$ (from (7.8))

So, $(\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR}$ is a non-negative LR flat fuzzy number.

(2) Since, s_1^i, s_2^i, s_3^i and s_4^i are unique real numbers so, $s_1^i + s_3^i, s_1^i + s_2^i + s_3^i$ will be unique real numbers and hence the i^{th} LR flat fuzzy number $(\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR}$ $= (s_1^i + s_3^i, s_1^i + s_2^i + s_3^i, s_3^i, s_4^i)_{LR}$, obtained from these real numbers, will also be a unique LR flat fuzzy number.

(3) $(\underline{a_t}, \overline{a_t}, a_t^L, a_t^R)_{LR} \oplus (\underline{c_i}, \overline{c_i}, c_i^L, c_i^R)_{LR}$

$= (\underline{a_t}, \overline{a_t}, a_t^L, a_t^R)_{LR} \oplus (s_1^i + s_3^i, s_1^i + s_2^i + s_3^i, s_3^i, s_4^i)_{LR}$

$= (\underline{a_t}, \overline{a_t}, a_t^L, a_t^R)_{LR} \oplus (\underline{a_i} - {a_i}^L - \underline{a_t} + {a_t}^L + {a_i}^L - {a_t}^L, \underline{a_i} - {a_i}^L - \underline{a_t} + {a_t}^L$
$+ \overline{a_i} - \underline{a_i} - \overline{a_t} + \underline{a_t} + {a_i}^L - {a_t}^L, a_i^L - a_t^L, a_i^R - a_t^R)_{LR}$

$= (\underline{a_t}, \overline{a_t}, a_t^L, a_t^R)_{LR} \oplus (\underline{a_i} - \underline{a_t}, \overline{a_i} - \overline{a_t}, a^L - a_t^L, a^R - a_t^R)_{LR}$

$= (\underline{a_i}, \overline{a_i}, a_i^L, a_i^R)_{LR}.$

7.4.3 Proposed JMD feasibility criteria

Since, the fuzzy normal time ($\widetilde{NT_{ij}}$) of an activity (i, j) will always be greater than the fuzzy crash time ($\widetilde{CT_{ij}}$) of the same activity and also, the specific fuzzy time ($\widetilde{PCT}$) in which a decision maker want to complete the project should be greater than the sum of fuzzy crash time of all the activities lying on the critical path ($\sum\limits_{(i,j)\in A} \widetilde{CT_{ij}^{CP}}$). So, for satisfying these conditions in Step 1 of the existing method [30] it is claimed that the problem will be feasible if and only if $(CT_{ij})^U_{\alpha=0} \leq (NT_{ij})^L_{\alpha=0}\ \forall\ (i, j) \in A$ and $\sum\limits_{(i,j)\in A}(CT_{ij}^{CP})^U_{\alpha=0} \leq (PCT)^L_{\alpha=0}$. However, there may exist several LR flat fuzzy numbers, representing $\widetilde{NT_{ij}}$, $\widetilde{CT_{ij}}$, $\widetilde{PCT}$ and $\sum\limits_{(i,j)\in A} \widetilde{CT_{ij}^{CP}}$, for which these conditions will be satisfied but it is not possible to find any non-negative LR flat fuzzy number $\widetilde{S_{ij}}$ and $\widetilde{S}$ such that $\widetilde{NT_{ij}} = \widetilde{CT_{ij}} \oplus \widetilde{S_{ij}}$ and $\widetilde{PCT} = \sum\limits_{(i,j)\in A} \widetilde{CT_{ij}^{CP}} \oplus \widetilde{S}$. So, due to the reason, pointed out in Section 7.3.3, it is not genuine to use this feasibility criteria. Hence, in this section a new feasibility criteria, named as JMD feasible criteria, is proposed.

1. Since, the crash time of an activity will always be less than the normal time of the same activity so if both the activity times are represented by fuzzy numbers then there should exist one non-negative fuzzy number $\widetilde{S}$ such that fuzzy crash time $\oplus\ \widetilde{S}$ = fuzzy normal time. But, none of the researchers have cared about this property in choosing the data e.g., Chen and Tsai [30] have represented the fuzzy normal time of the activity $(1, 2)$ by the triangular fuzzy number (13, 14, 15) and the fuzzy crash time of the same activity by triangular fuzzy number (4, 6, 6). But, there does not exist any non-negative triangular fuzzy number $\widetilde{S}$ such that (4, 6, 6) $\oplus\ \widetilde{S}$ = (13, 14, 15). Similarly, the same shortcoming is also occurring in the fuzzy normal time and fuzzy crash time

of some other activities are shown in Table 7.2.

Table 7.2: Existing fuzzy data [30]

Activity (i,j)	Fuzzy normal time $\widetilde{NT}_{ij}$	Fuzzy crash time $\widetilde{CT}_{ij}$	The incremental direct cost per unit decrease in activity time $\widetilde{C}_{ij}$
(1, 2)	$(13,14,15)$	$(4,6,6)$	$(80,100,120)$
(1, 3)	$(10,12,13)$	$(7,8,9)$	$(190,200,220)$
(2, 5)	$(16,18,19)$	$(10,14,14)$	$(90,100,130)$
(2, 4)	$(5,6,8)$	$(3,4,5)$	$(160,200,210)$
(3, 4)	$(3,4,5)$	$(2,2,3)$	$(180,200,220)$
(4, 5)	$(7,8,10)$	$(4,6,7)$	$(80,100,110)$
(5, 6)	$(9,12,14)$	$(6,8,8)$	$(90,100,120)$

However, if the fuzzy normal time $(\widetilde{NT}_{ij})$ and fuzzy crash time $(\widetilde{CT}_{ij})$ of all the activities (i,j) are chosen in such a manner that the condition, $\widetilde{NT}_{ij} \succeq \widetilde{CT}_{ij}$ is satisfied. Then, as discussed in Section 7.4.1, there will always exist non-negative fuzzy numbers $\widetilde{S}_{ij}$ such that $\widetilde{NT}_{ij} = \widetilde{CT}_{ij} \oplus \widetilde{S}_{ij}$.

So, for the feasibility of the problem the values of $\widetilde{NT}_{ij}$ and $\widetilde{CT}_{ij}$ should be chosen in such a manner so that the condition $\widetilde{NT}_{ij} \succeq \widetilde{CT}_{ij}$ is satisfied.

2. Since, the specific fuzzy time $(\widetilde{PCT})$ in which the decision maker want to complete the project should be less than the optimal initial fuzzy project completion time $(\widetilde{NT}_n)$ and greater than the optimal crash fuzzy project completion time $(\widetilde{CT}_n)$. i.e., if $\widetilde{PCT}$, $\widetilde{NT}_n$ and $\widetilde{CT}_n$ are represented by LR flat fuzzy numbers $\widetilde{PCT}= (\underline{pct}, \overline{pct}, pct^L, pct^R)_{LR}$, $\widetilde{NT}_n = (\underline{nt_n}, \overline{nt_n}, nt_n^L, nt_n^R)_{LR}$ and $\widetilde{CT}_n = (\underline{ct_n}, \overline{ct_n}, ct_n^L, ct_n^R)_{LR}$ respectively then the condition $\widetilde{CT}_n \preceq \widetilde{PCT} \preceq \widetilde{NT}_n$ i.e., $\underline{ct_n} - ct_n^L \leq \underline{pct} - pct^L \leq \underline{nt_n} - nt_n^L$, $ct_n^L \leq pct^L \leq nt_n^L$, $\overline{ct_n} - \underline{ct_n} \leq \overline{pct} - \underline{pct} \leq \overline{nt_n} - \underline{nt_n}$, $ct_n^R \leq pct^R \leq nt_n^R$, proposed in Section 7.4.1, should be satisfied.

So, if the decision maker want to complete the project within specific fuzzy time $\widetilde{PCT}$ for which these conditions are not satisfied then it is not possible to complete the project within specific fuzzy time.

7.4.4 Proposed JMD method for finding the unique optimal initial fuzzy project completion time

The aim of decision maker is to complete the project within a specific fuzzy time ($\widetilde{PCT}$) which is less than the optimal initial fuzzy completion time ($\widetilde{NT_n}$) of project. So, the specified fuzzy time in which the decision maker want to complete the project should be less than the optimal initial fuzzy project completion time.

Since, Mehar's method, proposed in Chapter 6, for finding the optimal initial fuzzy project completion time is based on proposed Mehar's ranking approach and in Section 7.3.4, it is pointed out that it is not genuine to use the proposed Mehar's ranking approach so it is also not genuine to use Mehar's method based on proposed Mehar's ranking approach for finding the optimal initial fuzzy project completion time.

So, in this section, a new method, named as JMD method for finding the optimal initial fuzzy project completion time, is proposed for finding the optimal initial fuzzy project completion time.

The steps of the proposed method are as follows:

Step 1 Since, the optimal initial completion time of the project in crisp environment is obtained by solving the crisp linear programming problem ($P_{7.5}$) so the optimal initial completion time of the project in fuzzy environment can be obtained by solving the fuzzy linear programming problem ($P_{7.6}$) [93].

Minimize (NT_n)

subject to

$$NT_j \geq NT_{ij} + NT_i, \qquad (P_{7.5})$$

$NT_1 = 0,$

NT_j are real numbers,

NT_{ij} are non-negative real numbers, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, $(i, j) \in A$

where,

NT_i: Normal time for the node i,

NT_j: Normal time for the node j,

NT_{ij}: Normal time of the activity (i, j)

Minimize $(\widetilde{NT_n})$

subject to

$$\widetilde{NT_j} \succeq \widetilde{NT_{ij}} \oplus \widetilde{NT_i}, \qquad (P_{7.6})$$

$\widetilde{NT_1} = \widetilde{0},$

$\widetilde{NT_j}$ are fuzzy numbers,

$\widetilde{NT_{ij}}$ are non-negative fuzzy numbers, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, $(i, j) \in A$

where,

$\widetilde{NT_i}$: Fuzzy normal time for the node i,

$\widetilde{NT_j}$: Fuzzy normal time for the node j,

$\widetilde{NT_{ij}}$: Fuzzy normal time of the activity (i, j).

Step 2 Assuming $\widetilde{NT_j} = (\underline{nt_j}, \overline{nt_j}, nt_j^L, nt_j^R)_{LR}$ and $\widetilde{NT}_{ij} = (\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR}$, the fuzzy linear programming problem $(P_{7.6})$ can be written as:

Minimize $(\underline{nt_n}, \overline{nt_n}, nt_n^L, nt_n^R)_{LR}$

subject to

$(\underline{nt_j}, \overline{nt_j}, nt_j^L, nt_j^R)_{LR} \succeq (\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR} \oplus (\underline{nt_i}, \overline{nt_i}, nt_i^L, nt_i^R)_{LR},$

$$(\underline{nt_1}, \overline{nt_1}, nt_1^L, nt_1^R)_{LR} = (0, 0, 0, 0)_{LR}, \qquad (P_{7.7})$$

$(\underline{nt_j}, \overline{nt_j}, nt_j^L, nt_j^R)_{LR}$ are LR flat fuzzy numbers,

$(\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR}$ are non-negative LR flat fuzzy numbers, $i = 1, 2, ..., n$,

$j = 1, 2, ..., n$, $(i, j) \in A$.

Step 3 Using the arithmetic operations, defined in Section 5.1.2, the fuzzy linear programming problem $(P_{7.7})$ can be written as:

Minimize $(\underline{nt_n}, \overline{nt_n}, nt_n^L, nt_n^R)_{LR}$

subject to

$$(\underline{nt_j}, \overline{nt_j}, nt_j^L, nt_j^R)_{LR} \succeq (\underline{nt_{ij}} + \underline{nt_i}, \overline{nt_{ij}} + \overline{nt_i}, nt_{ij}^L + nt_i^L, nt_{ij}^R + nt_i^R)_{LR}, \qquad (P_{7.8})$$

$(\underline{nt_1}, \overline{nt_1}, nt_1^L, nt_1^R)_{LR} = (0, 0, 0, 0)_{LR}$,

$(\underline{nt_j}, \overline{nt_j}, nt_j^L, nt_j^R)_{LR}$ are LR flat fuzzy numbers,

$(\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR}$ are non-negative LR flat fuzzy numbers, $i = 1, 2, ..., n$,

$j = 1, 2, ..., n$, $(i, j) \in A$.

Step 4 Using Definition 5.4 and Definition 5.6, the fuzzy linear programming problem $(P_{7.8})$ can be written as:

Minimize $(\underline{nt_n}, \overline{nt_n}, nt_n^L, nt_n^R)_{LR}$

subject to

$$(\underline{nt_j}, \overline{nt_j}, nt_j^L, nt_j^R)_{LR} \succeq (\underline{nt_{ij}} + \underline{nt_i}, \overline{nt_{ij}} + \overline{nt_i}, nt_{ij}^L + nt_i^L, nt_{ij}^R + nt_i^R)_{LR}, \qquad (P_{7.9})$$

$\underline{nt_1} = 0, \quad \overline{nt_1} = 0, \quad nt_1^L = 0, \quad nt_1^R = 0$,

$\underline{nt_j}, \overline{nt_j}$ are real numbers, $\overline{nt_j} \geq \underline{nt_j}$, $nt_j^L \geq 0$, $nt_j^R \geq 0$, $j = 1, 2, ..., n$,

$\underline{nt_{ij}} - nt_{ij}^L \geq 0$, $\overline{nt_{ij}} - \underline{nt_{ij}} \geq 0$, $nt_{ij}^L \geq 0$, $nt_{ij}^R \geq 0$, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$,

$(i, j) \in A$.

Step 5 Using the method, proposed in Section 7.4.1, for converting the fuzzy inequalities into crisp inequalities, the fuzzy linear programming problem $(P_{7.9})$ can be written as:

Minimize $(\underline{nt_n}, \overline{nt_n}, nt_n^L, nt_n^R)_{LR}$

subject to

$\underline{nt_j} - nt_j^L \geq \underline{nt_{ij}} + \underline{nt_i} - nt_{ij}^L - nt_i^L$, $\overline{nt_j} - \underline{nt_j} \geq \overline{nt_{ij}} + \overline{nt_i} - \underline{nt_{ij}} - \underline{nt_i}$, $nt_j^L \geq nt_{ij}^L + nt_i^L$, $nt_j^R \geq nt_{ij}^R + nt_i^R$,

$\underline{nt_1} = 0, \quad \overline{nt_1} = 0, \quad nt_1^L = 0, \quad nt_1^R = 0,$ $(P_{7.10})$

$\underline{nt_j}, \overline{nt_j}$ are real numbers, $\overline{nt_j} \geq \underline{nt_j}$, $nt_j^L \geq 0$, $nt_j^R \geq 0$, $j = 1, 2, ..., n$

$\underline{nt_{ij}} - nt_{ij}^L \geq 0$, $\overline{nt_{ij}} - \underline{nt_{ij}} \geq 0$, $nt_{ij}^L \geq 0$, $nt_{ij}^R \geq 0$, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, $(i, j) \in A$.

Step 6 Suppose the fuzzy linear programming problem $(P_{7.10})$ have h feasible solutions then there is need to find such feasible solution, out of all the feasible solutions, corresponding to which the value of objective function is minimum i.e., to find the minimum of LR flat fuzzy numbers, representing the values of objective function corresponding to all the feasible solutions i.e., if $\{(\underline{nt_j})^t, (\overline{nt_j})^t, (nt_j^L)^t, (nt_j^R)^t\}$ is the t^{th} feasible solution of fuzzy linear programming problem $(P_{7.10})$ then the goal is to find $\underset{1 \leq t \leq h}{\text{minimum}}\{((\underline{nt_n})^t, (\overline{nt_n})^t, (nt_n^L)^t, (nt_n^R)^t)_{LR}\}$ which can be obtained by using the method, proposed in Section 7.4.2, i.e., by using the method proposed in Section 7.4.2, the optimal solution of the fuzzy linear programming problem $(P_{7.10})$ can be obtained as follows:

Step 6(a) Solve the crisp linear programming problem $(P_{7.11})$

Minimize $(\underline{nt_n} - nt_n^L)$

subject to $(P_{7.11})$

Constraints of problem $(P_{7.10})$

Step 6(b) Solve the crisp linear programming problem $(P_{7.12})$

Minimize (nt_n^L)

subject to $(P_{7.12})$

Constraints of problem $(P_{7.10})$ with additional constraint

$\underline{nt_n} - nt_n^L = A_1$

where, A_1 is the optimal value of crisp linear programming problem $(P_{7.11})$

Step 6(c) Solve crisp linear programming problem $(P_{7.13})$

Minimize $(\overline{nt_n} - \underline{nt_n})$

subject to $(P_{7.13})$

Constraints of problem $(P_{7.12})$ with additional constraint

$nt_n^L = A_2$

where, A_2 is the optimal value of crisp linear programming problem $(P_{7.12})$

Step 6(d) Solve crisp linear programming problem $(P_{7.14})$

Minimize (nt_n^R)

subject to $(P_{7.14})$

Constraints of problem $(P_{7.13})$ with additional constraint

$\overline{nt_n} - \underline{nt_n} = A_3$

where, A_3 is the optimal value of crisp linear programming problem $(P_{7.13})$

Step 7 Find the optimal initial fuzzy completion time of the project $(\widetilde{NT_n})$ by putting the optimal values of $\underline{nt_n}, \overline{nt_n}, nt_n^L$ and nt_n^R, obtained from Step 6, in $\widetilde{NT_n} = (\underline{nt_n}, \overline{nt_n}, nt_n^L, nt_n^R)_{LR}$.

7.4.5 Proposed JMD method for finding the unique optimal crash fuzzy project completion time

The decision maker can reduce the optimal initial fuzzy completion time of the project up to the optimal crash fuzzy completion time $(\widetilde{CT_n})$ of the project which can be obtained by using the method, proposed in Section 7.4.4, by replacing the fuzzy linear programming problem ($P_{7.6}$) from the fuzzy linear programming problem ($P_{7.15}$)

Minimize $(\widetilde{CT_n})$

subject to

$$\widetilde{CT_j} \succeq \widetilde{CT_{ij}} \oplus \widetilde{CT_i}, \qquad (P_{7.15})$$

$\widetilde{CT_1} = \widetilde{0}$,

$\widetilde{CT_j}$ are fuzzy numbers,

$\widetilde{CT_{ij}}$ are non-negative fuzzy numbers, $i = 1, 2, ..., n,\ j = 1, 2, ..., n,\ (i, j) \in A$

where,

$\widetilde{CT_i}$: Fuzzy crash time for the node i,

$\widetilde{CT_j}$: Fuzzy crash time for the node j,

$\widetilde{CT_{ij}}$: Fuzzy crash time of the activity (i, j).

7.4.6 Proposed JMD method for finding the unique minimum fuzzy crashing cost for completing the project within specific fuzzy time

In this section, a new method for finding the unique minimum fuzzy crashing cost for completing the project within specific fuzzy time, named as JMD method for finding the unique minimum fuzzy crashing cost for completing the project within specific fuzzy time, is proposed.

The minimum fuzzy crashing cost for completing the project within a specific fuzzy time $\widetilde{PCT}$, for which the condition $\widetilde{CT_n} \preceq \widetilde{PCT} \preceq \widetilde{NT_n}$ is satisfied, can be obtained by using the following steps:

Step 1 If instead of using fuzzy linear programming problem ($P_{7.1}$), obtained from crisp linear programming problem ($P_{7.16}$), fuzzy linear programming problem ($P_{7.18}$), obtained from crisp linear programming problem ($P_{7.17}$) which is an alternative form of crisp linear programming problem ($P_{7.16}$), is used for finding the minimum fuzzy crashing cost for completing the project within specific fuzzy time then the shortcomings of the existing method [30], pointed out in second and third point of Section 7.2, will be resolved. So, formulate the chosen problem into fuzzy linear programming problem ($P_{7.18}$).

$$\text{Minimize} \sum_{(i,j)\in A} \left(C_{ij}\ (NT_{ij} - X_{ij})\right)$$

subject to

$$\left.\begin{array}{l} X_{ij} \geq CT_{ij}, \\ X_{ij} \leq NT_{ij}, \\ T_n \leq PCT, \\ T_i + X_{ij} - T_j \leq 0, \\ T_j, X_{ij} \geq 0,\ i = 1, 2, ..., n,\ j = 1, 2, ..., n,\ (i, j) \in A. \end{array}\right. \qquad (P_{7.16})$$

$$\text{Minimize} \sum_{(i,j)\in A} \left(C_{ij}\ Y_{ij}\right)$$

subject to

$$\left.\begin{array}{l} X_{ij} \geq CT_{ij}, \\ X_{ij} \leq NT_{ij}, \\ Y_{ij} + X_{ij} = NT_{ij}, \\ T_n = PCT, \\ T_1 = 0, \\ T_j \geq T_i + X_{ij}, \\ T_j, X_{ij}, Y_{ij} \geq 0,\ i = 1, 2, ..., n,\ j = 1, 2, ..., n,\ (i, j) \in A. \end{array}\right. \qquad (P_{7.17})$$

$$\text{Minimize} \sum_{(i,j)\in A} (\widetilde{C_{ij}} \otimes \widetilde{Y_{ij}})$$

subject to

$$\widetilde{X_{ij}} \succeq \widetilde{CT_{ij}},$$
$$\widetilde{X_{ij}} \preceq \widetilde{NT_{ij}}, \qquad (P_{7.18})$$
$$\widetilde{Y_{ij}} \oplus \widetilde{X_{ij}} = \widetilde{NT_{ij}},$$
$$\widetilde{T_n} = \widetilde{PCT},$$
$$\widetilde{T_1} = \widetilde{0},$$
$$\widetilde{T_j} \succeq \widetilde{T_i} \oplus \widetilde{X_{ij}},$$

$\widetilde{T_j}, \widetilde{X_{ij}}, \widetilde{Y_{ij}}$ are non-negative LR flat fuzzy numbers, $i = 1, 2, ..., n,\ j = 1, 2, ..., n,$ $(i, j) \in A$.

where,

$\widetilde{Y_{ij}}$: Reduction in fuzzy normal time ($\widetilde{NT_{ij}}$) of the activity for completing the project within specific fuzzy time $\widetilde{PCT}$.

Step 2 Assuming $\widetilde{NT_{ij}} = (\underline{nt_{ij}}, \overline{nt_{ij}}, nt^L_{ij}, nt^R_{ij})_{LR}$, $\widetilde{CT_{ij}} = (\underline{ct_{ij}}, \overline{ct_{ij}}, ct^L_{ij}, ct^R_{ij})_{LR}$, $\widetilde{C_{ij}} = (\underline{c_{ij}}, \overline{c_{ij}}, c^L_{ij}, c^R_{ij})_{LR}$, $\widetilde{Y_{ij}} = (\underline{y_{ij}}, \overline{y_{ij}}, y^L_{ij}, y^R_{ij})_{LR}$, $\widetilde{X_{ij}} = (\underline{x_{ij}}, \overline{x_{ij}}, x^L_{ij}, x^R_{ij})_{LR}$, $\widetilde{PCT} = (\underline{pct}, \overline{pct}, pct^L, pct^R)_{LR}$ and $\widetilde{T_j} = (\underline{t_j}, \overline{t_j}, t^L_j, t^R_j)_{LR}$ the fuzzy linear programming problem $(P_{7.18})$ can be written as:

$$\text{Minimize} \sum_{(i,j)\in A} [(\underline{c_{ij}}, \overline{c_{ij}}, c^L_{ij}, c^R_{ij})_{LR} \otimes (\underline{y_{ij}}, \overline{y_{ij}}, y^L_{ij}, y^R_{ij})_{LR}]$$

subject to

$$(\underline{x_{ij}}, \overline{x_{ij}}, x^L_{ij}, x^R_{ij})_{LR} \succeq (\underline{ct_{ij}}, \overline{ct_{ij}}, ct^L_{ij}, ct^R_{ij})_{LR},$$
$$(\underline{x_{ij}}, \overline{x_{ij}}, x^L_{ij}, x^R_{ij})_{LR} \preceq (\underline{nt_{ij}}, \overline{nt_{ij}}, nt^L_{ij}, nt^R_{ij})_{LR},$$
$$(\underline{y_{ij}}, \overline{y_{ij}}, y^L_{ij}, y^R_{ij})_{LR} \oplus (\underline{x_{ij}}, \overline{x_{ij}}, x^L_{ij}, x^R_{ij})_{LR} = (\underline{nt_{ij}}, \overline{nt_{ij}}, nt^L_{ij}, nt^R_{ij})_{LR},$$
$$(\underline{t_n}, \overline{t_n}, t^L_n, t^R_n)_{LR} = (\underline{pct}, \overline{pct}, pct^L, pct^R)_{LR}, \qquad (P_{7.19})$$
$$(\underline{t_1}, \overline{t_1}, t^L_1, t^R_1)_{LR} = (0, 0, 0, 0)_{LR},$$

$(\underline{t_j}, \overline{t_j}, t_j^L, t_j^R)_{LR} \succeq (\underline{t_i}, \overline{t_i}, t_i^L, t_i^R)_{LR} \oplus (\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR}$,

$(\underline{t_j}, \overline{t_j}, t_j^L, t_j^R)_{LR}$, $(\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR}$, $(\underline{y_{ij}}, \overline{y_{ij}}, y_{ij}^L, y_{ij}^R)_{LR}$ are non-negative LR flat fuzzy numbers, $i = 1, 2, ..., n,\ j = 1, 2, ..., n,\ (i, j) \in A$.

Step 3 Using the arithmetic operations, defined in Section 5.1.2, the fuzzy linear programming problem $(P_{7.19})$ can be written as:

Minimize $\big(\big(\sum_{(i,j)\in A} (\underline{c_{ij}}\,\underline{y_{ij}}),\ \sum_{(i,j)\in A} (\overline{c_{ij}}\,\overline{y_{ij}}),\ \sum_{(i,j)\in A} (\underline{c_{ij}}\,y_{ij}^L + \underline{y_{ij}}\,c_{ij}^L - c_{ij}^L\,y_{ij}^L),\ \sum_{(i,j)\in A} (\overline{c_{ij}}\,y_{ij}^R + \overline{y_{ij}}\,c_{ij}^R + c_{ij}^R\,y_{ij}^R)\big)_{LR}\big)$

subject to

$(\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR} \succeq (\underline{ct_{ij}}, \overline{ct_{ij}}, ct_{ij}^L, ct_{ij}^R)_{LR}$,

$(\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR} \preceq (\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR}$,

$(\underline{y_{ij}} + \underline{x_{ij}}, \overline{y_{ij}} + \overline{x_{ij}}, y_{ij}^L + x_{ij}^L, y_{ij}^R + x_{ij}^R)_{LR} = (\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR}$,

$(\underline{t_n}, \overline{t_n}, t_n^L, t_n^R)_{LR} = (\underline{pct}, \overline{pct}, pct^L, pct^R)_{LR}$, $(P_{7.20})$

$(\underline{t_1}, \overline{t_1}, t_1^L, t_1^R)_{LR} = (0, 0, 0, 0)_{LR}$,

$(\underline{t_j}, \overline{t_j}, t_j^L, t_j^R)_{LR} \succeq (\underline{t_i} + \underline{x_{ij}}, \overline{t_i} + \overline{x_{ij}}, t_i^L + x_{ij}^L, t_i^R + x_{ij}^R)_{LR}$,

$(\underline{t_j}, \overline{t_j}, t_j^L, t_j^R)_{LR}$, $(\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR}$, $(\underline{y_{ij}}, \overline{y_{ij}}, y_{ij}^L, y_{ij}^R)_{LR}$ are non-negative LR flat fuzzy numbers, $i = 1, 2, ..., n,\ j = 1, 2, ..., n,\ (i, j) \in A$.

Step 4 Using Definition 5.4, Definition 5.5 and Definition 5.6, the fuzzy linear programming problem $(P_{7.20})$ can be written as:

Minimize $\big(\big(\sum_{(i,j)\in A} (\underline{c_{ij}}\,\underline{y_{ij}}),\ \sum_{(i,j)\in A} (\overline{c_{ij}}\,\overline{y_{ij}}),\ \sum_{(i,j)\in A} (\underline{c_{ij}}\,y_{ij}^L + \underline{y_{ij}}\,c_{ij}^L - c_{ij}^L\,y_{ij}^L),\ \sum_{(i,j)\in A} (\overline{c_{ij}}\,y_{ij}^R + \overline{y_{ij}}\,c_{ij}^R + c_{ij}^R\,y_{ij}^R)\big)_{LR}\big)$

subject to

$(\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR} \succeq (\underline{ct_{ij}}, \overline{ct_{ij}}, ct_{ij}^L, ct_{ij}^R)_{LR}$,

$(\underline{x_{ij}}, \overline{x_{ij}}, x_{ij}^L, x_{ij}^R)_{LR} \preceq (\underline{nt_{ij}}, \overline{nt_{ij}}, nt_{ij}^L, nt_{ij}^R)_{LR}$,

$\underline{y_{ij}} + \underline{x_{ij}} = \underline{nt_{ij}}, \quad \overline{y_{ij}} + \overline{x_{ij}} = \overline{nt_{ij}}, \quad y_{ij}^L + x_{ij}^L = nt_{ij}^L, \quad y_{ij}^R + x_{ij}^R = nt_{ij}^R$,

$\underline{t_n} = \underline{pct}, \quad \overline{t_n} = \overline{pct}, \quad t_n^L = pct^L, \quad t_n^R = pct^R, \qquad (P_{7.21})$

$\underline{t_1} = 0, \quad \overline{t_1} = 0, \quad t_1^L = 0, \quad t_1^R = 0,$

$(\underline{t_j}, \overline{t_j}, t_j^L, t_j^R)_{LR} \succeq (\underline{t_i} + \underline{x_{ij}}, \overline{t_i} + \overline{x_{ij}}, t_i^L + x_{ij}^L, t_i^R + x_{ij}^R)_{LR},$

$\underline{t_j} - t_j^L \geq 0, \overline{t_j} - \underline{t_j} \geq 0, t_j^L \geq 0, t_j^R \geq 0,$

$\underline{x_{ij}} - x_{ij}^L \geq 0, \overline{x_{ij}} - \underline{x_{ij}} \geq 0, x_{ij}^L \geq 0, x_{ij}^R \geq 0,$

$\underline{y_{ij}} - y_{ij}^L \geq 0, \overline{y_{ij}} - \underline{y_{ij}} \geq 0, y_{ij}^L \geq 0, y_{ij}^R \geq 0.$

Step 5 Using the method, proposed in Section 7.4.1 for converting the fuzzy inequalities into crisp inequalities, the fuzzy linear programming problem $(P_{7.21})$ can be written as:

Minimize $\big((\sum_{(i,j)\in A}(\underline{c_{ij}}\,\underline{y_{ij}}), \sum_{(i,j)\in A}(\overline{c_{ij}}\,\overline{y_{ij}}), \sum_{(i,j)\in A}(\underline{c_{ij}}\,y_{ij}^L + \underline{y_{ij}}\,c_{ij}^L - c_{ij}^L\,y_{ij}^L), \sum_{(i,j)\in A}(\overline{c_{ij}}\,y_{ij}^R +$
$\overline{y_{ij}}\,c_{ij}^R + c_{ij}^R\,y_{ij}^R))_{LR}\big)$

subject to

$\underline{x_{ij}} - x_{ij}^L \geq \underline{ct_{ij}} - ct_{ij}^L, \quad \overline{x_{ij}} - \underline{x_{ij}} \geq \overline{ct_{ij}} - \underline{ct_{ij}}, \quad x_{ij}^L \geq ct_{ij}^L, \quad x_{ij}^R \geq ct_{ij}^R,$

$\underline{x_{ij}} - x_{ij}^L \leq \underline{nt_{ij}} - nt_{ij}^L, \quad \overline{x_{ij}} - \underline{x_{ij}} \leq \overline{nt_{ij}} - \underline{nt_{ij}}, \quad x_{ij}^L \leq nt_{ij}^L, \quad x_{ij}^R \leq nt_{ij}^R,$

$\underline{y_{ij}} + \underline{x_{ij}} = \underline{nt_{ij}}, \quad \overline{y_{ij}} + \overline{x_{ij}} = \overline{nt_{ij}}, \quad y_{ij}^L + x_{ij}^L = nt_{ij}^L, \quad y_{ij}^R + x_{ij}^R = nt_{ij}^R,$

$\underline{t_n} = \underline{pct}, \quad \overline{t_n} = \overline{pct}, \quad t_n^L = pct^L, \quad t_n^R = pct^R, \qquad (P_{7.22})$

$\underline{t_1} = 0, \quad \overline{t_1} = 0, \quad t_1^L = 0, \quad t_1^R = 0,$

$\underline{t_j} - t_j^L \geq \underline{t_i} + \underline{x_{ij}} - t_i^L - x_{ij}^L, \quad \overline{t_j} - \underline{t_j} \geq \overline{t_i} + \overline{x_{ij}} - \underline{t_i} - \underline{x_{ij}}, \quad t_j^L \geq t_i^L + x_{ij}^L, \quad t_j^R \geq t_i^R + x_{ij}^R,$

$\underline{t_j} - t_j^L \geq 0, \overline{t_j} - \underline{t_j} \geq 0, t_j^L \geq 0, t_j^R \geq 0,$

$\underline{x_{ij}} - x_{ij}^L \geq 0, \overline{x_{ij}} - \underline{x_{ij}} \geq 0, x_{ij}^L \geq 0, x_{ij}^R \geq 0,$

$\underline{y_{ij}} - y_{ij}^L \geq 0, \overline{y_{ij}} - \underline{y_{ij}} \geq 0, y_{ij}^L \geq 0, y_{ij}^R \geq 0.$

Step 6 Suppose the fuzzy linear programming problem $(P_{7.22})$ have h feasible solutions then there is need to find such feasible solution, out of all the possible feasible solutions, corresponding to which the value of objective function is minimum i.e., there is need to find the minimum of LR flat fuzzy numbers representing the values of objective function corresponding to all the feasible solutions i.e., if $\{(\underline{x_{ij}})^t,\ (\overline{x_{ij}})^t,\ (\underline{y_{ij}})^t,\ (\overline{y_{ij}})^t,\ (\underline{t_j})^t,\ (\overline{t_j})^t,\ (x^L_{ij})^t,\ (y^L_{ij})^t,\ (t^L_j)^t, (x^R_{ij})^t,\ (y^R_{ij})^t, (t^R_j)^t\}$ is the t^{th} feasible solution of fuzzy linear programming problem $(P_{7.22})$ then the goal is to find $\underset{1\le t\le h}{\text{minimum}}\{(\sum\limits_{(i,j)\in A}((\underline{c_{ij}})^t\ (\underline{y_{ij}})^t),\ \sum\limits_{(i,j)\in A}((\overline{c_{ij}})^t\ (\overline{y_{ij}})^t),$ $\sum\limits_{(i,j)\in A}((\underline{c_{ij}})^t\ (y^L_{ij})^t + (\underline{y_{ij}})^t\ (c^L_{ij})^t - (c^L_{ij})^t\ (y^L_{ij})^t),\ \sum\limits_{(i,j)\in A}((\overline{c_{ij}})^t\ (y^R_{ij})^t + (\overline{y_{ij}})^t\ (c^R_{ij})^t + (c^R_{ij})^t\ (y^R_{ij})^t))_{LR}\}$ which can be obtained by using the method, proposed in Section 7.4.2, i.e., by using the method, proposed in Section 7.4.2, the optimal solution of the fuzzy linear programming problem $(P_{7.22})$ can be obtained as follows:

Step 6(a) Solve the crisp linear programming problem $(P_{7.23})$

Minimize $[\sum\limits_{(i,j)\in A}(\underline{c_{ij}}\ \underline{y_{ij}}) - \sum\limits_{(i,j)\in A}(\underline{c_{ij}}\ y^L_{ij} + \underline{y_{ij}}\ c^L_{ij} - c^L_{ij}\ y^L_{ij})]$

subject to $\qquad (P_{7.23})$

Constraints of problem $(P_{7.22})$.

Step 6(b) Solve the crisp linear programming problem $(P_{7.24})$

Minimize $[\sum\limits_{(i,j)\in A}(\underline{c_{ij}}\ y^L_{ij} + \underline{y_{ij}}\ c^L_{ij} - c^L_{ij}\ y^L_{ij})]$

subject to

Constraints of problem $(P_{7.22})$ with the following additional constraint

$$\sum_{(i,j)\in A}(\underline{c_{ij}}\ \underline{y_{ij}}) - \sum_{(i,j)\in A}(\underline{c_{ij}}\ y^L_{ij} + \underline{y_{ij}}\ c^L_{ij} - c^L_{ij}\ y^L_{ij}) = A_4 \qquad (P_{7.24})$$

where, A_4 is the optimal value of crisp linear programming problem $(P_{7.23})$.

Step 6(c) Solve the crisp linear programming problem $(P_{7.25})$

Minimize $[\sum\limits_{(i,j)\in A}(\overline{c_{ij}}\ \overline{y_{ij}}) - \sum\limits_{(i,j)\in A}(\underline{c_{ij}}\ \underline{y_{ij}})]$

subject to

Constraints of problem $(P_{7.24})$ with the following additional constraint

$$\sum_{(i,j)\in A}(\underline{c_{ij}}\ y^L_{ij}+\underline{y_{ij}}\ c^L_{ij}-c^L_{ij}\ y^L_{ij})=A_5 \qquad (P_{7.25})$$

where, A_5 is the optimal value of crisp linear programming problem $(P_{7.24})$.

Step 6(d) Solve the crisp linear programming problem $(P_{7.26})$

Minimize $[\sum_{(i,j)\in A}(\overline{c_{ij}}\ y^R_{ij}+\overline{y_{ij}}\ c^R_{ij}+c^R_{ij}\ y^R_{ij})]$

subject to

Constraints of problem $(P_{7.25})$ with the following additional constraint

$$\sum_{(i,j)\in A}(\overline{c_{ij}}\ \overline{y_{ij}})-\sum_{(i,j)\in A}(\underline{c_{ij}}\ \underline{y_{ij}})=A_6 \qquad (P_{7.26})$$

where, A_6 is the optimal value of crisp linear programming problem $(P_{7.25})$.

Step 7 Find the fuzzy optimal solution $\{\widetilde{Y_{ij}},\ \widetilde{X_{ij}},\ \widetilde{T_j}\}$ of the fuzzy linear programming problem $(P_{7.18})$, by putting the optimal values of $\underline{y_{ij}},\overline{y_{ij}},y^L_{ij},y^R_{ij},\underline{x_{ij}},\overline{x_{ij}},x^L_{ij},x^R_{ij}$ and $\underline{t_j},\overline{t_j},t^L_j,t^R_j$, obtained from Step 6($d$), in $\widetilde{Y_{ij}}=(\underline{y_{ij}},\overline{y_{ij}},y^L_{ij},y^R_{ij})_{LR}$, $\widetilde{X_{ij}}=(\underline{x_{ij}},\overline{x_{ij}},x^L_{ij},x^R_{ij})_{LR}$ and $\widetilde{T_j}=(\underline{t_j},\overline{t_j},t^L_j,t^R_j)_{LR}$ respectively.

Step 8 Find the fuzzy optimal value of the fuzzy linear programming problem $(P_{7.18})$, by putting the optimal values of $\widetilde{Y_{ij}}$ in $\sum_{(i,j)\in A}(\widetilde{C_{ij}}\ \otimes\ \widetilde{Y_{ij}})$.

Remark 7.1: If all the parameters are represented by LR fuzzy numbers then there is no need of Step 6(c). Also, A_6, used in the Step 6(d), will be replaced by A_5.

7.5 Advantages of proposed JMD methods over existing methods

In this section, advantages of the JMD methods, proposed in Section 7.4, over the existing methods, presented in Section 7.1, Section 7.3.1 and Section 7.3.2 are discussed.

7.5.1 Advantages of proposed JMD method for converting the fuzzy inequality constraints into crisp inequality constraints

In Section 7.3.3.3, it is pointed out that although the fuzzy solution $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$, obtained by using the methods, presented in Section 7.3.1, satisfies the fuzzy constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$. But, it is not always possible to find a non-negative LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$. However, in Proposition 7.1 it is proved that the fuzzy solution $\widetilde{x} = (\underline{x}, \overline{x}, x^L, x^R)_{LR}$, obtained by using the proposed JMD method, will satisfy the fuzzy constraint $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \preceq (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$ as well as there will always exist a non-negative unique LR flat fuzzy number $\widetilde{s} = (\underline{s}, \overline{s}, s^L, s^R)_{LR}$ such that $(\underline{a}, \overline{a}, a^L, a^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{b}, \overline{b}, b^L, b^R)_{LR} \oplus (\underline{s}, \overline{s}, s^L, s^R)_{LR} = (\underline{c}, \overline{c}, c^L, c^R)_{LR} \otimes (\underline{x}, \overline{x}, x^L, x^R)_{LR} \oplus (\underline{d}, \overline{d}, d^L, d^R)_{LR}$.

Hence, it is better to use the proposed JMD method for converting the fuzzy inequality constraints into crisp inequality constraints as compared to the existing methods presented in Section 7.3.1.

7.5.2 Advantages of proposed JMD method for finding the minimum (or maximum) of fuzzy numbers

In Section 7.3.4.4, it is pointed out that if $\widetilde{a}_t$ is the minimum (or maximum) of n fuzzy numbers $\widetilde{a}_1$, $\widetilde{a}_2$,..., $\widetilde{a}_n$, obtained by using the methods, presented in Section 7.3.2, then $\widetilde{a}_t$ is neither necessarily a unique fuzzy number nor it is always possible to find n non-negative fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$,..., $\widetilde{s_n}$ such that $\widetilde{a_t} \oplus \widetilde{s_i} = \widetilde{a_i}$ (or $\widetilde{a_t} = \widetilde{a_i} \oplus \widetilde{s_i}$), $i = 1, 2, ..., n$. Also, it is pointed out that if $\widetilde{a_t}$ is the minimum (or maximum) of n fuzzy numbers $\widetilde{a_1}$, $\widetilde{a_2}$, ..., $\widetilde{a_n}$, obtained by using Mehar's ranking approach proposed in previous chapter, then $\widetilde{a_t}$ is always a unique fuzzy number but it is not always possible to find n non-negative fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$, ..., $\widetilde{s_n}$ such that $\widetilde{a_t} \oplus \widetilde{s_i} = \widetilde{a_i}$ (or $\widetilde{a_t} = \widetilde{a_i} \oplus \widetilde{s_i}$), $i = 1, 2, ..., n$. However, in Proposition 7.2 it is proved that if $\widetilde{a_t}$ is the minimum (maximum) of n fuzzy numbers $\widetilde{a_1}$, $\widetilde{a_2}$,..., $\widetilde{a_n}$, obtained by using the proposed method, then $\widetilde{a_t}$ will always be a unique LR flat fuzzy number and also there will always exist n non-negative LR flat fuzzy numbers $\widetilde{s_1}$, $\widetilde{s_2}$,..., $\widetilde{s_n}$ such that $\widetilde{a_t} \oplus \widetilde{s_i} = \widetilde{a_i}$ (or $\widetilde{a_t} = \widetilde{a_i} \oplus \widetilde{s_i}$), $i = 1, 2, ..., n$.

Hence, it is better to use the proposed JMD method for finding the minimum (or maximum) of fuzzy numbers as compared to the existing methods presented in Section 7.3.2 and proposed Mehar's ranking approach .

7.5.3 Advantages of proposed JMD method for finding the minimum fuzzy crashing cost for completing the project within specific fuzzy time

In this section, the advantages of the proposed JMD method for finding the minimum fuzzy crashing cost for completing the project within specific fuzzy time over the existing method [30] are discussed.

1. Since, by using the proposed JMD method directly fuzzy number, representing the optimal fuzzy times of the activities and minimum fuzzy crashing cost for completing the project within a specific fuzzy time are obtained so, the shortcoming of the existing method [30], pointed out in the first point of the Section 7.2, is resolved.

2. Since, in the proposed JMD method instead of using fuzzy linear programming problem ($P_{7.1}$) the proposed fuzzy linear programming problem ($P_{7.18}$) is used and in the proposed fuzzy linear programming problem ($P_{7.18}$) all the parameters are represented by LR flat fuzzy numbers so, the shortcoming of the existing method [30], pointed out in the second point of the Section 7.2, is resolved.

3. Since, in the proposed JMD method the distributive property is not used. So, the shortcoming of the existing method [30], pointed out in the third point of Section 7.2, is resolved.

4. To obtain the fuzzy optimal solution by using the existing method [30] firstly the lower and upper bound corresponding to different values of α are calculated and then the obtained lower and upper bounds are used to construct the fuzzy number i.e., for obtaining the optimal fuzzy activity times and minimum fuzzy crashing cost for completing the project within a specific fuzzy time by using the existing method [30] there is need to repeat all the steps of the existing method again and again for different values of α. While, in the proposed JMD method there is no need to repeat the process again and again for different values of α and directly a fuzzy number, representing the fuzzy optimal solution, is obtained.

7.6 Illustrative example

Since, in the first point of Section 7.4.3, it is pointed that there is no physical meaning of the existing data, presented in Table 7.2 so the existing data is modified in such a manner that there should always exist a non-negative fuzzy number $\widetilde{S}$ such that fuzzy crash time $\oplus \widetilde{S}$ = fuzzy normal time and the existing problem [30] with modified data, presented in Example 7.1, is solved by the proposed method.

Example 7.1: [30] Using the modified data, presented in Table 7.3, find the optimal initial fuzzy project completion time $(\widetilde{NT_6})$, optimal crash fuzzy completion time $(\widetilde{CT_6})$ and minimum fuzzy crashing cost $(\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes \widetilde{Y_{ij}}))$ for completing the project, shown in Figure 7.2, within specific fuzzy time $\widetilde{PCT} = (39, 10, 4)_{LR}$.

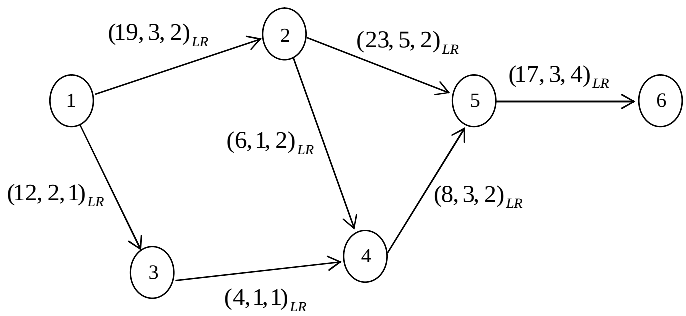

Figure 7.2 Structure of the project network

Table 7.3: Modified fuzzy data

Activity (i, j)	Fuzzy normal time $\widetilde{NT}_{ij}$	Fuzzy crash time $\widetilde{CT}_{ij}$	The incremental direct cost per unit decrease in activity time $\widetilde{C_{ij}}$
(1, 2)	$(19, 3, 2)_{LR}$	$(6, 2, 0)_{LR}$	$(100, 20, 20)_{LR}$
(1, 3)	$(12, 2, 1)_{LR}$	$(8, 1, 1)_{LR}$	$(200, 10, 20)_{LR}$
(2, 5)	$(23, 5, 2)_{LR}$	$(12, 4, 0)_{LR}$	$(100, 10, 30)_{LR}$
(2, 4)	$(6, 1, 2)_{LR}$	$(4, 1, 1)_{LR}$	$(200, 40, 10)_{LR}$
(3, 4)	$(4, 1, 1)_{LR}$	$(2, 0, 1)_{LR}$	$(200, 20, 10)_{LR}$
(4, 5)	$(8, 3, 2)_{LR}$	$(4, 2, 1)_{LR}$	$(100, 20, 10)_{LR}$
(5, 6)	$(17, 3, 4)_{LR}$	$(8, 2, 0)_{LR}$	$(100, 10, 20)_{LR}$

where, $L(x) = R(x) = \max\{0, 1 - x\}$

7.6.1 Optimal initial fuzzy project completion time

Using the method, proposed in Section 7.4.4, the optimal initial fuzzy project completion time for the problem, chosen in Example 7.1, can be obtained as follows:

Step 1 Using the fuzzy linear programming problem $(P_{7.6})$ the chosen problem can be formulated into the following fuzzy linear programming problem:

Minimize $\widetilde{NT_6}$

subject to

$\widetilde{NT_6} \succeq \widetilde{NT_{56}} \oplus \widetilde{NT_5}$, $\widetilde{NT_5} \succeq \widetilde{NT_{45}} \oplus \widetilde{NT_4}$,

$\widetilde{NT_5} \succeq \widetilde{NT_{25}} \oplus \widetilde{NT_2}$, $\widetilde{NT_4} \succeq \widetilde{NT_{24}} \oplus \widetilde{NT_2}$,

$\widetilde{NT_4} \succeq \widetilde{NT_{34}} \oplus \widetilde{NT_3}$, $\widetilde{NT_3} \succeq \widetilde{NT_{13}} \oplus \widetilde{NT_1}$,

$\widetilde{NT_2} \succeq \widetilde{NT_{12}} \oplus \widetilde{NT_1}$,

$\widetilde{NT_1} = \widetilde{0}$,

$\widetilde{NT_1}$, $\widetilde{NT_2}$, $\widetilde{NT_3}$, $\widetilde{NT_4}$, $\widetilde{NT_5}$ and $\widetilde{NT_6}$ are LR fuzzy numbers,

$\widetilde{NT_{12}}$, $\widetilde{NT_{13}}$, $\widetilde{NT_{34}}$, $\widetilde{NT_{24}}$, $\widetilde{NT_{25}}$, $\widetilde{NT_{45}}$ and $\widetilde{NT_{56}}$ are non-negative LR fuzzy numbers.

Step 2 Assuming $\widetilde{NT_i} = (nt_i, nt_i^L, nt_i^R)_{LR}$; $i = 1, 2, 3, 4, 5, 6$, the fuzzy linear programming problem, obtained in Step 1, can be written as:

Minimize $(nt_6, nt_6^L, nt_6^R)_{LR}$

subject to

$(nt_6, nt_6^L, nt_6^R)_{LR} \succeq (17, 3, 4)_{LR} \oplus (nt_5, nt_5^L, nt_5^R)_{LR}$,

$(nt_5, nt_5^L, nt_5^R)_{LR} \succeq (8, 3, 2)_{LR} \oplus (nt_4, nt_4^L, nt_4^R)_{LR}$,

$(nt_5, nt_5^L, nt_5^R)_{LR} \succeq (23, 5, 2)_{LR} \oplus (nt_2, nt_2^L, nt_2^R)_{LR}$,

$(nt_4, nt_4^L, nt_4^R)_{LR} \succeq (6, 1, 2)_{LR} \oplus (nt_2, nt_2^L, nt_2^R)_{LR}$,

$(nt_4, nt_4^L, nt_4^R)_{LR} \succeq (4, 1, 1)_{LR} \oplus (nt_3, nt_3^L, nt_3^R)_{LR}$,

$(nt_3, nt_3^L, nt_3^R)_{LR} \succeq (12, 2, 1)_{LR} \oplus (nt_1, nt_1^L, nt_1^R)_{LR}$,

$(nt_2, nt_2^L, nt_2^R)_{LR} \succeq (19, 3, 2)_{LR} \oplus (nt_1, nt_1^L, nt_1^R)_{LR}$,

$(nt_1, nt_1^L, nt_1^R)_{LR} = (0, 0, 0)_{LR}$,

$(nt_1, nt_1^L, nt_1^R)_{LR}$, $(nt_2, nt_2^L, nt_2^R)_{LR}$, $(nt_3, nt_3^L, nt_3^R)_{LR}$, $(nt_4, nt_4^L, nt_4^R)_{LR}$, $(nt_5,$

$nt_5^L, nt_5^R)_{LR}$ and $(nt_6, nt_6^L, nt_6^R)_{LR}$ are LR fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 5.1.2, the fuzzy linear programming problem, obtained in Step 2, can be written as:

Minimize $(nt_6, nt_6^L, nt_6^R)_{LR}$

subject to

$(nt_6, nt_6^L, nt_6^R)_{LR} \succeq (17 + nt_5, 3 + nt_5^L, 4 + nt_5^R)_{LR}$,

$(nt_5, nt_5^L, nt_5^R)_{LR} \succeq (8 + nt_4, 3 + nt_4^L, 2 + nt_4^R)_{LR}$,

$(nt_5, nt_5^L, nt_5^R)_{LR} \succeq (23 + nt_2, 5 + nt_2^L, 2 + nt_2^R)_{LR}$,

$(nt_4, nt_4^L, nt_4^R)_{LR} \succeq (6 + nt_2, 1 + nt_2^L, 2 + nt_2^R)_{LR}$,

$(nt_4, nt_4^L, nt_4^R)_{LR} \succeq (4 + nt_3, 1 + nt_3^L, 1 + nt_3^R)_{LR}$,

$(nt_3, nt_3^L, nt_3^R)_{LR} \succeq (12 + nt_1, 2 + nt_1^L, 1 + nt_1^R)_{LR}$,

$(nt_2, nt_2^L, nt_2^R)_{LR} \succeq (19 + nt_1, 3 + nt_1^L, 2 + nt_1^R)_{LR}$,

$(nt_1, nt_1^L, nt_1^R)_{LR} = (0, 0, 0)_{LR}$,

$(nt_1, nt_1^L, nt_1^R)_{LR}$, $(nt_2, nt_2^L, nt_2^R)_{LR}$, $(nt_3, nt_3^L, nt_3^R)_{LR}$, $(nt_4, nt_4^L, nt_4^R)_{LR}$, $(nt_5, nt_5^L, nt_5^R)_{LR}$ and $(nt_6, nt_6^L, nt_6^R)_{LR}$ are LR fuzzy numbers.

Step 4 Using Definition 5.4 and Definition 5.6, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Minimize $(nt_6, nt_6^L, nt_6^R)_{LR}$

subject to

$(nt_6, nt_6^L, nt_6^R)_{LR} \succeq (17 + nt_5, 3 + nt_5^L, 4 + nt_5^R)_{LR}$,

$(nt_5, nt_5^L, nt_5^R)_{LR} \succeq (8 + nt_4, 3 + nt_4^L, 2 + nt_4^R)_{LR}$,

$(nt_5, nt_5^L, nt_5^R)_{LR} \succeq (23 + nt_2, 5 + nt_2^L, 2 + nt_2^R)_{LR}$,

$(nt_4, nt_4^L, nt_4^R)_{LR} \succeq (6 + nt_2, 1 + nt_2^L, 2 + nt_2^R)_{LR}$,

$(nt_4, nt_4^L, nt_4^R)_{LR} \succeq (4 + nt_3, 1 + nt_3^L, 1 + nt_3^R)_{LR}$,

$(nt_3, nt_3^L, nt_3^R)_{LR} \succeq (12 + nt_1, 2 + nt_1^L, 1 + nt_1^R)_{LR},$

$(nt_2, nt_2^L, nt_2^R)_{LR} \succeq (19 + nt_1, 3 + nt_1^L, 2 + nt_1^R)_{LR},$

$nt_1 = 0,\ nt_1^L = 0,\ nt_1^R = 0,$

$nt_1, nt_2, nt_3, nt_4, nt_5,\ nt_6$ are real numbers,

$nt_1^L, nt_2^L, nt_3^L, nt_4^L, nt_5^L,\ nt_6^L \geq 0,$

$nt_1^R, nt_2^R, nt_3^R, nt_4^R, nt_5^R,\ nt_6^R \geq 0.$

Step 5 Using the method, proposed in Section 7.4.1, for converting fuzzy inequalities into crisp inequalities, the fuzzy linear programming problem, obtained in Step 4, can be written as:

Minimize $(nt_6, nt_6^L, nt_6^R)_{LR}$

subject to

$nt_6 - nt_6^L \geq 14 + nt_5 - nt_5^L,\ \ nt_6^L \geq 3 + nt_5^L,\ \ nt_6^R \geq 4 + nt_5^R,$

$nt_5 - nt_5^L \geq 5 + nt_4 - nt_4^L,\ \ nt_5^L \geq 3 + nt_4^L,\ \ nt_5^R \geq 2 + nt_4^R,$

$nt_5 - nt_5^L \geq 18 + nt_2 - nt_2^L,\ \ nt_5^L \geq 5 + nt_2^L,\ \ nt_5^R \geq 2 + nt_2^R,$

$nt_4 - nt_4^L \geq 5 + nt_2 - nt_2^L,\ \ nt_4^L \geq 1 + nt_2^L,\ \ nt_4^R \geq 2 + nt_2^R,$

$nt_4 - nt_4^L \geq 3 + nt_3 - nt_3^L,\ \ nt_4^L \geq 1 + nt_3^L,\ \ nt_4^R \geq 1 + nt_3^R,$

$nt_3 - nt_3^L \geq 10 + nt_1 - nt_1^L,\ \ nt_3^L \geq 2 + nt_1^L,\ \ nt_3^R \geq 1 + nt_1^R,$

$nt_2 - nt_2^L \geq 16 + nt_1 - nt_1^L,\ \ nt_2^L \geq 3 + nt_1^L,\ \ nt_2^R \geq 2 + nt_1^R,$

$nt_1 = 0,\ nt_1^L = 0,\ nt_1^R = 0,$

$nt_1, nt_2, nt_3, nt_4, nt_5,\ nt_6$ are real numbers,

$nt_1^L, nt_2^L, nt_3^L, nt_4^L, nt_5^L,\ nt_6^L \geq 0,$

$nt_1^R, nt_2^R, nt_3^R, nt_4^R, nt_5^R,\ nt_6^R \geq 0.$

Step 6 Using the method, proposed in Section 7.4.4, the optimal solution of the fuzzy linear programming problem, obtained in Step 5, can be obtained as follows:

Step 6(a) Solve the following crisp linear programming problem:

Minimize $(nt_6 - nt_6^L)$

subject to

Constraints of the problem obtained in Step 5

Step 6(b) Since, the optimal value of crisp linear programming problem, obtained in Step 6(a) is 48, so solve the following crisp linear programming problem:

Minimize (nt_6^L)

subject to

Constraints of the problem obtained in Step 5 with the following additional constraint

$nt_6 - nt_6^L = 48$

Step 6(c) Since, the optimal value of crisp linear programming problem, obtained in Step 6(b) is 11, so solve the following crisp linear programming problem:

Minimize nt_6^R

subject to

Constraints of the problem obtained in Step 5 with the following additional constraints

$nt_6 - nt_6^L = 48$

$nt_6^L = 11$

Step 7 Since, the optimal values of nt_6, nt_6^L and nt_6^R, obtained from Step 6, are 59, 11 and 10 respectively. So, the optimal initial fuzzy completion time of the project is $\widetilde{NT}_6 = (nt_6, nt_6^L, nt_6^R)_{LR} = (59, 11, 10)_{LR}$.

7.6.2 Optimal crash fuzzy project completion time

Using the method, proposed in Section 7.4.5, the optimal crash fuzzy completion time for the problem, chosen in Example 7.1, can be obtained as follows:

Step 1 Using the fuzzy linear programming problem ($P_{7.15}$) the chosen problem can be formulated into the following fuzzy linear programming problem:

Minimize $\widetilde{CT_6}$

subject to

$\widetilde{CT_6} \succeq \widetilde{CT_{56}} \oplus \widetilde{CT_5}$, $\quad \widetilde{CT_5} \succeq \widetilde{CT_{45}} \oplus \widetilde{CT_4}$,

$\widetilde{CT_5} \succeq \widetilde{CT_{25}} \oplus \widetilde{CT_2}$, $\quad \widetilde{CT_4} \succeq \widetilde{CT_{24}} \oplus \widetilde{CT_2}$,

$\widetilde{CT_4} \succeq \widetilde{CT_{34}} \oplus \widetilde{CT_3}$, $\quad \widetilde{CT_3} \succeq \widetilde{CT_{13}} \oplus \widetilde{CT_1}$,

$\widetilde{CT_2} \succeq \widetilde{CT_{12}} \oplus \widetilde{CT_1}$,

$\widetilde{CT_1} = \widetilde{0}$,

$\widetilde{CT_1}$, $\widetilde{CT_2}$, $\widetilde{CT_3}$, $\widetilde{CT_4}$, $\widetilde{CT_5}$, $\widetilde{CT_6}$ are LR fuzzy numbers,

$\widetilde{CT_{12}}$, $\widetilde{CT_{13}}$, $\widetilde{CT_{34}}$, $\widetilde{CT_{24}}$, $\widetilde{CT_{25}}$, $\widetilde{CT_{45}}$, $\widetilde{CT_{56}}$ are non-negative LR fuzzy numbers.

Step 2 Assuming $\widetilde{CT_i} = (ct_i, ct_i^L, ct_i^R)_{LR}$; $i = 1, 2, 3, 4, 5, 6$, the fuzzy linear programming problem, obtained in Step 1, can be written as:

Minimize $(ct_6, ct_6^L, ct_6^R)_{LR}$

subject to

$(ct_6, ct_6^L, ct_6^R)_{LR} \succeq (8, 2, 0)_{LR} \oplus (ct_5, ct_5^L, ct_5^R)_{LR}$,

$(ct_5, ct_5^L, ct_5^R)_{LR} \succeq (4, 2, 1)_{LR} \oplus (ct_4, ct_4^L, ct_4^R)_{LR}$,

$(ct_5, ct_5^L, ct_5^R)_{LR} \succeq (12, 4, 0)_{LR} \oplus (ct_2, ct_2^L, ct_2^R)_{LR}$,

$(ct_4, ct_4^L, ct_4^R)_{LR} \succeq (4, 1, 1)_{LR} \oplus (ct_2, ct_2^L, ct_2^R)_{LR}$,

$(ct_4, ct_4^L, ct_4^R)_{LR} \succeq (2, 0, 1)_{LR} \oplus (ct_3, ct_3^L, ct_3^R)_{LR}$,

$(ct_3, ct_3^L, ct_3^R)_{LR} \succeq (8, 1, 1)_{LR} \oplus (ct_1, ct_1^L, ct_1^R)_{LR}$,

$(ct_2, ct_2^L, ct_2^R)_{LR} \succeq (6, 2, 0)_{LR} \oplus (ct_1, ct_1^L, ct_1^R)_{LR}$,

$(ct_1, ct_1^L, ct_1^R)_{LR} = (0, 0, 0)_{LR}$,

$(ct_1, ct_1^L, ct_1^R)_{LR}$, $(ct_2, ct_2^L, ct_2^R)_{LR}$, $(ct_3, ct_3^L, ct_3^R)_{LR}$, $(ct_4, ct_4^L, ct_4^R)_{LR}$, $(ct_5, ct_5^L, ct_5^R)_{LR}$, $(ct_6, ct_6^L, ct_6^R)_{LR}$ are LR fuzzy numbers.

Step 3 Using the arithmetic operations, defined in Section 5.1.2, the fuzzy linear programming problem, obtained in Step 2, can be written as:

Minimize $(ct_6, ct_6^L, ct_6^R)_{LR}$

subject to

$(ct_6, ct_6^L, ct_6^R)_{LR} \succeq (8 + ct_5, 2 + ct_5^L, 0 + ct_5^R)_{LR}$,

$(ct_5, ct_5^L, ct_5^R)_{LR} \succeq (4 + ct_4, 2 + ct_4^L, 1 + ct_4^R)_{LR}$,

$(ct_5, ct_5^L, ct_5^R)_{LR} \succeq (12 + ct_2, 4 + ct_2^L, 0 + ct_2^R)_{LR}$,

$(ct_4, ct_4^L, ct_4^R)_{LR} \succeq (4 + ct_2, 1 + ct_2^L, 1 + ct_2^R)_{LR}$,

$(ct_4, ct_4^L, ct_4^R)_{LR} \succeq (2 + ct_3, 0 + ct_3^L, 1 + ct_3^R)_{LR}$,

$(ct_3, ct_3^L, ct_3^R)_{LR} \succeq (8 + ct_1, 1 + ct_1^L, 1 + ct_1^R)_{LR}$,

$(ct_2, ct_2^L, ct_2^R)_{LR} \succeq (6 + ct_1, 2 + ct_1^L, 0 + ct_1^R)_{LR}$,

$(ct_1, ct_1^L, ct_1^R)_{LR} = (0, 0, 0)_{LR}$,

$(ct_1, ct_1^L, ct_1^R)_{LR}$, $(ct_2, ct_2^L, ct_2^R)_{LR}$, $(ct_3, ct_3^L, ct_3^R)_{LR}$, $(ct_4, ct_4^L, ct_4^R)_{LR}$, $(ct_5, ct_5^L, ct_5^R)_{LR}$, $(ct_6, ct_6^L, ct_6^R)_{LR}$ are LR fuzzy numbers.

Step 4 Using Definition 5.4 and Definition 5.6, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Minimize $(ct_6, ct_6^L, ct_6^R)_{LR}$

subject to

$(ct_6, ct_6^L, ct_6^R)_{LR} \succeq (8 + ct_5, 2 + ct_5^L, 0 + ct_5^R)_{LR}$,

$(ct_5, ct_5^L, ct_5^R)_{LR} \succeq (4 + ct_4, 2 + ct_4^L, 1 + ct_4^R)_{LR},$

$(ct_5, ct_5^L, ct_5^R)_{LR} \succeq (12 + ct_2, 4 + ct_2^L, 0 + ct_2^R)_{LR},$

$(ct_4, ct_4^L, ct_4^R)_{LR} \succeq (4 + ct_2, 1 + ct_2^L, 1 + ct_2^R)_{LR},$

$(ct_4, ct_4^L, ct_4^R)_{LR} \succeq (2 + ct_3, 0 + ct_3^L, 1 + ct_3^R)_{LR},$

$(ct_3, ct_3^L, ct_3^R)_{LR} \succeq (8 + ct_1, 1 + ct_1^L, 1 + ct_1^R)_{LR},$

$(ct_2, ct_2^L, ct_2^R)_{LR} \succeq (6 + ct_1, 2 + ct_1^L, 0 + ct_1^R)_{LR},$

$ct_1 = 0,\ ct_1^L = 0,\ ct_1^R = 0,$

$ct_1, ct_2, ct_3, ct_4, ct_5,\ ct_6$ are real numbers,

$ct_1^L, ct_2^L, ct_3^L, ct_4^L, ct_5^L,\ ct_6^L \geq 0,$

$ct_1^R, ct_2^R, ct_3^R, ct_4^R, ct_5^R,\ ct_6^R \geq 0.$

Step 5 Using the method, proposed in Section 7.4.1, for converting fuzzy inequality into crisp inequalities, the fuzzy linear programming problem, obtained in Step 4, can be written as:

Minimize $(ct_6, ct_6^L, ct_6^R)_{LR}$

subject to

$ct_6 - ct_6^L \geq 6 + ct_5 - ct_5^L,\ \ ct_6^L \geq 2 + ct_5^L,\ \ ct_6^R \geq 0 + ct_5^R,$

$ct_5 - ct_5^L \geq 2 + ct_4 - ct_4^L,\ \ ct_5^L \geq 2 + ct_4^L,\ \ ct_5^R \geq 1 + ct_4^R,$

$ct_5 - ct_5^L \geq 8 + ct_2 - ct_2^L,\ \ ct_5^L \geq 4 + ct_2^L,\ \ ct_5^R \geq 0 + ct_2^R,$

$ct_4 - ct_4^L \geq 3 + ct_2 - ct_2^L,\ \ ct_4^L \geq 1 + ct_2^L,\ \ ct_4^R \geq 1 + ct_2^R,$

$ct_4 - ct_4^L \geq 2 + ct_3 - ct_3^L,\ \ ct_4^L \geq 0 + ct_3^L,\ \ ct_4^R \geq 1 + ct_3^R,$

$ct_3 - ct_3^L \geq 7 + ct_1 - ct_1^L,\ \ ct_3^L \geq 1 + ct_1^L,\ \ ct_3^R \geq 1 + ct_1^R,$

$ct_2 - ct_2^L \geq 4 + ct_1 - ct_1^L,\ \ ct_2^L \geq 2 + ct_1^L,\ \ ct_2^R \geq 0 + ct_1^R,$

$ct_1 = 0,\ ct_1^L = 0,\ ct_1^R = 0,$

$ct_1, ct_2, ct_3, ct_4, ct_5,\ ct_6$ are real numbers,

$ct_1^L, ct_2^L, ct_3^L, ct_4^L, ct_5^L,\ ct_6^L \geq 0,$

$ct_1^R, ct_2^R, ct_3^R, ct_4^R, ct_5^R,\ ct_6^R \geq 0.$

Step 6 Using the method, proposed in Section 7.4.4, the optimal solution of the fuzzy linear programming problem, obtained in Step 5, can be obtained as follows:

Step 6(a) Solve the following crisp linear programming problem:

Minimize $(ct_6 - ct_6^L)$

subject to

Constraints of the problem obtained in Step 5.

Step 6(b) Since, the optimal value of crisp linear programming problem, obtained in Step 6(a) is 18, so solve the following crisp linear programming problem:

Minimize (ct_6^L)

subject to

Constraints of the problem obtained in Step 5 with the following additional constraint

$ct_6 - ct_6^L = 18$

Step 6(c) Since, the optimal value of crisp linear programming, obtained in Step 6(b) is 8, so solve the following crisp linear programming problem:

Minimize (ct_6^R)

subject to

Constraints of the problem obtained in Step 5 with the following additional constraints

$ct_6 - ct_6^L = 18$

$ct_6^L = 8$

Step 7 Since, the optimal values of ct_6, ct_6^L and ct_6^R, obtained from Step 6, are 26, 8 and 3 respectively. So, the optimal crash fuzzy completion time of the project is $\widetilde{CT_6} = (ct_6, ct_6^L, ct_6^R)_{LR} = (26, 8, 3)_{LR}$.

7.6.3 Minimum fuzzy crashing cost for completing the project within specific fuzzy time

Using the method, proposed in Section 7.4.6, the minimum fuzzy crashing cost for completing the project within specific fuzzy time $(39, 10, 4)_{LR}$ can be obtained as follows:

Step 1 Using the proposed fuzzy linear programming problem $(P_{7.18})$, the chosen problem can be formulated into the following fuzzy linear programming problem:

$$\text{Minimize } (\widetilde{C_{12}} \otimes \widetilde{Y_{12}} \oplus \widetilde{C_{13}} \otimes \widetilde{Y_{13}} \oplus \widetilde{C_{25}} \otimes \widetilde{Y_{25}} \oplus \widetilde{C_{24}} \otimes \widetilde{Y_{24}} \oplus \widetilde{C_{34}} \otimes \widetilde{Y_{34}} \oplus \widetilde{C_{45}} \otimes \widetilde{Y_{45}} \oplus \widetilde{C_{56}} \otimes \widetilde{Y_{56}})$$

subject to

$$\widetilde{X_{12}} \succeq \widetilde{CT_{12}}, \qquad \widetilde{X_{12}} \preceq \widetilde{NT_{12}},$$

$$\widetilde{X_{13}} \succeq \widetilde{CT_{13}}, \qquad \widetilde{X_{13}} \preceq \widetilde{NT_{13}},$$

$$\widetilde{X_{25}} \succeq \widetilde{CT_{25}}, \qquad \widetilde{X_{25}} \preceq \widetilde{NT_{25}},$$

$$\widetilde{X_{24}} \succeq \widetilde{CT_{24}}, \qquad \widetilde{X_{24}} \preceq \widetilde{N_{24}},$$

$$\widetilde{X_{34}} \succeq \widetilde{CT_{34}}, \qquad \widetilde{X_{34}} \preceq \widetilde{NT_{34}},$$

$$\widetilde{X_{45}} \succeq \widetilde{CT_{45}}, \qquad \widetilde{X_{45}} \preceq \widetilde{NT_{45}},$$

$$\widetilde{X_{56}} \succeq \widetilde{CT_{56}}, \qquad \widetilde{X_{56}} \preceq \widetilde{NT_{56}},$$

$$\widetilde{Y_{12}} \oplus \widetilde{X_{12}} = \widetilde{NT_{12}}, \qquad \widetilde{T_2} \succeq \widetilde{T_1} \oplus \widetilde{X_{12}},$$

$$\widetilde{Y_{13}} \oplus \widetilde{X_{13}} = \widetilde{NT_{13}}, \qquad \widetilde{T_3} \succeq \widetilde{T_1} \oplus \widetilde{X_{13}},$$

$$\widetilde{Y_{25}} \oplus \widetilde{X_{25}} = \widetilde{NT_{25}}, \qquad \widetilde{T_5} \succeq \widetilde{T_2} \oplus \widetilde{X_{25}},$$

$$\widetilde{Y_{24}} \oplus \widetilde{X_{24}} = \widetilde{NT_{24}}, \qquad \widetilde{T_4} \succeq \widetilde{T_2} \oplus \widetilde{X_{24}},$$

$$\widetilde{Y_{34}} \oplus \widetilde{X_{34}} = \widetilde{NT_{34}}, \qquad \widetilde{T_4} \succeq \widetilde{T_3} \oplus \widetilde{X_{34}},$$

$\widetilde{Y_{45}} \oplus \widetilde{X_{45}} = \widetilde{NT_{45}}, \qquad \widetilde{T_5} \succeq \widetilde{T_4} \oplus \widetilde{X_{45}},$

$\widetilde{Y_{56}} \oplus \widetilde{X_{56}} = \widetilde{NT_{56}}, \qquad \widetilde{T_6} \succeq \widetilde{T_5} \oplus \widetilde{X_{56}},$

$\widetilde{T_6} = \widetilde{PCT}, \qquad \widetilde{T_1} = \widetilde{0},$

$\widetilde{T_1}$, $\widetilde{T_2}$, $\widetilde{T_3}$, $\widetilde{T_4}$, $\widetilde{T_5}$, $\widetilde{T_6}$, $\widetilde{X_{12}}$, $\widetilde{X_{13}}$, $\widetilde{X_{25}}$, $\widetilde{X_{24}}$, $\widetilde{X_{34}}$, $\widetilde{X_{45}}$, $\widetilde{X_{56}}$, $\widetilde{Y_{12}}$, $\widetilde{Y_{13}}$, $\widetilde{Y_{25}}$, $\widetilde{Y_{24}}$, $\widetilde{Y_{34}}$, $\widetilde{Y_{45}}$, $\widetilde{Y_{56}}$ are non-negative LR fuzzy numbers.

Step 2 Assuming $\widetilde{T_1} = (t_1, t_1^L, t_1^R)_{LR}$, $\widetilde{T_2} = (t_2, t_2^L, t_2^R)_{LR}$, $\widetilde{T_3} = (t_3, t_3^L, t_3^R)_{LR}$, $\widetilde{T_4} = (t_4, t_4^L, t_4^R)_{LR}$, $\widetilde{T_5} = (t_5, t_5^L, t_5^R)_{LR}$, $\widetilde{T_6} = (t_6, t_6^L, t_6^R)_{LR}$, $\widetilde{X_{12}} = (x_{12}, x_{12}^L, x_{12}^R)_{LR}$, $\widetilde{X_{13}} = (x_{13}, x_{13}^L, x_{13}^R)_{LR}$, $\widetilde{X_{25}} = (x_{25}, x_{25}^L, x_{25}^R)_{LR}$, $\widetilde{X_{24}} = (x_{24}, x_{24}^L, x_{24}^R)_{LR}$, $\widetilde{X_{34}} = (x_{34}, x_{34}^L, x_{34}^R)_{LR}$, $\widetilde{X_{45}} = (x_{45}, x_{45}^L, x_{45}^R)_{LR}$, $\widetilde{X_{56}} = (x_{56}, x_{56}^L, x_{56}^R)_{LR}$, $\widetilde{Y_{12}} = (y_{12}, y_{12}^L, y_{12}^R)_{LR}$, $\widetilde{Y_{13}} = (y_{13}, y_{13}^L, y_{13}^R)_{LR}$, $\widetilde{Y_{25}} = (y_{25}, y_{25}^L, y_{25}^R)_{LR}$, $\widetilde{Y_{24}} = (y_{24}, y_{24}^L, y_{24}^R)_{LR}$, $\widetilde{Y_{34}} = (y_{34}, y_{34}^L, y_{34}^R)_{LR}$, $\widetilde{Y_{45}} = (y_{45}, y_{45}^L, y_{45}^R)_{LR}$ and $\widetilde{Y_{56}} = (y_{56}, y_{56}^L, y_{56}^R)_{LR}$, the fuzzy linear programming problem, obtained in Step 1, can be written as:

Minimize $\big((100,20,20)_{LR} \otimes (y_{12}, y_{12}^L, y_{12}^R)_{LR} \oplus (200,10,20)_{LR} \otimes (y_{13}, y_{13}^L, y_{13}^R)_{LR} \oplus (100,10,30)_{LR} \otimes (y_{25}, y_{25}^L, y_{25}^R)_{LR} \oplus (200,40,10)_{LR} \otimes (y_{24}, y_{24}^L, y_{24}^R)_{LR} \oplus (200,20,10)_{LR} \otimes (y_{34}, y_{34}^L, y_{34}^R)_{LR} \oplus (100,20,10)_{LR} \otimes (y_{45}, y_{45}^L, y_{45}^R)_{LR} \oplus (100,10,20)_{LR} \otimes (y_{56}, y_{56}^L, y_{56}^R)_{LR}\big)$

subject to

$(x_{12}, x_{12}^L, x_{12}^R)_{LR} \succeq (6,2,0)_{LR}, \qquad (x_{12}, x_{12}^L, x_{12}^R)_{LR} \preceq (19,3,2)_{LR},$

$(x_{13}, x_{13}^L, x_{13}^R)_{LR} \succeq (8,1,1)_{LR}, \qquad (x_{13}, x_{13}^L, x_{13}^R)_{LR} \preceq (12,2,1)_{LR},$

$(x_{25}, x_{25}^L, x_{25}^R)_{LR} \succeq (12,4,0)_{LR}, \qquad (x_{25}, x_{25}^L, x_{25}^R)_{LR} \preceq (23,5,2)_{LR},$

$(x_{24}, x_{24}^L, x_{24}^R)_{LR} \succeq (4,1,1)_{LR}, \qquad (x_{24}, x_{24}^L, x_{24}^R)_{LR} \preceq (6,1,2)_{LR},$

$(x_{34}, x_{34}^L, x_{34}^R)_{LR} \succeq (2,0,1)_{LR}, \qquad (x_{34}, x_{34}^L, x_{34}^R)_{LR} \preceq (4,1,1)_{LR},$

$(x_{45}, x_{45}^L, x_{45}^R)_{LR} \succeq (4,2,1)_{LR}, \qquad (x_{45}, x_{45}^L, x_{45}^R)_{LR} \preceq (8,3,2)_{LR},$

$(x_{56}, x_{56}^L, x_{56}^R)_{LR} \succeq (8,2,0)_{LR}, \qquad (x_{56}, x_{56}^L, x_{56}^R)_{LR} \preceq (17,3,4)_{LR},$

$(y_{12}, y_{12}^L, y_{12}^R)_{LR} \oplus (x_{12}, x_{12}^L, x_{12}^R)_{LR} = (19,3,2)_{LR},$

$(y_{13}, y_{13}^L, y_{13}^R)_{LR} \oplus (x_{13}, x_{13}^L, x_{13}^R)_{LR} = (12, 2, 1)_{LR},$

$(y_{25}, y_{25}^L, y_{25}^R)_{LR} \oplus (x_{25}, x_{25}^L, x_{25}^R)_{LR} = (23, 5, 2)_{LR},$

$(y_{24}, y_{24}^L, y_{24}^R)_{LR} \oplus (x_{24}, x_{24}^L, x_{24}^R)_{LR} = (6, 1, 2)_{LR},$

$(y_{34}, y_{34}^L, y_{34}^R)_{LR} \oplus (x_{34}, x_{34}^L, x_{34}^R)_{LR} = (4, 1, 1)_{LR},$

$(y_{45}, y_{45}^L, y_{45}^R)_{LR} \oplus (x_{45}, x_{45}^L, x_{45}^R)_{LR} = (8, 3, 2)_{LR},$

$(y_{56}, y_{56}^L, y_{56}^R)_{LR} \oplus (x_{56}, x_{56}^L, x_{56}^R)_{LR} = (17, 3, 4)_{LR},$

$(t_6, t_6^L, t_6^R)_{LR} = (39, 10, 4)_{LR},$

$(t_1, t_1^L, t_1^R)_{LR} = (0, 0, 0)_{LR},$

$(t_2, t_2^L, t_2^R)_{LR} \succeq (t_1, t_1^L, t_1^R)_{LR} \oplus (x_{12}, x_{12}^L, x_{12}^R)_{LR},$

$(t_3, t_3^L, t_3^R)_{LR} \succeq (t_1, t_1^L, t_1^R)_{LR} \oplus (x_{13}, x_{13}^L, x_{13}^R)_{LR},$

$(t_5, t_5^L, t_5^R)_{LR} \succeq (t_2, t_2^L, t_2^R)_{LR} \oplus (x_{25}, x_{25}^L, x_{25}^R)_{LR},$

$(t_4, t_4^L, t_4^R)_{LR} \succeq (t_2, t_2^L, t_2^R)_{LR} \oplus (x_{24}, x_{24}^L, x_{24}^R)_{LR},$

$(t_4, t_4^L, t_4^R)_{LR} \succeq (t_3, t_3^L, t_3^R)_{LR} \oplus (x_{34}, x_{34}^L, x_{34}^R)_{LR},$

$(t_5, t_5^L, t_5^R)_{LR} \succeq (t_4, t_4^L, t_4^R)_{LR} \oplus (x_{45}, x_{45}^L, x_{45}^R)_{LR},$

$(t_6, t_6^L, t_6^R)_{LR} \succeq (t_5, t_5^L, t_5^R)_{LR} \oplus (x_{56}, x_{56}^L, x_{56}^R)_{LR},$

$(t_1, t_1^L, t_1^R)_{LR}$, $(t_2, t_2^L, t_2^R)_{LR}$, $(t_3, t_3^L, t_3^R)_{LR}$, $(t_4, t_4^L, t_4^R)_{LR}$, $(t_5, t_5^L, t_5^R)_{LR}$, $(t_6, t_6^L, t_6^R)_{LR}$, $(x_{12}, x_{12}^L, x_{12}^R)_{LR}$, $(x_{13}, x_{13}^L, x_{13}^R)_{LR}$, $(x_{25}, x_{25}^L, x_{25}^R)_{LR}$, $(x_{24}, x_{24}^L, x_{24}^R)_{LR}$, $(x_{34}, x_{34}^L, x_{34}^R)_{LR}$, $(x_{45}, x_{45}^L, x_{45}^R)_{LR}$, $(x_{56}, x_{56}^L, x_{56}^R)_{LR}$, $(y_{12}, y_{12}^L, y_{12}^R)_{LR}$, $(y_{13}, y_{13}^L, y_{13}^R)_{LR}$, $(y_{25}, y_{25}^L, y_{25}^R)_{LR}$, $(y_{24}, y_{24}^L, y_{24}^R)_{LR}$, $(y_{34}, y_{34}^L, y_{34}^R)_{LR}$, $(y_{45}, y_{45}^L, y_{45}^R)_{LR}$, $(y_{56}, y_{56}^L, y_{56}^R)_{LR}$ are non-negative LR fuzzy numbers.

Step 3 Using the arithmetic operations, the fuzzy linear programming problem, obtained in Step 2, can be written as:

Minimize $\big((100y_{12} + 200y_{13} + 100y_{25} + 200y_{24} + 200y_{34} + 100y_{45} + 100y_{56}, 80y_{12}^L + 20y_{12} + 190y_{13}^L + 10y_{13} + 90y_{25}^L + 10y_{25} + 160y_{24}^L + 40y_{24} + 180y_{34}^L + 20y_{34} + 80y_{45}^L +$

$20y_{45} + 90y_{56}^L + 10y_{56}, 120y_{12}^R + 20y_{12} + 220y_{13}^R + 20y_{13} + 130y_{25}^R + 30y_{25} + 210y_{24}^R +$

$10y_{24} + 210y_{34}^R + 10y_{34} + 110y_{45}^R + 10y_{45} + 120y_{56}^R + 20y_{56})_{LR}\big)$

subject to

$$(x_{12}, x_{12}^L, x_{12}^R)_{LR} \succeq (6, 2, 0)_{LR}, \qquad (x_{12}, x_{12}^L, x_{12}^R)_{LR} \preceq (19, 3, 2)_{LR},$$

$$(x_{13}, x_{13}^L, x_{13}^R)_{LR} \succeq (8, 1, 1)_{LR}, \qquad (x_{13}, x_{13}^L, x_{13}^R)_{LR} \preceq (12, 2, 1)_{LR},$$

$$(x_{25}, x_{25}^L, x_{25}^R)_{LR} \succeq (12, 4, 0)_{LR}, \qquad (x_{25}, x_{25}^L, x_{25}^R)_{LR} \preceq (23, 5, 2)_{LR},$$

$$(x_{24}, x_{24}^L, x_{24}^R)_{LR} \succeq (4, 1, 1)_{LR}, \qquad (x_{24}, x_{24}^L, x_{24}^R)_{LR} \preceq (6, 1, 2)_{LR},$$

$$(x_{34}, x_{34}^L, x_{34}^R)_{LR} \succeq (2, 0, 1)_{LR}, \qquad (x_{34}, x_{34}^L, x_{34}^R)_{LR} \preceq (4, 1, 1)_{LR},$$

$$(x_{45}, x_{45}^L, x_{45}^R)_{LR} \succeq (4, 2, 1)_{LR}, \qquad (x_{45}, x_{45}^L, x_{45}^R)_{LR} \preceq (8, 3, 2)_{LR},$$

$$(x_{56}, x_{56}^L, x_{56}^R)_{LR} \succeq (8, 2, 0)_{LR}, \qquad (x_{56}, x_{56}^L, x_{56}^R)_{LR} \preceq (17, 3, 4)_{LR},$$

$$(y_{12} + x_{12}, y_{12}^L + x_{12}^L, y_{12}^R + x_{12}^R)_{LR} = (19, 3, 2)_{LR},$$

$$(y_{13} + x_{13}, y_{13}^L + x_{13}^L, y_{13}^R + x_{13}^R)_{LR} = (12, 2, 1)_{LR},$$

$$(y_{25} + x_{25}, y_{25}^L + x_{25}^L, y_{25}^R + x_{25}^R)_{LR} = (23, 5, 2)_{LR},$$

$$(y_{24} + x_{24}, y_{24}^L + x_{24}^L, y_{24}^R + x_{24}^R)_{LR} = (6, 1, 2)_{LR},$$

$$(y_{34} + x_{34}, y_{34}^L + x_{34}^L, y_{34}^R + x_{34}^R)_{LR} = (4, 1, 1)_{LR},$$

$$(y_{45} + x_{45}, y_{45}^L + x_{45}^L, y_{45}^R + x_{45}^R)_{LR} = (8, 3, 2)_{LR},$$

$$(y_{56} + x_{56}, y_{56}^L + x_{56}^L, y_{56}^R + x_{56}^R)_{LR} = (17, 3, 4)_{LR},$$

$$(t_6, t_6^L, t_6^R)_{LR} = (39, 10, 4)_{LR},$$

$$(t_1, t_1^L, t_1^R)_{LR} = (0, 0, 0)_{LR},$$

$$(t_2, t_2^L, t_2^R)_{LR} \succeq (t_1 + x_{12}, t_1^L + x_{12}^L, t_1^R + x_{12}^R)_{LR},$$

$$(t_3, t_3^L, t_3^R)_{LR} \succeq (t_1 + x_{13}, t_1^L + x_{13}^L, t_1^R + x_{13}^R)_{LR},$$

$$(t_5, t_5^L, t_5^R)_{LR} \succeq (t_2 + x_{25}, t_2^L + x_{25}^L, t_2^R + x_{25}^R)_{LR},$$

$$(t_4, t_4^L, t_4^R)_{LR} \succeq (t_2 + x_{24}, t_2^L + x_{24}^L, t_2^R + x_{24}^R)_{LR},$$

$$(t_4, t_4^L, t_4^R)_{LR} \succeq (t_3 + x_{34}, t_3^L + x_{34}^L, t_3^R + x_{34}^R)_{LR},$$

$(t_5, t_5^L, t_5^R)_{LR} \succeq (t_4 + x_{45}, t_4^L + x_{45}^L, t_4^R + x_{45}^R)_{LR},$

$(t_6, t_6^L, t_6^R)_{LR} \succeq (t_5 + x_{56}, t_5^L + x_{56}^L, t_5^R + x_{56}^R)_{LR},$

$(t_1, t_1^L, t_1^R)_{LR}$, $(t_2, t_2^L, t_2^R)_{LR}$, $(t_3, t_3^L, t_3^R)_{LR}$, $(t_4, t_4^L, t_4^R)_{LR}$, $(t_5, t_5^L, t_5^R)_{LR}$, $(t_6, t_6^L, t_6^R)_{LR}$, $(x_{12}, x_{12}^L, x_{12}^R)_{LR}$, $(x_{13}, x_{13}^L, x_{13}^R)_{LR}$, $(x_{25}, x_{25}^L, x_{25}^R)_{LR}$, $(x_{24}, x_{24}^L, x_{24}^R)_{LR}$, $(x_{34}, x_{34}^L, x_{34}^R)_{LR}$, $(x_{45}, x_{45}^L, x_{45}^R)_{LR}$, $(x_{56}, x_{56}^L, x_{56}^R)_{LR}$, $(y_{12}, y_{12}^L, y_{12}^R)_{LR}$, $(y_{13}, y_{13}^L, y_{13}^R)_{LR}$, $(y_{25}, y_{25}^L, y_{25}^R)_{LR}$, $(y_{24}, y_{24}^L, y_{24}^R)_{LR}$, $(y_{34}, y_{34}^L, y_{34}^R)_{LR}$, $(y_{45}, y_{45}^L, y_{45}^R)_{LR}$, $(y_{56}, y_{56}^L, y_{56}^R)_{LR}$ are non-negative LR fuzzy numbers.

Step 4 Using the Definition 5.4, Definition 5.5 and Definition 5.6, the fuzzy linear programming problem, obtained in Step 3, can be written as:

Minimize $\big((100y_{12} + 200y_{13} + 100y_{25} + 200y_{24} + 200y_{34} + 100y_{45} + 100y_{56}, 80y_{12}^L + 20y_{12} + 190y_{13}^L + 10y_{13} + 90y_{25}^L + 10y_{25} + 160y_{24}^L + 40y_{24} + 180y_{34}^L + 20y_{34} + 80y_{45}^L + 20y_{45} + 90y_{56}^L + 10y_{56}, 120y_{12}^R + 20y_{12} + 220y_{13}^R + 20y_{13} + 130y_{25}^R + 30y_{25} + 210y_{24}^R + 10y_{24} + 210y_{34}^R + 10y_{34} + 110y_{45}^R + 10y_{45} + 120y_{56}^R + 20y_{56})_{LR}\big)$

subject to

$(x_{12}, x_{12}^L, x_{12}^R)_{LR} \succeq (6, 2, 0)_{LR}, \quad (x_{12}, x_{12}^L, x_{12}^R)_{LR} \preceq (19, 3, 2)_{LR},$

$(x_{13}, x_{13}^L, x_{13}^R)_{LR} \succeq (8, 1, 1)_{LR}, \quad (x_{13}, x_{13}^L, x_{13}^R)_{LR} \preceq (12, 2, 1)_{LR},$

$(x_{25}, x_{25}^L, x_{25}^R)_{LR} \succeq (12, 4, 0)_{LR}, \quad (x_{25}, x_{25}^L, x_{25}^R)_{LR} \preceq (23, 5, 2)_{LR},$

$(x_{24}, x_{24}^L, x_{24}^R)_{LR} \succeq (4, 1, 1)_{LR}, \quad (x_{24}, x_{24}^L, x_{24}^R)_{LR} \preceq (6, 1, 2)_{LR},$

$(x_{34}, x_{34}^L, x_{34}^R)_{LR} \succeq (2, 0, 1)_{LR}, \quad (x_{34}, x_{34}^L, x_{34}^R)_{LR} \preceq (4, 1, 1)_{LR},$

$(x_{45}, x_{45}^L, x_{45}^R)_{LR} \succeq (4, 2, 1)_{LR}, \quad (x_{45}, x_{45}^L, x_{45}^R)_{LR} \preceq (8, 3, 2)_{LR},$

$(x_{56}, x_{56}^L, x_{56}^R)_{LR} \succeq (8, 2, 0)_{LR}, \quad (x_{56}, x_{56}^L, x_{56}^R)_{LR} \preceq (17, 3, 4)_{LR},$

$y_{12} + x_{12} = 19,\ y_{12}^L + x_{12}^L = 3,\ y_{12}^R + x_{12}^R = 2,$

$y_{13} + x_{13} = 12,\ y_{13}^L + x_{13}^L = 2,\ y_{13}^R + x_{13}^R = 1,$

$y_{25} + x_{25} = 23,\ y_{25}^L + x_{25}^L = 5,\ y_{25}^R + x_{25}^R = 2,$

$y_{24} + x_{24} = 6,\ y^L_{24} + x^L_{24} = 1,\ y^R_{24} + x^R_{24} = 2,$

$y_{34} + x_{34} = 4,\ y^L_{34} + x^L_{34} = 1,\ y^R_{34} + x^R_{34} = 1,$

$y_{45} + x_{45} = 8,\ y^L_{45} + x^L_{45} = 3,\ y^R_{45} + x^R_{45} = 2,$

$y_{56} + x_{56} = 17,\ y^L_{56} + x^L_{56} = 3,\ y^R_{56} + x^R_{56} = 4,$

$t_6 = 39,\ t^L_6 = 10,\ t^R_6 = 4,$

$t_1 = 0,\ t^L_1 = 0,\ t^R_1 = 0,$

$(t_2, t^L_2, t^R_2)_{LR} \succeq (t_1 + x_{12}, t^L_1 + x^L_{12}, t^R_1 + x^R_{12})_{LR},$

$(t_3, t^L_3, t^R_3)_{LR} \succeq (t_1 + x_{13}, t^L_1 + x^L_{13}, t^R_1 + x^R_{13})_{LR},$

$(t_5, t^L_5, t^R_5)_{LR} \succeq (t_2 + x_{25}, t^L_2 + x^L_{25}, t^R_2 + x^R_{25})_{LR},$

$(t_4, t^L_4, t^R_4)_{LR} \succeq (t_2 + x_{24}, t^L_2 + x^L_{24}, t^R_2 + x^R_{24})_{LR},$

$(t_4, t^L_4, t^R_4)_{LR} \succeq (t_3 + x_{34}, t^L_3 + x^L_{34}, t^R_3 + x^R_{34})_{LR},$

$(t_5, t^L_5, t^R_5)_{LR} \succeq (t_4 + x_{45}, t^L_4 + x^L_{45}, t^R_4 + x^R_{45})_{LR},$

$(t_6, t^L_6, t^R_6)_{LR} \succeq (t_5 + x_{56}, t^L_5 + x^L_{56}, t^R_5 + x^R_{56})_{LR},$

$t_1 - t^L_1, t_2 - t^L_2, t_3 - t^L_3, t_4 - t^L_4, t_5 - t^L_5, t_6 - t^L_6, x_{12} - x^L_{12}, x_{13} - x^L_{13}, x_{24} - x^L_{24},$
$x_{25} - x^L_{25}, x_{24} - x^L_{24}, x_{34} - x^L_{34}, x_{45} - x^L_{45}, x_{56} - x^L_{56}, y_{12} - y^L_{12}, y_{13} - y^L_{13}, y_{24} - y^L_{24}, y_{25} -$
$y^L_{25}, y_{24} - y^L_{24}, y_{34} - y^L_{34}, y_{45} - y^L_{45}, y_{56} - y^L_{56} \geq 0,$

$t^L_1, t^R_1, t^L_2, t^R_2, t^L_3, t^R_3, t^L_4, t^R_4, t^L_5, t^R_5, t^L_6, t^R_6, x^L_{12}, x^R_{12}, x^L_{13}, x^R_{13}, x^L_{25}, x^R_{25}, x^L_{24}, x^R_{24}, x^L_{34},$
$x^R_{34}, x^L_{45}, x^R_{45}, x^L_{56}, x^R_{56}, y^L_{12}, y^R_{12}, y^L_{13}, y^R_{13}, y^L_{25}, y^R_{25}, y^L_{24}, y^R_{24}, y^L_{34}, y^R_{34}, y^L_{45}, y^R_{45}, y^L_{56}, y^R_{56} \geq 0.$

Step 5 Using the method, proposed in Section 7.4.1, for converting the fuzzy inequality into crisp inequalities, the fuzzy linear programming problem, obtained in Step 4, can be written as:

Minimize $\big((100y_{12} + 200y_{13} + 100y_{25} + 200y_{24} + 200y_{34} + 100y_{45} + 100y_{56}, 80y^L_{12} +$
$20y_{12} + 190y^L_{13} + 10y_{13} + 90y^L_{25} + 10y_{25} + 160y^L_{24} + 40y_{24} + 180y^L_{34} + 20y_{34} + 80y^L_{45} +$
$20y_{45} + 90y^L_{56} + 10y_{56}, 120y^R_{12} + 20y_{12} + 220y^R_{13} + 20y_{13} + 130y^R_{25} + 30y_{25} + 210y^R_{24} +$

$10y_{24} + 210y_{34}^R + 10y_{34} + 110y_{45}^R + 10y_{45} + 120y_{56}^R + 20y_{56})_{LR})$

subject to

$x_{12} - x_{12}^L \geq 4,\ x_{12}^L \geq 2,\ x_{12}^R \geq 0, \qquad x_{12} - x_{12}^L \leq 16,\ x_{12}^L \leq 3,\ x_{12}^R \leq 2,$

$x_{13} - x_{13}^L \geq 7,\ x_{13}^L \geq 1,\ x_{13}^R \geq 1, \qquad x_{13} - x_{13}^L \leq 10,\ x_{13}^L \leq 2,\ x_{13}^R \leq 1,$

$x_{25} - x_{25}^L \geq 8,\ x_{25}^L \geq 4,\ x_{25}^R \geq 0, \qquad x_{25} - x_{25}^L \leq 18,\ x_{25}^L \leq 5,\ x_{25}^R \leq 2,$

$x_{24} - x_{24}^L \geq 3,\ x_{24}^L \geq 1,\ x_{24}^R \geq 1, \qquad x_{24} - x_{24}^L \leq 5,\ x_{24}^L \leq 1,\ x_{24}^R \leq 2,$

$x_{34} - x_{34}^L \geq 2,\ x_{34}^L \geq 0,\ x_{34}^R \geq 1, \qquad x_{34} - x_{34}^L \leq 3,\ x_{34}^L \leq 1,\ x_{34}^R \leq 1,$

$x_{45} - x_{45}^L \geq 2,\ x_{45}^L \geq 2,\ x_{45}^R \geq 1, \qquad x_{45} - x_{45}^L \leq 5,\ x_{45}^L \leq 3,\ x_{45}^R \leq 2,$

$x_{56} - x_{56}^L \geq 6,\ x_{56}^L \geq 2,\ x_{56}^R \geq 0, \qquad x_{56} - x_{56}^L \leq 14,\ x_{56}^L \leq 3,\ x_{56}^R \leq 4,$

$y_{12} + x_{12} = 19,\ y_{12}^L + x_{12}^L = 3,\ y_{12}^R + x_{12}^R = 2,$

$y_{13} + x_{13} = 12,\ y_{13}^L + x_{13}^L = 2,\ y_{13}^R + x_{13}^R = 1,$

$y_{25} + x_{25} = 23,\ y_{25}^L + x_{25}^L = 5,\ y_{25}^R + x_{25}^R = 2,$

$y_{24} + x_{24} = 6,\ y_{24}^L + x_{24}^L = 1,\ y_{24}^R + x_{24}^R = 2,$

$y_{34} + x_{34} = 4,\ y_{34}^L + x_{34}^L = 1,\ y_{34}^R + x_{34}^R = 1,$

$y_{45} + x_{45} = 8,\ y_{45}^L + x_{45}^L = 3,\ y_{45}^R + x_{45}^R = 2,$

$y_{56} + x_{56} = 17,\ y_{56}^L + x_{56}^L = 3,\ y_{56}^R + x_{56}^R = 4,$

$t_6 = 39,\ t_6^L = 10,\ t_6^R = 4,$

$t_1 = 0,\ t_1^L = 0,\ t_1^R = 0,$

$t_2 - t_2^L \geq t_1 + x_{12} - t_1^L - x_{12}^L,\ t_2^L \geq t_1^L + x_{12}^L,\ t_2^R \geq t_1^R + x_{12}^R,$

$t_3 - t_3^L \geq t_1 + x_{13} - t_1^L - x_{13}^L,\ t_3^L \geq t_1^L + x_{13}^L,\ t_3^R \geq t_1^R + x_{13}^R,$

$t_5 - t_5^L \geq t_2 + x_{25} - t_2^L - x_{25}^L,\ t_5^L \geq t_2^L + x_{25}^L,\ t_5^R \geq t_2^R + x_{25}^R,$

$t_4 - t_4^L \geq t_2 + x_{24} - t_2^L - x_{24}^L,\ t_4^L \geq t_2^L + x_{24}^L,\ t_4^R \geq t_2^R + x_{24}^R,$

$t_4 - t_4^L \geq t_3 + x_{34} - t_3^L - x_{34}^L,\ t_4^L \geq t_3^L + x_{34}^L,\ t_4^R \geq t_3^R + x_{34}^R,$

$t_5 - t_5^L \geq t_4 + x_{45} - t_4^L - x_{45}^L,\ t_5^L \geq t_4^L + x_{45}^L,\ t_5^R \geq t_4^R + x_{45}^R,$

$t_6 - t_6^L \geq t_5 + x_{56} - t_5^L - x_{56}^L,\ t_6^L \geq t_5^L + x_{56}^L,\ t_6^R \geq t_5^R + x_{56}^R,$

$t_1 - t_1^L, t_2 - t_2^L, t_3 - t_3^L, t_4 - t_4^L, t_5 - t_5^L, t_6 - t_6^L, x_{12} - x_{12}^L, x_{13} - x_{13}^L, x_{24} - x_{24}^L, x_{25} -$

$x_{25}^L, x_{24} - x_{24}^L, x_{34} - x_{34}^L, x_{45} - x_{45}^L, x_{56} - x_{56}^L, y_{12} - y_{12}^L, y_{13} - y_{13}^L, y_{24} - y_{24}^L, y_{25} -$

$y_{25}^L, y_{24} - y_{24}^L, y_{34} - y_{34}^L, y_{45} - y_{45}^L, y_{56} - y_{56}^L \geq 0,$

$t_1^L, t_1^R, t_2^L, t_2^R, t_3^L, t_3^R, t_4^L, t_4^R, t_5^L, t_5^R, t_6^L, t_6^R, x_{12}^L, x_{12}^R, x_{13}^L, x_{13}^R, x_{25}^L, x_{25}^R, x_{24}^L, x_{24}^R, x_{34}^L, x_{34}^R,$

$x_{45}^L, x_{45}^R, x_{56}^L, x_{56}^R, y_{12}^L, y_{12}^R, y_{13}^L, y_{13}^R, y_{25}^L, y_{25}^R, y_{24}^L, y_{24}^R, y_{34}^L, y_{34}^R, y_{45}^L, y_{45}^R, y_{56}^L, y_{56}^R \geq 0.$

Step 6 Using the method, proposed in Section 7.4.5, the optimal solution of the fuzzy linear programming problem, obtained in Step 5, can be obtained as follows:

Step 6(a) Solve the crisp linear programming problem:

Minimize $\big(80y_{12} - 80y_{12}^L + 190y_{13} - 190y_{13}^L + 90y_{25} - 90y_{25}^L + 160y_{24} - 160y_{24}^L +$ $180y_{34} - 180y_{34}^L + 80y_{45} - 80y_{45}^L + 90y_{56} - 90y_{56}^L\big)$

subject to

Constraints of the problem obtained in Step 5.

Step 6(b) Since, the optimal value of crisp linear programming, obtained in Step 6(a) is 2000, so solve the crisp linear programming problem:

Minimize $\big(20y_{12} + 80y_{12}^L + 10y_{13} + 190y_{13}^L + 10y_{25} + 90y_{25}^L + 40y_{24} + 160y_{24}^L + 20y_{34} +$ $180y_{34}^L + 20y_{45} + 80y_{45}^L + 10y_{56} + 90y_{56}^L\big)$

subject to

Constraints of the problem obtained in Step 5 with the following additional constraint

$80y_{12} - 80y_{12}^L + 190y_{13} - 190y_{13}^L + 90y_{25} - 90y_{25}^L + 160y_{24} - 160y_{24}^L + 180y_{34} -$ $180y_{34}^L + 80y_{45} - 80y_{45}^L + 90y_{56} - 90y_{56}^L = 2000.$

Step 6(c) Since, the optimal value of crisp linear programming, obtained in Step

6(b) is 410, so solve the crisp linear programming problem:

Minimize $\left(20y_{12} + 120y_{12}^R + 20y_{13} + 220y_{13}^R + 30y_{25} + 130y_{25}^R + 10y_{24} + 210y_{24}^R + 10y_{34} + 210y_{34}^R + 10y_{45} + 110y_{45}^R + 20y_{56} + 120y_{56}^R\right)$

subject to

Constraints of the problem obtained in Step 5 with the following additional constraints

$80y_{12} - 80y_{12}^L + 190y_{13} - 190y_{13}^L + 90y_{25} - 90y_{25}^L + 160y_{24} - 160y_{24}^L + 180y_{34} - 180y_{34}^L + 80y_{45} - 80y_{45}^L + 90y_{56} - 90y_{56}^L = 2000,$

$20y_{12} + 80y_{12}^L + 10y_{13} + 190y_{13}^L + 10y_{25} + 90y_{25}^L + 40y_{24} + 160y_{24}^L + 20y_{34} + 180y_{34}^L + 20y_{45} + 80y_{45}^L + 10y_{56} + 90y_{56}^L = 410.$

Step 7 The optimal solution of the crisp linear programming problem, obtained in Step 6(c) is $y_{12} = 12$, $y_{12}^L = 0$, $y_{12}^R = 2$, $y_{25} = 5$, $y_{25}^L = 1$, $y_{25}^R = 0$, $y_{45} = 0$, $y_{45}^L = 0$, $y_{45}^R = 1$, $y_{56} = 3$, $y_{56}^L = 0$, $y_{56}^R = 3$, $y_{13} = 0$, $y_{13}^L = 0$, $y_{13}^R = 0$, $y_{24} = 0$, $y_{24}^L = 0$, $y_{24}^R = 0$, $y_{34} = 0$, $y_{34}^L = 0$, $y_{34}^R = 0$, $x_{12} = 7$, $x_{12}^L = 3$, $x_{12}^R = 0$, $x_{13} = 12$, $x_{13}^L = 2$, $x_{13}^R = 1$, $x_{25} = 18$, $x_{25}^L = 4$, $x_{25}^R = 2$, $x_{24} = 6$, $x_{24}^L = 1$, $x_{24}^R = 2$, $x_{34} = 4$, $x_{34}^L = 1$, $x_{34}^R = 1$, $x_{45} = 8$, $x_{45}^L = 3$, $x_{45}^R = 1$, $x_{56} = 14$, $x_{56}^L = 3$, $x_{56}^R = 1$, $t_1 = 0$, $t_1^L = 0$, $t_1^R = 0$, $t_2 = 7$, $t_2^L = 3$, $t_2^R = 0$, $t_3 = 12$, $t_3^L = 2$, $t_3^R = 1$, $t_4 = 17$, $t_4^L = 4$, $t_4^R = 2$, $t_5 = 25$, $t_5^L = 7$, $t_5^R = 3$, $t_6 = 39$, $t_6^L = 10$, $t_6^R = 4$.

Step 8: Using the optimal values of y_{ij}, y_{ij}^L, y_{ij}^R, x_{ij}, x_{ij}^L, x_{ij}^R, t_j, t_j^L, t_j^R, the fuzzy optimal values of $\widetilde{Y_{ij}} = (y_{ij}, y_{ij}^L, y_{ij}^R)_{LR}$, $\widetilde{X_{ij}} = (x_{ij}, x_{ij}^L, x_{ij}^R)_{LR}$ and $\widetilde{T_j} = (t_j, t_j^L, t_j^R)_{LR}$ are $\widetilde{Y_{12}} = (12, 0, 2)_{LR}$, $\widetilde{Y_{25}} = (5, 1, 0)_{LR}$, $\widetilde{Y_{45}} = (0, 0, 1)_{LR}$, $\widetilde{Y_{56}} = (3, 0, 3)_{LR}$, $\widetilde{Y_{13}} = \widetilde{Y_{24}} = \widetilde{Y_{34}} = (0, 0, 0)_{LR}$, $\widetilde{X_{12}} = (7, 3, 0)_{LR}$, $\widetilde{X_{13}} = (12, 2, 1)_{LR}$, $\widetilde{X_{25}} = (18, 4, 2)_{LR}$, $\widetilde{X_{24}} = (6, 1, 2)_{LR}$, $\widetilde{X_{34}} = (4, 1, 1)_{LR}$, $\widetilde{X_{45}} = (8, 3, 1)_{LR}$, $\widetilde{X_{56}} = (14, 3, 1)_{LR}$, $\widetilde{T_1} = (0, 0, 0)_{LR}$, $\widetilde{T_2} = (7, 3, 0)_{LR}$, $\widetilde{T_3} = (12, 2, 1)_{LR}$, $\widetilde{T_4} = (17, 4, 2)_{LR}$, $\widetilde{T_5} = (25, 7, 3)_{LR}$ and $\widetilde{T_6} =$

$(39, 10, 4)_{LR}$.

Step 9: Putting the fuzzy optimal values of $\widetilde{Y_{12}}$, $\widetilde{Y_{13}}$, $\widetilde{Y_{25}}$, $\widetilde{Y_{24}}$, $\widetilde{Y_{34}}$, $\widetilde{Y_{45}}$ and $\widetilde{Y_{56}}$ in $(100, 20, 20)_{LR} \otimes (y_{12}, y_{12}^L, y_{12}^R)_{LR} \oplus (200, 10, 20)_{LR} \otimes (y_{13}, y_{13}^L, y_{13}^R)_{LR} \oplus (100, 10, 30)_{LR} \otimes (y_{25}, y_{25}^L, y_{25}^R)_{LR} \oplus (200, 40, 10)_{LR} \otimes (y_{24}, y_{24}^L, y_{24}^R)_{LR} \oplus (200, 20, 10)_{LR} \otimes (y_{34}, y_{34}^L, y_{34}^R)_{LR} \oplus (100, 20, 10)_{LR} \otimes (y_{45}, y_{45}^L, y_{45}^R)_{LR} \oplus (100, 10, 20)_{LR} \otimes (y_{56}, y_{56}^L, y_{56}^R)_{LR}$, the minimum fuzzy crashing cost for completing the project within the specific fuzzy time $\widetilde{(PCT)} = (39, 10, 4)_{LR}$ is $(2000, 410, 1160)_{LR}$.

7.6.3.1 Physical interpretation of results

In this section, the minimum fuzzy crashing cost, obtained by using the proposed method, is physically interpreted. Similarly, the obtained optimal fuzzy activity times $\widetilde{X_{ij}}$ can also be physically interpreted.

Using the proposed method the minimum fuzzy crashing cost is $(2000, 410, 1160)_{LR}$, which can be physically interpreted as follows:

(1) The least amount of minimum crashing cost is 1590.

(2) The most possible amount of minimum crashing cost is 2000 .

(3) The greatest amount of minimum crashing cost is 3160.

i.e., the minimum crashing cost will always be greater than 1590 and less than 3160 and maximum chances are that the minimum crashing cost will be 2000.

The variation in minimum crashing cost with respect to chances are shown in Figure 7.3.

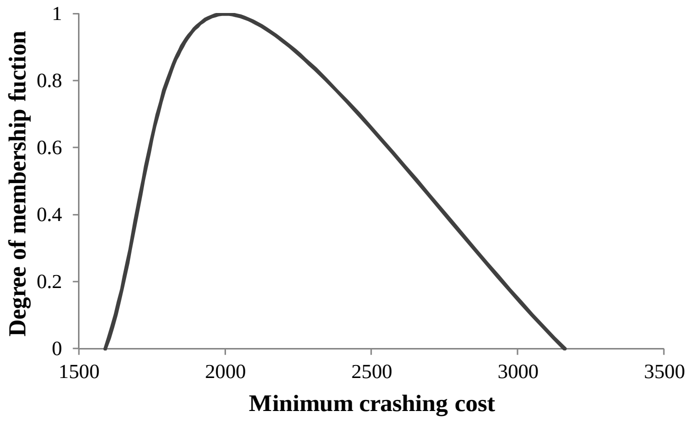

Figure 7.3 Membership function of *LR* fuzzy number representing minimum fuzzy crashing cost

7.6.4 Variation in minimum fuzzy crashing cost with respect to the specific fuzzy project completion time

To show the variation in minimum fuzzy crashing cost with respect to the specific fuzzy projet completion time, the minimum fuzzy crashing cost for the problem, chosen in Example 7.1, corresponding to different specific fuzzy times are shown in Table 7.4

Table 7.4: Fuzzy optimal solution and fuzzy optimal value of the chosen problem corresponding to different specific fuzzy times

$\widetilde{PCT}$	$(26,8,3)_{LR}$	$(34,9,4)_{LR}$	$(39,10,4)_{LR}$	$(41,10,4)_{LR}$	$(59,11,10)_{LR}$
$\sum_{(i,j)\in A}(\widetilde{C_{ij}} \otimes \widetilde{Y_{ij}})$	$(4200,820,1680)_{LR}$	$(2500,550,1260)_{LR}$	$(2000,410,1160)_{LR}$	$(1800,390,1070)_{LR}$	$(0,0,0)_{LR}$
$\widetilde{X_{ij}}$	$\widetilde{X_{12}}=(6,2,0)_{LR}$, $\widetilde{X_{13}}=(10,2,1)_{LR}$, $\widetilde{X_{25}}=(12,4,2)_{LR}$, $\widetilde{X_{24}}=(6,1,2)_{LR}$, $\widetilde{X_{34}}=(3,1,1)_{LR}$, $\widetilde{X_{45}}=(5,3,1)_{LR}$, $\widetilde{X_{56}}=(8,2,0)_{LR}$	$\widetilde{X_{12}}=(7,3,0)_{LR}$, $\widetilde{X_{13}}=(12,2,1)_{LR}$, $\widetilde{X_{25}}=(18,4,2)_{LR}$, $\widetilde{X_{24}}=(6,1,2)_{LR}$, $\widetilde{X_{34}}=(4,1,1)_{LR}$, $\widetilde{X_{45}}=(8,3,1)_{LR}$, $\widetilde{X_{56}}=(9,2,1)_{LR}$	$\widetilde{X_{12}}=(7,3,0)_{LR}$, $\widetilde{X_{13}}=(12,2,1)_{LR}$, $\widetilde{X_{25}}=(18,4,2)_{LR}$, $\widetilde{X_{24}}=(6,1,2)_{LR}$, $\widetilde{X_{34}}=(4,1,1)_{LR}$, $\widetilde{X_{45}}=(8,3,1)_{LR}$, $\widetilde{X_{56}}=(14,3,1)_{LR}$	$\widetilde{X_{12}}=(6,2,0)_{LR}$, $\widetilde{X_{13}}=(12,2,1)_{LR}$, $\widetilde{X_{25}}=(23,5,2)_{LR}$, $\widetilde{X_{24}}=(6,1,2)_{LR}$, $\widetilde{X_{34}}=(4,1,1)_{LR}$, $\widetilde{X_{45}}=(8,3,1)_{LR}$, $\widetilde{X_{56}}=(12,3,1)_{LR}$	$\widetilde{X_{12}}=(19,3,2)_{LR}$, $\widetilde{X_{13}}=(12,2,1)_{LR}$, $\widetilde{X_{25}}=(23,5,2)_{LR}$, $\widetilde{X_{24}}=(6,1,2)_{LR}$, $\widetilde{X_{34}}=(4,1,1)_{LR}$, $\widetilde{X_{45}}=(8,3,2)_{LR}$, $\widetilde{X_{56}}=(17,3,4)_{LR}$
$\widetilde{T}_j$	$\widetilde{T}_1=(0,0,0)_{LR}$, $\widetilde{T}_2=(6,2,0)_{LR}$, $\widetilde{T}_3=(10,2,1)_{LR}$, $\widetilde{T}_4=(13,3,2)_{LR}$, $\widetilde{T}_5=(18,6,3)_{LR}$, $\widetilde{T}_6=(26,8,3)_{LR}$,	$\widetilde{T}_1=(0,0,0)_{LR}$, $\widetilde{T}_2=(7,3,0)_{LR}$, $\widetilde{T}_3=(12,2,1)_{LR}$, $\widetilde{T}_4=(17,4,2)_{LR}$, $\widetilde{T}_5=(25,7,3)_{LR}$, $\widetilde{T}_6=(34,9,4)_{LR}$	$\widetilde{T}_1=(0,0,0)_{LR}$, $\widetilde{T}_2=(7,3,0)_{LR}$, $\widetilde{T}_3=(12,2,1)_{LR}$, $\widetilde{T}_4=(17,4,2)_{LR}$, $\widetilde{T}_5=(25,7,3)_{LR}$, $\widetilde{T}_6=(39,10,4)_{LR}$	$\widetilde{T}_1=(0,0,0)_{LR}$, $\widetilde{T}_2=(6,2,0)_{LR}$, $\widetilde{T}_3=(12,2,1)_{LR}$, $\widetilde{T}_4=(16,3,2)_{LR}$, $\widetilde{T}_5=(29,7,3)_{LR}$, $\widetilde{T}_6=(41,10,4)_{LR}$	$\widetilde{T}_1=(0,0,0)_{LR}$, $\widetilde{T}_2=(19,3,2)_{LR}$, $\widetilde{T}_3=(12,2,1)_{LR}$, $\widetilde{T}_4=(25,4,4)_{LR}$, $\widetilde{T}_5=(42,8,6)_{LR}$, $\widetilde{T}_6=(59,11,10)_{LR}$
$\widetilde{Y_{ij}}$	$\widetilde{Y_{12}}=(13,1,2)_{LR}$, $\widetilde{Y_{13}}=(2,0,0)_{LR}$, $\widetilde{Y_{25}}=(11,1,0)_{LR}$, $\widetilde{Y_{24}}=(0,0,0)_{LR}$, $\widetilde{Y_{34}}=(1,0,0)_{LR}$, $\widetilde{Y_{45}}=(3,0,1)_{LR}$, $\widetilde{Y_{56}}=(9,1,4)_{LR}$	$\widetilde{Y_{12}}=(12,0,2)_{LR}$, $\widetilde{Y_{13}}=(0,0,0)_{LR}$, $\widetilde{Y_{25}}=(5,1,0)_{LR}$, $\widetilde{Y_{24}}=(0,0,0)_{LR}$, $\widetilde{Y_{34}}=(0,0,0)_{LR}$, $\widetilde{Y_{45}}=(0,0,1)_{LR}$, $\widetilde{Y_{56}}=(8,1,3)_{LR}$	$\widetilde{Y_{12}}=(12,0,2)_{LR}$, $\widetilde{Y_{13}}=(0,0,0)_{LR}$, $\widetilde{Y_{25}}=(5,1,0)_{LR}$, $\widetilde{Y_{24}}=(0,0,0)_{LR}$, $\widetilde{Y_{34}}=(0,0,0)_{LR}$, $\widetilde{Y_{45}}=(0,0,1)_{LR}$, $\widetilde{Y_{56}}=(3,0,3)_{LR}$	$\widetilde{Y_{12}}=(13,1,2)_{LR}$, $\widetilde{Y_{13}}=(0,0,0)_{LR}$, $\widetilde{Y_{25}}=(0,0,0)_{LR}$, $\widetilde{Y_{24}}=(0,0,0)_{LR}$, $\widetilde{Y_{34}}=(0,0,0)_{LR}$, $\widetilde{Y_{45}}=(0,0,1)_{LR}$, $\widetilde{Y_{56}}=(5,0,3)_{LR}$	$\widetilde{Y_{12}}=(0,0,0)_{LR}$, $\widetilde{Y_{13}}=(0,0,0)_{LR}$, $\widetilde{Y_{25}}=(0,0,0)_{LR}$, $\widetilde{Y_{24}}=(0,0,0)_{LR}$, $\widetilde{Y_{34}}=(0,0,0)_{LR}$, $\widetilde{Y_{45}}=(0,0,0)_{LR}$, $\widetilde{Y_{56}}=(0,0,0)_{LR}$

In crisp project network problems, on decreasing the project completion time the crashing cost increases. On the same direction, it is obvious from Table 7.4 that on decreasing the specific fuzzy project completion time the fuzzy crashing cost is increasing.

7.7 Proposed JMD representation of LR flat fuzzy numbers

In this section, a new representation of LR flat fuzzy numbers, named as JMD representation of LR flat fuzzy numbers, is proposed and it is shown that it is better to represent the parameters of fuzzy linear programming problems by proposed JMD representation of LR flat fuzzy numbers as compared to the existing representation of LR flat fuzzy numbers.

7.7.1 Basic definitions

In this section, some definitions are proposed.

Definition 7.1 Let $(\underline{a}, \overline{a}, a^L, a^R)_{LR}$ be an LR flat fuzzy number then its JMD representation is $(a, a^L, a^M, a^R)_{LR}^{JMD}$, where $a = \underline{a} - a^L$, $a^M = \overline{a} - \underline{a}$.

Definition 7.2 A JMD LR flat fuzzy number $\widetilde{A} = (a, a^L, a^M, a^R)_{LR}^{JMD}$ is said to be non-negative JMD LR flat fuzzy number if and only if $a \geq 0$, $a^L \geq 0$, $a^M \geq 0$ and $a^R \geq 0$.

Definition 7.3 A JMD LR flat fuzzy number $\widetilde{A} = (a, a^L, a^M, a^R)_{LR}^{JMD}$ is said to be an unrestricted JMD LR flat fuzzy number if and only if a is a real number, $a^L \geq 0$, $a^M \geq 0$ and $a^R \geq 0$.

Definition 7.4 Two JMD LR flat fuzzy numbers $\widetilde{A} = (a_1, a_1^L, a_1^M, a_1^R)_{LR}^{JMD}$ and $\widetilde{B} = (a_2, a_2^L, a_2^M, a_2^R)_{LR}^{JMD}$ are said to be equal i.e., $\widetilde{A} = \widetilde{B}$ if and only if $a_1 = a_2$,

$a_1^L = a_2^L$, $a_1^M = a_2^M$ and $a_1^R = a_2^R$.

Definition 7.5 Let $\widetilde{A} = (a, a^L, a^M, a^R)_{LR}^{JMD}$ be a JMD LR flat fuzzy number and α be a real number in the interval $[0,1]$ then the crisp set $A_\alpha = \{x \in X : \mu_{\widetilde{A}}(x) \geq \alpha\} = [A_\alpha^L,\ A_\alpha^U] = [(a + a^L) - a^L L^{-1}(\alpha),\ (a + a^L + a^M) + a^R R^{-1}(\alpha)]$ is said to be α-cut of $\widetilde{A}$.

7.7.2 Arithmetic operations

In this section, some arithmetic operations between two JMD LR flat fuzzy numbers are proposed.

Let $\widetilde{A_1} = (a_1, a_1^L, a_1^M, a_1^R)_{LR}^{JMD}$ and $\widetilde{A_2} = (a_2, a_2^L, a_2^M, a_2^R)_{LR}^{JMD}$ be two JMD LR flat fuzzy numbers. Then,

(i) $\widetilde{A_1} \oplus \widetilde{A_2} = (a_1 + a_2, a_1^L + a_2^L, a_1^M + a_2^M, a_1^R + a_2^R)_{LR}^{JMD}$

(ii) $$\lambda\widetilde{A_1} = \begin{cases} (\lambda a_1, \lambda a_1^L, \lambda a_1^M, \lambda a_1^R)_{LR}^{JMD} & \lambda \geq 0 \\ (\lambda(a_1 + a_1^L + a_1^M + a_1^R), -\lambda a_1^R, -\lambda a_1^M, -\lambda a_1^L)_{RL}^{JMD} & \lambda \leq 0 \end{cases}$$

Let $\widetilde{A_1} = (a_1, a_1^L, a_1^M, a_1^R)_{LR}^{JMD}$ and $\widetilde{A_2} = (a_2, a_2^L, a_2^M, a_2^R)_{LR}^{JMD}$ be two non-negative JMD LR flat fuzzy numbers. Then,

(iii) $\widetilde{A_1} \otimes \widetilde{A_2} = (a_1 a_2,\ a_1 a_2^L + a_1^L a_2 + a_1^L a_2^L,\ a_1 a_2^M + a_1^L a_2^M + a_1^M a_2 + a_1^M a_2^L + a_1^M a_2^M,\ a_1 a_2^R + a_1^L a_2^R + a_1^M a_2^R + a_1^R a_2 + a_1^R a_2^L + a_1^R a_2^M + a_1^R a_2^R)_{LR}^{JMD}$

Remark 7.2: If $a^M = 0$ then a JMD LR flat fuzzy number $(a, a^L, a^M, a^R)_{LR}^{JMD}$ is said to be a JMD LR fuzzy number and is denoted as $(a, a^L, 0, a^R)_{LR}^{JMD}$.

7.7.3 Advantages of proposed JMD representation of LR flat fuzzy numbers over existing representation of LR flat fuzzy numbers

To show the advantages of proposed JMD representation of LR flat fuzzy numbers over existing representation of LR flat fuzzy numbers, the number of crisp constraints of a crisp linear programming problem obtained from such a fuzzy linear programming problem in which all the parameters are represented by existing representation of LR flat fuzzy numbers and the number of constraints of a crisp linear programming problem obtained from the same fuzzy linear programming problem by representing all the parameters as JMD representation of LR flat fuzzy numbers are compared in Table 7.5

Table 7.5: Comparison of proposed JMD and existing representation of LR flat fuzzy numbers

Details of crisp constraints in the crisp linear programming problem obtained by using any method with existing representation of LR flat fuzzy number $((\underline{a}, \overline{a}, a^L, a^R)_{LR})$	Details of crisp constraints in the crisp linear programming problem obtained by using any method with JMD representation of LR flat fuzzy number $((a, a^L, a^M, a^R)_{LR}^{JMD})$
Crisp constraints corresponding to one non-negative LR flat fuzzy variable $((\underline{a}, \overline{a}, a^L, a^R)_{LR})$ are $\underline{a}, \overline{a}, a^L, a^R, \underline{a} - a^L, \overline{a} - \underline{a} \geq 0,$	Crisp constraints corresponding to one non-negative JMD LR flat fuzzy variable $((a, a^L, a^M, a^R)_{LR}^{JMD})$ are $a^L, a^M, a^R, a \geq 0$
Number of crisp constraints corresponding to one non-negative LR flat fuzzy variable $((\underline{a}, \overline{a}, a^L, a^R)_{LR})$ are 6	Number of crisp constraints corresponding to one non-negative JMD LR flat fuzzy variable $((a, a^L, a^M, a^R)_{LR}^{JMD})$ are 4
Number of crisp constraints corresponding to (u) non-negative LR flat fuzzy variables are $6 \times u$	Number of crisp constraints corresponding to (u) non-negative JMD LR flat fuzzy variables are $4 \times u$
Crisp constraints corresponding to one unrestricted LR flat fuzzy variable $((\underline{a}, \overline{a}, a^L, a^R)_{LR})$ are $a^L \geq 0, a^R \geq 0,$ $\overline{a} - \underline{a} \geq 0,$ $\underline{a'} \geq 0,$ $\underline{a''} \geq 0,$ $\overline{a'} \geq 0,$ $\overline{a''} \geq 0,$ $\underline{a} = \underline{a'} - \underline{a''},$ $\overline{a} = \overline{a'} - \overline{a''}$	Crisp constraints corresponding to one unrestricted JMD LR flat fuzzy variable $((a, a^L, a^M, a^R)_{LR}^{JMD})$are $a^L \geq 0,$ $a^M \geq 0,$ $a^R \geq 0,$ $a' \geq 0,$ $a'' \geq 0,$ $a = a' - a''$
Number of unrestricted crisp constraints corresponding to one unrestricted LR flat fuzzy variables are 9	Number of unrestricted crisp constraints corresponding to one unrestricted JMD LR flat fuzzy variables are 6
Number of unrestricted crisp constraints corresponding to (v) unrestricted LR flat fuzzy variables are $9 \times v$	Number of unrestricted crisp constraints corresponding to (v) unrestricted JMD LR flat fuzzy variables are $6 \times v$
Total number of crisp constraints $= 6 \times u + 9 \times v$	Total number of crisp constraints $= 4 \times u + 6 \times v$

It is obvious from the results, shown in Table 7.5, that if any fuzzy linear programming problem in which parameters are represented by existing representation of LR flat fuzzy numbers and the same fuzzy linear programming in which parameters are represented by JMD representation of LR flat fuzzy numbers are converted into crisp linear programming problems. Then, total number of crisp constraints in that crisp linear programming problem which will be obtained by that fuzzy linear programming problem in which all the parameters are represented by existing representation of LR flat fuzzy numbers will be more than the number of constraints in the crisp linear programming problem obtained from that fuzzy linear programming problem in which all the parameters are represented by JMD representation of LR flat fuzzy numbers. Hence, it is better to use the proposed JMD representation of LR flat fuzzy numbers for representing the parameters of fuzzy linear programming problems as compared to the existing representation of LR flat fuzzy numbers.

7.8 Conclusions

On the basis of the presented study, the following conclusions can be drawn:

(i) It is better to use proposed JMD method as compared to the existing methods for converting the fuzzy inequality constraints into crisp inequality constraints.

(ii) It is better to use proposed JMD method as compared to the existing methods for finding the minimum and maximum of fuzzy numbers.

(iii) It is better to use proposed JMD method as compared to the existing methods and the methods proposed in previous chapter for finding the unique optimal initial fuzzy project completion time as well as crash fuzzy project completion time.

(iv) It is better to use proposed JMD method as compared to the existing method [30] for finding the minimum fuzzy crashing cost for completing the project within specific fuzzy time.

(v) It is better to use JMD representation of LR flat fuzzy numbers for representing the parameters of a fuzzy linear programming problems as compared to use the existing representation of LR flat fuzzy numbers

Chapter 8

Future Scope

Chen [24] proposed a method to find the optimal fuzzy project completion time of such project network problems in which time of all the activities are represented by different type of LR flat fuzzy numbers. Chen and Hsueh [28] pointed out that it is difficult to apply the existing method [24] and proposed a simple approach for the same.

It is not genuine to use any of the existing methods [24,28] due to the following reasons:

As discussed in Chapter 3, that the optimal fuzzy project completion time corresponding to all the critical paths should be same. However, on solving the problems, chosen in Example 3.1 of Chapter 3, by using the existing methods [24,28] more than one fuzzy numbers representing the optimal fuzzy project completion time are obtained i.e., the shortcomings, discussed in Chapter 3, are also occurring in the existing methods [24, 28].

In future, it may be tried to resolve the shortcomings of the existing methods [24, 28]. Also, it may be tried to develop methods for finding the fuzzy optimal solution of such project crashing problems in which the parameters are represented by different type of LR flat fuzzy numbers.

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