SOME APPROACHES FOR SOLVING FUZZY DATA ENVELOPMENT ANALYSIS MODELS

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SUBMITTED BY

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CERTIFICATE

This is to certify that the thesis entitled, "**Some Approaches for Solving Fuzzy Data Envelopment Analysis Models**", submitted by Ms. Bindu Bhardwaj, in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the LM Thapar School of Management, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisor

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DECLARATION

It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.

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DEDICATED

TO

MY SUPERVISOR

&

MY KIDS

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Abstract

In this thesis, flaws in the existing fuzzy CCR DEA model as well as in some of the existing methods for solving fuzzy CCR DEA model are pointed out. Also, to resolve the flaws new fuzzy CCR DEA model and new approaches for solving it are proposed.

1.2 Organization of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Hatami-Marbini et al. [40] proposed a method to solve fuzzy CCR DEA model for evaluating the best relative fuzzy efficiency of decision making units (DMUs). In this chapter, it is pointed that the product of fuzzy numbers, used by Hatami-Marbini et al. [40], in their proposed method, is incorrect and hence, the method, proposed by Hatami-Marbini et al. [40], is not valid. To resolve this flaw, the method proposed by Hatami Marbini et al. [40], is modified by using correct product of fuzzy numbers.

Chapter 3

Wang et al. [106] replaced the crisp output data and crisp input data of the crisp CCR DEA model with fuzzy data and proposed two methods for solving this fuzzy CCR DEA model. In this chapter, it is pointed out that the fuzzy CCR DEA model, proposed by Wang et al.[106], is not valid and hence the methods, proposed by Wang et al. [106] for evaluating the fuzzy efficiency of DMUs, are also not valid. Also, a new fuzzy CCR DEA model as well as a method to solve this fuzzy CCR DEA model is proposed.

Chapter 4

Wang and Chin [105] proposed an optimistic as well as pessimistic fuzzy CCR DEA model and an approach for solving it to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs. In this chapter, it is shown that the fuzzy CCR models, proposed by Wang and Chin [105], are

not valid and hence cannot be used to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs. To resolve the flaws of the fuzzy CCR DEA model, proposed by Wang and Chin [105], new fuzzy CCR DEA models are proposed. Also, a new approach is proposed to solve the proposed fuzzy CCR DEA models for evaluating the relative geometric fuzzy efficiency of DMUs.

Chapter 5

Puri and Yadav [83] proposed an optimistic as well as pessimistic intuitionistic fuzzy CCR DEA model and an approach to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). In this chapter, it is shown that the intuitionistic fuzzy CCR models, proposed by Puri and Yadav [83] are not valid and hence cannot be used to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). To resolve the flaws of the intuitionistic fuzzy CCR DEA model, proposed by Puri and Yadav [83], new intuitionistic fuzzy CCR DEA models are proposed. Also, a new approach is proposed to solve the proposed intuitionistic fuzzy CCR DEA models for evaluating the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs.

Chapter 6

Finally, in this chapter, some future directions has been suggested.

List of research papers

1) **Bindu Bhardwaj**, Amit Kumar, A Note on the Paper "A simplified novel technique for solving fully fuzzy linear programming". Journal of Optimization Theory and Applications, DOI: 10.1007/s10957-013-0505-3. **(Impact factor: 1.406)**
2) **Bindu Bhardwaj**, Amit Kumar, A Note on "A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem". Applied Mathematical Modelling. DOI: 10.1016/j.apm.2014.07.033 **(Impact factor: 2.195)**
3) **Bindu Bhardwaj**, Amit Kumar, Modified approach for solving fully fuzzy CCR DEA model, International Journal of Information and Decision Sciences (Communicated).
4) **Bindu Bhardwaj**, Amit Kumar, A new approach for solving fuzzy CCR DEA model, International Journal of Fuzzy Systems (Communicated).
5) **Bindu Bhardwaj**, Amit Kumar, A new approach for solving fully fuzzy CCR DEA model, Expert Systems with Applications (Communicated).
6) **Bindu Bhardwaj**, Amit Kumar, A new approach for solving intuitionistic fully fuzzy CCR DEA model, Expert Systems with Applications (Communicated).

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Chapter 1

Introduction

1.1 Introduction

The efficiency evaluation of every system is important to find its weakness so that subsequent improvements can be made. The same has been practiced and studied for a long time. It has been recognized that the efficiency evaluation is a very complex problem. Two basic approaches used for evaluating efficiency are the parametric and nonparametric approaches. The parametric approach requires an assumption on the relationship between inputs and outputs while the nonparametric approach does not require any such assumption on the functional form. Numerous methods for measuring and evaluating productive efficiency have been developed. Data envelopment analysis (DEA), a non-parametric approach proposed by Charnes et al. [11], is an approach to measure the relative efficiency of homogenous units called decision making units (DMUs) which consume the same type of inputs and produce the same type of outputs.

A common measure of relative efficiency when there are multiple inputs and multiple outputs is a ratio of weighted sum of outputs to the weighted sum of inputs. DEA uses this concept and generalizes multiple-inputs and multiple-outputs for each DMU into a single virtual input and single virtual output, respectively. Specifically, it determines a set of weights such that the efficiency of the target DMU, DMU_0 relative to the other DMUs is maximized. Thus, it provides an envelope for all considered DMUs rather than fitting a regression plane through the center of the data. Since, the pioneering work Charnes et al. [11], Data Envelopment Analysis (DEA) has been extensively used for evaluating the performance of many activities.

The reason of the DEA methodology being recommended widely in researches and applications is that it possess many merits which are not covered by traditional approaches [100]. For instance:

(1) It is not only capable of deriving a single aggregated measure of the relative efficiency of DMUs in terms of their utilization of input factors to produce the desired outputs, but also able to provide suggestions on the possibility of increasing outputs and/ or conserving inputs to make the inefficient DMU become efficient.

(2) In addition, DEA can handle non-commensurable multiple outputs and inputs without prior weight or prices.

Although, DEA methodology owns many advantages as stated above, it has some limitations which can be further improved [31]. The conventional DEA methods require accurate measurement of both the inputs and outputs. However, inputs and outputs of DMUs in real world problems are ever changeful. For example, for evaluating operation efficiencies of airlines, seat kilometers available, cargo-kilometers available, fuel and labor are regarded as inputs and passenger kilometers performed as the output. It is common sense that these inputs and output are easy to change because of weather, season, operating state and so on. Because DEA is a `boundary' method sensitive to outliers, it is very difficult to evaluate the efficiency of DMU with varying inputs and outputs by conventional DEA models. Some researchers have proposed several models to challenge how to deal with the variation of data in efficiency evaluation problems by stochastic frontier models [28]. On the other hand, in more general cases, the data for evaluation are often collected from investigation by polling where the natural language such as good, medium and bad are used to reect a kind of general situation of the investigated entities rather than a specific case [79]. In the above example, the expert can make a general Conclusions that airline A's passenger-kilometer is about 200 passenger-kilometers and fuel cost is high based on his rich experience.

Imprecise evaluations may be the result of unquantifiable, incomplete and non-obtainable information. In recent years, fuzzy set theory, established by Zadeh [113] has been proven to be useful as a way to quantify imprecise and vague data in DEA models. Fuzzy numbers are used to summarize the general situation of inputs and outputs and reacts the ambiguity of the expert's judgment. The center of a fuzzy number represents the most general case and the spread reacts some possibilities. The DEA model with fuzzy data, called “fuzzy DEA” models, can more realistically represent real world problems than the conventional DEA models. Fuzzy set theory also allows linguistic data to be used directly within the DEA model. Since, the original study by Sengupta [94, 95] there has been a continuous interest and increased development in fuzzy DEA (FDEA) literature. Several approaches have been developed and many new are coming for handling fuzzy input and output data in FDEA.

There are relatively a large number of papers in the fuzzy DEA literature. Since, summary development of the fuzzy DEA followed by a detailed description of the fuzzy DEA methods is easily available in the literature [25, 35]. So, to avoid the repetition of literature, the same is not described here and interested readers are requested to see the literature [25, 35]. However, research work done in four papers, published in last 6 years and base of this thesis work, is described in detailed manner in Chapter 2 to Chapter 5 of the thesis.

After reviewing the literature, it can be concluded that there are flaws in the existing fuzzy DEA models as well as in the existing methods for solving fuzzy DEA models. In this thesis, flaws in the existing fuzzy CCR DEA model as well as in some of the existing methods for solving fuzzy CCR DEA model are pointed out. Also, to resolve the flaws new fuzzy CCR DEA model and new approaches for solving it are proposed.

1.2 Organization of the thesis

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Puri and Yadav [83] proposed an optimistic as well as pessimistic intuitionistic fuzzy CCR DEA model and an approach to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). In this chapter, it is shown that the intuitionistic fuzzy CCR models, proposed by Puri and Yadav [83] are not valid and hence cannot be used to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). To resolve the flaws of the intuitionistic fuzzy CCR DEA model, proposed by Puri and Yadav [83], new intuitionistic fuzzy CCR DEA models are proposed. Also, a new approach is proposed to solve the proposed intuitionistic fuzzy CCR DEA models for evaluating the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs.

Chapter 6

Finally, in this chapter, some future directions has been suggested.

Chapter 2

Modified Approach for Solving Fully Fuzzy CCR DEA Model[1]

Hatami-Marbini et al. [40] proposed a method to solve fuzzy CCR DEA model for evaluating the best relative fuzzy efficiency of decision making units (DMUs). In this chapter, it is pointed that the product of fuzzy numbers, used by Hatami-Marbini et al. [40] in their proposed method, is incorrect and hence, the method, proposed by Hatami-Marbini et al. [40], is not valid. To resolve this flaw, the method proposed by Hatami Marbini et al. [40], is modified by using correct product of fuzzy numbers.

2.1 Preliminaries

In this section, some basic definitions and arithmetic operations of fuzzy numbers are reviewed [123].

2.1.1 Basic definitions

In this section, some basic definitions are reviewed [123].

Definition 2.1 If the universal set is defined as X then a fuzzy set $\tilde{A}$ of X can be defined by as a set of pairs of element x and $\mu_{\tilde{A}}(X)$, $\tilde{A} = \{((X, \mu_{\tilde{A}}(X))|x \in X)\}$, where, $\mu_{\tilde{A}}(x)$, a real number in interval $[0,1]$ represents the grade of membership function of x in $\tilde{A}$.

[1] The contents of this chapter are communicated for the possible publication in International Journal of Information and Decision Sciences.

Definition 2.2 A fuzzy set $\tilde{A}$ is said to be convex if

$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq$ minimum $\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \forall x_1, x_2 \in X, \lambda \in [0,1]$.

Definition 2.3 A fuzzy set $\tilde{A}$ in X is said to be a normal fuzzy set if there exist at least a $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

Definition 2.4 A fuzzy set $\tilde{A}$ is a fuzzy number if $\tilde{A}$ is normal and convex.

Definition 2.5 A fuzzy number $\tilde{A}$ is said to be non-negative (positive) if $\mu_A(x) = 0, \forall x < 0$ $(\forall x \leq 0)$.

Definition 2.6 A fuzzy number $\tilde{A} = (a^M, a^\alpha, a^\beta), a^\alpha \geq 0, a^\beta \geq 0$, is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{1}{a^\alpha}(x - a^M + a^\alpha), & a^M - a^\alpha \leq x < a^M \\ 1 & x = a^M \\ \frac{1}{a^\beta}(a^M - x + a^\beta), & a^M < x \leq a^M + a^\beta \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7 A triangular fuzzy number $\tilde{A} = (a^M, a^\alpha, a^\beta)$ is said to be non-negative triangular fuzzy number if and only if $a^M - a^\alpha \geq 0$.

Definition 2.8 A triangular fuzzy number $\tilde{A} = (a^M, a^\alpha, a^\beta)$ is said to be positive triangular fuzzy number if and only if $a^M - a^\alpha > 0$.

Definition 2.9 A fuzzy number $\tilde{A} = (a^M, a^N, a^\alpha, a^\beta), a^M \leq a^N, a^\alpha > 0, a^\beta > 0$, is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{1}{a^{\alpha}}(x - a^M + a^{\alpha}), & a^M - a^{\alpha} \le x < a^M \\ 1, & a^M \le x \le a^N \\ \frac{1}{a^{\beta}}(a^N - x + a^{\beta}), & a^N < x \le a^N + a^{\beta} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.10 A trapezoidal fuzzy number $\tilde{A} = (a^M, a^N, a^{\alpha}, a^{\beta})$ is said to be non-negative trapezoidal fuzzy number if and only if $a^M - a^{\alpha} \ge 0$.

Definition 2.11 A trapezoidal fuzzy number $\tilde{A} = (a^M, a^N, a^{\alpha}, a^{\beta})$ is said to be positive trapezoidal fuzzy number if and only if $a^M - a^{\alpha} > 0$.

Definition 2.12 [40] A function $\mathfrak{R} : F(R) \longrightarrow R$ is a ranking function, where $F(R)$ is a set of fuzzy numbers that is defined on the set of real numbers. This function maps each fuzzy number into real line.

Let $\tilde{A} = (a^M, a^{\alpha}, a^{\beta})$ and $\tilde{B} = (b^M, b^{\alpha}, b^{\beta})$ be two triangular fuzzy numbers. Then,

(i) $\tilde{A} \succcurlyeq \tilde{B}$ iff $\mathfrak{R}(\tilde{A}) \ge \mathfrak{R}(\tilde{B})$

(ii) $\tilde{A} \approx \tilde{B}$ iff $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$

where, $\mathfrak{R}(\tilde{A}) = a^M + \frac{a^{\beta} - a^{\alpha}}{4}$ and $\mathfrak{R}(\tilde{B}) = b^M + \frac{b^{\beta} - b^{\alpha}}{4}$.

Let $\tilde{A} = (a^M, a^N, a^{\alpha}, a^{\beta})$ and $\tilde{B} = (b^M, b^N, b^{\alpha}, b^{\beta})$ be two trapezoidal fuzzy numbers. Then,

(i) $\tilde{A} \succcurlyeq \tilde{B}$ iff $\mathfrak{R}(\tilde{A}) \ge \mathfrak{R}(\tilde{B})$

(ii) $\tilde{A} \approx \tilde{B}$ iff $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$

where, $\mathfrak{R}(\tilde{A}) = \frac{a^M + a^N}{2} + \frac{a^{\beta} - a^{\alpha}}{4}$ and $\mathfrak{R}(\tilde{B}) = \frac{b^M + b^N}{2} + \frac{b^{\beta} - b^{\alpha}}{4}$.

Remark 2.1 A triangular fuzzy number $\tilde{A} = (a^M, a^\alpha, a^\beta)$ is also represented as $\tilde{A} = (a, b, c)$, where, $a = a^M - a^\alpha, b = a^M, c = a^M + a^\beta$ and $a \le b \le c$. In Chapter 3, this representation is used.

Remark 2.2 A trapezoidal fuzzy number $\tilde{A} = (a^M, a^N, a^\alpha, a^\beta)$ is also represented as $\tilde{A} = (a, b, c, d)$, where, $a = a^M - a^\alpha, b = a^M, c = a^N, d = a^N + a^\beta$ and $a \le b \le c \le d$. In Chapter 4, this representation is used.

Remark 2.3 Let $\tilde{A} = (a, b, c, d)$ be a trapezoidal fuzzy number. Then, $\Re(\tilde{A}) = \frac{a+b+c+d}{4}$.

Remark 2.4 Let $\tilde{A} = (a, b, c)$ be a triangular fuzzy number. Then, $\Re(\tilde{A}) = \frac{a+2b+c}{4}$.

2.1.2 Arithmetic operations

In this section, arithmetic operations of triangular fuzzy numbers and trapezoidal fuzzy numbers are presented.

2.1.2.1 Arithmetic operations of triangular fuzzy numbers

Let $\tilde{A} = (a^M, a^\alpha, a^\beta)$ and $\tilde{B} = (b^M, b^\alpha, b^\beta)$ be two arbitrary triangular fuzzy numbers. Then,

(i) $\tilde{A} + \tilde{B} = (a^M + b^M, a^\alpha + b^\alpha, a^\beta + b^\beta)$

(ii) $\tilde{A} - \tilde{B} = (a^M - b^M, a^\alpha + b^\beta, a^\beta + b^\alpha)$

(iii) $\tilde{A}\tilde{B} = (a^M b^M,\ a^M b^\alpha + b^M a^\alpha - a^\alpha b^\alpha,\ a^N b^\beta + b^N a^\beta + a^\beta b^\beta)$, where $\tilde{A}$ and $\tilde{B}$ are non-negative triangular fuzzy numbers.

(iv) $\frac{\tilde{A}}{\tilde{B}} = \left(\frac{a^M}{b^M}, \frac{a^M}{b^M} - \frac{a^M - a^\alpha}{b^M + b^\beta}, \frac{a^M + a^\beta}{b^M - b^\alpha} - \frac{a^M}{b^M}\right)$, where $\tilde{A}$ is a non-negative triangular fuzzy number and $\tilde{B}$ is a positive triangular fuzzy number.

Let $\tilde{A} = (a_1, b_1, c_1) = (a^M - a^\alpha, a^M, a^M + a^\beta)$ and $\widetilde{B} = (a_2, b_2, c_2) = (b^M - b^\alpha, b^M, b^M + b^\beta)$ be two arbitrary triangular fuzzy numbers. Then,

(i) $\tilde{A} + \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

(ii) $\tilde{A} - \tilde{B} = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$

(iii) $\widetilde{A}\,\widetilde{B} = (a_1 a_2, b_1 b_2, c_1 c_2)$, where $\widetilde{A}$ and $\widetilde{B}$ are non-negative triangular fuzzy numbers.

(iv) $\frac{\tilde{A}}{\tilde{B}} = \left(\frac{a_1}{c_2}, \frac{b_1}{b_2}, \frac{c_1}{a_2}\right)$, where $\widetilde{A}$ is a non-negative triangular fuzzy number and $\widetilde{B}$ is a positive triangular fuzzy number.

2.1.2.2 Arithmetic operations of trapezoidal fuzzy numbers

Let $\tilde{A} = (a^M, a^N, a^\alpha, a^\beta)$ and $\widetilde{B} = (b^M, b^M, b^\alpha, b^\beta)$ be two arbitrary trapezoidal fuzzy numbers. Then,

(i) $\tilde{A} + \tilde{B} = (a^M + b^M, a^N + b^N, a^\alpha + b^\alpha, a^\beta + b^\beta)$

(ii) $\tilde{A} - \tilde{B} = (a^M - b^N, a^N - b^M, a^\alpha + b^\beta, a^\beta + b^\alpha)$

(iii) $\widetilde{A}\,\widetilde{B} = (a^M b^M, a^N b^N \quad a^M b^\alpha + b^M a^\alpha - a^\alpha b^\alpha, \; a^M b^\beta + b^M a^\beta + a^\beta b^\beta)$, where $\tilde{A}$ and $\widetilde{B}$ are non-negative trapezoidal fuzzy numbers.

(iv) $\frac{\widetilde{A}}{\widetilde{B}} = \left(\frac{a^M}{b^N}, \frac{a^N}{b^M}, \frac{a^M}{b^N} - \frac{a^M - a^\alpha}{b^N + b^\beta}, \frac{a^N + a^\beta}{b^M - b^\alpha} - \frac{a^N}{b^M}\right)$, where $\widetilde{A}$ is a non-negative trapezoidal fuzzy number and $\widetilde{B}$ is a positive trapezoidal fuzzy number.

Let $\tilde{A} = (a_1, b_1, c_1, d_1) = (a^M - a^\alpha, a^M, a^N, a^N + a^\beta)$ and $\widetilde{B} = (a_2, b_2, c_2, d_2) = (b^M - b^\alpha, b^M, b^N, b^N + b^\beta)$ be two arbitrary trapezoidal fuzzy numbers. Then,

(i) $\tilde{A} + \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

(ii) $\tilde{A} - \tilde{B} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$

(iii) $\widetilde{A}\,\widetilde{B} = (a_1a_2, b_1b_2, c_1c_2, d_1d_2)$, where $\tilde{A}$ and $\widetilde{B}$ are non-negative trapezoidal fuzzy numbers.

(iv) $\frac{\tilde{A}}{\tilde{B}} = \left(\frac{a_1}{d_2}, \frac{b_1}{c_2}, \frac{c_1}{a_2}, \frac{d_1}{a_2}\right)$, where $\widetilde{A}$ is a non-negative trapezoidal fuzzy number and $\widetilde{B}$ is a positive trapezoidal fuzzy number.

2.2 Existing fuzzy DEA model

If there are n DMUs, then the best relative efficiency (E_p) of p^{th} DMU can be obtained by solving the CCR DEA model-2.1 [11].

CCR DEA model-2.1

$$\text{Maximize} \left[E_p = \frac{\text{Virtual output of } p^{th} \text{ DMU}}{\text{Virtual input of } p^{th} \text{ DMU}}\right]$$

Subject to

$$\left[\frac{\text{Virtual output of } j^{th} \text{ DMU}}{\text{Virtual input of } j^{th} \text{ DMU}}\right] \leq 1, \qquad j = 1, \dots, n.$$

If each DMU have m inputs ($x_{ij};\ i = 1,2, \dots, m; j = 1,2, \dots, n$) and s outputs ($y_{rj}; r = 1,2, \dots s;\ j = 1,2, \dots, n$) then the CCR DEA model-2.1 can be transformed into the crisp CCR DEA model-2.2 [11].

Crisp CCR DEA model-2.2

$$\text{Maximize} \left[E_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad \forall j,$$

$$u_r, v_i \geq 0, \qquad \forall r, i.$$

where u_r $(r = 1, \ldots . s)$ and v_i $(i = 1, \ldots, m)$ are the weights assigned to the r^{th} output and i^{th} input, respectively.

The crisp CCR DEA model-2.2 can be transformed into the crisp CCR DEA model-2.3 [11].

CCR crisp DEA model-2.3

Maximize $\left[E_p = \sum_{r=1}^{s} u_r y_{rp}\right]$

Subject to

$$\sum_{i=1}^{m} v_i x_{ip} = 1 \,,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \,, \quad \forall j,$$

$$u_r, v_i \geq 0, \qquad \forall r, i.$$

where u_r $(r = 1, \ldots . s)$ and v_i $(i = 1, \ldots, m)$ are the weights assigned to the r^{th} output and i^{th} input, respectively.

Hatami Marbini et al. [40] pointed out that the crisp CCR DEA model-2.3, proposed by Charnes et al. [11] can be used only to evaluate the best relative efficiency (E_p) of p^{th} DMU only if the collected data is precise. However, if the collected data is imprecise or vague then the crisp CCR DEA model-2.3, proposed by Charnes et al. [11] cannot be used to obtain the best relative efficiency (E_p) of p^{th} DMU. To overcome this limitation of the crisp CCR DEA model-2.3, proposed by Charnes et al. [11], Hatami Marbini et al. [40] transformed the crisp CCR DEA model-2.3 into fuzzy CCR DEA model-2.4 by replacing the crisp parameters u_r, y_{rj}, v_i and x_{ij} with triangular fuzzy numbers $\tilde{u}_r = \left(u_r^M, u_r^\alpha, u_r^\beta\right)$, $\tilde{y}_{rj} = \left(y_{rj}^M, y_{rj}^\alpha, y_{rj}^\beta\right)$, $\tilde{v}_i\left(v_i^M, v_i^\alpha, v_i^\beta\right)$ and $\tilde{x}_{ij} = (x_{ij}^M, x_{ij}^\alpha, x_{ij}^\beta)$ respectively.

Fuzzy CCR DEA model-2.4

$$\text{Maximize}\left[\tilde{E}_p \approx \sum_{r=1}^{s}\left(u_r^M, u_r^\alpha, u_r^\beta\right) \otimes \left(y_{rp}^M, y_{rp}^\alpha, y_{rp}^\beta\right)\right]$$

Subject to

$$\sum_{i=1}^{m}\left(v_i^M, v_i^\alpha, v_i^\beta\right) \otimes \left(x_{ip}^M, x_{ip}^\alpha, x_{ip}^\beta\right) \approx (1,0,0)$$

$$\sum_{r=1}^{s}\left(u_r^M, u_r^\alpha, u_r^\beta\right) \otimes \left(y_{rj}^M, y_{rj}^\alpha, y_{rj}^\beta\right) - \sum_{i=1}^{m}\left(v_i^M, v_i^\alpha, v_i^\beta\right) \otimes \left(x_{ij}^M, x_{ij}^\alpha, x_{ij}^\beta\right) \preccurlyeq (0,0,0), \forall j,$$

$$\left(u_r^M, u_r^\alpha, u_r^\beta\right), \left(v_i^M, v_i^\alpha, v_i^\beta\right) \succcurlyeq (0,0,0), \forall r, i.$$

$\left(u_r^M, u_r^\alpha, u_r^\beta\right)$ and $\left(v_i^M, v_i^\alpha, v_i^\beta\right)$ are non-negative triangular fuzzy numbers.

2.3 Existing method

Hatami-Marbini et al. [40], proposed the following method to solve the fuzzy CCR DEA model-2.4 for evaluating the best relative fuzzy efficiency of p^{th} DMU.

Step 1: The fuzzy CCR DEA model-2.4 can be transformed into the fuzzy CCR DEA model-2.5.

Fuzzy CCR DEA model-2.5

$$\text{Maximize}\left[\tilde{E}_p \approx \sum_{r=1}^{s}\left(u_r^M\left(y_{rp}^M + y_{rp}^\beta - y_{rp}^\alpha\right), u_r^\beta\, y_{rp}^M, u_r^\alpha y_{rp}^M\right)\right]$$

$$\sum_{i=1}^{m}\left(v_i^M\left(x_{ip}^M + x_{ip}^\beta - x_{ip}^\alpha\right), v_i^\beta x_{ip}^M, v_i^\alpha x_{ip}^M\right) = 1,$$

$$\sum_{r=1}^{s}\left(u_r^M\left(y_{rj}^M + y_{rj}^\beta - y_{rj}^\alpha\right), u_r^\beta\, y_{rj}^M, u_r^\alpha y_{rj}^M\right)$$

$$-\sum_{i=1}^{m}\left(v_i^M\left(x_{ij}^M + x_{ij}^\beta - x_{ij}^\alpha\right), v_i^\beta x_{ij}^M, v_i^\alpha x_{ij}^M\right) \preccurlyeq (0,0,0), \forall j,$$

$$\left(u_r^M, u_r^\alpha, u_r^\beta\right), \left(v_i^M, v_i^\alpha, v_i^\beta\right) \succcurlyeq (0,0,0), \forall r, i.$$

$\left(u_r^M, u_r^\alpha, u_r^\beta\right)$ and $\left(v_i^M, v_i^\alpha, v_i^\beta\right)$ are non-negative triangular fuzzy numbers.

Step 2: The fuzzy CCR DEA model-2.5 can be transformed into the crisp CCR DEA model-2.6.

Crisp CCR DEA model-2.6

$$\text{Maximize}\left[E_p = \Re(\tilde{E}_p) = \sum_{r=1}^{s}\begin{bmatrix} u_r^M\left(y_{rp}^M + \left(\frac{1}{4}\right)y_{rp}^\beta - \left(\frac{1}{4}\right)y_{rp}^\alpha\right) \\ + u_r^\beta\left(\left(\frac{1}{4}\right)y_{rp}^M\right) - u_r^\alpha\left(\left(\frac{1}{4}\right)y_{rp}^M\right)\end{bmatrix}\right]$$

$$\sum_{i=1}^{m}\left[v_i^M\left(x_{ip}^M + \left(\frac{1}{4}\right)x_{ip}^\beta - \left(\frac{1}{4}\right)x_{ip}^\alpha\right) + v_i^\beta\left(\left(\frac{1}{4}\right)x_{ip}^M\right) - v_i^\alpha\left(\left(\frac{1}{4}\right)x_{ip}^M\right)\right] = 1,$$

$$\sum_{r=1}^{s}\left[u_r^M\left(y_{rj}^M + \left(\frac{1}{4}\right)y_{rj}^\beta - \left(\frac{1}{4}\right)y_{rj}^\alpha\right) + u_r^\beta\left(\left(\frac{1}{4}\right)y_{rj}^M\right) - u_r^\alpha\left(\left(\frac{1}{4}\right)y_{rj}^M\right)\right] -$$

$$\sum_{i=1}^{m}\left[v_i^M\left(x_{ij}^M + \left(\frac{1}{4}\right)x_{ij}^\beta - \left(\frac{1}{4}\right)x_{ij}^\alpha\right) + v_i^\beta\left(\left(\frac{1}{4}\right)x_{ij}^M\right) - v_i^\alpha\left(\left(\frac{1}{4}\right)x_{ij}^M\right)\right] \le 0 \ \forall j,$$

$$u_r^M - u_r^\alpha \ge 0 \ \forall r,$$

$$u_r^M - \left(\frac{1}{4}\right)u_r^\alpha + \left(\frac{1}{4}\right)u_r^\beta \ge 0, \forall r,$$

$$v_i^M - v_i^\beta \ge 0, \qquad \forall i,$$

$$v_i^M - \left(\frac{1}{4}\right)v_i^\alpha + \left(\frac{1}{4}\right)v_i^\beta \ge 0, \qquad \forall i,$$

$$u_r^\alpha \ge 0, u_r^\beta \ge 0, \forall r, v_i^\alpha \ge 0, \ v_i^\beta \ge 0, \ \forall i.$$

Step 3: Find the optimal value crisp optimal value $E_p = \Re(\tilde{E}_p)$, representing ranking value of best relative fuzzy efficiency of p^{th} DMU, of the crisp CCR DEA model 2.6.

2.4 Flaws in the existing method

If $\tilde{A} = (a^M, a^\alpha, a^\beta)$ and $\widetilde{B} = (b^M, b^\alpha, b^\beta)$ are two non-negative triangular fuzzy numbers then

$$\widetilde{A} \otimes \widetilde{B} = (a^M b^M, \ a^M b^\alpha + b^M a^\alpha - a^\alpha b^\alpha, \ a^M b^\beta + b^M a^\beta + a^\beta b^\beta).$$

However, it is obvious from Step 1 of the existing method [40], described in Section 2.3, that Hatami-Marbini et al. [40], have used the incorrect product $\widetilde{A} \otimes \widetilde{B} = (a^M(b^M + b^\beta - b^\alpha), \ a^\beta b^M, \ a^\alpha b^M)$ to transform the fuzzy CCR DEA model-2.4 into the fuzzy CCR DEA model-2.5. Hence, the method, proposed by Hatami-Marbini et al. [40], is not valid in its present form.

2.5 Modified method

In this section, the existing method [40] is modified.

Step 1: Using the product of triangular fuzzy numbers, defined in Section 2.1.2.1, the fuzzy CCR DEA model-2.4 can be transformed into the fuzzy CCR DEA model-2.7.

Fuzzy CCR DEA model-2.7

$$\text{Maximize}\left[\tilde{E}_p \approx \sum_{r=1}^{s}\left(u_r^M y_{rp}^M,\ u_r^M y_{rp}^\alpha + y_{rp}^M u_r^\alpha - u_r^\alpha y_{rp}^\alpha,\ u_r^M y_{rp}^\beta + y_{rp}^M u_r^\beta + u_r^\beta y_{rp}^\beta\right)\right]$$

Subject to

$$\sum_{i=1}^{m}\left(v_i^M x_{ip}^M,\ v_i^M x_{ip}^\alpha + x_{ip}^M v_i^\alpha - v_i^\alpha x_{ip}^\alpha,\ v_i^M x_{ip}^\beta + x_{ip}^M v_i^\beta + v_i^\beta x_{ip}^\beta\right) \approx 1,$$

$$\sum_{r=1}^{s}\left(u_r^M y_{rj}^M,\ u_r^M y_{rj}^\alpha + y_{rj}^M u_r^\alpha - u_r^\alpha y_{rj}^\alpha,\ u_r^M y_{rj}^\beta + y_{rj}^M u_r^\beta + u_r^\beta y_{rj}^\beta\right)$$

$$\preccurlyeq \sum_{i=1}^{m}\left(v_i^M x_{ij}^M,\ v_i^M x_{ij}^\alpha + x_{ij}^M v_i^\alpha - v_i^\alpha x_{ij}^\alpha,\ v_i^M x_{ij}^\beta + x_{ij}^M v_i^\beta + v_i^\beta x_{ij}^\beta\right), \forall j,$$

$$\left(u_r^M, u_r^\alpha, u_r^\beta\right), \left(v_i^M, v_i^\alpha, v_i^\beta\right) \succcurlyeq (0,0,0), \forall r, i.$$

$\left(u_r^M, u_r^\alpha, u_r^\beta\right)$ and $\left(v_i^M, v_i^\alpha, v_i^\beta\right)$ are non-negative triangular fuzzy numbers.

Step 2: Using Definition 2.12, the fuzzy CCR model-2.7 can be transformed into crisp CCR DEA model-2.8.

Crisp CCR DEA model-2.8

$$\text{Maximize}\left[\Re(\tilde{E}_p) = \Re\left(\sum_{r=1}^{s}\left(u_r^M y_{rp}^M,\ u_r^M y_{rp}^\alpha + y_{rp}^M u_r^\alpha - u_r^\alpha y_{rp}^\alpha,\ \ u_r^M y_{rp}^\beta + y_{rp}^M u_r^\beta + u_r^\beta y_{rp}^\beta\right)\right)\right]$$

Subject to

$$\Re\left(\sum_{i=1}^{m}\left(v_i^M x_{ip}^M,\ \ v_i^M x_{ip}^\alpha + x_{ip}^M v_i^\alpha - v_i^\alpha x_{ip}^\alpha,\ \ v_i^M x_{ip}^\beta + x_{ip}^M v_i^\beta + v_i^\beta x_{ip}^\beta\right)\right) = \Re(1,1,1)$$

$$\Re\left(\sum_{r=1}^{s}\left(u_r^M y_{rj}^M,\ \ u_r^M y_{rj}^\alpha + y_{rj}^M u_r^\alpha - u_r^\alpha y_{rj}^\alpha,\ \ u_r^M y_{rj}^\beta + y_{rj}^M u_r^\beta + u_r^\beta y_{rj}^\beta\right)\right)$$

$$\leq \Re\left(\sum_{i=1}^{m}\left(v_i^M x_{ij}^M,\ \ v_i^M x_{ij}^\alpha + x_{ij}^M v_i^\alpha - v_i^\alpha x_{ij}^\alpha,\ \ v_i^M x_{ij}^\beta + x_{ij}^M v_i^\beta + v_i^\beta x_{ij}^\beta\right)\right), \forall j,$$

$\Re\left(u_r^M, u_r^\alpha, u_r^\beta\right) \geq \Re(0,0,0), \Re\left(v_i^M, v_i^\alpha, v_i^\beta\right) \geq \Re(0,0,0), \forall r, i.$

$u_r^M - u_r^\alpha \geq 0\ \forall r,$

$v_i^M - v_i^\beta \geq 0, \qquad \forall i,$

$u_r^\alpha \geq 0, u_r^\beta \geq 0, \forall r, v_i^\alpha \geq 0,\ v_i^\beta \geq 0,\ \forall i.$

Step 3: The crisp CCR DEA model-2.8 can be transformed into crisp CCR DEA model-2.9.

Crisp CCR DEA model-2.9

$$\text{Maximize}\left[\Re(\tilde{E}_p) = \sum_{r=1}^{s}\Re\left(u_r^M y_{rp}^M,\ u_r^M y_{rp}^\alpha + y_{rp}^M u_r^\alpha - u_r^\alpha y_{rp}^\alpha,\ \ u_r^M y_{rp}^\beta + y_{rp}^M u_r^\beta + u_r^\beta y_{rp}^\beta\right)\right]$$

Subject to

$$\sum_{i=1}^{m}\Re\left(v_i^M x_{ip}^M,\ \ v_i^M x_{ip}^\alpha + x_{ip}^M v_i^\alpha - v_i^\alpha x_{ip}^\alpha,\ \ v_i^M x_{ip}^\beta + x_{ip}^M v_i^\beta + v_i^\beta x_{ip}^\beta\right) = \Re(1,1,1)$$

$$\sum_{r=1}^{s} \Re\left(u_r^M y_{rj}^M,\ u_r^M y_{rj}^\alpha + y_{rj}^M u_r^\alpha - u_r^\alpha y_{rj}^\alpha,\ u_r^M y_{rj}^\beta + y_{rj}^M u_r^\beta + u_r^\beta y_{rj}^\beta\right)$$

$$\leq \Re\left(\sum_{i=1}^{m}\left(v_i^M x_{ij}^M,\ v_i^M x_{ij}^\alpha + x_{ij}^M v_i^\alpha - v_i^\alpha x_{ij}^\alpha,\ v_i^M x_{ij}^\beta + x_{ij}^M v_i^\beta + v_i^\beta x_{ij}^\beta\right)\right), \forall j,$$

$$\Re\left(u_r^M, u_r^\alpha, u_r^\beta\right) \geq \Re(0,0,0), \Re\left(v_i^M, v_i^\alpha, v_i^\beta\right) \geq \Re(0,0,0), \forall r, i.$$

$$u_r^M - u_r^\alpha \geq 0\ \forall r,$$

$$v_i^M - v_i^\beta \geq 0, \qquad \forall i,$$

$$u_r^\alpha \geq 0, u_r^\beta \geq 0\ , \forall r, v_i^\alpha \geq 0,\ v_i^\beta \geq 0,\ \forall i.$$

Step 4: Using the ranking formula $\Re\left(a^M, a^\alpha, a^\beta\right) = a^M + \frac{a^\beta - a^\alpha}{4}$ and Definition 2.7, the crisp CCR DEA model-2.9 can be transformed into the crisp CCR DEA model-2.10.

CCR crisp DEA model-2.10

$$\text{Maximize}\left[E_p = \sum_{r=1}^{s}\left(u_r^M y_{rp}^M + \frac{1}{4}\left[\left(u_r^M y_{rp}^\beta + y_{rp}^M u_r^\beta + u_r^\beta y_{rp}^\beta\right) - \left(v_i^M x_{ip}^\alpha + x_{ip}^M v_i^\alpha - v_i^\alpha x_{ip}^\alpha\right)\right]\right)\right]$$

Subject to

$$\sum_{i=1}^{m}\left(v_i^M x_{ip}^M + \frac{1}{4}\left[\left(v_i^M x_{ip}^\beta + x_{ip}^M v_i^\beta + v_i^\beta x_{ip}^\beta\right) - \left(v_i^M x_{ip}^\alpha + x_{ip}^M v_i^\alpha - v_i^\alpha x_{ip}^\alpha\right)\right]\right) = 1,$$

$$\sum_{r=1}^{s}\left(u_r^M y_{rj}^M + \frac{1}{4}\left[\left(u_r^M y_{rj}^{\beta} + y_{rj}^M u_r^{\beta} + u_r^{\beta} y_{rj}^{\beta}\right) - \left(u_r^M y_{rj}^{\alpha} + y_{rj}^M u_r^{\alpha} - u_r^{\alpha} y_{rj}^{\alpha}\right)\right]\right)$$

$$- \sum_{i=1}^{m}\left(v_i^M x_{ij}^M + \frac{1}{4}\left[\left(v_i^M x_{ij}^{\beta} + x_{ij}^M v_i^{\beta} + v_i^{\beta} x_{ij}^{\beta}\right) - \left(v_i^M x_{ij}^{\alpha} + x_{ij}^M v_i^{\alpha} - v_i^{\alpha} x_{ij}^{\alpha}\right)\right]\right) \leq 0, \forall j,$$

$$u_r^M - u_r^{\alpha} \geq 0, \forall r,$$

$$u_r^M - \left(\frac{1}{4}\right) u_r^{\alpha} + \left(\frac{1}{4}\right) u_r^{\beta} \geq 0, \forall r,$$

$$v_i^M - v_i^{\beta} \geq 0, \quad \forall i,$$

$$v_i^M - \left(\frac{1}{4}\right) v_i^{\alpha} + \left(\frac{1}{4}\right) v_i^{\beta} \geq 0, \quad \forall i,$$

$$u_r^{\alpha} \geq 0, u_r^{\beta} \geq 0, \forall r, v_i^{\alpha} \geq 0, v_i^{\beta} \geq 0, \forall i.$$

Step 5: Find the optimal value $E_p = \Re(\tilde{E}_p)$, representing ranking value of relative fuzzy efficiency of p^{th} DMU, of the crisp CCR DEA model 2.6.

2.6 Exact best relative fuzzy efficiency of existing problem

Hatami-Marbini et al. [40] solved existing problem [31] to illustrate his proposed approach. However, as discussed in Section 2.3 that there are flaws in the existing method [40]. So, the results of this problem, obtained by Hatami Marbini et al. [40], is also not exact. In this section, the exact results of the same problem is obtained by the modified method.

2.6.1 Problem description

Hatami Marbini et al. [40] evaluated the best relative efficiency of each of five DMUs, each with two outputs and two inputs, to illustrate their proposed approach by considering the existing input and output data, represented by triangular fuzzy numbers, shown in Table 2.1.

Table 2.1. Input and output data

(DMUs)	Input 1	Input 2	Output 1	Output 2
1	(4, 0.5, 0.5)	(2.1, 0.2, 0.2)	(2.6, 0.2, 0.2)	(4.1, 0.3, 0.3)
2	(2.9, 0, 0)	(1.5, 0.1, 0.1)	(2.2, 0, 0)	(3.5, 0.2, 0.2)
3	(4.9, 0.5, 0.5)	(2.6, 0.4, 0.4)	(3.2, 0.5, 0.5)	(5.1, 0.8, 0.8)
4	(4.1, 0.7, 0.7)	(2.3, 0.1, 0.1)	(2.9, 0.4, 0.4)	(5.7, 0.2, 0.2)
5	(6.5, 0.6, 0.6)	(4.1, 0.5, 0.5)	(5.1, 0.7, 0.7)	(7.4, 0.9, 0.9)

2.6.2 Fuzzy CCR DEA models

Hatami Marbini et al. [40] solved the fuzzy CCR DEA model-2.11 to 2.15 to evaluate the best relative fuzzy efficiency of DMU_1 to DMU_5 respectively.

Fuzzy CCR DEA model-2.11

Maximize $\left[\tilde{E}_1 \approx (2.6, 0.2, 0.2)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (4.1, 0.3, 0.3)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right]$

Subject to

$$(4, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.1, 0.2, 0.2)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1,$$

$$\left[(2.6, 0.2, 0.2)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (4.1, 0.3, 0.3)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right] - \left[(4, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.1, 0.2, 0.2)\left(v_2^M, v_2^\alpha, v_2^\beta\right)\right] \preccurlyeq (0, 0,0),$$

$$\left[(2.2, 0, 0)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (3.5, 0.2, 0.2)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right] - \left[(2.9, 0, 0)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (1.5, 0.1, 0.1)\left(v_2^M, v_2^\alpha, v_2^\beta\right)\right] \preccurlyeq (0, 0,0),$$

$$\left[(3.2, 0.5, 0.5)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (5.1, 0.8, 0.8)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right] - \left[(4.9, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.6, 0.4, 0.4)\left(v_2^M, v_2^\alpha, v_2^\beta\right)\right] \preccurlyeq (0, 0,0),$$

$$\left[(2.9, 0.4, 0.4)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (5.7, 0.2, 0.2)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right] - \left[(4.1, 0.7, 0.7)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.3, 0.1, 0.1)\left(v_2^M, v_2^\alpha, v_2^\beta\right)\right] \preccurlyeq (0, 0,0),$$

$$\left[(5.1, 0.7, 0.7)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (7.4, 0.9, 0.9)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right] - \left[(6.5, 0.6, 0.6)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (4.1, 0.5, 0.5)\left(v_2^M, v_2^\alpha, v_2^\beta\right)\right] \preccurlyeq (0, 0,0),$$

$\left(u_1^M, u_1^\alpha, u_1^\beta\right) \succcurlyeq (0,0,0), \left(u_2^M, u_2^\alpha, u_2^\beta\right) \succcurlyeq (0,0,0),$

$\left(v_1^M, v_1^\alpha, v_1^\beta\right) \succcurlyeq (0,0,0), \left(v_2^M, v_2^\alpha, v_2^\beta\right) \succcurlyeq (0,0,0),$

$\left(u_1^M, u_1^\alpha, u_1^\beta\right), \left(u_2^M, u_2^\alpha, u_2^\beta\right), \left(v_1^M, v_1^\alpha, v_1^\beta\right)$ and $\left(v_2^M, v_2^\alpha, v_2^\beta\right)$ are non-negative triangular fuzzy numbers.

Fuzzy CCR DEA model-2.12

Maximize $\left[\tilde{E}_2 \approx (2.2, 0, 0)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (3.5, 0.2, 0.2)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right]$

Subject to

$$(2.9, 0, 0)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (1.5, 0.1, 0.1)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1,$$

and

All the constraints of fuzzy CCR DEA model-2.11 except the first constraint

"$(4, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.1, 0.2, 0.2)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1$".

Fuzzy CCR DEA model-2.13

Maximize $\left[\tilde{E}_3 \approx (3.2, 0.5, 0.5)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (5.1, 0.8, 0.8)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right]$

Subject to

$$(4.9, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.6, 0.4, 0.4)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1,$$

and

All the constraints of fuzzy CCR DEA model-2.11 except the first constraint

"$(4, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.1, 0.2, 0.2)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1$".

Fuzzy CCR DEA model-2.14

Maximize $\left[\tilde{E}_4 \approx (2.9, 0.4, 0.4)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (5.7, 0.2, 0.2)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right]$

Subject to

$$(4.1, 0.7, 0.7)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.3, 0.1, 0.1)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1,$$

and

All the constraints of fuzzy CCR DEA model-2.11 except the first constraint

"$(4, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.1, 0.2, 0.2)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1$"

Fuzzy CCR DEA model-2.15

Maximize $\left[\tilde{E}_5 \approx\ (5.1, 0.7, 0.7)\left(u_1^M, u_1^\alpha, u_1^\beta\right) + (7.4, 0.9, 0.9)\left(u_2^M, u_2^\alpha, u_2^\beta\right)\right]$

Subject to

$(6.5, 0.6, 0.6)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (4.1, 0.5, 0.5)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1,$

and

All the constraints of fuzzy CCR DEA model-2.11 except the first constraint

"$(4, 0.5, 0.5)\left(v_1^M, v_1^\alpha, v_1^\beta\right) + (2.1, 0.2, 0.2)\left(v_2^M, v_2^\alpha, v_2^\beta\right) \approx 1$"

2.6.3 Exact best relative fuzzy efficiency of $\mathbf{DMU_1}$

Using the method, modified in Section 2.5, the exact best relative fuzzy efficiency of DMU_1 can be obtained as follows:

Step 1: Using Step 1 of the modified method, fuzzy CCR DEA model-2.11 can be transformed into fuzzy CCR DEA model-2.16.

Fuzzy CCR DEA model-2.16

$$\text{Maximize } \left[\tilde{E}_1 \approx \left(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2\, u_1^M + 2.8\, u_1^\beta\right) + \left(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\ \right)\right]$$

Subject to

$$\left(4v_1^M, 0.5v_1^M + 3.5\, v_1^\alpha, 0.5\, v_1^M + 4.5v_1^\beta\right) + \left(2.1v_2^M, 0.2v_2^M + 1.9\, v_2^\alpha, 0.2v_2^M + 2.3v_2^\beta\right) \approx 1,$$

$$\left(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2u_1^M + \ 2.8\, u_1^\beta\right) + \left(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\right)$$

$$- \left(4v_1^M, 0.5v_1^M + 3.5v_1^\alpha, 4.5v_1^\beta + 0.5v_1^M\right)$$

$$- \left(2.1v_2^M, 0.2v_2^M + 1.9v_2^\alpha\ , 0.2v_2^M + 2.3v_2^\beta\right) \preccurlyeq (0,0,0),$$

$$\left(2.2u_1^M, 2.2\, u_1^\alpha, 2.2u_1^\beta\right) + \left(3.5u_2^M, 0.2u_2^M + 3.3u_2^\alpha\ , 0.2u_2^M + \ 3.7u_2^\beta\right) - \left(2.9v_1^M, 2.9v_1^\alpha, 2.9v_1^\beta\right)$$

$$- \left(1.5v_2^M, 0.1v_2^M + 1.4v_2^\alpha, 0.1v_2^M + 1.6v_2^\beta\right) \preccurlyeq (0,0,0),$$

$$\left(3.2\, u_1^M, 0.5u_1^M + 2.7u_1^\alpha, 0.5u_1^M + 3.7u_1^\beta\right) + \left(5.1u_2^M, 0.8\, u_2^M + 4.3u_2^\alpha, 0.8u_2^M + \ 5.9u_2^\beta\right)$$

$$- \left(4.9v_1^M, 0.5v_1^M + 4.4v_1^\alpha, 0.5v_1^M + 5.4v_1^\beta\ \right)$$

$$- \left(2.6v_2^M, 0.4v_2^M + 2.2\, v_2^\alpha, 0.4v_2^M + 3.0v_2^\beta\right) \preccurlyeq (0,0,0),$$

$$\left(2.9u_1^M, 0.4u_1^M + 2.5u_1^\alpha, , 0.4u_1^M + 3.3u_1^\beta\right) + \left(5.7u_2^M, 0.2u_2^M + 5.5\, u_2^\alpha, 0.2u_2^M + 5.9u_2^\beta\right)$$

$$- \left(4.1\, v_1^M, 0.7v_1^M + 3.4v_1^\alpha, 0.7v_1^M + 4.8v_1^\beta\right)$$

$$- \left(2.3v_2^M, 0.1v_2^M + 2.2v_2^\alpha, 0.1v_2^M + 2.4v_2^\beta\right) \preccurlyeq (0,0,0),$$

$$\left(5.1u_1^M, 0.7u_1^M + 4.4\, u_1^\alpha, 0.7u_1^M + 5.8u_1^\beta\right) + \left(7.4u_2^M, 0.9u_2^M + 6.5u_2^\alpha, 0.9u_2^M + 8.3u_2^\beta\right)$$

$$- \left(6.5v_1^M, 0.6v_1^M + 5.9v_1^\alpha\ , 0.6v_1^M + 7.1v_1^\beta\right)$$

$$- \left(4.1v_2^M, 0.5v_2^M + 3.6v_2^\alpha, 0.5v_2^M + \ 4.6v_2^\beta\right) \preccurlyeq (0,0,0),$$

$$\left(u_1^M, u_1^\alpha, u_1^\beta\right) \succcurlyeq (0,0,0), \left(u_2^M, u_2^\alpha, u_2^\beta\right) \succcurlyeq (0,0,0),$$

$\left(v_1^M, v_1^\alpha, v_1^\beta\right) \succcurlyeq (0,0,0), \left(v_2^M, v_2^\alpha, v_2^\beta\right) \succcurlyeq (0,0,0),$

$\left(u_1^M, u_1^\alpha, u_1^\beta\right), \left(u_2^M, u_2^\alpha, u_2^\beta\right), \left(v_1^M, v_1^\alpha, v_1^\beta\right)$ and $\left(v_2^M, v_2^\alpha, v_2^\beta\right)$ are non-negative triangular fuzzy numbers.

Step 2: Using Step 2 of the modified method, the fuzzy CCR DEA model-2.16 can be transformed into crisp CCR DEA model-2.17.

Crisp CCR DEA model-2.17

$$\text{Maximize } \Big[\Re\big(\tilde{E}_1\big)$$

$$= \Re\Big(\Big(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2\, u_1^M + 2.8\, u_1^\beta\Big)$$

$$+ \Big(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\,\Big)\Big)\Big]$$

Subject to

$$\Re\Big(\Big(4v_1^M, 0.5v_1^M + 3.5\, v_1^\alpha, 0.5\, v_1^M + 4.5v_1^\beta\Big) + \Big(2.1v_2^M, 0.2v_2^M + 1.9\, v_2^\alpha, 0.2v_2^M + 2.3v_2^\beta\Big)\Big)$$

$$= \Re(1,1,1),$$

$$\Re\Big(\Big(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2u_1^M + 2.8\, u_1^\beta\Big) + \Big(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\Big)$$

$$- \Big(4v_1^M, 0.5v_1^M + 3.5v_1^\alpha, 4.5v_1^\beta + 0.5v_1^M\Big)$$

$$- \Big(2.1v_2^M, 0.2v_2^M + 1.9v_2^\alpha\,, 0.2v_2^M + 2.3v_2^\beta\Big)\Big) \le \Re(0,0,0),$$

$$\Re\left(\left(2.2u_1^M, 2.2\,u_1^\alpha, 2.2u_1^\beta\right) + \left(3.5u_2^M, 0.2u_2^M + 3.3u_2^\alpha\,, 0.2u_2^M + \ 3.7u_2^\beta\right)\right.$$

$$\left. - \left(2.9v_1^M, 2.9v_1^\alpha, 2.9v_1^\beta\right) - \left(1.5v_2^M, 0.1v_2^M + 1.4v_2^\alpha, 0.1v_2^M + 1.6v_2^\beta\right)\right)$$

$$\leq \Re(0,0,0),$$

$$\Re\left(\left(3.2\,u_1^M, 0.5u_1^M + 2.7u_1^\alpha, 0.5u_1^M + 3.7u_1^\beta\right) + \left(5.1u_2^M, 0.8\,u_2^M + 4.3u_2^\alpha, 0.8u_2^M + \ 5.9u_2^\beta\right)\right.$$

$$- \left(4.9v_1^M, 0.5v_1^M + 4.4v_1^\alpha, 0.5v_1^M + 5.4v_1^\beta\right)$$

$$\left. - \ \left(2.6v_2^M, 0.4v_2^M + 2.2\,v_2^\alpha, 0.4v_2^M + 3.0v_2^\beta\right)\right) \leq \Re(0,0,0),$$

$$\Re\left(\left(2.9u_1^M, 0.4u_1^M + 2.5u_1^\alpha,, 0.4u_1^M + 3.3u_1^\beta\right) + \left(5.7u_2^M, 0.2u_2^M + 5.5\,u_2^\alpha, 0.2u_2^M + 5.9u_2^\beta\right)\right.$$

$$- \left(4.1\,v_1^M, 0.7v_1^M + 3.4v_1^\alpha, 0.7v_1^M + 4.8v_1^\beta\right)$$

$$\left. - \left(2.3v_2^M, 0.1v_2^M + 2.2v_2^\alpha, 0.1v_2^M + 2.4v_2^\beta\right)\right) \leq \Re(0,0,0),$$

$$\Re\left(\left(5.1u_1^M, 0.7u_1^M + 4.4\,u_1^\alpha, 0.7u_1^M + 5.8u_1^\beta\right) + \left(7.4u_2^M, 0.9u_2^M + 6.5u_2^\alpha, 0.9u_2^M + 8.3u_2^\beta\right)\right.$$

$$- \left(6.5v_1^M, 0.6v_1^M + 5.9v_1^\alpha\,, 0.6v_1^M + 7.1v_1^\beta\right)$$

$$\left. - \left(4.1v_2^M, 0.5v_2^M + 3.6v_2^\alpha, 0.5v_2^M + \ 4.6v_2^\beta\right)\right) \leq \Re(0,0,0),$$

$$\Re\left(u_1^M, u_1^\alpha, u_1^\beta\right) \geq \Re(0,0,0), \Re\left(u_2^M, u_2^\alpha, u_2^\beta\right) \geq \Re(0,0,0),$$

$$\Re\left(v_1^M, v_1^\alpha, v_1^\beta\right) \geq \Re(0,0,0), \Re\left(v_2^M, v_2^\alpha, v_2^\beta\right) \geq \Re(0,0,0),$$

$$u_1^M - u_1^\alpha \geq 0, u_2^M - u_2^\alpha \geq 0, v_1^M - v_1^\alpha \geq 0, v_2^M - v_2^\alpha \geq 0,$$

$$u_r^\alpha \geq 0, u_r^\beta \geq 0\,, \forall r, r = 1,2, v_i^\alpha \ \geq 0,\ v_i^\beta \geq 0,\ \forall i, i = 1,2.$$

Step 3: Using Step 3 of the modified method, the fuzzy CCR DEA model-2.17 can be transformed into crisp CCR DEA model-2.18.

Crisp CCR DEA model-2.18

Maximize $\Big[\Re(\tilde{E}_1)$

$$= \Re\Big(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2\, u_1^M + 2.8\, u_1^\beta\Big)$$

$$+ \Re\Big(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\Big)\Big]$$

Subject to

$$\Re\Big(4v_1^M, 0.5v_1^M + 3.5\, v_1^\alpha, 0.5\, v_1^M + 4.5v_1^\beta\Big) + \Re\Big(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2\, u_1^M + 2.8\, u_1^\beta\Big)$$

$$+ \Re\Big(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\ \Big)\Big(2.1v_2^M, 0.2v_2^M + 1.9\, v_2^\alpha, 0.2v_2^M$$

$$+ 2.3v_2^\beta\Big) = \Re(1,1,1),$$

$$\Re\Big(2.6u_1^M, 0.2u_1^M + 2.4u_1^\alpha, 0.2u_1^M +\ 2.8\, u_1^\beta\Big) + \Re\Big(4.1u_2^M, 0.3u_2^M + 3.8u_2^\alpha, 0.3u_2^M + 4.4u_2^\beta\Big)$$

$$- \Re\Big(4v_1^M, 0.5v_1^M + 3.5v_1^\alpha, 4.5v_1^\beta + 0.5v_1^M\Big)$$

$$- \Re\Big(2.1v_2^M, 0.2v_2^M + 1.9v_2^\alpha\ , 0.2v_2^M + 2.3v_2^\beta\Big)\ \leq \Re(0,0,0),$$

$$\Re\Big(2.2u_1^M, 2.2\, u_1^\alpha, 2.2u_1^\beta\Big) + \Re\Big(3.5u_2^M, 0.2u_2^M + 3.3u_2^\alpha\ , 0.2u_2^M +\ 3.7u_2^\beta\Big)$$

$$- \Re\Big(2.9v_1^M, 2.9v_1^\alpha, 2.9v_1^\beta\Big) - \Re\Big(1.5v_2^M, 0.1v_2^M + 1.4v_2^\alpha, 0.1v_2^M + 1.6v_2^\beta\Big)$$

$$\leq \Re(0,0,0),$$

$$\Big(\Re\Big(3.2\,u_1^M, 0.5u_1^M + 2.7u_1^\alpha, 0.5u_1^M + 3.7u_1^\beta\Big) + \Re\Big(5.1u_2^M, 0.8\,u_2^M + 4.3u_2^\alpha, 0.8u_2^M + \ 5.9u_2^\beta\Big)$$

$$- \Re\Big(4.9v_1^M, 0.5v_1^M + 4.4v_1^\alpha, 0.5v_1^M + 5.4v_1^\beta\Big)$$

$$- \Re\Big(2.6v_2^M, 0.4v_2^M + 2.2\,v_2^\alpha, 0.4v_2^M + 3.0v_2^\beta\Big)\Big) \le \Re(0,0,0),$$

$$\Re\Big(2.9u_1^M, 0.4u_1^M + 2.5u_1^\alpha,, 0.4u_1^M + 3.3u_1^\beta\Big) + \Re\Big(5.7u_2^M, 0.2u_2^M + 5.5\,u_2^\alpha, 0.2u_2^M + 5.9u_2^\beta\Big)$$

$$- \Re\Big(4.1\,v_1^M, 0.7v_1^M + 3.4v_1^\alpha, 0.7v_1^M + 4.8v_1^\beta\Big)$$

$$- \Re\Big(2.3v_2^M, 0.1v_2^M + 2.2v_2^\alpha, 0.1v_2^M + 2.4v_2^\beta\Big)$$

$$\le \Re(0,0,0), \Re\Big(5.1u_1^M, 0.7u_1^M + 4.4\,u_1^\alpha, 0.7u_1^M + 5.8u_1^\beta\Big)$$

$$+ \Re\Big(7.4u_2^M, 0.9u_2^M + 6.5u_2^\alpha, 0.9u_2^M + 8.3u_2^\beta\Big)$$

$$- \Re\Big(6.5v_1^M, 0.6v_1^M + 5.9v_1^\alpha\ , 0.6v_1^M + 7.1v_1^\beta\Big)$$

$$- \Re\Big(4.1v_2^M, 0.5v_2^M + 3.6v_2^\alpha, 0.5v_2^M + \ 4.6v_2^\beta\Big) \le \Re(0,0,0),$$

$$\Re\Big(u_1^M, u_1^\alpha, u_1^\beta\Big) \ge \Re(0,0,0), \Re\Big(u_2^M, u_2^\alpha, u_2^\beta\Big) \ge \Re(0,0,0),$$

$$\Re\Big(v_1^M, v_1^\alpha, v_1^\beta\Big) \ge \Re(0,0,0), \Re\Big(v_2^M, v_2^\alpha, v_2^\beta\Big) \ge \Re(0,0,0),$$

$$u_1^M - u_1^\alpha \ge 0, u_2^M - u_2^\alpha \ge 0, v_1^M - v_1^\alpha \ge 0, v_2^M - v_2^\alpha \ge 0,$$

$$u_r^\alpha \ge 0, u_r^\beta \ge 0\ , \forall r, r = 1,2, v_i^\alpha \ \ge 0,\ \ v_i^\beta \ge 0,\ \ \forall i, i = 1,2.$$

Step 4: Using Step 4 of the modified method, the fuzzy CCR DEA model-2.18 can be transformed into crisp CCR DEA model-2.19.

Crisp CCR DEA model-2.19

$$\text{Maximize}\left[E_1 = \left(\frac{10.4\,u_1^M - 2.4u_1^\alpha + 2.8\,u_1^\beta + 16.4u_2^M - 3.8u_2^\alpha + 4.4u_2^\beta}{4}\right)\right]$$

Subject to

$$16\,v_1^M - 3.5\,v_1^\alpha + 4.5v_1^\beta + 8.4v_2^M - 1.9\,v_2^\alpha + 2.3v_2^\beta = 1,$$

$$10.4u_1^M - 2.4\,u_1^\alpha + 2.8\,u_1^\beta + 16.4u_2^M - 3.8u_2^\alpha + 4.4u_2^\beta - 18v_1^M + 3.5v_1^\alpha - 4.5v_1^\beta - 8.4v_2^M + 1.9v_2^\alpha - 2.3v_2^\beta \le 0,$$

$$8.8u_1^M - 2.2\,u_1^\alpha + 2.2u_1^\beta + 14u_2^M - 3.3u_2^\alpha + 3.7u_2^\beta - 11.6v_1^M + 2.9v_1^\alpha - 2.9v_1^\beta - 6v_2^M + 1.4v_2^\alpha - 1.6v_2^\beta \le 0,$$

$$12.8u_1^M - 2.7u_1^\alpha + 3.7u_1^\beta + 20.4u_2^M - 4.3u_2^\alpha + 5.9u_2^\beta - 19.6v_1^M + 4.4v_1^\alpha - 5.4v_1^\beta - 10.4v_2^M - 2.2\,v_2^\alpha - 3.0v_2^\beta \le 0,$$

$$11.6u_1^M - 2.5u_1^\alpha + 3.3u_1^\beta + 22.8u_2^M - 5.5\,u_2^\alpha + 5.9u_2^\beta - 16.4v_1^M + 3.4v_1^\alpha - 4.8v_1^\beta - 9.2v_2^M + 2.2v_2^\alpha - 2.4v_2^\beta \le 0,$$

$$20.4u_1^M - 4.4\,u_1^\alpha + 5.8u_1^\beta + 29.6u_2^M - 6.5u_2^\alpha + 8.3u_2^\beta - 26v_1^M + 5.9v_1^\alpha - 7.1v_1^\beta - 16.4v_2^M + 3.6v_2^\alpha - 4.6v_2^\beta \le 0,$$

$$u_1^M - u_1^\alpha \ge 0, u_2^M - u_2^\alpha \ge 0, v_1^M - v_1^\alpha \ge 0, v_2^M - v_2^\alpha \ge 0.$$

$$u_1^M - \left(\frac{1}{4}\right)u_1^\alpha + \left(\frac{1}{4}\right)u_1^\beta \ge 0, u_2^M - \left(\frac{1}{4}\right)u_2^\alpha + \left(\frac{1}{4}\right)u_2^\beta \ge 0,$$

$$v_1^M - \left(\frac{1}{4}\right)v_1^\alpha + \left(\frac{1}{4}\right)v_1^\beta \ge 0, v_2^M - \left(\frac{1}{4}\right)v_2^\alpha + \left(\frac{1}{4}\right)v_2^\beta \ge 0,$$

$u_r^{\alpha} \geq 0, u_r^{\beta} \geq 0, \forall r, r = 1,2, v_i^{\alpha} \geq 0, v_i^{\beta} \geq 0, \forall i, i = 1,2.$

Step 5: The optimal value $E_1 = \Re(\tilde{E}_1)$, representing the ranking value of fuzzy efficiency of DMU_1 is 0.23.

2.6.4 Exact best relative fuzzy efficiency of remaining DMUs

The ranking value of exact best relative fuzzy efficiency of DMU_1, DMU_2, DMU_3, DMU_4 and DMU_5, evaluated by solving the fuzzy CCR DEA model-2.12 to 2.15 with the modified method, is shown in Table 2.2.

Table 2.2 The ranking value of exact best relative fuzzy efficiency of DMUs

DMUs	**Ranking value of best relative fuzzy efficiency**
1	0.23
2	0.25
3	0.245
4	0.25
5	0.25

It is obvious from Table 2.2 that $\Re(\tilde{E}_2) = \Re(\tilde{E}_4) = \Re(\tilde{E}_5) > R(\tilde{E}_3) > R(\tilde{E}_1)$. Therefore, $\tilde{E}_2 \approx \tilde{E}_4 \approx \tilde{E}_5 \succ \tilde{E}_3 \succ \tilde{E}_1$.

2.7 Conclusions

On the basis the present study, it can be concluded that there are flaws in the method, proposed by Hatami-Marbini et al. [40], and hence should not be used for evaluating the best relative fuzzy efficiency of DMUs. Also, to resolve the flaws, the method, proposed by Hatami-

Marbini et al. [40], is modified. Further, the exact best relative fuzzy efficiency of the DMUs, considered by Hatami-Marbini et al. [40], is evaluated by using the modified method.

Chapter 3

A New Approach for Solving Proposed Fuzzy CCR DEA Model[2]

The existing fuzzy CCR DEA model, used in the previous chapter to evaluate the best relative fuzzy efficiency of DMUs, are obtained by replacing the crisp parameters of crisp CCR DEA model-2.3 with the fuzzy parameters. Wang et al. [106] replaced the crisp output data and crisp input data of the crisp CCR DEA model-2.2 with fuzzy data instead of crisp CCR DEA model-2.3 and proposed two methods for solving this fuzzy CCR DEA model. In this chapter, it is pointed out that the fuzzy CCR DEA model, proposed by Wang et al.[106] is not valid and hence the methods, proposed by Wang et al.[106], for evaluating the fuzzy efficiency of DMUs are also not valid. Also, a new fuzzy CCR DEA model as well as a method to solve this fuzzy CCR DEA model is proposed.

3.1 Existing fuzzy CCR DEA model

Wang et al. [106] transformed the crisp CCR DEA model-2.2 into fuzzy CCR DEA model-3.1 by replacing the crisp input data (x_{ij}) and crisp output data (y_{rj}) with triangular fuzzy numbers $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$ respectively.

Fuzzy CCR DEA model-3.1

$$\text{Maximize}\left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \frac{\sum_{r=1}^{s} u_r\left(y_{rp}^L, y_{rp}^M, y_{rp}^U\right)}{\sum_{i=1}^{m} v_i\left(x_{ip}^L, x_{ip}^M, x_{ip}^U\right)}\right]$$

[2] The contents of this chapter are communicated for the possible publication in International Journal of Fuzzy Systems.

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^U\right) = \frac{\sum_{r=1}^{s} u_r \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right)}{\sum_{i=1}^{m} v_i \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right)} \preccurlyeq (1,1,1), \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \ldots, m, \qquad r = 1, \ldots, s.$$

3.2 Existing methods

Wang et al. [106] proposed the following two methods for solving the fuzzy CCR DEA model-3.1.

3.2.1 First method

Wang et al. [106] proposed the following method for solving fuzzy CCR DEA model-3.1.

Step 1: Using the product of triangular fuzzy numbers, defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.1 can be transformed into fuzzy CCR DEA model-3.2.

Fuzzy CCR DEA model-3.2

$$\text{Maximize} \left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \frac{\left(\sum_{r=1}^{s} u_r y_{rp}^L, \sum_{r=1}^{s} u_r\, y_{rp}^M, \sum_{r=1}^{s} u_r\, y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i\, x_{ip}^L, \sum_{i=1}^{m} v_i\, x_{ip}^M, \sum_{i=1}^{m} v_i\, x_{ip}^U\right)} \right]$$

Subject to

$$\tilde{E}_p \approx \left(E_j^L, E_j^M, E_j^U\right) \approx \frac{\left(\sum_{r=1}^{s} u_r y_{rj}^L, \sum_{r=1}^{s} u_r\, y_{rj}^M, \sum_{r=1}^{s} u_r\, y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i\, x_{ij}^L, \sum_{i=1}^{m} v_i\, x_{ij}^M, \sum_{i=1}^{m} v_i\, x_{ij}^U\right)} \preccurlyeq (1,1,1),, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \ldots, m, \qquad r = 1, \ldots, s.$$

Step 2: Using division of triangular fuzzy number, defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.2 can be transformed into fuzzy CCR DEA model-3.3.

Fuzzy CCR DEA model-3.3

$$\text{Maximize}\left[\ \tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\left[\tilde{E}_p \approx \left(E_j^L, E_j^M, E_j^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i\, x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L}\right)\right] \preccurlyeq (1,1,1),, \forall j,$$

$$u_r, v_i \ \geq 0, \qquad i = 1, \dots, m, \qquad r = \ 1, \dots, s.$$

Step 3: Using the relation $(a^L, a^M, a^U) \ \leq \ (b^L, b^M, b^U) \Rightarrow \ a^L \leq \ b^L, \ a^M \leq \ b^M, \ a^U \leq \ b^U,$

the fuzzy CCR DEA model-3.3 can be transformed into fuzzy CCR DEA model-3.4.

Fuzzy CCR DEA model-3.4

$$\text{Maximize}\left[\ \tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i\, x_{ij}^U} \leq 1, \forall j \ ,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \leq 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} \leq 1, \forall j,$$

$$u_r, v_i \ \geq 0, \qquad i = 1, \dots, m, \qquad r = \ 1, \dots, s.$$

Step 4: Since, $E_j^L \leq E_j^M \leq E_j^U$, So, if $E_j^U \leq 1$ then automatically condition $E_j^L \leq E_j^M \leq 1$ will be satisfied, so, the fuzzy CCR DEA model-3.4 can transformed into fuzzy CCR DEA model-3.5.

Fuzzy CCR DEA model-3.5

$$\text{Maximize}\left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} \leq 1, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Step 5: Transform the fuzzy CCR DEA model-3.5 into crisp CCR DEA model-3.6 to 3.8.

Crisp CCR DEA model-3.6

$$\text{Maximize}\left[E_p^L = \frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} \leq 1, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Crisp CCR DEA model-3.7

$$\text{Maximize}\left[E_p^M = \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} \leq 1, \forall j,$$

$$u_r, v_i \ \geq 0, \qquad i = 1, \dots, m, \qquad r = \ 1, \dots, s.$$

Crisp CCR DEA model-3.8

Maximize $\left[E_p^U \ = \frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}\right]$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} \leq 1, \forall j,$$

$$u_r, v_i \ \geq 0, \qquad i = 1, \dots, m, \qquad r = \ 1, \dots, s.$$

Step 6: The crisp CCR DEA models 3.6 to 3.8 can be transformed into crisp CCR DEA models 3.9 to 3.11 respectively.

Crisp CCR DEA model-3.9

Maximize $\left[E_p^L \ = \sum_{r=1}^{s} u_r y_{rp}^L \right]$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^U \ = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \leq 0, \forall j$$

$$u_r, v_i \ \geq 0, \qquad i = 1, \dots, m, \qquad r = \ 1, \dots, s.$$

Crisp CCR DEA model-3.10

$$\text{Maximize}\left[E_p^M = \sum_{r=1}^{s} u_r y_{rp}^M\right]$$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^M = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \leq 0, \forall j,$$

$$u_r, v_i \;\geq 0, \qquad i = 1, \dots, m, \qquad r = \;1, \dots, s.$$

Crisp CCR DEA model-3.11

$$\text{Maximize}\left[E_p^U = \sum_{r=1}^{s} u_r y_{rp}^U\right]$$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^L = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i\, x_{ij}^L \leq 0, \forall j,$$

$$u_r, v_i \;\geq 0, \qquad i = 1, \dots, m, \qquad r = \;1, \dots, s.$$

Step 7: Find the optimal value E_p^L, E_p^M, E_p^U of crisp CCR DEA models 3.9 to 3.11 respectively.

Step 8: Using the optimal values of E_p^L, E_p^M, E_p^U, obtained in Step 6, the fuzzy optimal value (best relative fuzzy efficiency of DMU$_p$) of fuzzy CCR DEA model-3.1is $\left(E_p^L, E_p^M, E_p^U\right)$.

3.2.2 Second method

Wang et al. [106] proposed the following method for solving fuzzy CCR DEA model-3.1.

Step 1: Using the product of triangular fuzzy numbers, defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.1 can be transformed into fuzzy CCR DEA model-3.12.

Fuzzy CCR DEA model-3.12

$$\text{Maximize}\left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \frac{\left(\sum_{r=1}^{s} u_r y_{rp}^L, \sum_{r=1}^{s} u_r\, y_{rp}^M, \sum_{r=1}^{s} u_r\, y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i\, x_{ip}^L, \sum_{i=1}^{m} v_i\, x_{ip}^M, \sum_{i=1}^{m} v_i\, x_{ip}^U\right)}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^U\right) \approx \frac{\left(\sum_{r=1}^{s} u_r y_{rj}^L, \sum_{r=1}^{s} u_r\, y_{rj}^M, \sum_{r=1}^{s} u_r\, y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_{ij} x_{ij}^L, \sum_{i=1}^{m} v_i\, x_{ij}^M, \sum_{i=1}^{m} v_i\, x_{ij}^U\right)}, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Step 2: Using the division of triangular fuzzy numbers, defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.12 can be transformed into fuzzy CCR DEA model-3.13.

Fuzzy CCR DEA model-3.13

$$\text{Maximize}\left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\left[\tilde{E}_p \approx \left(E_j^L, E_j^M, E_j^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i\, x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L}\right)\right] \preccurlyeq (1,1,1), \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Step 3: Using the relation $(a^L, a^M, a^U) \preccurlyeq (b^L, b^M, b^U) \Rightarrow a^L \leq b^L,\ a^M \leq b^M,\ a^U \leq b^U,$

the fuzzy CCR DEA model-3.13 can be transformed into fuzzy CCR DEA model-3.14.

Fuzzy CCR DEA model-3.14

$$\text{Maximize}\left[\ \tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i\, x_{ij}^U} \le 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \le 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} \le 1, \forall j,$$

$$u_r, v_i \ \ge 0, \qquad i = 1, \dots, m, \qquad r = \ 1, \dots, s.$$

Step 4: As for fuzzy CCR DEA model-3.14, since there are no restrictions imposed subjectively on the support of the fuzzy numbers $\tilde{1}$, its lower and upper bounds are therefore viewed as free. Based upon this point of view fuzzy CCR DEA model-3.14 can be transformed into fuzzy CCR DEA model-3.15.

Fuzzy CCR DEA model-3.15

$$\text{Maximize}\left[\ \tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \le 1, \forall j.$$

$u_r, v_i \geq 0, \qquad i = 1, \ldots, m, \qquad r = 1, \ldots, s.$

Step 5: Transform the fuzzy CCR DEA model-3.15 into crisp CCR DEA models-3.16 to 3.18.

Crisp CCR DEA model-3.16

Maximize $\left[E_p^L = \frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}\right]$

Subject to

$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \leq 1, \forall j,$

$u_r, v_i \geq 0, \qquad i = 1, \ldots, m, \qquad r = 1, \ldots, s.$

Crisp CCR DEA model-3.17

Maximize $\left[E_p^M = \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}\right]$

Subject to

$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \leq 1, \forall j,$

$u_r, v_i \geq 0, \qquad i = 1, \ldots, m, \qquad r = 1, \ldots, s.$

Crisp CCR DEA model-3.18

Maximize $\left[E_p^U = \frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}\right]$

Subject to

$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \leq 1, \forall j,$

$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$

Step 6: The crisp CCR DEA models-3.16 to 3.18 can be transformed into crisp CCR DEA models-3.19 to 3.21 respectively.

Crisp CCR DEA model-3.19

$$\text{Maximize} \left[E_p^L = \sum_{r=1}^{s} u_r y_{rp}^L \right]$$

Subject to

$$\sum_{i=1}^{m} v_i \, x_{ip}^U = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj}^M - \sum_{i=1}^{m} v_i \, x_{ij}^M \leq 0, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Crisp CCR DEA model-3.20

$$\text{Maximize} \left[E_p^M = \sum_{r=1}^{s} u_r y_{rp}^M \right]$$

Subject to

$$\sum_{i=1}^{m} v_i \, x_{ip}^M = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj}^M - \sum_{i=1}^{m} v_i \, x_{ij}^M \leq 0, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Crisp CCR DEA model-3.21

$$\text{Maximize}\left[E_p^U = \sum_{r=1}^{s} u_r y_{rp}^U\right]$$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^L = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj}^M - \sum_{i=1}^{m} v_i\, x_{ij}^M \leq 0, \forall j,$$

$$u_r, v_i \geq 0, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Step 7: Find the optimal value E_p^L, E_p^M, E_p^U of crisp CCR DEA models 3.19 to 3.21 respectively.

Step 8: Using the optimal values of E_p^L, E_p^M, E_p^U, obtained in Step 6, the fuzzy optimal value (best relative fuzzy efficiency of DMU_p) of fuzzy CCR DEA model 3.1 is $\left(E_p^L, E_p^M, E_p^U\right)$.

3.3 Flaws in the existing method

In this section, the flaws in the methods, proposed by Wang et al. [106] are pointed out.

1. In the fuzzy CCR DEA model-3.1, u_r, v_i are crisp numbers. Therefore, if these numbers are represented as triangular fuzzy numbers $u_r = (u_r^L,\ u_r^M, u_r^U)$ and $v_i = (v_i^L,\ v_i^M, v_i^U)$, then the condition, $u_r^L = u_r^M = u_r^U, v_i^L = v_i^M = v_i^U$, should always be satisfied. However, in both the methods, proposed by Wang et al. [106] the values of E_p^L, E_p^M, E_p^U are obtained by solving the crisp CCR DEA models-3.19 to 3.21 independently. Since, all these models are solved independently, so, the optimal values of u_r and v_i, obtained on solving these models, will not necessarily be same, i.e., if $u_r^L, v_i^L, u_r^M, v_i^M$ and u_r^U, v_i^U

represents the optimal values of u_r and v_i, obtained on solving crisp CCR DEA models-3.19 to 3.21 respectively, then the condition $u_r^L = u_r^M = u_r^U$ and $v_i^L = v_i^M = v_i^U$ will not necessarily be satisfied. Hence, using the methods, proposed by Wang et al.[106] the obtained optimal solution $\{u_r^L, v_i^L, u_r^M, v_i^M, u_r^U, v_i^U\}$ is not an optimal solution of fuzzy CCR DEA model-3.1 i.e., using the method, proposed by Wang et al. [106] it is not possible to find a crisp optimal solution $\{u_r, v_i\}$ of fuzzy CCR DEA model-3.1.

2. It is well known that for a triangular fuzzy number (a, b, c), the property $a \leq b \leq c$ should always be satisfied. However, it is obvious from Step 6 of the existing method, presented in Section 3.2, that the values of E_p^L, E_p^M, E_p^U are obtained by solving three independent crisp CCR DEA models so the restriction $E_p^L \leq E_p^M \leq E_p^U$ may or may not be satisfied.

3. 4 Flaws in the existing fuzzy DEA model

It is obvious from Step 2 and Step 3 of the existing method [106], presented in Section 3.2, that Wang et al. [106] have transformed the fuzzy constraints

$$\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L}\right) \preccurlyeq (1,1,1)$$

in to the crisp constraints.

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U} \leq 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i x_{ij}^M} \leq 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i \, x_{ij}^L} \leq 1, \forall j,$$

Since, $\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L}\right) \preccurlyeq (1,1,1), \forall j$. So, there should exist a non-negative triangular fuzzy number (S^L, S^M, S^U), $S^L \leq S^M \leq S^U$ such that

$$\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i \, x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i \, x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i \, x_{ij}^L}\right) + (S^L, S^M, S^U) = (1,1,1), \forall j.$$

However, the following clearly indicates that there will never exist a non-negative triangular fuzzy number (S^L, S^M, S^U), $S^L \leq S^M \leq S^U$ and hence, the constraint $\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L}\right) \preccurlyeq (1,1,1), \forall j$ cannot be transformed into the constraints

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i \, x_{ij}^U} \leq 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i \, x_{ij}^M} \leq 1, \forall j,$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i \, x_{ij}^L} \leq 1, \forall j.$$

$$\left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i \, x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i \, x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i \, x_{ij}^L}\right) \preccurlyeq (1,1,1)$$

$$\Rightarrow \left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i \, x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i \, x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i \, x_{ij}^L}\right) + (S^L, S^M, S^U) = (1,1,1)$$

$$\Rightarrow \frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i \, x_{ij}^U} + S^L = 1, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i \, x_{ij}^M} + S^M = 1, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i \, x_{ij}^L} + S^U = 1$$

$$\Rightarrow \frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i\, x_{ij}^U} = 1 - S^L, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} = 1 - S^M, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L} = 1 - S^U.$$

Now,

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i\, x_{ij}^U} \leq \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i\, x_{ij}^M} \leq \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i\, x_{ij}^L}$$

$$\Rightarrow 1 - S^L \leq 1 - S^M \leq 1 - S^U$$

$$\Rightarrow -S^L \leq -S^M \leq -S^U$$

$$\Rightarrow S^L \geq S^M \geq S^U$$

$\Rightarrow (S^L, S^M, S^U)$ is not a triangular fuzzy number.

3.5 Proposed fuzzy CCR DEA model

Wang et al. [] transformed the crisp CCR DEA model-2.2 into fuzzy CCR DEA model-3.1 by replacing the crisp input data (x_{ij}) and crisp output data (y_{rj}) with triangular fuzzy numbers $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$ respectively.

In this section, to resolve the flaws of the existing fuzzy DEA model, pointed out in Section 3.4, a modified fuzzy CCR DEA model is proposed.

If there are n DMUs then the best relative crisp efficiency (E_p) of p^{th} DMU can be obtained by solving the CCR DEA model-3.22

CCR DEA model-3.22

Maximize $\left[E_p = \frac{\text{Virtual output of } p^{th} \text{ DMU}}{\text{Virtual input of } p^{th} \text{ DMU}}\right]$

Subject to

Virtual output of j^{th} DMU $\leq$ Virtual input of j^{th} DMU, $\forall\, j$.

If each DMU have m inputs (x_{ij}; $i = 1,2, \dots, m; j = 1,2, \dots, n$) and s outputs ($y_{rj}; r = 1,2, \dots s$; $j = 1,2, \dots, n$) then CCR DEA model-3.22 is transformed into crisp CCR DEA model-3.23

Crisp CCR DEA model-3.23

$$\text{Maximize}\left[E_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}\right]$$

Subject to

$$\sum_{r=1}^{s} u_r y_{rj} \leq \sum_{i=1}^{m} v_i x_{ij} \qquad \forall j,$$

$$u_r, v_i \geq 0, \forall r, i.$$

where u_r $(r = 1, \dots. s)$ and v_i $(i = 1, \dots, m)$ are the weights assigned to the r^{th} output and i^{th} input, respectively.

Replacing the crisp input data (x_{ij}) and crisp output data (y_{rj}) by triangular fuzzy numbers $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$ respectively, the CCR DEA model-3.23 can be transformed into fuzzy CCR DEA model-3.24.

Fuzzy CCR DEA model-3.24

$$\text{Maximize}\left[\tilde{E}_j \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \frac{\sum_{r=1}^{s} u_r\left(y_{rp}^L, y_{rp}^M, y_{rp}^U\right)}{\sum_{i=1}^{m} v_i \left(x_{ip}^L, x_{ip}^M, x_{ip}^U\right)]}\right]$$

Subject to

$$\sum_{r=1}^{s} u_r\left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right) \preccurlyeq \sum_{i=1}^{m} v_i \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right), \qquad j = 1, \dots, n,$$

$u_r, v_i \geq 0, \quad i = 1, \dots, m, \quad r = 1, \dots, s.$

3.6 Proposed approach

In Section 3.4, it is shown that the fuzzy CCR DEA model-3.5, proposed by Wang et al. [106] are not valid and hence cannot be used for evaluating the best relative fuzzy efficiency of DMUs. In this section, a new approach is proposed to evaluate the best relative fuzzy efficiency of DMUs by using the fuzzy CCR DEA model-3.24.

Step 1: Using the product of triangular fuzzy numbers, defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.24 can be transformed into fuzzy CCR DEA model-3.25.

Fuzzy CCR DEA model-3.25

$$\text{Maximize}\left[\tilde{E}_j \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \frac{\sum_{r=1}^{s}\left(u_r y_{rp}^L, u_r y_{rp}^M, u_r y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i x_{ip}^L, v_i x_{ip}^M, v_i x_{ip}^U\right)}\right]$$

Subject to

$$\sum_{r=1}^{s}\left(u_r y_{rj}^L, u_r y_{rj}^M, u_r y_{rj}^U\right) \preccurlyeq \sum_{i=1}^{m}\left(v_i x_{ij}^L, v_i x_{ij}^M, v_i x_{ij}^U\right) \ , \forall j,$$

$u_r, v_i \geq 0, \quad i = 1, \dots, m, \quad r = 1, \dots, s.$

Step 2: Using the division of triangular fuzzy numbers defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.25 can be transformed into fuzzy CCR DEA model-3.26.

Fuzzy CCR DEA model-3.26

$$\text{Maximize}\left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\sum_{r=1}^{s}\left(u_r y_{rj}^L, u_r y_{rj}^M, u_r y_{rj}^U\right) \preccurlyeq \sum_{i=1}^{m}\left(v_i x_{ij}^L, v_i x_{ij}^M, v_i x_{ij}^U\right), \forall j,$$

$u_r, v_i \geq 0, \quad i = 1, \dots, m, \quad r = 1, \dots, s.$

Step 3: Using the relation $(a^L, a^M, a^U) \preccurlyeq (b^L, b^M, b^U) \Rightarrow a^L \leq b^L, a^M \leq b^M, a^U \leq b^U$,

the fuzzy CCR DEA model-3.26 can be transformed into fuzzy CCR DEA model-3.27.

Fuzzy CCR DEA model-3.27

$$\text{Maximize}\left[\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right) \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right)\right]$$

Subject to

$$\sum_{r=1}^{s} u_r y_{rj}^L \leq \sum_{i=1}^{m} v_i x_{ij}^L, \quad \forall j,$$

$$\sum_{r=1}^{s} u_r y_{rj}^M \leq \sum_{i=1}^{m} v_i x_{ij}^M, \quad \forall j,$$

$$\sum_{r=1}^{s} u_r y_{rj}^U \leq \sum_{i=1}^{m} v_i x_{ij}^U \quad , \forall j,$$

$u_r, v_i \geq 0, \; i = 1, \dots, m, \; r = 1, \dots, s.$

Step 4: The fuzzy optimal value $\left(\tilde{E}_p \approx \left(E_p^L, E_p^M, E_p^U\right)\right)$, representing the best relative fuzzy efficiency of p^{th} DMU, can be obtained by solving the fuzzy CCR DEA model-3.27 as follows:

Step 4(a) Find the optimal value $\left(E_p^L\right)$ of the crisp CCR DEA model-3.28a by solving crisp CCR DEA model-3.28b equivalent to crisp CCR DEA model-3.28a

Crisp CCR DEA model-3.28a

$$\text{Maximize}\left[E_p^L = \frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U} \leq 1,$$

and

All the constraints of fuzzy crisp CCR DEA model-3.27.

Crisp CCR DEA model-3.28b

$$\text{Maximize}\left[E_p^L = \sum_{r=1}^{s} u_r y_{rp}^L\right]$$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^U = 1,$$

$$\sum_{r=1}^{s} u_r y_{rp}^L \leq \sum_{i=1}^{m} v_i\, x_{ip}^U,$$

and

All the constraints of fuzzy crisp CCR DEA model-3.27.

Step 4(b) Find the optimal value $\left(E_p^M\right)$ of the crisp CCR DEA model-3.29a by solving crisp CCR DEA model-3.29b equivalent to crisp CCR DEA model-3.29a.

Crisp CCR DEA model-3.29a

$$\text{Maximize}\left[E_p^M = \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U} = E_p^L,$$

$$E_p^L \leq \frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M} \leq 1,$$

and

All the constraints of fuzzy crisp CCR DEA model-3.27.

Crisp CCR DEA model-3.29b

$$\text{Maximize}\left[E_p^L = \sum_{r=1}^{s} u_r y_{rp}^M\right]$$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^M = 1,$$

$$\sum_{r=1}^{s} u_r y_{rp}^L = E_p^L\left(\sum_{i=1}^{m} v_i\, x_{ip}^U\right),$$

$$E_p^L\left(\sum_{i=1}^{m} v_i\, x_{ip}^M\right) \leq \sum_{r=1}^{s} u_r y_{rp}^M$$

and

All the constraints of fuzzy crisp CCR DEA model-3.27.

Step 4(c) Find the optimal value $\left(E_p^U\right)$ of the crisp CCR DEA model-3.30a by solving crisp CCR DEA model-3.30b equivalent to crisp CCR DEA model-3.30a

Crisp CCR DEA model-3.30a

$$\text{Maximize}\left[E_p^U = \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r y_{rp}^L}{\sum_{i=1}^{m} v_i\, x_{ip}^U} = E_p^L$$

$$\frac{\sum_{r=1}^{s} u_r y_{rp}^M}{\sum_{i=1}^{m} v_i\, x_{ip}^M} = E_p^M,$$

$$E_p^M \le \frac{\sum_{r=1}^{s} u_r y_{rp}^U}{\sum_{i=1}^{m} v_i\, x_{ip}^L} \le 1,$$

and

All the constraints of fuzzy CCR DEA model-3.27.

Crisp CCR DEA model-3.30b

$$\text{Maximize}\left[E_p^U = \sum_{r=1}^{s} u_r y_{rp}^U\right]$$

Subject to

$$\sum_{i=1}^{m} v_i\, x_{ip}^L = 1,$$

$$\sum_{r=1}^{s} u_r y_{rp}^U = E_p^L\left(\sum_{i=1}^{m} v_i\, x_{ip}^L\right),$$

$$\sum_{r=1}^{s} u_r y_{rp}^L = E_p^M \left(\sum_{i=1}^{m} v_i \, x_{ip}^U \right),$$

$E_p^M \left(\sum_{i=1}^{m} v_i \, x_{ip}^L \right) \le \sum_{r=1}^{s} u_r y_{rp}^U \le \sum_{i=1}^{m} v_i \, x_{ip}^L$,

and

All the constraints of fuzzy CCR DEA model-3.27.

Step 5: Using the optimal values of E_p^L, E_p^M, E_p^U, obtained in Step 4, the fuzzy optimal value (best relative fuzzy efficiency of DMU_P) is $\left(E_p^L, E_p^M, E_p^U\right)$.

3.7 Exact fuzzy efficiency of real life problem

Wang et al. [106] solved a problem to illustrate his proposed approach. However, as discussed in Section 3.3 that there are flaws in the existing method [106]. So, the results of this problem, obtained by Wang et al. [106] is not exact. In this section, the exact results of the same problem is obtained by the proposed method.

3.7.1 Problem description

Consider a performance assessment problem in China where eight manufacturing enterprises (DMUs) are to be evaluated in terms of two inputs and two outputs. The eight manufacturing enterprises all manufacture the same type of products but with different qualities. Both the gross output value (GOV) and product quality (PQ) are considered as outputs. Manufacturing cost (MC) and the number of employees (NOE) are considered as inputs. The data about the GOV and MC are uncertain due to the unavailability at the time of assessment and are therefore estimated as fuzzy numbers. The product quality is assessed by customers using fuzzy linguistic terms such as Excellent, Very Good, Average, Poor and Very Poor. The assessment results by customers are weighted and averaged. Table 3.1 shows the input and output data for the eight manufacturing enterprises.

Table 3.1: Input and output data for eight manufacturing enterprises

Enterprises				
(DMUs)	**Inputs (Two)**		**Outputs (two)**	
	MC	NOE	GOV	PQ
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)
C	(2510, 2570, 2610)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)

3.7.2 Proposed fuzzy CCR DEA models

The best relative fuzzy efficiency of DMU_A to DMU_H respectively can be obtained by solving fuzzy CCR DEA model-3.31 to 3.39.

Fuzzy CCR DEA model-3.31

$$\text{Maximize } \left[\tilde{E}_A \approx (E_A^L, E_A^M, E_A^U) \approx \left[\frac{(14500, 14790, 14860)u_1 + (3.1, 4.1, 4.9)u_2}{(2120, 2170, 2210)v_1 + (1870,1870,1870)v_2}\right]\right]$$

Subject to

$$(14500u_1 + 3.1u_2, 14790u_1 + 4.1u_2, 14860u_1 + 4.9u_2)$$

$$\preccurlyeq (2120v_1 + 1870v_2, 2170v_1 + 1870v_2, 2210v_1 + 1870v_2),$$

$$(12470u_1 + 1.2u_2, 12720u_1 + 2.1u_2, 12790u_1 + 3.0u_2)$$
$$\preccurlyeq (1420v_1 + 1340v_2, 1460v_1 + 1340v_2, 1500v_1 + 1340v_2),$$
$$(17900u_1 + 3.3u_2, 18260u_1 + 4.3u_2, 18400u_1 + 5.0u_2)$$
$$\preccurlyeq (2510v_1 + 2360v_2, 2570v_1 + 2360v_2, 2610v_1 + 2360v_2),$$
$$(14970u_1 + 2.7u_2, 15270u_1 + 3.7u_2, 15400u_1 + 4.6u_2)$$
$$\preccurlyeq (2300v_1 + 2020v_2, 2350v_1 + 2020v_2, 2400v_1 + 2020v_2),$$
$$(13980u_1 + 1.0u_2, 14260u_1 + 1.8u_2, 14330u_1 + 2.7u_2)$$
$$\preccurlyeq (1480v_1 + 1550v_2, 1520v_1 + 1550v_2, 1560v_1 + 1550v_2),$$
$$(14030u_1 + 1.6u_2, 14310u_1 + 2.6u_2, 14400u_1 + 3.6u_2)$$
$$\preccurlyeq (1990v_1 + 1760v_2, 2030v_1 + 1760v_2, 2100v_1 + 1760v_2),$$
$$(16540u_1 + 2.4u_2, 16870u_1 + 3.4u_2, 17000u_1 + 4.4u_2)$$
$$\preccurlyeq (2200v_1 + 1980v_2, 2260v_1 + 1980v_2, 2300v_1 + 1980v_2),$$
$$(17600u_1 + 2.6u_2, 17960u_1 + 3.6u_2, 18100u_1 + 4.6u_2)$$
$$\preccurlyeq (2400v_1 + 2250v_2, 2460v_1 + 2250v_2, 2520v_1 + 2250v_2),$$
$$u_1, u_2, v_1, v_2 \geq 0.$$

Fuzzy CCR DEA model-3.32

$$\text{Maximize } \left[\tilde{E}_B \approx (E_B^L, E_B^M, E_B^U) \approx \left[\frac{(12470, 12720, 12790)u_1 + (1.2, 2.1, 3.0)u_2}{(1420, 1460, 1500)v_1 + (1340,1340,1340)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

Fuzzy CCR DEA model-3.33

$$\text{Maximize } \left[\tilde{E}_C \approx (E_C^L, E_C^M, E_C^U) \approx \left[\frac{(17900, 18260, 18400)u_1 + (3.3, 4.3, 5.0)u_2}{(2510, 2570, 2610)v_1 + (2360,2360,2360)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

Fuzzy CCR DEA model-3.34

$$\text{Maximize}\ \left[\tilde{E}_D\ \approx (E_D^L, E_D^M, E_D^U) \approx \left[\frac{(14970, 15270, 15400)u_1 + (2.7, 3.7, 4.6)u_2}{(2300, 2350, 2400)v_1 + (2020,2020,2020)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

Fuzzy CCR DEA model-3.35

$$\text{Maximize}\ \left[\tilde{E}_E\ \approx (E_E^L, E_E^M, E_E^U) \approx \left[\frac{(13980, 14260, 14330)u_1 + (1.0, 1.8, 2.7)u_2}{(1480, 1520, 1560)v_1 + (1550,150,150)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

Fuzzy CCR DEA model-3.36

$$\text{Maximize}\ \left[\tilde{E}_F \approx (E_F^L, E_F^M, E_F^U) \approx \left[\frac{(14030, 14310, 14400)u_1 + (1.6, 2.6, 3.6)u_2}{(1990, 2030, 2100)v_1 + (1760,1760,1760)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

Fuzzy CCR DEA model-3.37

$$\text{Maximize}\ \left[\tilde{E}_G \approx (E_G^L, E_G^M, E_G^U) \approx \left[\frac{(16540, 16870, 17000)u_1 + (2.4, 3.4, 4.4)u_2}{(2200, 2260, 2300)v_1 + (1980,1980,1980)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

Fuzzy CCR DEA model-3.38

$$\text{Maximize } \left[\tilde{E}_H \approx (E_H^L, E_H^M, E_H^U) \approx \left[\frac{(17600, 17960, 18100)u_1 + (2.6, 3.6, 4.6)u_2}{(2400, 2460, 2520)v_1 + (2250,2250,2250)v_2}\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.31.

3.7.3 Exact best relative fuzzy efficiency of DMU$_A$

Using the approach, proposed in Section 3.6, the exact best relative fuzzy efficiency of DMU$_\text{A}$ can be obtained as follows:

Step 1: Using the arithmetic operations of triangular fuzzy numbers, defined in Section 2.1.2.1, the fuzzy CCR DEA model-3.31 can be transformed into fuzzy CCR DEA model-3.39.

Fuzzy CCR DEA model-3.39

$$\text{Maximize } \left[\tilde{E}_A \approx (E_A^L, E_A^M, E_A^U) \approx \left[\frac{(14500u_1 + 3.1u_2, 14790u_1 + 4.1u_2, 14860u_1 + 4.9u_2)}{(2120v_1 + 4.9u_2, 2170v_1 + 1870v_2, 2210v_1 + 1870v_2)}\right]\right]$$

Subject to

$$(14500u_1 + 3.1u_2, 14790u_1 + 4.1u_2, 14860u_1 + 4.9u_2) \preccurlyeq (2120v_1 + 1870v_2, 2170v_1 + 1870v_2, 2210v_1 + 1870v_2),$$

$$(12470u_1 + 1.2u_2, 12720u_1 + 2.1u_2, 12790u_1 + 3.0u_2) \preccurlyeq (1420v_1 + 1340v_2, 1460v_1 + 1340v_2, 1500v_1 + 1340v_2),$$

$$(17900u_1 + 3.3u_2, 18260u_1 + 4.3u_2, 18400u_1 + 5.0u_2) \preccurlyeq (2510v_1 + 2360v_2, 2570v_1 + 2360v_2, 2610v_1 + 2360v_2),$$

$$(14970u_1 + 2.7u_2, 15270u_1 + 3.7u_2, 15400u_1 + 4.6u_2) \preccurlyeq (2300v_1 + 2020v_2, 2350v_1 + 2020v_2, 2400v_1 + 2020v_2),$$

$$(13980u_1 + 1.0u_2, 14260u_1 + 1.8u_2, 14330u_1 + 2.7u_2)$$

$$\preccurlyeq (1480v_1 + 1550v_2, 1520v_1 + 1550v_2, 1560v_1 + 1550v_2),$$

$$(14030u_1 + 1.6u_2, 14310u_1 + 2.6u_2, 14400u_1 + 3.6u_2)$$

$$\preccurlyeq (1990v_1 + 1760v_2, 2030v_1 + 1760v_2, 2100v_1 + 1760v_2),$$

$$(16540u_1 + 2.4u_2, 16870u_1 + 3.4u_2, 17000u_1 + 4.4u_2)$$

$$\preccurlyeq (2200v_1 + 1980v_2, 2260v_1 + 1980v_2, 2300v_1 + 1980v_2),$$

$$(17600u_1 + 2.6u_2, 17960u_1 + 3.6u_2, 18100u_1 + 4.6u_2)$$

$$\preccurlyeq (2400v_1 + 2250v_2, 2460v_1 + 2250v_2, 2520v_1 + 2250v_2),$$

$$u_1, u_2, v_1, v_2 \geq 0.$$

Step 2: The fuzzy CCR DEA model-3.39 can be transformed into fuzzy CCR DEA model-3.40.

Fuzzy CCR DEA model-3.40

$$\text{Maximize } \left[\tilde{E}_A \approx (E_A^L, E_A^M, E_A^U) \approx \left[\frac{(14500u_1 + 3.1u_2, 14790u_1 + 4.1u_2, 14860u_1 + 4.9u_2)}{(2120v_1 + 4.9u_2, 2170v_1 + 1870v_2, 2210v_1 + 1870v_2)}\right]\right]$$

Subject to

$14500u_1 + 3.1u_2 \leq 2120v_1 + 1870v_2,$ $14790u_1 + 4.1u_2 \leq 2170v_1 + 1870v_2,$

$14860u_1 + 4.9u_2 \leq 2210v_1 + 1870v_2,$ $12470u_1 + 1.2u_2 \leq 1420v_1 + 1340v_2,$

$12720u_1 + 2.1u_2 \leq 1460v_1 + 1340v_2,$ $12790u_1 + 3.0u_2 \leq 1500v_1 + 1340v_2,$

$17900u_1 + 3.3u_2 \leq 2510v_1 + 2360v_2,$ $18260u_1 + 4.3u_2 \leq 2570v_1 + 2360v_2,$

$18400u_1 + 5.0u_2 \leq 2610v_1 + 2360v_2,$ $14970u_1 + 2.7u_2 \leq 300v_1 + 2020v_2,$

$15270u_1 + 3.7u_2 \leq 2350v_1 + 2020v_2,$ $15400u_1 + 4.6u_2 \leq 2400v_1 + 2020v_2,$

$13980u_1 + 1.0u_2 \leq 1480v_1 + 1550v_2,$ $14260u_1 + 1.8u_2 \leq 1520v_1 + 1550v_2,$

$14330u_1 + 2.7u_2 \leq 1560v_1 + 1550v_2,$ $14030u_1 + 1.6u_2 \leq 1990v_1 + 1760v_2,$

$14310u_1 + 2.6u_2 \leq 2030v_1 + 1760v_2,$ $14400u_1 + 3.6u_2 \leq 2100v_1 + 1760v_2,$

$16540u_1 + 2.4u_2 \leq 2200v_1 + 1980v_2,$ $16870u_1 + 3.4u_2 \leq 2260v_1 + 1980v_2,$

$17000u_1 + 4.4u_2 \leq 2300v_1 + 1980v_2,$ $17600u_1 + 2.6u_2 \leq 2400v_1 + 2250v_2,$

$17960u_1 + 3.6u_2 \leq 2460v_1 + 2250v_2,$ $18100u_1 + 4.6u_2 \leq 2520v_1 + 2250v_2.$

$u_1, u_2, v_1, v_2 \geq 0.$

Step 3: The fuzzy CCR DEA model-3.40 can be transformed into fuzzy CCR DEA model-3.41.

Fuzzy CCR DEA model-3.41

$$\text{Maximize } \left[\tilde{E}_A \approx (E_A^L, E_A^M, E_A^U) \right.$$

$$\left. \approx \left[\left(\frac{14500u_1 + 3.1u_2}{2210v_1 + 1870v_2}, \frac{14790u_1 + 4.1u_2}{2170v_1 + 1870v_2}, \frac{14860u_1 + 4.9u_2}{2120v_1 + 4.9u_2}\right)\right]\right]$$

Subject to

All the constraints of fuzzy CCR DEA model-3.40.

Step 4: The fuzzy optimal value $\left(\tilde{E}_A \approx (E_A^L, E_A^M, E_A^U)\right)$, representing the best relative fuzzy efficiency of A^{th} DMU, can be obtained by solving the CCR fuzzy DEA model-3.41 as follows

Step (4a): The optimal value $\left(E_p^L\right)$ of the crisp CCR DEA model-3.43 and hence of crisp CCR DEA model-3.42 is 0.812.

Crisp CCR DEA model-3.42

$$\text{Maximize } \left[E_A^L = \frac{14500u_1 + 3.1u_2}{2210v_1 + 1870v_2}\right]$$

Subject to

$$\frac{14500u_1 + 3.1u_2}{2210v_1 + 1870v_2} \leq 1,$$

All the constraints of fuzzy CCR DEA model-3.40.

Crisp CCR DEA model-3.43

Maximize $[E_A^L = 14500u_1 + 3.1u_2]$

Subject to

$2210v_1 + 1870v_2 = 1,$

$14500u_1 + 3.1u_2 \leq 2210v_1 + 1870v_2,$

and

All the constraints of fuzzy CCR DEA model-3.40.

Step (4b): The optimal value $\left(E_p^M\right)$ of the crisp CCR DEA model-3.45 and hence of crisp CCR DEA model-3.44 is 0.829.

Crisp CCR DEA model-3.44

Maximize $\left[E_A^M = \dfrac{14790u_1 + 4.1u_2}{2170v_1 + 1870v_2}\right]$

Subject to

$$\frac{14500u_1 + 3.1u_2}{2210v_1 + 1870v_2} = 0.812,$$

$$0.812 \leq \frac{14790u_1 + 4.1u_2}{2170v_1 + 1870v_2} \leq 1,$$

All the constraints of fuzzy CCR DEA model-3.40.

Crisp CCR DEA model-3.45

Maximize $[E_A^M = 14790u_1 + 4.1u_2]$

Subject to

$2170v_1 + 1870v_2 = 1,$

$14500u_1 + 3.1u_2 = 0.812(2210v_1 + 1870v_2),$

$0.812(2170v_1 + 1870v_2) \leq 14790u_1 + 4.1u_2 \leq 2170v_1 + 1870v_2,$

All the constraints of fuzzy CCR DEA model-3.41.

Step (4c): The optimal value $\left(E_p^U\right)$ of the crisp CCR DEA model-3.47 and hence of crisp CCR DEA model-3.46 is 0.833.

Crisp CCR DEA model-3.46

Maximize $\left[E_A^U = \dfrac{14860u_1 + 4.9u_2}{2120v_1 + 4.9u_2}\right]$

Subject to

$\dfrac{14500u_1 + 3.1u_2}{2210v_1 + 1870v_2} = 0.812,$

$\dfrac{14790u_1 + 4.1u_2}{2170v_1 + 1870v_2} = 0.829,$

$0.829 \leq \dfrac{14860u_1 + 4.9u_2}{2120v_1 + 4.9u_2} \leq 1,$

and

All the constraints of fuzzy CCR DEA model-3.40.

Crisp CCR DEA model-3.47

Maximize $[E_A^U = 14860u_1 + 4.9u_2]$

Subject to

$2120v_1 + 4.9u_2 = 1,$

$14500u_1 + 3.1u_2 = 0.812(2210v_1 + 1870v_2),$

$14790u_1 + 4.1u_2 = 0.829(2170v_1 + 1870v_2),$

$0.829(2120v_1 + 4.9u_2) \leq 14860u_1 + 4.9u_2 \leq 2120v_1 + 4.9u_2,$

All the constraints of fuzzy CCR DEA model-3.40.

Step 5: Using the optimal values of E_A^L, E_A^M, E_A^U, obtained in Step 4, the fuzzy optimal value (best relative fuzzy efficiency of DMU_A of fuzzy CCR DEA model-3.31 is $(E_A^L, E_A^M, E_A^U) = (0.812, 0.829, 0.833)$.

3.7.4 Results

The exact best relative fuzzy efficiency of all the DMUs, evaluated on solving fuzzy CCR DEA model-3.31 to 3.38 by the proposed method, are shown in Table 3.2. Also, the ranking value corresponding to best relative fuzzy efficiency, obtained by using the ranking formula defined in Remark 2.4, is shown in Table 3.2.

Table 3.2 Exact best relative fuzzy efficiencies of DMUs

DMUs	**Exact best relative fuzzy efficiency of j^{th} DMU ($\tilde{E}_j$)**	**Ranking Value ($\Re\left(\tilde{E}_j\right)$**
A	$(.812, .829, .833)$	.826
B	$(.975, 1, 1)$	.994
C	$(.797, .815, .825)$	.813
D	$(.776, .792, 799)$	.790
E	$(.973, 1, 1)$	.993
F	$(.835, .852, .857)$	.849
G	$(.875, .893, .900)$	.890
H	$(.820, .836, .843)$	.833

It is obvious from Table 3.2 that $\Re\left(\tilde{E}_B\right) > R\left(\tilde{E}_E\right) > R\left(\tilde{E}_G\right) > R\left(\tilde{E}_F\right) > R\left(\tilde{E}_H\right) > R\left(\tilde{E}_A\right) > R\left(\tilde{E}_C\right) > R\left(\tilde{E}_D\right)$. Therefore, $\tilde{E}_B \succ \tilde{E}_E \succ \tilde{E}_G \succ \tilde{E}_F \succ \tilde{E}_H \succ \tilde{E}_A \succ \tilde{E}_C \succ \tilde{E}_D$.

3.8 Conclusions

On the basis the present study, it can be concluded that there are flaws in the fuzzy CCR DEA model as well as in the method, proposed by Wang et al.[106] and hence neither the fuzzy CCR DEA model nor the method, proposed by Wang et al. [106] should be used for evaluating the best relative fuzzy efficiency of DMUs. Also, to resolve the flaws of the fuzzy CCR DEA model, a new fuzzy CCR model is proposed. Further, a new method is proposed to solve the proposed fuzzy CCR DEA model for evaluating the best relative fuzzy efficiency of DMUs. Further, the exact best relative fuzzy efficiency of the DMUs, considered by Wang et al. [106] is evaluated by using the proposed method.

Chapter 4

A New Approach for Solving Proposed Fully Fuzzy CCR DEA Model[3]

Wang and Chin [105] proposed an optimistic as well as pessimistic fuzzy CCR DEA model and an approach for solving it to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs. In this chapter, it is shown that the fuzzy CCR models, proposed by Wang and Chin [105] are not valid and hence cannot be used to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs. To resolve the flaws of the fuzzy CCR DEA model, proposed by Wang and Chin [105] new fuzzy CCR DEA models are proposed. Also, a new approach is proposed to solve the proposed fuzzy CCR DEA models for evaluating the relative geometric crisp efficiency of DMUs.

4.1 Existing fuzzy DEA models

Wang and Chin [105] proposed the optimistic fuzzy CCR DEA model-4.1 and pessimistic fuzzy CCR DEA model-4.1 to evaluate the best relative fuzzy efficiency $\left(\tilde{E}_p^{BEST}\right)$ and worst relative fuzzy efficiency $\left(\tilde{E}_p^{WORST}\right)$ respectively of p^{th} DMU by incorporating the following modifications in the fuzzy CCR DEA model-4.1.

[3] The contents of this chapter are communicated for the possible publication in Expert Systems with Applications. The flaw pointed out in this chapter is published in Applied Mathematical Modelling.

1. Used trapezoidal fuzzy numbers instead of triangular fuzzy numbers to represent the input and output data.
2. Used trapezoidal fuzzy numbers instead of crisp numbers to represent the weights.

Optimistic fuzzy CCR DEA model-4.1

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right)\right.$$

$$\left.\approx \frac{\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)\left(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U\right)]}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)\left(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U\right)]} \preccurlyeq (1,1,1,1), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.1

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right)\right.$$

$$\left.\approx \frac{\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)\left(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U\right)]}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)\left(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)\left(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U\right)]} \succcurlyeq (1,1,1,1), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m,\ r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

4.2 Existing method

Wang and Chin [105] proposed the following method to evaluate best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, geometric crisp efficiency of DMUs.

Step 1: Using the product of trapezoidal fuzzy numbers, defined in Section 2.1.2.1, the optimistic fuzzy CCR DEA model-4.1 and the pessimistic fuzzy DEA model-4.1 can be transformed into optimistic fuzzy CCR DEA model-4.2 and the pessimistic fuzzy CCR DEA model-4.2 respectively.

Optimistic fuzzy CCR DEA model-4.2

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right)} \preccurlyeq (1,1,1,1), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m,\ r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.2

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right)} \succcurlyeq (1,1,1,1), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 2: The optimistic fuzzy CCR DEA model-4.2 and the pessimistic fuzzy CCR DEA model-4.2 can be transformed into optimistic fuzzy CCR DEA model-4.3 and the pessimistic fuzzy CCR DEA model-4.3 respectively.

Optimistic fuzzy CCR DEA model-4.3

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right) \approx \frac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L, \sum_{r=1}^{s} u_r^M y_{rj}^M, \sum_{r=1}^{s} u_r^N y_{rj}^N, \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L, \sum_{i=1}^{m} v_i^M x_{ij}^M, \sum_{i=1}^{m} v_i^N x_{ij}^N, \sum_{i=1}^{m} v_i^U x_{ij}^U\right)} \preccurlyeq (1,1,1,1), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.3

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right) \approx \frac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\tilde{E}_j \approx \left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L, \sum_{r=1}^{s} u_r^M y_{rj}^M, \sum_{r=1}^{s} u_r^N y_{rj}^N, \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L, \sum_{i=1}^{m} v_i^M x_{ij}^M, \sum_{i=1}^{m} v_i^N x_{ij}^N, \sum_{i=1}^{m} v_i^U x_{ij}^U\right)} \succcurlyeq (1,1,1,1), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 3: The optimistic fuzzy CCR DEA model-4.3 and the pessimistic fuzzy CCR DEA model-4.3 can be transformed into optimistic crisp CCR DEA model-4.4 and the pessimistic crisp CCR DEA model-4.4 respectively.

Optimistic crisp CCR DEA model-4.4

$$\text{Minimize}\left[\Re\left(\tilde{E}_p^{BEST}\right) = \Re\left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right) = \Re\left[\frac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]\right]$$

Subject to

$$\Re(\tilde{E}_j) = \Re(E_j^L, E_j^M, E_j^N, E_j^U) = \Re\left[\frac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L, \sum_{r=1}^{s} u_r^M y_{rj}^M, \sum_{r=1}^{s} u_r^N y_{rj}^N, \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L, \sum_{i=1}^{m} v_i^M x_{ij}^M, \sum_{i=1}^{m} v_i^N x_{ij}^N, \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}\right]$$

$$\leq \Re(1,1,1,1), \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.$$

Pessimistic crisp CCR DEA model-4.4

$$\text{Minimize}\left[\Re(\tilde{E}_p^{WORST}) = \Re(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST})\right.$$

$$\left. = \Re\left[\frac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]\right]$$

Subject to

$$\Re(\tilde{E}_j) = \Re(E_j^L, E_j^M, E_j^N, E_j^U) = \Re\left[\frac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L, \sum_{r=1}^{s} u_r^M y_{rj}^M, \sum_{r=1}^{s} u_r^N y_{rj}^N, \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L, \sum_{i=1}^{m} v_i^M x_{ij}^M, \sum_{i=1}^{m} v_i^N x_{ij}^N, \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}\right]$$

$$\geq \Re(1,1,1,1), \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.$$

Step 4: The optimistic crisp CCR DEA model-4.4 and the pessimistic crisp CCR DEA model-4.4 can be transformed into optimistic crisp CCR DEA model-4.5 and the pessimistic crisp CCR DEA model-4.5 respectively.

Optimistic crisp CCR DEA model-4.5

$$\text{Maximize}\left[\Re\left(\tilde{E}_p^{BEST}\right) = \Re\left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right)\right.$$

$$\left. = \left[\frac{\Re\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\Re\left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]\right]$$

Subject to

$$\Re\left(\tilde{E}_j\right) = \Re\left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\Re\left(\sum_{r=1}^{s} u_r^L y_{rj}^L, \sum_{r=1}^{s} u_r^M y_{rj}^M, \sum_{r=1}^{s} u_r^N y_{rj}^N, \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\Re\left(\sum_{i=1}^{m} v_i^L x_{ij}^L, \sum_{i=1}^{m} v_i^M x_{ij}^M, \sum_{i=1}^{m} v_i^N x_{ij}^N, \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}$$

$$\leq \Re(1,1,1,1), \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Pessimistic crisp CCR DEA model-4.5

$$\text{Minimize}\left[\Re\left(\tilde{E}_p^{WORST}\right) = \Re\left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right)\right.$$

$$\left. = \frac{\Re\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\Re\left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\Re\left(\tilde{E}_j\right) = \Re\left(E_j^L, E_j^M, E_j^N, E_j^U\right) = \frac{\Re\left(\sum_{r=1}^{s} u_r^L y_{rj}^L, \sum_{r=1}^{s} u_r^M y_{rj}^M, \sum_{r=1}^{s} u_r^N y_{rj}^N, \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\Re\left(\sum_{i=1}^{m} v_i^L x_{ij}^L, \sum_{i=1}^{m} v_i^M x_{ij}^M, \sum_{i=1}^{m} v_i^N x_{ij}^N, \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}$$

$$\geq \Re(1,1,1,1), \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U, \qquad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Step 5: The optimistic crisp CCR DEA model-4.5 and pessimistic crisp CCR DEA model-4.5 can be transformed into optimistic crisp CCR DEA model-4.6 and pessimistic crisp CCR DEA model-4.6 respectively.

Optimistic crisp CCR DEA model-4.6

$$\text{Maximize}\left[\Re\left(\tilde{E}_p^{BEST}\right) = \frac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L + \sum_{r=1}^{s} u_r^M y_{rp}^M + \sum_{r=1}^{s} u_r^N y_{rp}^N + \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L + \sum_{i=1}^{m} v_i^M x_{ip}^M + \sum_{i=1}^{m} v_i^N x_{ip}^N + \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\Re\left(\frac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L + \sum_{r=1}^{s} u_r^M y_{rj}^M + \sum_{r=1}^{s} u_r^N y_{rj}^N + \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L + \sum_{i=1}^{m} v_i^M x_{ij}^M + \sum_{i=1}^{m} v_i^N x_{ij}^N + \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}\right) \leq 1,, \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U \quad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Pessimistic crisp CCR DEA model-4.6

$$\text{Minimize}\left[\Re\left(\tilde{E}_p^{WORST}\right) = \frac{\left(\sum_{r=1}^{s} u_r^L y_{rp}^L + \sum_{r=1}^{s} u_r^M y_{rp}^M + \sum_{r=1}^{s} u_r^N y_{rp}^N + \sum_{r=1}^{s} u_r^U y_{rp}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ip}^L + \sum_{i=1}^{m} v_i^M x_{ip}^M + \sum_{i=1}^{m} v_i^N x_{ip}^N + \sum_{i=1}^{m} v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\Re\left(\frac{\left(\sum_{r=1}^{s} u_r^L y_{rj}^L + \sum_{r=1}^{s} u_r^M y_{rj}^M + \sum_{r=1}^{s} u_r^N y_{rj}^N + \sum_{r=1}^{s} u_r^U y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i^L x_{ij}^L + \sum_{i=1}^{m} v_i^M x_{ij}^M + \sum_{i=1}^{m} v_i^N x_{ij}^N + \sum_{i=1}^{m} v_i^U x_{ij}^U\right)}\right) \geq 1, \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U \quad i = 1, \dots, m, \qquad r = 1, \dots, s.$$

Step 6: The optimistic crisp CCR DEA model-4.6 and pessimistic crisp CCR DEA model-4.6 can be transformed into optimistic crisp CCR DEA model-4.7 and pessimistic crisp CCR DEA model-4.7 respectively.

Optimistic crisp CCR DEA model-4.7

$$\text{Maximize}\left[\Re\left(\tilde{E}_p^{BEST}\right)=\sum_{r=1}^{s}u_r^L y_{rp}^L+\sum_{r=1}^{s}u_r^M y_{rp}^M+\sum_{r=1}^{s}u_r^N y_{rp}^N+\sum_{r=1}^{s}u_r^U y_{rp}^U\right]$$

Subject to

$$\sum_{i=1}^{m}v_i^L x_{ip}^L+\sum_{i=1}^{m}v_i^M x_{ip}^M+\sum_{i=1}^{m}v_i^N x_{ip}^N+\sum_{i=1}^{m}v_i^U x_{ip}^U=1$$

$$\left(\sum_{r=1}^{s}u_r^L y_{rj}^L+\sum_{r=1}^{s}u_r^M y_{rj}^M+\sum_{r=1}^{s}u_r^N y_{rj}^N+\sum_{r=1}^{s}u_r^U y_{rj}^U\right)$$
$$-\left(\sum_{i=1}^{m}v_i^L x_{ij}^L+\sum_{i=1}^{m}v_i^M x_{ij}^M+\sum_{i=1}^{m}v_i^N x_{ij}^N+\sum_{i=1}^{m}v_i^U x_{ij}^U\right)\leq 0,\forall,j,$$

$$0\leq u_r^L\leq u_r^M\leq u_r^N\leq u_r^U;0\leq v_i^L\leq v_i^M\leq v_i^N\leq v_i^U\quad i=1,\dots,m,\qquad r=1,\dots,s.$$

Pessimistic crisp CCR DEA model-4.7

$$\text{Minimize}\left[\Re\left(\tilde{E}_p^{WORST}\right)=\sum_{r=1}^{s}u_r^L y_{rp}^L+\sum_{r=1}^{s}u_r^M y_{rp}^M+\sum_{r=1}^{s}u_r^N y_{rp}^N+\sum_{r=1}^{s}u_r^U y_{rp}^U\right]$$

Subject to

$$\sum_{i=1}^{m}v_i^L x_{ip}^L+\sum_{i=1}^{m}v_i^M x_{ip}^M+\sum_{i=1}^{m}v_i^N x_{ip}^N+\sum_{i=1}^{m}v_i^U x_{ip}^U=1$$

$$\left(\sum_{r=1}^{s}u_r^L y_{rj}^L+\sum_{r=1}^{s}u_r^M y_{rj}^M+\sum_{r=1}^{s}u_r^N y_{rj}^N+\sum_{r=1}^{s}u_r^U y_{rj}^U\right)$$
$$-\left(\sum_{i=1}^{m}v_i^L x_{ij}^L+\sum_{i=1}^{m}v_i^M x_{ij}^M+\sum_{i=1}^{m}v_i^N x_{ij}^N+\sum_{i=1}^{m}v_i^U x_{ij}^U\right)\geq 0,\forall,j,$$

$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U \;\; i = 1, \ldots, m, \qquad r = 1, \ldots, s.$

Step 4: Find the optimal value $\left(\Re\left(\tilde{E}_p^{BEST}\right)\right)$, representing the best relative crisp efficiency of p^{th} DMU, by solving optimistic crisp CCR DEA model-4.7.

Step 5: Find the optimal value $\left(\Re\left(\tilde{E}_p^{WORST}\right)\right)$, representing the worst relative crisp efficiency of p^{th} DMU, by solving pessimistic crisp CCR DEA model-4.7.

Step 6: Find the relative geometric average crisp efficiency $E_P^{GEOMETRIC}$ of p^{th} DMU by putting the values E_p^{BEST} and E_p^{WORST}, obtained in Step 4 and Step 5, in $E_P^{GEOMETRIC} = \sqrt{E_p^{BEST} \times E_p^{WORST}}$.

4.3 Flaws in the existing method

If (a^L, a^M, a^N, a^U) & (b^L, b^M, b^N, b^U) are two trapezoidal fuzzy numbers then

$$\Re\left(\frac{(a^L, a^M, a^N, a^U)}{(b^L, b^M, b^N, b^U)}\right) = \Re\left(\frac{a^L}{b^U}, \frac{a^M}{b^N}, \frac{a^N}{b^M}, \frac{a^U}{b^L}\right) = \frac{\frac{a^L}{b^U} + \frac{a^M}{b^N} + \frac{a^N}{b^M} + \frac{a^U}{b^L}}{4}$$

and

$$\frac{\Re(a^L, a^M, a^N, a^U)}{\Re(b^L, b^M, b^N, b^U)} = \frac{a^L + a^M + a^N + a^U}{b^L + b^M + b^N + b^U}$$

It is obvious that

$$\Re\left(\frac{(a^L, a^M, a^N, a^U)}{(b^L, b^M, b^N, b^U)}\right) \neq \frac{\Re(a^L, a^M, a^N, a^U)}{\Re(b^L, b^M, b^N, b^U)}$$

However, Wang and Chin have used the property

$$\Re\left(\frac{(a^L, a^M, a^N, a^U)}{(b^L, b^M, b^N, b^U)}\right) = \frac{\Re(a^L, a^M, a^N, a^U)}{\Re(b^L, b^M, b^N, b^U)}$$

in Step 4 of their proposed method, therefore, the method proposed by Wang and Chin [105] is not valid.

4.4 Proposed fuzzy DEA model

Wang and Chin [105] transformed the crisp CCR DEA model 2.2 into fuzzy CCR DEA model 4.1 by replacing the crisp input data (x_{ij}), crisp output data (y_{rj}) and weights u_r and v_i with trapezoidal fuzzy numbers $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U)$, $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U)$ and $(u_r^L, u_r^M, u_r^N, u_r^U)$, $(v_i^L, v_i^M, v_i^N, v_i^U)$ respectively.

However, the flaws, pointed out in Section 3.3 of Chapter 3, is also existing in the optimistic fuzzy CCR DEA model-4.1 and pessimistic fuzzy CCR DEA model-4.2 proposed by Wang and Chin [105] .

In this section, to resolve the flaws of the existing optimistic fuzzy CCR DEA model-4.1 and pessimistic fuzzy CCR DEA model-4.2, proposed by Wang and Chin [105] new optimistic fuzzy CCR DEA model-4.8 and pessimistic fuzzy CCR DEA model-4.8 are obtained by replacing the crisp weights u_r and v_i of the fuzzy DEA model, proposed in Section 3.5 of Chapter 3, with trapezoidal fuzzy numbers $(u_r^L, u_r^M, u_r^N, u_r^U)$, $(v_i^L, v_i^M, v_i^N, v_i^U)$ respectively.

Optimistic fuzzy CCR DEA model-4.8

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right) \approx \frac{\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U)]}\right]$$

Subject to

$$\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U) \preccurlyeq \sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U)\ , \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.8

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right) \approx \frac{\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)(y_{rp}^L, y_{rp}^M, y_{rp}^N, y_{rp}^U)}{\sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)(x_{ip}^L, x_{ip}^M, x_{ip}^N, x_{ip}^U)]}\right]$$

Subject to

$$\sum_{r=1}^{s}(u_r^L, u_r^M, u_r^N, u_r^U)(y_{rj}^L, y_{rj}^M, y_{rj}^N, y_{rj}^U) \succcurlyeq \sum_{i=1}^{m}\left(v_i^L, v_i^M, v_i^N, v_i^U\right)(x_{ij}^L, x_{ij}^M, x_{ij}^N, x_{ij}^U), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

4.5 Proposed approach

In this section, a new approach is proposed to evaluate the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs by using the optimistic fuzzy CCR DEA model-4.8 and pessimistic fuzzy CCR DEA model-4.8.

Step 1: Using the product of trapezoidal fuzzy numbers, defined in Section 2.1.2.2, the optimistic fuzzy CCR DEA model-4.8 and pessimistic fuzzy CCR DEA model-4.8 can be transformed into optimistic fuzzy CCR DEA model-4.9 and pessimistic fuzzy CCR DEA model-4.9 respectively.

Optimistic fuzzy CCR DEA model-4.9

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right) \preccurlyeq \sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.9

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right) \approx \frac{\sum_{r=1}^{s}\left(u_r^L y_{rp}^L, u_r^M y_{rp}^M, u_r^N y_{rp}^N, u_r^U y_{rp}^U\right)}{\sum_{i=1}^{m}\left(v_i^L x_{ip}^L, v_i^M x_{ip}^M, v_i^N x_{ip}^N, v_i^U x_{ip}^U\right)}\right]$$

Subject to

$$\sum_{r=1}^{s}\left(u_r^L y_{rj}^L, u_r^M y_{rj}^M, u_r^N y_{rj}^N, u_r^U y_{rj}^U\right) \succcurlyeq \sum_{i=1}^{m}\left(v_i^L x_{ij}^L, v_i^M x_{ij}^M, v_i^N x_{ij}^N, v_i^U x_{ij}^U\right), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \dots, m,\ r = 1, \dots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 2: Using the division of trapezoidal fuzzy numbers, defined in Section 2.1.2.2, the optimistic fuzzy CCR DEA model-4.9 and pessimistic fuzzy CCR DEA model-4.9 can be transformed into optimistic fuzzy CCR DEA model-4.10 and pessimistic fuzzy CCR DEA model-4.10 respectively.

Optimistic fuzzy CCR DEA model-4.10

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right.$$

$$\left.\approx \left(\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}, \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right)\right]$$

Subject to

$$\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)$$

$$\preccurlyeq \left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m,\ r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.10

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right)\right.$$

$$\left.\approx \left(\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}, \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right)\right]$$

Subject to

$$\left(\sum_{r=1}^{s} u_r^L y_{rp}^L, \sum_{r=1}^{s} u_r^M y_{rp}^M, \sum_{r=1}^{s} u_r^N y_{rp}^N, \sum_{r=1}^{s} u_r^U y_{rp}^U\right)$$

$$\succcurlyeq \left(\sum_{i=1}^{m} v_i^L x_{ip}^L, \sum_{i=1}^{m} v_i^M x_{ip}^M, \sum_{i=1}^{m} v_i^N x_{ip}^N, \sum_{i=1}^{m} v_i^U x_{ip}^U\right), \forall, j,$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1, \ldots, m,\ r = 1, \ldots, s,$ are non-negative trapezoidal fuzzy numbers.

Step 3: Using the relation $(a^L, a^M, a^N, a^U) \leq (b^L, b^M, b^N, b^U)$, $a^L \leq b^L$, $a^M \leq b^M, a^N \leq b^N, a^U \leq b^U,$ the optimistic fuzzy CCR DEA model-4.10 and pessimistic fuzzy CCR DEA model-4.10 can be transformed into optimistic fuzzy CCR DEA model-4.11 and pessimistic fuzzy CCR DEA model-4.11 respectively.

Optimistic fuzzy CCR DEA model-4.11

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right)\right.$$

$$\left.\approx \left(\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}, \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right)\right]$$

Subject to

$$\sum_{r=1}^{s} u_r^L y_{rj}^L \leq \sum_{i=1}^{m} v_i^L x_{ij}^L, \forall, j,$$

$$\sum_{r=1}^{s} u_r^M y_{rj}^M \leq \sum_{i=1}^{m} v_i^M x_{ij}^M, \forall, j,$$

$$\sum_{r=1}^{s} u_r^N y_{rp}^N \leq \sum_{i=1}^{m} v_i^N x_{ip}^N, \forall, j,$$

$$\sum_{r=1}^{s} u_r^U y_{rp}^U \leq \sum_{i=1}^{m} v_i^U x_{ip}^U, \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U \;\; i = 1, \dots, m, \;\; r = \; 1, \dots, s.$$

Pessimistic fuzzy CCR DEA model-4.11

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right)\right.$$

$$\left.\approx \left(\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}, \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}, \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}, \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right)\right]$$

Subject to

$$\sum_{r=1}^{s} u_r^L y_{rj}^L \geq \sum_{i=1}^{m} v_i^L x_{ij}^L, \forall, j,$$

$$\sum_{r=1}^{s} u_r^M y_{rj}^M \geq \sum_{i=1}^{m} v_i^M x_{ij}^M, \forall, j,$$

$$\sum_{r=1}^{s} u_r^N y_{rp}^N \geq \sum_{i=1}^{m} v_i^N x_{ip}^N, \forall, j,$$

$$\sum_{r=1}^{s} u_r^U y_{rp}^U \geq \sum_{i=1}^{m} v_i^U x_{ip}^U, \forall, j,$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U \;\; i = 1, \dots, m, \;\; r = \; 1, \dots, s.$$

Step 4: The fuzzy optimal value $\left(\tilde{E}_p^{BEST} \approx \left(\left(\left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right)\right)\right)$, representing the best relative fuzzy efficiency of p^{th} DMU, as well as the fuzzy optimal value $\left(\tilde{E}_p^{WORST} \approx \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right)\right)$, representing the worst relative fuzzy efficiency of

p^{th} DMU, can be obtained by solving the optimistic fuzzy CCR DEA model-4.11 and pessimistic fuzzy CCR DEA model-4.11 as follows:

Step 4(a) Find the optimal value $\left(E_{p1}^{BEST}\right)$ and $\left(E_{p1}^{WORST}\right)$ of the optimistic crisp CCR DEA model-4.12a and pessimistic CCR DEA model-4.12a by solving optimistic crisp CCR DEA model-4.12b and pessimistic CCR DEA model-4.12b equivalent to optimistic crisp CCR DEA model-4.12a and pessimistic CCR DEA model-4.12a respectively .

Optimistic crisp CCR DEA model-4.12a

$$\text{Maximize}\left[E_{p1}^{BEST} = \frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U} \leq 1,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Optimistic crisp CCR DEA model-4.12a

$$\text{Maximize}\left[E_{p1}^{BEST} = \sum_{r=1}^{s} u_r^L y_{rp}^L\right]$$

Subject to

$$\sum_{i=1}^{m} v_i^U x_{ip}^U = 1, \qquad \sum_{r=1}^{s} u_r^L y_{rp}^L \geq \sum_{i=1}^{m} v_i^U x_{ip}^U,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.12a

$$\text{Minimize}\left[E_{p1}^{WORST}=\frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U\,x_{ip}^U}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U\,x_{ip}^U}\geq 1,$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.12b

$$\text{Minimize}\left[E_{p1}^{WORST}=\sum_{r=1}^{s}u_r^L y_{rp}^L\right]$$

Subject to

$$\sum_{i=1}^{m}v_i^U\,x_{ip}^U\;=1,\;\sum_{r=1}^{s}u_r^L y_{rp}^L\geq\sum_{i=1}^{m}v_i^U\,x_{ip}^U,$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Step 4(b): Find the optimal value $\left(E_{p2}^{BEST}\right)$ and $\left(E_{p2}^{WORST}\right)$ of the optimistic crisp CCR DEA model-4.13a and pessimistic CCR DEA model-4.13a by solving optimistic crisp CCR DEA model-4.13b and pessimistic CCR DEA model-4.13b equivalent to optimistic crisp CCR DEA model-4.13a and pessimistic CCR DEA model-4.13a respectively .

Optimistic crisp CCR DEA model-4.13a

$$\text{Maximize}\left[E_{p2}^{BEST} = \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U} = E_{p1}^{BEST},$$

$$E_{p1}^{BEST} \leq \frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N},$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Optimistic crisp CCR DEA model-4.13b

$$\text{Maximize}\left[E_{p2}^{BEST} = \sum_{r=1}^{s} u_r^M y_{rp}^M\right]$$

Subject to

$$\sum_{i=1}^{m} v_i^N x_{ip}^N = 1,$$

$$\sum_{r=1}^{s} u_r^L y_{rp}^L = E_{p1}^{BEST}\left(\sum_{i=1}^{m} v_i^U x_{ip}^U\right),$$

$$E_{p1}^{BEST}\left(\sum_{i=1}^{m} v_i^N x_{ip}^N\right) \leq \sum_{r=1}^{s} u_r^M y_{rp}^M \leq \sum_{i=1}^{m} v_i^M x_{ip}^M,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.13a

$$\text{Minimize}\left[E_{p2}^{WORST}=\frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s}u_r^L y_{rp}^L}{\sum_{i=1}^{m}v_i^U x_{ip}^U}=E_{p1}^{WORST},$$

$$E_{p1}^{WORST}\le\frac{\sum_{r=1}^{s}u_r^M y_{rp}^M}{\sum_{i=1}^{m}v_i^N x_{ip}^N},$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.13b

$$\text{Maximize}\left[E_{p2}^{WORST}=\sum_{r=1}^{s}u_r^M y_{rp}^M\right]$$

Subject to

$$\sum_{i=1}^{m}v_i^N x_{ip}^N=1,$$

$$\sum_{r=1}^{s}u_r^L y_{rp}^L=E_{p1}^{WORST}\left(\sum_{i=1}^{m}v_i^U x_{ip}^U\right),$$

$$E_{p1}^{WORST}\left(\sum_{i=1}^{m}v_i^N x_{ip}^N\right)\le\sum_{r=1}^{s}u_r^M y_{rp}^M,$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Step 4(c): Find the optimal value $\left(E_{p3}^{BEST}\right)$ and $\left(E_{p3}^{WORST}\right)$ of the optimistic crisp CCR DEA model-4.14a and pessimistic CCR DEA model-4.14a by solving optimistic crisp CCR DEA model-4.14b and pessimistic CCR DEA model-4.14b equivalent to optimistic crisp CCR DEA model-4.14a and pessimistic CCR DEA model-4.14a respectively .

Optimistic crisp CCR DEA model-4.14a

$$\text{Maximize}\left[E_{p3}^{BEST} = \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U} = E_{p1}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N} = E_{p2}^{BEST},$$

$$E_{p2}^{BEST} \leq \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M} \leq 1,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Optimistic crisp CCR DEA model-4.14b

$$\text{Maximize}\left[E_{p3}^{BEST} = \sum_{r=1}^{s} u_r^N y_{rp}^N\right]$$

Subject to

$$\sum_{i=1}^{m} v_i^M x_{ip}^M = 1,$$

$$\sum_{r=1}^{s} u_r^L y_{rp}^L = E_{p1}^{BEST}\left(\sum_{i=1}^{m} v_i^U x_{ip}^U\right),$$

$$\sum_{r=1}^{s} u_r^M y_{rp}^M = E_{p2}^{BEST}\left(\sum_{i=1}^{m} v_i^N x_{ip}^N\right),$$

$$E_{p2}^{BEST}\left(\sum_{i=1}^{m} v_i^M x_{ip}^M\right) \leq \sum_{r=1}^{s} u_r^N y_{rp}^N \leq \sum_{i=1}^{m} v_i^M x_{ip}^M,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.14a

$$\text{Minimize}\left[E_{p3}^{WORST} = \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M} \geq 1,$$

$$\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U} = E_{p1}^{WORST},$$

$$\frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N} = E_{p2}^{WORST},$$

$$E_{p2}^{WORST} \leq \frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M},$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.14b

$$\text{Minimize}\left[E_{p3}^{WORST} = \sum_{r=1}^{s} u_r^N y_{rp}^N\right]$$

Subject to

$$\sum_{i=1}^{m} v_i^M x_{ip}^M = 1,$$

$$\sum_{r=1}^{s} u_r^L y_{rp}^L = E_{p1}^{WORST}\left(\sum_{i=1}^{m} v_i^U x_{ip}^U\right),$$

$$\sum_{r=1}^{s} u_r^M y_{rp}^M = E_{p2}^{WORST}\left(\sum_{i=1}^{m} v_i^N x_{ip}^N\right),$$

$$E_{p2}^{WORST}\left(\sum_{i=1}^{m} v_i^M x_{ip}^M\right) \leq \sum_{r=1}^{s} u_r^N y_{rp}^N,$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Step (4d): Find the optimal value $\left(E_{p4}^{BEST}\right)$ and $\left(E_{p4}^{WORST}\right)$ of the optimistic crisp CCR DEA model-4.15a and pessimistic CCR DEA model-4.15a by solving optimistic crisp CCR DEA model-4.15b and pessimistic CCR DEA model-4.15b equivalent to optimistic crisp CCR DEA model-4.15a and pessimistic CCR DEA model-4.15a respectively .

Optimistic crisp CCR DEA model-4.15a

$$\text{Maximize}\left[E_{p4}^{BEST} = \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L}\right]$$

Subject to

$$\frac{\sum_{r=1}^{s} u_r^L y_{rp}^L}{\sum_{i=1}^{m} v_i^U x_{ip}^U} = E_{p1}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_r^M y_{rp}^M}{\sum_{i=1}^{m} v_i^N x_{ip}^N} = E_{p2}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M x_{ip}^M} = E_{p3}^{BEST},$$

$$E_{p3}^{BEST} \leq \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L x_{ip}^L} \leq 1,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Optimistic crisp CCR DEA model-4.15b

$$\text{Maximize}\left[E_{p4}^{BEST} = \sum_{r=1}^{s} u_r^U y_{rp}^U\right]$$

Subject to

$$\sum_{i=1}^{m} v_i^L x_{ip}^L = 1,$$

$$\sum_{r=1}^{s} u_r^L y_{rp}^L = E_{p1}^{BEST}\left(\sum_{i=1}^{m} v_i^U\, x_{ip}^U\right),$$

$$\sum_{r=1}^{s} u_r^M y_{rp}^M = E_{p2}^{BEST}\left(\sum_{i=1}^{m} v_i^N\, x_{ip}^N\right),$$

$$\sum_{r=1}^{s} u_r^N y_{rp}^N = E_{p3}^{BEST}(\sum_{i=1}^{m} v_i^M\, x_{ip}^M),$$

$$E_{p3}^{BEST}\left(\sum_{i=1}^{m} v_i^L\, x_{ip}^L\right) \leq \sum_{r=1}^{s} u_r^U y_{rp}^U \leq \sum_{i=1}^{m} v_i^L\, x_{ip}^L,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.15a

Minimize $\left[E_{p4}^{WORST} = \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L\, x_{ip}^L}\right]$

Subject to

$$\sum_{r=1}^{s} u_r^L y_{rp}^L = E_{p1}^{WORST}\left(\sum_{i=1}^{m} v_i^U\, x_{ip}^U\right),$$

$$\sum_{r=1}^{s} u_r^M y_{rp}^M = E_{p2}^{WORST}\left(\sum_{i=1}^{m} v_i^N\, x_{ip}^N\right),$$

$$\frac{\sum_{r=1}^{s} u_r^N y_{rp}^N}{\sum_{i=1}^{m} v_i^M\, x_{ip}^M} = E_{p3}^{WORST},$$

$$E_{p3}^{WORST} \leq \frac{\sum_{r=1}^{s} u_r^U y_{rp}^U}{\sum_{i=1}^{m} v_i^L\, x_{ip}^L} \leq 1,$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Pessimistic crisp CCR DEA model-4.15b

$$\text{Minimize}\left[E_{p4}^{WORST} = \sum_{r=1}^{s} u_r^U y_{rp}^U\right]$$

Subject to

$\sum_{i=1}^{m} v_i^L x_{ip}^L = 1,$

$$\sum_{r=1}^{s} u_r^M y_{rp}^M = E_{p2}^{WORST}\left(\sum_{i=1}^{m} v_i^N x_{ip}^N\right),$$

$$\sum_{r=1}^{s} u_r^N y_{rp}^N = E_{p3}^{WORST}(\sum_{i=1}^{m} v_i^M x_{ip}^M),$$

$$\sum_{r=1}^{s} u_r^N y_{rp}^N = E_{p3}^{WORST}\left(\sum_{i=1}^{m} v_i^M x_{ip}^M\right)$$

$$E_{p3}^{WORST}\left(\sum_{i=1}^{m} v_i^U x_{ip}^U\right) \leq \sum_{r=1}^{s} u_r^U y_{rp}^U \leq 1,$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.11.

Step 5: Using the values of $E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}$ and $E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}$, obtained in Step (4a) to Step (4d), find the fuzzy optimal value $\left(\tilde{E}_p^{BEST} = \left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}, E_{p4}^{BEST}\right)\right)$ of optimistic fuzzy DEA model-4.11, representing the best relative fuzzy efficiency of p^{th} DMU, as well as pessimistic fuzzy optimal value $\left(\tilde{E}_p^{WORST} = \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}, E_{p4}^{WORST}\right)\right)$ of pessimistic fuzzy DEA model-4.11, representing the worst relative fuzzy efficiency of p^{th} DMU.

Step 5: Find the crisp optimal value $\left(E_p^{BEST} = \Re\left(\tilde{E}_p^{BEST}\right)\right)$, representing the best relative crisp efficiency of p^{th} DMU.

Step 6: Find the crisp optimal value $\left(E_p^{WORST} = \Re\left(\tilde{E}_p^{WORST}\right)\right)$, representing the worst relative crisp efficiency of p^{th} DMU.

Step 7: Find the relative geometric crisp efficiency $E_P^{GEOMETRIC}$ of p^{th} DMU by putting the values E_p^{BEST} and E_p^{WORST}, obtained in Step 5 and Step 6, in $E_P^{GEOMETRIC} = \sqrt{E_p^{BEST} \times E_p^{WORST}}$.

4.6 Exact fuzzy efficiency of real life problem

Wang and Chin [105] evaluated the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of by considering eight manufacturing enterprises (DMUs) of China with two inputs and two outputs shown in Table 4.1 and using the optimistic fuzzy CCR DEA model-4.1 as well as pessimistic fuzzy CCR DEA model-4.1. The eight manufacturing enterprises, all manufacture the same type of products but with different qualities. Both the gross output value (GOV) and product quality (PQ) are considered as outputs. Manufacturing cost (MC) and the number of employees (NOE) are considered as inputs. The data about the GOV and MC are uncertain due to the unavailability at the time of assessment and are therefore estimated as fuzzy numbers. The product quality is assessed by customers using fuzzy linguistic terms such as Excellent, Very Good, Average, Poor and Very Poor. The assessment results by customers are weighted and averaged.

However, as discussed in Section 4.4, that the optimistic fuzzy CCR DEA model-4.1 as well as pessimistic fuzzy CCR DEA model-4.1 are not valid. Therefore, the best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of 8 manufacturing enterprises, evaluated by Wang and Chin [105], is not exact. In this section, to illustrate the proposed method the exact best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence, relative geometric crisp efficiency of Enterprise A is evaluated by using the proposed method.

Table 4.1: Input and output data for eight manufacturing enterprises [105]

(DMUs)	**Inputs (Two)**		**Outputs (two)**	
	MC	NOE	GOV	PQ
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)
C	(2510, 2570, 2610)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)

4.6.1 Proposed fuzzy CCR DEA models

The best relative fuzzy efficiency and worst relative fuzzy efficiency of $\mathrm{DMU_A}$ to $\mathrm{DMU_H}$ respectively can be obtained by solving the optimistic fuzzy CCR DEA models-4.16 to 4.23 and pessimistic CCR DEA models-4.16 to 4.23.

Optimistic fuzzy CCR DEA model-4.16

$$\text{Maximize } \left[\tilde{E}_A^{BEST} = (E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST}) \approx \left[\frac{(14500, 14790, 14860)(u_1^L, u_1^M, u_1^U) + (3.1, 4.1, 4.9)(u_2^L, u_2^M, u_2^U)}{(2120, 2170, 2210)(v_1^L, v_1^M, v_1^U) + (1870,1870,1870)(v_2^L, v_2^M, v_2^U)}\right]\right]$$

Subject to

$$(14500, 14790, 14860)(u_1^L, u_1^M, u_1^U) + (3.1, 4.1, 4.9)(u_2^L, u_2^M, u_2^U) \precsim (2120, 2170, 2210)(v_1^L, v_1^M, v_1^U) + (1870,1870,1870)(v_2^L, v_2^M, v_2^U)$$

$$(12470, 12720, 12790)(u_1^L, u_1^M, u_1^U) + (1.2, 2.1, 3.0)(u_2^L, u_2^M, u_2^U) \precsim (1420, 1460, 1500)(v_1^L, v_1^M, v_1^U) + (1340,1340,1340)(v_2^L, v_2^M, v_2^U)$$

$$(17900, 18260, 18400)(u_1^L, u_1^M, u_1^U) + (3.3, 4.3, 4.0)(u_2^L, u_2^M, u_2^U) \precsim (2510, 2570, 2610)(v_1^L, v_1^M, v_1^U) + (2360,2360,2360)(v_2^L, v_2^M, v_2^U)$$

$$(14970, 15270, 15400)(u_1^L, u_1^M, u_1^U) + (2.7, 3.7, 4.6)(u_2^L, u_2^M, u_2^U) \precsim (2300, 2350, 2400)(v_1^L, v_1^M, v_1^U) + (2020,2020,2020)(v_2^L, v_2^M, v_2^U)$$

$$(13980, 14260, 14330)(u_1^L, u_1^M, u_1^U) + (1.0, 1.8, 2.7)(u_2^L, u_2^M, u_2^U) \precsim (1480, 1520, 1560)(v_1^L, v_1^M, v_1^U) + (1550,1550,1550)(v_2^L, v_2^M, v_2^U)$$

$$(14030, 14310, 14400)(u_1^L, u_1^M, u_1^U) + (1.6, 2.6, 3.6)(u_2^L, u_2^M, u_2^U) \precsim (1990, 2030, 2100)(v_1^L, v_1^M, v_1^U) + (1760,1760,1760)(v_2^L, v_2^M, v_2^U)$$

$$(16540, 16870, 17000)(u_1^L, u_1^M, u_1^U) + (2.4, 3.4, 4.4)(u_2^L, u_2^M, u_2^U)$$

$$\preccurlyeq (2200, 2260, 2300)(v_1^L, v_1^M, v_1^U) + (1980,1980,1980)(v_2^L, v_2^M, v_2^U)$$

$$(17600, 17960, 18100)(u_1^L, u_1^M, u_1^U) + (2.6, 3.6, 4.6)(u_2^L, u_2^M, u_2^U)$$

$$\preccurlyeq (2400, 2460, 2520)(v_1^L, v_1^M, v_1^U) + (2250,2250,2250)(v_2^L, v_2^M, v_2^U),$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1,2,\ r = 1,2,$ are non-negative trapezoidal fuzzy numbers.

Pessimistic fuzzy CCR DEA model-4.16

$$\text{Minimize}\left[\tilde{E}_A^{WORST} = (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST})\right.$$

$$\left.\approx \left[\frac{(14500, 14790, 14860)(u_1^L, u_1^M, u_1^U) + (3.1, 4.1, 4.9)(u_2^L, u_2^M, u_2^U)}{(2120, 2170, 2210)(v_1^L, v_1^M, v_1^U) + (1870,1870,1870)(v_2^L, v_2^M, v_2^U)}\right]\right]$$

Subject to

$$(14500, 14790, 14860)(u_1^L, u_1^M, u_1^U) + (3.1, 4.1, 4.9)(u_2^L, u_2^M, u_2^U)$$

$$\succcurlyeq (2120, 2170, 2210)(v_1^L, v_1^M, v_1^U) + (1870,1870,1870)(v_2^L, v_2^M, v_2^U)$$

$$(12470, 12720, 12790)(u_1^L, u_1^M, u_1^U) + (1.2, 2.1, 3.0)(u_2^L, u_2^M, u_2^U)$$

$$\succcurlyeq (1420, 1460, 1500)(v_1^L, v_1^M, v_1^U) + (1340,1340,1340)(v_2^L, v_2^M, v_2^U)$$

$$(17900, 18260, 18400)(u_1^L, u_1^M, u_1^U) + (3.3, 4.3, 4.0)(u_2^L, u_2^M, u_2^U)$$

$$\preccurlyeq (2510, 2570, 2610)(v_1^L, v_1^M, v_1^U) + (2360,2360,2360)(v_2^L, v_2^M, v_2^U)$$

$$(14970, 15270, 15400)(u_1^L, u_1^M, u_1^U) + (2.7, 3.7, 4.6)(u_2^L, u_2^M, u_2^U)$$

$$\succcurlyeq (2300, 2350, 2400)(v_1^L, v_1^M, v_1^U) + (2020,2020,2020)(v_2^L, v_2^M, v_2^U)$$

$$(13980, 14260, 14330)(u_1^L, u_1^M, u_1^U) + (1.0, 1.8, 2.7)(u_2^L, u_2^M, u_2^U)$$

$$\succcurlyeq (1480, 1520, 1560)(v_1^L, v_1^M, v_1^U) + (1550,1550,1550)(v_2^L, v_2^M, v_2^U)$$

$(14030, 14310, 14400)(u_1^L, u_1^M, u_1^U) + (1.6, 2.6, 3.6)(u_2^L, u_2^M, u_2^U)$

$$\succcurlyeq (1990, 2030, 2100)(v_1^L, v_1^M, v_1^U) + (1760,1760,1760)(v_2^L, v_2^M, v_2^U)$$

$(16540, 16870, 17000)(u_1^L, u_1^M, u_1^U) + (2.4, 3.4, 4.4)(u_2^L, u_2^M, u_2^U)$

$$\succcurlyeq (2200, 2260, 2300)(v_1^L, v_1^M, v_1^U) + (1980,1980,1980)(v_2^L, v_2^M, v_2^U)$$

$(17600, 17960, 18100)(u_1^L, u_1^M, u_1^U) + (2.6, 3.6, 4.6)(u_2^L, u_2^M, u_2^U)$

$$\succcurlyeq (2400, 2460, 2520)(v_1^L, v_1^M, v_1^U) + (2250,2250,2250)(v_2^L, v_2^M, v_2^U),$$

$(u_r^L, u_r^M, u_r^N, u_r^U)$ and $\left(v_i^L, v_i^M, v_i^N, v_i^U\right), i = 1,2,\ r = 1,2,$ are non-negative trapezoidal fuzzy numbers.

Optimistic fuzzy CCR DEA model-4.17

Maximize $\left[\tilde{E}_B^{BEST} = (E_{B1}^{BEST}, E_{B2}^{BEST}, E_{B3}^{BEST})\right.$

$$\left.\approx \left[\frac{(12470, 12720, 12790)(u_1^L, u_1^M, u_1^U) + (1.2, 2.1, 3.0)u_2}{(1420, 1460, 1500)v_1 + (1340,1340,1340)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16.

Pessimistic fuzzy CCR DEA model-4.17

Minimize $\left[\tilde{E}_B^{WORST} = (E_{B1}^{WORST}, E_{B2}^{WORST}, E_{B3}^{WORST})\right.$

$$\left.\approx \left[\frac{(12470, 12720, 12790)(u_1^L, u_1^M, u_1^U) + (1.2, 2.1, 3.0)u_2}{(1420, 1460, 1500)v_1 + (1340,1340,1340)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16

Optimistic fuzzy CCR DEA model-4.18

$$\text{Maximize } \left[\tilde{E}_C^{BEST} = (E_{C1}^{BEST}, E_{C2}^{BEST}, E_{C3}^{BEST}) \approx \left[\frac{(17900, 18260, 18400)(u_1^L, u_1^M, u_1^U) + (3.3, 4.3, 4.0)u_2}{(2510, 2570, 2610)v_1 + (2360,2360,2360)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16

Pessimistic fuzzy CCR DEA model-4.18

$$\text{Minimize } \left[\tilde{E}_C^{WORST} = (E_{C1}^{WORST}, E_{C2}^{WORST}, E_{C3}^{WORST}) \approx \left[\frac{(17900, 18260, 18400)(u_1^L, u_1^M, u_1^U) + (3.3, 4.3, 4.0)u_2}{(2510, 2570, 2610)v_1 + (2360,2360,2360)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16.

Optimistic fuzzy CCR DEA model-4.19

$$\text{Maximize } \left[\tilde{E}_D^{BEST} = (E_{D1}^{BEST}, E_{D2}^{BEST}, E_{D3}^{BEST}) \approx \left[\frac{(14970, 15270, 15400)(u_1^L, u_1^M, u_1^U) + (2.7, 3.7, 4.6)u_2}{(2300, 2350, 2400)v_1 + (2020,2020,2020)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16

Pessimistic fuzzy CCR DEA model-4.19

$$\text{Minimize}\left[\tilde{E}_D^{WORST} = (E_{D1}^{WORST}, E_{D2}^{WORST}, E_{D3}^{WORST})\right.$$

$$\left.\approx\left[\frac{(14970, 15270, 15400)(u_1^L, u_1^M, u_1^U) + (2.7, 3.7, 4.6)u_2}{(2300, 2350, 2400)v_1 + (2020,2020,2020)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16

Optimistic fuzzy CCR DEA model-4.20

$$\text{Maximize}\left[\tilde{E}_E^{BEST} = (E_{E1}^{BEST}, E_{E2}^{BEST}, E_{E3}^{BEST})\right.$$

$$\left.\approx\left[\frac{(13980, 14260, 14330)(u_1^L, u_1^M, u_1^U) + (1.0, 1.8, 2.7)u_2}{(1480, 1520, 1560)v_1 + (1550,150,150)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16

Pessimistic fuzzy CCR DEA model-4.20

$$\text{Minimize}\left[\tilde{E}_E^{WORST} = (E_{E1}^{WORST}, E_{E2}^{WORST}, E_{E3}^{WORST})\right.$$

$$\left.\approx\left[\frac{(13980, 14260, 14330)(u_1^L, u_1^M, u_1^U) + (1.0, 1.8, 2.7)u_2}{(1480, 1520, 1560)v_1 + (1550,150,150)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16

Optimistic fuzzy CCR DEA model-4.22

$$\text{Maximize}\left[\tilde{E}_F^{BEST} = (E_{F1}^{BEST}, E_{F2}^{BEST}, E_{F3}^{BEST})\right.$$

$$\left.\approx \left[\frac{(14030, 14310, 14400)(u_1^L, u_1^M, u_1^U) + (1.6, 2.6, 3.6)u_2}{(1990, 2030, 2100)v_1 + (1760,1760,1760)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16

Pessmistic fuzzy CCR DEA model-4.22

$$\text{Minimize}\left[\tilde{E}_F^{WORST} = (E_{F1}^{WORST}, E_{F2}^{WORST}, E_{F3}^{WORST})\right.$$

$$\left.\approx \left[\frac{(14030, 14310, 14400)(u_1^L, u_1^M, u_1^U) + (1.6, 2.6, 3.6)u_2}{(1990, 2030, 2100)v_1 + (1760,1760,1760)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16

Optimistic fuzzy CCR DEA model-4.23

$$\text{Maximize}\left[\tilde{E}_G^{BEST} = (E_{G1}^{BEST}, E_{G2}^{BEST}, E_{G3}^{BEST})\right.$$

$$\left.\approx \left[\frac{(16540, 16870, 17000)(u_1^L, u_1^M, u_1^U) + (2.4, 3.4, 4.4)u_2}{(2200, 2260, 2300)v_1 + (1980,1980,1980)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16

Pessimistic fuzzy CCR DEA model-4.23

$$\text{Minimize}\left[\tilde{E}_G^{WORST} = (E_{G1}^{WORST}, E_{G2}^{WORST}, E_{G3}^{WORST})\right.$$

$$\left.\approx \left[\frac{(16540, 16870, 17000)(u_1^L, u_1^M, u_1^U) + (2.4, 3.4, 4.4)u_2}{(2200, 2260, 2300)v_1 + (1980,1980,1980)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16

Optimistic fuzzy CCR DEA model-4.23

$$\text{Maximize}\left[\tilde{E}_H^{BEST} = (E_{H1}^{BEST}, E_{H2}^{BEST}, E_{H3}^{BEST})\right.$$

$$\left.\approx \left[\frac{(17600, 17960, 18100)(u_1^L, u_1^M, u_1^U) + (2.6, 3.6, 4.6)u_2}{(2400, 2460, 2520)v_1 + (2250,2250,2250)v_2}\right]\right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.16

Pessimistic fuzzy CCR DEA model-4.23

$$\text{Minimize}\left[\tilde{E}_H^{WORST} = (E_{H1}^{WORST}, E_{H2}^{WORST}, E_{H3}^{WORST})\right.$$

$$\left.\approx \left[\frac{(17600, 17960, 18100)(u_1^L, u_1^M, u_1^U) + (2.6, 3.6, 4.6)u_2}{(2400, 2460, 2520)v_1 + (2250,2250,2250)v_2}\right]\right]$$

Subject to

All the constraints of pessimistic fuzzy CCR DEA model-4.16

4.6.2 Exact relative geometric crisp efficiency of DMU$_A$

Using the method, proposed in Section 4.5, the exact best relative fuzzy efficiency as well as worst relative fuzzy efficiency and hence relative geometric crisp efficiency of DMU$_A$ can be obtained as follows:

Step 1: Using the product of triangular fuzzy numbers, defined in Section 2.1.2.1, the optimistic fuzzy CCR DEA model-4.16 and pessimistic fuzzy CCR DEA model-4.16 can be transformed into optimistic fuzzy CCR DEA model-4.24 and pessimistic fuzzy CCR DEA model-4.24 respectively.

Optimistic fuzzy CCR DEA model-4.24

$$\text{Maximize }\left[\tilde{E}_A^{BEST} = (E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST})\right.$$

$$\left.\approx \left[\frac{(14500u_1^L + 3.1u_2^L, 14790u_1^L + 4.1u_2^L, 14860u_1^L + 4.9u_2^L)}{(2120v_1^L + 4.9v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U)}\right]\right]$$

Subject to

$$(14500u_1^L + 3.1u_2^L, 14790u_1^M + 4.1u_2^M, 14860u_1^U + 4.9u_2^U)$$
$$\preccurlyeq (2120v_1^L + 1870v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U)$$

$$(12470u_1^L + 1.2u_2^L, 12720u_1^M + 2.1u_2^M, 12790u_1^U + 3.0u_2^U)$$
$$\preccurlyeq (1420v_1^L + 1340v_2^L, 1460v_1^M + 1340v_2^M, 1500v_1^U + 1340v_2^U)$$

$$(17900u_1^L + 3.3u_2, 18260u_1^M + 4.3u_2^M, 18400u_1^U + 4.0u_2^U)$$
$$\preccurlyeq (2510v_1^L + 2360v_2^L, 2570v_1^M + 2360v_2^M, 2610v_1^U + 2360v_2^U)$$

$$(14970u_1^L + 2.7u_2^L, 15270u_1^M + 3.7u_2^M, 15400u_1^U + 4.6u_2^U)$$
$$\preccurlyeq (2300v_1^L + 2020v_2^L, 2350v_1^M + 2020v_2^M, 2400v_1^U + 2020v_2^U)$$

$$(13980u_1^L + 1.0u_2^L, 14260u_1^M + 1.8u_2^M, 14330u_1^U + 2.7u_2^U)$$
$$\preccurlyeq (1480v_1^L + 1550v_2^L, 1520v_1^M + 1550v_2^M, 1560v_1^U + 1550v_2^U)$$
$$(14030u_1^L + 1.6u_2^L, 14310u_1^M + 2.6u_2^M, 14400u_1^U + 3.6u_2^U)$$
$$\preccurlyeq (1990v_1^L + 1760v_2^L, 2030v_1^M + 1760v_2^M, 2100v_1^U + 1760v_2^U)$$
$$(16540u_1^L + 2.4u_2^L, 16870u_1^M + 3.4u_2^M, 17000u_1^U + 4.4u_2^U)$$
$$\preccurlyeq (2200v_1^L + 1980v_2^L, 2260v_1^M + 1980v_2^M, 2300v_1^U + 1980v_2^U)$$
$$(17600u_1^L + 2.6u_2^L, 17960u_1^M + 3.6u_2^M, 18100u_1^U + 4.6u_2^U)$$
$$\preccurlyeq (2400v_1^L + 2250v_2^L, 2460v_1^M + 2250v_2^M, 2520v_1^U + 2250v_2^U),$$
$$0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, \qquad i = 1,2, \qquad r = 1,2,$$

Pessimistic fuzzy CCR DEA model-4.24

$$\text{Minimize}\left[\tilde{E}_A^{WORST} = (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST})\right.$$
$$\left.\approx \left[\frac{(14500u_1^L + 3.1u_2^L, 14790u_1^L + 4.1u_2^L, 14860u_1^L + 4.9u_2^L)}{(2120v_1^L + 4.9v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U)}\right]\right]$$

Subject to

$$(14500u_1^L + 3.1u_2^L, 14790u_1^M + 4.1u_2^M, 14860u_1^U + 4.9u_2^U)$$
$$\succcurlyeq (2120v_1^L + 1870v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U)$$
$$(12470u_1^L + 1.2u_2^L, 12720u_1^M + 2.1u_2^M, 12790u_1^U + 3.0u_2^U)$$
$$\succcurlyeq (1420v_1^L + 1340v_2^L, 1460v_1^M + 1340v_2^M, 1500v_1^U + 1340v_2^U)$$
$$(17900u_1^L + 3.3u_2, 18260u_1^M + 4.3u_2^M, 18400u_1^U + 4.0u_2^U)$$
$$\succcurlyeq (2510v_1^L + 2360v_2^L, 2570v_1^M + 2360v_2^M, 2610v_1^U + 2360v_2^U)$$
$$(14970u_1^L + 2.7u_2^L, 15270u_1^M + 3.7u_2^M, 15400u_1^U + 4.6u_2^U)$$
$$\succcurlyeq (2300v_1^L + 2020v_2^L, 2350v_1^M + 2020v_2^M, 2400v_1^U + 2020v_2^U)$$

$$(13980u_1^L + 1.0u_2^L, 14260u_1^M + 1.8u_2^M, 14330u_1^U + 2.7u_2^U)$$
$$\succcurlyeq (1480v_1^L + 1550v_2^L, 1520v_1^M + 1550v_2^M, 1560v_1^U + 1550v_2^U)$$
$$(14030u_1^L + 1.6u_2^L, 14310u_1^M + 2.6u_2^M, 14400u_1^U + 3.6u_2^U)$$
$$\succcurlyeq (1990v_1^L + 1760v_2^L, 2030v_1^M + 1760v_2^M, 2100v_1^U + 1760v_2^U)$$
$$(16540u_1^L + 2.4u_2^L, 16870u_1^M + 3.4u_2^M, 17000u_1^U + 4.4u_2^U)$$
$$\succcurlyeq (2200v_1^L + 1980v_2^L, 2260v_1^M + 1980v_2^M, 2300v_1^U + 1980v_2^U)$$
$$(17600u_1^L + 2.6u_2^L, 17960u_1^M + 3.6u_2^M, 18100u_1^U + 4.6u_2^U)$$
$$\succcurlyeq (2400v_1^L + 2250v_2^L, 2460v_1^M + 2250v_2^M, 2520v_1^U + 2250v_2^U),$$
$$0 \le u_r^L \le u_r^M \le u_r^N \le u_r^U; 0 \le v_i^L \le v_i^M \le v_i^N \le v_i^U, \quad i = 1,2, \quad r = 1,2,$$

Step 2: The optimistic fuzzy CCR DEA model-4.24 and pessimistic fuzzy CCR DEA model-4.24 can be transformed into optimistic fuzzy CCR DEA model-4.25 and pessimistic fuzzy CCR DEA model-4.25 respectively.

Optimistic fuzzy CCR DEA model-4.25

$$\text{Maximize} \left[\tilde{E}_A^{BEST} = (E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST}) \right.$$
$$\left. \approx \left[\frac{(14500u_1^L + 3.1u_2^L, 14790u_1^L + 4.1u_2^L, 14860u_1^L + 4.9u_2^L)}{(2120v_1^L + 4.9v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U)} \right] \right]$$

Subject to

$$14500u_1^L + 3.1u_2^L \le 2120v_1^L + 1870v_2^L, \quad 14790u_1^M + 4.1u_2^M \le 2170v_1^M + 1870v_2^M,$$
$$14860u_1^U + 4.9u_2^U \le 2210v_1^U + 1870v_2^U, \quad 12470u_1^L + 1.2u_2^L \le 1420v_1^L + 1340v_2^L,$$
$$12720u_1^M + 2.1u_2^M \le 1460v_1^M + 1340v_2^M, \quad 12790u_1^U + 3.0u_2^U \le 1500v_1^U + 1340v_2^U,$$
$$17900u_1^L + 3.3u_2 \le 2510v_1^L + 2360v_2^L, \quad 18260u_1^M + 4.3u_2^M \le 2570v_1^M + 2360v_2^M,$$

$18400u_1^U + 4.0u_2^U \leq 2610v_1^U + 2360v_2^U, \qquad 14970u_1^L + 2.7u_2^L \leq 2300v_1^L + 2020v_2^L,$

$15270u_1^M + 3.7u_2^M \leq 2350v_1^M + 2020v_2^M, \qquad 15400u_1^U + 4.6u_2^U \leq 2400v_1^U + 2020v_2^U,$

$13980u_1^L + 1.0u_2^L \leq 1480v_1^L + 1550v_2^L, \qquad 14260u_1^M + 1.8u_2^M \leq 1520v_1^M + 1550v_2^M,$

$14330u_1^U + 2.7u_2^U \leq 1560v_1^U + 1550v_2^U, \qquad 14030u_1^L + 1.6u_2^L \leq 1990v_1^L + 1760v_2^L,$

$14310u_1^M + 2.6u_2^M \leq 2030v_1^M + 1760v_2^M, \qquad 14400u_1^U + 3.6u_2^U \leq 2100v_1^U + 1760v_2^U,$

$16540u_1^L + 2.4u_2^L \leq 2200v_1^L + 1980v_2^L, \qquad 16870u_1^M + 3.4u_2^M \leq 2260v_1^M + 1980v_2^M,$

$17000u_1^U + 4.4u_2^U \leq 2300v_1^U + 1980v_2^U, \qquad 17600u_1^L + 2.6u_2^L \leq 2400v_1^L + 2250v_2^L,$

$17960u_1^M + 3.6u_2^M \leq 2460v_1^M + 2250v_2^M, \qquad 18100u_1^U + 4.6u_2^U \leq 2520v_1^U + 2250v_2^U.$

$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U, \qquad i = 1,2, \qquad r = 1,2,$

Pessimistic fuzzy CCR DEA model-4.25

$$\text{Minimize} \left[\tilde{E}_A^{WORST} = (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST}) \right.$$

$$\left. \approx \left[\frac{(14500u_1^L + 3.1u_2^L, 14790u_1^L + 4.1u_2^L, 14860u_1^L + 4.9u_2^L)}{(2120v_1^L + 4.9v_2^L, 2170v_1^M + 1870v_2^M, 2210v_1^U + 1870v_2^U)} \right] \right]$$

Subject to

$14500u_1^L + 3.1u_2^L \geq 2120v_1^L + 1870v_2^L, \qquad 14790u_1^M + 4.1u_2^M \geq 2170v_1^M + 1870v_2^M,$

$14860u_1^U + 4.9u_2^U \geq 2210v_1^U + 1870v_2^U, \qquad 12470u_1^L + 1.2u_2^L \geq 1420v_1^L + 1340v_2^L,$

$12720u_1^M + 2.1u_2^M \geq 1460v_1^M + 1340v_2^M, \qquad 12790u_1^U + 3.0u_2^U \geq 1500v_1^U + 1340v_2^U,$

$17900u_1^L + 3.3u_2 \geq 2510v_1^L + 2360v_2^L, \qquad 18260u_1^M + 4.3u_2^M \geq 2570v_1^M + 2360v_2^M,$

$18400u_1^U + 4.0u_2^U \geq 2610v_1^U + 2360v_2^U, \qquad 14970u_1^L + 2.7u_2^L \geq 2300v_1^L + 2020v_2^L,$

$15270u_1^M + 3.7u_2^M \geq 2350v_1^M + 2020v_2^M, \qquad 15400u_1^U + 4.6u_2^U \geq 2400v_1^U + 2020v_2^U,$

$13980u_1^L + 1.0u_2^L \geq 1480v_1^L + 1550v_2^L, \qquad 14260u_1^M + 1.8u_2^M \geq 1520v_1^M + 1550v_2^M,$

$14330u_1^U + 2.7u_2^U \geq 1560v_1^U + 1550v_2^U, \qquad 14030u_1^L + 1.6u_2^L \geq 1990v_1^L + 1760v_2^L,$

$$14310u_1^M + 2.6u_2^M \geq 2030v_1^M + 1760v_2^M, \qquad 14400u_1^U + 3.6u_2^U \geq 2100v_1^U + 1760v_2^U,$$

$$16540u_1^L + 2.4u_2^L \geq 2200v_1^L + 1980v_2^L, \qquad 16870u_1^M + 3.4u_2^M \geq 2260v_1^M + 1980v_2^M,$$

$$17000u_1^U + 4.4u_2^U \geq 2300v_1^U + 1980v_2^U, \qquad 17600u_1^L + 2.6u_2^L \geq 2400v_1^L + 2250v_2^L,$$

$$17960u_1^M + 3.6u_2^M \geq 2460v_1^M + 2250v_2^M, \qquad 18100u_1^U + 4.6u_2^U \geq 2520v_1^U + 2250v_2^U.$$

$$0 \leq u_r^L \leq u_r^M \leq u_r^N \leq u_r^U; 0 \leq v_i^L \leq v_i^M \leq v_i^N \leq v_i^U, \qquad i = 1,2, \qquad r = 1,2,$$

Step 3: The optimistic fuzzy CCR DEA model-4.25 and pessimistic fuzzy CCR DEA model-4.25 can be transformed into optimistic fuzzy CCR DEA model-4.26 and pessimistic fuzzy CCR DEA model-4.26 respectively.

Optimistic fuzzy CCR DEA model-4.26

$$\text{Maximize} \left[\tilde{E}_A^{BEST} = (E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST}) \right.$$

$$\left. \approx \left[\left(\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U}, \frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M}, \frac{14860u_1^U + 4.9u_2^U}{2120v_1^L + 4.9v_2^L} \right) \right] \right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Pessimistic fuzzy CCR DEA model-4.26

$$\text{Minimize} \left[\tilde{E}_A^{WORST} = (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST}) \right.$$

$$\left. \approx \left[\left(\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U}, \frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M}, \frac{14860u_1^U + 4.9u_2^U}{2120v_1^L + 4.9v_2^L} \right) \right] \right]$$

Subject to

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Step 4: The fuzzy optimal value $\left(\tilde{E}_A^{BEST} \approx \left(\left((E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST})\right)\right)\right)$, representing the best relative fuzzy efficiency of A^{th} DMU, as well as the fuzzy optimal value $\left(\tilde{E}_A^{WORST} \approx (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST})\right)$, representing the worst relative fuzzy efficiency of A^{th} DMU, can be obtained by solving the optimistic fuzzy CCR DEA model-4.26 and pessimistic fuzzy CCR DEA model-4.26 as follows:

Step 4(a) The optimal value (E_{A1}^{BEST}) and (E_{A1}^{WORST}) of the optimistic crisp CCR DEA model-4.27a and pessimistic CCR DEA model-4.27a by solving optimistic crisp CCR DEA model-4.27b and pessimistic CCR DEA model-4.27b equivalent to optimistic crisp CCR DEA model-4.27a and pessimistic CCR DEA model-4.27a are 0.812 and 1 respectively.

Optimistic crisp CCR DEA model-4.27a

Maximize $\left[E_{A1}^{BEST} = \dfrac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U}\right]$

Subject to

$$\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U} \le 1,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.16.

Optimistic crisp CCR DEA model-4.27b

Maximize $[E_{A1}^{BEST} = 14500u_1^L + 3.1u_2^L]$

Subject to

$$2210v_1^U + 1870v_2^U = 1$$

$14500u_1^L + 3.1u_2^L \leq 2210v_1^U + 1870v_2^U,$

and

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Pessimistic crisp CCR DEA model-4.27a

Minimize $\left[E_{A1}^{WORST} = \dfrac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U}\right]$

Subject to

$\dfrac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U} \geq 1,$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.25.

Pessimistic crisp CCR DEA model-4.27b

Minimize $[E_{A1}^{WORST} = 14500u_1^L + 3.1u_2^L]$

Subject to

$2210v_1^U + 1870v_2^U = 1$

$14500u_1^L + 3.1u_2^L \geq 2210v_1^U + 1870v_2^U,$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.25.

Step 4(b) The optimal value (E_{A2}^{BEST}) and (E_{A2}^{WORST}) of the optimistic crisp CCR DEA model-4.28a and pessimistic CCR DEA model-4.28a by solving optimistic crisp CCR DEA model-4.28b and pessimistic CCR DEA model-4.28b equivalent to optimistic crisp CCR DEA model-4.28a and pessimistic CCR DEA model-4.28a are 0.833 and 1.046 respectively.

Optimistic crisp CCR DEA model-4.28a

Maximize $\left[E_{A2}^{BEST} = \frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M}\right]$

Subject to

$$\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U} = 0.812$$

$$0.812 \leq \frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M} \leq 1$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Optimistic crisp CCR DEA model-4.28b

Maximize $[E_{A2}^{BEST} = 14790u_1^M + 4.1u_2^M]$

Subject to

$2170v_1^M + 1870v_2^M = 1,$

$14500u_1^L + 3.1u_2^L = (0.812)(2210v_1^U + 1870v_2^U),$

$(0.812)(2170v_1^M + 1870v_2^M) \leq 14790u_1^M + 4.1u_2^M \leq 2170v_1^M + 1870v_2^M$

and

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Pessimistic crisp CCR DEA model-4.28a

Minimize $\left[E_{A2}^{WORST} = \frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M}\right]$

Subject to

$$\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U} = 1,$$

$$1 \leq \frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M}$$

All the constraints of pessimistic fuzzy CCR DEA model-4.25.

Pessimistic crisp CCR DEA model-4.28b

Minimize$[E_{A2}^{BEST} = 14790u_1^M + 4.1u_2^M]$

Subject to

$2170v_1^M + 1870v_2^M = 1,$

$14500u_1^L + 3.1u_2^L = 2210v_1^U + 1870v_2^U,$

$2170v_1^M + 1870v_2^M \leq 14790u_1^M + 4.1u_2^M.$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.25.

Step 4(c) The optimal value (E_{A2}^{BEST}) and (E_{A2}^{WORST}) of the optimistic crisp CCR DEA model-4.29a and pessimistic CCR DEA model-4.29a by solving optimistic crisp CCR DEA model-4.29b and pessimistic CCR DEA model-4.29b equivalent to optimistic crisp CCR DEA model-4.29a and pessimistic CCR DEA model-4.29a are 0.854 and 1.072 respectively.

Optimistic crisp CCR DEA model-4.29a

Maximize $\left[E_{A3}^{BEST} = \frac{14860u_1^U + 4.9u_2^U}{2120v_1^L + 4.9v_2^L}\right]$

Subject to

$$\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U} = 0.812$$

$$\frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M} = 0.833$$

$$0.833 \leq \frac{14860u_1^U + 4.9u_2^U}{2120v_1^L + 4.9v_2^L} \leq 1,$$

and

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Optimistic crisp CCR DEA model-4.29b

Maximize $[E_{A3}^{BEST} = 14860u_1^U + 4.9u_2^U]$

Subject to

$2120v_1^L + 4.9v_2^L = 1,$

$14500u_1^L + 3.1u_2^L = (0.812)(2210v_1^U + 1870v_2^U)$

$14790u_1^M + 4.1u_2^M = (0.833)(2170v_1^M + 1870v_2^M),$

$(0.833)(2120v_1^L + 4.9v_2^L) \leq 14860u_1^U + 4.9u_2^U \leq 2120v_1^L + 4.9v_2^L.$

and

All the constraints of optimistic fuzzy CCR DEA model-4.25.

Pessimistic crisp CCR DEA model-4.29b

Minimize $\left[E_{A3}^{WORST} = \frac{14860u_1^U + 4.9u_2^U}{2120v_1^L + 4.9v_2^L}\right]$

Subject to

$$\frac{14500u_1^L + 3.1u_2^L}{2210v_1^U + 1870v_2^U} = 1,$$

$$\frac{14790u_1^M + 4.1u_2^M}{2170v_1^M + 1870v_2^M} = 1.046$$

$$1.046 \leq \frac{14860u_1^U + 4.9u_2^U}{2120v_1^L + 4.9v_2^L},$$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.25.

Pessimistic crisp CCR DEA model-4.29b

Minimize $[E_{A3}^{WORST} = 14860u_1^U + 4.9u_2^U]$

Subject to

$2120v_1^L + 4.9v_2^L = 1,$

$14500u_1^L + 3.1u_2^L = 2210v_1^U + 1870v_2^U$

$14790u_1^M + 4.1u_2^M = (1.046)(2170v_1^M + 1870v_2^M),$

$(1.046)(2120v_1^L + 4.9v_2^L) \leq 14860u_1^U + 4.9u_2^U \leq 2120v_1^L + 4.9v_2^L.$

and

All the constraints of pessimistic fuzzy CCR DEA model-4.25.

Step 5: Using the values of $E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST}$ and $E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST}$, obtained in Step (4a) to Step (4c), the fuzzy optimal value $\left(\tilde{E}_A^{BEST} = (E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST})\right)$ of optimistic fuzzy DEA model-4.16, representing the best relative fuzzy efficiency of A^{th} DMU, is $\left(\tilde{E}_A^{BEST} = (E_{A1}^{BEST}, E_{A2}^{BEST}, E_{A3}^{BEST}) = (0.812, 0.833, 0.854)\right)$, as well as pessimistic fuzzy optimal value $\left(\tilde{E}_A^{WORST} = (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST})\right)$ of pessimistic fuzzy DEA model-4.16, representing the worst relative fuzzy efficiency of A^{th} DMU, is $\left(\tilde{E}_A^{WORST} = (E_{A1}^{WORST}, E_{A2}^{WORST}, E_{A3}^{WORST}) = (1, 1.046, 1.072)\right)$.

Step 6: The crisp optimal value$\left(E_A^{BEST} = \Re(\tilde{E}_A^{BEST}) = \Re(0.812, 0.833, 0.854)\right)$, representing the best relative crisp efficiency of A^{th} DMU is 0.833.

Step 7: The crisp optimal value$\left(E_A^{WORST} = \Re\left(\tilde{E}_A^{WORST}\right) = \Re(1,1.046,1.072)\right)$, representing the worst relative crisp efficiency of A^{th} DMU is 1.041.

Step 8: The geometric average crisp efficiency $E_A^{GEOMETRIC}$ of A^{th} DMU by putting the values E_A^{BEST} and E_A^{WORST}, obtained in Step 5 and Step 6, in $E_A^{GEOMETRIC} = \sqrt{E_A^{BEST} \times E_A^{WORST}}$ is 0.931.

4.6.3 Results

The exact best relative fuzzy efficiency, exact worst relative fuzzy efficiency and relative geometric crisp efficiency of all the DMUs evaluated on solving fuzzy CCR DEA by the proposed method are shown in Table 4.2.

Table 4.2 Exact best relative fuzzy efficiency, exact worst relative fuzzy efficiency and relative geometric crisp efficiency of DMUs.

$\mathbf{DMU}_j$	**Best relative fuzzy efficiency**	**Worst relative fuzzy efficiency**	**Relative geometric crisp efficiency**
A	$(.81238, .833217, .85407)$	$(1, 1.04625, 1.07227))$	.931
B	$(.97498, 1,1)$	$(1, 1.12689, 1.28793)$	1.062
C	$(.79661, .815201, .83732)$	$(1, 1.01738, 1.055204)$	.913
D	$(.77643, .79636, 81927)$	$(1, 1, 1.028724)$	.898
E	$(.97303, 1, 1)$	$(1, 1, 1)$	.996
F	$(.83517, .85653, .879205)$	$(1, 1, 1.104026)$	.938
G	$(.87519, .89757, .92263)$	$(1, 1.07384, 1.1585)$	.938
H	$(.819529, .84089, .86447)$	$(1, 1.009602, 1.08548)$	.929

It is obvious from Table 3.2 that $\Re\left(\tilde{E}_B\right) > R\left(\tilde{E}_E\right) > R\left(\tilde{E}_G\right) = \Re\left(\tilde{E}_F\right) > R\left(\tilde{E}_H\right) > R\left(\tilde{E}_A\right) > R\left(\tilde{E}_C\right) > R\left(\tilde{E}_D\right)$. Therefore, $\tilde{E}_B \succ \tilde{E}_E \succ \tilde{E}_G \approx \tilde{E}_F \succ \tilde{E}_A \succ \tilde{E}_H \succ \tilde{E}_C \succ \tilde{E}_D$.

4.7 Conclusions

On the basis the present study, it can be concluded that there are flaws in the optimistic as well as pessimistic fuzzy CCR DEA model and in the method proposed by Wang et al. [105] and hence, neither the fuzzy CCR DEA model proposed by, Wang et al. [105] nor the method proposed by Wang et al. [105] should be used for evaluating the best relative geometric crisp efficiency of DMUs. Also, to resolve the flaws of the fuzzy CCR DEA models, new fuzzy CCR DEA models are proposed. Further, a new approach is proposed to solve the proposed fuzzy CCR DEA models for evaluating the best relative geometric crisp efficiency of DMUs.

Chapter 5

A New Approach for Solving Proposed Intuitionistic Fully Fuzzy CCR DEA Model[4]

Puri and Yadav [83] proposed an optimistic as well as pessimistic intuitionistic fuzzy CCR DEA model and an approach to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). In this chapter, it is shown that the intuitionistic fuzzy CCR models, proposed by Puri and Yadav [83] are not valid and hence cannot be used to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of decision making units (DMUs). To resolve the flaws of the intuitionistic fuzzy CCR DEA model, proposed by Puri and Yadav [83], new intuitionistic fuzzy CCR DEA models are proposed. Also, a new approach is proposed to solve the proposed intuitionistic fuzzy CCR DEA models for evaluating the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs.

5.1 Preliminaries

Puri and Yadav [83] pointed out that the main characteristic of the fuzzy set theory is that in fuzzy sets/numbers, the degree of non-membership of an element is equal to one minus the

[4] The contents of this chapter are communicated for the possible publication in Expert Systems with Applications.

degree of its membership and thus, the sum of membership and non-membership degrees of an element is equal to one. However, in real life applications, we deal with the information which is sometimes vague or inexact or insufficient and therefore, there is possibility that the sum of the membership and non-membership degrees of an element may come out to be less than one. It means there remains some degree of hesitation. Certainly, fuzzy sets theory is not appropriate to deal with such type of problems/situations; rather intuitionistic fuzzy set (IFS) theory is more suitable. Intuitionistic fuzzy set (IFS) [5] is an extension of fuzzy set and have been found to be highly useful to deal with vagueness. IFS considers both the degrees of membership (acceptance) and non-membership (rejection) of an element such that the sum of both values is less than or equal to 1.

5.1.1 Basic definitions

In this section, some basic definitions are reviewed [83].

Definition 5.1 Let X be a universe of disclosure. Then an IFS $\tilde{A}$ in X is given by $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x),\ v_{\tilde{A}}(x)\rangle : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0,1]$ and $v_{\tilde{A}} : X \to [0,1]$ with the condition, $0 \leq \mu_{\tilde{A}}(x) + v_{\tilde{A}}(x) \leq 1, \forall x \in X$ represent the membership and non-membership functions respectively and the numbers $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ represent the membership degree and non-member-ship degree of the element x being in $\tilde{A}$. Further, $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - v_{\tilde{A}}(x), \forall x \in X$ is a hesitation degree of x being in $\tilde{A}$. It is obvious that $0 \leq \pi_{\tilde{A}}(x) \leq 1, \forall x \in X$. If $\pi_{\tilde{A}}(x) = 0, \forall x \in X$ then $\tilde{A}$ becomes a fuzzy number.

Definition 5.2 Let $\tilde{A}$ be an IFS defined on the set of real numbers $\mathbb{R}$ with its membership function $\mu_{\tilde{A}}$ and non-membership function $v_{\tilde{A}}$, then $\tilde{A}$ is said to be an intuitionistic fuzzy numbers (IFN) if

(i) It is normal, i.e., there is any $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$ (so $v_{\tilde{A}}(x_0) = 0$).

(ii) It is convex for $\mu_{\tilde{A}}(x)$, i.e., $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$, $\forall x_1, x_2 \in \mathbb{R}$, $\lambda \in [0,1]$.

(iii) It is concave for $v_{\tilde{A}^1}(x)$, i.e., $v_{\tilde{A}}((\lambda x_1 + (1-\lambda)x_2) \leq \max(v_{\tilde{A}}(x_1), v_{\tilde{A}}(x_2))$, $\forall x_1, x_2 \in \mathbb{R}$, $\lambda \in [0,1]$.

Definition 5.3 An intuitionistic fuzzy number $\tilde{A}$ is said to be a triangular intuitionistic fuzzy number if the membership function $\mu_{\tilde{A}}$ and non-membership function $v_{\tilde{A}}$ is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2, \\ 1, & x = a_{2,}, \\ \frac{x-a_3}{a_2-a_3}, & a_2 \leq x < a_3, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } v_{\tilde{A}}(x) = \begin{cases} \frac{x-a_2}{a_4-a_2}, & a_4 \leq x < a_2, \\ 0, & x = a_{2,}, \\ \frac{x-a_2}{a_5-a_2} & a_2 \leq x < a_5, \\ 1, & \text{otherwise.} \end{cases}$$

where $a_4, a_1, a_2, a_3, a_5 \in \mathbb{R}$ such that $a_4 \leq a_1 \leq a_2 \leq a_3 \leq a_5$. It is denoted by $(a_1, a_2, a_3; a_4, a_2, a_5)$.

A triangular intuitionistic fuzzy number $\tilde{A} = (a_1, a_2, a_3; a_4, a_2, a_5)$ becomes triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ if $a_1 = a_4, a_3 = a_5$ and $v_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x), \forall x \in \mathbb{R}$ and it becomes a real number a if $a_4 = a_1 = a_2 = a_3 = a_5 = a$.

Definition 5.4 A triangular intuitionistic fuzzy number $\tilde{A} = (a_1, a_2, a_3; a_4, a_2, a_5)$ is said to be non-negative triangular intuitionistic fuzzy number, represented by $\tilde{A} \succcurlyeq (0,0,0; 0,0,0)$, if and only if $a_4 \geq 0$.

5.1.2 Arithmetic operations of triangular intuitionistic fuzzy numbers

Let $\tilde{A} = (a_1, a_2, a_3; a_4, a_2, a_5)$ and $\tilde{B} = (b_1, b_2, b_3; b_4, b_2, b_5)$ be two triangular intuitionistic fuzzy numbers. Then, *(i)* $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3;\ a_4 + b_4, a_2 + b_2, a_5 + b_5)$.

(ii) $\tilde{A} \otimes \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3;\ a_4 b_4, a_2 b_2, a_5\, b_5)$ for $\tilde{A}$ and $\tilde{B}$ are non-negative triangular intuitionistic fuzzy numbers.

(iii) $k\tilde{A} = \begin{cases} (ka_1, ka_2, ka_3; ka_4, ka_2, ka_5), k \geq 0, \\ (kb_3, kb_2, kb_1\, ; kb_5, kb_2, kb_4), k \leq 0. \end{cases}$

(iv) $\frac{\tilde{A}}{\tilde{B}} = \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1};\ \frac{a_4}{b_5}, \frac{a_2}{b_2}, \frac{a_5}{b_4}\right)$, $\tilde{A}^1$ is a non-negative triangular I.F.N and $\tilde{B}$is a positive triangular intuitionistic fuzzy number.

5.1.3. Comparison of triangular intuitionistic fuzzy numbers

Puri and Yadav [83] used the following method for comparing triangular intuitionistic fuzzy numbers.

If $\tilde{A} = (a_1, a_2, a_3; a_4, a_2, a_5)$ and $\tilde{B} = (b_1, b_2, b_3; b_4, b_2, b_5)$ are two triangular intuitionistic fuzzy numbers. Then,

(i) $\tilde{A} \succcurlyeq \tilde{B} \Leftrightarrow \Re(\tilde{A}) \geq \Re(\tilde{B})$

(ii) $\tilde{A} \preccurlyeq \tilde{B} \Leftrightarrow \Re(\tilde{A}) \leq \Re(\tilde{B})$

(iii) $\tilde{A} \approx \tilde{B} \Leftrightarrow \Re(\tilde{A}) = \Re(\tilde{B})$

where,

$\Re(\tilde{A}) = \left(\frac{a_1 + a_3 + 4a_2 + a_4 + a_5}{8}\right)$ and $\Re\left(\tilde{\tilde{B}}\right) = \left(\frac{b_1 + b_3 + 4b_2 + b_4 + b_5}{8}\right)$

5.2 Existing intuitionistic fuzzy DEA models

Puri and Yadav [83] proposed the optimistic intuitionistic fuzzy CCR DEA model-5.1 and pessimistic intuitionistic fuzzy CCR DEA model-5.1, respectively, to evaluate the best relative intuitionistic fuzzy efficiency and the worst relative intuitionistic fuzzy efficiency of p^{th} DMU.

Optimistic intuitionistic fuzzy CCR DEA model-5.1

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}; x_3^{ip};\ x_4^{ip}, x_2^{ip}, x_5^{ip}\right) \approx (1,1,1;1,1,1),$$

$$\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right)$$

$$-\sum_{i=1}^{m}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ij}, x_2^{ij}, x_3^{ij};\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right) \preccurlyeq (0,0,0;0,0,0), \forall j,$$

$$= 1,2,3, \dots, n,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq \varepsilon, \forall r = 1,2,3, \dots, s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right) \succcurlyeq \varepsilon, \forall i = 1,2,3, \dots, m, \varepsilon > 0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.1

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}; x_3^{ip};\ x_4^{ip}, x_2^{ip}, x_5^{ip}\right) \approx (1,1,1;1,1,1),$$

$$\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rk},y_2^{rk},y_3^{rk};\ y_4^{rk},y_2^{rk},y_5^{rk}\right)$$

$$-\sum_{i=1}^{m}\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\otimes\left(x_1^{ik},x_2^{ik},x_3^{ik};\ x_4^{ik},x_2^{ik},x_5^{ik}\right)\ \succcurlyeq(0,0,0;0,0,0),\forall j,$$

$$=1,2,3,\dots,n,$$

$$(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\succcurlyeq\ \varepsilon,\forall r=1,2,3,\dots,s,$$

$$\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\ \succcurlyeq\varepsilon,\forall i=1,2,3,\dots,m,\varepsilon>0.$$

5.3 Existing method

Puri and Yadav [83] proposed the following method to evaluate the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of DMUs.

Step 1: Using the arithmetic operations of triangular intuitionistic fuzzy numbers, presented in Section 5.1.1, the optimistic intuitionistic fuzzy CCR DEA model-5.1 and the pessimistic intuitionistic fuzzy CCR DEA model-5.1 can be transformed into the optimistic intuitionistic fuzzy CCR DEA model-5.2 and the pessimistic intuitionistic fuzzy CCR DEA model-5.2 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.2

$$\text{Maximize }\left[\tilde{E}_p^{BEST}\approx\sum_{r=1}^{s}\left(u_1^r y_1^{rp},u_2^r y_2^{rp},u_3^r y_3^{rp};u_4^r y_4^r,u_2^r y_2^{rp},u_5^r y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i x_1^{ip}, v_2^i x_2^{ip}, v_3^i x_3^{ip}; v_4^i x_4^{ip}, v_2^i x_2^{ip}, v_5^i x_5^{ip}\right) \approx (1,1,1;1,1,1),$$

$$\sum_{r=1}^{s}\left(u_1^r y_1^{rk}, u_2^r y_2^{rk}, u_3^r y_3^{rk};\ u_4^r y_4^{rk}, u_2^r y_2^{rk}, u_5^r y_5^{rk}\right) - \sum_{i=1}^{m}\left(v_1^i x_1^{ik}, v_2^i x_2^{ik}, v_3^i x_3^{ik}; v_4^i x_4^{ik}, v_2^i x_2^{ik}, v_5^i x_5^{ik}\right)$$

$$\preccurlyeq (0,0,0;0,0,0), \forall j = 1,2,3, \dots, n,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,3, \dots, s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1,2,3, \dots, m, \varepsilon > 0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.2

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \sum_{r=1}^{s}\left(u_1^r y_1^{rp}, u_2^r y_2^{rp}, u_3^r y_3^{rp}; u_4^r y_4^{r}, u_2^r y_2^{rp}, u_5^r y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i x_1^{ip}, v_2^i x_2^{ip}, v_3^i x_3^{ip}; v_4^i x_4^{ip}, v_2^i x_2^{ip}, v_5^i x_5^{ip}\right) \approx (1,1,1;1,1,1),$$

$$\sum_{r=1}^{s}\left(u_1^r y_1^{rj}, u_2^r y_2^{rj}, u_3^r y_3^{rj};\ u_4^r y_4^{rj}, u_2^r y_2^{rj}, u_5^r y_5^{rj}\right) - \sum_{i=1}^{m}\left(v_1^i x_1^{ij}, v_2^i x_2^{ij}, v_3^i x_3^{ij}; v_4^i x_4^{ij}, v_2^i x_2^{ij}, v_5^i x_5^{ij}\right)$$

$$\succcurlyeq (0,0,0;0,0,0), \forall j = 1,2,3, \dots, n,$$

$$(u_1^r, u_2^r, u_3^r, u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,3, \dots, s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1,2,3, \dots, m, \varepsilon > 0.$$

Step 2: Using Definition 5.4 and the method for comparing triangular intuitionistic fuzzy numbers, presented in Section 5.1.2, the optimistic intuitionistic fuzzy CCR DEA model-5.2 and the pessimistic intuitionistic fuzzy CCR DEA model-5.2 can be transformed into optimistic crisp CCR DEA model-5.3 and the pessimistic crisp CCR DEA model-5.3 respectively.

Optimistic crisp CCR DEA model-5.3

$$\text{Maximize } \left[\Re\left(\tilde{E}_p^{BEST}\right) = E_p^{BEST} = \sum_{r=1}^{s} \Re\left(u_1^r y_1^{rp}, u_2^r y_2^{rp}, u_3^r y_3^{rp}; u_4^r y_4^r, u_2^r y_2^{rp}, u_5^r y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}} \Re\left(v_1^i x_1^{ip}, v_2^i x_2^{ip}, v_3^i x_3^{ip}; v_4^i x_4^{ip}, v_2^i x_2^{ip}, v_5^i x_5^{ip}\right) = \Re(1,1,1;1,1,1),$$

$$\sum_{r=1}^{s} \Re\left(u_1^r y_1^{rk}, u_2^r y_2^{rk};\ u_3^r y_3^{rk}, u_4^r y_4^{rk}, u_2^r y_2^{rk}, u_5^r y_5^{rk}\right)$$

$$- \sum_{i=1}^{m} \Re\left(v_1^i x_1^{ik}, v_2^i x_2^{ik}, v_3^i x_3^{ik}; v_4^i x_4^{ik}, v_2^i x_2^{ik}, v_5^i x_5^{ik}\right) \le \Re(0,0,0;0,0,0), \forall j$$

$$= 1,2,3, \dots, n,$$

$$\varepsilon \le u_4^r \le u_1^r \le u_2^r \le u_3^r \le u_5^r \ \ \forall, r = 1,2,3, \dots, s,$$

$$\varepsilon \le v_4^i \le v_1^i \le v_2^i \le v_3^i \le v_5^i \ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Pessimistic crisp CCR DEA model-5.3

$$\text{Minimize} \left[\Re\left(\tilde{E}_p^{WORST}\right) = E_p^{WORST} = \left(\sum_{r=1}^{s} \Re\left(u_1^r y_1^{rp}, u_2^r y_2^{rp}, u_3^r y_3^{rp}; u_4^r y_4^r, u_2^r y_2^{rp}, u_5^r y_5^{rp}\right)\right)\right]$$

Subject to

$$\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}} \Re\left(v_1^i x_1^{ip}, v_2^i x_2^{ip}, v_3^i x_3^{ip}; v_4^i x_4^{ip}, v_2^i x_2^{ip}, v_5^i x_5^{ip}\right)\right) = \Re(1,1,1;1,1,1),$$

$$\left(\sum_{r=1}^{s} \Re\left(u_1^r y_1^{rk}, u_2^r y_2^{rk}, u_3^r y_3^{rk}, u_4^r y_4^{rk}, u_2^r y_2^{rk}, u_5^r y_5^{rk}\right)\right.$$

$$\left. - \sum_{i=1}^{m} \Re\left(v_1^i x_1^{ik}, v_2^i x_2^{ik}, v_3^i x_3^{ik}; v_4^i x_4^{ik}, v_2^i x_2^{ik}, v_5^i x_5^{ik}\right)\right) \geq \Re(0,0,0;0,0,0), \forall j$$

$$= 1,2,3, \dots, n,$$

$$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r \ \ \forall, r = 1,2,3, \dots, s$$

$$\varepsilon \leq v_4^i \leq v_1^i \leq v_2^i \leq v_3^i \leq v_5^i \ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Step 3: The optimistic crisp CCR DEA model-5.3 and the pessimistic crisp CCR DEA model-5.3 can be transformed into optimistic crisp CCR DEA model-5.4 and the pessimistic crisp CCR DEA model-5.4 respectively.

Optimistic crisp CCR DEA model-5.4

$$\text{Maximize } \left[\Re\left(\tilde{E}_p^{BEST}\right) = E_p^{BEST} = \sum_{r=1}^{s}\left(\frac{u_1^r y_1^{rp} + u_3^r y_3^{rp} + 4u_2^r y_2^{rp} + u_4^r y_4^{rp} + u_5^r y_5^{rp}}{8}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(\frac{v_1^i x_1^{ip} + \ v_2^i x_2^{ip} + \ v_3^i x_3^{ip} + v_4^i x_4^{ip} + v_2^i x_2^{ip} + v_5^i x_5^{ip}}{8}\right) = 1,$$

$$\sum_{r=1}^{s}\left(\frac{u_1^r y_1^{rk} + \ u_3^r y_3^{rk} + 4u_2^r y_2^{rk} + u_4^r y_4^{rk} + u_5^r y_5^{rk}}{8}\right)$$

$$- \sum_{i=1}^{m}\left(\frac{v_1^i x_1^{ik} + v_3^i x_3^{ik} + 4v_2^i x_2^{ik} + v_4^i x_4^{ik} + v_5^i x_5^{ik}}{8}\right) \leq 0, \forall j = 1,2,3, \dots, n,$$

$$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r \ \ \forall, r = 1,2,3, \dots, s$$

$$\varepsilon \le v_4^i \le v_1^i \le v_2^i \le v_3^i \le v_5^i \ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Pessimistic crisp CCR DEA model-5.4

$$\text{Minimize}\left[\Re\left(\tilde{E}_p^{WORST}\right) = E_p^{WORST} = \sum_{r=1}^{s}\left(\frac{u_1^r y_1^{rp} + u_3^r y_3^{rp} + 4u_2^r y_2^{rp} + u_4^r y_4^{rp} + u_5^r y_5^{rp}}{8}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(\frac{v_1^i x_1^{ip} + \ v_2^i x_2^{ip} + \ v_3^i x_3^{ip} + v_4^i x_4^{ip} + v_2^i x_2^{ip} + v_5^i x_5^{ip}}{8}\right) = 1$$

$$\sum_{r=1}^{s}\left(\frac{u_1^r y_1^{rk} + \ u_3^r y_3^{rk} + 4u_2^r y_2^{rk} + u_4^r y_4^{rk} + u_5^r y_5^{rk}}{8}\right)$$

$$-\sum_{i=1}^{m}\left(\frac{v_1^i x_1^{ik} + v_3^i x_3^{ik} + 4v_2^i x_2^{ik} + v_4^i x_4^{ik} + v_5^i x_5^{ik}}{8}\right) \ge 0, \forall j = 1,2,3, \dots, n,$$

$$\varepsilon \le u_4^r \le u_1^r \le u_2^r \le u_3^r \le u_5^r \ \ \forall, r = 1,2,3, \dots, s,$$

$$\varepsilon \le v_4^i \le v_1^i \le v_2^i \le v_3^i \le v_5^i \ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Step 4: Find the optimal value $\left(E_p^{BEST}\right)$, representing the relative best crisp efficiency of p^{th} DMU, by solving optimistic crisp CCR DEA model-5.4.

Step 5: Find the optimal value $\left(E_p^{WORST}\right)$, representing the relative worst crisp efficiency of p^{th} DMU, by solving pessimistic crisp CCR DEA model-5.4.

Step 6: Find the relative geometric crisp efficiency $E_P^{GEOMETRIC}$ of p^{th} DMU by putting the values E_p^{BEST} and E_p^{WORST}, obtained in Step 4 and Step 5, in $E_P^{GEOMETRIC} = \sqrt{E_p^{BEST} \times E_p^{WORST}}$.

5.4 Flaws in the existing intuitionistic fuzzy CCR DEA models

In this section, it is pointed out that Puri and Yadav [83] have considered a mathematical incorrect assumption to obtain the optimistic intuitionistic fuzzy CCR DEA model 5.1 as well as pessimistic intuitionistic fuzzy CCR DEA model 5.1 hence these intuitionistic fuzzy CCR DEA mode, proposed by Puri and Yadav [83], are not valid and cannot be used to evaluate the relative geometric crisp efficiency of DMUs.

5.4.1 Origin of existing intuitionistic fuzzy CCR DEA models

The intuitionistic fuzzy CCR DEA models, proposed by Puri and Yadav [83], can be obtained as follows:

Step 1: If there are n DMUs then the best relative crisp efficiency $\left(E_p^{BEST}\right)$ and the worst relative crisp efficiency $\left(E_p^{WORST}\right)$ of the p^{th} DMU can be obtained by solving the optimistic CCR DEA model-5.5 and pessimistic CCR DEA model-5.5 respectively.

Optimistic CCR DEA model-5.5

$$\text{Maximize}\left[E_p^{BEST}=\frac{\text{Virtual output of } p^{th} \text{ DMU}}{\text{Virtual input of } p^{th} \text{ DMU}}\right]$$

Subject to

$$\text{Virtual output of } j^{th} \text{ DMU} - \text{Virtual input of } j^{th} \text{ DMU} \leq 0, \forall j.$$

Pessimistic crisp CCR DEA model-5.5

$$\text{Minimize}\left[E_p^{WORST}=\frac{\text{Virtual output of } p^{th} \text{ DMU}}{\text{Virtual input of } p^{th} \text{ DMU}}\right]$$

Subject to

Virtual output of j^{th} DMU − Virtual input of j^{th} DMU $\geq 0, \forall j$.

If each DMU have m inputs ($x_{ij};\ i = 1,2,\dots,m; j = 1,2,\dots,n$) and s outputs ($y_{rj}; r = 1,2,\dots s;\ j = 1,2,\dots,n$) then the optimistic crisp CCR DEA model-5.5 and the pessimistic CCR DEA model-5.5 can be transformed into the optimistic crisp CCR DEA model-5.6 and the pessimistic CCR DEA model-5.6 respectively.

Optimistic crisp CCR DEA model-5.6

$$\text{Maximize}\left[E_p^{BEST} = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}\right]$$

Subject to

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad \forall j,$$

$$u_r, v_i \geq 0, \qquad \forall r, i.$$

Pessimistic crisp CCR DEA model-5.6

$$\text{Minimize}\left[E_p^{WORST} = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}\right]$$

Subject to

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0 \qquad \forall j,$$

$$u_r, v_i \geq 0, \qquad \forall r, i.$$

Replacing x_{ij}, y_{rj}, u_r and v_i by intuitionistic fuzzy numbers $\left(x_1^{ij}, x_2^{ij}, x_3^{ij};\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right)$, $\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right)$, $(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)$ and $\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)$respectively, the optimistic crisp CCR DEA model-5.6 and the pessimistic crisp CCR DEA model-5.6 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.7 and pessimistic intuitionistic fuzzy CCR DEA model-5.7 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.7

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \left(\frac{\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}, x_3^{ip};\ x_4^{ip}, x_2^{ip}, x_5^{ip}\right)}\right)\right]$$

Subject to

$$\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right) -$$

$$\sum_{i=1}^{m}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ij}, x_2^{ij}, x_3^{ij};\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right) \preccurlyeq (0,0,0;0,0,0), \forall j = 1,2,3,\dots,n,$$

$$(u_1^r, u_2^r, u_3^r; u_4^r, u_2^r, u_5^r) \succcurlyeq \varepsilon, \forall r = 1,2,3,\dots,s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right) \succcurlyeq \varepsilon, \forall i = 1,2,3,\dots,m, \varepsilon > 0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.7

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(\frac{\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}, x_3^{ip}\ ;\ x_4^{ip}, x_2^{ip}, x_5^{ip}\right)}\right)\right]$$

Subject to

$$\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj}, y_4^{rj}, y_2^{rj}, y_5^{rj}\right) -$$

$$\sum_{i=1}^{m}(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i)\otimes(x_1^{ij}, x_2^{ij}, x_3^{ij}\ ;\ x_4^{ij}, x_2^{ij}, x_5^{ij})(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r$$

$$= 1,2,3, \dots, s,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r), (v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i)\ \succcurlyeq \varepsilon, \forall i = 1,2,3, \dots, m, \varepsilon > 0.$$

Step 2: The optimistic intuitionistic fuzzy CCR DEA model-5.7 and pessimistic intuitionistic fuzzy CCR DEA model-5.7 can be transformed into optimistic crisp CCR DEA model-5.8 and pessimistic crisp CCR DEA model-5.8 respectively.

Optimistic crisp CCR DEA model-5.8

$$\text{Maximize } \left[\Re(\tilde{E}_p^{BEST}) \approx \Re\left(\frac{\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp})}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i)\otimes(x_1^{ip}, x_2^{ip}, x_3^{ip}\ ;\ x_4^{ip}, x_2^{ip}, x_5^{ip})}\right)\right]$$

Subject to

$$\Re\left(\begin{array}{c}\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes(y_1^{rj}, y_2^{rj}, y_3^{rj}, y_4^{rj}, y_2^{rj}, y_5^{rj}) - \\ \sum_{i=1}^{m}(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i)\otimes(x_1^{ij}, x_2^{ij}, x_3^{ij}\ ;\ x_4^{ij}, x_2^{ij}, x_5^{ij})\end{array}\right) \leq \varepsilon, \qquad \forall j = 1,2,3, \dots, n,$$

$$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r\ \ \forall, r = 1,2,3, \dots, s\ ,$$

$$\varepsilon \leq v_4^i \leq v_1^i \leq v_2^i \leq v_3^i \leq v_5^i\ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Pessimistic crisp CCR DEA model-5.8

$$\text{Minimize} \left[\Re(\tilde{E}_p^{WORST}) \approx \Re\left(\frac{\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp})}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i)\otimes(x_1^{ip}, x_2^{ip}, x_3^{ip}\ ;\ x_4^{ip}, x_2^{ip}, x_5^{ip})}\right)\right]$$

Subject to

$$\Re\left(\begin{array}{c}\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rj},y_2^{rj},y_3^{rj};\ y_4^{rj},y_2^{rj},y_5^{rj}\right)-\\ \sum_{i=1}^{m}\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\otimes\left(x_1^{ij},x_2^{ij},x_3^{ij}\ ;\ x_4^{ij},x_2^{ij},x_5^{ij}\right)\end{array}\right)\geq\varepsilon,\forall j=1,2,3,\dots,n,$$

$$\varepsilon\leq u_4^r\leq u_1^r\leq u_2^r\leq u_3^r\leq u_5^r\ \ \forall,r=1,2,3,\dots,s\ ,$$

$$\varepsilon\leq v_4^i\leq v_1^i\leq v_2^i\leq v_3^i\leq v_5^i\ \ \forall,i=1,2,3,\dots,m,\qquad \varepsilon>0.$$

Step 4: The optimistic crisp CCR DEA model-5.8 and pessimistic crisp CCR DEA model-5.8 can be transformed into optimistic crisp CCR DEA model-5.9 and pessimistic crisp CCR DEA model-5.9 respectively.

Optimistic crisp CCR DEA model-5.9

$$\text{Maximize }\left[\Re\left(\tilde{E}_p^{BEST}\right)=\frac{\Re\left(\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rp},y_2^{rp},y_3^{rp};\ y_4^{rp},y_2^{rp},y_5^{rp}\right)\right)}{\Re\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\otimes\left(x_1^{ip},x_2^{ip},x_3^{ip}\ ;\ x_4^{ip},x_2^{ip},x_5^{ip}\right)\right)}\right]$$

Subject to

$$\Re\left(\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rj},y_2^{rj},y_3^{rj};\ y_4^{rj},y_2^{rj},y_5^{rj}\right)\right)-$$

$$\Re\left(\sum_{i=1}^{m}\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\otimes\left(x_1^{ij},x_2^{ij},x_3^{ij}\ ;\ x_4^{ij},x_2^{ij},x_5^{ij}\right)\right)\leq\varepsilon,\forall j=1,2,3,\dots,n,$$

$$\varepsilon\leq u_4^r\leq u_1^r\leq u_2^r\leq u_3^r\leq u_5^r\ \ \forall,r=1,2,3,\dots,s\ ,$$

$$\varepsilon\leq v_4^i\leq v_1^i\leq v_2^i\leq v_3^i\leq v_5^i\ \ \forall,i=1,2,3,\dots,m,\qquad \varepsilon>0.$$

Pessimistic crisp CCR DEA model-5.9

$$\text{Minimize}\left[\Re\left(\tilde{E}_p^{WORST}\right)=\frac{\Re\left(\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rp},y_2^{rp},y_3^{rp};\ y_4^{rp},y_2^{rp},y_5^{rp}\right)\right)}{\Re\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i)\otimes\left(x_1^{ip},x_2^{ip},x_3^{ip}\ ;\ x_4^{ip},x_2^{ip},x_5^{ip}\right)\right)}\right]$$

Subject to

$$\Re\left(\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rj},y_2^{rj},y_3^{rj};\ y_4^{rj},y_2^{rj},y_5^{rj}\right)\right)-$$

$$\Re\left(\sum_{i=1}^{m}\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\otimes\left(x_1^{ij},x_2^{ij},x_3^{ij}\ ;\ x_4^{ij},x_2^{ij},x_5^{ij}\right)\right)\geq\varepsilon,\forall j=1,2,3,\dots,n,$$

$$\varepsilon\leq u_4^r\leq u_1^r\leq u_2^r\leq u_3^r\leq u_5^r\ \ \forall,r=1,2,3,\dots,s\ ,$$

$$\varepsilon\leq v_4^i\leq v_1^i\leq v_2^i\leq v_3^i\leq v_5^i\ \ \forall,i=1,2,3,\dots,m,\qquad\varepsilon>0.$$

Step 5: The optimistic crisp CCR DEA model-5.9 and pessimistic crisp CCR DEA model-5.9 can be transformed into optimistic crisp CCR DEA model-5.10 and pessimistic crisp CCR DEA model-5.10 respectively.

Optimistic crisp CCR DEA model-5.10

$$\text{Maximize}\left[\Re\left(\tilde{E}_p^{BEST}\right)=\Re\left(\sum_{r=1}^{s}(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\otimes\left(y_1^{rp},y_2^{rp},y_3^{rp};\ y_4^{rp},y_2^{rp},y_5^{rp}\right)\right)\right]$$

Subject to

$$\Re\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\otimes\left(x_1^{ip},x_2^{ip},x_3^{ip};\ x_4^{ip},x_2^{ip},x_5^{ip}\right)\right)=1$$

$$\Re\left(\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right)\right)$$

$$-\Re\left(\sum_{i=1}^{m}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ij}, x_2^{ij}, x_3^{ij}\ ;\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right)\right) \leq \varepsilon, \forall j$$

$$= 1,2,3, \dots, n,$$

$$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r \ \ \forall, r = 1,2,3, \dots, s\ ,$$

$$\varepsilon \leq v_4^i \leq v_1^i \leq v_2^i \leq v_3^i \leq v_5^i \ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Pessimistic crisp CCR DEA model-5.10

$$\text{Minimize } \left[\Re\left(\tilde{E}_p^{WORST}\right) = \Re\left(\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)\right)\right]$$

Subject to

$$\Re\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}; x_3^{ip}\ , x_4^{ip}, x_2^{ip}, x_5^{ip}\right)\right) = 1$$

$$\Re\left(\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right)\right)$$

$$-\Re\left(\sum_{i=1}^{m}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ij}, x_2^{ij}, x_3^{ij}\ ;\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right)\right) \leq \varepsilon, \forall j$$

$$= 1,2,3, \dots, n,$$

$$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r \ \ \forall, r = 1,2,3, \dots, s$$

$$\varepsilon \leq v_4^i \leq v_1^i \leq v_2^i \leq v_3^i \leq v_5^i \ \ \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$$

Step 6: The optimistic crisp CCR DEA model-5.10 and pessimistic crisp CCR DEA model-5.10 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.11 and pessimistic intuitionistic fuzzy CCR DEA model-5.11 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.11

$$\text{Maximize } \left[\tilde{E}_p^{BEST} \approx \sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r, u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}, x_3^{ip}\ ;\ x_4^{ip}, x_2^{ip}, x_5^{ip}\right) \approx (1,1,1;1,1,1)$$

$$\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right)$$

$$-\sum_{i=1}^{m}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ij}, x_2^{ij}, x_3^{ij}\ ;\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right) \preccurlyeq \varepsilon, \forall j = 1,2,3,\dots,n,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,3,\dots,s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1,2,3,\dots,m, \varepsilon > 0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.11

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rp}, y_2^{rp}, y_3^{rp};\ y_4^{rp}, y_2^{rp}, y_5^{rp}\right)\right]$$

Subject to

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ip}, x_2^{ip}, x_3^{ip}\ ;\ x_4^{ip}, x_2^{ip}, x_5^{ip}\right) \approx (1,1,1,1)$$

$$\sum_{r=1}^{s}(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r)\otimes\left(y_1^{rj}, y_2^{rj}, y_3^{rj};\ y_4^{rj}, y_2^{rj}, y_5^{rj}\right)$$

$$-\sum_{i=1}^{m}\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\otimes\left(x_1^{ij}, x_2^{ij}, x_3^{ij};\ x_4^{ij}, x_2^{ij}, x_5^{ij}\right) \succcurlyeq \varepsilon, \forall j = 1,2,3, \dots, n,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,3, \dots, s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1,2,3, \dots, m, \varepsilon > 0.$$

5.4.2 Mathematical incorrect assumptions

If $(a_1, a_2, a_3; a_4, a_2, a_5)$ & $(b_1, b_2, b_3; b_4, b_2, b_5)$ are two triangular intuitionistic fuzzy numbers then

$$\Re\left(\frac{(a_1, a_2, a_3; a_4, a_2, a_5)}{(b_1, b_2, b_3; b_4, b_2, b_5)}\right) = \Re\left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}; \frac{a_4}{b_5}, \frac{a_2}{b_2}, \frac{a_5}{b_4}\right) = \frac{\frac{a_1}{b_3} + \frac{a_3}{b_1} + \frac{4a_2}{b_2} + \frac{a_4}{b_5} + \frac{a_5}{b_4}}{8}$$

and

$$\frac{\Re(a_1, a_2, a_3; a_4, a_2, a_5)}{\Re(b_1, b_2, b_3; b_4, b_2, b_5)} = \frac{a_1 + a_3 + 4a_2 + a_4 + a_5}{b_1 + b_3 + 4b_2 + b_4 + b_5}$$

It is obvious that

$$\Re\left(\frac{(a_1, a_2, a_3; a_4, a_2, a_5)}{(b_1, b_2, b_3; b_4, b_2, b_5)}\right) \neq \frac{\Re(a_1, a_2, a_3; a_4, a_2, a_5)}{\Re(b_1, b_2, b_3; b_4, b_2, b_5)}$$

However, Puri and Yadav have used the property

$$\Re\left(\frac{(a_1, a_2, a_3; a_4, a_2, a_5)}{(b_1, b_2, b_3; b_4, b_2, b_5)}\right) = \frac{\Re(a_1, a_2, a_3; a_4, a_2, a_5)}{\Re(b_1, b_2, b_3; b_4, b_2, b_5)}$$

in Step 4 of their proposed method to transform the optimistic crisp CCR DEA model-5.8 and pessimistic crisp CCR DEA model-5.8 into optimistic crisp CCR DEA model-5.9 and pessimistic crisp CCR DEA model-5.9 respectively. Therefore, the optimistic intuitionistic fuzzy CCR DEA model-5.1 and pessimistic intuitionistic fuzzy CCR DEA model-5.1, proposed by Puri and Yadav, are not valid.

5.5 Proposed intuitionistic fuzzy DEA CCR models

In Section 5.4, it is shown that the optimistic intuitionistic fuzzy CCR DEA model-5.1 and pessimistic intuitionistic fuzzy CCR DEA model-5.1, proposed by Puri and Yadav [83], are not valid and hence cannot be used for evaluating the relative geometric crisp efficiency of DMUs. In this section, new intuitionistic fuzzy CCR DEA models are proposed.

Replacing the trapezoidal fuzzy number with triangular intuitionistic fuzzy number in the optimistic fuzzy CCR DEA model-5.8 and pessimistic fuzzy CCR DEA model-5.8, proposed in Section-5.4 of Chapter 5, these models are transformed into optimistic intuitionistic fuzzy CCR DEA model 5.12 and the pessimistic intuitionistic fuzzy CCR DEA models-5.12

5.6 Proposed approach

In this section, a new approach is proposed to evaluate the best relative intuitionistic fuzzy efficiency, worst relative intuitionistic fuzzy efficiency and hence, relative geometric efficiency of DMUs by using the optimistic intuitionistic fuzzy CCR DEA model-5.12 and pessimistic intuitionistic fuzzy CCR DEA model-5.12.

Step 1: Using the product of triangular intuitionistic fuzzy numbers, defined in Section 5.1.2, the optimistic intuitionistic fuzzy CCR DEA model-5.12 and pessimistic intuitionistic fuzzy CCR DEA model-5.12 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.13 and pessimistic intuitionistic fuzzy CCR DEA model-5.13 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.13

$$\text{Maximize}\left[\tilde{E}_p^{BEST} \approx \frac{\sum_{r=1}^{s}\left(u_1^r y_1^{rp}, u_2^r y_2^{rp}, u_3^r y_3^{rp};\ u_4^r y_4^{rp}, u_2^r y_2^{rp}, u_5^r y_5^{rp}\right)}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i x_1^{ip}, v_2^i x_2^{ip}, v_3^i x_3^{ip}; v_4^i x_4^{ip}, v_2^i x_2^{ip}, v_5^i x_5^{ip}\right)}\right]$$

Subject to

$$\sum_{r=1}^{s}\left(u_1^r y_1^{rj}, u_2^r y_2^{rj}, u_3^r y_3^{rj};\ u_4^r y_4^{rj}, u_2^r y_2^{rj}, u_5^r y_5^{rj}\right) \preccurlyeq$$

$$\sum_{i=1}^{m}\left(v_1^i x_1^{ij}, v_2^i x_2^{ij}, v_3^i x_3^{ij}; v_4^i x_4^{ij}, v_2^i x_2^{ij}, v_5^i x_5^{ij}\right)\ \ \forall j = 1,2,3,\dots,n,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,3,\dots,s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1,2,3,\dots,m, \varepsilon > 0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.13

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \frac{\sum_{r=1}^{s}\left(u_1^r y_1^{rp}, u_2^r y_2^{rp}, u_3^r y_3^{rp}, u_4^r y_4^{rp}, u_2^r y_2^{rp}, u_5^r y_5^{rp}\right)}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}}\left(v_1^i x_1^{ip}, v_2^i x_2^{ip}, v_3^i x_3^{ip}; v_4^i x_4^{ip}, v_2^i x_2^{ip}, v_5^i x_5^{ip}\right)}\right]$$

Subject to

$$\sum_{r=1}^{s}\left(u_1^r y_1^{rj}, u_2^r y_2^{rj}, u_3^r y_3^{rj};\ u_4^r y_4^{rj}, u_2^r y_2^{rj}, u_5^r y_5^{rj}\right) \succcurlyeq$$

$$\sum_{i=1}^{m}\left(v_1^i x_1^{ij}, v_2^i x_2^{ij}, v_3^i x_3^{ij}; v_4^i x_4^{ij}, v_2^i x_2^{ij}, v_5^i x_5^{ij}\right)\ \ \forall j = 1,2,3,\dots,n,$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,3,\dots,s,$$

$$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1,2,3,\dots,m, \varepsilon > 0.$$

Step 2: Using the relation $(a_1, a_2, a_3; a_4, a_2, a_5) \preccurlyeq (b_1, b_2, b_3; b_4, b_2, b_5), \Rightarrow a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3$, $a_4 \leq b_4$, $a_5 \leq b_5$ and division of triangular intuitionistic fuzzy numbers , defined in Section 5.1.2, the optimistic intuitionistic fuzzy CCR DEA model-5.13 and pessimistic intuitionistic fuzzy CCR DEA model-5.13 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.14 and pessimistic intuitionistic fuzzy CCR DEA model-5.14 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.14

$$\text{Maximize} \left[\tilde{E}_p^{BEST} \approx \left(\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}}, \frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}}, \frac{\sum_{r=1}^{s} u_3^r y_3^{rp}}{\sum_{i=1}^{m} v_1^i x_1^{ip}}; \frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}}, \frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}}, \frac{\sum_{r=1}^{s} u_5^r y_5^{rp}}{\sum_{i=1}^{m} v_4^i x_4^{ip}} \right) \right]$$

Subject to

$$\sum_{r=1}^{s} u_1^r y_1^{rj} \leq \sum_{i=1}^{m} v_1^i x_1^{ij} \quad \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_2^r y_2^{rj} \leq \sum_{i=1}^{m} v_2^i x_2^{ij} \quad \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_3^r y_3^{rj} \leq \sum_{i=1}^{m} v_3^i x_3^{ij} \quad \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_4^r y_4^{rj} \leq \sum_{i=1}^{m} v_4^i x_4^{ij} \quad \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_5^r y_5^{rj} \leq \sum_{i=1}^{m} v_5^i x_5^{ij} \quad \forall j = 1,2,3, \dots, n,$$

$\varepsilon \le u_4^r \le u_1^r \le u_2^r \le u_3^r \le u_5^r \;\; \forall, r = 1,2,3, \dots, s,$

$\varepsilon \le v_4^i \le v_1^i \le v_2^i \le v_3^i \le v_5^i \;\; \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$

Pessimistic intuitionistic fuzzy CCR DEA model-5.14

$$\text{Minimize}\left[\tilde{E}_p^{WORST} \approx \left(\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}}, \frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}}, \frac{\sum_{r=1}^{s} u_3^r y_3^{rp}}{\sum_{i=1}^{m} v_1^i x_1^{ip}}; \frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}}, \frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}}, \frac{\sum_{r=1}^{s} u_5^r y_5^{rp}}{\sum_{i=1}^{m} v_4^i x_4^{ip}}\right)\right]$$

Subject to

$$\sum_{r=1}^{s} u_1^r y_1^{rj} \ge \sum_{i=1}^{m} v_1^i x_1^{ij} \;\; \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_2^r y_2^{rj} \ge \sum_{i=1}^{m} v_2^i x_2^{ij} \;\; \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_3^r y_3^{rj} \ge \sum_{i=1}^{m} v_3^i x_3^{ij} \;\; \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_4^r y_4^{rj} \ge \sum_{i=1}^{m} v_4^i x_4^{ij} \;\; \forall j = 1,2,3, \dots, n,$$

$$\sum_{r=1}^{s} u_5^r y_5^{rj} \ge \sum_{i=1}^{m} v_5^i x_5^{ij} \;\; \forall j = 1,2,3, \dots, n,$$

$\varepsilon \le u_4^r \le u_1^r \le u_2^r \le u_3^r \le u_5^r \;\; \forall, r = 1,2,3, \dots, s,$

$\varepsilon \le v_4^i \le v_1^i \le v_2^i \le v_3^i \le v_5^i \;\; \forall, i = 1,2,3, \dots, m, \qquad \varepsilon > 0.$

Step 3: The intuitionistic fuzzy optimal value $\left(\tilde{E}_p^{BEST} = \begin{pmatrix} E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}; \\ E_{p4}^{BEST}, E_{p2}^{BEST}, E_{p5}^{BEST} \end{pmatrix}\right)$, representing the best relative intuitionistic fuzzy efficiency of p^{th} DMU, as well as the intuitionistic fuzzy optimal value $\left(\tilde{E}_p^{WORST} = \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}; E_{p4}^{WORST}, E_{p2}^{WORST}, E_{p5}^{WORST}\right)\right)$, representing the worst relative intuitionistic fuzzy efficiency of p^{th} DMU, can be obtained by solving the optimistic intuitionistic fuzzy CCR DEA model-5.14 and pessimistic intuitionistic fuzzy CCR DEA model-5.14 as follows:

Step 3(a): Find the optimal value $\left(E_{p4}^{BEST}\right)$ and $\left(E_{p4}^{WORST}\right)$ of the optimistic crisp CCR DEA model-5.15 and pessimistic CCR DEA model-5.15 respectively.

Optimistic crisp CCR DEA model-5.15

Maximize $\left[E_{p4}^{BEST} = \frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}}\right]$

Subject to constraints of optimistic intuitionistic fuzzy CCR DEA model-5.14 with the following additional constraint

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} \leq 1.$$

Pessimistic crisp CCR DEA model-5.15

Minimize $\left[E_{p4}^{WORST} = \frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}}\right]$

Subject to

All the constraints of pessimistic intuitionistic fuzzy CCR DEA model-5.14 with the following additional constraint

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} \geq 1,$$

Step 3(b): Find the optimal value $\left(E_{p1}^{BEST}\right)$ and $\left(E_{p1}^{WORST}\right)$ of the optimistic crisp CCR DEA model-5.16 and pessimistic CCR DEA model-5.16 respectively.

Optimistic crisp CCR DEA model-5.16

$$\text{Maximize}\left[E_{p1}^{BEST} = \frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}}\right]$$

Subject to

All Constraints of optimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{BEST},$$

$$E_{p4}^{BEST} \leq \frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} \leq 1.$$

Pessimistic crisp CCR DEA model-5.16

$$\text{Minimize}\left[E_{p1}^{WORST} = \frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}}\right]$$

Subject to

All the constraints of pessimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{WORST},$$

$$E_{p4}^{WORST} \leq \frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}}.$$

Step 3(c): Find the optimal value $\left(E_{p2}^{BEST}\right)$ and $\left(E_{p2}^{WORST}\right)$ of the optimistic crisp CCR DEA model-5.17 and pessimistic CCR DEA model-5.17 respectively.

Optimistic crisp CCR DEA model-5.17

Maximize $\left[E_{p2}^{BEST} = \sum_{r=1}^{s} \frac{u_2^r y_2^{rp}}{v_2^i x_2^{ip}}\right]$

Subject to

Constraints of optimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} = E_{p1}^{BEST},$$

$$E_{p1}^{BEST} \leq \sum_{r=1}^{s} \frac{u_2^r y_2^{rp}}{v_2^i x_2^{ip}} \leq 1.$$

Pessimistic crisp CCR DEA model-5.17

$$\text{Minimize}\left[E_{p2}^{WORST} = \sum_{r=1}^{s} \frac{u_2^r y_2^{rp}}{v_2^i x_2^{ip}}\right]$$

Subject to

Constraints of pessimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{WORST}$$

$$\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} = E_{p1}^{WORST},$$

$$E_{p1}^{WORST} \le \sum_{r=1}^{s} \frac{u_2^r y_2^{rp}}{v_2^i x_2^{ip}}.$$

Step 3(d): Find the optimal value $\left(E_{p3}^{BEST}\right)$ and $\left(E_{p3}^{WORST}\right)$ of the optimistic crisp CCR DEA model-5.18 and pessimistic CCR DEA model-5.18 respectively.

Optimistic crisp CCR DEA model-5.18

$$\text{Maximize}\left[E_{p3}^{BEST} = \sum_{r=1}^{s} \frac{u_3^r y_3^{rp}}{v_1^i x_1^{ip}}\right]$$

Subject to

Constraints of optimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} = E_{p1}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}} = E_{p2}^{BEST},$$

$$E_{p2}^{BEST} \leq \sum_{r=1}^{s} \frac{u_3^r y_3^{rp}}{v_1^i x_1^{ip}} \leq 1.$$

Pessimistic crisp CCR DEA model-5.18

$$\text{Minimize} \left[E_{p3}^{WORST} = \sum_{r=1}^{s} \frac{u_3^r y_3^{rp}}{v_1^i x_1^{ip}} \right]$$

Subject to

Constraints of optimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{WORST}$$

$$\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} = E_{p1}^{WORST},$$

$$\frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}} = E_{p2}^{WORST},$$

$$E_{p2}^{WORST} \leq \sum_{r=1}^{s} \frac{u_3^r y_3^{rp}}{v_1^i x_1^{ip}}.$$

Step 3(e): Find the optimal value $\left(E_{p5}^{BEST}\right)$ and $\left(E_{p5}^{WORST}\right)$ of the optimistic crisp CCR DEA model-5.19 and pessimistic CCR DEA model-5.19 respectively.

Optimistic crisp CCR DEA model-5.19

Maximize $\left[E_{p5}^{BEST} = \sum_{r=1}^{s} \frac{u_5^r y_5^{rp}}{v_4^i x_4^{ip}}\right]$

Subject to

All the constraints of optimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} = E_{p1}^{BEST},$$

$$\frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}} = E_{p2}^{BEST},$$

$$E_{p4}^{BEST} \leq \sum_{r=1}^{s} \frac{u_5^r y_5^{rp}}{v_4^i x_4^{ip}} \leq 1.$$

Pessimistic crisp CCR DEA model-5.19

Minimize $\left[E_{p5}^{WORST} = \sum_{r=1}^{s} \frac{u_5^r y_5^{rp}}{v_4^i x_4^{ip}}\right]$

Subject to

All the constraints of pessimistic crisp CCR DEA model-5.14 with the following additional constraints:

$$\frac{\sum_{r=1}^{s} u_4^r y_4^{rp}}{\sum_{i=1}^{m} v_5^i x_5^{ip}} = E_{p4}^{WORST}$$

$$\frac{\sum_{r=1}^{s} u_1^r y_1^{rp}}{\sum_{i=1}^{m} v_3^i x_3^{ip}} = E_{p1}^{WORST},$$

$$\frac{\sum_{r=1}^{s} u_2^r y_2^{rp}}{\sum_{i=1}^{m} v_2^i x_2^{ip}} = E_{p2}^{WORST},$$

$$E_{p3}^{WORST} \leq \sum_{r=1}^{s} \frac{u_5^r y_5^{rp}}{v_4^i x_4^{ip}}.$$

Step 4: Using the values of $E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}; E_{p4}^{BEST}, E_{p5}^{BEST}$ and $E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}; E_{p4}^{WORST}, E_{p2}^{WORST}, E_{p5}^{WORST}$, obtained in Step (3a) to Step (3e), find the intuitionistic fuzzy optimal value $\left(\tilde{E}_p^{BEST} = \left(E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}; E_{p4}^{BEST}, E_{p5}^{BEST}\right)\right)$ of optimistic intuitionistic fuzzy DEA model-5.14, representing the best relative intuitionistic fuzzy efficiency of p^{th} DMU, as well as intuitionistic fuzzy optimal value $\left(\tilde{E}_p^{WORST} = \left(E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}; E_{p4}^{WORST}, E_{p2}^{WORST}, E_{p5}^{WORST}\right)\right)$ of pessimistic intuitionistic fuzzy DEA model-5.14, representing the worst relative intuitionistic fuzzy efficiency of p^{th} DMU.

Step 5: Find the crisp optimal value $\left(E_p^{BEST} = \Re\left(\tilde{E}_p^{BEST}\right)\right)$, representing the best relative crisp efficiency of p^{th} DMU.

Step 6: Find the crisp optimal value $\left(E_p^{WORST} = \Re\left(\tilde{E}_p^{WORST}\right)\right)$, representing the worst relative crisp efficiency of p^{th} DMU.

Step 7: Find the relative geometric crisp efficiency ($E_P^{GEOMETRIC}$) of p^{th} DMU by putting the values E_p^{BEST} and E_p^{WORST}, obtained in Step 5 and Step 6, in $E_P^{GEOMETRIC} = \sqrt{E_p^{BEST} \times E_p^{WORST}}$.

5.7 Exact relative geometric crisp efficiency of real life problem

Puri and Yadav [83] evaluated the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of 16 branches of State Bank of Patiala, Punjab, India by considering the input and output data shown in Table 5.1 and using the optimistic intuitionistic fuzzy CCR DEA model-5.1 as well as pessimistic intuitionistic fuzzy CCR DEA model-5.1. However, as discussed in Section 5.3, that the optimistic intuitionistic fuzzy CCR DEA model-5.1 as well as pessimistic intuitionistic fuzzy CCR DEA model-5.1are not valid. Therefore, the best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of 16 branches of State Bank of Patiala, Punjab, India , evaluated by Puri and Yadav [83], is not exact.

In this section, to illustrate the proposed method the exact intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of these Banks is evaluated by using the proposed approach.

Table 5.1 Input-Output for 16 bank branches

Sr.no.	Branch Name	Inputs			Outputs	
		Labour	OE	IE	II	OI
1	Amritsar (M.M)	(15,17,19;13,17,21)	(.39, .44, .46; .26, .44, .63)	9.16	1.85	0.33
2	Patti	(14, 16, 18;12,16,20)	(.46, .54, .65, .38, .54, .73)	2.68	0.67	0.41
3	Amritsar E.M.N.	(9, 11, 13; 6, 11, 15)	(.47, .54, .57; .42, .54, .60)	4.24	10.73	0.49
4	Verka	(10, 13, 15; 8, 13,17)	(.19, .21, .24; .18, .21, .25)	2.08	1	0.084
5	Amritsar C.W.G.	(11, 14, 16; 9,14,19)	(.25, .29, .31; .23, .29, .33)	4.73	2.88	0.2
6	Amritsar K.J.S	(18,21,24; 15,21,27)	(.28, .29, .30; .24,.29,.31)	4.58	4.97	0.85

7	Mananwala A. D. B	(11, 14, 17; 9, 14,19)	(.21, .23, .24; .19, .23, .25)	1.33	3.94	0.28
8	Baba Bakala	(9, 12, 13; 7, 12, 16)	(.56, .60, .62; .52, .60, .64)	2.98	2.16	0.3
9	Chabhal	(6, 9, 12; 6, 9, 12)	(.16, .17, .19; .13, .17, .20)	1.48	2.58	0.15
10	Lohaka	(5, 7, 9; 3, 7, 10)	(.10, .11, .12; .08, .11, .13)	0.75	1	0.13
11	Harsha Chhina	(4, 6, 8; 3, 6, 9)	(.15, .18, .20; .12, .18, .21)	0.87	1.73	0.13
12	Jhanjhoti	(8, 10, 13; 6, 10, 14)	(.18, .20, .22; .15, .20, .23)	0.9	1.26	0.069
13	Taran Taran	(9, 11, 13; 7, 11, 15)	(.19, .22, .24; .16, .22, .26)	1.51	3.53	0.26
14	Amritsar (Textile)	(9, 11, 14; 8, 11, 15)	(.20, .21, .23; .18, .21, .24)	2.98	5.98	0.27
15	Rayya Mandi	(7, 10, 12; 6, 10, 14)	(.21, .22, .24; .17, .22, .26)	1.99	1.79	0.16
16	Amritsar (Civil Lines)	(6, 8, 10; 5, 8, 12)	(.25, .39, .44, .19, .39, .47)	3.64	7.1	0.37

(OE; Operating Expenses, IE; Interest Expenses, II; Interest Income, OI; Other Income)

5.7.1 Proposed intuitionistic fuzzy CCR DEA models

The best relative intuitionistic fuzzy efficiency as well as worst best relative intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of DMU_1 (1^{st} Branch) can be obtained by solving optimistic intuitionistic fuzzy CCR DEA model-5.20 and pessimistic intuitionistic fuzzy CCR DEA model-5.20.

Optimistic intuitionistic fuzzy CCR DEA model-5.20

$$\text{Maximize}\left[\tilde{E}_A^{BEST} \approx ((E_A^{11})^{BEST},(E_A^{12})^{BEST},(E_A^{13})^{BEST};(E_A^{14})^{BEST},(E_A^{12})^{BEST},(E_A^{15})^{BEST}) \right.$$

$$\left. = \left(\frac{\begin{array}{c}(1.85)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.33)\ (u_1^{21},u_2^{22},u_3^{23};\ u_4^{24},u_2^{22},u_5^{25})\end{array}}{\begin{array}{c}(15,17,19;13,17,21)(v_1^{11},v_2^{12},v_3^{13};\ v_4^{14},v_2^{12},v_5^{15})+\\ (.39,.44,.46;\ .26,.44,.63)(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25})+\\ (9.16)(v_1^{31},v_2^{32},v_3^{33};\ v_4^{34},v_2^{32},v_5^{35})\end{array}} \right)\right]$$

Subject to

$$\begin{bmatrix}(1.85)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.33)\ (u_1^{21},u_2^{22},u_3^{23};\ u_4^{24},u_2^{22},u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(15,17,19;13,17,21)\left(v_1^{11},v_2^{12},v_3^{13};\ v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.39,.44,.46;\ .26,.44,.63)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (9.16)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(0.67)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.41)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(14,16,18;12,16,20)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.46,.54,.65,.38,.54,.73)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (2.68)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(10.73)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.49)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(9,11,13;\ 6,11,15)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.47,.54,.57;\ .42,.54,.60)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (4.24)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.00)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\(.084)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(10,13,15;\ 8,13,17)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.19,.21,.24;\ .18,.21,.25)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(2.08)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(2.88)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.20)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(11,14,16;\ 9,14,19)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.25,.29,.31;\ .23,.29,.33)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(4.73)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(4.97)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.85)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(18,21,24;\ 15,21,27)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.28,.29,.30;.24,.29,.31)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(4.58)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(3.94)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.28)\ (u_1^{21},u_2^{22},u_3^{23};\ u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(11,14,17;\ 9,14,19)\left(v_1^{11},v_2^{12},v_3^{13};\ v_4^{14},v_2^{12},v_5^{15}\right)+\\(.21,.23,.24;\ .19,.23,.25)\left(v_1^{21},v_2^{22},v_3^{23};\ v_4^{24},v_2^{22},v_5^{25}\right)+\\(1.33)\left(v_1^{31},v_2^{32},v_3^{33};\ v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(2.16)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.30)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(9,12,13;\ 7,12,16)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.56,.60,.62;\ .52,.60,.64)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(2.98)\left(v_1^{31},v_2^{32},v_3^{33};\ v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(2.58)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.15)(u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(6,9,12;\ 6,9,12)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.16,.17,.19;\ .13,.17,.20)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(1.48)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.00)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.13)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(5,7,9;\ 3,7,10)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.10,.11,.12;\ .08,.11,.13)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(0.75)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.73)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.13)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(4,6,8;\ 3,6,9)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.15,.18,.20;\ .12,.18,.21)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(0.87)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.26)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.069)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(8,10,13;\ 6,10,14)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.18,.20,.22;\ .15,.20,.23)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(0.90)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(3.53)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.26)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(9,11,13;\ 7,11,15)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.19,.22,.24;\ .16,.22,.26)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(1.51)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(5.98)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.27)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(9,11,14;\ 8,11,15)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.20,.21,.23;\ .18,.21,.24)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(2.98)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.79)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.16)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(7,10,12;\ 6,7,14)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.21,.22,.24;\ .17,.22,.26)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(1.99)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(7.10)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.37)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$

$$\begin{bmatrix}(6,8,10;\ 5,8,12)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.25,.39,.44,.19,.39,.47)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(3.64)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\succcurlyeq\ \varepsilon,\forall r=1,2,\left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\ \succcurlyeq\varepsilon,\forall i=1,2,3,\varepsilon>0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.20

$$\text{Minimize}\left[\tilde{E}_A^{BEST}\approx((E_A^{11})^{BEST},(E_A^{12})^{BEST},(E_A^{13})^{BEST};(E_A^{14})^{BEST},(E_A^{12})^{BEST},(E_A^{15})^{BEST})\right.$$

$$\left.=\left(\frac{\begin{matrix}(1.85)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.33)\ (u_1^{21},u_2^{22},u_3^{23};\ u_4^{24},u_2^{22},u_5^{25})\end{matrix}}{\begin{matrix}(15,17,19;13,17,21)(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15})+\\(.39,.44,.46;\ .26,.44,.63)(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25})+\\(9.16)(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35})\end{matrix}}\right)\right]$$

Subject to

$$\begin{bmatrix}(1.85)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.33)\ (u_1^{21},u_2^{22},u_3^{23};\ u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(15,17,19;13,17,21)\left(v_1^{11},v_2^{12},v_3^{13};\ v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.39,.44,.46;\ .26,.44,.63)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (9.16)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(0.67)\left(u_1^{11},u_2^{12},u_3^{13},u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.41)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(14,16,18;12,16,20)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.46,.54,.65,.38,.54,.73)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (2.68)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(10.73)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.49)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(9,11,13;\ 6,11,15)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.47,.54,.57;\ .42,.54,.60)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (4.24)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.00)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\ (.084)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(10,13,15;\ 8,13,17)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.19,.21,.24;\ .18,.21,.25)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (2.08)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(2.88)\left(u_1^{11},u_2^{12},u_3^{13};u_4^{14},u_2^{12},u_5^{15}\right)+\\ (0.20)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(11,14,16;\ 9,14,19)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\ (.25,.29,.31;\ .23,.29,.33)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\ (4.73)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix} (4.97)\left(u_1^{11}, u_2^{12}, u_3^{13}; u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.85)\ (u_1^{21}, u_2^{22}, u_3^{23}; u_4^{24}, u_2^{22}, u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (18, 21, 24;\ 15, 21, 27)\left(v_1^{11}, v_2^{12}, v_3^{13}; v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.28, .29, .30; .24, .29, .31)\left(v_1^{21}, v_2^{22}, v_3^{23}; v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (4.58)\left(v_1^{31}, v_2^{32}, v_3^{33}; v_4^{34}, v_2^{32}, v_5^{35}\right) \end{bmatrix}$$

$$\begin{bmatrix} (3.94)\left(u_1^{11}, u_2^{12}, u_3^{13}; u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.28)\ (u_1^{21}, u_2^{22}, u_3^{23};\ u_4^{24}, u_2^{22}, u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (11, 14, 17;\ 9, 14, 19)\left(v_1^{11}, v_2^{12}, v_3^{13};\ v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.21, .23, .24;\ .19, .23, .25)\left(v_1^{21}, v_2^{22}, v_3^{23};\ v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (1.33)\left(v_1^{31}, v_2^{32}, v_3^{33};\ v_4^{34}, v_2^{32}, v_5^{35}\right) \end{bmatrix}$$

$$\begin{bmatrix} (2.16)\left(u_1^{11}, u_2^{12}, u_3^{13};\ u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.30)\ (u_1^{21}, u_2^{22}, u_3^{23}; u_4^{24}, u_2^{22}, u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (9, 12, 13;\ 7, 12, 16)\left(v_1^{11}, v_2^{12}, v_3^{13}; v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.56, .60, .62;\ .52, .60, .64)\left(v_1^{21}, v_2^{22}, v_3^{23}; v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (2.98)\left(v_1^{31}, v_2^{32}, v_3^{33};\ v_4^{34}, v_2^{32}, v_5^{35}\right) \end{bmatrix}$$

$$\begin{bmatrix} (2.58)\left(u_1^{11}, u_2^{12}, u_3^{13}; u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.15)(u_1^{21}, u_2^{22}, u_3^{23}; u_4^{24}, u_2^{22}, u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (6, 9, 12;\ 6, 9, 12)\left(v_1^{11}, v_2^{12}, v_3^{13}; v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.16, .17, .19;\ .13, .17, .20)\left(v_1^{21}, v_2^{22}, v_3^{23}; v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (1.48)\left(v_1^{31}, v_2^{32}, v_3^{33}; v_4^{34}, v_2^{32}, v_5^{35}\right) \end{bmatrix}$$

$$\begin{bmatrix} (1.00)\left(u_1^{11}, u_2^{12}, u_3^{13};\ u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.13)\ (u_1^{21}, u_2^{22}, u_3^{23}; u_4^{24}, u_2^{22}, u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (5, 7, 9;\ 3, 7, 10)\left(v_1^{11}, v_2^{12}, v_3^{13}; v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.10, .11, .12;\ .08, .11, .13)\left(v_1^{21}, v_2^{22}, v_3^{23}; v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (0.75)\left(v_1^{31}, v_2^{32}, v_3^{33}; v_4^{34}, v_2^{32}, v_5^{35}\right) \end{bmatrix}$$

$$\begin{bmatrix}(1.73)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.13)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(4,6,8;\ 3,6,9)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.15,.18,.20;\ .12,.18,.21)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(0.87)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.26)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.069)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(8,10,13;\ 6,10,14)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.18,.20,.22;\ .15,.20,.23)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(0.90)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(3.53)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.26)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(9,11,13;\ 7,11,15)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.19,.22,.24;\ .16,.22,.26)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(1.51)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(5.98)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.27)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(9,11,14;\ 8,11,15)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.20,.21,.23;\ .18,.21,.24)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(2.98)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(1.79)\left(u_1^{11},u_2^{12},u_3^{13};\ u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.16)\ (u_1^{21},u_2^{22},u_3^{23};u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\succcurlyeq$$

$$\begin{bmatrix}(7,10,12;\ 6,7,14)\left(v_1^{11},v_2^{12},v_3^{13};v_4^{14},v_2^{12},v_5^{15}\right)+\\(.21,.22,.24;\ .17,.22,.26)\left(v_1^{21},v_2^{22},v_3^{23};v_4^{24},v_2^{22},v_5^{25}\right)+\\(1.99)\left(v_1^{31},v_2^{32},v_3^{33};v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}(7.10)\left(u_1^{11}, u_2^{12}, u_3^{13};\ u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.37)\ (u_1^{21}, u_2^{22}, u_3^{23}; u_4^{24}, u_2^{22}, u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(6, 8, 10;\ 5, 8, 12)\left(v_1^{11}, v_2^{12}, v_3^{13}; v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.25, .39, .44, .19, .39, .47)\left(v_1^{21}, v_2^{22}, v_3^{23}; v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (3.64)\left(v_1^{31}, v_2^{32}, v_3^{33}; v_4^{34}, v_2^{32}, v_5^{35}\right)\end{bmatrix}$$

$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1{,}2,$

$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right)\ \succcurlyeq \varepsilon, \forall i = 1{,}2{,}3, \varepsilon > 0.$

5.7.2 Exact relative geometric crisp efficiency of DMU_1 (Amritsar Branch, M.M.)

Using the method, proposed in Section 5.6, the exact relative optimistic intuitionistic fuzzy efficiency as well as pessimistic intuitionistic fuzzy efficiency and hence, relative geometric crisp efficiency of DMU_1 can be obtained as follows:

Step 1: Using the product of triangular intuitionistic fuzzy numbers, defined in Section 5.1.2, the optimistic intuitionistic fuzzy CCR DEA model-5.20 and pessimistic intuitionistic fuzzy CCR DEA model-5.20 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.21 and pessimistic intuitionistic fuzzy CCR DEA model-5.21 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.21

$$\text{Maximize}\left[\tilde{E}_1^{BEST} \approx (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}, E_{12}^{BEST}, E_{15}^{BEST})\right.$$

$$\left. = \left(\frac{\begin{gathered}(1.85u_1^{11}, 1.85\, u_2^{12}, 1.85u_3^{13}; 1.85u_4^{14}, 1.85u_2^{12}, 1.85u_5^{15}) + \\ (0.33u_1^{21}, 0.33u_2^{22}, 0.33\, u_3^{23};\ 0.33\, u_4^{24}, 0.33u_2^{22}, 0.33u_5^{25})\end{gathered}}{\begin{gathered}(15v_1^{11}, 17v_2^{12}, 19v_3^{13}; 13v_4^{14}, 17v_2^{12}, 21v_5^{15}) + \\ (.39v_1^{21}, .44v_2^{22}, .46v_3^{23};\ .26\, v_4^{24}, .44v_2^{22}, .63v_5^{25}) + \\ (9.16v_1^{31}, 9.16v_2^{32}, 9.16v_3^{33};\ 9.16v_4^{34}, 9.16\, v_2^{32}, 9.16v_5^{35})\end{gathered}} \right)\right]$$

Subject to

$$\begin{bmatrix}(1.85u_1^{11}, 1.85\, u_2^{12}, 1.85u_3^{13}; 1.85u_4^{14}, 1.85u_2^{12}, 1.85u_5^{15}) + \\ (0.33u_1^{21}, 0.33u_2^{22}, 0.33\, u_3^{23};\ 0.33\, u_4^{24}, 0.33u_2^{22}, 0.33u_5^{25})\end{bmatrix} \preccurlyeq \begin{bmatrix}(15v_1^{11}, 17v_2^{12}, 19v_3^{13}; 13v_4^{14}, 17v_2^{12}, 21v_5^{15}) + \\ (.39v_1^{21}, .44v_2^{22}, .46v_3^{23};\ .26\, v_4^{24}, .44v_2^{22}, .63v_5^{25}) + \\ (9.16v_1^{31}, 9.16v_2^{32}, 9.16v_3^{33};\ 9.16v_4^{34}, 9.16v_2^{32}, 9.16v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}(0.67u_1^{11}, 0.67u_2^{12}, 0.67u_3^{13};\ 0.67u_4^{14}, 0.67\, u_2^{12}, 0.67u_5^{15}) + \\ (0.41u_1^{21}, 0.41u_2^{22}, 0.41u_3^{23}; 0.41u_4^{24}, 0.41u_2^{22}, 0.41u_5^{25})\end{bmatrix} \preccurlyeq \begin{bmatrix}(14v_1^{11}, 16v_2^{12}, 18v_3^{13}; 12v_4^{14}, 16v_2^{12}, 20v_5^{15}) + \\ (.46v_1^{21}, .54v_2^{22}, .65v_3^{23};\ .38v_4^{24}, .54v_2^{22}, .73v_5^{25}) + \\ (2.68v_1^{31}, 2.68\, v_2^{32}, 2.68v_3^{33}; 2.68v_4^{34}, 2.68v_2^{32}, 2.68v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}(10.73u_1^{11}, 10.73u_2^{12}, 10.73u_3^{13}; 10.73u_4^{14}, 10.73u_2^{12}, 10.73u_5^{15}) + \\ (0.49u_1^{21}, 0.49u_2^{22}, 0.49u_3^{23}; 0.49u_4^{24}, 0.49\, u_2^{22}, 0.49u_5^{25})\end{bmatrix} \preccurlyeq \begin{bmatrix}(9v_1^{11}, 11v_2^{12}, 13v_3^{13};\ 6v_4^{14}, 11v_2^{12}, 15v_5^{15}) + \\ (.47v_1^{21}, .54v_2^{22}, .57v_3^{23};\ .42v_4^{24}, .54v_2^{22}, .60v_5^{25}) + \\ (4.24v_1^{31}, 4.24v_2^{32}, 4.24v_3^{33};\ 4.24v_4^{34}, 4.24v_2^{32}, 4.24v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix} \left(1.00u_1^{11}, 1.00u_2^{12}, 1.00\ u_3^{13}; 1.00u_4^{14}, 1.00u_2^{12}, 1.00u_5^{15}\right) + \\ (.084u_1^{21}, .084u_2^{22}, .084u_3^{23}; .084u_4^{24}, .084u_2^{22}, .084u_5^{25}) \end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix} (10v_1^{11}, 13v_2^{12}, 15v_3^{13};\ 8v_4^{14}, 13v_2^{12}, 17v_5^{15}) + \\ (.19v_1^{21}, .21v_2^{22}, .24v_3^{23};\ .18\ v_4^{24}, .21v_2^{22}, .25v_5^{25}) + \\ (2.08v_1^{31}, 2.08v_2^{32}, 2.08v_3^{33}; 2.08v_4^{34}, 2.08v_2^{32}, 2.08v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(2.88u_1^{11}, 2.88u_2^{12}, 2.88u_3^{13}; 2.88u_4^{14}, 2.88u_2^{12}, 2.88u_5^{15}\right) + \\ (0.20u_1^{21}, 0.20\ u_2^{22}, 0.20u_3^{23}; 0.20u_4^{24}, 0.20u_2^{22}, 0.20u_5^{25}) \end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix} (11v_1^{11}, 14v_2^{12}, 16v_3^{13};\ 9v_4^{14}, 14\ v_2^{12}, 19v_5^{15}) + \\ (.25v_1^{21}, .29v_2^{22}, .31v_3^{23};\ .23v_4^{24}, .29v_2^{22}, .33v_5^{25}) + \\ (4.73v_1^{31}, 4.73v_2^{32}, 4.73v_3^{33}; 4.73v_4^{34}, 4.73v_2^{32}, 4.73v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(4.97u_1^{11}, 4.97u_2^{12}, 4.97\ u_3^{13}; 4.97u_4^{14}, 4.97u_2^{12}, 4.97u_5^{15}\right) + \\ (0.85u_1^{21}, 0.85u_2^{22}, 0.85u_3^{23}; 0.85u_4^{24}, 0.85u_2^{22}, 0.85u_5^{25}) \end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix} (18v_1^{11}, 21v_2^{12}, 24v_3^{13};\ 15v_4^{14}, 21v_2^{12}, 27v_5^{15}) + \\ (.28v_1^{21}, .29v_2^{22}, .30v_3^{23}; .24v_4^{24}, .29v_2^{22}, .31v_5^{25}) + \\ (4.58v_1^{31}, 4.58v_2^{32}, 4.58v_3^{33}; 4.58\ v_4^{34}, 4.58v_2^{32}, 4.58v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(3.94u_1^{11}, 3.94u_2^{12}, 3.94\ u_3^{13}; 3.94u_4^{14}, 3.94u_2^{12}, 3.94u_5^{15}\right) + \\ (0.28u_1^{21}, 0.28u_2^{22}, 0.28u_3^{23}; 0.28u_4^{24}, 0.28u_2^{22}, 0.28u_5^{25}) \end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix} (11v_1^{11}, 14v_2^{12}, 17v_3^{13};\ 9v_4^{14}, 14v_2^{12}, 19v_5^{15}) + \\ (.21v_1^{21}, .23v_2^{22}, .24v_3^{23};\ .19v_4^{24}, .23\ v_2^{22}, .25v_5^{25}) + \\ (1.33v_1^{31}, 1.33\ v_2^{32}, 1.33v_3^{33}; 1.33v_4^{34}, 1.33v_2^{32}, 1.33v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(2.16u_1^{11}, 2.16\ u_2^{12}, 2.16\ u_3^{13};\ 2.16u_4^{14}, 2.16u_2^{12}, 2.16\ u_5^{15}\right) + \\ (0.30u_1^{21}, 0.30\ u_2^{22}, 0.30u_3^{23};\ 0.30u_4^{24}, 0.30u_2^{22}, 0.30u_5^{25}) \end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix} (9\ v_2^{12}, 12v_2^{12}, 13v_3^{13};\ 7v_4^{14}, 12v_2^{12}, 16\ v_5^{15}) + \\ (.56v_1^{21}, .60v_2^{22}, .62v_3^{23};\ .52v_4^{24}, .60v_2^{22}, .64v_5^{25}) + \\ (2.98v_1^{31}, 2.98v_2^{32}, 2.98v_3^{33};\ 2.98v_4^{34}, 2.98\ v_2^{32}, 2.98v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix}\left(2.58u_1^{11}, 2.58u_2^{12}, 2.58u_3^{13};\ 2.58u_4^{14}, 2.58u_2^{12}, 2.58u_5^{15}\right) + \\ (0.15u_1^{21}, 0.15u_2^{22}, 0.15u_3^{23};\ 0.15u_4^{24}, 0.15u_2^{22}, 0.15u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(6v_1^{11}, 9v_2^{12}, 12v_3^{13};\ 6v_4^{14}, 9v_2^{12}, 12\ v_5^{15}) + \\ (.16v_1^{21}, .17v_2^{22}, .19v_3^{23};\ .13v_4^{24}, .17v_2^{22}, .20v_5^{25}) + \\ (1.48v_1^{31}, 1.48v_2^{32}, 1.48\ v_3^{33}; 1.48v_4^{34}, 1.48v_2^{32}, 1.48v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(1.00u_1^{11}, 1.00u_2^{12}, 1.00u_3^{13};\ 1.00u_4^{14}, 1.00u_2^{12}, 1.00u_5^{15}\right) + \\ (0.13u_1^{21}, 0.13\ u_2^{22}, 0.13u_3^{23};\ 0.13u_4^{24}, 0.13u_2^{22}, 0.13u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(5v_1^{11}, 7v_2^{12}, 9v_3^{13};\ 3v_4^{14}, 7v_2^{12}, 10v_5^{15}) + \\ (.10v_1^{21}, .11v_2^{22}, .12v_3^{23};\ .08v_4^{24}, .11\ v_2^{22}, .13v_5^{25}) + \\ (0.75v_1^{31}, 0.75v_2^{32}, 0.75v_3^{33};\ 0.75v_4^{34}, 0.75v_2^{32}, 0.75v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(1.73u_1^{11}, 1.73u_2^{12}, 1.73u_3^{13};\ 1.73u_4^{14}, 1.73u_2^{12}, 1.73u_5^{15}\right) + \\ (0.13u_1^{21}, 0.13\ u_2^{22}, 0.13u_3^{23};\ 0.13u_4^{24}, 0.13u_2^{22}, 0.13u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(4v_1^{11}, 6v_2^{12}, 8v_3^{13};\ 3v_4^{14}, 6v_2^{12}, 9v_5^{15}) + \\ (.15v_1^{21}, .18v_2^{22}, .20v_3^{23};\ .12v_4^{24}, .18v_2^{22}, .21v_5^{25}) + \\ (0.87v_1^{31}, 0.87v_2^{32}, 0.87v_3^{33};\ 0.87v_4^{34}, 0.87v_2^{32}, 0.87v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(1.26u_1^{11}, 1.26u_2^{12}, 1.26u_3^{13};\ 1.26u_4^{14}, 1.26u_2^{12}, 1.26u_5^{15}\right) + \\ (0.069u_1^{21}, 0.069u_2^{22}, 0.069u_3^{23}; 0.069u_4^{24}, 0.069u_2^{22}, 0.069u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(8v_1^{11}, 10v_2^{12}, 13v_3^{13};\ 6v_4^{14}, 10v_2^{12}, 14\ v_5^{15}) + \\ (.18v_1^{21}, .20v_2^{22}, .22v_3^{23};\ .15v_4^{24}, .20v_2^{22}, .23v_5^{25}) + \\ (0.90v_1^{31}, 0.90v_2^{32}, 0.90\ v_3^{33};\ 0.90v_4^{34}, 0.90v_2^{32}, 0.90v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(3.53u_1^{11}, 3.53u_2^{12}, 3.53u_3^{13};\ 3.53u_4^{14}, 3.53u_2^{12}, 3.53u_5^{15}\right) + \\ (0.26u_1^{21}, 0.26u_2^{22}, 0.26u_3^{23};\ 0.26u_4^{24}, 0.26u_2^{22}, 0.26\ u_5^{25})\end{bmatrix} \preccurlyeq$$

$$\begin{bmatrix}(9v_1^{11}, 11v_2^{12}, 13\ v_3^{13};\ 7v_4^{14}, 11v_2^{12}, 15v_5^{15}) + \\ (.19v_1^{21}, .22\ v_2^{22}, .24\ v_3^{23};\ .16v_4^{24}, .22v_2^{22}, .26v_5^{25}) + \\ (1.51v_1^{31}, 1.51v_2^{32}, 1.51v_3^{33};\ 1.51v_4^{34}, 1.51v_2^{32}, 1.51v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}(5.98,5.98,5.98;\ 5.98,5.98,5.98)\left(u_1^{11},u_2^{12},u_3^{13},u_4^{14},u_2^{12},u_5^{15}\right)+\\(0.27,0.27,0.27;\ 0.27,0.27,0.27)\ (u_1^{21},u_2^{22},u_3^{23},u_4^{24},u_2^{22},u_5^{25})\end{bmatrix}\preccurlyeq$$
$$\begin{bmatrix}(9,11,14;\ 8,11,15)\left(v_1^{11},v_2^{12},v_3^{13},v_4^{14},v_2^{12},v_5^{15}\right)+\\(.20,.21,.23;\ .18,.21,.24)\left(v_1^{21},v_2^{22},v_3^{23},v_4^{24},v_2^{22},v_5^{25}\right)+\\(2.98,2.98,2.98;\ 2.98,2.98,2.98)\left(v_1^{31},v_2^{32},v_3^{33},v_4^{34},v_2^{32},v_5^{35}\right)\end{bmatrix}$$

$$\begin{bmatrix}\left(1.79u_1^{11},1.79u_2^{12},1.79u_3^{13};1.79u_4^{14},1.79u_2^{12},1.79u_5^{15}\right)+\\(0.16u_1^{21},0.16u_2^{22},0.16u_3^{23};\ 0.16u_4^{24},0.16u_2^{22},0.16u_5^{25})\end{bmatrix}\preccurlyeq$$
$$\begin{bmatrix}(7v_1^{11},10v_2^{12},12v_3^{13};\ 6v_4^{14},10v_2^{12},14v_5^{15})+\\(.21v_1^{21},.22v_2^{22},.24v_3^{23};\ .17v_4^{24},.22v_2^{22},.26v_5^{25})+\\(1.99v_1^{31},1.99v_2^{32},1.99v_3^{33};\ 1.99v_4^{34},1.99v_2^{32},1.99v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(7.10u_1^{11},7.10u_2^{12},7.10u_3^{13};\ 7.10u_4^{14},7.10u_2^{12},7.10u_5^{15}\right)+\\(0.37u_1^{21},0.37u_2^{22},0.37u_3^{23};\ 0.37u_4^{24},0.37u_2^{22},0.37u_5^{25})\end{bmatrix}\preccurlyeq$$
$$\begin{bmatrix}(6v_1^{11},8v_2^{12},10v_3^{13};\ 5\ v_4^{14},8v_2^{12},12v_5^{15})+\\(.25v_1^{21},.39v_2^{22},.44v_3^{23},.19v_4^{24},.39v_2^{22},.47\ v_5^{25})+\\(3.64v_1^{31},3.64v_2^{32},3.64v_3^{33};\ 3.64v_4^{34},3.64v_2^{32},3.64v_5^{35})\end{bmatrix}$$

$$(u_1^r,u_2^r,u_3^r;\ u_4^r,u_2^r,u_5^r)\succcurlyeq\ \varepsilon,\forall r=1,2,\qquad \left(v_1^i,v_2^i,v_3^i;v_4^i,v_2^i,v_5^i\right)\ \succcurlyeq\varepsilon,\forall i=1,2,3,\varepsilon>0.$$

Pessimistic intuitionistic fuzzy CCR DEA model-5.21

$$\text{Minimize}\left[\tilde{E}_1^{BEST}\approx(E_{11}^{BEST},E_{12}^{BEST},E_{13}^{BEST};E_{14}^{BEST}\ ,E_{12}^{BEST},E_{15}^{BEST})\right.$$

$$=\left.\left(\frac{\begin{array}{c}\left(1.85u_1^{11},1.85\ u_2^{12},1.85u_3^{13};1.85u_4^{14},1.85u_2^{12},1.85u_5^{15}\right)+\\(0.33u_1^{21},0.33u_2^{22},0.33\ u_3^{23};\ 0.33\ u_4^{24},0.33u_2^{22},0.33u_5^{25})\end{array}}{\begin{array}{c}(15v_1^{11},17v_2^{12},19v_3^{13};13v_4^{14},17v_2^{12},21v_5^{15})+\\(.\,39v_1^{21},.44v_2^{22},.46v_3^{23};\ .26\ v_4^{24},.44v_2^{22},.63v_5^{25})+\\(9.16v_1^{31},9.16v_2^{32},9.16v_3^{33};\ 9.16v_4^{34},9.16\ v_2^{32},9.16v_5^{35})\end{array}}\right)\right]$$

Subject to

$$\begin{bmatrix}\left(1.85u_1^{11}, 1.85\,u_2^{12}, 1.85u_3^{13}; 1.85u_4^{14}, 1.85u_2^{12}, 1.85u_5^{15}\right) + \\ (0.33u_1^{21}, 0.33u_2^{22}, 0.33\,u_3^{23};\; 0.33\,u_4^{24}, 0.33u_2^{22}, 0.33u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(15v_1^{11}, 17v_2^{12}, 19v_3^{13}; 13v_4^{14}, 17v_2^{12}, 21v_5^{15}) + \\ \left(.39v_1^{21}, .44v_2^{22}, .46v_3^{23};\; .26\,v_4^{24}, .44v_2^{22}, .63v_5^{25}\right) + \\ (9.16v_1^{31}, 9.16v_2^{32}, 9.16v_3^{33};\; 9.16v_4^{34}, 9.16v_2^{32}, 9.16v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(0.67u_1^{11}, 0.67u_2^{12}, 0.67u_3^{13};\; 0.67u_4^{14}, 0.67\,u_2^{12}, 0.67u_5^{15}\right) + \\ (0.41u_1^{21}, 0.41u_2^{22}, 0.41u_3^{23}; 0.41u_4^{24}, 0.41u_2^{22}, 0.41u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(14v_1^{11}, 16v_2^{12}, 18v_3^{13}; 12v_4^{14}, 16v_2^{12}, 20v_5^{15}) + \\ \left(.46v_1^{21}, .54v_2^{22}, .65v_3^{23};\; .38v_4^{24}, .54v_2^{22}, .73v_5^{25}\right) + \\ (2.68v_1^{31}, 2.68\,v_2^{32}, 2.68v_3^{33}; 2.68v_4^{34}, 2.68v_2^{32}, 2.68v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(10.73u_1^{11}, 10.73u_2^{12}, 10.73u_3^{13}; 10.73u_4^{14}, 10.73u_2^{12}, 10.73u_5^{15}\right) + \\ (0.49u_1^{21}, 0.49u_2^{22}, 0.49u_3^{23}; 0.49u_4^{24}, 0.49\,u_2^{22}, 0.49u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(9v_1^{11}, 11v_2^{12}, 13v_3^{13};\; 6v_4^{14}, 11v_2^{12}, 15v_5^{15}) + \\ (.47v_1^{21}, .54v_2^{22}, .57v_3^{23};\; .42v_4^{24}, .54v_2^{22}, .60v_5^{25}) + \\ (4.24v_1^{31}, 4.24v_2^{32}, 4.24v_3^{33};\; 4.24v_4^{34}, 4.24v_2^{32}, 4.24v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(1.00u_1^{11}, 1.00u_2^{12}, 1.00\,u_3^{13}; 1.00u_4^{14}, 1.00u_2^{12}, 1.00u_5^{15}\right) + \\ (.084u_1^{21}, .084u_2^{22}, .084u_3^{23}; .084u_4^{24}, .084u_2^{22}, .084u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(10v_1^{11}, 13v_2^{12}, 15v_3^{13};\; 8v_4^{14}, 13v_2^{12}, 17v_5^{15}) + \\ (.19v_1^{21}, .21v_2^{22}, .24v_3^{23};\; .18\,v_4^{24}, .21v_2^{22}, .25v_5^{25}) + \\ (2.08v_1^{31}, 2.08v_2^{32}, 2.08v_3^{33}; 2.08v_4^{34}, 2.08v_2^{32}, 2.08v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix}\left(2.88u_1^{11}, 2.88u_2^{12}, 2.88u_3^{13}; 2.88u_4^{14}, 2.88u_2^{12}, 2.88u_5^{15}\right) + \\ (0.20u_1^{21}, 0.20\,u_2^{22}, 0.20u_3^{23}; 0.20u_4^{24}, 0.20u_2^{22}, 0.20u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(11v_1^{11}, 14v_2^{12}, 16v_3^{13};\; 9v_4^{14}, 14\,v_2^{12}, 19v_5^{15}) + \\ (.25v_1^{21}, .29v_2^{22}, .31v_3^{23};\; .23v_4^{24}, .29v_2^{22}, .33v_5^{25}) + \\ (4.73v_1^{31}, 4.73v_2^{32}, 4.73v_3^{33}; 4.73v_4^{34}, 4.73v_2^{32}, 4.73v_5^{35})\end{bmatrix}$$

$$\begin{bmatrix} \left(4.97u_1^{11}, 4.97u_2^{12}, 4.97\, u_3^{13}; 4.97u_4^{14}, 4.97u_2^{12}, 4.97u_5^{15}\right) + \\ (0.85u_1^{21}, 0.85u_2^{22}, 0.85u_3^{23}; 0.85u_4^{24}, 0.85u_2^{22}, 0.85u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (18v_1^{11}, 21v_2^{12}, 24v_3^{13};\ 15v_4^{14}, 21v_2^{12}, 27v_5^{15}) + \\ (.28v_1^{21}, .29v_2^{22}, .30v_3^{23}; .24v_4^{24}, .29v_2^{22}, .31v_5^{25}) + \\ (4.58v_1^{31}, 4.58v_2^{32}, 4.58v_3^{33}; 4.58\, v_4^{34}, 4.58v_2^{32}, 4.58v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(3.94u_1^{11}, 3.94u_2^{12}, 3.94\, u_3^{13}; 3.94u_4^{14}, 3.94u_2^{12}, 3.94u_5^{15}\right) + \\ (0.28u_1^{21}, 0.28u_2^{22}, 0.28u_3^{23}; 0.28u_4^{24}, 0.28u_2^{22}, 0.28u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (11v_1^{11}, 14v_2^{12}, 17v_3^{13};\ 9v_4^{14}, 14v_2^{12}, 19v_5^{15}) + \\ (.21v_1^{21}, .23v_2^{22}, .24v_3^{23};\ .19v_4^{24}, .23\, v_2^{22}, .25v_5^{25}) + \\ (1.33v_1^{31}, 1.33\, v_2^{32}, 1.33v_3^{33}; 1.33v_4^{34}, 1.33v_2^{32}, 1.33v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(2.16u_1^{11}, 2.16\, u_2^{12}, 2.16\, u_3^{13};\ 2.16u_4^{14}, 2.16u_2^{12}, 2.16\, u_5^{15}\right) + \\ (0.30u_1^{21}, 0.30\, u_2^{22}, 0.30u_3^{23};\ 0.30u_4^{24}, 0.30u_2^{22}, 0.30u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (9\, v_2^{12}, 12v_2^{12}, 13v_3^{13};\ 7v_4^{14}, 12v_2^{12}, 16\, v_5^{15}) + \\ (.56v_1^{21}, .60v_2^{22}, .62v_3^{23};\ .52v_4^{24}, .60v_2^{22}, .64v_5^{25}) + \\ (2.98v_1^{31}, 2.98v_2^{32}, 2.98v_3^{33};\ 2.98v_4^{34}, 2.98\, v_2^{32}, 2.98v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(2.58u_1^{11}, 2.58u_2^{12}, 2.58u_3^{13};\ 2.58u_4^{14}, 2.58u_2^{12}, 2.58u_5^{15}\right) + \\ (0.15u_1^{21}, 0.15u_2^{22}, 0.15u_3^{23};\ 0.15u_4^{24}, 0.15u_2^{22}, 0.15u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (6v_1^{11}, 9v_2^{12}, 12v_3^{13};\ 6v_4^{14}, 9v_2^{12}, 12\, v_5^{15}) + \\ (.16v_1^{21}, .17v_2^{22}, .19v_3^{23};\ .13v_4^{24}, .17v_2^{22}, .20v_5^{25}) + \\ (1.48v_1^{31}, 1.48v_2^{32}, 1.48\, v_3^{33}; 1.48v_4^{34}, 1.48v_2^{32}, 1.48v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(1.00u_1^{11}, 1.00u_2^{12}, 1.00u_3^{13};\ 1.00u_4^{14}, 1.00u_2^{12}, 1.00u_5^{15}\right) + \\ (0.13u_1^{21}, 0.13\, u_2^{22}, 0.13u_3^{23};\ 0.13u_4^{24}, 0.13u_2^{22}, 0.13u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (5v_1^{11}, 7v_2^{12}, 9v_3^{13};\ 3v_4^{14}, 7v_2^{12}, 10v_5^{15}) + \\ (.10v_1^{21}, .11v_2^{22}, .12v_3^{23};\ .08v_4^{24}, .11\, v_2^{22}, .13v_5^{25}) + \\ (0.75v_1^{31}, 0.75v_2^{32}, 0.75v_3^{33};\ 0.75v_4^{34}, 0.75v_2^{32}, 0.75v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(1.73u_1^{11}, 1.73u_2^{12}, 1.73u_3^{13};\ 1.73u_4^{14}, 1.73u_2^{12}, 1.73u_5^{15}\right) + \\ (0.13u_1^{21}, 0.13\ u_2^{22}, 0.13u_3^{23};\ 0.13u_4^{24}, 0.13u_2^{22}, 0.13u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (4v_1^{11}, 6v_2^{12}, 8v_3^{13};\ 3v_4^{14}, 6v_2^{12}, 9v_5^{15}) + \\ (.15v_1^{21}, .18v_2^{22}, .20v_3^{23};\ .12v_4^{24}, .18v_2^{22}, .21v_5^{25}) + \\ (0.87v_1^{31}, 0.87v_2^{32}, 0.87v_3^{33};\ 0.87v_4^{34}, 0.87v_2^{32}, 0.87v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(1.26u_1^{11}, 1.26u_2^{12}, 1.26u_3^{13};\ 1.26u_4^{14}, 1.26u_2^{12}, 1.26u_5^{15}\right) + \\ (0.069u_1^{21}, 0.069u_2^{22}, 0.069u_3^{23}; 0.069u_4^{24}, 0.069u_2^{22}, 0.069u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (8v_1^{11}, 10v_2^{12}, 13v_3^{13};\ 6v_4^{14}, 10v_2^{12}, 14\ v_5^{15}) + \\ (.18v_1^{21}, .20v_2^{22}, .22v_3^{23};\ .15v_4^{24}, .20v_2^{22}, .23v_5^{25}) + \\ (0.90v_1^{31}, 0.90v_2^{32}, 0.90\ v_3^{33};\ 0.90v_4^{34}, 0.90v_2^{32}, 0.90v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} \left(3.53u_1^{11}, 3.53u_2^{12}, 3.53u_3^{13};\ 3.53u_4^{14}, 3.53u_2^{12}, 3.53u_5^{15}\right) + \\ (0.26u_1^{21}, 0.26u_2^{22}, 0.26u_3^{23};\ 0.26u_4^{24}, 0.26u_2^{22}, 0.26\ u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (9v_1^{11}, 11v_2^{12}, 13\ v_3^{13};\ 7v_4^{14}, 11v_2^{12}, 15v_5^{15}) + \\ (.19v_1^{21}, .22\ v_2^{22}, .24\ v_3^{23};\ .16v_4^{24}, .22v_2^{22}, .26v_5^{25}) + \\ (1.51v_1^{31}, 1.51v_2^{32}, 1.51v_3^{33};\ 1.51v_4^{34}, 1.51v_2^{32}, 1.51v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix} (5.98, 5.98, 5.98;\ 5.98, 5.98, 5.98)\left(u_1^{11}, u_2^{12}, u_3^{13}, u_4^{14}, u_2^{12}, u_5^{15}\right) + \\ (0.27, 0.27, 0.27;\ 0.27, 0.27, 0.27)\ (u_1^{21}, u_2^{22}, u_3^{23}, u_4^{24}, u_2^{22}, u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (9, 11, 14;\ 8, 11, 15)\left(v_1^{11}, v_2^{12}, v_3^{13}, v_4^{14}, v_2^{12}, v_5^{15}\right) + \\ (.20, .21, .23;\ .18, .21, .24)\left(v_1^{21}, v_2^{22}, v_3^{23}, v_4^{24}, v_2^{22}, v_5^{25}\right) + \\ (2.98, 2.98, 2.98;\ 2.98, 2.98, 2.98)\left(v_1^{31}, v_2^{32}, v_3^{33}, v_4^{34}, v_2^{32}, v_5^{35}\right) \end{bmatrix}$$

$$\begin{bmatrix} \left(1.79u_1^{11}, 1.79u_2^{12}, 1.79u_3^{13}; 1.79u_4^{14}, 1.79u_2^{12}, 1.79u_5^{15}\right) + \\ (0.16u_1^{21}, 0.16u_2^{22}, 0.16u_3^{23};\ 0.16u_4^{24}, 0.16u_2^{22}, 0.16u_5^{25}) \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} (7v_1^{11}, 10v_2^{12}, 12v_3^{13};\ 6v_4^{14}, 10v_2^{12}, 14v_5^{15}) + \\ (.21v_1^{21}, .22v_2^{22}, .24v_3^{23};\ .17v_4^{24}, .22v_2^{22}, .26v_5^{25}) + \\ (1.99v_1^{31}, 1.99v_2^{32}, 1.99v_3^{33};\ 1.99v_4^{34}, 1.99v_2^{32}, 1.99v_5^{35}) \end{bmatrix}$$

$$\begin{bmatrix}(7.10u_1^{11}, 7.10u_2^{12}, 7.10u_3^{13};\ 7.10u_4^{14}, 7.10u_2^{12}, 7.10u_5^{15}) + \\ (0.37u_1^{21}, 0.37u_2^{22}, 0.37u_3^{23};\ 0.37u_4^{24}, 0.37u_2^{22}, 0.37u_5^{25})\end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix}(6v_1^{11}, 8v_2^{12}, 10v_3^{13};\ 5\ v_4^{14}, 8v_2^{12}, 12v_5^{15}) + \\ (.25v_1^{21}, .39v_2^{22}, .44v_3^{23}, .19v_4^{24}, .39v_2^{22}, .47\ v_5^{25}) + \\ (3.64v_1^{31}, 3.64v_2^{32}, 3.64v_3^{33};\ 3.64v_4^{34}, 3.64v_2^{32}, 3.64v_5^{35})\end{bmatrix}$$

$$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,$$

$$(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i)\ \succcurlyeq \varepsilon, \forall i = 1,2,3, \varepsilon > 0.$$

Step 2: Using the addition of triangular intuitionistic fuzzy numbers, defined in Section 5.1.2, the optimistic intuitionistic fuzzy CCR DEA model-5.21 and pessimistic intuitionistic fuzzy CCR DEA model-5.21 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.22 and pessimistic intuitionistic fuzzy CCR DEA model-5.22respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.22

$$\text{Maximize}\left[\tilde{E}_1^{BEST} \approx (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}\ , E_{12}^{BEST}, E_{15}^{BEST})\right.$$

$$\left. = \left(\frac{\begin{pmatrix}1.85u_1^{11} + 0.33u_1^{21}, 1.85\ u_2^{12} + 0.33u_2^{22}, 1.85u_3^{13} +\ 0.33\ u_3^{23}; \\ 1.85u_4^{14} +\ 0.33\ u_4^{24}, 1.85u_2^{12} + 0.33u_2^{22}, 1.85u_5^{15} + 0.33u_5^{25}\end{pmatrix}}{\begin{pmatrix}15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}, 17v_2^{12} +\ .44v_2^{22} +\ 9.16v_2^{32}, \\ 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}; 13v_4^{14} + .26\ v_4^{24} + 9.16v_4^{34}, \\ 17v_2^{12} +\ .44v_2^{22} +\ 9.16\ v_2^{32}, 21v_5^{15} +\ .63v_5^{25} + 9.16v_5^{35}\end{pmatrix}} \right)\right]$$

Subject to

$$\begin{pmatrix}1.85u_1^{11} + 0.33u_1^{21}, 1.85\ u_2^{12} + 0.33u_2^{22}, 1.85u_3^{13} +\ 0.33\ u_3^{23}, \\ 1.85u_4^{14} +\ 0.33\ u_4^{24}, 1.85u_2^{12} + 0.33u_2^{22}, 1.85u_5^{15} + 0.33u_5^{25}\end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}, 17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32}, \\ 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}; 13v_4^{14} + .26\ v_4^{24} + 9.16v_4^{34}, \\ 17v_2^{12} + \ .44v_2^{22} + \ 9.16\ v_2^{32}, 21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35} \end{pmatrix},$$

$$\begin{pmatrix} 0.67u_1^{11} + 0.41u_1^{21}, 0.67u_2^{12} + 0.41u_2^{22}, 0.67u_3^{13} + 0.41u_3^{23}; \\ 0.67u_4^{14} + 0.41u_4^{24}, 0.67\ u_2^{12} + 0.41u_2^{22}, 0.67u_5^{15} + 0.41u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 14v_1^{11} + .46v_1^{21} + 2.68v_1^{31}, 16v_2^{12} + .54v_2^{22} + \ 2.68\ v_2^{32}, \\ 18v_3^{13} + .65v_3^{23} + 2.68v_3^{33}; 12v_4^{14} + .38v_4^{24} + 2.68v_4^{34}, \\ 16v_2^{12} + .54v_2^{22} + 2.68v_2^{32}, 20v_5^{15} + .73v_5^{25} + 2.68v_5^{35} \end{pmatrix},$$

$$\begin{pmatrix} 10.73u_1^{11} + 0.49u_1^{21}, 10.73u_2^{12} + 0.49u_2^{22}, 10.73u_3^{13} + \ 0.49u_3^{23}; \\ 10.73u_4^{14} + 0.49u_4^{24}, 10.73u_2^{12} + 0.49\ u_2^{22}, 10.73u_5^{15} + 0.49u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .47v_1^{21} + 4.24v_1^{31}, 11v_2^{12} + .54v_2^{22} + 4.24v_2^{32}, \\ 13v_3^{13} + .57v_3^{23} + 4.24v_3^{33};\ 6v_4^{14} + .42v_4^{24} + 4.24v_4^{34}, \\ 11v_2^{12} + \ .54v_2^{22} + \ 4.24v_2^{32}, 15v_5^{15} + \ .60v_5^{25} + \ 4.24v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.00u_1^{11} + .084u_1^{21}, 1.00u_2^{12} + .084u_2^{22}, 1.00\ u_3^{13} + .084u_3^{23}; \\ 1.00u_4^{14} + .084u_4^{24}, 1.00u_2^{12} + .084u_2^{22}, 1.00u_5^{15} + .084u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 10v_1^{11} + .19v_1^{21} + 2.08v_1^{31}, 13v_2^{12} + .21v_2^{22} + 2.08v_2^{32}, \\ 15v_3^{13} + .24v_3^{23} + 2.08v_3^{33};\ 8v_4^{14} + .18v_4^{24} + 2.08v_4^{34}, \\ 13v_2^{12} + .21v_2^{22} + \ 2.08v_2^{32}, 17v_5^{15} + \ .25v_5^{25} + \ 2.08v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21},; \\ 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 11v_1^{11} + .25v_1^{21} + 4.73v_1^{31}, 14v_2^{12} + .29v_2^{22} + 4.73v_2^{32}, \\ 16v_3^{13} + .31v_3^{23} + 4.73v_3^{33};\ 9v_4^{14} + .23v_4^{24} + 4.73v_4^{34}, \\ 14v_2^{12} + .29v_2^{22} + 4.73v_2^{32}, 19v_5^{15} + \ .33v_5^{25} + \ 4.73v_4^{34} \end{pmatrix},$$

$$\begin{pmatrix} 4.97u_1^{11} + .85u_1^{21}, 4.97u_1^{11} + .85u_1^{21}, 497u_1^{11} + .85u_1^{21},; \\ 4.97u_1^{11} + .85u_1^{21}, 4.97u_1^{11} + .85u_1^{21}, 4.97u_1^{11} + .85u_1^{21} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 18v_1^{11} + .28v_1^{21} + 4.58v_1^{31}, 21v_2^{12} + .29v_2^{22} + 4.58v_2^{32}, \\ 24v_3^{13} + .30v_3^{23} + 4.58v_3^{33};\ 15v_4^{14} + .24v_4^{24} + 4.58v_4^{34}, \\ 21v_2^{12} + .29v_2^{22} + \ 4.58v_2^{32}, 27v_5^{15} + \ .31v_5^{25} + \ 4.58v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 3.94u_1^{11} + 0.28u_1^{21}, 3.94u_2^{12} + 0.28u_2^{22}, 3.94\ u_3^{13} + 0.28u_3^{23}; \\ 3.94u_4^{14} + 0.28u_4^{24}, 3.94u_2^{12} + 0.28u_2^{22}, 3.94u_5^{15} + 0.28u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 11v_1^{11} + .21v_1^{21} + 1.33v_1^{31}, 14v_2^{12} + .23v_2^{22} + 1.33v_2^{32}, \\ 17v_3^{13} + .24v_3^{23} + 1.33v_3^{33};\ 9v_4^{14} + .19v_4^{24} + 1.33v_4^{34}, \\ 14v_2^{12} + .23v_2^{22} + \ 1.33v_2^{32}, 19v_5^{15} + \ .25v_5^{25} + \ 1.33v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 2.16u_1^{11} + 0.30u_1^{21}, 2.16u_2^{12} + 0.30u_2^{22}, 2.16\ u_3^{13} + 0.30u_3^{23}; \\ 2.16u_4^{14} + 0.30u_4^{24}, 2.16u_2^{12} + 0.30u_2^{22}, 2.16u_5^{15} + 0.30u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .56v_1^{21} + 2.98v_1^{31}, 12v_2^{12} + .60v_2^{22} + 2.98v_2^{32}, \\ 13v_3^{13} + .62v_3^{23} + 2.98v_3^{33};\ 7v_4^{14} + .52v_4^{24} + 2.98v_4^{34}, \\ 12v_2^{12} + .60v_2^{22} + \ 2.98v_2^{32}, 16v_5^{15} + \ .64v_5^{25} + 2.98v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 2.58u_1^{11} + 0.15u_1^{21}, 2.58u_2^{12} + 0.15u_2^{22}, 2.58\ u_3^{13} + 0.15u_3^{23}; \\ 2.58u_4^{14} + 0.15u_4^{24}, 2.58u_2^{12} + 0.15u_2^{22}, 2.58u_5^{15} + 0.15u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 6v_1^{11} + .16v_1^{21} + 1.48v_1^{31}, 9v_2^{12} + .17v_2^{22} + 1.48v_2^{32}, \\ 12v_3^{13} + .19v_3^{23} + 1.48v_3^{33};\ 6v_4^{14} + .13v_4^{24} + 1.48v_4^{34}, \\ 9v_2^{12} + .17v_2^{22} + \ 1.48v_2^{32}, 12v_5^{15} + \ .20v_5^{25} + 1.48v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.00u_1^{11} + 0.13u_1^{21}, 1.00u_2^{12} + 0.13u_2^{22}, 1.00\ u_3^{13} + 0.13u_3^{23}; \\ 1.00u_4^{14} + 0.13u_4^{24}, 1.00u_2^{12} + 0.13u_2^{22}, 1.00u_5^{15} + 0.13u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 5v_1^{11} + .10v_1^{21} + .75v_1^{31}, 7v_2^{12} + .11v_2^{22} + .75v_2^{32}, \\ 9v_3^{13} + .12v_3^{23} + .75v_3^{33};\ 3v_4^{14} + .08v_4^{24} + .75v_4^{34}, \\ 7v_2^{12} + .11v_2^{22} + \ .75v_2^{32}, 10v_5^{15} + \ .13v_5^{25} + .75v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.73u_1^{11} + 0.13u_1^{21}, 1.73u_2^{12} + 0.13u_2^{22}, 1.73\, u_3^{13} + 0.13u_3^{23}; \\ 1.73u_4^{14} + 0.13u_4^{24}, 1.73u_2^{12} + 0.13u_2^{22}, 1.73u_5^{15} + 0.13u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 4v_1^{11} + .15v_1^{21} + .87v_1^{31}, 6v_2^{12} + .18v_2^{22} + .87v_2^{32}, \\ 8v_3^{13} + .20v_3^{23} + .87v_3^{33};\ 3v_4^{14} + .12v_4^{24} + .87v_4^{34}, \\ 6v_2^{12} + .18v_2^{22} + .87v_2^{32}, 9v_5^{15} + .21v_5^{25} + .87v_2^{32} \end{pmatrix}$$

$$\begin{pmatrix} 1.26u_1^{11} + 0.069u_1^{21}, 1.26u_2^{12} + 0.069u_2^{22}, 1.26\, u_3^{13} + 0.069u_3^{23}; \\ 1.26u_4^{14} + 0.069u_4^{24}, 1.26u_2^{12} + 0.069u_2^{22}, 1.26u_5^{15} + 0.069u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 8v_1^{11} + .18v_1^{21} + .90v_1^{31}, 10v_2^{12} + .20v_2^{22} + .90v_2^{32}, \\ 13v_3^{13} + .22v_3^{23} + .90v_3^{33};\ 6v_4^{14} + .15v_4^{24} + .90v_4^{34}, \\ 10v_2^{12} + .20v_2^{22} + .90v_2^{32}, 14v_5^{15} + .23v_5^{25} + .90v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 3.53u_1^{11} + 0.26u_1^{21}, 3.53u_2^{12} + 0.26u_2^{22}, 3.53\, u_3^{13} + 0.26u_3^{23}; \\ 3.53u_4^{14} + 0.26u_4^{24}, 3.53u_2^{12} + 0.26u_2^{22}, 3.53u_5^{15} + 0.26u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .19v_1^{21} + 1.51v_1^{31}, 11v_2^{12} + .22v_2^{22} + 1.51v_2^{32}, \\ 13v_3^{13} + .24v_3^{23} + 1.51v_3^{33};\ 7v_4^{14} + .16v_4^{24} + 1.51v_4^{34}, \\ 11v_2^{12} + .22v_2^{22} + 1.51v_2^{32}, 15v_5^{15} + .26v_5^{25} + 1.51v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 5.98u_1^{11} + 0.27u_1^{21}, 5.98u_2^{12} + 0.27u_2^{22}, 5.98\, u_3^{13} + 0.27u_3^{23}; \\ 5.98u_4^{14} + 0.27u_4^{24}, 5.98u_2^{12} + 0.27, 5.98u_5^{15} + 0.27u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .20v_1^{21} + 2.98v_1^{31}, 11v_2^{12} + .21v_2^{22} + 2.98v_2^{32}, \\ 14v_3^{13} + .23v_3^{23} + 2.98v_3^{33};\ 8v_4^{14} + .18v_4^{24} + 2.98v_4^{34}, \\ 11v_2^{12} + .21v_2^{22} + 2.98v_2^{32}, 15v_5^{15} + .24v_5^{25} + 2.98v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.79u_1^{11} + 0.16u_1^{21}, 1.79u_2^{12} + 0.16u_2^{22}, 1.79\, u_3^{13} + 0.16u_3^{23}; \\ 1.79u_4^{14} + 0.16u_4^{24}, 1.79u_2^{12} + 0.16, 1.79u_5^{15} + 0.16u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 7v_1^{11} + .21v_1^{21} + 1.99v_1^{31}, 10v_2^{12} + .22v_2^{22} + 1.99v_2^{32}, \\ 12v_3^{13} + .24v_3^{23} + 1.99v_3^{33}; 6v_4^{14} + .17v_4^{24} + 1.99v_4^{34}, \\ 7v_2^{12} + .22v_2^{22} + 1.99v_2^{32}, 14v_5^{15} + .26v_5^{25} + 1.99v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 7.10u_1^{11} + 0.37u_1^{21}, 7.10u_2^{12} + 0.37u_2^{22}, 7.10\, u_3^{13} + 0.37u_3^{23}; \\ 7.10u_4^{14} + 0.37u_4^{24}, 7.10u_2^{12} + 0.37, 7.10u_5^{15} + 0.37u_5^{25} \end{pmatrix} \preccurlyeq$$

$$\begin{pmatrix} 6v_1^{11} + .25v_1^{21} + 3.64v_1^{31}, 8v_2^{12} + .39v_2^{22} + 3.64v_2^{32}, \\ 10v_3^{13} + .44v_3^{23} + 3.64v_3^{33}; 5v_4^{14} + .19v_4^{24} + 3.64v_4^{34}, \\ 8v_2^{12} + .39v_2^{22} + 3.64v_2^{32}, 12v_5^{15} + .47v_5^{25} + 3.64v_2^{32} \end{pmatrix},$$

$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq \varepsilon, \forall r = 1,2,$

$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right) \succcurlyeq \varepsilon, \forall i = 1,2,3, \varepsilon > 0.$

Pessimistic intuitionistic fuzzy CCR DEA model-5.22

$$\text{Minimize}\left[\tilde{E}_1^{BEST} \approx (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}, E_{12}^{BEST}, E_{15}^{BEST})\right.$$

$$= \left[\left(\frac{\begin{pmatrix} 1.85u_1^{11} + 0.33u_1^{21}, 1.85\, u_2^{12} + 0.33u_2^{22}, 1.85u_3^{13} + 0.33\, u_3^{23}; \\ 1.85u_4^{14} + 0.33\, u_4^{24}, 1.85u_2^{12} + 0.33u_2^{22}, 1.85u_5^{15} + 0.33u_5^{25} \end{pmatrix}}{\begin{pmatrix} 15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}, 17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}, \\ 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}; 13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}, \\ 17v_2^{12} + .44v_2^{22} + 9.16\, v_2^{32}, 21v_5^{15} + .63v_5^{25} + 9.16v_5^{35} \end{pmatrix}}\right)\right]$$

Subject to

$$\begin{pmatrix} 1.85u_1^{11} + 0.33u_1^{21}, 1.85\, u_2^{12} + 0.33u_2^{22}, 1.85u_3^{13} + 0.33\, u_3^{23}; \\ 1.85u_4^{14} + 0.33\, u_4^{24}, 1.85u_2^{12} + 0.33u_2^{22}, 1.85u_5^{15} + 0.33u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}, 17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}, \\ 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}; 13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}, \\ 17v_2^{12} + .44v_2^{22} + 9.16\, v_2^{32}, 21v_5^{15} + .63v_5^{25} + 9.16v_5^{35} \end{pmatrix},$$

$$\begin{pmatrix} 0.67u_1^{11} + 0.41u_1^{21}, 0.67u_2^{12} + 0.41u_2^{22}, 0.67u_3^{13} + 0.41u_3^{23}; \\ 0.67u_4^{14} + 0.41u_4^{24}, 0.67\ u_2^{12} + 0.41u_2^{22}, 0.67u_5^{15} + 0.41u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 14v_1^{11} + .46v_1^{21} + 2.68v_1^{31}, 16v_2^{12} + .54v_2^{22} + \ 2.68\ v_2^{32}, \\ 18v_3^{13} + .65v_3^{23} + 2.68v_3^{33}; 12v_4^{14} + .38v_4^{24} + 2.68v_4^{34}, \\ 16v_2^{12} + .54v_2^{22} + 2.68v_2^{32}, 20v_5^{15} + .73v_5^{25} + 2.68v_5^{35} \end{pmatrix},$$

$$\begin{pmatrix} 10.73u_1^{11} + 0.49u_1^{21}, 10.73u_2^{12} + 0.49u_2^{22}, 10.73u_3^{13} + \ 0.49u_3^{23}; \\ 10.73u_4^{14} + 0.49u_4^{24}, 10.73u_2^{12} + 0.49\ u_2^{22}, 10.73u_5^{15} + 0.49u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .47v_1^{21} + 4.24v_1^{31}, 11v_2^{12} + .54v_2^{22} + 4.24v_2^{32}, \\ 13v_3^{13} + .57v_3^{23} + 4.24v_3^{33};\ 6v_4^{14} + .42v_4^{24} + 4.24v_4^{34}, \\ 11v_2^{12} + \ .54v_2^{22} + \ 4.24v_2^{32}, 15v_5^{15} + \ .60v_5^{25} + \ 4.24v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.00u_1^{11} + .084u_1^{21}, 1.00u_2^{12} + .084u_2^{22}, 1.00\ u_3^{13} + .084u_3^{23}; \\ 1.00u_4^{14} + .084u_4^{24}, 1.00u_2^{12} + .084u_2^{22}, 1.00u_5^{15} + .084u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 10v_1^{11} + .19v_1^{21} + 2.08v_1^{31}, 13v_2^{12} + .21v_2^{22} + 2.08v_2^{32}, \\ 15v_3^{13} + .24v_3^{23} + 2.08v_3^{33};\ 8v_4^{14} + .18v_4^{24} + 2.08v_4^{34}, \\ 13v_2^{12} + .21v_2^{22} + \ 2.08v_2^{32}, 17v_5^{15} + \ .25v_5^{25} + \ 2.08v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21}, ; \\ 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21}, 2.88u_1^{11} + .20u_1^{21} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 11v_1^{11} + .25v_1^{21} + 4.73v_1^{31}, 14v_2^{12} + .29v_2^{22} + 4.73v_2^{32}, \\ 16v_3^{13} + .31v_3^{23} + 4.73v_3^{33};\ 9v_4^{14} + .23v_4^{24} + 4.73v_4^{34}, \\ 14v_2^{12} + .29v_2^{22} + 4.73v_2^{32}, 19v_5^{15} + \ .33v_5^{25} + \ 4.73v_4^{34} \end{pmatrix},$$

$$\begin{pmatrix} 18v_1^{11} + .28v_1^{21} + 4.58v_1^{31}, 21v_2^{12} + .29v_2^{22} + 4.58v_2^{32}, \\ 24v_3^{13} + .30v_3^{23} + 4.58v_3^{33};\ 15v_4^{14} + .24v_4^{24} + 4.58v_4^{34}, \\ 21v_2^{12} + .29v_2^{22} + \ 4.58v_2^{32}, 27v_5^{15} + \ .31v_5^{25} + \ 4.58v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 3.94u_1^{11} + 0.28u_1^{21}, 3.94u_2^{12} + 0.28u_2^{22}, 3.94\ u_3^{13} + 0.28u_3^{23}; \\ 3.94u_4^{14} + 0.28u_4^{24}, 3.94u_2^{12} + 0.28u_2^{22}, 3.94u_5^{15} + 0.28u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 11v_1^{11} + .21v_1^{21} + 1.33v_1^{31}, 14v_2^{12} + .23v_2^{22} + 1.33v_2^{32}, \\ 17v_3^{13} + .24v_3^{23} + 1.33v_3^{33};\ 9v_4^{14} + .19v_4^{24} + 1.33v_4^{34}, \\ 14v_2^{12} + .23v_2^{22} +\ 1.33v_2^{32}, 19v_5^{15} +\ .25v_5^{25} +\ 1.33v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 2.16u_1^{11} + 0.30u_1^{21}, 2.16u_2^{12} + 0.30u_2^{22}, 2.16\ u_3^{13} + 0.30u_3^{23}; \\ 2.16u_4^{14} + 0.30u_4^{24}, 2.16u_2^{12} + 0.30u_2^{22}, 2.16u_5^{15} + 0.30u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .56v_1^{21} + 2.98v_1^{31}, 12v_2^{12} + .60v_2^{22} + 2.98v_2^{32}, \\ 13v_3^{13} + .62v_3^{23} + 2.98v_3^{33};\ 7v_4^{14} + .52v_4^{24} + 2.98v_4^{34}, \\ 12v_2^{12} + .60v_2^{22} +\ 2.98v_2^{32}, 16v_5^{15} +\ .64v_5^{25} + 2.98v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 2.58u_1^{11} + 0.15u_1^{21}, 2.58u_2^{12} + 0.15u_2^{22}, 2.58\ u_3^{13} + 0.15u_3^{23}; \\ 2.58u_4^{14} + 0.15u_4^{24}, 2.58u_2^{12} + 0.15u_2^{22}, 2.58u_5^{15} + 0.15u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 6v_1^{11} + .16v_1^{21} + 1.48v_1^{31}, 9v_2^{12} + .17v_2^{22} + 1.48v_2^{32}, \\ 12v_3^{13} + .19v_3^{23} + 1.48v_3^{33};\ 6v_4^{14} + .13v_4^{24} + 1.48v_4^{34}, \\ 9v_2^{12} + .17v_2^{22} +\ 1.48v_2^{32}, 12v_5^{15} +\ .20v_5^{25} + 1.48v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.00u_1^{11} + 0.13u_1^{21}, 1.00u_2^{12} + 0.13u_2^{22}, 1.00\ u_3^{13} + 0.13u_3^{23}; \\ 1.00u_4^{14} + 0.13u_4^{24}, 1.00u_2^{12} + 0.13u_2^{22}, 1.00u_5^{15} + 0.13u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 5v_1^{11} + .10v_1^{21} + .75v_1^{31}, 7v_2^{12} + .11v_2^{22} + .75v_2^{32}, \\ 9v_3^{13} + .12v_3^{23} + .75v_3^{33};\ 3v_4^{14} + .08v_4^{24} + .75v_4^{34}, \\ 7v_2^{12} + .11v_2^{22} +\ .75v_2^{32}, 10v_5^{15} +\ .13v_5^{25} + .75v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.73u_1^{11} + 0.13u_1^{21}, 1.73u_2^{12} + 0.13u_2^{22}, 1.73\ u_3^{13} + 0.13u_3^{23}; \\ 1.73u_4^{14} + 0.13u_4^{24}, 1.73u_2^{12} + 0.13u_2^{22}, 1.73u_5^{15} + 0.13u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 4v_1^{11} + .15v_1^{21} + .87v_1^{31}, 6v_2^{12} + .18v_2^{22} + .87v_2^{32}, \\ 8v_3^{13} + .20v_3^{23} + .87v_3^{33};\ 3v_4^{14} + .12v_4^{24} + .87v_4^{34}, \\ 6v_2^{12} + .18v_2^{22} +\ .87v_2^{32}, 9v_5^{15} +\ .21v_5^{25} + .87v_2^{32} \end{pmatrix}$$

$$\begin{pmatrix} 1.26u_1^{11} + 0.069u_1^{21}, 1.26u_2^{12} + 0.069u_2^{22}, 1.26\ u_3^{13} + 0.069u_3^{23}; \\ 1.26u_4^{14} + 0.069u_4^{24}, 1.26u_2^{12} + 0.069u_2^{22}, 1.26u_5^{15} + 0.069u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 8v_1^{11} + .18v_1^{21} + .90v_1^{31}, 10v_2^{12} + .20v_2^{22} + .90v_2^{32}, \\ 13v_3^{13} + .22v_3^{23} + .90v_3^{33};\ 6v_4^{14} + .15v_4^{24} + .90v_4^{34}, \\ 10v_2^{12} + .20v_2^{22} +\ .90v_2^{32}, 14v_5^{15} +\ .23v_5^{25} + .90v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 3.53u_1^{11} + 0.26u_1^{21}, 3.53u_2^{12} + 0.26u_2^{22}, 3.53\ u_3^{13} + 0.26u_3^{23}; \\ 3.53u_4^{14} + 0.26u_4^{24}, 3.53u_2^{12} + 0.26u_2^{22}, 3.53u_5^{15} + 0.26u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .19v_1^{21} + 1.51v_1^{31}, 11v_2^{12} + .22v_2^{22} + 1.51v_2^{32}, \\ 13v_3^{13} + .24v_3^{23} + 1.51v_3^{33};\ 7v_4^{14} + .16v_4^{24} + 1.51v_4^{34}, \\ 11v_2^{12} + .22v_2^{22} +\ 1.51v_2^{32}, 15v_5^{15} +\ .26v_5^{25} + 1.51v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 5.98u_1^{11} + 0.27u_1^{21}, 5.98u_2^{12} + 0.27u_2^{22}, 5.98\ u_3^{13} + 0.27u_3^{23}; \\ 5.98u_4^{14} + 0.27u_4^{24}, 5.98u_2^{12} + 0.27, 5.98u_5^{15} + 0.27u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 9v_1^{11} + .20v_1^{21} + 2.98v_1^{31}, 11v_2^{12} + .21v_2^{22} + 2.98v_2^{32}, \\ 14v_3^{13} + .23v_3^{23} + 2.98v_3^{33};\ 8v_4^{14} + .18v_4^{24} + 2.98v_4^{34}, \\ 11v_2^{12} + .21v_2^{22} +\ 2.98v_2^{32}, 15v_5^{15} +\ .24v_5^{25} + 2.98v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 1.79u_1^{11} + 0.16u_1^{21}, 1.79u_2^{12} + 0.16u_2^{22}, 1.79\ u_3^{13} + 0.16u_3^{23}; \\ 1.79u_4^{14} + 0.16u_4^{24}, 1.79u_2^{12} + 0.16, 1.79u_5^{15} + 0.16u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 7v_1^{11} + .21v_1^{21} + 1.99v_1^{31}, 10v_2^{12} + .22v_2^{22} + 1.99v_2^{32}, \\ 12v_3^{13} + .24v_3^{23} + 1.99v_3^{33}; 6v_4^{14} + .17v_4^{24} + 1.99v_4^{34}, \\ 7v_2^{12} + .22v_2^{22} +\ 1.99v_2^{32}, 14v_5^{15} +\ .26v_5^{25} + 1.99v_2^{32} \end{pmatrix},$$

$$\begin{pmatrix} 7.10u_1^{11} + 0.37u_1^{21}, 7.10u_2^{12} + 0.37u_2^{22}, 7.10\ u_3^{13} + 0.37u_3^{23}; \\ 7.10u_4^{14} + 0.37u_4^{24}, 7.10u_2^{12} + 0.37, 7.10u_5^{15} + 0.37u_5^{25} \end{pmatrix} \succcurlyeq$$

$$\begin{pmatrix} 6v_1^{11} + .25v_1^{21} + 3.64v_1^{31}, 8v_2^{12} + .39v_2^{22} + 3.64v_2^{32}, \\ 10v_3^{13} + .44v_3^{23} + 3.64v_3^{33}; 5v_4^{14} + .19v_4^{24} + 3.64v_4^{34}, \\ 8v_2^{12} + .39v_2^{22} +\ 3.64v_2^{32}, 12v_5^{15} +\ .47v_5^{25} + 3.64v_2^{32} \end{pmatrix},$$

$(u_1^r, u_2^r, u_3^r;\ u_4^r, u_2^r, u_5^r) \succcurlyeq\ \varepsilon, \forall r = 1,2,$

$\left(v_1^i, v_2^i, v_3^i; v_4^i, v_2^i, v_5^i\right) \succcurlyeq \varepsilon, \forall i = 1,2,3, \varepsilon > 0.$

Step 2: Using the relation $(a_1, a_2, a_3; a_4, a_2, a_5) \preccurlyeq (b_1, b_2, b_3; b_4, b_2, b_5) \Longrightarrow a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3, a_4 \leq b_4, a_5 \leq b_5$ and the division of triangular intuitionistic fuzzy numbers, defined in Section 5.1.2, the optimistic intuitionistic fuzzy CCR DEA model-5.22 and pessimistic intuitionistic fuzzy CCR DEA model-5.22 can be transformed into optimistic intuitionistic fuzzy CCR DEA model-5.23 and pessimistic intuitionistic fuzzy CCR DEA model-5.23 respectively.

Optimistic intuitionistic fuzzy CCR DEA model-5.23

$$\text{Maximize}\left[\tilde{E}_1^{BEST} \approx (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}, E_{12}^{BEST}, E_{15}^{BEST})\right.$$

$$\left.\approx \begin{pmatrix} \frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}}, \frac{1.85\, u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}}, \frac{1.85u_3^{13} + 0.33\, u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}}; \\ \frac{1.85u_4^{14} + 0.33\, u_4^{24}}{21v_5^{15} + .63v_5^{25} + 9.16v_5^{35}}, \frac{1.85u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + .44v_2^{22} + 9.16\, v_2^{32}}, \frac{1.85u_5^{15} + 0.33u_5^{25}}{13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}} \end{pmatrix}\right]$$

Subject to

$$1.85u_1^{11} + 0.33u_1^{21} \leq 15v_1^{11} + .39v_1^{21} + 9.16v_1^{31},$$

$$1.85u_1^{12} + 0.33u_1^{22} \leq 17v_2^{12} + .44v_2^{22} + 9.16v_2^{32},$$

$$1.85u_1^{13} + 0.33u_1^{23} \leq 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33},$$

$$1.85u_1^{14} + 0.33u_1^{24} \leq 13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34},$$

$$1.85u_1^{15} + 0.33u_1^{25} \leq 21v_5^{15} + .63v_5^{25} + 9.16v_5^{35}$$

$$0.67u_1^{11} + 0.41u_1^{21} \leq 14v_1^{11} + .46v_1^{21} + 2.68v_1^{31},$$

$$0.67u_1^{12} + 0.41u_1^{22} \leq 16v_2^{12} + .54v_2^{22} + 2.68\, v_2^{32}$$

$$0.67u_1^{13} + 0.41u_1^{23} \leq 18v_3^{13} + .65v_3^{23} + 2.68v_3^{33},$$

$$0.67u_1^{14} + 0.41u_1^{24} \leq 12v_4^{14} + .38v_4^{24} + 2.68v_4^{34},$$

$$0.67u_1^{15} + 0.41u_1^{25} \leq 20v_5^{15} + .73v_5^{25} + 2.68v_5^{35},$$

$$10.73u_1^{11} + 0.49u_1^{21} \leq 9v_1^{11} + .47v_1^{21} + 4.24v_1^{31},$$

$$10.73u_1^{12} + 0.49u_1^{22} \leq 11v_2^{12} + .54v_2^{22} + 4.24v_2^{32},$$

$$10.73u_1^{13} + 0.49u_1^{23} \leq 13v_3^{13} + .57v_3^{23} + 4.24v_3^{33};$$

$$10.73u_1^{14} + 0.49u_1^{24} \leq 6v_4^{14} + .42v_4^{24} + 4.24v_4^{34},$$

$$10.73u_1^{15} + 0.49u_1^{25} \leq 15v_5^{15} + .60v_5^{25} + 4.24v_2^{35},$$

$$1.00u_1^{11} + .084u_1^{21} \leq 10v_1^{11} + .19v_1^{21} + 2.08v_1^{31},$$

$$1.00u_1^{12} + .084u_1^{22} \leq 13v_2^{12} + .21v_2^{22} + 2.08v_2^{32},$$

$$1.00u_1^{13} + .084u_1^{23} \leq 15v_3^{13} + .24v_3^{23} + 2.08v_3^{33};$$

$$1.00u_1^{14} + .084u_1^{24} \leq 8v_4^{14} + .18v_4^{24} + 2.08v_4^{34},$$

$$1.00u_1^{15} + .084u_1^{25} \leq 17v_5^{15} + .25v_5^{25} + 2.08v_2^{35},$$

$$2.88u_1^{11} + .20u_1^{21} \leq 11v_1^{11} + .25v_1^{21} + 4.73v_1^{31},$$

$$2.88u_1^{12} + .20u_1^{22} \leq 14v_2^{12} + .29v_2^{22} + 4.73v_2^{32},$$

$$2.88u_1^{11} + .20u_1^{21} \leq 16v_3^{13} + .31v_3^{23} + 4.73v_3^{33};$$

$$2.88u_1^{14} + .20u_1^{24} \leq 9v_4^{14} + .23v_4^{24} + 4.73v_4^{34},$$

$$2.88u_1^{15} + .20u_1^{25} \leq 19v_5^{15} + .33v_5^{25} + 4.73v_4^{35}$$

$$4.97u_1^{11} + .85u_1^{21} \leq 18v_1^{11} + .28v_1^{21} + 4.58v_1^{31},$$

$$4.97u_1^{12} + .85u_1^{22} \leq 21v_2^{12} + .29v_2^{22} + 4.58v_2^{32},$$

$$4.97u_1^{13} + .85u_1^{23} \leq 24v_3^{13} + .30v_3^{23} + 4.58v_3^{33};$$

$$4.97u_1^{14} + .85u_1^{24} \leq 15v_4^{14} + .24v_4^{24} + 4.58v_4^{34},$$

$$4.97u_1^{15} + .85u_1^{25} \leq 27v_5^{15} + .31v_5^{25} + 4.58v_2^{35},$$

$$3.94u_1^{11} + 0.28u_1^{21} \leq 11v_1^{11} + .21v_1^{21} + 1.33v_1^{31},$$

$$3.94u_1^{12} + 0.28u_1^{22} \leq 14v_2^{12} + .23v_2^{22} + 1.33v_2^{32},$$

$$3.94u_1^{13} + 0.28u_1^{23} \leq 17v_3^{13} + .24v_3^{23} + 1.33v_3^{33};$$

$$3.94u_1^{14} + 0.28u_1^{24} \leq 9v_4^{14} + .19v_4^{24} + 1.33v_4^{34},$$

$$3.94u_1^{15} + 0.28u_1^{25} \leq 19v_5^{15} + .25v_5^{25} + 1.33v_2^{35}$$

$$2.16u_1^{11} + 0.30u_1^{21} \leq 9v_1^{11} + .56v_1^{21} + 2.98v_1^{31},$$

$$2.16u_1^{12} + 0.30u_1^{22} \leq 12v_2^{12} + .60v_2^{22} + 2.98v_2^{32},$$

$$2.16u_1^{13} + 0.30u_1^{23} \leq 13v_3^{13} + .62v_3^{23} + 2.98v_3^{33};$$

$$2.16u_1^{14} + 0.30u_1^{24} \leq 7v_4^{14} + .52v_4^{24} + 2.98v_4^{34},$$

$$2.16u_1^{15} + 0.30u_1^{25} \leq 16v_5^{15} + .64v_5^{25} + 2.98v_2^{35},$$

$$2.58u_1^{11} + 0.15u_1^{21} \leq 6v_1^{11} + .16v_1^{21} + 1.48v_1^{31},$$

$$2.58u_1^{12} + 0.15u_1^{22} \leq 9v_2^{12} + .17v_2^{22} + 1.48v_2^{32},$$

$$2.58u_1^{13} + 0.15u_1^{23} \leq 12v_3^{13} + .19v_3^{23} + 1.48v_3^{33};$$

$$2.58u_1^{14} + 0.15u_1^{24} \leq 6v_4^{14} + .13v_4^{24} + 1.48v_4^{34},$$

$$2.58u_1^{15} + 0.15u_1^{25} \leq 12v_5^{15} + \ .20v_5^{25} + 1.48v_2^{35},$$

$$1.00u_1^{11} + 0.13u_1^{21} \leq 5v_1^{11} + .10v_1^{21} + .75v_1^{31},$$

$$1.00u_1^{12} + 0.13u_1^{22} \leq 7v_2^{12} + .11v_2^{22} + .75v_2^{32},$$

$$1.00u_1^{13} + 0.13u_1^{23} \leq 9v_3^{13} + .12v_3^{23} + .75v_3^{33};$$

$$1.00u_1^{14} + 0.13u_1^{24} \leq 3v_4^{14} + .08v_4^{24} + .75v_4^{34},$$

$$1.00u_1^{15} + 0.13u_1^{25} \leq 10v_5^{15} + \ .13v_5^{25} + .75v_2^{35}$$

$$1.73u_1^{11} + 0.13u_1^{21} \leq 4v_1^{11} + .15v_1^{21} + .87v_1^{31},$$

$$1.73u_1^{12} + 0.13u_1^{22} \leq 6v_2^{12} + .18v_2^{22} + .87v_2^{32},$$

$$1.73u_1^{13} + 0.13u_1^{23} \leq 8v_3^{13} + .20v_3^{23} + .87v_3^{33};$$

$$1.73u_1^{14} + 0.13u_1^{24} \leq 3v_4^{14} + .12v_4^{24} + .87v_4^{34},$$

$$1.73u_1^{15} + 0.13u_1^{25} \leq 9v_5^{15} + \ .21v_5^{25} + .87v_2^{35}$$

$$1.26u_1^{11} + 0.069u_1^{21} \leq 8v_1^{11} + .18v_1^{21} + .90v_1^{31},$$

$$1.26u_1^{12} + 0.069u_1^{22} \leq 10v_2^{12} + .20v_2^{22} + .90v_2^{32},$$

$$1.26u_1^{13} + 0.069u_1^{23} \leq 13v_3^{13} + .22v_3^{23} + .90v_3^{33};$$

$$1.26u_1^{14} + 0.069u_1^{24} \leq 6v_4^{14} + .15v_4^{24} + .90v_4^{34},$$

$$1.26u_1^{15} + 0.069u_1^{25} \leq 14v_5^{15} + \ .23v_5^{25} + .90v_2^{35}$$

$$3.53u_1^{11} + 0.26u_1^{21} \leq 9v_1^{11} + .19v_1^{21} + 1.51v_1^{31},$$

$$3.53u_1^{12} + 0.26u_1^{22} \leq 11v_2^{12} + .22v_2^{22} + 1.51v_2^{32},$$

$$3.53u_1^{13} + 0.26u_1^{23} \leq \ 13v_3^{13} + .24v_3^{23} + 1.51v_3^{33}$$

$$3.53u_1^{14} + 0.26u_1^{24} \leq \ 7v_4^{14} + .16v_4^{24} + 1.51v_4^{34},$$

$$3.53u_1^{15} + 0.26u_1^{25} \leq 15v_5^{15} + \ .26v_5^{25} + 1.51v_2^{35}$$

$$5.98u_1^{11} + 0.27u_1^{21} \leq 9v_1^{11} + .20v_1^{21} + 2.98v_1^{31},$$

$$5.98u_1^{12} + 0.27u_1^{22} \leq 11v_2^{12} + .21v_2^{22} + 2.98v_2^{32},$$

$$5.98u_1^{13} + 0.27u_1^{23} \leq 14v_3^{13} + .23v_3^{23} + 2.98v_3^{33};$$

$$5.98u_1^{14} + 0.27u_1^{24} \leq 8v_4^{14} + .18v_4^{24} + 2.98v_4^{34},$$

$$5.98u_1^{15} + 0.27u_1^{25} \leq 15v_5^{15} + \ .24v_5^{25} + 2.98v_2^{35}$$

$$1.79u_1^{11} + 0.16u_1^{21} \leq 7v_1^{11} + .21v_1^{21} + 1.99v_1^{31},$$

$$1.79u_1^{12} + 0.16u_1^{22} \leq 10v_2^{12} + .22v_2^{22} + 1.99v_2^{32},$$

$$1.79u_1^{13} + 0.16u_1^{23} \leq \ 12v_3^{13} + .24v_3^{23} + 1.99v_3^{33};$$

$$1.79u_1^{14} + 0.16u_1^{24} \leq 6v_4^{14} + .17v_4^{24} + 1.99v_4^{34},$$

$$1.79u_1^{15} + 0.16u_1^{25} \leq 14v_5^{15} + \ .26v_5^{25} + 1.99v_2^{35},$$

$$7.10u_1^{11} + 0.37u_1^{21} \leq \ 6v_1^{11} + .25v_1^{21} + 3.64v_1^{31},$$

$$7.10u_2^{12} + 0.37u_2^{22} \leq 8v_2^{12} + .39v_2^{22} + 3.64v_2^{32},$$

$$7.10\ u_3^{13} + 0.37u_3^{23} \leq 10v_3^{13} + .44v_3^{23} + 3.64v_3^{33},$$

$$7.10u_4^{14} + 0.37u_4^{24} \leq 5v_4^{14} + .19v_4^{24} + 3.64v_4^{34},$$

$$7.10u_5^{15} + 0.37u_5^{25} \leq 12v_5^{15} + \ .47v_5^{25} + 3.64v_2^{35}$$

$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r \ \forall, r = 1,2,$

$\varepsilon \leq v_4^i \leq v_1^i \leq v_2^i \leq v_3^i \leq v_5^i \ \forall, i = 1,2,3, \qquad \varepsilon > 0.$

Pessimistic intuitionistic fuzzy CCR DEA model-5.23

$$\text{Minimize}\left[\tilde{E}_1^{BEST} \approx (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}\ , E_{12}^{BEST}, E_{15}^{BEST})\right.$$

$$\left.\approx\left(\begin{array}{c}\frac{1.85u_1^{11}+0.33u_1^{21}}{19v_3^{13}+.46v_3^{23}+9.16v_3^{33}},\frac{1.85\,u_2^{12}+0.33u_2^{22}}{17v_2^{12}+\ .44v_2^{22}+\ 9.16v_2^{32}},\frac{1.85u_3^{13}+\ 0.33\,u_3^{23}}{15v_1^{11}+.39v_1^{21}+9.16v_1^{31}};\\ \frac{1.85u_4^{14}+\ 0.33\,u_4^{24}}{21v_5^{15}+\ .63v_5^{25}+9.16v_5^{35}},\frac{1.85u_2^{12}+0.33u_2^{22}}{17v_2^{12}+\ .44v_2^{22}+\ 9.16\,v_2^{32}},\frac{1.85u_5^{15}+0.33u_5^{25}}{13v_4^{14}+.26\,v_4^{24}+9.16v_4^{34}}\end{array}\right)\right]$$

Subject to

$$1.85u_1^{11}+0.33u_1^{21}\geq 15v_1^{11}+.39v_1^{21}+9.16v_1^{31},$$

$$1.85u_1^{12}+0.33u_1^{22}\geq 17v_2^{12}+\ .44v_2^{22}+\ 9.16v_2^{32},$$

$$1.85u_1^{13}+0.33u_1^{23}\geq 19v_3^{13}+.46v_3^{23}+9.16v_3^{33},$$

$$1.85u_1^{14}+0.33u_1^{24}\geq 13v_4^{14}+.26\,v_4^{24}+9.16v_4^{34},$$

$$1.85u_1^{15}+0.33u_1^{25}\geq 21v_5^{15}+\ .63v_5^{25}+9.16v_5^{35}$$

$$0.67u_1^{11}+0.41u_1^{21}\geq 14v_1^{11}+.46v_1^{21}+2.68v_1^{31},$$

$$0.67u_1^{12}+0.41u_1^{22}\geq 16v_2^{12}+.54v_2^{22}+\ 2.68\,v_2^{32}$$

$$0.67u_1^{13}+0.41u_1^{23}\geq 18v_3^{13}+.65v_3^{23}+2.68v_3^{33},$$

$$0.67u_1^{14}+0.41u_1^{24}\geq 12v_4^{14}+.38v_4^{24}+2.68v_4^{34},$$

$$0.67u_1^{15}+0.41u_1^{25}\geq 20v_5^{15}+.73v_5^{25}+2.68v_5^{35},$$

$$10.73u_1^{11}+0.49u_1^{21}\geq 9v_1^{11}+.47v_1^{21}+4.24v_1^{31},$$

$$10.73u_1^{12} + 0.49u_1^{22} \geq 11v_2^{12} + .54v_2^{22} + 4.24v_2^{32},$$

$$10.73u_1^{13} + 0.49u_1^{23} \geq 13v_3^{13} + .57v_3^{23} + 4.24v_3^{33};$$

$$10.73u_1^{14} + 0.49u_1^{24} \geq 6v_4^{14} + .42v_4^{24} + 4.24v_4^{34},$$

$$10.73u_1^{15} + 0.49u_1^{25} \geq 15v_5^{15} + \ .60v_5^{25} + \ 4.24v_2^{35},$$

$$1.00u_1^{11} + .084u_1^{21} \geq 10v_1^{11} + .19v_1^{21} + 2.08v_1^{31},$$

$$1.00u_1^{12} + .084u_1^{22} \geq 13v_2^{12} + .21v_2^{22} + 2.08v_2^{32},$$

$$1.00u_1^{13} + .084u_1^{23} \geq 15v_3^{13} + .24v_3^{23} + 2.08v_3^{33};$$

$$1.00u_1^{14} + .084u_1^{24} \geq 8v_4^{14} + .18v_4^{24} + 2.08v_4^{34},$$

$$1.00u_1^{15} + .084u_1^{25} \geq 17v_5^{15} + \ .25v_5^{25} + \ 2.08v_2^{35},$$

$$2.88u_1^{11} + .20u_1^{21} \geq 11v_1^{11} + .25v_1^{21} + 4.73v_1^{31},$$

$$2.88u_1^{12} + .20u_1^{22} \geq 14v_2^{12} + .29v_2^{22} + 4.73v_2^{32},$$

$$2.88u_1^{11} + .20u_1^{21} \geq 16v_3^{13} + .31v_3^{23} + 4.73v_3^{33};$$

$$2.88u_1^{14} + .20u_1^{24} \geq 9v_4^{14} + .23v_4^{24} + 4.73v_4^{34},$$

$$2.88u_1^{15} + .20u_1^{25} \geq 19v_5^{15} + \ .33v_5^{25} + \ 4.73v_4^{35}$$

$$4.97u_1^{11} + .85u_1^{21} \geq 18v_1^{11} + .28v_1^{21} + 4.58v_1^{31},$$

$$4.97u_1^{12} + .85u_1^{22} \geq 21v_2^{12} + .29v_2^{22} + 4.58v_2^{32},$$

$$4.97u_1^{13} + .85u_1^{23} \geq 24v_3^{13} + .30v_3^{23} + 4.58v_3^{33};$$

$$4.97u_1^{14} + .85u_1^{24} \geq 15v_4^{14} + .24v_4^{24} + 4.58v_4^{34},$$

$$4.97u_1^{15} + .85u_1^{25} \geq 27v_5^{15} + \ .31v_5^{25} + \ 4.58v_2^{35},$$

$$3.94u_1^{11} + 0.28u_1^{21} \geq 11v_1^{11} + .21v_1^{21} + 1.33v_1^{31},$$

$$3.94u_1^{12} + 0.28u_1^{22} \geq 14v_2^{12} + .23v_2^{22} + 1.33v_2^{32},$$

$$3.94u_1^{13} + 0.28u_1^{23} \geq 17v_3^{13} + .24v_3^{23} + 1.33v_3^{33};$$

$$3.94u_1^{14} + 0.28u_1^{24} \geq 9v_4^{14} + .19v_4^{24} + 1.33v_4^{34},$$

$$3.94u_1^{15} + 0.28u_1^{25} \geq 19v_5^{15} + \ .25v_5^{25} + \ 1.33v_2^{35}$$

$$2.16u_1^{11} + 0.30u_1^{21} \geq 9v_1^{11} + .56v_1^{21} + 2.98v_1^{31},$$

$$2.16u_1^{12} + 0.30u_1^{22} \geq 12v_2^{12} + .60v_2^{22} + 2.98v_2^{32},$$

$$2.16u_1^{13} + 0.30u_1^{23} \geq 13v_3^{13} + .62v_3^{23} + 2.98v_3^{33};$$

$$2.16u_1^{14} + 0.30u_1^{24} \geq 7v_4^{14} + .52v_4^{24} + 2.98v_4^{34},$$

$$2.16u_1^{15} + 0.30u_1^{25} \geq 16v_5^{15} + \ .64v_5^{25} + 2.98v_2^{35},$$

$$2.58u_1^{11} + 0.15u_1^{21} \geq 6v_1^{11} + .16v_1^{21} + 1.48v_1^{31},$$

$$2.58u_1^{12} + 0.15u_1^{22} \geq \ 9v_2^{12} + .17v_2^{22} + 1.48v_2^{32},$$

$$2.58u_1^{13} + 0.15u_1^{23} \geq 12v_3^{13} + .19v_3^{23} + 1.48v_3^{33};$$

$$2.58u_1^{14} + 0.15u_1^{24} \geq 6v_4^{14} + .13v_4^{24} + 1.48v_4^{34},$$

$$2.58u_1^{15} + 0.15u_1^{25} \geq 12v_5^{15} + \ .20v_5^{25} + 1.48v_2^{35},$$

$$1.00u_1^{11} + 0.13u_1^{21} \geq 5v_1^{11} + .10v_1^{21} + .75v_1^{31},$$

$$1.00u_1^{12} + 0.13u_1^{22} \geq 7v_2^{12} + .11v_2^{22} + .75v_2^{32},$$

$$1.00u_1^{13} + 0.13u_1^{23} \geq 9v_3^{13} + .12v_3^{23} + .75v_3^{33};$$

$$1.00u_1^{14} + 0.13u_1^{24} \geq 3v_4^{14} + .08v_4^{24} + .75v_4^{34},$$

$$1.00u_1^{15} + 0.13u_1^{25} \geq 10v_5^{15} + \ .13v_5^{25} + .75v_2^{35}$$

$$1.73u_1^{11} + 0.13u_1^{21} \geq 4v_1^{11} + .15v_1^{21} + .87v_1^{31},$$

$$1.73u_1^{12} + 0.13u_1^{22} \geq 6v_2^{12} + .18v_2^{22} + .87v_2^{32},$$

$$1.73u_1^{13} + 0.13u_1^{23} \geq 8v_3^{13} + .20v_3^{23} + .87v_3^{33};$$

$$1.73u_1^{14} + 0.13u_1^{24} \geq 3v_4^{14} + .12v_4^{24} + .87v_4^{34},$$

$$1.73u_1^{15} + 0.13u_1^{25} \geq 9v_5^{15} + \ .21v_5^{25} + .87v_2^{35}$$

$$1.26u_1^{11} + 0.069u_1^{21} \geq 8v_1^{11} + .18v_1^{21} + .90v_1^{31},$$

$$1.26u_1^{12} + 0.069u_1^{22} \geq 10v_2^{12} + .20v_2^{22} + .90v_2^{32},$$

$$1.26u_1^{13} + 0.069u_1^{23} \geq 13v_3^{13} + .22v_3^{23} + .90v_3^{33};$$

$$1.26u_1^{14} + 0.069u_1^{24} \geq 6v_4^{14} + .15v_4^{24} + .90v_4^{34},$$

$$1.26u_1^{15} + 0.069u_1^{25} \geq 14v_5^{15} + \ .23v_5^{25} + .90v_2^{35}$$

$$3.53u_1^{11} + 0.26u_1^{21} \geq 9v_1^{11} + .19v_1^{21} + 1.51v_1^{31},$$

$$3.53u_1^{12} + 0.26u_1^{22} \geq 11v_2^{12} + .22v_2^{22} + 1.51v_2^{32},$$

$$3.53u_1^{13} + 0.26u_1^{23} \geq \ 13v_3^{13} + .24v_3^{23} + 1.51v_3^{33}$$

$$3.53u_1^{14} + 0.26u_1^{24} \geq \ 7v_4^{14} + .16v_4^{24} + 1.51v_4^{34},$$

$$3.53u_1^{15} + 0.26u_1^{25} \geq 15v_5^{15} + \ .26v_5^{25} + 1.51v_2^{35}$$

$$5.98u_1^{11} + 0.27u_1^{21} \geq 9v_1^{11} + .20v_1^{21} + 2.98v_1^{31},$$

$$5.98u_1^{12} + 0.27u_1^{22} \geq 11v_2^{12} + .21v_2^{22} + 2.98v_2^{32},$$

$$5.98u_1^{13} + 0.27u_1^{23} \geq 14v_3^{13} + .23v_3^{23} + 2.98v_3^{33};$$

$$5.98u_1^{14} + 0.27u_1^{24} \geq 8v_4^{14} + .18v_4^{24} + 2.98v_4^{34},$$

$$5.98u_1^{15} + 0.27u_1^{25} \geq 15v_5^{15} + \ .24v_5^{25} + 2.98v_2^{35}$$

$$1.79u_1^{11} + 0.16u_1^{21} \geq 7v_1^{11} + .21v_1^{21} + 1.99v_1^{31},$$

$$1.79u_1^{12} + 0.16u_1^{22} \geq 10v_2^{12} + .22v_2^{22} + 1.99v_2^{32},$$

$$1.79u_1^{13} + 0.16u_1^{23} \geq \ 12v_3^{13} + .24v_3^{23} + 1.99v_3^{33};$$

$$1.79u_1^{14} + 0.16u_1^{24} \geq 6v_4^{14} + .17v_4^{24} + 1.99v_4^{34},$$

$$1.79u_1^{15} + 0.16u_1^{25} \geq 14v_5^{15} + \ .26v_5^{25} + 1.99v_2^{35},$$

$$7.10u_1^{11} + 0.37u_1^{21} \geq \ 6v_1^{11} + .25v_1^{21} + 3.64v_1^{31},$$

$$7.10u_2^{12} + 0.37u_2^{22} \geq 8v_2^{12} + .39v_2^{22} + 3.64v_2^{32},$$

$$7.10\ u_3^{13} + 0.37u_3^{23} \geq 10v_3^{13} + .44v_3^{23} + 3.64v_3^{33},$$

$$7.10u_4^{14} + 0.37u_4^{24} \geq 5v_4^{14} + .19v_4^{24} + 3.64v_4^{34},$$

$$7.10u_5^{15} + 0.37u_5^{25} \geq 12v_5^{15} + \ .47v_5^{25} + 3.64v_2^{35},$$

$\varepsilon \leq u_4^r \leq u_1^r \leq u_2^r \leq u_3^r \leq u_5^r \ \forall, r = 1,2,$

$\varepsilon \leq v_4^i \leq v_1^i \leq v_2^i \leq v_3^i \leq v_5^i \ \forall, i = 1,2,3, \qquad \varepsilon > 0.$

Step 3: The intuitionistic fuzzy optimal value $\left(\tilde{E}_1^{BEST} = (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}, E_{12}^{BEST}, E_{15}^{BEST})\right)$, representing the best relative intuitionistic fuzzy efficiency of 1^{st} DMU, as well as the intuitionistic fuzzy optimal value $\left(\tilde{E}_1^{WORST} = (E_{11}^{WORST}, E_{12}^{WORST}, E_{13}^{WORST}; E_{14}^{WORST}, E_{12}^{WORST}, E_{15}^{WORST})\right)$, representing the worst relative intuitionistic fuzzy efficiency of I^{st} DMU, can be obtained by solving the optimistic intuitionistic fuzzy CCR DEA model-5.23 and pessimistic intuitionistic fuzzy CCR DEA model-5.23 as follows:

Step 3(a): The optimal value (E_{14}^{BEST}) and (E_{14}^{WORST}) of the optimistic crisp CCR DEA model-5.24a and pessimistic crisp CCR DEA model-5.24a by solving optimistic crisp CCR DEA model-5.24b and pessimistic crisp CCR DEA model-5.24b equivalent to optimistic crisp CCR DEA model-5.24a and pessimistic crisp CCR DEA model-5.24a are 0.292 and 1 respectively .

Optimistic crisp CCR DEA model-5.24a

Maximize $\left[E_{14}^{BEST} = \frac{1.85u_4^{14} + \ 0.33\ u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}}\right]$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.38 with the following additional constraint:

$$\frac{1.85u_4^{14} + \ 0.33\ u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} \leq 1.$$

Optimistic crisp CCR DEA model-5.24b

Maximize$[E_{14}^{BEST} = 1.85u_4^{14} + \ 0.33\ u_4^{24}]$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR DEA model-5.24 with the following additional constraint:

$$21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35} = 1,$$

$1.85u_4^{14} + \ 0.33\ u_4^{24} \leq 21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}$.

Pessimistic crisp CCR DEA model-5.24a

Minimize $\left[E_{14}^{WORST} = \frac{1.85u_4^{14} + \ 0.33\ u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}}\right]$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraint:

$$\frac{1.85u_4^{14} + \ 0.33\ u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} \geq 1.$$

Pessimistic crisp CCR DEA model-5.24b

Minimize$[E_{14}^{WORST} = 1.85u_4^{14} + \ 0.33\ u_4^{24}]$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.24 with the following additional constraints:

$21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35} = 1$,

$1.85u_4^{14} + \ 0.33\ u_4^{24} \geq 21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}$.

Step (3b): The optimal value (E_{11}^{BEST}) and (E_{11}^{WORST}) of the optimistic crisp CCR DEA model-5.25a and pessimistic CCR DEA model-5.25a by solving optimistic crisp CCR DEA model-5.25b and pessimistic CCR DEA model-5.25b equivalent to optimistic crisp CCR DEA model-5.25a and pessimistic CCR DEA model-5.25a are 0.416 and 1 respectively .

Optimistic crisp CCR DEA model-5.25a

Maximize $\left[E_{11}^{BEST} = \dfrac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}}\right]$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$\frac{1.85u_4^{14} + \ 0.33\, u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} = 0.292,$$

$$0.292 \leq \frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} \leq 1.$$

Optimistic crisp CCR DEA model-5.25b

Maximize$[E_{11}^{BEST} = 1.85u_1^{11} + 0.33u_1^{21}]$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$19v_3^{13} + .46v_3^{23} + 9.16v_3^{33} = 1,$$

$$1.85u_4^{14} + \ 0.33\, u_4^{24} = 0.292\left(21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}\right),$$

$$0.292(19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}) \leq 1.85u_1^{11} + 0.33u_1^{21} \leq 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}.$$

Pessimistic crisp CCR DEA model-5.25a

Minimize $\left[E_{11}^{WORST} = \dfrac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}}\right]$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$\frac{1.85u_4^{14} + \ 0.33\, u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} = 1,$$

$$1 \leq \frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}}.$$

Pessimistic crisp CCR DEA model-5.25b

Minimize$[E_{11}^{WORST} = 1.85u_1^{11} + 0.33u_1^{21}]$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$19v_3^{13} + .46v_3^{23} + 9.16v_3^{33} = 1,$$

$$1.85u_4^{14} + \ 0.33\, u_4^{24} = 21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35},$$

$$19v_3^{13} + .46v_3^{23} + 9.16v_3^{33} \leq 1.85u_1^{11} + 0.33u_1^{21}.$$

Step (3c): The optimal value (E_{12}^{BEST}) and (E_{12}^{WORST}) of the optimistic crisp CCR DEA model-5.26a and pessimistic CCR DEA model-5.26a by solving optimistic crisp CCR DEA model-5.26b and pessimistic CCR DEA model-5.26b equivalent to optimistic crisp CCR DEA model-5.26a and pessimistic CCR DEA model-5.26b are 0.464 and 1 respectively .

Optimistic crisp CCR DEA model-5.26a

$$\text{Maximize}\left[E_{12}^{BEST} = \frac{1.85\,u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + \,.44v_2^{22} + \,9.16v_2^{32}}\right]$$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraint:

$$\frac{1.85u_4^{14} + \,0.33\,u_4^{24}}{21v_5^{15} + \,.63v_5^{25} + 9.16v_5^{35}} = 0.292,$$

$$\frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} = .416,$$

$$0.416 \leq \frac{1.85\,u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + \,.44v_2^{22} + \,9.16v_2^{32}} \leq 1.$$

Optimistic crisp CCR DEA model-5.26b

$$\text{Maximize}[E_{12}^{BEST} = 1.85\,u_2^{12} + 0.33u_2^{22}]$$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$17v_2^{12} + \,.44v_2^{22} + \,9.16v_2^{32} = 1,$$

$$1.85u_4^{14} + \,0.33\,u_4^{24} = 0.292\left(21v_5^{15} + \,.63v_5^{25} + 9.16v_5^{35}\right),$$

$$1.85u_1^{11} + 0.33u_1^{21} = 0.416(19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}),$$

$$0.416(17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}) \leq 1.85\, u_2^{12} + 0.33u_2^{22} \leq 17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}.$$

Pessimistic crisp CCR DEA model-5.26a

$$\text{Minimize}\left[E_{12}^{BEST} = \frac{1.85\, u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}}\right]$$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.38 with the following additional constraint:

$$\frac{1.85u_4^{14} + 0.33\, u_4^{24}}{21v_5^{15} + .63v_5^{25} + 9.16v_5^{35}} = 1,$$

$$\frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} = 1,$$

$$0.416 \leq \frac{1.85\, u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}}$$

Pessimistic crisp CCR DEA model-5.40b

$$\text{Minimize}[E_{12}^{BEST} = 1.85\, u_2^{12} + 0.33u_2^{22}]$$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$17v_2^{12} + .44v_2^{22} + 9.16v_2^{32} = 1,$$

$$1.85u_4^{14} + 0.33\, u_4^{24} = 21v_5^{15} + .63v_5^{25} + 9.16v_5^{35},$$

$$1.85u_1^{11} + 0.33u_1^{21} = 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33},$$

$$17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32} \leq 1.85\ u_2^{12} + 0.33u_2^{22}.$$

Step (3d): The optimal value (E_{13}^{BEST}) and (E_{13}^{WORST}) of the optimistic crisp CCR DEA model-5.27a and pessimistic CCR DEA model-5.27a by solving optimistic crisp CCR DEA model-5.27b and pessimistic CCR DEA model-5.27b equivalent to optimistic crisp CCR DEA model-5.27a and pessimistic CCR DEA model-5.27a are 0.464 and 1 respectively .

Optimistic crisp CCR DEA model-5.27a

$$\text{Maximize}\left[E_{13}^{BEST} = \frac{1.85u_3^{13} + \ 0.33\ u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}}\right]$$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$\frac{1.85u_4^{14} + \ 0.33\ u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} = 0.292,$$

$$\frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} = 0.416,$$

$$\frac{1.85\ u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32}} = 0.464,$$

$$0.464 \leq \frac{1.85u_3^{13} + \ 0.33\ u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}} \leq 1.$$

Optimistic crisp CCR DEA model-5.27b

$$\text{Maximize}[E_{13}^{BEST} = 1.85u_3^{13} + \ 0.33\ u_3^{23}]$$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraint:

$15v_1^{11} + .39v_1^{21} + 9.16v_1^{31} = 1,$

$1.85u_4^{14} + 0.33\, u_4^{24} = 0.292\big(21v_5^{15} + .63v_5^{25} + 9.16v_5^{35}\big),$

$1.85u_1^{11} + 0.33u_1^{21} = 0.416(19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}),$

$1.85\, u_2^{12} + 0.33u_2^{22} = 0.464(17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}),$

$0.464(15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}) \leq 1.85u_3^{13} + 0.33\, u_3^{23} \leq 15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}.$

Pessimistic crisp CCR DEA model-5.27a

$$\text{Minimize}\left[E_{13}^{WORST} = \frac{1.85u_3^{13} + 0.33\, u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}}\right]$$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraint:

$$\frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} = 1,$$

$$\frac{1.85u_4^{14} + 0.33\, u_4^{24}}{21v_5^{15} + .63v_5^{25} + 9.16v_5^{35}} = 1,$$

$$\frac{1.85\, u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + .44v_2^{22} + 9.16v_2^{32}} = 1,$$

$$1 \le \frac{1.85u_3^{13} + \ 0.33\ u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}}.$$

Pessimistic crisp CCR DEA model-5.27b

Minimize$[E_{13}^{WORST} = 1.85u_3^{13} + \ 0.33\ u_3^{23}]$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$15v_1^{11} + .39v_1^{21} + 9.16v_1^{31} = 1$$

$$1.85u_1^{11} + 0.33u_1^{21} = 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}$$

$$1.85u_4^{14} + \ 0.33\ u_4^{24} = 21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}$$

$$1.85\ u_2^{12} + 0.33u_2^{22} = 17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32}$$

$$15v_1^{11} + .39v_1^{21} + 9.16v_1^{31} \le 1.85u_3^{13} + \ 0.33\ u_3^{23}$$

Step 3: The optimal value (E_{15}^{BEST}) and (E_{15}^{WORST}) of the optimistic crisp CCR DEA model-5.28a and pessimistic CCR DEA model-5.28a by solving optimistic crisp CCR DEA model-5.28b and pessimistic CCR DEA model-5.28b equivalent to optimistic crisp CCR DEA model-5.28a and pessimistic CCR DEA model-5.28a are 0.464 and 1 respectively .

Optimistic crisp CCR DEA model-5.28a

Maximize $\left[E_{15}^{BEST} = \frac{1.85u_5^{15} + 0.33u_5^{25}}{13v_4^{14} + .26\ v_4^{24} + 9.16v_4^{34}}\right]$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$\frac{1.85u_4^{14} + \ 0.33\, u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} = 0.292,$$

$$\frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} = 0.416,$$

$$\frac{1.85\, u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32}} = 0.464,$$

$$\frac{1.85u_3^{13} + \ 0.33\, u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}} = 0.464,$$

$$0.464 \leq \frac{1.85u_5^{15} + 0.33u_5^{25}}{13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}} \leq 1.$$

Optimistic crisp CCR DEA model-5.28b

Maximize$\left[E_{15}^{BEST} = 1.85u_5^{15} + 0.33u_5^{25}\right]$

Subject to

Constraints of optimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34} = 1,$$

$$1.85u_1^{11} + 0.33u_1^{21} = 0.416(19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}),$$

$$1.85u_4^{14} + \ 0.33\, u_4^{24} = 0.292\left(21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}\right),$$

$$1.85\, u_2^{12} + 0.33u_2^{22} = 0.464(17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32}),$$

$1.85u_3^{13} + \ 0.33\, u_3^{23} = 0.464(15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}),$

$0.464(13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}) \leq 1.85u_5^{15} + 0.33u_5^{25} \leq 13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}.$

Pessimistic crisp CCR DEA model-5.28a

$$\text{Minimize}\left[E_{15}^{WORST} = \frac{1.85u_5^{15} + 0.33u_5^{25}}{13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}}\right]$$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraints:

$$\frac{1.85u_1^{11} + 0.33u_1^{21}}{19v_3^{13} + .46v_3^{23} + 9.16v_3^{33}} = 1,$$

$$\frac{1.85u_4^{14} + \ 0.33\, u_4^{24}}{21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35}} = 1,$$

$$\frac{1.85\, u_2^{12} + 0.33u_2^{22}}{17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32}} = 1,$$

$$\frac{1.85u_3^{13} + \ 0.33\, u_3^{23}}{15v_1^{11} + .39v_1^{21} + 9.16v_1^{31}} = 1,$$

$$1 \leq \frac{1.85u_5^{15} + 0.33u_5^{25}}{13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}}.$$

Pessimistic crisp CCR DEA model-5.28b

$$\text{Minimize}\left[E_{15}^{WORST} = \frac{1.85u_5^{15} + 0.33u_5^{25}}{13v_4^{14} + .26\, v_4^{24} + 9.16v_4^{34}}\right]$$

Subject to

Constraints of pessimistic intuitionistic fuzzy CCR fuzzy DEA model-5.23 with the following additional constraint:

$13v_4^{14} + .26\ v_4^{24} + 9.16v_4^{34} = 1,$

$1.85u_1^{11} + 0.33u_1^{21} = 19v_3^{13} + .46v_3^{23} + 9.16v_3^{33},$

$1.85u_4^{14} + \ 0.33\ u_4^{24} = 21v_5^{15} + \ .63v_5^{25} + 9.16v_5^{35},$

$1.85\ u_2^{12} + 0.33u_2^{22} = 17v_2^{12} + \ .44v_2^{22} + \ 9.16v_2^{32},$

$1.85u_3^{13} + \ 0.33\ u_3^{23} = 15v_1^{11} + .39v_1^{21} + 9.16v_1^{31},$

$13v_4^{14} + .26\ v_4^{24} + 9.16v_4^{34} \leq 1.85u_5^{15} + 0.33u_5^{25}.$

Step 4: Using the values of $E_{p1}^{BEST}, E_{p2}^{BEST}, E_{p3}^{BEST}; E_{p4}^{BEST}, E_{p5}^{BEST}$ and $E_{p1}^{WORST}, E_{p2}^{WORST}, E_{p3}^{WORST}; E_{p4}^{WORST}, E_{p2}^{WORST}, E_{p5}^{WORST}$, obtained in Step (3a) to Step (3e), the intuitionistic fuzzy optimal value $\left(\tilde{E}_1^{BEST} = (E_{11}^{BEST}, E_{12}^{BEST}, E_{13}^{BEST}; E_{14}^{BEST}, E_{12}^{BEST}, E_{15}^{BEST})\right)$ of optimistic intuitionistic fuzzy DEA model-5.23, representing the best relative intuitionistic fuzzy efficiency of Ist DMU, is $(0.416,0.464,0.464; 0.292,0.464,0.464)$as well as intuitionistic fuzzy optimal value $\left(\tilde{E}_1^{WORST} = (E_{11}^{WORST}, E_{12}^{WORST}, E_{13}^{WORST}; E_{14}^{WORST}, E_{12}^{WORST}, E_{15}^{WORST})\right)$ of pessimistic intuitionistic fuzzy DEA model-5.23, representing the worst relative intuitionistic fuzzy efficiency of Ist DMU is $(1,1,1; 1,1,1)$.

Step 5: The crisp optimal value $\left(E_1^{BEST} = \Re(\tilde{E}_1^{BEST}) = \Re(0.416,0.464,0.464; 0.292,0.464,0.464)\right)$, representing the best relative crisp efficiency of Ist DMU is 0.437.

Step 6: The crisp optimal value $\left(E_1^{WORST} = \Re\left(\tilde{E}_1^{WORST}\right) = \Re(1,1,1;1,1,1)\right)$, representing the worst relative crisp efficiency of I^{st} DMU is 1.

5.7.3 Results

The best relative intuitionistic fuzzy efficiency as well as worst relative intuitionistic fuzzy efficiency and hence, the relative geometric crisp efficiency of all the DMUs, obtained by the proposed approach, are shown in Table 5.2.

Table 5.2 Best relative intuitionistic fuzzy efficiency, worst relative intuitionistic fuzzy efficiency and the relative geometric crisp efficiency of all the DMUs.

DMUs	Best relative intuitionistic fuzzy efficiency	Worst relative intuitionistic fuzzy efficiency	Relative crisp geometric efficiency
1	$(.416, .464, .464; .292, .464, .464)$	$(1, 1, 1; 1, 1, 1)$	.661
2	$(.742, .751, .763; .727, .751, .768)$	$(1, 1.243, 1.532;\ \ 1, 1.243, 1.352)$	.970
3	$(.940, .946, .949; .937, .947, .958)$	$(1.023, 1.245, 2.034; 1, 1.243, 3.65$	1.225
4	$(.238, .278, .305; .216, .278, .325)$	$(1,1,1.554; 1, 1, 1.554)$	.544
5	$(.345, .351, .353; .339, .351, .360)$	$(1, 1, 1; 1, 1, 1)$	.591
6	$(1, 1, 1; 1, 1, 1)$	$(1,1, 1.582; 1,1, 1.603)$	1.07
7	$(1, 1, 1; 1, 1, 1)$	$(1, 2.63,3.96; 1, 2.63, 10.21)$	1.825
8	$(.525, .525, .525; .525, .525, .525)$	$(1, 1.234, 1.952; 1, 1.234,2.018)$	.809
9	$(.623, .630, .635; .604, .630, .639)$	$(1, 2.349, 3.809; 1, 2.349, 6.023)$	1.268
10	$(.825, .830, .838; .823, .830, .842)$	$(1, 1, 1; 1, 1, 5.508)$	1.074

11	$(.716, .718, .723; .710, .718, .728)$	$(1, 2.938, 3.809; 1, 2.938, 4.432)$	1.405
12	$(.473, .473, .473; .473, .473, .473)$	$(1, 1, 1.247; 1,1, 1.513)$	.720
13	$(.820, .827, .830; .818, .827, .837)$	$(1, 3.253, 3.997; 1, 3.253, 10.908)$	1.758
14	$(.679, .685, .690; .672, .685, .693)$	$(1, 3.040, 4.411; 1, 3.040, 4.411)$	1.402
15	$(.732, .735, .739; 727, 735, 748)$	$(1, 1.853, 2.092; 1, 1.853, 3.453)$	1.173
16	$(.733, .737, .740; .730, .737, .742)$	$(1, 3.973, 5.298; 1, 3.973, 8.416)$	1.706

5.8 Conclusions

On the basis the present study, it can be concluded that there are flaws in the optimistic as well as pessimistic intuitionistic fuzzy CCR DEA model and in the method proposed by Puri and Yadav [83] and hence neither the intuitionistic fuzzy CCR DEA model proposed by, Puri and Yadav [83] nor the method proposed by Puri and Yadav [83] should be used for evaluating the best relative geometric crisp efficiency of DMUs. Also, to resolve the flaws of the intuitionistic fuzzy CCR DEA models, new intuitionistic fuzzy CCR DEA models are proposed. Further, a new approach is proposed to solve the proposed intuitionistic fuzzy CCR DEA models for evaluating the best relative geometric crisp efficiency of DMUs.

Chapter 6

Future Scope

The following may be treated as future directions:

1. The fuzzy/intuitionistic fuzzy CCR DEA models, proposed in thesis, are valid only if positive fuzzy/intuitionistic fuzzy numbers represents input and output data. However, in real life problem, there exist several real life problems in which some of the input data and/ or output data may be negative e.g., investment returns of mutual funds may be negative. Some researchers have developed some fuzzy DEA models to deal with negative data. However, after a deep study of these methods, it is concluded that there are some flaws in these methods. In future, it may be tried to develop new fuzzy/ intuitionistic fuzzy DEA models to deal with negative data.
2. In the thesis, the flaws in the exiting fuzzy/ intuitionistic fuzzy CCR DEA models are pointed out and to resolve the flaws, new fuzzy/ intuitionistic fuzzy CCR DEA models are proposed. There are several other fuzzy DEA models which are extension of fuzzy CCR DEA models. It can be easily verified that the flaws, pointed out in the fuzzy/ intuitionistic fuzzy CCR DEA models, also exists in the other fuzzy DEA models. In future, new fuzzy/ intuitionistic fuzzy DEA models may be developed to resolve the flaws of other existing fuzzy DEA models.
3. The approaches, proposed in this thesis, may be used to evaluate the exact best relative geometric crisp efficiency of real life problems.

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