FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

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CERTIFICATE

I hereby certify that the work presented in the thesis entitled " MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW" which is being presented for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Mahesh Kumar Sharma.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

The fixed charge bi-criterion transportation problem with restricted flow which is an extension of the fixed charge bi-criterion transportation problem has been studied in the present work. In this type of problem, there is a restriction on the total flow. In the fixed charge bi-criterion transportation problem a fixed cost called the setup cost is incurred for every origin. In the bi-criterion transportation problem the cost of transportation is directly proportional to the number of units transported, but on account of quantity discounts, price breaks etc .

The fixed-charge transportation problem (FCTP) is an extension of the classical transportation problem in which a fixed cost is incurred, independent of the amount transported, along with a variable cost that is proportional to the amount shipped. The introduction of fixed costs in addition to variable costs results in the objective function being a step function. Therefore, fixed-charge problems are usually solved using sophisticated analytical or computer software. This thesis deviates from that approach. It presents a simple algorithm for the solution of small fixed-charge problems with restricted flow. We present numerical examples to illustrate applications of the proposed method.

The present thesis consists of three chapters. The first chapter is introductory in nature. In which classical transportation problem, time minimizing transportation problem and brief survey on literature related to the topic has been discussed. In the second chapter the Fixed charge bi-criterion transportation problem with restricted flow given by Thirwani et al. (1997) is reviewed to find the cost-time trade-off pair. In chapter three Multi-index bi-criterion transportation problem with restricted flow has been formulated and separated into two related problems.

CONTENTS

CHAPTER-1

INTRODUCTION

Linear programming, or LP, is a method of allocating resources in an optimal way. It is one of the most widely used operation research tools and has been a decision making aid in almost all manufacturing industries and in financial and service organizations.

In the term linear programming refers to the mathematical programming. In the context, it refers to a planning process that allocates resources labor, material, machines, capital in the best possible (optimal) way so that costs are minimized or profits are maximized. In LP these resources are known as decision variables. The criterion for selecting the best values of decision variables (e.g. minimizing costs) is known as objective function. Limitations on resource availability from what is known as constraint set.

The word linear indicates that the criterion for selecting the best values of decision variables can be described by a linear function of these variables; that is, a mathematical function involving only the first power of variables with no cross products.

Classical transportation problem is a particular class of linear programming, which is associated with day-to-day activities in our real life and mainly deals with logistics. It helps in solving problems on distribution and transportation of resources from one place to another. The goods are transported from a set of m origins (e.g., factories) to a set of n destinations (e.g., shops) to meet the specific requirements. In other words, transportation problems deal with the transportation of a single product manufactured at different plants (supply origins) to a number of different warehouses (demand destinations). The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. And, it can be formulated as

$$\text{Minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} \le a_i, \quad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j, \quad j = 1, 2, ..., n$$

$$x_{ij} \ge 0 \text{ for all } i \text{ and } j.$$

Where i = 1,2,...,m are the origins

j = 1,2,..., n are the destinations

a_i = amount available at the i^{th} origin.

b_j = demand of the j^{th} destination.

x_{ij} = amount transported from the i^{th} origin to the j^{th} destination.

c_{ij} = variable cost for transporting one unit from the i^{th} origin to the j^{th} destination.

$$\text{if} \quad \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_j$$

Then the problem is a balanced transportation problem, otherwise it is unbalanced.

The aim is to minimize the objective function satisfying the above constraints. In classical transportation problem in linear programming the traditional objective is to minimize the total cost.

1.1. TIME MINIMIZING TRANSPORTATION PROBLEM

In a time minimizing transportation problem, the time of transporting goods from *m* origins to *n* destinations is minimized, satisfying certain conditions in respect of availabilities and source requirement at the destination.

Time minimization transportation problem can be formulated as

Minimize $$\left[\max_{(i,j)} t_{ij} \,/\, x_{ij} > 0\right]$$

subject to $$\sum_{j=1}^{n} x_{ij} = a_i \,,\ (i = 1,2,..., m),$$

$$\sum_{i=1}^{m} x_{ij} = b_j \,,\ (j = 1,2,..., n).$$

Here t_{ij} =time of transporting goods from i^{th} origin.

a_i = amount available at the i^{th} origin.

b_j = demand of the j^{th} destination.

For any given feasible solution, $X=[x_{ij}]$ satisfying the above constraints, the time of transportation is the maximum of t_{ij}'s among the cells in which there are positive allocations, i.e. corresponding to the solution *X*, the time of transportation is

$$Z = [\max t_{ij} \;/\; x_{ij} > 0]$$

The aim is to minimize the transportation problem. The time of transportation remains independent of the amount of commodity sent as long as $x_{ij} > 0$. It is assumed that

(i) The carriers have sufficient capacity to carry goods from an origin to a destination in a single trip.

(ii) They start simultaneously from the respective origin.

The time minimizing transportation problems are of importance when it is required to transport perishable goods. Sometimes there may exist emergency situations such as fire services, ambulance services, police services etc. when the time of transportation is of greater importance than the cost of transportation.

1.2. MULTI-OBJECTIVE TRANSPORTATION PROBLEM

The multi-objective transportation model is set to solve the transportation problem simultaneously associated with several objectives. Normally, existing multi-objective transportation models use a minimization of the total cost objective as one of their objectives. The other objectives may concern about quantity of goods delivered, underused capacity, energy consumption, total delivery time, etc.

Kasana and Kumar (2003) formulate the multi-objective transportation problem as follows:

Consider *m* origins and *n* destinations and also the quantities available at each origin and the quantities to be transported to each destination. The total quantities required at the destinations may differ from the total quantities available at the origins. For such situations, the problem is balanced by introducing fictitious origin or destination; whichever is needed in order to get precisely the same quantities at the origins and the destinations. Specifically, a balanced transportation problem is considered as it amounts to no loss of generality.

Suppose xij = amount transported from the i^{th} origin to the j^{th} destination and for each fixed k: $k = 0,1,\ldots,p\text{-}1$,α^k_{ij}, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$ be the units of parameter required for transporting one unit of the quantity from origin *i* to destination *j*. what is to be determined is the routing from origin *i* to destination *j* satisfying *p* objectives. The starting object is termed as primary and the other are classified as secondary.

The primary objective is to minimize

$$Z_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^0{}_{ij} x_{ij} \tag{1.1}$$

and for k= 1,2,. . .,(p-1), also to minimize

$$x_k = \max\{\alpha^k{}_{ij} : x_{ij} \geq 0,\ i = 1,2, . . . ,m,\ j = 1,2,. . .,n\} \tag{1.2}$$

in order of the priorities to be assigned under the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i\ ,\ (i = 1,2,...,m), \tag{1.3}$$

$$\sum_{i=1}^{m} x_{ij} = b_j\ ,\ (j = 1,2,...,n). \tag{1.4}$$

The problem formulated above has p objective functions given by equations (1.1) and (1.2).

Keep on viewing in this way we will discuss Fixed charge bi-criterion transportation problem with restricted flow and Multi-Index Fixed charge bi-criterion transportation problem transportation problem with restricted flow have been discussed in succeeding chapters, Fixed charge bi-criterion transportation problem with restricted flow is an extension of the fixed charge bi-criterion transportation problem. Multi-Index Fixed charge bi-criterion transportation problem is an extension of the fixed charge bi-criterion transportation problem and may be thought of as a block in which the layers in all the directions from restricted transportation problem.

1.3. LITRATURE SURVAY

The Extremum Difference method (EDM) concerning initial basic feasible solution (BFS) of a transportation problem has been conceived by Kasana and Kumar. This is simpler than Vogal approximation method (VAM). The algorithm is based on the principle: if an allocation is not made in the lowest cost cell of a row or column having the largest extremum difference, then the cost penalty per unit cost will be higher for any other choices of rows or column with other extreme difference. This causes in the objective function value.

Transportation problem (TP) was firstly developed by Hitchcock (1941).it usually aims to minimize the total transportation cost. Other objectives that can be set are a minimization of the total delivery time, a maximization of profit, etc. from the investigation; the entire existing objectives in single objective transportation model are represented by quantitative information. This may cause the negligence of some crucial points which cannot be described by quantitative data.

The time minimizing transportation problem has been studied by Hammer (1969), Garfinkal and Rao (1971) and Szware (1971). Hammer (1969) and Szware (1971) used labeling techniques to solve the problem of time minimizing. Garfinkel and Rao (1971) solved the problem by introducing a sufficiently large cost M on certain routes.

Sometimes there may exist emergency situations such as fire services, ambulance services, police services etc. when the time of transportation is of greater importance than the cost of transportation. Several methods for minimizing the time of transportation are also developed. Then Bhatia et al. (1975) developed a technique for minimizing time in a transportation problem. The procedure involved finite number of iterations and is based on moving from one basic feasible solution to another till the last solution is arrived at. The algorithm given by them consists of determination of an initial basic feasible solution which can be found by the methods applicable in the case of the common cost minimizing transportation problem and finding an adjacent better basic feasible solution, the procedure is repeated until no better adjacent basic feasible solution can be found. This repeated procedure deals with the determination of a cell not in the basis which, when introduced, will either reduce the time of transportation or reduce the allocation in at least one of the cells Є Q, where Q is the set of cell with positive allocations and corresponding time equal to the time of transportation.

The transportation with two-objectives known as the bi-criterion problem. It has been studied by many workers. In this type of problem one objective is primary and other is secondary. The primary objective is to minimize the total cost and secondary is to minimize the time of transportation.

In a classical transportation problem the cost of transportation is directly proportional to the number of units of the commodity transported. But in real world situations when a commodity is transported, a fixed cost is incurred in objective function. The fixed cost may represent the cost of renting a vehicle, landing fees in an airport, set up costs for machines in a manufacturing environment etc. such type of situation is formulated into a problem. This problem with bi-criterion transportation problem is called Fixed charge bi-criterion transportation problem.

The fixed charge transportation problem was originally formulated by Dantzig and Hirsch (1954). Then K.G.Murty(1968) solved the fixed charge problem by ranking the extreme points. After that several procedures for solving Fixed charge transportation problems were developed.

Also Basu et. al.(1994)developed an algorithm for the optimum time-cost trade-off in a fixed charge linear transportation problem giving same priority to cost and time. The fixed-charge transportation problem (FCTP) is an extension of the classical transportation problem in which a fixed cost is incurred, independent of the amount transported, along with a variable cost that is proportional to the amount shipped. The introduction of fixed costs in addition to variable costs results in the objective function being a step function. Therefore, fixed-charge problems are usually solved using sophisticated analytical or computer software.

The transportation problem considered in the classical transportation problem is generally a two-dimensional linear transportation problem. After fixed charge bi-criterion transportation problem Deepa Thirwani et al. (1997) review this algorithm and the gives the algorithm on Fixed charge bi-criterion transportation problem with restricted flow is introduced. Fixed charge bi-criterion transportation problem with restricted flow which is an extension of the fixed charge bi-criterion transportation problem. In this type of problem, there is a restriction on the total flow. In the fixed charge bi-criterion transportation problem a fixed cost called the setup cost is incurred for every origin. In the bi-criterion transportation problem the cost of transportation is directly proportional to the number of units transported, but on account of quantity discounts, price breaks etc. in this problem we keep on solving with initial basic feasible solution to an optimal solution. When we reach to the optimality then we stop the criteria.

Haley (1962) considered the Multi-index transportation problem and presented an algorithm to solve the Multi-index transportation problem where there are three indices. The method of solution is an extension of the Modi-method. Multi-level fixed charge problems are mathematical optimization problems in which the separable portion of the objective function is the sum of piecewise continuous functions of a single variable. Haley (1963) gave the theorems justifying the method and an extension of the necessary conditions laid down by Schell. He also described the application of the technique to two special transportation problem and showed that the ‘Three axial sums’ problem of Schell can be written as ‘Three planar sums’. Haley (1965) laid down a set of necessary conditions for a feasible solution to exist. He also proved that these conditions are sufficient. Morovek and Vlach (1967) presented the necessary conditions for the existence of the solution of the Multi-index transportation problem. Arnold and Soland (1969) described a branch and bound algorithm that finds a global solution to multi-level fixed charge problem. The algorithm has the feature that a good feasible solution is generated at the start. Moreover, at each step of the algorithm an additional feasible solution may be generated for comparison with the best solution found previously. Graham Smith (1971) gave the further necessary conditions for the existence of a solution to the Multi-index transportation problem.

1.4. PRESENT WORK

In the present thesis a fixed charge bi-criterion transportation problem, wherein there is a restriction on the total flow, is studied. An algorithm to find the efficient cost-time trade off pairs in a fixed charge bi-criterion transportation problem with restricted flow is presented. A related fixed charge bi-criterion transportation problem is formulated and the efficient cost-time trade off pairs to the given problem is shown to be derivable from this related problem.

CHAPTER-2

FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

2.1. INTRODUCTION

The Fixed charge bi-criterion transportation problem is an extension of Fixed charge bi-criterion transportation problem. In Fixed charge bi-criterion transportation problem a fixed cost is incurred for every origin. In the classical transportation problem the cost of transportation is directly proportional to the units transported but on account of quantity discounts, price breaks etc. the transportation may not be linear. Thus in real world situations, when a commodity is transported, a fixed cost is incurred in the objective function. The fixed cost may represent the cost of renting a vehicle, lending fees in an airport, setup costs for machines in a manufacturing environment, a new facility cost money to be constructed etc. it also costs money to operate.

The fixed charge bi-criterion transportation problem is given by

$$\text{Minimize} \quad \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} F_i, \ \ Max[t_{ij} \ / \ x_{ij} \geq 0] \right\}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$x_{ij} \geq 0, \ for\, i = 1, 2, ..., m; \ j = 1, 2, ..., n.$$

Where i = {1, 2 . . . m} is the set of origins

j = {1, 2 . . . n} is the set of destinations

x_{ij} = amount transported from the i^{th} origin to the j^{th} destination.

c_{ij}= variable cost for transporting one unit from the i^{th} origin to the j^{th} destination.

t_{ij}= time of transporting the product from the i^{th} origin to the j^{th} destination and is independent of the amount transported so long as $x_{ij} > 0$.

F_i = the fixed cost associated with the i^{th} origin.

a_i= amount available at the i^{th} origin.

b_j = demand of the j^{th} destination.

The total flow in the problem is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

2.2. FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

The Fixed charge bi-criterion transportation problem with restricted flow is an extension of Fixed charge bi-criterion transportation problem which is more complex to solve. If in the fixed charge bi-criterion transportation problem total availability is not equal to the total demand, then some of the source and/or destination constraints are satisfied as inequations. Sometimes, situations may arise where one wishes to keep reserve stocks at the sources, say for emergencies, thereby restricting the total transportation flow to a known specified level, say $P\ (< \min(\sum_{i \in I} a_i, \sum_{j \in j} b_j))$. This flow constraint breaks the transportation flow to a transportation structure of the problem. A technique for solving fixed charge bi-criterion transportation problem with restricted flow is developed.

2.3. FORMULATION OF FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

Thirwan *et al*.(1997) formulated the fixed charge bi-criterion transportation problem as

Minimize $$\left\{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} F_i ,[\underset{\substack{i\in I\\ j\in J}}{\text{Max}}[t_{ij} / x_{ij} \geq 0]\right\}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \; i \in I$$

$$\sum_{i=1}^{m} x_{ij} \leq b_j, \; j \in J$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P\left(< \min(\sum_{i\in I} a_i, \sum_{j\in J} b_j)\right)$$

$$x_{ij} \geq 0 \;\; i \in I, j \in J$$

where $i = \{1,2 , . . . , m\}$ is set of origins

$j = \{1,2 , . . . , n\}$ is set of destinations

x_{ij} = amount transported from the i^{th} origin to the j^{th} destination.

c_{ij}= variable cost for transporting one unit from the i^{th} origin to the j^{th} destination.

t_{ij}= time of transporting the product from the i^{th} origin to the j^{th} destination and is independent of the amount transported so long as $x_{ij} > 0$.

F_i = the fixed cost associated with the i^{th} origin.

a_i= amount available at the i^{th} origin.

b_j = demand of the j^{th} destination.

2.4. SOLUTION PROCEDURE

Fixed charge bi-criterion transportation problem (FCBTP) is separated into two problems (P1) and (S1) where

$$(P1) : \text{Minimize the cost function } \{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} F_i \}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i, \ i \in I$$

$$\sum_{i=1}^{m} x_{ij} \le b_j, \ j \in J$$

$$\sum_{i \in I} \sum_{j \in J} x_{ij} = P\left(< \min(\sum_{i \in I} a_i , \sum_{j \in J} b_j)\right)$$

$$x_{ij} \ge 0 \ \ i \in I, j \in J$$

and

$$(S1) : \text{Minimize the time function } [\underset{\substack{i \in I \\ j \in J}}{\text{Max}}[t_{ij} \ / \ x_{ij} \ge 0]]$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i, \ i \in I$$

$$\sum_{i=1}^{m} x_{ij} \le b_j, \ j \in J$$

$$\sum_{i \in I} \sum_{j \in J} x_{ij} = P\left(< \min(\sum_{i \in I} a_i , \sum_{j \in J} b_j)\right)$$

$$x_{ij} \ge 0 \ \ i \in I, j \in J$$

For formulation of F_i (i = 1, 2,... m) we assume that F_i (i = 1, 2,... m) has p number of steps so that

$$F_i = \sum_{i=1}^{p} \delta_{il} F_{il}, \; i = 1, 2, ..., m$$

Where $$\delta_{il} = \begin{cases} 1 \text{ if } \sum_{j=1}^{n} x_{il} > A_{il}, \; i = 1,2,...m \; ; \; l = 1,2,..., p \\ 0 \text{ otherwise} \end{cases}$$

Here $0 = A_{i1} < A_{i2} < ... < A_{ip}$.

A_{i1} ,A_{i2} , ..., A_{ip} $(i = 1,2,.. ., m)$ are constants and F_{il}($l= 1,2, . . . , p$; $i = 1,2, . . . , m$) are fixed costs.

The flow constraint in the problem (P1) implies that a total ($\sum a_i$ - P) of source reserves has to be kept at the various sources and a total ($\sum bj$ - P) of destination slacks to be retained at the various destinations. Therefore, an extra destination to receive the source reserves and an extra source to fill up the destination slacks are introduced. Hence, the related fixed charge bi-criterion transportation problem (RFCBTP) associated with (FCBTP) is

$$\text{(RFCBTP)}: \text{ Minimize } \left\{ \sum_{i\in I'} \sum_{j\in J'} c'_{ij} \, y_{ij} + \sum_{i\in I'} F_i, Max\,[t'_{ij} / \, y_{ij} > 0] \right\}$$

Subject to

$$\sum_{j\in J'} y_{ij} = a'_i \qquad i \in I'$$

$$\sum_{i\in I'} y_{ij} = b'_j \qquad j \in J'$$

$$\sum_{i\in I'} \sum_{j\in J'} y_{ij} = P$$

$$y_{ij} \geq 0 \quad i \in I', \qquad j \in J'$$

where $I' = \{1, 2, \ ..., \ m+1\} \ = \ I \ \cup \{m+1\}$

$$J' = \{1, 2, \ ..., \ n \ +1\} \ = \ J \cup \{ \ n+1\}$$

$$a'_i = a_i, \quad i \in I \ , \ a'_{m+1} = (\sum_{j \in J} b_j - P)$$

$$b'_j = b_j, \ \ j \in J \ , \ \ b'_{n+1} = (\ \sum_{i \in I} a_i - P \ )$$

$$c'_{ij} = c_{ij} (i, j) \in I \times J$$

$$t'_{ij} = t_{ij} (i, j) \in I \times J$$

$$c'_{i,n+1} = c'_{m+1, j} = 0, \ \ i \in I, j \in J$$

$$t'_{i,n+1} = t'_{m+1, j} = 0, \ \ i \in I, j \in J$$

$$F_{m+1} = 0,$$

$$c'_{m+1,n+1} = M, \quad t'_{m+1,n+1} > \max_{\substack{i \in I \\ j \in J}} [t_{ij} \setminus y_{ij} > 0]$$

Where M is a large positive number.

(RFCBTP) is separated into two problems (RP1) and (RS1).

$$\text{(RP1)}: \ \text{Minimize} \ \{ \sum_{i \in I'} \sum_{j \in J'} c'_{ij} \, y_{ij} + \sum_{i \in I'} F_i \}$$

Subject to

$$\sum_{j \in J'} y_{ij} = a'_i \qquad i \in I'$$

$$\sum_{i \in I'} y_{ij} = b'_j \qquad j \in J'$$

$$\sum_{i \in I'} \sum_{j \in J'} y_{ij} = P$$

$$y_{ij} \geq 0 \quad i \in I', \qquad j \in J'$$

$$(RS1): \text{ Minimize } T = \max_{\substack{i\in I \\ j\in J}} [t_{ij} / y_{ij} > 0]$$

subject to

$$\sum_{j\in J'} y_{ij} = a'_i \qquad i \in I'$$

$$\sum_{i\in I'} y_{ij} = b'_j \qquad j \in J'$$

$$\sum_{i\in I'} \sum_{j\in J'} y_{ij} = P$$

$$y_{ij} \geq 0 \quad i \in I', \qquad j \in J'$$

DEFINITION:- A basic feasible solution $\{y_{ij}\}$, $i \in I'$, $j \in J'$ to (RFCBTP) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$.

THEOREM 1:- Every corner feasible solution of (RFCBTP) provides a basic feasible solution to (FCBTP) and conversely.

Proof: - Let $\{y_{ij}\}$ be a cfs to (RFCBTP)

Define

$$x_{ij} = y_{ij} \qquad (i,j) \in I\times J$$

$\{x_{ij}\}$ so defined can be established to be a basic feasible solution to (FCBTP).

Conversely, given $\{x_{ij}\}$ to be a basic feasible solution to (FCBTP) then $\{y_{ij}\}$,

$(i,j) \in I' \times J'$

where $I' = \{1,2,\ldots,m+1\}$, $J' = \{1,2,\ldots,n+1\}$ defined by the transformation

$$x_{ij} = y_{ij} \qquad (i,j) \in I\times J$$

$$y_{i,n+1} = a_i - \sum_J x_{ij}, \quad i \in I$$

$$y_{m+1,j} = b_j - \sum_I x_{ij}, \quad j \in J$$

$$y_{m+1,n+1} = 0$$

Can be shown to be a cfs to (RFCBTP).

Remarks1:-

A cfs of (RFCBTP) is also a cfs of (RP1).

Remarks2:-

The value of the objective function of (RP1) at a corner feasible solution is equal to the value of the objective function of (P1) at its corresponding basic feasible solution. Value of the objective function of (RP1) is

$$= \sum_{i \in I'} \sum_{j \in J'} c'_{ij} y_{ij} + \sum_{i \in I'} F_i$$

$$= \sum_{i \in I'} \sum_{j \in J} c'_{ij} y_{ij} + \sum_{i \in I'} c'_{i,n+1} y_{i,n+1} + \sum_{i \in I} F_i \text{(because } F_{m+1} = 0)$$

$$= \sum_{i \in I} \sum_{j \in J} c'_{ij} y_{ij} + \sum_{j \in J'} c'_{m+1,j} y_{m+1,j} + \sum_{i \in I'} c'_{i,n+1} y_{i,n+1} + \sum_{i \in I'} F_i$$

$$= \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} + \sum_{i \in I} F_i (\because c'_{m+1,j} = c'_{i,n+1} = 0, y_{m+1,n+1} = 0)$$

=value of the objective function of (P1)

Remarks3:-

A non corner feasible solution to (RFCBTP) cannot provide a feasible solution to (FCBTP).

THEOREM2:- An optimal solution to (RP1) has to be a corner feasible solution.

Proof: - If possible, let there exist an optimal solution $\{y'_{ij}\}$ to (RP1).which is not a cfs. Let the optimal cost for (RP1) be C^1 which is a large positive number. Now, consider an optimal solution $\{x^0_{ij}\}$ say to (P1), with corresponding optimal cost C^0. Let $\{y^0_{ij}\}$ be the corresponding cfs to (RP1). The cost corresponding to cfs $\{y^0_{ij}\}$ is also C^0.clearly $C^0 < C^1$ which contradicts the fact that $\{y^0_{ij}\}$ is an optimal solution to (RP1). So, no non cfs to (RP1) can be an optimal solution.

THEOREM3:- There is a one to one correspondence between optimal solutions to (P1) and optima among the corner feasible solutions to (RP1).

proof : - Let $\{x^0_{ij}\}$ be an optimal solution to (P1) yielding optimal cost C^0 and $\{y^0_{ij}\}$ be the corresponding cfs to (RP1). The cost corresponding to the solution $\{y^0_{ij}\}$ is C^0 . if possible, let $\{y^0_{ij}\}$ be not an optimal cfs to (RP1). So there exist an optimal cfs $\{y^*_{ij}\}$, say to (RP1) with the corresponding cost $C^* < C^0$. Let $\{x^*_{ij}\}$ be the basic feasible solution to (P1) corresponding to the cfs $\{y^*_{ij}\}$. Optimal objective value of (P1) at optimal solution $\{x^*_{ij}\}$ is equal to $C^* < C^0$ which contradicts the fact that $\{x^0_{ij}\}$ is the optimal solution to (P1). Hence $\{y^0_{ij}\}$ must be an optimal cfs to (RP1).

Similarly an optimal cfs to (RP1) will provide an optimal solution to (P1).

Conclusion: **-** Optimizing (P1) is exactly equivalent to optimizing (RP1) which is a fixed charge transportation problem whose optimal solution is at an extreme point.

2.5. ALGORITHM

Step 1:

Given the fixed charge bi-criterion transportation problem. Separate into two problems (P1) and (S1). Let the flow be restricted to P. Introduce an additional row and an additional column with availability = $\sum a_i$ - P and demand = $\sum bj$ - P. From the problem (RP1). Find its basic feasible solution y_{ij}^1.Let B be the corresponding basis.

Step 2:

(a) Calculate the fixed cost of the current basic feasible solution and denote this by F^1 (current) where

$$F^1(\text{current}) = \sum_{i \in I} F_i$$

(b) Find $c_{ij} - u_i^1 - v_j^1$ for all $(i, j) \notin$ B and denote it by $(c_{ij})_1$ where u_i^1, v_j^1 are the dual variable for $i \in I', j \in J'$

Step 3:

(a) Find $A_{ij}^1 = (c_{ij})_1 \times (E_{ij})_1$

Where A_{ij}^1 is the vriable cost on introducing a non basic cell (i, j) with value $(E_{ij})_1$ (*for all* $(i, j) \notin B$) into the basis

(b) Find F_{ij}^1(Difference) = change in fixed cost

$$= F_{ij}^1(NB) - F^1(\text{current})$$

where $F_{ij}^1(NB)$ is the total fixed cost involved on introducing the variable x_{ij} with value $(E_{ij})_1$ (for all $(i, j) \notin$ B) into the current basis to form a new basis.

(c) Find $\Delta^1_{ij} = F^1_{ij}$(Difference)$+A^1_{ij}\ \forall\ (i, j) \notin B$

If all $\Delta^1_{ij} \geq 0,$ then it is not possible to decrease the total cost i.e. (varible cost+fixed cost). Go to step-4.

But, if there exists at least one $\Delta^1_{ij} < 0$, find

$\min\{\Delta^1_{ij}, \Delta^1_{ij} < 0, (i, j) \notin B\}$

$= \Delta_{pq}$, therefore, the cell (p,q) will enter the basis, Go to step-2

Step 4:

Let C^1 be the optimal cost of (RP1) yielded by the basic feasible solution $\{y^1_{ij}\}$

Find $T^1 = \underset{\substack{i \in I' \\ j \in J'}}{\text{Max}}\{t_{ij} \,/\, y^1_{ij} > 0\}$ from problem (RS1)

Then, the corresponding pair (C^1, T^1) is the first best cost-time trade of pair for the problem (RFCBTP) and subsequently for the problem (FCBTP). to find the next best cost-time trade off pair, go to step-5.

Step5:

Define $c^1_{ij} = \begin{cases} M & \text{if } t_{ij} \geq T^1 \\ c_{ij} & \text{if } t_{ij} < T^1 \end{cases}$

where M is a sufficiently large positive number. Form the corresponding fixed charge transportation problem with variable cost c^1_{ij} & Repeat the above process till, we get an infeasible solution.

The complete set of cost-time trade off pairs of (FCBTP) at the end of qth iteration will be given by

$(C^1, T^1), (C^2, T^2), \ldots, (C^q, T^q)$

where $C^1 < C^2 < \ldots < C^q$

and $T^1 > T^2 > \ldots > T^q$

REMARK4:-

The pair (C^1, T^q) with minimum cost and minimum time is the ideal pair which cannot be achieved in practice except in some trivial case.

REMARK5:-

The choice of c^1_{ij} given in Step-5 will ensure the infeasibility of the basic feasible solution after a finite number of iterations.

2.6. NUMERICAL EXAMPLE

Fixed Charge Bi-Criterion Transportation Problem with restricted flow is:

Minimize the cost function $\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} + \sum_{i=1}^{3} F_i$

subject to

$$\sum_{j=1}^{3} x_{ij} \le a_i \ , i = 1,2,3$$

$$\sum_{i=1}^{3} x_{ij} \le b_j \ , j = 1,2,3$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij} = P(< \min(\sum_{i=1}^{3} a_i, \sum_{j=1}^{3} b_j))$$

$$xij \ge 0 \quad i = 1,2,3; \ j = 1,2,3$$

Table 1 gives the values of variable cost C_{ij} *(i=1, 2, 3; j=1, 2, 3)* and

Table 2 gives the values of t_{ij} *(i=1, 2, 3; j=1, 2, 3).*

Table-1

$j\rightarrow$ $i\downarrow$	1	2	3	a_i
1	5	9	9	19
2	4	6	2	10
3	2	1	1	11
b_j	5	8	15	

Table-2

$j\rightarrow$ $i\downarrow$	1	2	3	a_i
1	15	8	2	19
2	10	13	11	10
3	6	9	17	11
b_j	5	8	15	

The fixed costs are

$F_{11}=100$, $F_{12}=50$, $F_{13}=50$

$F_{21}=150$, $F_{22}=50$, $F_{23}=50$

$F_{31}=200$, $F_{32}=100$, $F_{33}=50$

The total cost which is to be minimized is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} + \sum_{i=1}^{3} F_i$$

where

$$F_i = \sum_{i=1}^{3} \delta_{il} F_{il} \qquad \text{for } i = 1, 2, 3$$

$$\text{Where } \delta_{i1} = \begin{cases} 1 & \text{if } \sum_{j=1}^{3} x_{ij} > 0 \quad \text{for } i = 1,2,3 \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_{i2} = \begin{cases} 1 & \text{if } \sum_{j=1}^{3} x_{ij} > 7 \quad \text{for } i = 1,2,3 \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_{i3} = \begin{cases} 1 & \text{if } \sum_{j=1}^{3} x_{ij} > 10 \quad \text{for } i = 1,2,3 \\ 0 & \text{otherwise.} \end{cases}$$

Let the restricted flow P=25

where $P = 25 < \min\left(\sum_{i=1}^{3} ai = 40, \sum_{j=1}^{3} bj = 28 \right)$

Introducing a dummy source and a dummy destination in Table-1 with

$c_{i4} = 0, \quad i=1, 2, 3$

$c_{4j} = 0, \quad j=1, 2, 3$

c_{44} =M, M is a large positive number

$$a_4 = \sum_{j=1}^{3} b_j - P = 28 - 25 = 3$$

And

$$b_4 = \sum_{i=1}^{3} a_i - P = 40 - 25 = 15$$

Form the corresponding (RP1)

Similarly on introducing a dummy source and a dummy destination in Table-2 with

$t_{i4} = 0, \quad i=1, 2, 3$

$t_{4j} = 0, \quad j=1, 2, 3$

$$t_{44} = \max_{\substack{i \in I \\ j \in J}} t_{ij} = 15$$

Taking $t_{44} = 18$ with $a_4 = 3$ and $b_4 = 15$, form the corresponding (RS1).

Solve problem (RP1). On introducing source reserve and slacks destination in Table-1 and Table-2 we get the Table-3 and table-4 respectively.

Table-3

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	a_i
1	5	9	9	0	19
2	4	6	2	0	10
3	2	1	1	0	11
4	0	0	0	M	3
bj	5	8	15	15	

Table-4

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	a_i
1	15	8	2	0	19
2	10	13	11	0	10
3	6	9	17	0	11
4	0	0	0	18	3
bj	5	8	15	15	

A basic feasible solution of problem (RP1) is given in table-5. An initial solution is calculated by using Extremum Difference method.

The right hand side value of table-5 gives the total fixed cost of current solution.

Table-5

Destination j→ Origin i↓	1	2	3	4	F^1(current)
1	5 (4)	9	9	0 (15)	100
2	4 (1)	6	2 (9)	0	200
3	2	1 (5)	1 (6)	0	350
4	0	0 (3)	0	M	0
					650

Applying step 2 (b), the values $(c_{ij} - u_i - v_j)$ are calculated for all (i,j) B, which are given in table-6.

Table-6

ij	12	13	22	24	31	34	41	43
$c_{ij} - u_i - v_j$	6	6	4	1	-1	2	-2	0

Applying step 3(a), the values of A_{ij}^1 are calculated, which are given in table-7.

Table-7

ij	12	13	22	24	31	34	41	43
A_{ij}^1	24	24	20	1	-1	2	-2	0

Applying step 3(b), the following results are obtained which are tabulated in table-8.

F_{ij}(Difference) = F_{ij} (NB) - F_i (Current)

Table-8

$ij\rightarrow$ $i\downarrow$	12	13	22	24	31	34	41	43
1	100	100	100	100	100	100	100	100
2	200	200	200	200	200	200	200	200
3	350	350	350	350	350	300	350	350
F_{ij} (NB)	650	650	650	650	650	600	650	650
F_{ij}(Difference)	0	0	0	0	0	-50	0	0

Applying step 3(c), the values of Δ_{ij}^{1} are obtained, which are given in table-9.

Table-9

ij	12	13	22	24	31	34	41	43
Δ_{ij}^{1}	24	24	20	1	-1	-48	-2	0

In table-9, it is observed that

$$\text{Min } \{\Delta_{ij}^{1}, \Delta_{ij}^{1} < 0, (i, j) \notin B\} = -48 \text{ corresponding to the cell } (3, 4).$$

Therefore the variable to enter the basis is x_{34} and the new solution is given in Table-10.

Table-10

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F^{1}(current)
1	5 (5)	9	9	0 (14)	100
2	4	6	2 (10)	0	200
3	2	1 (5)	1 (5)	0 (1)	300
4	0	0 (3)	0	M	0
					600

Applying step 2(b), the following results are obtained as shown in table-11.

Table-11

ij	12	13	21	22	24	31	41	43
$c_{ij} - u_i - v_j$	8	8	-2	4	-1	-3	-4	0

Applying step 3(a), the values of A_{ij}^1 are calculated, which are given in table-12.

Table-12

ij	12	13	21	22	24	31	41	43
A_{ij}^1	40	40	-2	20	-1	-3	-4	0

Applying step 3(b), the following results are obtained which are tabulated in table-13

Table-13

ij→ i↓	12	13	21	22	24	31	41	43
1	150	150	100	100	100	100	100	100
2	200	200	200	200	200	200	200	200
3	200	200	350	300	350	350	350	300
F_{ij} (NB)	550	550	650	600	650	650	650	600
F_{ij}(Difference)	-50	-50	50	0	50	50	50	0

Applying step 3(c), the values of Δ_{ij}^1 are obtained, which are given in table-14.

Table-14

ij	12	13	21	22	24	31	41	43
Δ_{ij}^1	-10	-10	48	20	49	47	46	0

In table-14, it is observed that

Min $\{\Delta_{ij}^{1}, \Delta_{ij}^{1} < 0, (i, j) \notin B\}$ = -10 corresponding to the cell (1, 3).

Therefore the variable to enter the basis is x_{13} and the new solution is given in Table-15.

Table-15

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F^1(current)
1	5 (5)	9	9 (5)	0 (9)	100
2	4	6	2 (10)	0	200
3	2	1 (5)	1	0 (6)	300
4	0	0 (3)	0	M	0

600

Applying step 2 (b), the following results are obtained as shown in table-16.

Table-16

ij	12	21	22	24	31	33	41	43
$c_{ij}-u_i-v_j$	8	6	12	7	-3	-8	-4	-8

Applying step 3(a), the values of A_{ij}^{1} are calculated, which are given in table-17.

Table-17

ij	12	21	22	24	31	33	41	43
A_{ij}^{1}	40	30	60	63	-15	-40	-12	-24

Applying step 3(b), the following results are obtained which are tabulated in table-18.

Table-18

$ij\rightarrow$ $i\downarrow$	12	21	22	24	31	33	41	43
1	200	150	200	200	100	100	100	100
2	200	200	200	150	200	200	200	200
3	0	200	0	200	300	300	300	300
F_{ij} (NB)	400	550	400	550	600	600	600	600
F_{ij}(Difference)	-200	-50	-200	-50	0	0	0	0

Applying step 3(c), the values of Δ_{ij}^{1} are obtained, which are given in table-19.

Table-19

ij	12	21	22	24	31	33	41	43
Δ_{ij}^{1}	-192	-44	-188	-43	-3	-8	-4	-8

In table-19, it is observed that

$$\text{Min } \{\Delta_{ij}^{1}, \Delta_{ij}^{1} < 0, (i, j) \notin B\} = -192 \text{ corresponding to the cell } (1, 2).$$

Therefore the variable to enter the basis is x_{12} and the new solution is given in Table-20.

Table-20

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F_i
1	5 (5)	9 (5)	9 (5)	0 (4)	200
2	4	6	2 (10)	0	200
3	2	1	1	0 (11)	0
4	0	0 (3)	0	M	0
					400

Applying step 2(b), the following results are obtained as shown in table-21.

Table-21

ij	21	22	24	31	32	33	41	43
$c_{ij} - u_i - v_j$	6	4	7	-3	-8	-8	4	0

Applying step 3(a), the values of A_{ij}^1 are calculated, which are given in table-22.

Table-22

ij	21	22	24	31	32	33	41	43
A_{ij}^1	30	20	28	-15	-40	-40	12	0

Applying step 3(b), the following results are obtained which are tabulated in table-23.

Table-23

$ij\rightarrow$ $i\downarrow$	21	22	24	31	32	33	41	43
1	200	200	200	150	150	150	200	200
2	200	200	150	200	200	200	200	200
3	0	0	0	200	200	200	0	0
F_{ij} (NB)	400	400	350	550	550	550	400	400
F_{ij}(Difference)	0	0	-50	150	150	150	0	0

Applying step 3(c), the values of Δ_{ij}^1 are obtained, which are given in table-24.

Table-24

ij	21	22	24	31	32	33	41	43
Δ_{ij}^1	30	20	-22	135	110	110	12	0

In table-24, it is observed that

Min $\{\Delta_{ij}^1, \Delta_{ij}^1 < 0, (i, j) \notin B\} = -22$ corresponding to the cell (2, 4).

Therefore the variable to enter the basis is x_{24} and the new solution is given in Table-25.

Table-25

Destination *j*→ Origin *i*↓	1	2	3	4	F^1(current)
1	5 (5)	9 (5)	9 (9)	0	200
2	4	6	2 (6)	0 (4)	150
3	2	1	1	0 (11)	0
4	0	0 (3)	0	M	0
					350

Applying step 2 (b), the following results are obtained as shown in table-26.

Table-26

ij	14	21	22	31	32	33	41	43
$c_{ij} - u_i - v_j$	-7	6	4	4	-1	-1	4	0

Applying step 3(a), the values of A_{ij}^1 are calculated, which are given in table-27.

Table-27

Ij	14	21	22	31	32	33	41	43
A_{ij}^1	-28	30	20	20	-5	-6	12	0

Applying step 3(b), the following results are obtained which are tabulated in table-28.

Table-28

$ij\rightarrow$ $i\downarrow$	14	21	22	31	32	33	41	43
1	200	200	200	200	200	200	200	200
2	200	150	150	150	150	150	150	150
3	0	0	0	200	200	200	0	0
F_{ij} (NB)	400	350	350	550	550	550	350	350
F_{ij}(Difference)	50	0	0	200	200	200	0	0

Applying step 3(c), the values of Δ_{ij}^{1} are obtained, which are given in table-29.

Table-29

ij	14	21	22	31	32	33	41	43
Δ_{ij}^{1}	22	30	20	220	195	194	12	0

Here, all $\Delta_{ij}^{1} \geq 0$ (for all $i,j \notin$ B). Now, applying step 4 we get

Minimum cost = C^1=163+350=513

and the corresponding time=T^1=15

Hence, the first cost – time trade-off pair is (C^1,T^1) = (513,15).Now, applying step 5 the given problem is modified and the basic feasible solution is given in table 30.

Table-30

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F_i
1	M	9	9 (4)	0 (15)	100
2	4	6	2 (10)	0	200
3	2 (2)	1 (8)	M (1)	0	350
4	0 (3)	0	0	M	0
					650

Applying step 2 (b), the following results are obtained as shown in table-31.

Table-31

ij	11	12	21	22	24	34	42	43
$c_{ij}-u_i-v_j$	-11	M-1	M	M+3	7	9-M	1	2-M

Applying step 3(a), the values of A_{ij}^2 are calculated, which are given in table-32.

Table-32

ij	11	12	21	22	24	34	42	43
A_{ij}^2	-22	4M-4	2M	8M+24	70	9-M	3	2-M

Applying step 3(b), the following results are obtained which are tabulated in table-33.

Table-33

ij→ i↓	11	12	21	22	24	34	42	43
1	100	100	100	100	200	100	100	100
2	200	200	200	200	0	200	200	200
3	350	350	350	350	350	300	350	350
F_{ij} (NB)	650	650	650	650	550	600	650	650
F_{ij}(Difference)	0	0	0	0	-100	-50	0	0

Applying step 3(c), the values of Δ_{ij}^2 are obtained, which are given in table-34.

Table-34

ij	11	12	21	22	24	34	42	43
Δ_{ij}^2	-22	4M-4	2M	8M+24	-30	-41-M	3	2-M

In table-34, it is observed that

Min $\{\Delta_{ij}^2, \Delta_{ij}^2 < 0, i,j \notin B\}$= -41-M corresponding to the cell (3,4).

Therefore the variable to enter the basis is x_{34} and the new solution is given in Table-35.

Table-35

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F_i
1	M	9	9 (5)	0 (14)	100
2	4	6	2 (10)	0	200
3	2 (2)	1 (8)	M	0 (1)	300
4	0 (3)	0	0	M	0
					600

Applying step 2 (b), the following results are obtained as shown in table-36.

Table-36

ij	11	12	21	22	24	33	42	43
$c_{ij} - u_i - v_j$	M-2	8	9	12	7	M-9	1	-7

Applying step 3(a), the values of A_{ij}^2 are calculated, which are given in table-37.

Table-37

Ij	11	12	21	22	24	33	42	43
A_{ij}^2	2M-4	64	18	96	70	M-9	3	-7

Applying step 3(b), the following results are obtained which are tabulated in table-38.

Table-38

$ij\rightarrow$ $i\downarrow$	11	12	21	22	24	33	42	43
1	100	200	100	200	200	100	100	100
2	200	200	200	200	0	200	200	200
3	300	200	300	2000	300	350	300	350
F_{ij} (NB)	600	600	600	600	500	650	600	650
F_{ij}(Difference)	0	0	0	0	-100	50	0	50

Applying step 3(c), the values of Δ_{ij}^2 are obtained, which are given in table-39.

Table-39

ij	11	12	21	22	24	33	42	43
Δ_{ij}^2	2M-4	64	18	96	-30	M+41	3	43

In table-39, it is observed that

Min $\{\Delta_{ij}^2, \Delta_{ij}^2 < 0, i,j \notin B\}$= -30 corresponding to the cell (2,4).

Therefore the variable to enter the basis is x_{24} and the new solution is given in Table-40.

Table-40

Destination $j\rightarrow$ Origin $i\downarrow$	1	2	3	4	F_i
1	M	9	9 (15)	0 (4)	200
2	4	6	2	0 (10)	0
3	2 (2)	1 (8)	M	0 (1)	300
4	0 (3)	0	0	M	0
					500

Applying step 2 (b), the following results are obtained as shown in table-41.

Table-41

ij	11	12	21	22	23	33	42	43
$c_{ij} - u_i - v_j$	M-2	8	2	5	-7	M-9	1	-7

Applying step 3(a), the values of A_{ij}^2 are calculated, which are given in table-42.

Table-42

ij	11	12	21	22	23	33	42	43
A_{ij}^2	2M-4	32	4	40	-70	M-9	3	-7

Applying step 3(b), the following results are obtained which are tabulated in table-43.

Table-43

ij→ i↓	11	12	21	22	23	33	42	43
1	200	200	200	200	100	200	200	200
2	0	0	150	200	200	0	0	0
3	300	200	300	200	300	350	300	350
F_{ij} (NB)	500	400	650	600	600	550	500	550
F_{ij}(Difference)	0	-100	150	100	100	50	0	50

Applying step 3(c), the values of Δ_{ij}^2 are obtained, which are given in table-44.

Table-44

ij	11	12	21	22	23	33	42	43
Δ_{ij}^2	2M-4	-68	154	140	30	M+43	3	43

In table-44, it is observed that

$$\text{Min}\ \{\Delta_{ij}^2,\ \Delta_{ij}^2 < 0,\ i,j \notin B\} = -68$$ corresponding to the cell (1,2).

Therefore the variable to enter the basis is x_{12} and the new solution is given in Table-45.

Table-45

Destination *j* Origin *i*	1	2	3	4	F_i
1	M	9 (4)	9 (15)	0	200
2	4	6	2	0 (10)	0
3	2 (2)	1 (4)	M	0 (5)	200
4	0 (3)	0	0	M	0
					400

Applying step 2 (b), the following results are obtained as shown in table-46.

Table-46

ij	11	14	21	22	23	33	42	43
$c_{ij} - u_i - v_j$	M-10	-8	2	5	1	M-1	1	1

Applying step 3(a), the values of A_{ij}^2 are calculated, which are given in table-47.

Table-47

ij	11	14	21	22	23	33	42	43
A_{ij}^2	2M-20	-32	4	20	4	4M-4	3	3

Applying step 3(b), the following results are obtained which are tabulated in table-48.

Table-48

$ij\rightarrow$ $i\downarrow$	11	14	21	22	23	33	42	43
1	200	200	200	200	200	200	200	200
2	0	0	150	150	150	0	0	0
3	200	300	200	200	200	200	200	200
F_{ij} (NB)	400	500	550	550	550	400	400	400
F_{ij}(Difference)	0	100	150	150	150	0	0	0

Applying step 3(c), the values of Δ_{ij}^2 are obtained, which are given in table-49.

Table-49

ij	11	14	21	22	23	33	42	43
Δ_{ij}^2	2M-20	68	154	170	154	4M-4	3	3

Here, all $\Delta_{ij}^2 \geq 0$(for all $i,j \notin$ B). Now, applying step 4 we get

Minimum cost = C^2=179+400=579

and the corresponding time=T^2=9

Hence, the second cost – time trade-off pair is (C^2,T^2) = (579,9).

After second iteration, it is observed that the solution is infeasible. Hence two cost – time trade-off pairs are obtained as follows:

(513,15), (579,9)

The result shows that the minimum cost is 513 which corresponds to the pair (513,15) and the minimum time is 9 which corresponds to the pair (579,9). So, the ideal solution corresponds to the pair is (513,9).

Table 50

Trade-off pairs	Ideal solution	Distance (D1) between ideal solution and trade-off pair	(D1)opt
(513,15) (579,9)	(513,9)	6 66	6

The distance of the trade-off pairs from the ideal solution is presented in table 50.

2.7. CONCLUSION

The Fixed charge by criterion transportation problem with restricted flow by Deepa Thirwan et al. (1996) is reviewed and an improved cost-time trade off pair has been obtained.

CHAPTER-3

MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

3.1. MULTI-INDEX BI-CRITERION TRANSPORTATION PROBLEM

Multi-index bi-criterion transportation problem with restricted flow is an extension of Multi-index bi-criterion transportation problem. The multi-index problem can be described as minimizing the cost of moving a set of p different commodities ($k = 1, 2, ..., p$) from n origins ($i = 1, 2, ..., n$) to m destinations ($j = 1, 2, ..., m$). The equations then give rise to the conditions on the amount of the various types of combination that is available and required.

An ordinary transportation problem can be written in the form of a two-dimensional table for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$. it can be shown in a table which is given below. Each cell represents one of the y_{ij}'s. When these y_{ij}'s are summed along the rows of the table they must equal the b_i and when they are summed down the columns they must equal the a_j.

$j\rightarrow$ $i\downarrow$	1	2		m	
1	y_{11}	y_{12}		y_{1m}	b_1
2	y_{21}	y_{22}		y_{2m}	b_2
n	y_{n1}	y_{n2}		y_{nm}	b_n
	a_1	a_2		a_m	

The solid problem can be set out as a three dimensional block for i=1, 2 ,. . . ,n; j=1,2,. . .,m; k=1, 2, . . . ,p. Each cell of this block represents one of the x_{ijk}'s .When these are summed along the rows (for constant *j* and *k*) they equal A_{jk}. When they are summed along the columns (for constant *k* and *i)* they equal B_{ki}. When they are summed down the heights (for constant *i* and *j*) they equal E_{ij}. The arrangement of x_{ijk}*'s* and the boundary conditions are shown in Figure-1 and Figure-2.

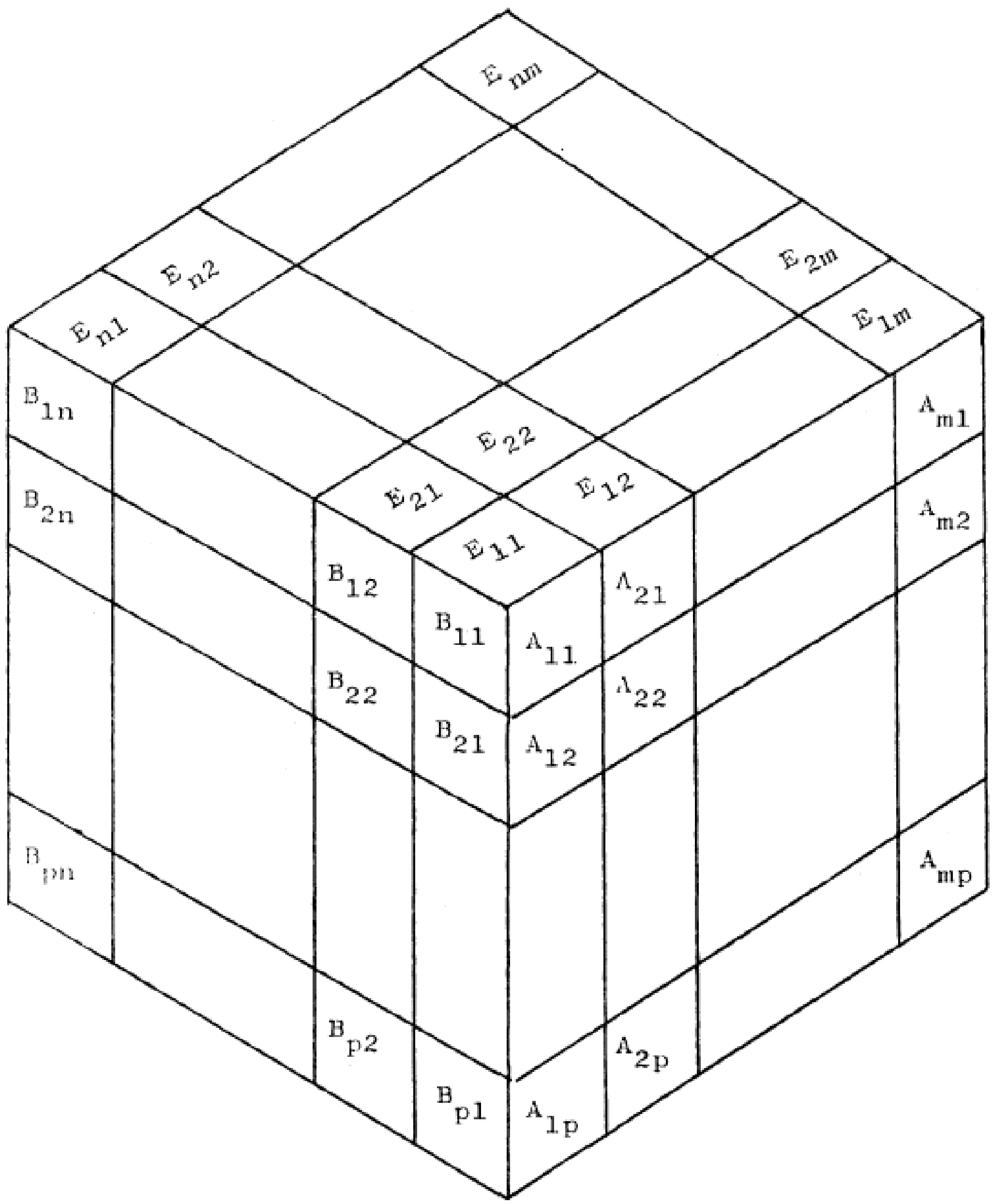

Figure-1

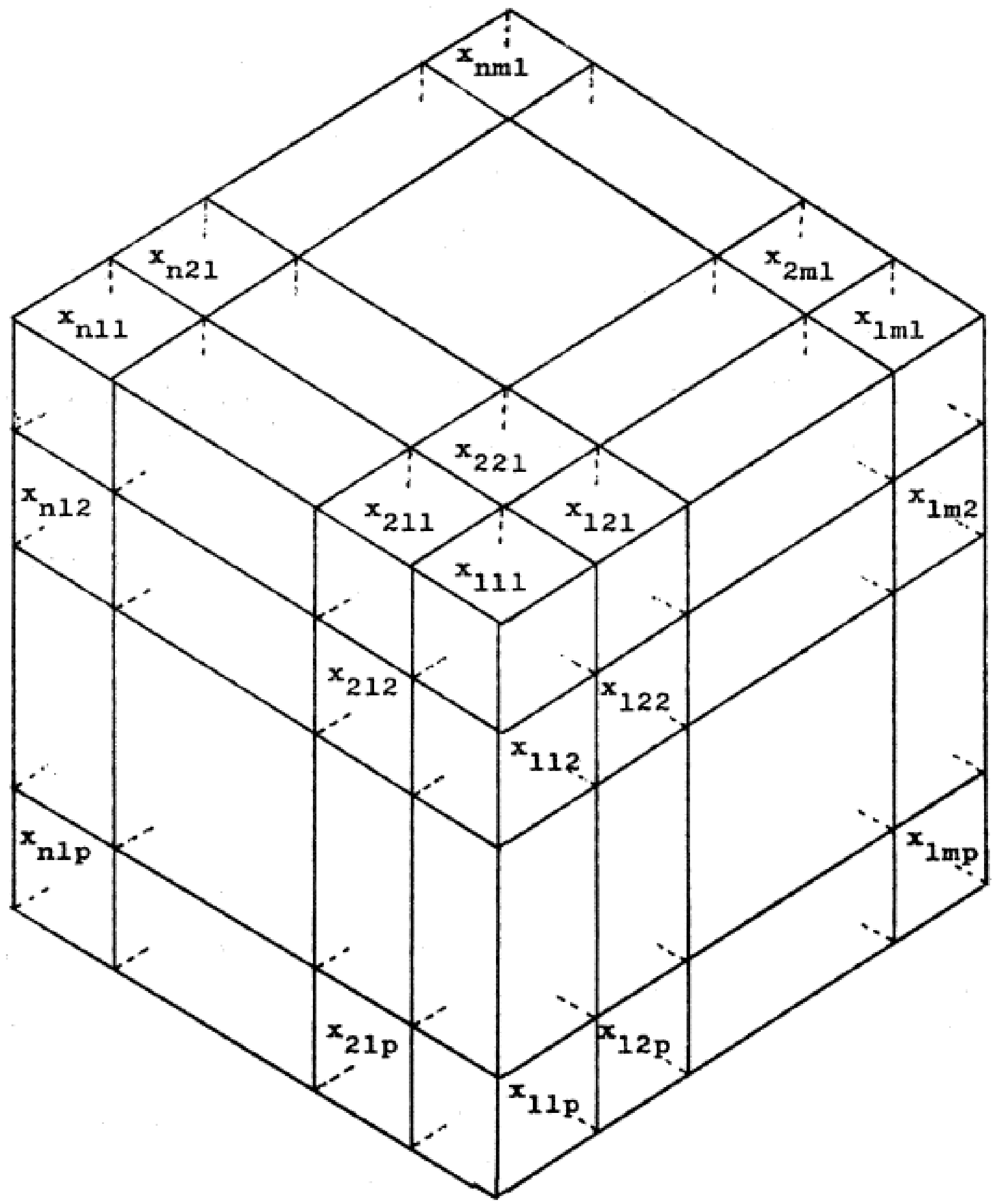

Figure-2

3.2. FORMULATION OF MULTI-INDEX BI-CRITERION TRANSPORTATION PROBLEM

The multi-index transportation problem in which there are *m* origins, *n* destinations and *p* type of commodities to be transported can be formulated as follows. In this problem there are two objectives one of minimizing the total cost and the other is to minimize the total time of transportation.

Minimize $z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}\, x_{ijk}$

And $z_2 = \max\{t_{ijk} : x_{ijk} > 0 \ (i = 1,2,\ldots,m;\ j = 1,2,\ldots,n; k = 1,2,\ldots,p)\}$

subject to $\sum_{i=1}^{m} x_{ijk} = A_{jk}$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

$$x_{ijk} \geq 0,\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n;\ k = 1,2,\ldots,p \ \ where$$

$$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki},\ \ \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij},\ \ \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p}\sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}$$

3.3. MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM

Multi index fixed charge bi-criterion transportation problem is an extension of the multi-index fixed charge transportation problem in which a fixed cost is incurred in the objective function. This problem can be formulated as:

Minimize $\left\{ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} C_{ijk}\, x_{ijk} + \sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik\,i},\ \underset{\substack{1\le i\le m\\ 1\le j\le n\\ 1\le k\le p}}{Max} \left[t_{ijk} / x_{ijk} > 0\right] \right\}$

subject to $\sum_{i=1}^{m} x_{ijk} = A_{jk}$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

$$x_{ijk} \geq 0,\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n;\ k = 1,2,\ldots,p$$

where

$$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki}, \ \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij}, \ \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p}\sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}$$

Here $i = 1,2, \ldots, m$ are the origins

$j = 1, 2, \ldots, n$ are the destinations

$k = 1, 2, \ldots, p$ are the various types of commodities

x_{ijk} is the amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination

c_{ijk} is the variable cost per unit amount of k^{th} type of commodity from the i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported , so long as $x_{ijk} > 0$.

F_{ik} is the fixed cost associated with origin i and commodity k.

$$F_{ik} = \sum_{j=1}^{n} F_{ijk}\delta_{ijk}, \ i = 1,2,\ldots,m; \ k = 1,2,\ldots, p$$

$$\begin{aligned} where \quad \delta_{ijk} &= 1, \ if \ \ x_{ijk} > 0 \\ &= 0, \ if \ \ x_{ijk} = 0 \end{aligned}$$

A_{jk} is the total quantity of k^{th} type of commodity to be sent to the j^{th} destination

B_{ki} is the total quantity of k^{th} type of commodity available at the i^{th} origin.

E_{ij} is the total quantity to be sent from i^{th} origin to the j^{th} destination.

3.4. MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

Multi-index bi-criterion transportation problem with restricted flow is an extension of Multi-index bi-criterion transportation problem. In this type of problem situation may arise where one wishes to reserve some stocks at the sources. In order to reserve the stocks at the sources thereby restricting the total transportation problem flow to known satisfied level, say

$$P_k = \left(< \min\left(\sum_{i=1}^{m} B_{ki}, \sum_{j=1}^{n} A_{jk}\right)\right) \qquad \text{for } k = 1, 2, \ldots, p$$

3.5. FORMULATION OF MULTI-INDEX FIXED CHARGE BI-CRITERION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

Minimize $$\left\{ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} C_{ijk}\, x_{ijk} + \sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik\,i},\ \underset{\substack{1\le i\le m\\ 1\le j\le n\\ 1\le k\le p}}{Max} \left[t_{ijk} / x_{ijk} > 0 \right] \right\}$$

subject to $$\sum_{i=1}^{m} x_{ijk} = A_{jk}$$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

$$\sum_{i\in I}\sum_{j\in J} x_{ijk} = P_k \left(< \min\left(\sum_{i=1}^{m} B_{ki}, \sum_{j=1}^{n} A_{jk}\right)\right) \ for\ k = 1, 2, \ldots, p$$

$$x_{ijk} \ge 0,\ \ i = 1, 2, \ldots m,\ j = 1, 2, \ldots, n, k = 1, 2, \ldots, p$$

where $$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki},\ \ \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij},\ \ \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p}\sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}$$

Here i = 1,2 , . . . , m are the origins

j = 1, 2, . . ., n are the destinations

$k = 1, 2, \ldots, p$ are the various types of commodities

x_{ijk} is the amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination

c_{ijk} is the variable cost per unit amount of k^{th} type of commodity from the i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported , so long as $x_{ijk} > 0$.

F_{ik} is the fixed cost associated with origin i and commodity k.

$$
\begin{aligned}
& F_{ik} = \sum_{j=1}^{n} F_{ijk}\delta_{ijk},\ i = 1,2,\ldots,m;\ k = 1,2,\ldots,p \\
\text{where}\quad & \delta_{ijk} = 1,\ if\ \ x_{ijk} > 0 \\
& \quad = 0,\ if\ \ x_{ijk} = 0
\end{aligned}
$$

A_{jk} is the total quantity of k^{th} type of commodity to be sent to the j^{th} destination

B_{ki} is the total quantity of k^{th} type of commodity available at the i^{th} origin.

E_{ij} is the total quantity to be sent from i^{th} origin to the j^{th} destination.

In order to solve (*P*) it is separated into two problems (*P1*) and (*P2*)

Problem (*P1*) is :

Minimize $$z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} c_{ijk}\, x_{ijk} + \sum_{i=1}^{m}\sum_{k=1}^{p} F_{ik}$$

subject to the above constraints.

Problem (*P2*) is :

Minimize $$T = \max_{\substack{1\le i\le m \\ 1\le j\le n \\ 1\le k\le p}} [t_{ijk} \,/\, x_{ijk} > 0]$$

subject to the above constraints.

For formulation of $F_{ik}(i=1,2,\ldots,m,k=1,2,\ldots,p)$

$$F_{ik} = \sum_{j=1}^{n} F_{ijk}\delta_{ijk}, \; i = 1,2,\ldots,m; \; k = 1,2,\ldots, p$$

$$\text{where} \quad \delta_{ijk} = 1, \; if \;\; x_{ijk} > 0$$
$$= 0, \; if \;\; x_{ijk} = 0$$

The flow constraint in the problem (P1) implies that a total ($\sum B_{ki} - P$ for $k=1,2,\ldots,p$) of source reserves has to be kept at the various sources and a total ($\sum Ajk$ – P for $k=1,2,\ldots,p$) of destination slacks to be retained at the various destinations. Therefore, an extra destination to receive the source reserves and an extra source to fill up the destination slacks are introduced. Hence, the related multi-index fixed charge bi-criterion transportation problem associated with multi-index fixed charge bi-criterion transportation problem (FCBTP) is

$$\text{Minimize} \quad \left\{ \sum_{i\in I'}\sum_{j\in J'}\sum_{k\in K'} C'_{ijk}\, y_{ijk} + \sum_{i\in I'}\sum_{k\in K'} F_{ik}, \max_{\substack{i\in I' \\ j\in J' \\ k\in k'}} \left[t'_{ijk} / y_{ijk} > 0 \right] \right\}$$

subject to

$$\sum_{i\in I'} y_{ijk} = A'_{jk} \quad j \in J', k \in K'$$
$$\sum_{j\in J'} y_{ijk} = B'_{ki} \quad i \in I', k \in K'$$
$$\sum_{k\in K'} y_{ijk} = E'_{ij} \quad j \in J', i \in I'$$
$$y_{ijk} \geq 0$$

where

$$\sum_{j\in J'} A'_{jk} = \sum_{i\in I'} B'_{ki}, \; \sum_{k\in K'} B'_{ki} = \sum_{j\in J'} E'_{ij}, \; \sum_{i\in I'} E'_{ij} = \sum_{k\in K'} A'_{jk}$$

$$\sum_{j\in J'}\sum_{k\in K'} A'_{jk} = \sum_{k\in K'}\sum_{i\in I'} B'_{ki} = \sum_{i\in I'}\sum_{j\in J'} E'_{ij}$$

$$\begin{aligned}
\textit{where}\quad & I' = \{1, 2, \ldots, m+1\} = I \cup \{m+1\} \\
& J' = \{1, 2, \ldots, n+1\} = J \cup \{n+1\} \\
& k' = \{1, 2, \ldots, p\} \\
& A'_{jk} = A_{jk},\ \ j \in J\ ,\ A'_{n+1,k} = (\sum_{j \in J} B_{ki} - P_k) \text{ for } k = 1, 2, \ldots, p \\
& B'_{ki} = B_{ki},\ \ i \in I\ ,\ B'_{k,m+1} = (\sum_{i \in I} A_{jk} - P_k) \text{for } k = 1, 2, \ldots, p
\end{aligned}$$

$$\begin{aligned}
& c'_{ijk} = c_{ijk} (i, j, k) \in I \times J \times K \\
& t'_{ijk} = t_{ijk} (i, j, k) \in I \times J \times K \\
& c'_{i,n+1,k} = c'_{m+1,j,k} = 0,\ \ i \in I, j \in J, k \in K \\
& t'_{i,n+1,k} = t'_{m+1,j,k} = 0,\ \ i \in I, j \in J, k \in K \\
& F_{m+1,k} = 0,\ \ \text{for } k = 1, 2, \ldots, p \\
& c'_{m+1,n+1,k} = M,\ \ \text{for } k = 1, 2, \ldots, p \\
& t'_{m+1,n+1,k} > \underset{\substack{i \in I \\ j \in J \\ k \in K}}{\text{Max}} [t_{ijk} \setminus y_{ijk} > 0]
\end{aligned}$$

Where M is a large positive integer.

Multi-index fixed-charge bi-criterion transportation problemwith restricted flow is seprated into two problems (MP1) and (MP2)

$$\text{(MP1)} \quad \text{Minimize} \quad \left\{ \sum_{i \in I'} \sum_{j \in J'} \sum_{k \in K'} C'_{ijk}\, y_{ijk} + \sum_{i \in I'} \sum_{k \in K'} F_{ik} \right\}$$

subject to $\sum_{i\in I'} y_{ijk} = A'_{jk} \quad j \in J', k \in K'$

$$\sum_{j\in J'} y_{ijk} = B'_{ki} \quad i \in I', k \in K'$$

$$\sum_{k\in K'} y_{ijk} = E'_{ij} \quad j \in J', i \in I'$$

$$y_{ijk} \geq 0$$

(MP2) Minimize $\underset{\substack{i\in I' \\ j\in J' \\ k\in k'}}{Max} \left[t'_{ijk} / y_{ijk} > 0 \right] \}$

subject to $\sum_{i\in I'} y_{ijk} = A'_{jk} \quad j \in J', k \in K'$

$$\sum_{j\in J'} y_{ijk} = B'_{ki} \quad i \in I', k \in K'$$

$$\sum_{k\in K'} y_{ijk} = E'_{ij} \quad j \in J', i \in I'$$

$$y_{ijk} \geq 0$$

where $\sum_{j\in J'} A'_{jk} = \sum_{i\in I'} B'_{ki}, \sum_{k\in K'} B'_{ki} = \sum_{j\in J'} E'_{ij}, \sum_{i\in I'} E'_{ij} = \sum_{k\in K'} A'_{jk}$

$$\sum_{j\in J'} \sum_{k\in K'} A'_{jk} = \sum_{k\in K'} \sum_{i\in I'} B'_{ki} = \sum_{i\in I'} \sum_{j\in J'} E'_{ij}$$

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