FUZZY LINEAR PROGRAMMING AND ITS APPLICATIONS

Thesis submitted in partial fulfillment of the requirement for

The award of the degree of

Masters of Science

In

Mathematics and Computing

Submitted by

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JULY 2009

School of Mathematics and Computer Applications

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INDIA

DEDICATED

TO

GOD, MY PARENTS AND MY SUPERVISOR

List of Research Papers

1. "A new approach for solving fuzzy transportation problem", Eighteenth International conference of Forum of Interdisciplinary Mathematics on Interdisciplinary Mathematical and Statistical Techniques, August 2-4, 2009, Jaypee University of Information Technology, Waknaghat, Distt. Solan, (Near Shimla), Himachal Pradesh, INDIA. (**Accepted**)
2. "A new approach for solving fuzzy travelling salseman problem", 12th International Conference on Rough Sets, Fuzzy Sets, Data Mining & Granular Computing (RSFDGrC 2009), December 16-18, 2009, I.I.T Delhi. (**Communicated**)
3. "A new approach for solving fuzzy assignment problem", International conference on Statistics, Probability, Operations Research, Computer Science and allied Areas in conjunction with 8th International Indian Statistical Association (IISA), January 4-8, 2010, Department of Mathematics, Andhra University, Visakhapatnam. (**Communicated**)
4. "Multiobjective transportation problem using linear membership function", Applied Computational Intelligence and Soft Computing. (**Communicated**)

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Fuzzy linear programming and its applications**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ACKNOWLEDGMENTS

First of all, I would like to thank the Almighty for granting perseverance. I would like to express my gratitude to **Dr. Amit Kumar**, **Lecturer**, **SMCA**, **Thapar University**, **Patiala**, for their patient guidance and support throughout this work. I was truly very fortunate to have the opportunity to work under him as a student. It was both an honor and a privilege to work with him. He also provides help in technical writing and presentation style and I found this guidance to be extremely valuable. I take this opportunity to express my sincere thanks to **Prof. S. S. Bhatia**, **Head SMCA**, **Thapar University**, **Patiala**, for their valuable support and help without which it would not have been possible for me to complete this work.

I am also thankful to all my friends who devoted their valuable time and helped me in all possible ways towards successful completion of this work. I do not find enough words with which I can express my feeling of thanks to the entire faculty and staff of SMCA, Thapar University, Patiala, for their help, inspiration and moral support which went a long way in successful completion of my work. I thank all those who have contributed directly or indirectly to this work.

Lastly, and more importantly, I would like to thank my parents for their years of unyielding love and encouragement. They have always wanted the best for me and I admire my parent's determination and sacrifice to put me through college.

Patiala

July 14, 2009 (Jolly Puri)

ABSTRACT

In many real life situations, the decision maker may not be in a position to specify the objective and/or constraint functions precisely but rather can specify them in a "fuzzy sense". In such situations, it is desirable to use some fuzzy linear programming type of modeling. The technique of fuzzy linear programming enlarges the range of applications of the linear programming method. It enables us to consider tolerances for values of decision model parameters in a more natural and direct way. It is of special importance in a situation when the necessity of taking into account tolerances for parameters is due to the impossibility of determining them precisely as well as the situation when some tolerances for parameters are consciously assumed by the decision maker.

This thesis is devoted to fuzzy linear programming and its applications. The main topics are various models of fuzzy linear programming problem, fuzzy transportation problem, fuzzy assignment problem, fuzzy travelling salesman problem and multiobjective transportation problem.

The chapter-wise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes basic definitions, operations and concepts used throughout the work.

Chapter 2 presents brief review of the work done in the area of fuzzy linear programming problem, fuzzy transportation problem, fuzzy assignment problem, fuzzy travelling salesman problem and multiobjective transportation problem.

In **Chapter 3**, decision making under fuzzy environment and various models of fuzzy linear programming problem have been studied.

In **Chapter 4**, fuzzy transportation problem has been studied. Three new methods have been proposed to obtain the initial fuzzy basic feasible solution of a particular type of fuzzy transportation problem. Also a new method has been proposed to find the fuzzy optimal solution from initial basic feasible solution. To illustrate the proposed methods, a real life problem has been solved.

In **Chapter 5**, fuzzy assignment problem which is a special type of fuzzy linear programming problem has been studied. A new algorithm has been proposed to find the fuzzy optimal solution for the fuzzy assignment problem and is illustrated by a numerical example.

In **Chapter 6**, fuzzy travelling salesman problem is studied. A new algorithm has been proposed using ranking to characterize the fuzzy optimality of travelling salesman problem. To illustrate the proposed algorithm a numerical has been solved.

In **Chapter 7**, the importance of the fuzzy programming approach, goal programming approach and their combination called the fuzzy goal programming approach to solve the multiobjective transportation problem has been discussed. A new approach has been proposed for the problem solved by Zangiabadi and Maleki [58]. Also the results of existing and proposed approaches are compared from different point of views.

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Chapter 1

INTRODUCTION

1.1 Introduction

During the last four decades, a large number of mathematical tools have been developed in and for Operational Research. They are primarily devices to find the optimal solutions after a problem has been modeled formally. Hardly any new tools have been developed for proper modeling of problems which are not well-structured and which can not easily be cast into classical mathematical models. Fuzzy set theory as proposed by Zadeh [57] promises to bridge part of this gap. It is much more adaptable to different problem structures and better suited to model human evaluation and decision making processes, than classical mathematics. For the linear programming problems (LPPs) in the crisp scenario, the aim is to maximize or minimize a linear objective function subject to linear constraints. But in many practical situations, the decision maker may not be in a position to specify the objective and/or constraint functions precisely but rather can specify them in a "fuzzy sense". In such situations, it is desirable to use some fuzzy linear programming type of modeling. The basic definitions [59] used throughout the work are as follows:

Definition 1.1

A crisp set or a classical set A is defined as a collection of distinct and distinguishable objects. The objects are called elements of A. A crisp set A, defined on the universal set *X*, can also be represented by $A = \{(x, \mu_A(x)); x \in X\}$ where $\mu_A : X \to \{0,1\}$ is called characteristic function defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 1.2

The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range $[0,1]$ i.e $\mu_{\tilde{A}} : X \to [0,1]$. The assigned values indicate the membership grade of the element in the set A.

The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set. $\mu_{\tilde{A}}(x)$ is the degree of membership of x in $\tilde{A}$. The closer the value of $\mu_{\tilde{A}}(x)$ is to 1, the more x belongs to A.

Definition 1.3

Let $\tilde{A}$ be a fuzzy set and α be a real number in the interval $[0,1]$. The crisp set A_α defined by $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ is called $\alpha-$ cut of $\tilde{A}$.

The crisp set $A_{\alpha^+} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\}$ is called strong $\alpha-$ cut of $\tilde{A}$.

Definition 1.4

The support of a fuzzy set $\tilde{A}$ is the crisp subset of X and is presented as:

$$\text{supp}(\tilde{A}) = \{\, x \in X \mid \mu_{\tilde{A}}(x) > 0\} .$$

Definition 1.5

A fuzzy set $\tilde{A}$, defined on the universal set X, is said to be convex if

$\mu_{\tilde{A}}(\alpha_1 x_1 + \alpha_2 x_2) \geq \min\{\, \mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2) \,\}$ $\forall\ x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \geq 0$,

$\alpha_1 + \alpha_2 = 1$.

Definition 1.6

A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

1. $\mu_{\tilde{A}} : R \to [0,1]$ is continuous.
2. $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, c] \cup [d, \infty)$.
3. Its strictly increasing on $[c, a]$ and strictly decreasing on $[b, d]$.
4. $\mu_{\tilde{A}}(x) = 1$ for all $x \in [a, b]$.

Definition 1.7

A fuzzy number $\tilde{A} = (a_l, a, a_u)$ is called a triangular fuzzy number if its membership function $\mu_{\tilde{A}}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_l}{a - a_l}, & a_l \le x \le a, \\ \dfrac{x - a_u}{a - a_u}, & a \le x \le a_u. \end{cases}$$

The triangular fuzzy number $\tilde{A}$ has the shape of a triangle as shown in Figure 1.1 given below:

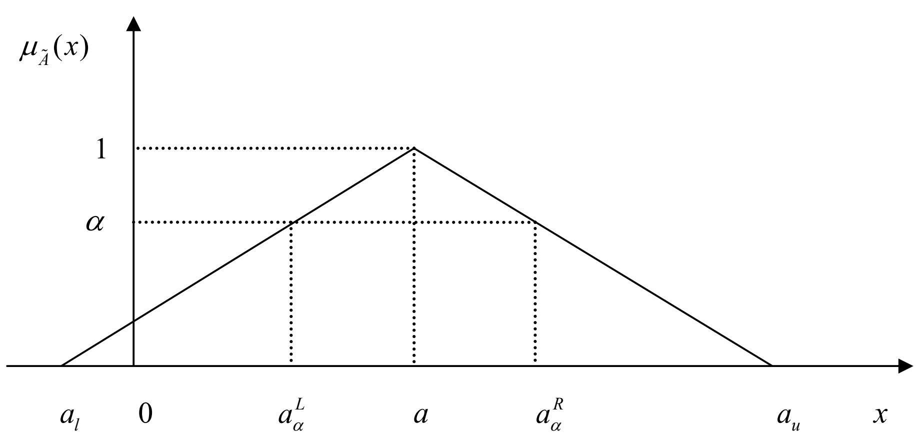

Figure 1.1 A Triangular fuzzy number $\tilde{A} = (a_l, a, a_u)$

Further, the α - cut of the triangular fuzzy number $\tilde{A} = (a_l, a, a_u)$ is the closed interval

$$A_\alpha = \left[a_\alpha^L, a_\alpha^R\right] = \left[a_l + (a - a_l)\alpha \ , \ a_u + (a - a_u)\alpha\right], \ \alpha \in (0,1].$$

The another representation of triangular fuzzy number is $\tilde{A} = (a-\alpha, a, a+\beta)$ where $a - \alpha = a_l$ and $a + \beta = a_u$ and $\tilde{A}$ can also be written in another form as $\tilde{A} = (a, \alpha, \beta)$.

1.2 Arithmetic operations between two triangular fuzzy numbers

Let $\tilde{A} = (a_l, a, a_u)$ and $\tilde{B} = (b_l, b, b_u)$ be two triangular fuzzy numbers, then using the α - cuts, A_α and B_α for $\alpha \in (0,1]$ one can compute $\tilde{A} * \tilde{B}$ where $*$ may be any operation.

Addition ($\oplus$):- $\quad \tilde{A} \oplus \tilde{B} = (a_l + b_l, a + b, a_u + b_u)$

Subtraction (Θ):- $\quad \tilde{A} \ \Theta \ \tilde{B} = (a_l - b_u, a - b, a_u - b_l)$

Scalar Multiplication ($k\tilde{A}$):- $k\tilde{A} = (ka_l, \ ka, \ ka_u)$ if $k > 0$ is a scalar and

$k\tilde{A} = (ka_u, \ ka, \ ka_l)$ if $k < 0$ is a scalar.

Symmetry (or mirror image) ($-\tilde{A}$):- $-\tilde{A} = (-a_u, -a, -a_l)$.

Multiplication ($\otimes$):- $\tilde{A} \otimes \tilde{B} = (a', b', c')$ where $a' = \min(a_l b_l, a_l b_u, b_l a_u, a_u b_u)$, $b' = ab$

and $c' = \max(a_l b_l, a_l b_u, b_l a_u, a_u b_u)$.

Division ($\oslash$):- $\quad \tilde{A} \oslash \tilde{B} = (a', b', c')$ where $a' = \min\left(\dfrac{a_l}{b_l}, \dfrac{a_l}{b_u}, \dfrac{a_u}{b_l}, \dfrac{a_u}{b_u}\right)$, $\quad b' = \dfrac{a}{b}$ and

$$c' = \max\left(\frac{a_l}{b_l}, \frac{a_l}{b_u}, \frac{a_u}{b_l}, \frac{a_u}{b_u}\right), \ b_l > 0 \ \text{ or } \ b_u < 0.$$

Remark – To find arithmetic operations between $\tilde{A} = (a, \alpha_1, \beta_1)$ and $\tilde{B} = (b, \alpha_2, \beta_2)$, first convert $\tilde{A}$ into $\tilde{A} = (a - \alpha_1, a, a + \beta_1)$ and $\tilde{B}$ into $\tilde{B} = (b - \alpha_2, b, b + \beta_2)$ and then use arithmetic operation given in section 1.2 on $\tilde{A}$ and $\tilde{B}$.

Definition 1.8

A fuzzy number $\tilde{A}=(a_l,\underline{a},\overline{a},a_u)$ is called a trapezoidal fuzzy number if its membership function is given by:

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a_l}{\underline{a}-a_l}, & a_l\le x\le \underline{a},\\ 1, & \underline{a}\le x\le \overline{a},\\ \dfrac{x-a_u}{\overline{a}-a_u}, & \overline{a}\le x\le a_u.\end{cases}$$

The trapezoidal fuzzy number $\tilde{A}$ has the shape of a trapezoid as shown in the Figure 1.2 given below:

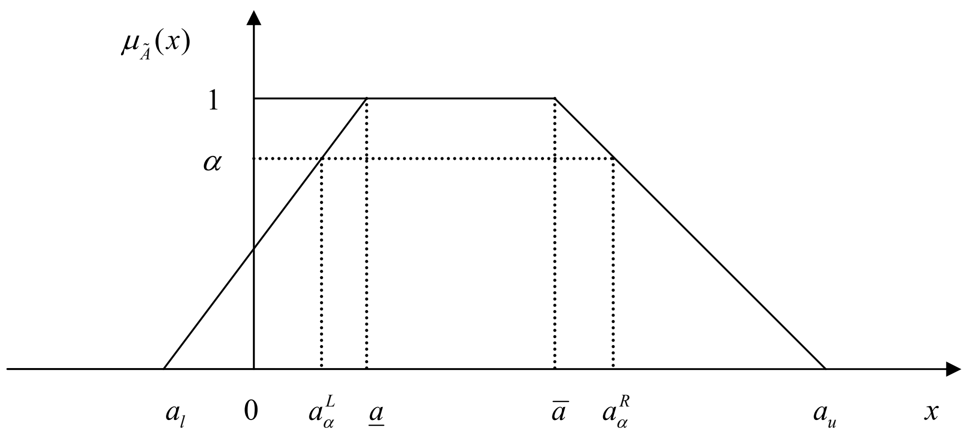

Figure 1.2 A Trapezoidal fuzzy number $\tilde{A}=(a_l,\underline{a},\overline{a},a_u)$

Further, the α- cut of the trapezoidal fuzzy number $\tilde{A}=(a_l,\underline{a},\overline{a},a_u)$ is the closed interval

$$A_\alpha=\left[a_\alpha^L,a_\alpha^R\right]=\left[a_l+(\underline{a}-a_l)\alpha\ ,\ a_u+(\overline{a}-a_u)\alpha\right],\ \alpha\in(0,1].$$

The another representation of trapezoidal fuzzy number is $\tilde{A}=(\underline{a}-\alpha,\underline{a},\overline{a},\overline{a}+\beta)$ where $\underline{a}-\alpha=a_l$ and $\overline{a}+\beta=a_u$ and $\tilde{A}$ can also be written in another form as $\tilde{A}=(\underline{a},\overline{a},\alpha,\beta)$.

1.3 Arithmetic operations between two trapezoidal fuzzy numbers

Let $\tilde{A}=(a_l,\underline{a},\overline{a},a_u)$ and $\tilde{B}=(b_l,\underline{b},\overline{b},b_u)$ be two trapezoidal fuzzy numbers, then using the α - cuts, A_α and B_α for $\alpha\in(0,1]$ one can compute $\tilde{A}*\tilde{B}$ where $*$ may be any operation.

Addition ($\oplus$):- $\tilde{A}\oplus\tilde{B}=(a_l+b_l,\ \underline{a}+\underline{b},\ \overline{a}+\overline{b},\ a_u+b_u)$

Subtraction (Θ):- $\tilde{A}\ \Theta\ \tilde{B}=(a_l-b_u,\underline{a}-\overline{b},\overline{a}-\underline{b},a_u-b_l)$

Scalar Multiplication ($k\tilde{A}$):- $k\tilde{A}=(ka_l,k\underline{a},k\overline{a},ka_u)$ if $k>0$ is a scalar and

$k\tilde{A}=(ka_u,\ k\overline{a},\ k\underline{a},\ ka_l)$ if $k<0$ is a scalar.

Symmetry (or mirror image) ($-\tilde{A}$):- $-\tilde{A}=(-a_u,-\overline{a},-\underline{a},-a_l)$

Multiplication ($\otimes$):- $\tilde{A}\otimes\tilde{B}=(a',b',c',d')$ where $a'=\min(a_lb_l,a_lb_u,b_la_u,a_ub_u)$,

$b'=\underline{a}\,\underline{b}$, $c'=\overline{a}\,\overline{b}$ and $d'=\max(a_lb_l,a_lb_u,b_la_u,a_ub_u)$.

Division ($\oslash$):- $\tilde{A}\oslash\tilde{B}=(a',b',c',d')$ where $a'=\min\left(\frac{a_l}{b_l},\frac{a_l}{b_u},\frac{a_u}{b_l},\frac{a_u}{b_u}\right)$, $b'=\frac{\underline{a}}{\underline{b}}$, $c'=\frac{\overline{a}}{\overline{b}}$

and $d'=\max\left(\frac{a_l}{b_l},\frac{a_l}{b_u},\frac{a_u}{b_l},\frac{a_u}{b_u}\right)$, $b_l>0$ or $b_u<0$.

1.4 Ranking Function

A convenient method for comparing of fuzzy number is by use of ranking function [40, 55]. A ranking function $\Re:F(R)\to R$, where $F(R)$ (a set of all fuzzy numbers defined on set of real numbers), maps each fuzzy number into a real number of $F(R)$.

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers in $F(R)$, then

$$(i)\ \tilde{a}\underset{\Re}{\geq}\tilde{b} \text{ if and only if } \Re(\tilde{a})\geq\Re(\tilde{b})$$

$$(ii)\ \tilde{a} \underset{\Re}{>} \tilde{b} \text{ if and only if } \Re(\tilde{a}) > \Re(\tilde{b})$$

$$(iii)\ \tilde{a} \underset{\Re}{=} \tilde{b} \text{ if and only if } \Re(\tilde{a}) = \Re(\tilde{b})$$

Let $\Re$ be any linear ranking function. Then,

i. $\tilde{a} \underset{\Re}{\geq} \tilde{b}$ if and only if $\tilde{a} - \tilde{b} \underset{\Re}{\geq} \tilde{0}$ if and only if $-\tilde{b} \underset{\Re}{\geq} -\tilde{a}$.

ii. If $\tilde{a} \underset{\Re}{\geq} \tilde{b}$ and $\tilde{c} \underset{\Re}{\geq} \tilde{d}$, then $\tilde{a} + \tilde{c} \underset{\Re}{\geq} \tilde{b} + \tilde{d}$.

1.4.1 Ranking function for triangular fuzzy number

For a triangular fuzzy number $\tilde{a} = (a_l, a, a_u)$ (or (a, α, β)), ranking function is given by $\Re(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf a_\alpha + \sup a_\alpha) d\alpha$, where a_α is $\alpha-$cut on $\tilde{a}$. This reduces to $\Re(\tilde{a}) = \frac{1}{4}(a_l + 2a + a_u) \left(\text{or } a + \frac{(\beta - \alpha)}{4}\right)$.

Then for triangular fuzzy number $\tilde{a} = (a, \alpha, \beta)$ and $\tilde{b} = (b, \gamma, \theta)$, We have $\tilde{a} \underset{\Re}{\geq} \tilde{b}$ if and only if $\Re(\tilde{a}) = \left[a + \frac{\beta - \alpha}{4}\right] \geq \left[b + \frac{\theta - \gamma}{4}\right] = \Re(\tilde{b})$

1.5 Fuzzy linear programming [60]

A fuzzy linear programming (FLP) problem is defined as follows:

$$(\text{P}) \qquad \begin{aligned} &\text{Max} \quad \tilde{z} \underset{\Re}{=} \tilde{c}x \\ &\text{subject to} \quad Ax = b, \\ &\qquad\qquad\quad x \geq 0. \end{aligned}$$

where $b \in R^m, x \in R^n, \tilde{c} \in (F(R))^n, A \in R^{m \times n}$, $\Re$ is linear ranking function.

Definition 1.9 [50]

A vector $x \in R^n$ is said to be feasible solution of fuzzy linear programming problem if and only if x satisfies the constraints of the problem.

Definition 1.10 [50]

If there are m equations in $(m+n)$ variables then to solve these equations put any n variables equal to zero and find the solution of m equations in m variables. If the obtained solution is unique then it is called basic solution, otherwise, it is called non-basic solution. The zero valued variables are called non-basic variables and the remaining variables are called basic variables.

Definition 1.11 [50]

A basic feasible solution x^* is an optimal solution for (P), if for all basic feasible solution x of (P), we have $\tilde{c}x^* \underset{\Re}{\geq} \tilde{c}x$.

Definition 1.12 [60]

$\tilde{z} \underset{\Re}{=} \tilde{c}x$ is said to be the fuzzy optimal value of problem (P) if x is the optimal solution of (P).

Definition 1.13 [6]

The linear multiobjective transportation problem is a special type of vector minimum problem in which constraints are of equality type and the objectives are conflicting in nature.

Definition 1.14 [6]

A feasible solution $\hat{x} = \{\hat{x}_{ij}\} \in X$ is said to be a nondominated solution of the multiobjective transportation problem in [6] if there is no other feasible solution $x = \{x_{ij}\} \in X$ such that $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij} \leq \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k \hat{x}_{ij}$ for all k, and $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij} < \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k \hat{x}_{ij}$ for at least one k.

Definition 1.15 [6]

An optimal compromise solution of the multiobjective transportation problem from [6] is a solution $\hat{x} = \left\{\hat{x}_{ij}\right\} \in X$ which is preferred by the decision maker to all other solutions, taking into consideration all criteria contained in the multiobjective functions. Hence, an optimal compromise solution has to be a nondominated solution according to the definition of nondominated solution.

Chapter 2

LITERATURE REVIEW

The concept of fuzzy set theory, first introduced by Zadeh [57], is used for solving different types of linear programming problems [2, 7, 11, 20, 31, 37, 42, 43, 56, 60]. Fuzzy set theory has been applied to many disciplines such as control theory and management sciences, mathematical modelling and industrial applications. The concept of fuzzy mathematical programming on general level was first proposed by Tanaka et al. [51] in the framework of the fuzzy decision of Bellman and Zadeh [4]. Chanas [9] showed an application of parametric programming techniques in fuzzy linear programming and obtained the set of solutions maximizing the objective function, being analytically dependent on a parameter. Delgado et al. [17] studied a general model for fuzzy linear programming problems which includes fuzziness both in the coefficients and in the accomplishment of the constraints. Fang and Hu [21] considered linear programming with fuzzy constraint coefficients. Buckley and Feuring [8] considered the extreme case of fully fuzzified linear programming problem with all the parameters and variables as fuzzy numbers. Maleki and et al [40] introduced a linear programming problem with fuzzy variables and proposed a new method for solving these problems using an auxiliary problem. Vasant and et al. [52] applied linear programming with fuzzy parameters for decision making in industrial production planning. Some authors used the concept of comparison of fuzzy numbers for solving fuzzy linear programming problems. In fact, most convenient methods are based on the concept of comparison of fuzzy numbers by use of ranking functions [40]. Ranking functions have been proposed by researchers to suit their requirements of the problem under consideration and conceivably there are no generally accepted criteria for application of ranking function. A review of some common methods for ranking fuzzy numbers can be seen in [55].

The transportation problem is one of the earliest applications of linear programming problems. The basic transportation problem was originally developed by Hitchcock [26]. Efficient methods of solution derived from the simplex algorithm were developed in 1947, primarily by Dantzig and then by Charnes and Cooper [14]. The transportation problem can be modeled as a standard linear programming problem, which can then be solved by the simplex method. However, because of its very special mathematical structure, it was recognized early that the simplex method applied to the transportation problem can be made quite efficient in terms of how to evaluate the necessary simplex-method information (variable to enter the basis, variable to leave the basis and optimality conditions). In 1963, Dantzig used the simplex method to the transportation problem as the Primal Simplex Transportation Method (PSTM). An initial basic feasible solution for the transportation problem can be obtained by using the North West corner rule, Row minima, Column minima, Matrix minima, or the Vogel Approximation Method (VAM). The Modified Distribution (MODI) Method is useful for finding the optimal solution for the transportation problem. Charnes and Cooper [14] developed the Stepping Stone Method (SSM) which provides an alternative way of determining the simplex-method information. The linear Interactive and Discrete Optimization (LINDO) [49], General Interactive Optimizer (GINO) [38] and TORA packages [50] as well as many other commercial and academic packages are useful to find the solution of the transportation problem. In general, transportation problems are solved with the assumptions that the coefficients or cost parameters are specified in a precise way i.e.

in crisp environment. In real life, there are many diverse situations due to uncertainty in judgements, lack of evidence etc. Sometimes it is not possible to get relevant precise data for the cost parameter. This type of imprecise data is not always well represented by random variable selected from a probability distribution. Fuzzy number may represent this data. So, fuzzy decision making method is needed here.

Zimmerman [60] showed that solutions obtained by fuzzy linear programming are always efficient. Subsequently, Zimmermann's fuzzy linear programming has developed into several fuzzy optimization methods for solving the transportation problems. Chanas et al. [11] presented a FLP model for solving transportation problems with crisp cost coefficients and fuzzy supply and demand values. Moreover, Chanas and Kuchta [12] proposed the concept of the optimal solution for the transportation problem with fuzzy coefficients expressed as $L-R$ fuzzy numbers, and developed an algorithm for obtaining the optimal solution. Chanas and Kuchta [13] designed an algorithm for solving the integer fuzzy transportation problem with fuzzy supply and demand volumes in the sense of maximizing the joint satisfaction of the fuzzy goal and constraints. Kikuchi [32] suggested that in many problems of transportation engineering and planning, the observed or derived values of the variables are approximate, yet the variables themselves must satisfy a set of rigid relationships dictated by physical principle. When the observed values do not satisfy the relationships, each value is adjusted until they satisfy the relationship. They proposed a simple adjustment method that finds the most appropriate set of crisp numbers. The method assumes that each observed value is an approximate number (or a fuzzy number) and the true value is found in the support of the membership function. For each of many possible sets of values that satisfy the relationships, the lowest membership grade is checked and the set whose lowest membership grade is the highest, is chosen as the best set of values for the problem. This process is performed using the fuzzy linear programming method. Their paper presents the model, the computational process and applications. Chanas and Kuchta [12] and Omar and Samir [45] discussed the solution algorithm for solving the transportation problem in fuzzy environment. Grzegorzewski [23] and Chanas [10] approximated the fuzzy number to its nearest interval. Liu and Kao [39] described a method for solving fuzzy transportation problems based on extension principle.

Fuzzy assignment problems have received great attention in recent years. For instance, Chen [16] proposed a fuzzy assignment model that did not consider the differences of individuals, and also proved some theorems. Wang [54] solved a similar model by graph theory. The author first presented his model on a network of which the arc values are fuzzy numbers, and then proposed a solution procedure to solve this network problem. Fortemps and Dubois [22] proposed a flexible assignment problem, which combines with fuzzy theory, multiple criteria decision-making and constraint-directed methodology. They also demonstrated and solved an example of fuzzy assignment problem. Sakawa et al. [47] dealt with actual problems on production and work force assignment of a housing material manufacturer, and formulated two-level linear and linear fractional programming problems according to profit and profitability maximization, respectively. By applying interactive fuzzy programming for two-level linear and linear fractional programming problems, they derived satisfactory solutions to the problems and compared the results.

In 1961, goal programming introduced by Charnes and Cooper [15]. Goal programming has been widely applied to solve different real-world problems which involve multiple objectives [24, 25, 28, 41, 56]. Lee and Moore [36] applied goal programming to find a solution for MOTP. Other authors used fuzzy goal programming technique to solve different types of multiobjective linear programming problems [2, 24, 43]. Abd El-Wahed and Lee [2] presented an interactive fuzzy goal programming technique for MOTP. Basically, the method of goal programming consists of formulating an objective function in which optimization comes as close as possible to the specified goals. In 1965 the concept of priority factors was developed, assigning different priority levels to goals and different weights for the goals at the same priority level. Ignizio and Lee [28, 34] have discussed the subject of goal programming which is an extension of linear programming.

A variety of approaches, such as lexicographic goal programming approach, interval goal programming approach, interactive algorithms, fuzzy programming approach, the step method, the utility function method have been developed by many researchers for the multiobjective linear programming problem. The interactive algorithms by Ringuest and Rinks [44] give more than k non-dominated and dominated solutions if there are k objectives. Thus the decision maker has to determine a compromise solution from the set of non-dominated solutions. For the larger problem, it is not easy to find the compromise solution by using the algorithm developed by Ringuest and Rinks [44] but, using the fuzzy programming method, one can easily find a compromise solution. The multiobjective transportation problem (MOTP) in a crisp environment was extensively studied in [29, 44]. Lee and Moore [36] studied the optimization of multi-objectives transportation problem. Zimmermann [60] first applied the fuzzy set theory concept with some suitable membership functions to solve linear programming problem with several objective functions. Diaz [18, 19] developed an algorithm for finding the solution of multi-objective transportation problem. Isermann [29] developed an algorithm for identifying all the non-dominated solutions, for a linear multi-objective transportation problem. Leberling [33] used hyperbolic membership function for multiobjective linear programming problem. He pointed out that his solution procedure with hyperbolic membership function always gives an efficient solution. Sakawa [46] and Sakawa et al. [48] proposed an interactive fuzzy decision making method using linear and non-linear membership functions to solve the multiobjective linear programming problem. Ringuest and Rinks [44] developed two interactive algorithm for solving multi-objective transportation problem.

Bit et al. [6] considered a k - objective transportation problem fuzzified by fuzzy numbers and used α-cut to obtain a transportation problem in the fuzzy sense expressed in linear programming form. Bit et al. [5] said that the algorithms proposed by Ringuest and Rinks [44], Diaz [18], and Bit et al. [6] are not applicable to multiobjective transportation problems when the relative importance or priority of the objectives are given . To overcome this difficulty, they proposed a new method. Bit et al. [5], Lee and Li [35], Jimenez and Verdegay [30], Li and Lai [37], Wahed [1] represented the fuzzy compromise programming approach to multi-objective transportation problem. Verma et al. [53] applied a fuzzy programming technique to solve multiobjective transportation probelm with some nonlinear membership functions. Hussien [27] studied the complete set of α-possibly efficient solutions of multiobjective transportation problem with possibilistic coefficients of the objective functions. Ammar and Youness [3] investigated the efficient solutions and stability of multiobjective transportation problem with fuzzy coefficient and/or fuzzy supply quantities and/or fuzzy demands quantities.

Chapter 3

VARIOUS MODELS OF FUZZY LINEAR PROGRAMMING PROBLEM

3.1 Introduction

For the linear programming problems (LPPs) in the crisp scenario, the aim is to maximize or minimize a linear objective function subject to linear constraints. But in many practical situations, the decision maker may not be in a position to specify the objective and/or constraint functions precisely but rather can specify them in a "fuzzy sense". In such situations, it is desirable to use some fuzzy linear programming type of modeling. This chapter aims to study various models [60] of fuzzy linear programming problem and consists of sections, namely, decision making under fuzzy environment and fuzzy linear programming, linear programming problems with fuzzy inequalities and crisp objective function.

3.2 Decision making under fuzzy environment and fuzzy linear programming

A fuzzy decision making model is characterized by a set of goals $G_i \ (i = 1, 2, ..., m)$, along with a set of constraints $C_j \ (j = 1, 2, ..., n)$, each of which is expressed by a fuzzy set on X. For such a model of decision making, Bellman and Zadeh [4] in their pioneering work , proposed that a fuzzy decision is determined by an appropriate aggregation of fuzzy sets $G_i \ (i = 1, 2, ..., m)$ and $C_j \ (j = 1, 2, ..., n)$. The main feature in this approach is the symmetry between goals and constraints. Keeping this in mind, they (Bellman and Zadeh) suggested the aggregation operator to be the fuzzy intersection. Thus a *fuzzy decision* D could be defined as the fuzzy set $D = (G_1 \cap G_2 \cap ... \cap G_p) \cap (C_1 \cap C_2 \cap ... \cap C_n)$ and $\mu_D : X \to [0,1]$ is given by

$$\mu_D = \min_{i,j} (\mu_{G_i}(x), \mu_{C_j}(x)) .$$

After knowing the fuzzy decision D, the decision $x^* \in X$ is said to be an *optimal decision* if $\mu_D(x^*) = \max_x \mu_D(x)$. Another method to solve a decision making model is choose an α s.t. $0 < \alpha < 1$ and find all the points $x^* \in X$ for which $\mu_D(x^*) \geq \alpha$. These decisions x^* will have at least α degree of membership value.

The classical linear programming problem aims to find the minimum or maximum of a linear objective function under some linear constraints. The most typical linear programming problem is stated as (L P)

$$\begin{aligned} &\text{Max} \quad c^T x \\ &\text{subject to,} \\ &\qquad A x \leq b, \\ &\qquad\;\; x \geq 0, \\ &\text{where } x \in R^n, c \in R^m, and\ A \in R^m \times R^n . \end{aligned}$$

In the decision making terminology, x is referred as a vector of decision variables, A as the constraint coefficient matrix, b as a vector of available resources and c as a vector of cost coefficients. When decision is to be made in fuzzy environment, many possible modifications of the above linear programming model exist, such as,

i. The decision maker might not really want to maximize or minimize the objective function, rather he might want to achieve some aspiration level which might not be even defined crisply. For example, the decision maker might want to "improve the present sales situation considerably".

ii. The entries of the vectors c, b and the matrix A may not be crisp but rather may be fuzzy numbers and the inequalities may be interpreted in terms of ranking of fuzzy numbers.

Thus the fuzzy linear programming models are not uniquely defined as it will very much depend upon the type of fuzziness and its specification as prescribed by the decision maker. Therefore the class of fuzzy linear programming problems can be broadly classified as:

i. linear programming problems with fuzzy resources and fuzzy coefficient, also termed as linear programming problems with fuzzy parameters, i.e. elements of c, b and A are fuzzy numbers,

ii. linear programming problems with fuzzy inequalities and crisp objective function,

iii. linear programming problems with crisp inequalities and fuzzy objective function, and

iv. linear programming problems with fuzzy inequalities and fuzzy objective function.

The class of fuzzy linear programming problems can also be classified as *symmetric* or *non symmetric*. The *symmetric* models are based on the definition of fuzzy decision as proposed by Bellman and Zadeh [4]. In symmetric models the basic feature is the symmetry of objectives and constraints. Bellman and Zadeh approach gives the decision set as a fuzzy set resulting from the intersection of the fuzzy sets corresponding to the objective and constraints. On the other hand, the *non symmetric* models keep distinction between the objective and constraints. Here usually two approaches are followed. In the first approach, a fuzzy set of *decisions* is determined and then the (crisp) objective function is "maximized" over this fuzzy set. This approach leads to a parametric linear programming problem. In the second approach, after determining the fuzzy set of decisions, a suitable membership function of the objective function is determined and then the problem is solved similar to the symmetric case. In the next sections, two approaches for solving LPPs with fuzzy inequalities and crisp objective function are given.

3.3 LPPs with fuzzy inequalities and crisp objective function

The general model of a linear programming problem with fuzzy inequalities and crisp objective function is as follows :

$$\begin{aligned} &\text{Max} \quad c^T x \\ &\text{subject to}, \\ &\qquad A_i x \lesssim b_i, (i = 1, 2, ..., m), \\ &\qquad x \geq 0, \end{aligned} \tag{3.1}$$

where $\lesssim$ is called " fuzzy less than or equal to" and is to be understood in terms of a suitably chosen membership function.

There are two approaches to solve LPPs with fuzzy inequalities and crisp objective function which are as follows:

3.3.1 Verdegay's Approach : a non symmetric model

Verdegay showed that the problem (3.1) is equivalent to a crisp parametric linear programming problem and therefore, parametric programming methods can be used to solve such fuzzy linear programming problems. Here the fuzzy constraints are transformed into crisp constraints by choosing appropriate membership function for each constraint. To motivate for a meaningful choice of membership function, it is argued that if $A_i x \leq b_i$ then it means the i^{th} constraint is absolutely satisfied, where as if $A_i x \geq b_i + p_i$, where p_i is the maximum tolerance from b_i, as determined by the decision maker, then it means the i^{th} constraint is absolutely violated. For

$A_i\, x \in (b_i\ , b_i + p_i)$, the membership function is monotonically decreasing. If this decrease is along a linear function then it makes sense to choose the membership function of the i^{th} constraint $(i = 1, 2, ..., m)$ as

$$\mu_i(A_i x) = \begin{cases} 1, & A_i x < b_i\,, \\ 1 - \dfrac{A_i x - b_i}{p_i}, & b_i \le A_i x < b_i + p_i\,, \\ 0\,, & A_i x > b_i + p_i\,. \end{cases} \tag{3.2}$$

where $A_i\, (i = 1, 2, ..., m)$ denotes the i^{th} row of A .

Now, for $\alpha \in [\,0\,,1]$ let $X_\alpha = \left\{\, x \in R^n : x \ge 0 \text{ and } \mu_i(A_i x) \ge \alpha\,,\ \ (i = 1, 2, ..., m)\right\}$

then the problem (3.1) is equivalent to

$$\begin{aligned} &\text{Max} \qquad c^T x \\ &\text{subject to}, \\ &\qquad\qquad x \in X_\alpha . \end{aligned}$$

Now substitute the expression for the membership functions $\mu_i(A_i x)$ and then the above problem becomes:

$$\begin{aligned} (\text{LP})_\alpha \qquad &\text{Max} \qquad c^T x \\ &\text{subject to}, \\ &\qquad A_i x \le b_i + (1-\alpha)\, p_i\,, (\,i = 1, 2, ..., m\,), \\ &\qquad\quad x \ge 0,\ \alpha \in [\,0\,,1]. \end{aligned}$$

which is equivalent to a standard parametric linear programming problem, with parameter $(1-\alpha)$. Thus the fuzzy linear programming problem (3.1) can be solved by solving an equivalent crisp parametric linear programming problem.

3.3.2 Werner's approach : a symmetric model

Werner's proposed that for the problems of the type (3.1), the objective function should be fuzzy. Further, to construct a membership function for objective function, he suggested to solve the following two linear programming problems

(LP(b)) and (LP($b+p$)).

(LP(b))

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{subject to}, \quad & \\ & Ax \le b, \\ & x \ge 0. \end{aligned}$$

(LP($b+p$))

$$\begin{aligned} \text{Max} \quad & c^T x \\ \text{subject to}, \quad & \\ & Ax \le b+p, \\ & x \ge 0. \end{aligned}$$

Here as before, $p=(p_1, p_2, ..., p_m)^T$ is the vector of tolerances for the m constraints of (3.1). Let Z_0 and Z_1 be optimal values of (LP(b)) and (LP($b+p$)) respectively. Now, construct a linear membership function for the objective function by using Z_0 and Z_1 as follows:

$$\mu_0(c^T x) = \begin{cases} 1, & c^T x > Z_1, \\ 1 - \dfrac{Z_1 - c^T x}{Z_1 - Z_0}, & Z_0 \le c^T x \le Z_1, \\ 0, & c^T x > Z_0. \end{cases}$$

The membership functions of the constraints are same as (3.2) in Verdegay's approach.

Now using the above membership functions, $\mu_i \ (i = 0,1,2,...,m)$ and following Bellman and Zadeh principle, the problem (3.1) is solved by solving the following crisp linear programming problem

$$\begin{aligned} \text{Max} \quad & \alpha \\ \text{subject to}, \quad & \mu_0(x) \ge \alpha, \\ & \mu_i(x) \ge \alpha, \quad (i=1,2,...,m), \\ & \alpha \in [0,1],\ x \ge 0. \end{aligned}$$

which on substitution for $\mu_i \ (i = 0,1,2,...,m)$ becomes

$$\begin{aligned} \text{Max} \quad & \alpha \\ \text{subject to}, \quad & c^T x \ \ge \ Z_1 - (1-\alpha)(Z_1 - Z_0), \\ & A_i x \ \le \ b_i + (1-\alpha) p_i, \\ & \alpha \in [0,1], x \ge 0. \end{aligned}$$

Chapter 4

FUZZY TRANSPORTATION PROBLEM

4.1 Introduction

The transportation problem refers to a special class of linear programming problems. In a typical problem, a product is to be transported from m sources to n destinations and their capacities are $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ respectively. In addition there is a penalty c_{ij} associated with transporting unit of product from i^{th} source to j^{th} destination. This penalty may be cost or delivery time. A variable x_{ij} represents the unknown quantity to be shipped from i^{th} source to j^{th} destination. Efficient algorithms have been developed for solving the transportation problem when the cost coefficients and the supply and demand quantities are known exactly. However, there are cases that these parameters may not be presented in a precise manner. For example, the unit shipping cost may vary in a time frame. The supplies and demands may be uncertain due to some uncontrollable factors. To deal quantitatively with imprecise information in decision making, Bellman and Zadeh [4] introduced the notion of fuzziness. In this chapter, three new methods have been proposed to find the initial basic feasible solution of the fuzzy transportation problem and a new method for the optimal solution and fuzzy optimal value.

4.2 Fuzzy transportation problem (FTP)

The formulation of the model and the mathematical model of FTP is given in the section 4.2.1 and an efficient algorithm has been introduced to find the fuzzy optimal solution of FTP.

4.2.1 Formulation

The formulation of the FTP is similar to the conventional transportation problem i.e. the objective function is to minimize the total fuzzy transportation cost and the constraints are the supply and demand available to each source and destination, respectively.

The mathematical model of fuzzy transportation problem when all the cost coefficients are fuzzy numbers and all supply and demand are crisp numbers is given by:

$$\text{Min } \tilde{z} \underset{\Re}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}\, x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = a_{ij}, \qquad i = 1, 2, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_{ij}, \qquad j = 1, 2, \ldots, n.$$

$$\sum_{i=1}^{m} a_{ij} = \sum_{j=1}^{n} b_{ij}$$

$$x_{ij} \geq 0 \;, \forall\, i, j.$$

where $\Re$ is a ranking function (i.e. $\Re : F(R) \to R$) which maps each fuzzy number into the real line;

a_i is the quantity of material available at source S_i $(i = 1, 2, \ldots, m)$;

b_j is the quantity of material required at destination D_j $(j = 1, 2, \ldots, n)$ and

$\tilde{c}_{ij}$ is fuzzy unit cost of transportation from source S_i to destination D_j.

Every fuzzy transportation problem can be represented by a fuzzy matrix of order m by n, called the fuzzy cost matrix or fuzzy effectiveness matrix. For $m = 3$, $n = 4$, the structure of fuzzy cost matrix is

Destination → Sources ↓	D_1	D_2	D_3	D_4	Availability or Supply ↓
S_1	$\tilde{c}_{11}$ x_{11}	$\tilde{c}_{12}$ x_{12}	$\tilde{c}_{13}$ x_{13}	$\tilde{c}_{14}$ x_{14}	a_1
S_2	$\tilde{c}_{21}$ x_{21}	$\tilde{c}_{22}$ x_{22}	$\tilde{c}_{23}$ x_{23}	$\tilde{c}_{24}$ x_{24}	a_2
S_3	$\tilde{c}_{31}$ x_{31}	$\tilde{c}_{32}$ x_{32}	$\tilde{c}_{33}$ x_{33}	$\tilde{c}_{34}$ x_{34}	a_3
Requirement or Demand →	b_1	b_2	b_3	b_4	

4.2.2 The fuzzy transportation algorithm

Step 1. Formulate the fuzzy transportation problem and arrange the data in matrix form i.e. the fuzzy cost matrix.

Step 2. Obtain an initial basic feasible solution (IBFS). Three different methods are proposed to obtain the IBFS which are as follows:

- Fuzzy North-West Corner Method,
- Fuzzy Least Cost Method and
- Fuzzy Vogel's Approximation (or Penalty) Method.

The algorithms of these methods for the FTP when all the cost coefficients are fuzzy numbers and all demand and supply are crisp numbers are as follows:

- **Fuzzy North-West Corner Method**

    1. Start with the cell at the north – west corner of fuzzy transportation matrix and allocate as much as possible equal to the minimum of demand and supply values for the first row and first column, i.e. $\min(a_1, b_1)$.

2. The row or column which is satisfied is crossed out and ignored for further consideration.
3. Repeat the procedure until the entire supply at various sources and demand at various destinations is satisfied.

   Remark. During the process of making allocation at a particular cell if supply equals demand, then next allocation of magnitude zero can be made in a cell either in the next row or column. This condition is known as degeneracy.

- **Fuzzy Least Cost Method**

The objective of fuzzy transportation problem is to minimize the total fuzzy transportation cost. To achieve this objective, transport as much as possible through those cells (of fuzzy transportation cost matrix) where the unit fuzzy transportation cost is lowest. Steps in the Least Cost Method are as follows:

1. Select the cell with the lowest unit cost (i.e. select the cell whose the fuzzy cost $\tilde{c}_{ij}$ has minimum rank) in the entire fuzzy transportation table (matrix) and allocate as much as possible to this cell and eliminate (line out) the row or column at every step in which either supply or demand is exhausted. If both row and column are satisfied simultaneously, only one is crossed out.

   In case the smallest fuzzy unit cost is not unique (i.e. rank of more than one fuzzy unit cost is same), then select the cell where maximum allocation can be made.
2. Repeat the procedure with the next lowest fuzzy unit cost among the remaining rows and columns of the fuzzy transportation table until the

entire supply at various sources and demand at various destinations are satisfied.

- **Fuzzy Vogel's Approximation Method**

Vogel's approximation method is a heuristic method in which each allocation is made on the basis of the penalty cost that would have been incurred if allocations in certain cells with minimum fuzzy unit transportation cost were missed. In this method, allocations are made so that the penalty cost is minimized. Steps in Vogel's approximation method are as follows:

1. Calculate penalties for each row (column) by taking the difference between the smallest and next smallest fuzzy unit transportation cost in the same row (column). This difference indicates the penalty cost which has to be paid if one fails to allocate to the cell with the minimum fuzzy transportation cost.
2. Select the row or column with the largest penalty (i.e. the row or column with largest rank of penalty cost) and allocate as much as possible in the cell having the least fuzzy cost in the selected row or column. If there is a tie in the values of penalties, it can be broken by selecting the cell where maximum allocation can be made.
3. Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of them is crossed out and the remaining row (column) with zero supply or demand should not be used in computing future penalties.
4. Repeat step 1 to 3 until the entire available supply at various sources and demand at various destinations are satisfied.

Step 3. The initial solution obtained by any of the three methods must satisfy the following conditions:

i. The solution must be feasible, i.e. it must satisfy all the supply and demand constraints. In other words,

$$\sum_{j=1}^{n} x_{ij} = a_i , \qquad i=1,2,\ldots,m,$$
$$\sum_{i=1}^{m} x_{ij} = b_j , \qquad j=1,2,\ldots,n.$$

ii. The number of positive allocations must be equal to $m+n-1$, when m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called non-degenerate basic feasible solution, otherwise , degenerate solution.

Step 4. Test the IBFS for optimality. The optimal solution is found by fuzzy u-v method which is given in the section 4.2.2.1.

4.2.2.1 Fuzzy u-v method

In this section, fuzzy u-v method has been introduced to find the optimal solution of a FTP with the help of IBFS. The steps for finding the optimal solution are as follows:

Step 1. Find the IBFS of FTP using fuzzy north-west corner method or fuzzy least cost method or fuzzy vogel's approximation method.

Step 2. Introduce the fuzzy dual variables $\tilde{u}_i$ and $\tilde{v}_j$ corresponding to each i^{th} row and j^{th} column, respectively. Write $\tilde{u}_i$ in front of each i^{th} row and $\tilde{v}_j$ at the bottom of each j^{th} column. Take any one of the $\tilde{u}_i$ or $\tilde{v}_j$ to be zero ranked fuzzy number.

Step 3. For basic cells, $\tilde{u}_i \oplus \tilde{v}_j = \tilde{c}_{ij}$. This relation assigns values to all $\tilde{u}_i$ and $\tilde{v}_j$.

Step 4. For non-basic cells find the rank of $\tilde{d}_{ij} = \tilde{u}_i \oplus \tilde{v}_j \Theta \ \tilde{c}_{ij} \ \forall i, j$ and write them in corner of the concerned cell.

Case (i) If $\tilde{d}_{ij} \underset{\Re}{\leq} 0 \ \forall i \,\&\, j$ then the obtained IBFS is fuzzy optimal solution.

Case (ii) If there exist at least one $\tilde{d}_{ij}$ such that $\tilde{d}_{ij} \underset{\Re}{>} 0$ then this IBFS is not optimal. Go to the step 5.

Step 5. In the FTP choose that $\tilde{d}_{ij}$ whose rank is most positive.

Step 6. Assign θ quantity in that cell, corresponding to which rank of $\tilde{d}_{ij}$ is most positive, and make a loop as follows:

Rule for making the loop: Start from θ cell and move horizontally and vertically to the nearest basic cell with the restriction that the turn of the loop must not lie in any non-basic cell (except θ cell). In this way, return to the θ cell to complete the loop.

Step 7. Add and subtract θ in cornered x_{ij} of the loop maintaining feasibility, and value of θ is fixed as the minimum of the entries from which θ has been subtracted.

Step 8. Inserting the fixed value of θ, we get next BFS which improves the fuzzy initial transportation cost. While inserting the value of θ one cell assumes 0 value. We shall not mention 0 value as this is the fuzzy leaving variable, i.e., this cell has become non-basic. This gives the improved basic feasible solution (BFS).

Step 9. Again, use the latest BFS, and repeat steps 1 to 8 until $\tilde{d}_{ij} \underset{\Re}{\leq} 0 \ \forall i \,\&\, j$.

Step 10. The obtained optimal solution and fuzzy optimal cost are x_{ij} and

$\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \, x_{ij}$ respectively.

Example 4.1 The data is collected from a trader which supplies a product to different companies after taking that product from different sources. There are three plants denoted as S1, S2, S3 which supplies the product TMT to four different centers denoted as D1, D2, D3, D4. This finished product TMT is made from raw material INGOT and BILLET. The approximate cost per ton (in rupees) of the product is represented by a triangular fuzzy number. Supply and demand are crisp numbers and The fuzzy transportation costs are shown in Table 4.1. Solve the problem to minimize the approximate cost.

Destinations → Sources ↓	D1 Ludhiana	D2 Delhi	D3 Himachal Pardesh(Kullu)	D4 Leh Ladakh	Supply In tons
S1 Mandi Gobindgarh (Fortune Metals)	(190,200,220)	(550,600,650)	(900,950,1000)	(3300,3500,4500)	4000
S2 Bhwarhi (Kamdhenu Saria)	(600,650,700)	(300,320,340)	(900,930,950)	(3400,3600,4600)	3200
S3 Raipur (Goel Group)	(2500,2600,2700)	(2000,2050,2100)	(2750,2800,2850)	(5000,5100,5200)	1800
Demand In tons	3000	3200	2200	2000	

Table 4.1

In this fuzzy transportation problem, per unit costs are fuzzy numbers whereas demand, supply are crisp numbers.

Destinations → Sources ↓	D1	D2	D3	D4	
S1	(190,200,220)	(550,600,650)	(900,950,1000)	(3300,3500,4500)	4000
S2	(600,650,700)	(300,320,340)	(900,930,950)	(3400,3600,4600)	3200
S3	(2500,2600,2700)	(2000,2050,2100)	(2750,2800,2850)	(5000,5100,5200)	1800
	3000	3200	2200	2000	

Table 4.2

Here costs are triangular fuzzy numbers of the form $(m-\alpha, m, m+\beta)$. Convert all the costs into the form (m, α, β) and find ranks of the costs using formula $m + \dfrac{(\beta-\alpha)}{4}$.

Destinations → Sources ↓	D1	D2	D3	D4	1800
S1	(200,10,20) Rank=**202.5**	(600,50,50) Rank=**600**	(950,50,50) Rank=**950**	(3500,200,1000) Rank=**3700**	4000
S2	(650,50,50) Rank=**650**	(320,20,20) Rank=**320**	(930,30,20) Rank=**927.5**	(3600,200,1000) Rank=**3800**	3200
S3	(2600,100,100) Rank=**2600**	(2050,50,50) Rank=**2050**	(2800,50,50) Rank=**2800**	(5100,100,100) Rank=**5100**	
3000	3200	2200	2000		

Table 4.3

From now onwards the bold numerics below the fuzzy costs represent the ranks of the corresponding fuzzy costs. This problem is unbalanced fuzzy transportation problem$\left(\sum a_i \neq \sum b_j\right)$. To obtain it's the fuzzy optimal solution; first convert it into fuzzy balanced transportation problem. Create a dummy row (dummy source) if total demand exceeds total supply, and a dummy column (dummy destination) if total supply exceeds total demand. The supply or demand at the dummy origin is equal to the symmetric difference of total supply and total demand. Here in this problem total demand exceeds total supply. So, create a dummy row (fictitious supply). The supply at the dummy origin is $10400-9000=1400$ tons.

(200,10,20) **202.5**	(600,50,50) **600**	(950,50,50) **950**	(3500,200,1000) **3700**	4000
(650,50,50) **650**	(320,20,20) **320**	(930,30,20) **927.5**	(3600,200,1000) **3800**	3200
(2600,100,100) **2600**	(2050,50,50) **2050**	(2800,50,50) **2800**	(5100,100,100) **5100**	1800
(0,0,0) **0**	(0,0,0) **0**	(0,0,0) **0**	(0,0,0) **0**	1400
3000	3200	2200	2000	

Table 4.4

The IBFS can be obtained by any of the following methods:

I. Fuzzy North-West Corner Rule

The final table after applying the algorithm of Fuzzy North-West Rule to a given problem is given by:

(200,10,20) **202.5** **3000**	(600,50,50) **600** **1000**	(950,50,50) **950**	(3500,200,1000) **3700**	4000
(650,50,50) **650**	(320,20,20) **320** **2200**	(930,30,20) **927.5** **1000**	(3600,200,1000) **3800**	3200
(2600,100,100) **2600**	(2050,50,50) **2050**	(2800,50,50) **2800** **1200**	(5100,100,100) **5100** **600**	1800
(0,0,0) **0**	(0,0,0) **0**	(0,0,0) **0**	(0,0,0) **0** **1400**	1400
3000	3200	2200	2000	

Table 4.5

Value of objective function at IBFS $=\sum_{i=1}^{4}\sum_{j=1}^{4}\tilde{c}_{ij}\,x_{ij} = (8980000,\ 9254000,\ 9548000)$.

Apply fuzzy u-v method on the above table by taking fuzzy costs of the form (m,α,β) to find an optimal solution.

In order to calculate the value's of $\tilde{u}_i$'s $(i=1,2,3,4)$ and $\tilde{v}_j$'s $(j=1,2,3,4)$ for each occupied cell, we arbitrarily assign $\tilde{u}_1=(0,0,0)$ to simplify calculations. Taking $\tilde{u}_1=(0,0,0)$, $\tilde{u}_2,\tilde{u}_3,\tilde{u}_4,\tilde{v}_1,\tilde{v}_2,\tilde{v}_3$ and $\tilde{v}_4$ can be computed immediately by using the relation $\tilde{c}_{ij}=\tilde{u}_i\oplus\tilde{v}_j$ for occupied cells (allocated cells) as shown below:

$\tilde{c}_{11}=\tilde{u}_1\oplus\tilde{v}_1 \quad \Rightarrow \tilde{v}_1=\tilde{c}_{11}\ \Theta\ \tilde{u}_1 \quad \Rightarrow \tilde{v}_1=(200,10,20)\,\Theta\,(0,0,0,)=(200,10,20)$

Similarly, others can easily be calculated.

$\tilde{u}_1=(0,0,0,),\tilde{u}_2=(-280,70,70),\tilde{u}_3=(1590,140,150),\tilde{u}_4=(-3510,240,250)$ and

$\tilde{v}_1 = (200,10,20)$, $\tilde{v}_2 = (600,50,50)$, $\tilde{v}_3 = (1210,100,90)$, $\tilde{v}_4 = (3510,250,240)$.

Now, the opportunity cost for each of the unoccupied cell is determined by using the relation $\tilde{d}_{ij} \underset{\Re}{=} \tilde{u}_i \oplus \tilde{v}_j \Theta\ \tilde{c}_{ij}$.

The opportunity cost of the cell $(1,3)$ is given by:

$$\tilde{d}_{13} \underset{\Re}{=} \tilde{u}_1 \oplus \tilde{v}_3 \Theta \tilde{c}_{13} \Rightarrow \tilde{d}_{13} \underset{\Re}{=} (0,0,0) \oplus (1210,100,90)\ \Theta\ (950,50,50) = (260,150,140).$$

Similarly, other's can be calculated and there value's are given by:

$\tilde{d}_{14} = (10,1250,440)$, $\Re(\tilde{d}_{14}) = -192.5 < 0$; $\tilde{d}_{21} = (-730,130,140)$, $\Re(\tilde{d}_{21}) = -727.5 < 0$;

$\tilde{d}_{24} = (-370,1320,510)$, $\Re(\tilde{d}_{24}) = -572.5 < 0$; $\tilde{d}_{31} = (-810,250,270)$, $\Re(\tilde{d}_{31}) = -805 < 0$;

$\tilde{d}_{32} = (140,240,250)$, $\Re(\tilde{d}_{32}) = 142.5 > 0$; $\tilde{d}_{41} = (-3310,250,270)$, $\Re(\tilde{d}_{41}) = -3305 < 0$;

$\tilde{d}_{42} = (-2910,290,300)$, $\Re(\tilde{d}_{42}) = -2907.5 < 0$; $\tilde{d}_{43} = (-2300,340,340)$, $\Re(\tilde{d}_{43}) = -2300 < 0$.

According to the optimality criterion for fuzzy cost minimizing fuzzy transportation problem, the current solution is not optimal, since the ranks of all opportunity costs of the unoccupied cells are not zero or negative. The value of $\Re(\tilde{d}_{13}) = 257.5$ which is most positive and this indicates that x_{13} is the entering variable and make a loop in order to find the leaving variable.

(200,10,20) **3000**	(600,50,50) **1000 -** θ	(950,50,50) θ	(3500,200,1000)
(650,50,50)	(320,20,20) **2200 +** θ	(930,30,20) **1000 -** θ	(3600,200,1000)
(2600,100,100)	(2050,50,50)	(2800,50,50) **1200**	(5100,100,100) **600**
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0) **1400**

Table 4.6

$\theta = \min\{(1000, 1000\} = 1000$. Thus, the leaving variable is x_{12}. The next table is given by

(200,10,20) **3000**	(600,50,50)	(950,50,50) **1000**	(3500,200,1000)
(650,50,50)	(320,20,20) **3200**	(930,30,20) **0**	(3600,200,1000)
(2600,100,100)	(2050,50,50)	(2800,50,50) **1200**	(5100,100,100) **600**
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0) **1400**

Table 4.7

In next iteration, the opportunity cost $\tilde{d}_{32} = (140, 240, 250)$ and the value of $\Re(\tilde{d}_{32}) = 142.5$, which is the most positive value among all the opportunity costs. This indicates that x_{32} is the entering variable and by making loop, we get that the leaving variable is x_{33}. The next table is shown in Table 4.8.

(200,10,20) **3000**	(600,50,50)	(950,50,50) **1000**	(3500,200,1000)
(650,50,50)	(320,20,20) **2000**	(930,30,20) **1200**	(3600,200,1000)
(2600,100,100)	(2050,50,50) **1200**	(2800,50,50)	(5100,100,100) **600**
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0) **1400**

Table 4.8

In next iteration, the ranks of opportunity cost for each of the unoccupied cell is negative. Therefore, the above iteration is the last iteration and further reduction of fuzzy cost of fuzzy transportation problem is not possible. Thus the final fuzzy optimal value by taking fuzzy cost of the form $(m-\alpha,\ m,\ m+\beta)$ is given by:

Fuzzy optimal value = $\sum_{i=1}^{4}\sum_{j=1}^{4} \tilde{c}_{ij}\, x_{ij} = (8550000, 8826000, 9120000).$

II. Fuzzy Least Cost Method

The final table after applying the algorithm of Fuzzy Least Cost Method to a given problem is given below and find initial basic feasible solution by taking fuzzy cost of the form $(m-\alpha,\ m,\ m+\beta)$.

(200,10,20) **202.5** **1600**	(600,50,50) **600**	(950,50,50) **950** **800**	(3500,200,1000) **3700** **200**	4000
(650,50,50) **650**	(320,20,20) **320** **1800**	(930,30,20) **927.5** **1400**	(3600,200,1000) **3800**	3200
(2600,100,100) **2600**	(2050,50,50) **2050**	(2800,50,50) **2800** **1200**	(5100,100,100) **5100** **1800**	1800
(0,0,0) **0**	(0,0,0) **0** **1400**	(0,0,0) **0**	(0,0,0) **0**	1400
3000	3200	2200	2000	

Table 4.9

Value of objective function at IBFS $=(12750000,\ 13118000,\ 13662000)$.

After applying the $u-v$ method, the optimal solution is shown in Table 4.10.

(200,10,20) **3000**	(600,50,50)	(950,50,50) **1000**	(3500,200,1000)
(650,50,50)	(320,20,20) **2000**	(930,30,20) **1200**	(3600,200,1000)
(2600,100,100)	(2050,50,50) **1200**	(2800,50,50)	(5100,100,100) **600**
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0) **1400**

Table 4.10

Thus the final fuzzy optimal value by taking fuzzy cost of the form $(m-\alpha,\ m,\ m+\beta)$ is given by:

Fuzzy optimal value = $\sum_{i=1}^{4}\sum_{j=1}^{4}\tilde{c}_{ij}\,x_{ij}=(8550000, 8826000, 9120000)$.

III. Fuzzy Vogel Approximation Method

The final table after applying the algorithm of Fuzzy Vogel approximation method to a given problem is given below and find initial basic feasible solution by taking fuzzy cost of the form $(m-\alpha,\ m,\ m+\beta)$.

(200,10,20) **202.5** **3000**	(600,50,50) **600**	(950,50,50) **950** **400**	(3500,200,1000) **3700** **600**	4000
(650,50,50) **650**	(320,20,20) **320** **1400**	(930,30,20) **927.5** **1800**	(3600,200,1000) **3800**	3200
(2600,100,100) **2600**	(2050,50,50) **2050** **1800**	(2800,50,50) **2800**	(5100,100,100) **5100**	1800
(0,0,0) **0**	(0,0,0) **0**	(0,0,0) **0**	(0,0,0) **0** **1400**	1400
3000	3200	2200	2000	

Table 4.11

Value of objective function at IBFS = $=(8550000,\ 8892000,\ 9726000)$.

After applying the $u-v$ method, the optimal solution is shown in Table 4.12.

(200,10,20) **3000**	(600,50,50)	(950,50,50) **1000**	(3500,200,1000)
(650,50,50)	(320,20,20) **2000**	(930,30,20) **1200**	(3600,200,1000)
(2600,100,100)	(2050,50,50) **1200**	(2800,50,50)	(5100,100,100) **600**
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0) **1400**

Table 4.12

Thus the final fuzzy optimal value by taking fuzzy cost of the form $(m-\alpha,\ m,\ m+\beta)$ is given by:

Fuzzy optimal value = $\sum_{i=1}^{4}\sum_{j=1}^{4}\tilde{c}_{ij}\,x_{ij} = (8550000, 8826000, 9120000)$.

4.3 Results and discussion

The results obtained by solving fuzzy transportation problem by proposed methods can be explained as follows:

1 Total cost of transportation is greater than 8550000 and less than 9120000.

2 Maximum number of persons are in favour that cost will be 8826000.

3 The membership function for the obtained result is shown in Figure 4.1.

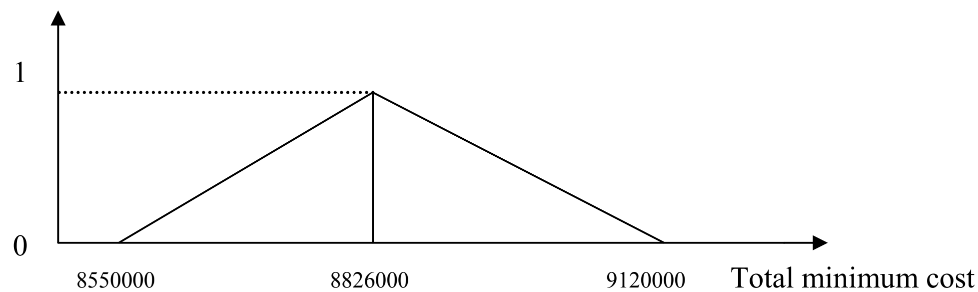

Figure 4.1

4 The percentage of persons increases when cost varies from 8550000 to 8826000 and decreases when cost varies from 8826000 to 9120000.

4.4 Conclusion

In this chapter, three new methods have been proposed to obtain the IBFS of a particular type fuzzy transportation problem. Also a new method has been proposed to find the fuzzy optimal solution from the IBFS. To illustrate the proposed methods the IBFS of a fuzzy transportation problem has been obtained by all three proposed methods and hence the optimal solution has been found using initial basic feasible solutions obtained from all the methods. It has been shown that the fuzzy optimal values does not depend on the chosen method for obtaining IBFS only the number of steps changes.

Chapter 5

FUZZY ASSIGNMENT PROBLEM

5.1 Introduction

The fuzzy assignment problem (FAP) is a special type of fuzzy linear programming problem and it is sub-class of fuzzy transportation problem. The fuzzy assignment problem can be stated as: Let n number of jobs be performed by n number of persons, where the costs depend on the specific assignments. Each job must be assigned to one and only one worker and each worker has to perform one and only one job. The problem is to find such an assignment that the total costs i.e., the sum of the single assignments, become a minimum. The fuzzy assignment problem can be applied to $n \times n$ fuzzy cost matrix $[\tilde{c}_{ij}]$, where $\tilde{c}_{ij}$ represents the fuzzy cost[1] associated with worker i $(i = 1, 2, \ldots, n)$ who has performed job j $(j = 1, 2, \ldots, n)$. The fuzzy assignment problem when costs are fuzzy numbers can also be modeled as 0-1 integer programming problem. Mathematically, a fuzzy assignment problem may be stated as follows:

$$
\begin{aligned}
&\text{Min} \underset{\Re}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}\, x_{ij} \\
&\text{s.t.} \ \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n, \\
&\qquad \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n, \\
&\qquad x_{ij} \in \{0, 1\} \quad \text{for } i, j = 1, 2, \ldots, n.
\end{aligned}
\tag{5.1}
$$

The fuzzy transportation problem becomes fuzzy assignment problem if all the

[1] In this chapter , approximate assignment cost is represented by a fuzzy number which is named as fuzzy cost

availabilities and demands are taken as 1. Thus, the fuzzy assignment problems can be solved by the method purposed for fuzzy transportation problem. In this chapter, a new algorithm has been proposed to find the fuzzy optimal solution for above type fuzzy assignment problem.

5.2 Unbalanced fuzzy assignment problem

If the number of jobs is not equal to the number of workers i.e. the fuzzy cost matrix is not square matrix then the problem becomes unbalanced fuzzy assignment problem. In such problems, to make the problem balanced introduce dummy job or dummy worker with zero ranked fuzzy cost (i.e. add dummy fuzzy cost $(0,0,0)$ having 0 rank in case of triangular fuzzy number).

Theorem 5.1 If in an assignment problem we add (or subtract) a constant to every element of a row (or column) of the fuzzy cost matrix, then an assignment plan which minimizes the total approximate cost for the new matrix, also minimizes the total approximate cost for the original fuzzy cost matrix i.e. if $x_{ij} = x_{ij}^*$ minimizes $\tilde{Z} \underset{\Re}{=} \sum\sum \tilde{c}_{ij} x_{ij}$ over all x_{ij} s.t. $\sum_i x_{ij} = \sum_j x_{ij} = 1$, $x_{ij} \geq 0$ then $x_{ij} = x_{ij}^*$ also minimizes $Z' \underset{\Re}{=} \sum\sum \tilde{c}'_{ij} x_{ij}$, where $\Re(\tilde{c}'_{ij}) = \Re(\tilde{c}_{ij} \pm \tilde{a}_i \pm \tilde{b}_j)$, $i, j = 1, 2, \ldots, m$ i.e. we have added (subtracted) numbers $\tilde{a}_1, \ldots, \tilde{a}_m$ to the respective rows and added (subtracted) numbers $\tilde{b}_1, \ldots, \tilde{b}_m$ to the respective columns.

Proof $$\Re(\tilde{Z}') = \Re\left(\sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{c}'_{ij} x_{ij}\right) = \Re\left(\sum_{i=1}^{m}\sum_{j=1}^{m} (\tilde{c}_{ij} \pm \tilde{a}_i \pm \tilde{b}_j)\, x_{ij}\right)$$

$$= \Re\left(\sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{c}_{ij} x_{ij} \pm \sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{a}_i x_{ij} \pm \sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{b}_j x_{ij}\right)$$

$$= \Re\left(\sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{c}_{ij} x_{ij}\right) \pm \Re\left(\sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{a}_i x_{ij}\right) \pm \Re\left(\sum_{i=1}^{m}\sum_{j=1}^{m} \tilde{b}_j x_{ij}\right)$$

$$= \Re(\tilde{Z}) \pm \Re\left(\sum_{i=1}^{m} \tilde{a}_i \sum_{j=1}^{m} x_{ij}\right) \pm \Re\left(\sum_{j=1}^{m} \tilde{b}_j \sum_{i=1}^{m} x_{ij}\right)$$

$$= \Re(\tilde{Z}) \pm \Re\left(\sum_{i=1}^{m} \tilde{a}_i .1\right) \pm \Re\left(\sum_{j=1}^{m} \tilde{b}_j .1\right)$$

$$\therefore \; \Re(\tilde{Z}') = \Re(\tilde{Z}) \pm \Re\left(\sum_{i=1}^{m} \tilde{a}_i\right) \pm \Re\left(\sum_{j=1}^{m} \tilde{b}_j\right)$$

Now, $\Re(\tilde{Z}')$ is minimum when $\Re(\tilde{Z})$ is minimum as $\sum \tilde{a}_i$ and $\sum \tilde{b}_j$ are constant quantities. Hence $x_{ij} = x_{ij}^*$ also minimizes $\Re(\tilde{Z}')$.

5.3 Proposed algorithm to solve fuzzy assignment problem

In this section, a new algorithm has been introduced to solve fuzzy assignment problem. The steps to solve balanced fuzzy assignment problems are as follows:

Step 1. Calculate rank of each fuzzy cost in fuzzy cost matrix. These ranks are used to compare two fuzzy numbers.

Step 2. Subtract the minimum element of each row in the fuzzy cost matrix from that row.

Step 3. Subtract the minimum element of each column in the fuzzy cost matrix from that column. This gives a fuzzy cost matrix which contains at least one fuzzy cost with zero rank in each row and in each column.

Step 4(a). Starting with first row, examine rows of the matrix obtained in step 3 successively until a row containing exactly one fuzzy cost with zero rank is found. Box this fuzzy cost as an assignment will be made there. Cut the column containing this assigned fuzzy cost with a vertical dotted line. This eliminates the confusion of making further assignments in that column. In this way examine all the rows one by one.

Step 4(b). After examining the rows completely, apply a similar procedure to columns successively. Starting from the first column examine all the uncrossed columns until a column containing exactly one fuzzy cost having zero rank is found. Box this fuzzy cost and cut the row containing this assigned zero with a horizontal line.

Step 4(c). After applying the above procedure (steps 4(a) and 4(b)) on rows and columns iteratively, any of the following two situations can be possible.

i. All fuzzy costs with rank zero have been boxed or discarded.

ii. The remaining zero ranked fuzzy costs lie at least two in each row and column.

In the first case, the maximum number of assignments have been assigned while in the second case, still some zero ranked fuzzy costs can be treated. Use trial and error method to break up such ties of zeros.

If the number of marked (assigned) zero ranked fuzzy cost is exactly m, then a complete optimal assignment is obtained. But if the number of assigned fuzzy costs (with zero rank) is less than m, then modify the cost matrix by creating some more zero ranked fuzzy costs in it.

Step 5. If a complete assignment is not possible in step 4, then select the fuzzy cost with minimum rank out of those which do not lie on any of the lines in the above matrix. Subtract this minimum fuzzy cost from all such uncrossed elements and add it to the elements which are placed at the intersection of horizontal and vertical lines. The entries lying on these lines but not on the intersection must be left unchanged. Form a new matrix in this way and check this matrix for step 2 and then apply step 3 and 4 to it. If still a complete optimal assignment is not available in this new matrix, then use the above steps iteratively.

Example 5.1 There are five workers and five jobs. In Table 5.1 the approximate efficiency (cost) of each worker to work on each job is given. Assign five jobs to five workers so as the total approximate efficiency (cost) is minimum.

Jobs → Workers ↓	A	B	C	D	E
1	(10,11,12)	(3,6,7)	(7,9,11)	(15,18,20)	(10,11,12)
2	(12,13,15)	(18,20,21)	(5,6,8)	(9,12,15)	(12,14,17)
3	(4,5,6)	(2,4,8)	(5,6,8)	(4,6,8)	(5,7,10)
4	(17,18,20)	(8,9,10)	(10,12,13)	(16,17,18)	(13,15,16)
5	(10,12,14)	(6,7,8)	(12,15,17)	(19,20,22)	(10,11,13)

Table 5.1

Solution. The optimal solution of the above fuzzy assignment problem can be obtained by using the proposed algorithm as follows:

Step 1. The ranks of fuzzy costs given in Table 5.1 are shown in Table 5.2.

(10,11,12) Rank = 11	(3,6,7) Rank = 5.5	(7,9,11) Rank = 9	(15,18,20) Rank =17.75	(10,11,12) Rank =11
(12,13,15) Rank = 13.25	(18,20,21) Rank =19.75	(5,6,8) Rank =6.25	(9,12,15) Rank = 12	(12,14,17) Rank = 14.25
(4,5,6) Rank = 5	(2,4,8) Rank = 4.5	(5,6,8) Rank = 6.25	(4,6,8) Rank = 6	(5,7,10) Rank = 7.25
(17,18,20) Rank = 18.25	(8,9,10) Rank = 9	(10,12,13) Rank = 11.75	(16,17,18) Rank = 17	(13,15,16) Rank = 14.75
(10,12,14) Rank = 12	(6,7,8) Rank = 7	(12,15,17) Rank = 14.75	(19,20,22) Rank = 20.25	(10,11,13) Rank = 11.25

Table 5.2

From now onwards, the ranks are denoted as bold numeric below fuzzy costs in the fuzzy cost matrix.

Step 2. The minimum element in the first row is (3,6,7). Subtract this element from the first row. Minimum element in the second row is (5,6,8) and subtract this minimum element from the second row. Do the similar procedure for the remaining rows. Now, repeat the same procedure for columns. The resultant matrix is shown in Table 5.3.

(-1,4,13) **5**	(-4,0,4) **0**	(0,3,8) **3.5**	(2,10,21) **10.75**	(-5,2,12) **2.75**
(0,6,14) **6.5**	(10,14,16) **13.5**	(-3,0,3) **0**	(-5,4,14) **4.25**	(-4,5,15) **5.25**
(-8,0,8) **0**	(-6,0,6) **0**	(-3,2,6) **1.75**	(-10,0,10) **0**	(-11,0,11) **0**
(3,8,16) **8.75**	(-2,0,2) **0**	(0,3,5) **2.75**	(0,6,14) **6.5**	(-5,3,11) **3**
(-2,4,12) **4.5**	(-2,0,2) **0**	(4,8,11) **7.75**	(5,11,20) **11.75**	(-6,1,10) **1.5**

Table 5.3

Step 3. In the first row, there is only single fuzzy cost (-4,0,4) having zero rank. So, make first assignment in the first row. Box (-4,0,4) in the first row and cut the corresponding column. In the second row, (-3,0,3) is the only zero ranked fuzzy cost. Thus, assign (-3,0,3) in the second row and cut the corresponding column. Now, in the third row, there is more than one zero ranked fuzzy cost. So, we cannot make assignment in the third row. In fourth and fifth row, no zero ranked fuzzy cost is left to make assignment. Now, in the first column, (-8,0,8) is the only zero ranked fuzzy cost. So, make assignment in the first column. Box (-8,0,8) and cut the corresponding row. At this step, all the zero ranked fuzzy costs are either boxed or crossed by line. The resultant matrix is shown in Table 5.4.

(-1,4,13) **5**	$\boxed{(-4,0,4)}$ **0**	(0,3,8) **3.5**	(2,10,21) **10.75**	(-5,2,12) **2.75**
(0,6,14) **6.5**	(10,14,16) **13.5**	$\boxed{(-3,0,3)}$ **0**	(-5,4,14) **4.25**	(-4,5,15) **5.25**
$\boxed{(-8,0,8)}$ **0**	(-6,0,6) **0**	(-3,2,6) **1.75**	(-10,0,10) **0**	(-11,0,11) **0**
(3,8,16) **8.75**	(-2,0,2) **0**	(0,3,5) **2.75**	(0,6,14) **6.5**	(-5,3,11) **3**
(-2,4,12) **4.5**	(-2,0,2) **0**	(4,8,11) **7.75**	(5,11,20) **11.75**	(-6,1,10) **1.5**

Table 5.4

It is obtained from Table 5.4, that each row and column does not contain boxed zero ranked fuzzy cost. Thus, the optimality is not reached.

Step 4. The minimum element from the uncovered elements in the Table 5.4 is

(-6,1,10). Subtract this element from the elements which are not covered with the vertical as well as horizontal lines. Add the same quantity to the elements which lie at the intersection of the lines and the remaining elements remain unchanged. The resultant matrix is shown in Table 5.5.

(-11,3,19) **3.5**	(-4,0,4) **0**	(0,3,8) **3.5**	(-8,9,27) **9.25**	(-15,1,18) **1.25**
(-10,5,20) **5**	(10,14,16) **13.5**	(-3,0,3) **0**	(-15,3,20) **2.75**	(-14,4,21) **3.75**
(-8,0,8) **0**	(-12,1,16) **1.5**	(-9,3,16) **3.25**	(-10,0,10) **0**	(-11,0,11) **0**
(-7,7,22) **7.25**	(-2,0,2) **0**	(0,3,5) **2.75**	(-10,5,20) **5**	(-15,2,17) **1.5**
(-12,3,18) **3**	(-2,0,2) **0**	(4,8,11) **7.75**	(-5,10,26) **10.25**	(-16,0,16) **0**

Table 5.5

After performing these four steps, first iteration is completed. Repeat steps 2 to 4 for the next iterations until each row and column contain exactly one boxed zero ranked fuzzy cost.

Step 5. In the second iteration, perform the steps 2 to 4 of the first iteration to find an optimal solution. The resultant matrix after making assignments is shown in Table 5.6.

(-11,3,19) 3.5	(-4,0,4) **0**	(0,3,8) **3.5**	(-8,9,27) **9.25**	(-15,1,18) **1.25**
(-10,5,20) 5	(10,14,16) **13.5**	(-3,0,3) **0**	(-15,3,20) **2.75**	(-14,4,21) **3.75**
(-8,0,8) **0**	(-12,1,16) **1.5**	(-9,3,16) **3.25**	(-10,0,10) **0**	(-11,0,11) **0**
(-7,7,22) **7.25**	(-2,0,2) **0**	(0,3,5) **2.75**	(-10,5,20) **5**	(-15,2,17) **1.5**
(-12,3,18) **3**	(-2,0,2) **0**	(4,8,11) **7.75**	(-5,10,26) **10.25**	(-16,0,16) **0**

Table 5.6

In the Table 5.6, still each row and each column does not contain exactly one boxed zero ranked fuzzy cost. Thus, repeat the procedure of 4th step of first iteration to Table 5.6. Step 4 of second iteration gives the Table 5.7.

(-31,0,34) **0.75**	(-4,0,4) **0**	(0,3,8) **3.5**	(-28,6,42) **6.5**	(-15,1,18) **1.25**
(-30,2,35) **2.25**	(10,14,16) **13.5**	(-3,0,3) **0**	(-35,0,35) **0**	(-14,4,21) **3.75**
(-8,0,8) **0**	(-27,4,36) **4.25**	(-24,6,36) **6**	(-10,5,20) **5**	(-26,3,31) **2.75**
(-27,4,37) **4.5**	(-2,0,2) **0**	(0,3,5) **2.75**	(-30,2,35) **2.25**	(-15,2,17) **1.5**
(-32,0,35) **0.25**	(-2,0,2) **0**	(4,8,11) **7.75**	(-25,7,41) **7.5**	(-16,0,16) **0**

Table 5.7

Now, second iteration is completed. Still optimal solution is not obtained. So, do the third iteration.

Step 6. In the third iteration, perform the steps 2 to 4 of the first iteration to find an optimal solution of the given problem. The resultant matrix after making assignments is shown in Table 5.8.

(-31,0,34) **0.75**	$\boxed{\text{(-4,0,4)}}$ **0**	(0,3,8) **3.5**	(-28,6,42) **6.5**	(-15,1,18) **1.25**
(-30,2,35) **2.25**	(10,14,16) **13.5**	$\boxed{\text{(-3,0,3)}}$ **0**	(-35,0,35) **0**	(-14,4,21) **3.75**
$\boxed{\text{(-8,0,8)}}$ **0**	(-27,4,36) **4.25**	(-24,6,36) **6**	(-10,0,10) **0**	(-26,3,31) **2.75**
(-27,4,37) **4.5**	(-2,0,2) **0**	(0,3,5) **2.75**	(-30,2,35) **2.25**	(-15,2,17) **1.5**
(-32,0,33) **0.25**	(-2,0,2) **0**	(4,8,11) **7.75**	(-25,7,41) **7.5**	$\boxed{\text{(-16,0,16)}}$ **0**

Table 5.8

In the Table 5.8, still optimal condition is not satisfied. Thus, the resultant matrix of step 4 of third iteration is shown in Table 5.9 and this completes the third iteration.

(-64,0,66) **0.5**	(-4,0,4) **0**	(-33,3,40) **3.25**	(-61,6,74) **6.25**	(-15,1,18) **1.25**
(-30,2,35) **2.25**	(-22,14,49) **13.75**	(-3,0,3) **0**	(-35,0,35) **0**	(-46,4,54) **4**
(-8,0,8) **0**	(-59,4,69) **4.5**	(-24,6,36) **6**	(-10,0,10) **0**	(-58,3,64) **3**
(-60,4,69) **4.25**	(-2,0,2) **0**	(-33,3,37) **2.5**	(-62,2,67) **2.25**	(-15,2,17) **1.5**
(-65,0,65) **0**	(-2,0,2) **0**	(-29,8,43) **7.5**	(-56,7,73) **7.75**	(-16,0,16) **0**

Table 5.9

Step 7. In the fourth iteration, perform the steps 2 to 4 of the first iteration to find an optimal solution. The resultant matrix after making assignments is shown in Table 5.10.

(-64,0,66) **0.5**	(-4,0,4) **0**	(-33,3,40) **3.25**	(-61,6,74) **6.25**	(-15,1,18) **1.25**
(-30,2,35) **2.25**	(-22,4,49)) **13.75**	(-3,0,3) **0**	(-35,0,35) **0**	(-46,4,54) **4**
(-8,0,8) **0**	(-59,4,69) **4.5**	(-24,6,36) **6**	(-10,0,10) **0**	(-58,3,64) **3**
(-60,4,69) **4.25**	(-2,0,2) **0**	(-33,3,37) **2.5**	(-62,2,67) **2.25**	(-15,2,17) **1.5**
(-65,0,65) **0**	(-2,0,2) **0**	(-29,8,43) **7.5**	(-56,7,73) **7.75**	(-16,0,16) **0**

Table 5.10

In the Table 5.10, still each row and each column do not contain exactly one boxed zero ranked fuzzy cost. Thus step 4 of fourth iteration is shown in Table 5.11:

(-130,0,130) **0**	(-4,0,4) **0**	(-99,3,104) **2.75**	(-127,6,138) **5.75**	(-81,1,82) **0.75**
(-30,2,35) **2.25**	(-86,14,115) **14.25**	(-3,0,3) **0**	(-35,0,35) **0**	(-46,4,54) **4**
(-8,0,8) **0**	(-123,4,135) **5**	(-24,6,36) **6**	(-10,0,10) **0**	(-58,3,64) **3**
(-126,4,133) **3.75**	(-2,0,2) **0**	(-99,3,101) **2**	(-128,2,131) **1.75**	(-81,2,81) **1**
(-65,0,65) **0**	(-66,0,68) **0.5**	(-29,8,43) **7.5**	(-56,7,73) **7.75**	(-16,0,16) **0**

Table 5.11

Now, third iteration is completed.

Step 8. In the fourth iteration, the resultant matrix after making assignments is shown in Table 5.12.

(-130,0,130) **0**	(-4,0,4) **0**	(-99,3,104) **2.75**	(-127,6,138) **5.75**	(-81,1,82) **0.75**
(-30,2,35) **2.25**	(-86,14,115) **14.25**	(-3,0,3) **0**	(-35,0,35) **0**	(-46,4,54) **4**
(-8,0,8) **0**	(-123,4,135) **5**	(-24,6,36) **6**	(-10,0,10) **0**	(-58,3,64) **3**
(-126,4,133) **3.75**	(-2,0,2) **0**	(-99,3,101) **2**	(-128,2,131) **1.75**	(-81,2,81) **1**
(-65,0,65) **0**	(-66,0,68) **0.5**	(-29,8,43) **7.5**	(-56,7,73) **7.75**	(-16,0,16) **0**

Table 5.12

In the Table 5.12, each row and each column contains exactly one boxed zero ranked fuzzy cost. Thus, this is the last iteration of a given fuzzy assignment problem. The Table 5.12 gives that job A should be assigned to 1st worker, job B should be assigned to 4th worker, job C should be assigned to 2nd worker, job D should be assigned to 3rd worker and job E should be assigned to 5th worker. Thus, the assignments are $x_{11} = 1, x_{23} = 1, x_{34} = 1, x_{42} = 1, x_{55} = 1$.

The optimal solution = $\sum_{i=1}^{5}\sum_{j=1}^{5} \tilde{c}_{ij}\, x_{ij}$

$$= (10,11,12) + (5,6,8) + (4,6,8) + (8,9,10) + (10,11,13).$$

$$= (37,43,51).$$

5.4 Results and discussion

The obtained results can be explained as follows:

1 Total minimum cost of assignment is greater than 37 and less than 51.

2 Maximum number of persons are in favour that total minimum assignment cost will be 43.

3 The membership function for the obtained result is shown in Figure 5.1.

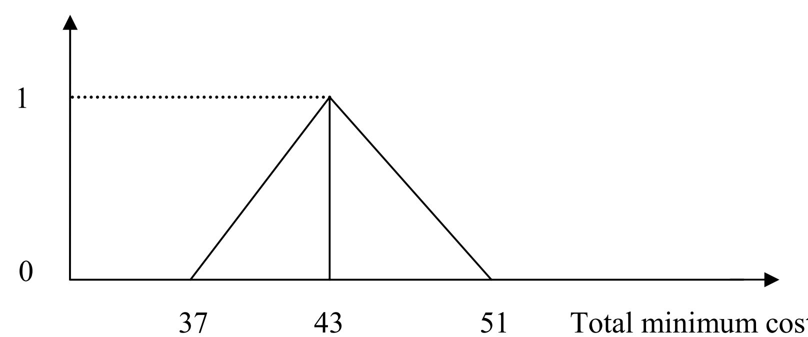

Figure 5.1

4 The percentage of persons increases when cost varies from 37 to 43 and decreases when cost varies from 43 to 51.

5.5 Conclusion

In this chapter, a new algorithm has been proposed to solve the fuzzy assignment problems occurring in real life situations. To illustrate the algorithm a numerical example has been solved in which approximate cost is represented by a triangular fuzzy number. If there is no uncertainty about the cost then the proposed algorithm gives the same result as in crisp assignment problem.

Chapter 6

FUZZY TRAVELLING SALESMAN PROBLEM

6.1 Introduction

One of the problems similar to that of fuzzy assignment problem is the fuzzy travelling salesman problem. This is an interesting problem in the field of operations research. In conventional travelling salesman problem, a travelling salesman wants to minimize the travelling cost (or travelling time or travelling distance) during his visit of n cities. In a fuzzy travelling salesman problem, a travelling salesman wants to minimize the approximate travelling cost[2]. Thus the problem is stated as follows:

A salesman has to visit n cities and return to the starting point. How should travelling salesman visit the cities such that the total approximate cost of travelling is minimum?

It is assumed that the starting city is included in the n cities to be visited. Since the person comes back to the starting point, any of the n cities can be a starting point. Therefore, for a given solution there are $n-1$ other solutions that are same. In this chapter, a new algorithm has been proposed to find the optimal solution for the fuzzy travelling salesman problem.

6.2 Mathematical programming formulation of the fuzzy travelling salesman problem

Mathematically, a fuzzy travelling salesman problem may be formulated as follows:

---

[2] In this chapter , approximate travelling salesman cost is represented by a fuzzy number which is named as fuzzy cost.

$$
\begin{aligned}
\text{Min} \quad & \tilde{z} \underset{\Re}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}\, x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, \ldots, n, \\
& \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, \ldots, n, \\
& x_{ij} + x_{ji} \le 1 \qquad \forall\ i, j, \\
& x_{ij} = 0 \text{ or } 1.
\end{aligned} \tag{6.1}
$$

The fuzzy cost matrix of travelling salesman problem of five cities may be represented as follows:

$\longrightarrow$ **Cities** $\downarrow$	**1**	**2**	**3**	**4**	**5**
1	$\tilde{\infty}$	$\tilde{c}_{12}$	$\tilde{c}_{13}$	$\tilde{c}_{14}$	$\tilde{c}_{15}$
2	$\tilde{c}_{21}$	$\tilde{\infty}$	$\tilde{c}_{23}$	$\tilde{c}_{24}$	$\tilde{c}_{25}$
3	$\tilde{c}_{31}$	$\tilde{c}_{22}$	$\tilde{\infty}$	$\tilde{c}_{34}$	$\tilde{c}_{35}$
4	$\tilde{c}_{41}$	$\tilde{c}_{42}$	$\tilde{c}_{43}$	$\tilde{\infty}$	$\tilde{c}_{45}$
5	$\tilde{c}_{51}$	$\tilde{c}_{52}$	$\tilde{c}_{53}$	$\tilde{c}_{54}$	$\tilde{\infty}$

6.3 Proposed algorithm of fuzzy travelling salesman problem

Step 1. Firstly solve the given problem by fuzzy assignment technique. If solution obtained contains a path which starts from given city and covers all the other cities exactly once and terminates again at starting city then, the optimal solution of travelling salesman problem is obtained. Otherwise go to step 2.

Step 2. After solving the given problem by fuzzy assignment technique, use the method of enumeration by assigning in a cell (say A) having minimum rank other than zero (or assigning in those non-zero ranked cells whose sum of ranks is minimum from that of the rank of A) of the matrix instead of assigning in cell having zero rank. Cut the column corresponding to this assignment. The remaining assignments can be made according to the same fuzzy assignment technique.

Step 3. Repeat step 2 until, the closed path is not obtained.

The proposed method has been explained through the example given below:

Example 6.1 There are five cities and a travelling salesman has to visit these five cities. In Table 6.1 the approximate travelling costs between different cities are given. Find a closed path, starting from city one and terminating at city one with the restriction that one city is travelled only once, i.e. solve the following travelling salesman problem to minimize the approximate travelling cost.

Cities $\rightarrow$ $\downarrow$	1	2	3	4	5
6	(∞,∞,∞)	(8,10,11)	(7,8,10)	(5,9,10)	(5,7,9)
7	(9,10,11)	(∞,∞,∞)	(8,10,11)	(3,5,8)	(5,6,7)
8	(7,8,10)	(9,10,11)	(∞,∞,∞)	(5,8,10)	(6,9,12)
9	(7,9,11)	(3,5,7)	(5,8,10)	(∞,∞,∞)	(4,6,8)
10	(5,7,8)	(8,9,10)	(8,9,10)	(4,6,9)	(∞,∞,∞)

Table 6.1

Solution: The optimal solution of the above fuzzy travelling salesman problem can be obtained by using the proposed algorithm as follows:

Step 1. Initially, the solution obtained by solving the given problem through fuzzy assignment technique discussed in previous chapter 5 is shown in Table 6.2.

(∞,∞,∞) Rank = ∞	(-8,4,15) Rank = **3.75**	$\boxed{(-7,0,7)}$ Rank = **0**	(-7,3,9) Rank = **2**	(-4,0,4) Rank = **0**
(-6,5,11) Rank = **3.75**	(∞,∞,∞) Rank = ∞	(-9,3,13) Rank = **2.5**	$\boxed{(-5,0,5)}$ Rank = **0**	(-7,0,7) Rank = **0**
$\boxed{(-8,0,8)}$ Rank = **0**	(-5,2,11) Rank = **2.5**	(∞,∞,∞) Rank = ∞	(-5,0,5) Rank = **0**	(-8,0,10) Rank = **0.5**
(-10,4,15) Rank = **3.25**	(-4,0,4) Rank = **0**	(-16,1,16) Rank = **0.5**	(∞,∞,∞) Rank = ∞	$\boxed{(-12,0,12)}$ Rank = **0**
(-13,1,11) Rank = **0**	$\boxed{(-4,0,4)}$ Rank = **0**	(-14,1,15) Rank = **0.75**	(-9,0,9) Rank = **0**	(∞,∞,∞) Rank = ∞

Table 6.2

Total minimum approximate cost for assignment solution = $\sum_{i=1}^{5}\sum_{j=1}^{5}\tilde{c}_{ij}\, x_{ij}$

$$=(7,8,10)+(3,5,8)+(7,8,10)+(4,6,8)+(4,6,8).$$

$$=(25,33,44).$$

It is clear from Table 6.2 that travelling salesman travels from city 1 to 3 and then return from city 3 to 1 i.e. travelling salesman is not visiting cities 2, 4 and 5. So, it is not an optimal solution of the fuzzy travelling salesman problem.

Step 2. To obtain the optimal solution, find the cell having minimum rank. It is cell c_{43} having rank $\Re(c_{43})=0.5$ and in this problem there does not exist cells for which sum of ranks is less than 0.5. So, make first assignment in the cell c_{43} and make other assignments according to the fuzzy assignment technique. The obtained values are shown in Table 6.3.

(∞,∞,∞) Rank = ∞	(-8,4,15) Rank = **3.75**	(-7,0,7) Rank = **0**	(-7,3,9) Rank = **2**	$\boxed{(-4,0,4)}$ Rank = **0**
(-6,5,11) Rank = **3.75**	(∞,∞,∞) Rank = ∞	(-9,3,13) Rank = **2.5**	$\boxed{(-5,0,5)}$ Rank = **0**	(-7,0,7) Rank = **0**
$\boxed{(-8,0,8)}$ Rank = **0**	(-5,2,11) Rank = **2.5**	(∞,∞,∞) Rank = ∞	(-5,0,5) Rank = **0**	(-8,0,10) Rank = **0.5**
(-10,4,15) Rank = **3.25**	(-4,0,4) Rank = **0**	$\boxed{(-16,1,16)}$ Rank = **0.5**	(∞,∞,∞) Rank = ∞	(-12,0,12) Rank = **0**
(-13,1,11) Rank = **0**	$\boxed{(-4,0,4)}$ Rank = **0**	(-14,1,15) Rank = **0.75**	(-9,0,9) Rank = **0**	(∞,∞,∞) Rank = ∞

Table 6.3

The path obtained from the Table 6.3 is $1\to5\to2\to4\to3\to1$ and it is a closed path which starts from city 1 and by covering all the cities exactly once terminates at the city 1. So, the solution obtained from Table 6.3 satisfies the optimality of the fuzzy travelling salesman problem. The last Table 6.4 is as follows:

(∞,∞,∞) Rank = ∞	(-8,4,15) Rank = **3.75**	(-7,0,7) Rank = **0**	(-7,3,9) Rank = **2**	$\boxed{(-4,0,4)}$ Rank = **0**
(-6,5,11) Rank = **3.75**	(∞,∞,∞) Rank = ∞	(-9,3,13) Rank = **2.5**	$\boxed{(-5,0,5)}$ Rank = **0**	(-7,0,7) Rank = **0**
$\boxed{(-8,0,8)}$ Rank = **0**	(-5,2,11) Rank = **2.5**	(∞,∞,∞) Rank = ∞	(-5,0,5) Rank = **0**	(-8,0,10) Rank = **0.5**
(-10,4,15) Rank = **3.25**	(-4,0,4) Rank = **0**	$\boxed{(-16,1,16)}$ Rank = **0.5**	(∞,∞,∞) Rank = ∞	(-12,0,12) Rank = **0**
(-13,1,11) Rank = **0**	$\boxed{(-4,0,4)}$ Rank = **0**	(-14,1,15) Rank = **0.75**	(-9,0,9) Rank = **0**	(∞,∞,∞) Rank = ∞

Table 6.4

Thus, the assignments are $x_{15}=1, x_{24}=1, x_{31}=1, x_{43}=1, x_{52}=1$.

Total minimum approximate cost for fuzzy travelling salesman problem

$$= \sum_{i=1}^{5}\sum_{j=1}^{5}\tilde{c}_{ij}\, x_{ij}$$

$$=(5,7,9)+(3,5,8)+(7,8,10)+(5,8,10)+(4,6,8)$$

$$=(24,34,45).$$

6.4 Results and discussion

The obtained can be explained as follows:

1 Total cost of travelling is greater than 24 and less than 45.

2 Maximum numbers of persons are in favour that cost will be 34.

3 The membership function for the obtained result is shown in Figure 6.1.

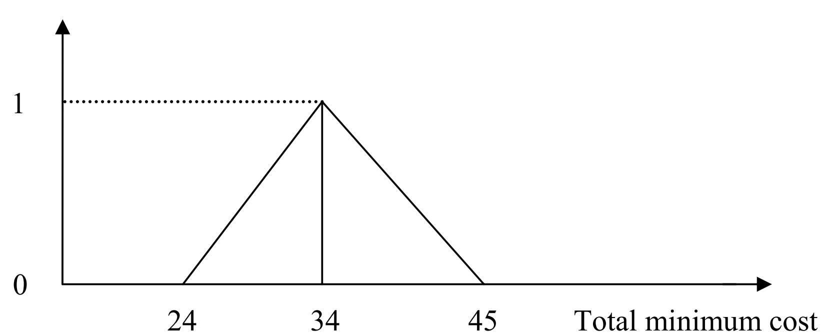

Figure 6.1

4 The percentage of persons increases when cost varies from 24 to 34 and decreases when cost varies from 34 to 45.

6.5 Conclusion

In this chapter, a new algorithm has been proposed to solve the fuzzy travelling salesman problems occurring in real life situation. To illustrate the algorithm a numerical example has been solved in which approximate cost is represented as a triangular fuzzy number. If there is no uncertainty about the cost i.e. if the cost is not fuzzy then the proposed algorithm gives the same result as in crisp travelling salesman problem.

Chapter 7

MULTIOBJECTIVE TRANSPORTATION PROBLEM

7.1 Introduction

In real life situations, the transportation problem usually involves multiple, incommensurable and conflicting objective functions. This kind of problem is called multiobjective transportation problem. Similar to a typical transportation problem, in a multiobjective transportation problem a product is to be transported from m sources to n destinations and their capacities are $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ respectively. In addition, there is a penalty c_{ij} associated with transporting a unit of product from i^{th} source to j^{th} destination. This penalty may be cost or delivery time or safety of delivery or etc. A variable x_{ij} represents the unknown quantity to be shipped from i^{th} source to j^{th} destination. A mathematical model of multiobjective transportation problem with r objectives, m sources and n destinations can be written as:

$$
\begin{aligned}
\text{Min} \quad & Z_r = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{r}\, x_{ij}, \qquad r = 1, 2, \ldots, k, \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m, \\
& \sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n, \\
& x_{ij} \geq 0, \qquad \text{for all } i, j.
\end{aligned}
\tag{7.1}
$$

The subscript on Z_r and superscript on c_{ij}^{r} are related to the r^{th} penalty criterion. Without loss of generality, it may be assumed that $a_i \geq 0$ and $b_j \geq 0 \quad \forall\, i, j$ and the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

The existing solution procedures of this problem can be divided into two categories. First, those that are generating all the sets of efficient solutions. The second category represents the procedures that are seeking the best compromise solution among the set of efficient solutions. From a practical point of view the knowledge of the set of efficient solutions is not always necessary. In such a case, a procedure is needed to determine a compromise solution. As a result, different approaches have been developed in the context of multiobjective transportation problem to find the compromise solution.

The various approaches [37] to solve MOLPP are:

- Lexicographic goal programming approach.
- Interval goal programming approach.
- Interactive procedure.
- Fuzzy programming approach.
- Fuzzy goal programming approach.

Each and every approach has its own drawbacks as well as its own strengths, and any rational choice as to which should be used is almost dependent on at least two vital considerations:

1. The type and size of the problem.
2. The characteristics of the ultimate decision maker(s).

The purpose of this chapter is to present a fuzzy programming approach to find an optimal compromise solution of a transportation problem with several objectives. Zangiabadi and Maleki [58] used a non-linear membership function to solve a multiobjective transportation problem. In this chapter, instead of using non-linear membership function, a linear membership function have been used to solve the same problem and it is shown that the obtained problem is linear and easy to solve as compare to the non-linear problem obtained by Zangiabadi and Maleki [58]. Also results of the proposed approach and the existing approach are compared.

7.2 Fuzzy goal programming approach for solving MOTP [58]

Fuzzy Goal Programming approach consists of two different approaches:

- Goal programming approach.
- Fuzzy programming approach.

7.2.1 Goal programming approach

Goal programming is widely used as a powerful tool for handling multiple criteria. Many times the multiple goals are in conflict and one can be achieved only at the expense of other. Therefore goals are arranged in order of importance and their contribution to organization's well-being. With such set-up, the problem can be solved by goal programming.

7.2.1.1 Why goal programming approach?

The goal programming model for multiple-objective linear programming has been used because:

1. The model development is relatively simple and straightforward.
2. Minor modifications may be employed so as to encompass the alternative approaches (e.g., fuzzy programming, nondominated or efficient solution methods, weighted objectives, etc.) to the multiple-objective linear programming problem.
3. The method of solution is quite simple and is, in fact, just a refinement to the two-phase simplex method.
4. The goal programming model, and variations thereof, have already found extensive implementation in actual problems since the early 1950s.
5. The model seems consistent with typical real-world problems.

7.2.1.2 Formulation of goal programming model [58]

The goal programming is to minimize distance between $Z=(Z_1, Z_2, \dots, Z_k)$ and targets or aspiration levels $\bar{Z}=(\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_k)$, which are determined by the decision maker. To do this, first positive and negative deviational variables introduced respectively as follows:

$$d_r^+ = \max(0, Z_r - \bar{Z}_r) = \frac{1}{2}\left\{(Z_r - \bar{Z}_r) + \left|Z_r - \bar{Z}_r\right|\right\}, \quad r=1,2,\dots,k,$$

and $d_r^- = \max(0, \bar{Z}_r - Z_r) = \frac{1}{2}\left\{(\bar{Z}_r - Z_r) + \left|\bar{Z}_r - Z_r\right|\right\}, \quad r=1,2,\dots,k$

Then, minimizing the distance between Z_r and $\bar{Z}_r$ leads to minimizing d_r^+ when $Z_r \le \bar{Z}_r$ is needed in a minimization problem. In this case, by using the min-max form of goal programming, the model (7.1) converts to the following linear programming model:

$$\begin{aligned}
\min \quad & \psi \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \qquad i=1,2,\dots,m, \\
& \sum_{i=1}^{m} x_{ij} = b_j, \qquad j=1,2,\dots,n, \\
& Z_r + d_r^- - d_r^+ = \bar{Z}_r, \quad r=1,2,\dots,k, \\
& \psi \ge d_r^+, \qquad r=1,2,\dots,k, \\
& d_r^+, d_r^- \ge 0, \quad d_r^+ d_r^- = 0, \quad r=1,2,\dots,k, \\
& x_{ij} \ge 0 \quad \text{for all } i, j.
\end{aligned} \tag{7.2}$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

7.2.2 Fuzzy programming approach [58]

Let L_r and U_r be the aspired level of achievements and the highest acceptable level of achievement for the r^{th} objective function, respectively. Based on the fuzzy programming approach, the following algorithm can be used to solve multiobjective transportation problem.

Step 1. Solve the multi-objective transportation problem as a single objective transportation problem, taking each time only one objective as objective function and ignoring all others.

Step 2. Compute the value of each objective function at each solution derived in Step1.

Step 3. From Step 2, find for each objective the best (L_r) and the worst (U_r) values corresponding to the set of solutions. By using these results as aspiration levels, the initial fuzzy model can be formulated as:

Find x_{ij} which satisfy:

$$\begin{aligned} & Z_r \lesssim L_r, \qquad r = 1, 2, \ldots, k, \\ & \sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m, \\ & \sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n, \\ & x_{ij} \geq 0, \qquad \text{for all } i, j. \end{aligned} \tag{7.3}$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied. The symbol "$\lesssim$" is the fuzzification of "$\leq$" and the notion $Z_r \lesssim L_r$ stands for "Z_r is substantially smaller than L_r".

Step 4. To solve (7.3), define a linear membership function $\mu_r(Z_r)$ corresponds to r^{th} objective function as:

$$\mu_r(Z_r)=\begin{cases}1 & ,Z_r \le L_r \\ 1-\dfrac{Z_r-L_r}{U_r-L_r} & ,L_r \le Z_r \le U_r \\ 0 & ,Z_r \ge U_r\end{cases} \tag{7.4}$$

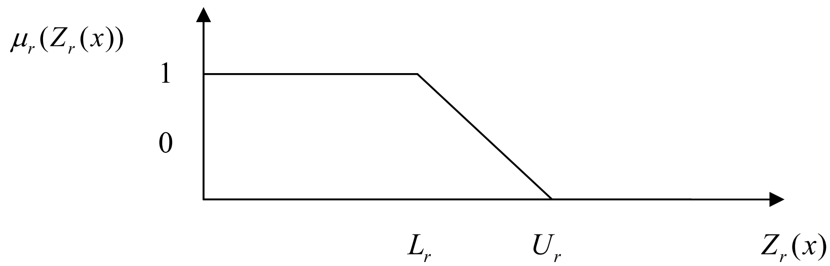

Figure 7.1 The membership function of Z_r

By using the min operator proposed by Zadeh with the membership function (7.4), model (7.3) converts to an equivalent crisp linear programming model as follows:

$$\begin{aligned}
&\text{Max} \quad \alpha, \\
&\text{s.t.} \quad \alpha \le 1-\frac{Z_r-L_r}{U_r-L_r}, \quad r=1,2,\ldots,k, \\
&\qquad \sum_{j=1}^{n} x_{ij}=a_i, \quad i=1,2,\ldots,m, \\
&\qquad \sum_{i=1}^{m} x_{ij}=b_j, \quad j=1,2,\ldots,n, \\
&\qquad \alpha \le 1, \\
&\qquad \alpha \ge 0,\ x_{ij} \ge 0 \ \text{ for all } i, j.
\end{aligned} \tag{7.5}$$

where the equilibrium condition $\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$ is satisfied.

7.2.3 Fuzzy goal programming approach

Several fuzzy approaches can be considered for solving multi-objective transportation problems. A fuzzy goal programming approach is one of these approaches which determine an optimal compromise solution for the multi-objective transportation problem.

In this, assume that each objective function has a fuzzy goal and also assign a special type of linear membership function to each objective function to describe each fuzzy goal. The approach focuses on minimizing the negative deviation variables from 1 to obtain a compromise solution of the multiobjective transportation problem. Note that the highest degree of the membership function is 1. Therefore by introducing the deviational variables $d_r^-, d_r^+ \geq 0$ corresponding to the r^{th} linear membership function defined in (7.4), the flexible membership goal with aspired level 1 can be presented as:

$$1 - \frac{Z_r - L_r}{U_r - L_r} + d_r^- - d_r^+ = 1,$$

where $d_r^- \, d_r^+ = 0$. Also any over-deviation from 1 indicates the full achievement of the membership value. Therefore to achieve the aspired levels of the fuzzy goal, it is enough to minimize its negative deviational variable from 1.

Applying the Min – Max form of goal programming to the fuzzy model of multiobjective transportation problem (7.3), with the linear membership function leads the following model:

$$\begin{aligned}
\text{min} \quad & \phi, \\
\text{s.t.} \quad & 1 - \frac{Z_r - L_r}{U_r - L_r} + d_r^- - d_r^+ = 1, \\
& \phi \geq d_r^-, \qquad r = 1, 2, \ldots, k, \\
& d_r^+ \, d_r^- = 0, \\
& \sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m, \\
& \sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n, \\
& \phi \leq 1, \phi \geq 0, \; x_{ij} \geq 0 \quad \text{for all } i, j.
\end{aligned} \tag{7.6}$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

Theorem 7.1 For the MOTP, model (7.5) is equivalent to model (7.6)

Proof. First of all, we rewrite the model (7.5) as:

$$\begin{aligned}
\text{Min} \quad & 1-\alpha, \\
\text{s.t.} \quad & 1-\alpha \le 1-\left\{1-\frac{Z_r-L_r}{U_r-L_r}\right\}, \quad r=1,2,\dots,k, \\
& \sum_{j=1}^{n} x_{ij}=a_i, \quad i=1,2,\dots,m, \\
& \sum_{i=1}^{m} x_{ij}=b_j, \quad j=1,2,\dots,n, \\
& \alpha \le 1, \\
& \alpha \ge 0,\ x_{ij} \ge 0 \ \text{ for all } i,j.
\end{aligned} \tag{7.7}$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied.

Take $\phi = 1-\alpha$, then we have:

$$\begin{aligned}
\text{min} \quad & \phi, \\
\text{s.t.} \quad & \phi \le 1-\left\{1-\frac{Z_r-L_r}{U_r-L_r}\right\}, \quad r=1,2,\dots,k, \\
& \sum_{j=1}^{n} x_{ij}=a_i, \quad i=1,2,\dots,m, \\
& \sum_{i=1}^{m} x_{ij}=b_j, \quad j=1,2,\dots,n, \\
& \phi \le 1, \\
& \phi \ge 0,\ x_{ij} \ge 0 \ \text{ for all } i,j.
\end{aligned} \tag{7.8}$$

Let us define the negative and positive deviational variables d_r^- and d_r^+ as:

$$d_r^- = \max\left\{0, 1-\left(1-\frac{Z_r-L_r}{U_r-L_r}\right)\right\} \quad \text{and} \quad d_r^+ = \max\left\{0, \left(1-\frac{Z_r-L_r}{U_r-L_r}\right)-1\right\}$$

Then from model (7.8), we have $\phi \ge d_r^-$, where

$$1-\frac{Z_r-L_r}{U_r-L_r} + d_r^- - d_r^+ = 1$$

Hence, the model (7.8) can be rewritten as:

$$
\begin{aligned}
\min \quad & \phi, \\
\text{s.t.} \quad & 1-\frac{Z_r - L_r}{U_r - L_r} + d_r^- - d_r^+ = 1, \\
& \sum_{j=1}^{n} x_{ij} = a_i \ , \qquad i = 1, 2, \ldots, m, \\
& \sum_{i=1}^{m} x_{ij} = b_j \ , \qquad j = 1, 2, \ldots, n, \\
& \phi \geq d_r^- \ , \qquad r = 1, 2, \ldots, k, \\
& d_r^+ \, d_r^- = 0, \\
& \phi \leq 1 \ , \\
& \phi \geq 0, \ x_{ij} \geq 0 \quad \text{for all } i, j.
\end{aligned}
\tag{7.9}
$$

where the equilibrium condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied. Model (7.9) is same as model (7.6) and the proof is completed.

7.2.3.1 The solution procedure for model (7.6)

To solve Multiobjective Transportation Problem based on the fuzzy goal programming technique, one can use the following steps:

Step 1. Solve the multiobjective transportation problem as a single objective transportation problem, taking each time only one objective as objective function and ignoring all others.

Step 2. Compute the value of each objective function at each solution derived in step 1.

Step 3. From step 2, find for each objective the best (L_r) and the worst (U_r) values corresponding to the set of solutions. L_r and U_r are the aspired level of achievement and the highest acceptable level of achievement for the r^{th} objective function, respectively.

Step 4. Build model (7.6) and solve it.

The solution obtained in step 4 will be the optimal compromise solution of multiobjective transportation problem model.

Example 7.1 To illustrate the efficiency of the proposed method, we consider the following numerical example presented by Verma et al. [53].

$$\begin{aligned}
\text{Min } Z_1 &= 16x_{11} + 19x_{12} + 12x_{13} + 22x_{21} + 13x_{22} + 19x_{23} + 14x_{31} \\
&\quad + 28x_{32} + 8x_{33}, \\
\text{Min } Z_1 &= 9x_{11} + 14x_{12} + 12x_{13} + 16x_{21} + 10x_{22} + 14x_{23} + 8x_{31} \\
&\quad + 20x_{32} + 6x_{33} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} = 14, \\
& x_{21} + x_{22} + x_{23} = 16, \\
& x_{31} + x_{32} + x_{33} = 12, \\
& x_{11} + x_{21} + x_{31} = 10, \\
& x_{12} + x_{22} + x_{32} = 15, \\
& x_{13} + x_{23} + x_{33} = 17, \; x_{ij} \geq 0, \; i = 1,2,3, \quad j = 1,2,3
\end{aligned} \tag{7.10}$$

Solution. In the following the proposed steps of the previous section are presented.

Step 1. The solution of each single objective transportation problem is:

$$X^1 = (x_{11}^1 = 9, x_{12}^1 = 0, x_{13}^1 = 5, x_{21}^1 = 1, x_{22}^1 = 15, x_{23}^1 = 0, x_{31}^1 = 0, x_{32}^1 = 0, x_{33}^1 = 12)'.$$

$$X^2 = (x_{11}^2 = 10, x_{12}^2 = 0, x_{13}^2 = 4, x_{21}^2 = 0, x_{22}^2 = 15, x_{23}^2 = 1, x_{31}^2 = 0, x_{32}^2 = 0, x_{33}^2 = 12)'.$$

Step 2. The objective function values are:

$$Z_1(X^1) = 517, Z_1(X^2) = 518, Z_2(X^1) = 379, Z_2(X^2) = 374.$$

Step 3. The upper and lower bounds of each objective function can be written as follows:

$$517 \leq Z_1 \leq 518 \text{ and } 374 \leq Z_2 \leq 379.$$

Hence $L_1 = 517$, $U_1 = 518$, $L_2 = 374$ and $U_2 = 379$.

The membership functions of the two objectives are shown in Figure 7.2 (a) and (b).

Figure 7.2 (a) The membership function of Z_1

and

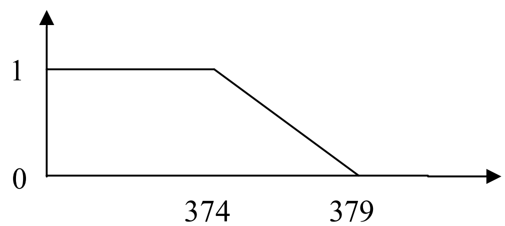

Figure 7.2 (b) The membership function of Z_2

Step 4. The model (7.6) is as follows:

$$
\begin{aligned}
\text{Min} \quad & \phi, \\
\text{s.t.} \quad & 1-\frac{Z_1-L_1}{U_1-L_1}+d_1^- - d_1^+ = 1, \\
& 1-\frac{Z_2-L_2}{U_2-L_2}+d_2^- - d_2^+ \\
& x_{11}+x_{12}+x_{13}=14, \\
& x_{21}+x_{22}+x_{23}=16, \\
& x_{31}+x_{32}+x_{33}=12, \\
& x_{11}+x_{21}+x_{31}=10, \\
& x_{12}+x_{22}+x_{32}=15, \\
& x_{13}+x_{23}+x_{33}=17, \\
& \phi \geq d_r^-, \qquad r=1,2, \\
& d_r^+ \, d_r^- = 0, \qquad r=1,2, \\
& \phi \leq 1, \phi \geq 0, \; x_{ij} \geq 0 \quad \text{for all } i, j
\end{aligned}
$$

where

$Z_1 = 16x_{11} + 19x_{12} + 12x_{13} + 22x_{21} + 13x_{22} + 19x_{23} + 14x_{31} + 28x_{32} + 8x_{33}$ and

$Z_2 = 9x_{11} + 14x_{12} + 12x_{13} + 16x_{21} + 10x_{22} + 14x_{23} + 8x_{31} + 20x_{32} + 6x_{33}$.

This is a linear programming problem which is solved by TORA and the results are:

$x_{11}^* = 9.5$, $x_{13}^* = 4.5$, $x_{21}^* = 0.5$, $x_{22}^* = 15$, $x_{23}^* = 0.5$, $x_{33}^* = 12$,
$d_1^- = 0.5$, $d_1^+ = 0$, $d_2^- = 0.5$, $d_2^+ = 0$, $\phi^* = 0.5$.
$z_1^* = 517.5$, $z_2^* = 376.5$,

and the other variables that are not in the above have a zero value .

Note that for the multiobjective transportation problem x^* is the same as compromise optimal solution obtained by Verma et al. [53].

7.2.4 Advantages of fuzzy goal programming approach

A major advantage of fuzzy goal programming is that it may be transformed into a conventional linear programming model. Another advantage is that the treatment of objectives (once they are transformed into goals via the aspiration level) is symmetric with the treatment of the constraints. Finally, the concept of an aspiration level subject to some allowable degradation seems consistent with actual decision making.

7.3 Results and discussion

The advantages of the proposed approach over existing approach is shown in Table 7.1.

Advantages	**Existing approach** [58]	**Proposed Approach**
Membership function	Non-linear	Linear
Converted problem	Non-linear	Linear
Solution procedure	Difficult	Easy
Optimal value	Same	Same

Table 7.1

7.4 Conclusion

In this chapter, the importance of the fuzzy programming approach, goal programming approach and their combination called the fuzzy goal programming approach to solve MOTP has been discussed. In the present work, the equivalence between fuzzy goal programming and fuzzy programming for solving MOTP is studied. A linear membership function is used and a linear optimization model is developed to solve MOTP.

Zangiabadi and Maleki [58] in there paper combine two approaches, fuzzy programming and goal programming, to develop a fuzzy goal programming approach with hyperbolic membership function for solving MOTP. They used non-linear membership function to solve MOTP. In this chapter, instead of using non-linear membership function (i.e. hyperbolic function) a linear membership function have been used to solve the same problem and it is shown that obtained problem is linear and easy to solve as compare to the non-linear problem obtained by Zangiabadi and Maleki [58]. It is also shown that the results obtained by the proposed approach are same as obtained by Zangiabadi and Maleki [58]. Thus, this chapter gives the comparison of existing approach and the proposed approach. The results of the proposed approach and existing approach are compared from different point of views and it is obvious from Table 7.1 that the proposed approach is better than an existing approach.

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