

**EFFICIENT METHODS TO FIND MAXIMAL AND MINIMAL
FUZZY SOLUTION OF SYSTEM OF FUZZY LINEAR
EQUATIONS AND ITS VARIANTS**

A Thesis

Submitted in fulfilment of the requirement
for the award of the Degree of

**DOCTOR OF PHILOSOPHY
IN
MATHEMATICS**

SUBMITTED BY

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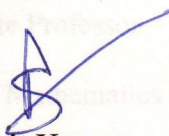
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August 2017**

CERTIFICATE

This is to certify that the thesis entitled, “Efficient Methods to find Maximal and Minimal Fuzzy Solution of System of Fuzzy Linear Equations and Its Variants” submitted by **Jeevan Jot Kaur** in the fulfilment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own and the ideas and references cited herein have been duly acknowledged.

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August 6, 2017

Abstract

In this thesis, limitations as well as flaws of all the existing methods to find maximal and minimal fuzzy solution of the system of fuzzy linear equations and its variants are pointed out. Furthermore, to overcome the limitations as well as to resolve the flaws of the existing methods, new methods (named as Mehar-Keerat methods) are proposed to find the maximal and minimal fuzzy solution of the system of fuzzy linear equations and its variants.

The thesis comprises six chapters. A brief outline of the chapters is as follows:

Chapter 1 Introduction

Chapter 1 is introductory in nature. In this chapter, a brief review of the work done in the literature to find maximal and minimal fuzzy solution of the system of fuzzy linear equations and its variants is presented.

Chapter 2 Mehar-Keerat method-I to find maximal and minimal fuzzy solution of system of fuzzy linear equations

Allahviranloo and Salahshour [26] pointed out that there are several methods to find the solution of system of fuzzy linear equations. However, there is no method in the literature to find maximal and minimal fuzzy solution of any type of the system of fuzzy linear equations. Furthermore, Allahviranloo and Salahshour [26] proposed a method to find maximal and minimal fuzzy solution of such system of fuzzy linear equations in which all the elements of the coefficient matrix are real numbers, the elements of the right hand side vector are triangular fuzzy numbers and the elements of the unknown vector are symmetric triangular fuzzy numbers. In this chapter, the limitations of this existing method are pointed

out. Also, to overcome these limitations, a new method (named as Mehar-Keerat method-I) is proposed. The proposed Mehar-Keerat method-I is illustrated with the help of numerical examples.

Chapter 3 Mehar-Keerat method-II to find maximal and minimal fuzzy solution of system of fully fuzzy linear equations

Allahviranloo et al. [29] pointed out that the existing method [26], cannot be used to find maximal and minimal fuzzy solution of the system of fully fuzzy linear equations. Furthermore, to overcome this limitation of the existing method [26], Allahviranloo et al. [29] extended the existing method [26], to find maximal and minimal fuzzy solution of such system of fully fuzzy linear equations in which all the elements of the unknown vector are non-negative symmetric triangular fuzzy numbers, whereas, all the elements of the coefficient matrix as well as all the elements of the right hand side vector are non-negative/non-positive triangular fuzzy numbers. In this chapter, limitations of the existing method [29] are pointed out. Also, to overcome the limitations of the existing method [29], a new method (named as Mehar-Keerat method-II) is proposed by extending the Mehar-Keerat method-I, proposed in Chapter 2. The proposed Mehar-Keerat method-II is illustrated with the help of numerical examples.

Chapter 4 Mehar-Keerat method-III to find maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations

Ghomashi et al. [75] extended the existing method [29], to find maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \tilde{x}_j + \tilde{d}_i, i=1, 2, \dots, m, \text{ in which } \tilde{x}_j \text{ are non-negative symmetric triangular}$$

fuzzy numbers and $\tilde{a}_{ij}, \tilde{c}_{ij}, \tilde{b}_i$ and $\tilde{d}_i$ are non-negative/non-positive triangular fuzzy numbers.

In this chapter, the flaws as well as limitations of the existing method [75] are pointed out. Also, to resolve the flaws as well as to overcome the limitations, a new method (named as Mehar-Keerat method-III) is proposed to find maximal and minimal fuzzy solution of such dual system of fully fuzzy linear equations in which all the parameters are trapezoidal fuzzy numbers. The proposed Mehar-Keerat method-III is illustrated with the help of numerical examples.

Chapter 5 Mehar-Keerat method-IV to find maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation

Dookhitram et al. [59] extended the existing method [29], to find maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$, where the elements of the matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are non-negative/non-positive triangular fuzzy numbers and the elements of unknown matrix $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers.

In this chapter, the limitations as well as flaws of the existing method [59] are pointed out. Also, to overcome the limitations as well as to resolve the flaws of the existing method [59], a new method (named as Mehar-Keerat method-IV) is proposed to find maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where the elements of $\tilde{A}, \tilde{B}, \tilde{X}$ and $\tilde{C}$ are trapezoidal fuzzy numbers. The proposed Mehar-Keerat method-IV is illustrated with the help of numerical examples.

Chapter 6 Future scope

There are few methods for solving the system of fuzzy complex linear equations. However, there is no method in the literature to find maximal and minimal fuzzy solution of any type of system of fuzzy complex linear equations. Also, after a deep study of the existing

methods for solving the system of fuzzy complex linear equations, it is noticed that in all the existing methods for solving the system of fuzzy complex linear equations, some mathematical incorrect assumptions are used. Therefore, it is not valid to use the existing methods. Furthermore, there exist several system of fuzzy complex linear equations and its variants which cannot be solved by any of the existing methods.

In this chapter, new methods (named as Mehar method, Keerat method and Mehar-Keerat method-V) are proposed to resolve the shortcomings as well as to overcome the limitations of the existing methods for solving the system of fuzzy complex linear equations. To propose methods to find maximal and minimal fuzzy solution of system of fuzzy complex linear equations and its variants may be considered as an open challenging research problem.

Appendix A

Gong et al. [77] proposed a method to solve the dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$ and claimed that their method is valid for all real matrices A and C of appropriate order. In this appendix, it is pointed out that this claim is not valid and the method, proposed by Gong et al. [77], is valid only if all the elements of A and C are non-negative real numbers. Also, it is pointed out that the dual fuzzy matrix equations, solved by Gong et al. [77] to illustrate their proposed method, have no significance as some of the elements of the matrices A and/or C are negative real numbers in these dual fuzzy matrix equations. Furthermore, it is pointed out that such a dual fuzzy matrix equation in which some/all the elements of A and/or C are negative real numbers can never exist and hence, an attempt to overcome this limitation of Gong et al. [77] method is meaningless.

2. List of Publications

2.1. List of papers published/communicated/to be communicated in Journals

1. Jeevan Jot Kaur, Amit Kumar: Mehar and Keerat method for solving system of fuzzy complex linear equations. Journal of Intelligent & Fuzzy Systems. (Published) **(SCI)** (Impact Factor: 1.812)
2. Jeevan Jot Kaur, Amit Kumar: A note on “Approximate solution of dual fuzzy matrix equations”. Information Sciences. (Published) **(SCI)** (Impact Factor: 4.832)
3. Jeevan Jot Kaur, Amit Kumar, Jagdeep Kaur: A note on “Solving fuzzy complex system of linear equations”. Information Sciences. (Communicated)
4. Jeevan Jot Kaur, Amit Kumar: Mehar fuzzy mathematical programming method to find maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation. Fuzzy Sets and Systems. (Communicated)
5. Jeevan Jot Kaur, Amit Kumar: Mehar method for solving fully fuzzy Sylvester matrix equation. SpringerPlus. (Communicated)
6. Jeevan Jot Kaur, Amit Kumar: A new method for solving system of fuzzy complex linear equations. IEEE Transactions on Fuzzy Systems. (Communicated)
7. Jeevan Jot Kaur, Amit Kumar: A new method to find maximal and minimal fuzzy solution of system of fuzzy linear equations. (Communicated)
8. Jeevan Jot Kaur, Amit Kumar: A new method to find maximal and minimal fuzzy solution of system of fully fuzzy linear equations. (Communicated)
9. Jeevan Jot Kaur, Amit Kumar: A new method to find maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations with unrestricted fuzzy parameters. (Communicated)

10. Jeevan Jot Kaur, Amit Kumar: Membership function of triangular and trapezoidal complex fuzzy numbers. (Communicated)
11. Jeevan Jot Kaur, Amit Kumar: A note on “Solutions of fuzzy LR algebraic linear systems using linear programs”. Applied Mathematical Modelling. (Communicated)

2.2. Papers presented in International Conferences

1. Jeevan Jot Kaur: A numerical method for solving fuzzy complex system of linear equations with fuzzy complex coefficients. International Conference on Modelling, Simulation and Optimizing Techniques (ICMSOT-2015) held on February 12-14, 2015, organized by PG Department of Mathematics, D.A.V. College, Jalandhar, Punjab.
2. Jeevan Jot Kaur: A numerical method for solving fuzzy complex system of linear equations with fuzzy complex variables. International Conference on Advances in Mathematical Sciences-2015 (ICAMS-2015) held on March 19-21, 2015, organized by Khalsa College, Patiala in collaboration with International Multidisciplinary Research Foundation.
3. Jeevan Jot Kaur: Graphical representation of triangular and trapezoidal complex fuzzy numbers. The Fifth International Conference on Mathematical Sciences (ICM-2016) held on March 21-24, 2016, organized by Department of Mathematical Sciences, United Arab Emirates University, Al-Ain, United Arab Emirates.

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Chapter 1

Introduction

The solution of a real world problem is obtained by solving its mathematical model. There exist several real world problems whose mathematical model represents the system of linear equations (1.1).

$$\sum_{j=1}^n a_{ij}x_j = b_i, i = 1, 2, \dots, m. \quad (1.1)$$

To find the solution of (1.1) and hence to find the solution of corresponding real world problem, the values of the parameters a_{ij} and b_i should be known. However, in many situations, only some vague knowledge about the parameters a_{ij} and/or b_i are available. One way, followed by several researchers, to deal with such situation is to represent the parameters a_{ij} and/or b_i and/or x_j as fuzzy numbers [165]. Since, for a real number a , the property $a - a = 0$ holds, therefore, the dual system of linear equations (1.2) can be transformed into the system of linear equations (1.3).

$$\sum_{j=1}^n a_{ij}x_j + b_i = \sum_{j=1}^n c_{ij}x_j + d_i, i = 1, 2, \dots, m. \quad (1.2)$$

$$\sum_{j=1}^n (a_{ij} - c_{ij})x_j = d_i - b_i, i = 1, 2, \dots, m. \quad (1.3)$$

as well as the dual system of matrix equations (1.4) can be transformed into the system of matrix equations (1.5).

$$\sum_{j=1}^n a_{ij}x_{jk} + b_{ik} = \sum_{j=1}^n c_{ij}x_{jk} + d_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.4)$$

$$\sum_{j=1}^n (a_{ij} - c_{ij})x_{jk} = d_{ik} - b_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.5)$$

However, for a fuzzy number $\tilde{A}$, the property $\tilde{A} - \tilde{A} = \tilde{0}$ is not satisfied. Therefore, the dual system of fuzzy linear equations (1.6), the dual system of fuzzy linear equations (1.7) and the dual system of fully fuzzy linear equations (1.8) (dual system of linear equations in which all the parameters are represented by fuzzy numbers) cannot be transformed into the system of fuzzy linear equations (1.9), system of fuzzy linear equations (1.10) and system of fully fuzzy linear equations (1.11) (system of linear equations in which all the parameters are represented by fuzzy numbers) respectively and different methods are proposed in the literature [7,27,33,77,97,106,113,117,127,142,144,163] for solving the dual system of fuzzy linear equations (1.6), the dual system of fuzzy linear equations (1.7) and the dual system of fully fuzzy linear equations (1.8).

$$\sum_{j=1}^n \tilde{a}_{ij} x_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} x_j + \tilde{d}_i, i = 1, 2, \dots, m. \quad (1.6)$$

$$\sum_{j=1}^n a_{ij} \tilde{x}_j + \tilde{b}_i = \sum_{j=1}^n c_{ij} \tilde{x}_j + \tilde{d}_i, i = 1, 2, \dots, m. \quad (1.7)$$

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \tilde{x}_j + \tilde{d}_i, i = 1, 2, \dots, m. \quad (1.8)$$

$$\sum_{j=1}^n (\tilde{a}_{ij} - \tilde{c}_{ij}) x_j = \tilde{d}_i - \tilde{b}_i, i = 1, 2, \dots, m. \quad (1.9)$$

$$\sum_{j=1}^n (a_{ij} - c_{ij}) \tilde{x}_j = \tilde{d}_i - \tilde{b}_i, i = 1, 2, \dots, m. \quad (1.10)$$

$$\sum_{j=1}^n (\tilde{a}_{ij} - \tilde{c}_{ij}) \tilde{x}_j = \tilde{d}_i - \tilde{b}_i, i = 1, 2, \dots, m. \quad (1.11)$$

Similarly, the dual system of fuzzy matrix equations (1.12), the dual system of fuzzy matrix equations (1.13) and the dual system of fully fuzzy matrix equations (1.14) (dual system of matrix equations in which all the parameters are represented by fuzzy numbers) cannot be transformed into the system of fuzzy matrix equations (1.15), the system of fuzzy matrix equations (1.16) and the system of fully fuzzy matrix equations (1.17) (system of

matrix equations in which all the parameters are represented by fuzzy numbers) respectively. However, as the dual system of fuzzy matrix equations (1.12), the dual system of fuzzy matrix equations (1.13) and the dual system of fully fuzzy matrix equations (1.14) are a collection of p independent dual system of fuzzy linear equations (1.6), the dual system of fuzzy linear equations (1.7) and the dual system of fully fuzzy linear equations (1.8) respectively. Therefore, the solution of these dual system of fuzzy matrix equations can be easily obtained by using the method to find the solution of the corresponding dual system of fuzzy/fully fuzzy linear equations.

$$\sum_{j=1}^n \tilde{a}_{ij} x_{jk} + \tilde{b}_{ik} = \sum_{j=1}^n \tilde{c}_{ij} x_{jk} + \tilde{d}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.12)$$

$$\sum_{j=1}^n a_{ij} \tilde{x}_{jk} + \tilde{b}_{ik} = \sum_{j=1}^n c_{ij} \tilde{x}_{jk} + \tilde{d}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.13)$$

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_{jk} + \tilde{b}_{ik} = \sum_{j=1}^n \tilde{c}_{ij} \tilde{x}_{jk} + \tilde{d}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.14)$$

$$\sum_{j=1}^n (\tilde{a}_{ij} - \tilde{c}_{ij}) x_{jk} = \tilde{d}_{ik} - \tilde{b}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.15)$$

$$\sum_{j=1}^n (a_{ij} - c_{ij}) \tilde{x}_{jk} = \tilde{d}_{ik} - \tilde{b}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.16)$$

$$\sum_{j=1}^n (\tilde{a}_{ij} - \tilde{c}_{ij}) \tilde{x}_{jk} = \tilde{d}_{ik} - \tilde{b}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p. \quad (1.17)$$

Allahviranloo and Salahshour [26] pointed out that there are several methods in the literature to find the solution of different types of system of fuzzy linear equations and its variants (dual system of fuzzy linear equations, fuzzy Sylvester equation, fuzzy matrix equations, fuzzy Sylvester matrix equation etc.). However, there is no method in the literature, to find maximal and minimal fuzzy solution of any type of system of fuzzy linear equations. Furthermore, Allahviranloo and Salahshour [26] proposed a method to find

maximal and minimal fuzzy solution of such system of fuzzy linear equations in which all the elements of unknown vector are symmetric triangular fuzzy numbers, all the elements of right hand side vector are triangular fuzzy numbers and all the elements of coefficient matrix are real numbers.

Allahviranloo et al. [29] pointed out that the existing method [26] cannot be used to find maximal and minimal fuzzy solution of such system of fuzzy linear equations in which some or all the elements of coefficient matrix are also triangular fuzzy numbers. To overcome this limitation of the existing method [26], Allahviranloo et al. [29] proposed a method, with the help of existing method [26], to find maximal and minimal fuzzy solution of such system of fully fuzzy linear equations in which all the elements of unknown vector are non-negative symmetric triangular fuzzy numbers and all the elements of coefficient matrix as well as right hand side vector are non-negative/non-positive triangular fuzzy numbers.

Ghomashi et al. [75] pointed out that none of the existing methods [26,29] can be used to find maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations. To overcome the limitations of the existing methods [26,29], Ghomashi et al. [75], proposed a method, with the help of existing methods [26,29], to find maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations $\tilde{A}\tilde{X} + \tilde{b} = \tilde{C}\tilde{X} + \tilde{d}$, in which all the elements of unknown vector $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers, whereas, all the elements of $\tilde{A}, \tilde{C}, \tilde{b}$ and $\tilde{d}$ are non-negative/non-positive triangular fuzzy numbers.

Dookhitram et al. [59] pointed out that the fuzzy Sylvester matrix equation has numerous applications in control theory, model reduction of linear time invariant systems, image restoration and block-diagonalization of matrices. However, none of the existing

methods [26,29,75] can be used to find maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation. To overcome the limitation of the existing methods [26,29,75], Dookhitram et al. [59] proposed a method, with the help of existing methods [26,29,75], to find maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$, in which all the elements of the unknown matrix $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers, whereas, all the elements of the fuzzy matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are non-negative/non-positive triangular fuzzy numbers.

To the best of my knowledge, only the existing methods [26,29,59,75] have been proposed in the literature [1-9,11-25,27,28,30,31,33-36,40-43,46-50,52-56,60,64,66-68,70-74,77-80,85-89,97-109,111-135,137,139-148,151,154-159,163,164,166] to find the maximal and minimal fuzzy solution of system of fuzzy linear equations and its variants (dual system of fuzzy linear equations, fuzzy Sylvester equation, fuzzy matrix equations, fuzzy Sylvester matrix equation etc.). In this thesis, limitations as well as flaws of all these existing methods [26,29,59,75] are pointed out. Furthermore, to overcome the limitations and to resolve the flaws of the existing methods [26,29,59,75], new methods (named as Mehar-Keerat methods) are proposed.

The chapter wise summary of the thesis is as follows.

Chapter 1 Introduction

Chapter 1 is introductory in nature. In this chapter, a brief review of the work done in the literature to find maximal and minimal fuzzy solution of the system of fuzzy linear equations and its variants is presented.

Chapter 2 Mehar-Keerat method-I to find maximal and minimal fuzzy solution of system of fuzzy linear equations

Allahviranloo and Salahshour [26] pointed out that there are several methods to find the solution of system of fuzzy linear equations. However, there is no method in the literature to find maximal and minimal fuzzy solution of any type of the system of fuzzy linear equations. Furthermore, Allahviranloo and Salahshour [26] proposed a method to find maximal and minimal fuzzy solution of such system of fuzzy linear equations in which all the elements of the coefficient matrix are real numbers, the elements of the right hand side vector are triangular fuzzy numbers and the elements of the unknown vector are symmetric triangular fuzzy numbers. In this chapter, the limitations of this existing method are pointed out. Also, to overcome these limitations, a new method (named as Mehar-Keerat method-I) is proposed. The proposed Mehar-Keerat method-I is illustrated with the help of numerical examples.

Chapter 3 Mehar-Keerat method-II to find maximal and minimal fuzzy solution of system of fully fuzzy linear equations

Allahviranloo et al. [29] pointed out that the existing method [26], cannot be used to find maximal and minimal fuzzy solution of the system of fully fuzzy linear equations. Furthermore, to overcome this limitation of the existing method [26], Allahviranloo et al. [29] extended the existing method [26], to find maximal and minimal fuzzy solution of such system of fully fuzzy linear equations in which all the elements of the unknown vector are non-negative symmetric triangular fuzzy numbers, whereas, all the elements of the coefficient matrix as well as all the elements of the right hand side vector are non-negative/non-positive triangular fuzzy numbers. In this chapter, limitations of the existing method [29] are pointed out. Also, to overcome the limitations of the existing method [29], a

new method (named as Mehar-Keerat method-II) is proposed by extending the Mehar-Keerat method-I, proposed in Chapter 2. The proposed Mehar-Keerat method-II is illustrated with the help of numerical examples.

Chapter 4 Mehar-Keerat method-III to find maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations

Ghomashi et al. [75] extended the existing method [29], to find maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \tilde{x}_j + \tilde{d}_i, i=1,2,\dots,m,$$
 in which $\tilde{x}_j$ are non-negative symmetric triangular fuzzy numbers and $\tilde{a}_{ij}, \tilde{c}_{ij}, \tilde{b}_i$ and $\tilde{d}_i$ are non-negative/non-positive triangular fuzzy numbers.

In this chapter, the flaws as well as limitations of the existing method [75] are pointed out. Also, to resolve the flaws as well as to overcome the limitations, a new method (named as Mehar-Keerat method-III) is proposed to find maximal and minimal fuzzy solution of such dual system of fully fuzzy linear equations in which all the parameters are trapezoidal fuzzy numbers. The proposed Mehar-Keerat method-III is illustrated with the help of numerical examples.

Chapter 5 Mehar-Keerat method-IV to find maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation

Dookhitram et al. [59] extended the existing method [29], to find maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$, where the elements of the matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are non-negative/non-positive triangular fuzzy

numbers and the elements of unknown matrix $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers.

In this chapter, the limitations as well as flaws of the existing method [59] are pointed out. Also, to overcome the limitations as well as to resolve the flaws of the existing method [59], a new method (named as Mehar-Keerat method-IV) is proposed to find maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where the elements of $\tilde{A}, \tilde{B}, \tilde{X}$ and $\tilde{C}$ are trapezoidal fuzzy numbers. The proposed Mehar-Keerat method-IV is illustrated with the help of numerical examples.

Chapter 6 Future scope

There are few methods for solving the system of fuzzy complex linear equations. However, there is no method in the literature to find maximal and minimal fuzzy solution of any type of system of fuzzy complex linear equations. Also, after a deep study of the existing methods for solving the system of fuzzy complex linear equations, it is noticed that in all the existing methods for solving the system of fuzzy complex linear equations, some mathematical incorrect assumptions are used. Therefore, it is not valid to use the existing methods. Furthermore, there exist several system of fuzzy complex linear equations and its variants which cannot be solved by any of the existing methods.

In this chapter, new methods (named as Mehar method, Keerat method and Mehar-Keerat method-V) are proposed to resolve the shortcomings as well as to overcome the limitations of the existing methods for solving the system of fuzzy complex linear equations. To propose methods to find maximal and minimal fuzzy solution of system of fuzzy complex linear equations and its variants may be considered as an open challenging research problem.

Appendix A

Gong et al. [77] proposed a method to solve the dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$ and claimed that their method is valid for all real matrices A and C of appropriate order. In this appendix, it is pointed out that this claim is not valid and the method, proposed by Gong et al. [77], is valid only if all the elements of A and C are non-negative real numbers. Also, it is pointed out that the dual fuzzy matrix equations, solved by Gong et al. [77] to illustrate their proposed method, have no significance as some of the elements of the matrices A and/or C are negative real numbers in these dual fuzzy matrix equations. Furthermore, it is pointed out that such a dual fuzzy matrix equation in which some/all the elements of A and/or C are negative real numbers can never exist and hence, an attempt to overcome this limitation of Gong et al. [77] method is meaningless.

Chapter 2

Mehar-Keerat method-I to find maximal and minimal fuzzy solution of system of fuzzy linear equations*

Allahviranloo and Salahshour [26] pointed out that there are several methods to find the solution of system of fuzzy linear equations. However, there is no method in the literature to find maximal and minimal fuzzy solution of any type of the system of fuzzy linear equations. Furthermore, Allahviranloo and Salahshour [26] proposed a method to find maximal and minimal fuzzy solution of such system of fuzzy linear equations in which all the elements of the coefficient matrix are real numbers, the elements of the right hand side vector are triangular fuzzy numbers and the elements of the unknown vector are symmetric triangular fuzzy numbers. In this chapter, the limitations of this existing method [26] are pointed out. Also, to overcome these limitations, a new method (named as Mehar-Keerat method-I) is proposed. The proposed Mehar-Keerat method-I is illustrated with the help of numerical examples.

2.1. Preliminaries

In this section, some basic definitions as well as arithmetic operations of triangular and trapezoidal fuzzy numbers are presented [92].

2.1.1. Basic definitions

In this section, some basic definitions are presented.

* The contents of this chapter are communicated for possible publication in Journal of Computational and Applied Mathematics.

Definition 2.1. Let X be a classical set of objects. Then, the set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \rightarrow [0,1]$ is called a fuzzy set in X . The function $\mu_{\tilde{A}}$ is called the membership function.

Definition 2.2. Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in (0,1]$. The α -level set of the fuzzy set $\tilde{A}$ is the crisp set, denoted as $\tilde{A}_\alpha$ and is defined as $\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 2.3. The support of a fuzzy set $\tilde{A}$, represented by $Supp(\tilde{A})$, within a universal set X is a crisp set that contains all the elements of X that have a non-zero membership value in $\tilde{A}$ i.e., $Supp(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) > 0\}$.

Definition 2.4. The height of a fuzzy set $\tilde{A}$, represented by $h(\tilde{A})$, is the largest membership grade obtained by any element in that set i.e., $h(\tilde{A}) = \sup \{\mu_{\tilde{A}}(x) \mid x \in X\}$.

Definition 2.5. A fuzzy set $\tilde{A}$ is called normal when $h(\tilde{A}) = 1$.

Definition 2.6. A fuzzy set $\tilde{A}$, defined on universal set of real numbers R , is said to be a fuzzy number if it has the following characteristics:

- (i) $\tilde{A}$ is convex, i.e., $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \forall x_1, x_2 \in R,$
 $\forall \lambda \in [0,1]$.
- (ii) $\tilde{A}$ is normal, i.e., $\exists x_0 \in R$ such that $\mu_{\tilde{A}}(x_0) = 1$.
- (iii) $\mu_{\tilde{A}}$ is piecewise continuous.

Definition 2.7. A fuzzy number $\tilde{A}$ is said to be non-negative fuzzy number if and only if $\mu_{\tilde{A}}(x) = 0$, for all $x < 0$ and is said to be non-positive fuzzy number if and only if $\mu_{\tilde{A}}(x) = 0$, for all $x > 0$.

Definition 2.8. A fuzzy number $\tilde{A} = (a - \sigma, a, a + \sigma')$ is said to be a triangular fuzzy number if its membership function $\mu_{\tilde{A}}$ is given by,

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - (a - \sigma)}{\sigma}, & a - \sigma \leq x < a \\ 1, & x = a \\ \frac{(a + \sigma') - x}{\sigma'}, & a < x \leq a + \sigma' \\ 0, & \text{otherwise} \end{cases}$$

A triangular fuzzy number $\tilde{A} = (a - \sigma, a, a + \sigma')$ is said to be symmetric triangular fuzzy number if $\sigma = \sigma'$ and is said to be non-symmetric triangular fuzzy number if $\sigma \neq \sigma'$.

Definition 2.9. Let $\tilde{A} = (a - \sigma, a, a + \sigma')$ be a triangular fuzzy number. Then, its r -cut $\tilde{A}_r$ is defined as follows:

$$\tilde{A}_r = [(a - \sigma) + r\sigma, (a + \sigma') - r\sigma'], 0 \leq r \leq 1.$$

Definition 2.10. A triangular fuzzy number $\tilde{A} = (a - \sigma, a, a + \sigma')$ is said to be non-negative (non-positive) triangular fuzzy number i.e., $\tilde{A} \geq 0$ ($\tilde{A} \leq 0$) if and only if $a - \sigma \geq 0$ ($a + \sigma' \leq 0$).

Definition 2.11. A triangular fuzzy number $\tilde{A} = (a - \sigma, a, a + \sigma')$ is said to be unrestricted triangular fuzzy number if $a - \sigma$ is an unrestricted real number.

Definition 2.12. A fuzzy number $\tilde{A} = (a^L - \sigma, a^L, a^U, a^U + \sigma')$ is said to be a trapezoidal fuzzy number if its membership function $\mu_{\tilde{A}}$ is given by,

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - (a^L - \sigma)}{\sigma}, & a^L - \sigma \leq x < a^L \\ 1, & a^L \leq x \leq a^U \\ \frac{a^U + \sigma' - x}{\sigma'}, & a^U < x \leq a^U + \sigma' \\ 0, & \text{otherwise} \end{cases}$$

A trapezoidal fuzzy number $\tilde{A} = (a^L - \sigma, a^L, a^U, a^U + \sigma')$ is said to be symmetric trapezoidal fuzzy number if $\sigma = \sigma'$ and is said to be non-symmetric trapezoidal fuzzy number if $\sigma \neq \sigma'$.

Definition 2.13. Let $\tilde{A} = (a^L - \sigma, a^L, a^U, a^U + \sigma')$ be a trapezoidal fuzzy number. Then, its r -cut $\tilde{A}_r$ is defined as follows:

$$\tilde{A}_r = \left[(a^L - \sigma) + r\sigma, (a^U + \sigma') - r\sigma' \right], 0 \leq r \leq 1.$$

Definition 2.14. A trapezoidal fuzzy number $\tilde{A} = (a^L - \sigma, a^L, a^U, a^U + \sigma')$ is said to be non-negative (non-positive) trapezoidal fuzzy number i.e., $\tilde{A} \geq 0$ ($\tilde{A} \leq 0$) if and only if $a^L - \sigma \geq 0$ ($a^U + \sigma' \leq 0$).

Definition 2.15. A trapezoidal fuzzy number $\tilde{A} = (a^L - \sigma, a^L, a^U, a^U + \sigma')$ is said to be unrestricted trapezoidal fuzzy number if $a^L - \sigma$ is an unrestricted real number.

2.1.2. Arithmetic operations

In this section, arithmetic operations of triangular and trapezoidal fuzzy numbers are presented.

2.1.2.1. Arithmetic operations of triangular fuzzy numbers

In this section, arithmetic operations of triangular fuzzy numbers are presented.

Let $\tilde{A}_1 = (a_1 - \alpha_1, a_1, a_1 + \alpha'_1)$ and $\tilde{A}_2 = (a_2 - \alpha_2, a_2, a_2 + \alpha'_2)$ be two triangular fuzzy numbers.

Then,

$$(i) \quad \tilde{A}_1 + \tilde{A}_2 = (a_1 - \alpha_1 + a_2 - \alpha_2, a_1 + a_2, a_1 + \alpha'_1 + a_2 + \alpha'_2).$$

$$(ii) \quad \tilde{A}_1 - \tilde{A}_2 = (a_1 - \alpha_1 - a_2 - \alpha'_2, a_1 - a_2, a_1 + \alpha'_1 - a_2 + \alpha_2).$$

$$(iii) \quad \lambda \tilde{A}_1 = \begin{cases} (\lambda(a_1 - \alpha_1), \lambda a_1, \lambda(a_1 + \alpha'_1)), & \lambda \geq 0 \\ (\lambda(a_1 + \alpha'_1), \lambda a_1, \lambda(a_1 - \alpha_1)), & \lambda < 0. \end{cases}$$

2.1.2.2. Arithmetic operations of trapezoidal fuzzy numbers

In this section, arithmetic operations of trapezoidal fuzzy numbers are presented.

Let $\tilde{A}_1 = (a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1)$ and $\tilde{A}_2 = (a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2)$ be two trapezoidal fuzzy numbers. Then,

$$(i) \quad \tilde{A}_1 + \tilde{A}_2 = (a_1^L - \alpha_1 + a_2^L - \alpha_2, a_1^L + a_2^L, a_1^U + a_2^U, a_1^U + \alpha'_1 + a_2^U + \alpha'_2).$$

$$(ii) \quad \tilde{A}_1 - \tilde{A}_2 = (a_1^L - \alpha_1 - a_2^U - \alpha'_2, a_1^L - a_2^U, a_1^U - a_2^L, a_1^U + \alpha'_1 - a_2^L + \alpha_2).$$

$$(iii) \quad \lambda \tilde{A}_1 = \begin{cases} (\lambda(a_1^L - \alpha_1), \lambda a_1^L, \lambda a_1^U, \lambda(a_1^U + \alpha'_1)), & \lambda \geq 0 \\ (\lambda(a_1^U + \alpha'_1), \lambda a_1^U, \lambda a_1^L, \lambda(a_1^L - \alpha_1)), & \lambda < 0. \end{cases}$$

2.2. Maximal and minimal fuzzy solution of system of fuzzy linear equations

Allahviranloo and Salahshour [26] pointed out that there are several methods to find the solution of system of fuzzy linear equations. However, there is no method in the literature [26] to find the maximal and minimal fuzzy solution of any type of system of fuzzy linear equations. Furthermore, Allahviranloo and Salahshour [26] proposed a method to find the maximal and minimal fuzzy solution of the system of fuzzy linear equations (2.1).

$$\begin{aligned} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \dots + a_{1n}\tilde{x}_n &= \tilde{b}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \dots + a_{2n}\tilde{x}_n &= \tilde{b}_2, \\ &\vdots \\ a_{m1}\tilde{x}_1 + a_{m2}\tilde{x}_2 + \dots + a_{mn}\tilde{x}_n &= \tilde{b}_m, \end{aligned}$$

$$\text{i.e., } A\tilde{X} = \tilde{b} \text{ or } \sum_{j=1}^n a_{ij}\tilde{x}_j = \tilde{b}_i \text{ for } i=1,2,\dots,m, \quad (2.1)$$

where a_{ij} is a real number, $\tilde{x}_j$ is a symmetric triangular fuzzy number and $\tilde{b}_i$ is a triangular fuzzy number.

In this section, the existing definitions [26] of maximal and minimal fuzzy solution of the system of fuzzy linear equations (2.1) are presented in the modified form by considering the assumption that $\tilde{x}_j$ is a non-symmetric triangular/trapezoidal fuzzy number.

Definition 2.16. The set $\{\tilde{X} = (x_j - \sigma_j, x_j, x_j + \sigma'_j), j = 1, 2, \dots, n : A\tilde{X} \subseteq \tilde{b}\}$ is called the maximal fuzzy solution of the system of fuzzy linear equations (2.1), if for any arbitrary set $\{\tilde{Z} = (z_j - \alpha_j, z_j, z_j + \alpha'_j), j = 1, 2, \dots, n : A\tilde{Z} \subseteq \tilde{b}\}$, we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_j = x_j, \alpha_j \leq \sigma_j, \alpha'_j \leq \sigma'_j$.

Definition 2.17. The set $\{\tilde{X} = (x_j - \sigma_j, x_j, x_j + \sigma'_j), j = 1, 2, \dots, n : A\tilde{X} \supseteq \tilde{b}\}$ is called the minimal fuzzy solution of the system of fuzzy linear equations (2.1), if for any arbitrary set $\{\tilde{Z} = (z_j - \alpha_j, z_j, z_j + \alpha'_j), j = 1, 2, \dots, n : A\tilde{Z} \supseteq \tilde{b}\}$, we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_j = x_j, \alpha_j \geq \sigma_j, \alpha'_j \geq \sigma'_j$.

Definition 2.18. If in the system of fuzzy linear equations (2.1), the elements $\tilde{x}_j$ of the unknown vector $\tilde{X}$ are non-symmetric trapezoidal fuzzy numbers, then the set $\{\tilde{X} = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j), j = 1, 2, \dots, n : A\tilde{X} \subseteq \tilde{b}\}$ is called the maximal fuzzy solution of the system of fuzzy linear equations (2.1), if for any arbitrary set $\{\tilde{Z} = (z_j^L - \alpha_j, z_j^L, z_j^U, z_j^U + \alpha'_j), j = 1, 2, \dots, n : A\tilde{Z} \subseteq \tilde{b}\}$, we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_j^L = x_j^L, z_j^U = x_j^U, \alpha_j \leq \sigma_j, \alpha'_j \leq \sigma'_j$.

Definition 2.19. If in the system of fuzzy linear equations (2.1), the elements $\tilde{x}_j$ of the unknown vector $\tilde{X}$ are non-symmetric trapezoidal fuzzy numbers, then the set $\{\tilde{X} = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j), j = 1, 2, \dots, n : A\tilde{X} \supseteq \tilde{b}\}$ is called the minimal fuzzy solution of the system of fuzzy linear equations (2.1), if for any arbitrary set $\{\tilde{Z} = (z_j^L - \alpha_j, z_j^L, z_j^U, z_j^U + \alpha'_j), j = 1, 2, \dots, n : A\tilde{Z} \supseteq \tilde{b}\}$, we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_j^L = x_j^L, z_j^U = x_j^U, \alpha_j \geq \sigma_j, \alpha'_j \geq \sigma'_j$.

Remark 2.1 [29]. The tolerable solution set and the controllable solution set for system of fuzzy linear equations (2.1) are respectively as follows:

$$X_{\forall\exists} = \{x' \in \mathbb{R}^n : (\forall A' \in A)(\exists b' \in b) \text{ s.t. } A'x' = b'\} = \{x' \in \mathbb{R}^n : Ax' \subseteq b\},$$

$$X_{\exists\forall} = \{x' \in \mathbb{R}^n : (\forall b' \in b)(\exists A' \in A) \text{ s.t. } A'x' = b'\} = \{x' \in \mathbb{R}^n : Ax' \supseteq b\}.$$

2.3. Allahviranloo and Salahshour method

Allahviranloo and Salahshour [26] proposed the following method to find the maximal and minimal fuzzy solution of the system of fuzzy linear equations (2.1).

Step 1: Assuming

$\tilde{x}_j = (x_j - \sigma_j, x_j, x_j + \sigma_j)$ and $\tilde{b}_i = (b_i - \beta_{i1}, b_i, b_i + \beta_{i2})$, the system of fuzzy linear equations (2.1) can be transformed into the system of fuzzy linear equations (2.2).

$$\sum_{j=1}^n a_{ij} (x_j - \sigma_j, x_j, x_j + \sigma_j) = (b_i - \beta_{i1}, b_i, b_i + \beta_{i2}), i = 1, 2, \dots, m. \quad (2.2)$$

Step 2: Replacing the triangular fuzzy numbers $(x_j - \sigma_j, x_j, x_j + \sigma_j)$ and $(b_i - \beta_{i1}, b_i, b_i + \beta_{i2})$ by the corresponding r -cuts $[x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j]$ and $[b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}]$, the system of fuzzy linear equations (2.2) can be transformed into the system of interval equations (2.3).

$$\sum_{j=1}^n a_{ij} [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] = [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], i = 1, 2, \dots, m. \quad (2.3)$$

Step 3: Since, a_{ij} is a real number, so $a_{ij} > 0$ or $a_{ij} = 0$ or $a_{ij} < 0$. Assuming $S = \{a_{ij} : a_{ij} \geq 0\}$ and $T = \{a_{ij} : a_{ij} < 0\}$, the system of interval equations (2.3) can be transformed into the system of interval equations (2.4).

$$\sum_{a_{ij} \in S} a_{ij} [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] + \sum_{a_{ij} \in T} a_{ij} [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j]$$

$$= [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], i = 1, 2, \dots, m. \quad (2.4)$$

Step 4: Using the property, $\lambda[a, b] = \begin{cases} [\lambda a, \lambda b] & \text{if } \lambda \geq 0, \\ [\lambda b, \lambda a] & \text{if } \lambda < 0. \end{cases}$

the system of interval equations (2.4) can be transformed into the system of interval equations (2.5).

$$\begin{aligned} & \sum_{a_{ij} \in S} [a_{ij}(x_j - \sigma_j + r\sigma_j), a_{ij}(x_j + \sigma_j - r\sigma_j)] + \sum_{a_{ij} \in T} [a_{ij}(x_j + \sigma_j - r\sigma_j), a_{ij}(x_j - \sigma_j + r\sigma_j)] \\ &= [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], \quad i = 1, 2, \dots, m. \end{aligned} \quad (2.5)$$

Step 5: Using the property, $\sum_{i=1}^m [a_i, b_i] = \left[\sum_{i=1}^m a_i, \sum_{i=1}^m b_i \right]$, the system of interval equations (2.5)

can be transformed into the system of interval equations (2.6).

$$\begin{aligned} & \left[\sum_{a_{ij} \in S} a_{ij}(x_j - \sigma_j + r\sigma_j), \sum_{a_{ij} \in S} a_{ij}(x_j + \sigma_j - r\sigma_j) \right] + \left[\sum_{a_{ij} \in T} a_{ij}(x_j + \sigma_j - r\sigma_j), \sum_{a_{ij} \in T} a_{ij}(x_j - \sigma_j + r\sigma_j) \right] \\ &= [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], \quad i = 1, 2, \dots, m. \end{aligned} \quad (2.6)$$

Step 6: Using the property, $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, the system of interval equations (2.6) can be transformed into the system of interval equations (2.7).

$$\begin{aligned} & \left[\sum_{a_{ij} \in S} a_{ij}(x_j - \sigma_j + r\sigma_j) + \sum_{a_{ij} \in T} a_{ij}(x_j + \sigma_j - r\sigma_j), \sum_{a_{ij} \in S} a_{ij}(x_j + \sigma_j - r\sigma_j) + \sum_{a_{ij} \in T} a_{ij}(x_j - \sigma_j + r\sigma_j) \right] \\ &= [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], \quad i = 1, 2, \dots, m. \end{aligned} \quad (2.7)$$

Step 7: Using the property, $[a_1, a_2] = [b_1, b_2] \Rightarrow a_1 = b_1$ and $a_2 = b_2$, the system of interval equations (2.7) can be transformed into the system of crisp linear equations (2.8) and (2.9).

$$\sum_{a_{ij} \in S} a_{ij}(x_j - \sigma_j + r\sigma_j) + \sum_{a_{ij} \in T} a_{ij}(x_j + \sigma_j - r\sigma_j) = b_i - \beta_{i1} + r\beta_{i1}, \quad i = 1, 2, \dots, m. \quad (2.8)$$

$$\sum_{a_{ij} \in S} a_{ij}(x_j + \sigma_j - r\sigma_j) + \sum_{a_{ij} \in T} a_{ij}(x_j - \sigma_j + r\sigma_j) = b_i + \beta_{i2} - r\beta_{i2}, \quad i = 1, 2, \dots, m. \quad (2.9)$$

Step 8: Solve the system of crisp linear equations (2.10), obtained by putting $r = 1$ in the system of crisp linear equations (2.8) and (2.9), to find the values of $x_j, j = 1, 2, \dots, n$.

$$\sum_{a_{ij} \in S} a_{ij} x_j + \sum_{a_{ij} \in T} a_{ij} x_j = b_i, i = 1, 2, \dots, m. \quad (2.10)$$

Step 9: Replacing σ_j of the i^{th} equation of the system of crisp linear equations (2.8) by σ_{i1} and σ_j of the i^{th} equation of the system of crisp linear equations (2.9) by σ_{i2} , these system of crisp linear equations are transformed into the system of crisp linear equations (2.11) and (2.12) respectively.

$$\sum_{a_{ij} \in S} a_{ij} (x_j - \sigma_{i1} + r\sigma_{i1}) + \sum_{a_{ij} \in T} a_{ij} (x_j + \sigma_{i1} - r\sigma_{i1}) = b_i - \beta_{i1} + r\beta_{i1}, i = 1, 2, \dots, m. \quad (2.11)$$

$$\sum_{a_{ij} \in S} a_{ij} (x_j + \sigma_{i2} - r\sigma_{i2}) + \sum_{a_{ij} \in T} a_{ij} (x_j - \sigma_{i2} + r\sigma_{i2}) = b_i + \beta_{i2} - r\beta_{i2}, i = 1, 2, \dots, m. \quad (2.12)$$

Step 10: Put the values of $x_j, j = 1, 2, \dots, n$, obtained in Step 8, in the system of crisp linear equations (2.11) and (2.12) and then solve these to find the values of σ_{i1} and σ_{i2} .

Step 11: Find

$$\alpha_L(r) = \underset{r}{\text{minimum}} \{ (1-r)\sigma_{i1}(r), (1-r)\sigma_{i2}(r) \}, i = 1, 2, \dots, m, 0 \leq r < 1,$$

$$\alpha_U(r) = \underset{r}{\text{maximum}} \{ (1-r)\sigma_{i1}(r), (1-r)\sigma_{i2}(r) \}, i = 1, 2, \dots, m, 0 \leq r < 1.$$

Step 12: Using the values of $x_j, j = 1, 2, \dots, n$, obtained in Step 8 and the values of $\alpha_L(r)$ and $\alpha_U(r)$, obtained in Step 11, the r -cuts of the maximal fuzzy symmetric solution $\tilde{X}_L$ and the minimal fuzzy symmetric solution $\tilde{X}_U$ of the system of fuzzy linear equations (2.1) are $[x_j - \alpha_L(r), x_j + \alpha_L(r)]$ and $[x_j - \alpha_U(r), x_j + \alpha_U(r)]$ respectively.

2.4. Maximal and minimal fuzzy solution of the existing problem

Allahviranloo and Salahshour [26] solved the system of fuzzy linear equations, presented in Example 2.1, to illustrate their proposed method. In this section, the same example is solved to illustrate the existing method [26].

Example 2.1 [26, Section 4, Example 4.1, pp. 4550]. Find the maximal and minimal fuzzy solution of the following system of fuzzy linear equations.

$$\begin{aligned}\tilde{x}_1 - \tilde{x}_2 &= (0, 1, 2), \\ \tilde{x}_1 + 3\tilde{x}_2 &= (4, 5, 7),\end{aligned}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are symmetric triangular fuzzy numbers.

Using the existing method [26], the maximal and minimal fuzzy solution of the system of fuzzy linear equations, presented in Example 2.1, can be obtained as follows:

Step 1: Assuming

$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1)$ and $\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2)$, the system of fuzzy linear equations, presented in Example 2.1 can be transformed into the system of fuzzy linear equations (2.13).

$$\begin{aligned}(x_1 - \sigma_1, x_1, x_1 + \sigma_1) - (x_2 - \sigma_2, x_2, x_2 + \sigma_2) &= (0, 1, 2), \\ (x_1 - \sigma_1, x_1, x_1 + \sigma_1) + 3(x_2 - \sigma_2, x_2, x_2 + \sigma_2) &= (4, 5, 7).\end{aligned}\tag{2.13}$$

Step 2: Replacing all the triangular fuzzy numbers of system of fuzzy linear equations (2.13) by their r -cuts, it can be transformed into the system of interval equations (2.14).

$$\begin{aligned}[x_1 - \sigma_1 + r\sigma_1, x_1 + \sigma_1 - r\sigma_1] - [x_2 - \sigma_2 + r\sigma_2, x_2 + \sigma_2 - r\sigma_2] &= [0 + r, 2 - r], \\ [x_1 - \sigma_1 + r\sigma_1, x_1 + \sigma_1 - r\sigma_1] + 3[x_2 - \sigma_2 + r\sigma_2, x_2 + \sigma_2 - r\sigma_2] &= [4 + r, 7 - 2r].\end{aligned}\tag{2.14}$$

Step 3: Using the property, $\lambda[a, b] = \begin{cases} [\lambda a, \lambda b], & \lambda \geq 0 \\ [\lambda b, \lambda a], & \lambda < 0. \end{cases}$

the system of interval equations (2.14) can be transformed into the system of interval equations (2.15).

$$\begin{aligned}[x_1 - \sigma_1 + r\sigma_1, x_1 + \sigma_1 - r\sigma_1] + [-x_2 - \sigma_2 + r\sigma_2, -x_2 + \sigma_2 - r\sigma_2] &= [r, 2 - r], \\ [x_1 - \sigma_1 + r\sigma_1, x_1 + \sigma_1 - r\sigma_1] + [3x_2 - 3\sigma_2 + 3r\sigma_2, 3x_2 + 3\sigma_2 - 3r\sigma_2] &= [4 + r, 7 - 2r].\end{aligned}\tag{2.15}$$

Step 4: Using the property,

$[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$ and $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$, the system of interval equations (2.15) can be transformed into the system of interval equations (2.16).

$$\begin{aligned} [x_1 - \sigma_1 + r\sigma_1 - x_2 - \sigma_2 + r\sigma_2, x_1 + \sigma_1 - r\sigma_1 - x_2 + \sigma_2 - r\sigma_2] &= [r, 2 - r], \\ [x_1 - \sigma_1 + r\sigma_1 + 3x_2 - 3\sigma_2 + 3r\sigma_2, x_1 + \sigma_1 - r\sigma_1 + 3x_2 + 3\sigma_2 - 3r\sigma_2] &= [4 + r, 7 - 2r]. \end{aligned} \quad (2.16)$$

Step 5: Using the property,

$[a_1, a_2] = [b_1, b_2] \Rightarrow a_1 = b_1$ and $a_2 = b_2$, the system of interval equations (2.16) can be transformed into the system of crisp linear equations (2.17) and (2.18).

$$\begin{aligned} x_1 - \sigma_1 + r\sigma_1 - x_2 - \sigma_2 + r\sigma_2 &= r, \\ x_1 - \sigma_1 + r\sigma_1 + 3x_2 - 3\sigma_2 + 3r\sigma_2 &= 4 + r, \end{aligned} \quad (2.17)$$

$$\begin{aligned} x_1 + \sigma_1 - r\sigma_1 - x_2 + \sigma_2 - r\sigma_2 &= 2 - r, \\ x_1 + \sigma_1 - r\sigma_1 + 3x_2 + 3\sigma_2 - 3r\sigma_2 &= 7 - 2r. \end{aligned} \quad (2.18)$$

Step 6: On solving the system of crisp linear equations (2.19), obtained by putting $r = 1$ in the system of crisp linear equations (2.17) and (2.18), the obtained values of x_1 and x_2 are 2 and 1 respectively.

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_1 + 3x_2 &= 5. \end{aligned} \quad (2.19)$$

Step 7: Replacing σ_1 and σ_2 of the i^{th} equation of the system of crisp linear equations (2.17) by σ_{i1} and replacing σ_1 and σ_2 of the i^{th} equation of the system of crisp linear equations (2.18) by σ_{i2} respectively, these system of crisp linear equations are transformed into the system of crisp linear equations (2.20) and (2.21).

$$\begin{aligned} x_1 - \sigma_{11} + r\sigma_{11} - x_2 - \sigma_{11} + r\sigma_{11} &= r, \\ x_1 - \sigma_{21} + r\sigma_{21} + 3x_2 - 3\sigma_{21} + 3r\sigma_{21} &= 4 + r. \end{aligned} \quad (2.20)$$

$$\begin{aligned} x_1 + \sigma_{12} - r\sigma_{12} - x_2 + \sigma_{12} - r\sigma_{12} &= 2 - r, \\ x_1 + \sigma_{22} - r\sigma_{22} + 3x_2 + 3\sigma_{22} - 3r\sigma_{22} &= 7 - 2r. \end{aligned} \quad (2.21)$$

Step 8: On solving the system of crisp linear equations (2.20) and (2.21) after putting the values of x_1 and x_2 , obtained in Step 6, the following values of $\sigma_{11}, \sigma_{21}, \sigma_{12}$ and σ_{22} are obtained.

$$\sigma_{11} = \frac{1}{2}, \sigma_{21} = \frac{1}{4}, \sigma_{12} = \frac{1}{2} \text{ and } \sigma_{22} = \frac{1}{2}.$$

Step 9: The values of

$$\alpha_L(r) = \underset{r}{\text{minimum}} \{ (1-r)\sigma_{11}(r), (1-r)\sigma_{21}(r), (1-r)\sigma_{12}(r), (1-r)\sigma_{22}(r) \}, 0 \leq r < 1 \text{ and}$$

$$\alpha_U(r) = \underset{r}{\text{maximum}} \{ (1-r)\sigma_{11}(r), (1-r)\sigma_{21}(r), (1-r)\sigma_{12}(r), (1-r)\sigma_{22}(r) \}, 0 \leq r < 1, \text{ are}$$

$$\frac{1-r}{4} \text{ and } \frac{1-r}{2} \text{ respectively.}$$

Step 10: Using the values of x_1 and x_2 , obtained in Step 6, and the values of $\alpha_L(r)$ and $\alpha_U(r)$, obtained in Step 9, the r -cuts of the obtained maximal fuzzy symmetric solution $\tilde{X}_L$ and minimal fuzzy symmetric solution $\tilde{X}_U$ of the system of fuzzy linear equations, presented in Example 2.1, are $\left[2 - \frac{1-r}{4}, 2 + \frac{1-r}{4} \right], \left[1 - \frac{1-r}{4}, 1 + \frac{1-r}{4} \right]$ and

$$\left[2 - \frac{1-r}{2}, 2 + \frac{1-r}{2} \right], \left[1 - \frac{1-r}{2}, 1 + \frac{1-r}{2} \right] \text{ respectively.}$$

2.5. Limitations of Allahviranloo and Salahshour method

The method, proposed by Allahviranloo and Salahshour [26], can be used only to find the maximal and minimal fuzzy solution of such system of fuzzy linear equations $A\tilde{X} = \tilde{b}$, in which all the elements of the unknown vector $\tilde{X}$ are symmetric triangular fuzzy numbers, all the elements of the coefficient matrix A are real numbers and all the elements of right hand side vector $\tilde{b}$ are triangular fuzzy numbers. For example, the existing system of fuzzy linear equations, presented in Example 2.1 [26, Section 4, Example 4.1, pp. 4550] and Example 2.2

[26, Section 4, Example 4.2, pp. 4551], are same type of system of fuzzy linear equations and hence can be solved by the existing method [26].

However, if in the system of fuzzy linear equations (2.1), the variables $\tilde{x}_j, j=1,2,\dots,n$, are assumed as non-symmetric triangular/trapezoidal fuzzy numbers instead of symmetric triangular fuzzy numbers, then, the maximal and minimal fuzzy solution of the transformed system of fuzzy linear equations cannot be obtained by the existing method [26]. For example, if in the existing system of fuzzy linear equations, presented in Example 2.1 [26, Section 4, Example 4.1, pp. 4550], the variables $\tilde{x}_j, j=1,2$ and in the existing system of fuzzy linear equations, presented in Example 2.2 [26, Section 4, Example 4.2, pp. 4551], the variables $\tilde{x}_j, j=1,2,3,4,5$, are assumed as non-symmetric triangular fuzzy numbers instead of symmetric triangular fuzzy numbers, then the maximal and minimal fuzzy solution of transformed system of fuzzy linear equations, presented in Example 2.3 and Example 2.4, cannot be obtained by the existing method [26].

Moreover, the existing method [26] cannot be used to find the maximal and minimal fuzzy solution of such system of fuzzy linear equations in which the elements of the unknown vector are either symmetric trapezoidal fuzzy numbers or non-symmetric trapezoidal fuzzy numbers. For example, the maximal and minimal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.5, cannot be obtained by the existing method [26].

Example 2.2 [26, Section 4, Example 4.2, pp. 4551]. Find the maximal and minimal fuzzy solution of the following system of fuzzy linear equations.

$$\begin{aligned} 6\tilde{x}_1 + \tilde{x}_2 + 3\tilde{x}_3 - \tilde{x}_4 + 6\tilde{x}_5 &= (1, 2, 3), \\ 5\tilde{x}_1 + 9\tilde{x}_2 + \tilde{x}_3 + 2\tilde{x}_4 + 3\tilde{x}_5 &= (6, 7, 8), \\ 2\tilde{x}_1 + 3\tilde{x}_2 + 9\tilde{x}_3 + 2\tilde{x}_4 + 3\tilde{x}_5 &= (5, 6, 7), \\ -\tilde{x}_1 + \tilde{x}_2 + 3\tilde{x}_3 + 8\tilde{x}_4 + 3\tilde{x}_5 &= (3, 4, 5), \\ \tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 + 9\tilde{x}_5 &= (2, 3, 4), \end{aligned}$$

where $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ and $\tilde{x}_5$ are symmetric triangular fuzzy numbers.

Example 2.3. Find the maximal and minimal fuzzy solution of the following system of fuzzy linear equations.

$$\begin{aligned}\tilde{x}_1 - \tilde{x}_2 &= (0, 1, 2), \\ \tilde{x}_1 + 3\tilde{x}_2 &= (4, 5, 7),\end{aligned}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-symmetric triangular fuzzy numbers.

Example 2.4. Find the maximal and minimal fuzzy solution of the following system of fuzzy linear equations.

$$\begin{aligned}6\tilde{x}_1 + \tilde{x}_2 + 3\tilde{x}_3 - \tilde{x}_4 + 6\tilde{x}_5 &= (1, 2, 3), \\ 5\tilde{x}_1 + 9\tilde{x}_2 + \tilde{x}_3 + 2\tilde{x}_4 + 3\tilde{x}_5 &= (6, 7, 8), \\ 2\tilde{x}_1 + 3\tilde{x}_2 + 9\tilde{x}_3 + 2\tilde{x}_4 + 3\tilde{x}_5 &= (5, 6, 7), \\ -\tilde{x}_1 + \tilde{x}_2 + 3\tilde{x}_3 + 8\tilde{x}_4 + 3\tilde{x}_5 &= (3, 4, 5), \\ \tilde{x}_1 + 2\tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 + 9\tilde{x}_5 &= (2, 3, 4),\end{aligned}$$

where $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ and $\tilde{x}_5$ are non-symmetric triangular fuzzy numbers.

Example 2.5. Find the maximal and minimal fuzzy solution of the following system of fuzzy linear equations.

$$\begin{aligned}\tilde{x}_1 - 3\tilde{x}_2 &= (-16, -3, 11, 15), \\ 2\tilde{x}_1 + \tilde{x}_2 &= (-5, 4, 12, 17),\end{aligned}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are symmetric/non-symmetric trapezoidal fuzzy numbers.

2.6. Proposed Mehar-Keerat method-I

In this section, to overcome the limitations of the existing method [26], discussed in Section 2.5, a new method (named as Mehar-Keerat method-I) is proposed to find the maximal and minimal fuzzy solution of the system of fuzzy linear equations (2.22),

$$\sum_{j=1}^n a_{ij} \tilde{x}_j = \tilde{b}_i, \quad i = 1, 2, \dots, m, \quad (2.22)$$

in which a_{ij} are real numbers and $\tilde{x}_j, \tilde{b}_i$ are trapezoidal fuzzy numbers.

The steps of proposed Mehar-Keerat method-I are as follows:

Step 1: Assuming

$\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j)$, $\tilde{b}_i = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2})$, the system of fuzzy linear equations (2.22) can be transformed into the system of fuzzy linear equations (2.23).

$$\sum_{j=1}^n a_{ij} (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}); i = 1, 2, \dots, m. \quad (2.23)$$

Step 2: Since, a_{ij} is real number, so $a_{ij} > 0$ or $a_{ij} = 0$ or $a_{ij} < 0$. Assuming $S = \{a_{ij} : a_{ij} \geq 0\}$ and $T = \{a_{ij} : a_{ij} < 0\}$, the system of fuzzy linear equations (2.23) can be transformed into the system of fuzzy linear equations (2.24).

$$\sum_{a_{ij} \in S} a_{ij} (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) + \sum_{a_{ij} \in T} a_{ij} (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}); i = 1, 2, \dots, m. \quad (2.24)$$

Step 3: Using the property,

$$\lambda(a^L - \alpha, a^L, a^U, a^U + \alpha') = \begin{cases} (\lambda(a^L - \alpha), \lambda a^L, \lambda a^U, \lambda(a^U + \alpha')) & \text{if } \lambda \geq 0, \\ (\lambda(a^U + \alpha'), \lambda a^U, \lambda a^L, \lambda(a^L - \alpha)) & \text{if } \lambda < 0. \end{cases}$$

the system of fuzzy linear equations (2.24) can be transformed into the system of fuzzy linear equations (2.25).

$$\begin{aligned} & \sum_{a_{ij} \in S} (a_{ij} (x_j^L - \sigma_j), a_{ij} x_j^L, a_{ij} x_j^U, a_{ij} (x_j^U + \sigma'_j)) + \sum_{a_{ij} \in T} (a_{ij} (x_j^U + \sigma'_j), a_{ij} x_j^U, a_{ij} x_j^L, a_{ij} (x_j^L - \sigma_j)) \\ & = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}); i = 1, 2, \dots, m. \end{aligned} \quad (2.25)$$

Step 4: Using the property,

$$\sum_{i=1}^n (a_i^L - \alpha_i, a_i^L, a_i^U, a_i^U + \alpha'_i) = \left(\sum_{i=1}^n a_i^L - \alpha_i, \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^U, \sum_{i=1}^n a_i^U + \alpha'_i \right), \text{ the system of fuzzy linear}$$

equations (2.25) can be transformed into the system of fuzzy linear equations (2.26).

$$\begin{aligned} & \left(\sum_{a_{ij} \in S} a_{ij} (x_j^L - \sigma_j), \sum_{a_{ij} \in S} a_{ij} x_j^L, \sum_{a_{ij} \in S} a_{ij} x_j^U, \sum_{a_{ij} \in S} a_{ij} (x_j^U + \sigma'_j) \right) + \left(\sum_{a_{ij} \in T} a_{ij} (x_j^U + \sigma'_j), \sum_{a_{ij} \in T} a_{ij} x_j^U, \sum_{a_{ij} \in T} a_{ij} x_j^L, \sum_{a_{ij} \in T} a_{ij} (x_j^L - \sigma_j) \right) \\ & = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}); i = 1, 2, \dots, m. \end{aligned} \quad (2.26)$$

Step 5: Using the property,

$$\begin{aligned} & (a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1) + (a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2) \\ &= (a_1^L - \alpha_1 + a_2^L - \alpha_2, a_1^L + a_2^L, a_1^U + a_2^U, a_1^U + \alpha'_1 + a_2^U + \alpha'_2), \end{aligned}$$

the system of fuzzy linear equations (2.26) can be transformed into the system of fuzzy linear equations (2.27).

$$\begin{aligned} & \left(\sum_{a_{ij} \in S} a_{ij} (x_j^L - \sigma_j) + \sum_{a_{ij} \in T} a_{ij} (x_j^U + \sigma'_j), \sum_{a_{ij} \in S} a_{ij} x_j^L + \sum_{a_{ij} \in T} a_{ij} x_j^U, \sum_{a_{ij} \in S} a_{ij} x_j^U + \sum_{a_{ij} \in T} a_{ij} x_j^L, \sum_{a_{ij} \in S} a_{ij} (x_j^U + \sigma'_j) + \sum_{a_{ij} \in T} a_{ij} (x_j^L - \sigma_j) \right) \\ &= (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}); i = 1, 2, \dots, m. \end{aligned} \quad (2.27)$$

Step 6: Using Definition 2.18 (definition of maximal fuzzy solution), the system of fuzzy linear equations (2.27) can be transformed into multi-objective mathematical programming problem (2.28).

Maximize $\sigma_j; j = 1, 2, \dots, n,$

Maximize $\sigma'_j; j = 1, 2, \dots, n,$

subject to

$$\begin{aligned} & \sum_{a_{ij} \in S} a_{ij} \sigma_j - \sum_{a_{ij} \in T} a_{ij} \sigma'_j \leq \beta_{i1}; i = 1, 2, \dots, m, \\ & \sum_{a_{ij} \in S} a_{ij} x_j^L + \sum_{a_{ij} \in T} a_{ij} x_j^U = b_i^L; i = 1, 2, \dots, m, \\ & \sum_{a_{ij} \in S} a_{ij} x_j^U + \sum_{a_{ij} \in T} a_{ij} x_j^L = b_i^U; i = 1, 2, \dots, m, \\ & \sum_{a_{ij} \in S} a_{ij} \sigma'_j - \sum_{a_{ij} \in T} a_{ij} \sigma_j \leq \beta_{i2}; i = 1, 2, \dots, m, \\ & x_j^L - \sigma_j \leq x_j^L \leq x_j^U \leq x_j^U + \sigma'_j \text{ and } \sigma_j \geq 0, \sigma'_j \geq 0. \end{aligned} \quad (2.28)$$

Step 7: Using Definition 2.19 (definition of minimal fuzzy solution), the system of fuzzy linear equations (2.27) can be transformed into multi-objective mathematical programming problem (2.29).

Minimize $\sigma_j; j = 1, 2, \dots, n,$

Minimize $\sigma'_j; j = 1, 2, \dots, n,$

subject to

$$\begin{aligned}
& \sum_{a_{ij} \in S} a_{ij} \sigma_j - \sum_{a_{ij} \in T} a_{ij} \sigma'_j \geq \beta_{i1}; i = 1, 2, \dots, m, \\
& \sum_{a_{ij} \in S} a_{ij} x_j^L + \sum_{a_{ij} \in T} a_{ij} x_j^U = b_i^L; i = 1, 2, \dots, m, \\
& \sum_{a_{ij} \in S} a_{ij} x_j^U + \sum_{a_{ij} \in T} a_{ij} x_j^L = b_i^U; i = 1, 2, \dots, m, \\
& \sum_{a_{ij} \in S} a_{ij} \sigma'_j - \sum_{a_{ij} \in T} a_{ij} \sigma_j \geq \beta_{i2}; i = 1, 2, \dots, m, \\
& x_j^L - \sigma_j \leq x_j^L \leq x_j^U \leq x_j^U + \sigma'_j \text{ and } \sigma_j \geq 0, \sigma'_j \geq 0.
\end{aligned} \tag{2.29}$$

Step 8: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (2.28) can be transformed into single-objective mathematical programming problem (2.30).

$$\begin{aligned}
& \text{Maximize } (\sigma_j + \sigma'_j); j = 1, 2, \dots, n, \\
& \text{subject to} \\
& \text{Constraints of Problem (2.28)}.
\end{aligned} \tag{2.30}$$

Step 9: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (2.29) can be transformed into single-objective mathematical programming problem (2.31).

$$\begin{aligned}
& \text{Minimize } (\sigma_j + \sigma'_j); j = 1, 2, \dots, n, \\
& \text{subject to} \\
& \text{Constraints of Problem (2.29)}.
\end{aligned} \tag{2.31}$$

Step 10: Find the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (2.30).

Step 11: Find the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (2.31).

Step 12: Using the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (2.30), obtained in Step 10, the maximal fuzzy solution of the system of fuzzy linear equations (2.22) is

$$\{\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j); j = 1, 2, \dots, n\}.$$

Step 13: Using the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (2.31), obtained in Step 11, the minimal fuzzy solution of the system of fuzzy linear equations (2.22) is

$$\{\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j); j = 1, 2, \dots, n\}.$$

Remark 2.2. The maximal and minimal fuzzy symmetric solution of the system of fuzzy linear equations (2.22) can be obtained by solving the mathematical programming problems (2.30) and (2.31) respectively, with the additional constraints $\sigma_j = \sigma'_j; j = 1, 2, \dots, n$.

2.7. Illustrative examples

In this section, to illustrate the proposed Mehar-Keerat method-I, the system of fuzzy linear equations, chosen in Example 2.3 and Example 2.5 which cannot be solved by the existing method [26], are solved by proposed Mehar-Keerat method-I. Also, the maximal and minimal fuzzy solution of the system of fuzzy linear equations, presented in Example 2.1, is obtained.

2.7.1. Maximal and minimal fuzzy solution of system of fuzzy linear equations chosen in Example 2.1 and Example 2.3

Using Mehar-Keerat method-I, proposed in Section 2.6, the maximal and minimal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.1 and Example 2.3, can be obtained as follows.

Step 1: Assuming

$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1)$ and $\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2)$, the system of fuzzy linear equations, chosen in Example 2.3, can be transformed into the system of fuzzy linear equations (2.32).

$$\begin{aligned}
(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) - (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) &= (0, 1, 2), \\
(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) + 3(x_2 - \sigma_2, x_2, x_2 + \sigma'_2) &= (4, 5, 7).
\end{aligned} \tag{2.32}$$

Step 2: Using Step 2 to Step 5 of the proposed Mehar-Keerat method-I, the system of fuzzy linear equations (2.32) can be transformed into the system of fuzzy linear equations (2.33).

$$\begin{aligned}
(x_1 - \sigma_1 - x_2 - \sigma'_2, x_1 - x_2, x_1 + \sigma'_1 - x_2 + \sigma_2) &= (0, 1, 2), \\
(x_1 - \sigma_1 + 3x_2 - 3\sigma_2, x_1 + 3x_2, x_1 + \sigma'_1 + 3x_2 + 3\sigma'_2) &= (4, 5, 7).
\end{aligned} \tag{2.33}$$

Step 3: Using Step 6 of the proposed Mehar-Keerat method-I, the system of fuzzy linear equations (2.33) can be transformed into multi-objective mathematical programming problem (2.34).

$$\begin{aligned}
&\text{Maximize } \sigma_1, \\
&\text{Maximize } \sigma'_1, \\
&\text{Maximize } \sigma_2, \\
&\text{Maximize } \sigma'_2, \\
&\text{subject to} \\
&\sigma_1 + \sigma'_2 \leq 1, \\
&x_1 - x_2 = 1, \\
&\sigma'_1 + \sigma_2 \leq 1, \\
&\sigma_1 + 3\sigma_2 \leq 1, \\
&x_1 + 3x_2 = 5, \\
&\sigma'_1 + 3\sigma'_2 \leq 2, \\
&x_1 - \sigma_1 \leq x_1 \leq x_1 + \sigma'_1, x_2 - \sigma_2 \leq x_2 \leq x_2 + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.
\end{aligned} \tag{2.34}$$

Step 4: Using Step 7 of the proposed Mehar-Keerat method-I, the system of fuzzy linear equations (2.33) can be transformed into multi-objective mathematical programming problem (2.35).

$$\begin{aligned}
& \text{Minimize } \sigma_1, \\
& \text{Minimize } \sigma'_1, \\
& \text{Minimize } \sigma_2, \\
& \text{Minimize } \sigma'_2, \\
& \text{subject to} \\
& \sigma_1 + \sigma'_2 \geq 1, \\
& x_1 - x_2 = 1, \\
& \sigma'_1 + \sigma_2 \geq 1, \\
& \sigma_1 + 3\sigma_2 \geq 1, \\
& x_1 + 3x_2 = 5, \\
& \sigma'_1 + 3\sigma'_2 \geq 2, \\
& x_1 - \sigma_1 \leq x_1 \leq x_1 + \sigma'_1, x_2 - \sigma_2 \leq x_2 \leq x_2 + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.
\end{aligned} \tag{2.35}$$

Step 5: Using Step 8 of the proposed Mehar-Keerat method-I, the multi-objective mathematical programming problem (2.34) can be transformed into single-objective mathematical programming problem (2.36).

$$\begin{aligned}
& \text{Maximize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
& \text{subject to} \\
& \text{Constraints of Problem (2.34)}.
\end{aligned} \tag{2.36}$$

Step 6: Using Step 9 of the proposed Mehar-Keerat method-I, the multi-objective mathematical programming problem (2.35) can be transformed into single-objective mathematical programming problem (2.37).

$$\begin{aligned}
& \text{Minimize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
& \text{subject to} \\
& \text{Constraints of Problem (2.35)}.
\end{aligned} \tag{2.37}$$

Step 7: The optimal solution $\{x_1, x_2, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (2.36)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = 1, \sigma'_1 = 1, \sigma_2 = 0, \sigma'_2 = 0, x_1 = 2, x_2 = 1.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{1}{4}, \sigma'_1 = \frac{1}{4}, \sigma_2 = \frac{1}{4}, \sigma'_2 = \frac{1}{4}, x_1 = 2, x_2 = 1.$$

(iii) without any additional constraint is

$$\sigma_1 = \frac{5}{8}, \sigma'_1 = \frac{7}{8}, \sigma_2 = \frac{1}{8}, \sigma'_2 = \frac{3}{8}, x_1 = 2, x_2 = 1.$$

Step 8: The optimal solution $\{x_1, x_2, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (2.37)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = 0, \sigma'_1 = 0, \sigma_2 = 1, \sigma'_2 = 1, x_1 = 2, x_2 = 1.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{1}{2}, \sigma'_1 = \frac{1}{2}, \sigma_2 = \frac{1}{2}, \sigma'_2 = \frac{1}{2}, x_1 = 2, x_2 = 1.$$

(iii) without any additional constraint is

$$\sigma_1 = 0, \sigma'_1 = 0, \sigma_2 = 1, \sigma'_2 = 1, x_1 = 2, x_2 = 1.$$

Step 9: Using optimal solution, obtained in Step 7, the maximal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.1

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\begin{aligned}\tilde{x}_1 &= (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = (1, 2, 3), \\ \tilde{x}_2 &= (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = (1, 1, 1).\end{aligned}$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\begin{aligned}\tilde{x}_1 &= (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = \left(\frac{7}{4}, 2, \frac{9}{4}\right), \\ \tilde{x}_2 &= (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = \left(\frac{3}{4}, 1, \frac{5}{4}\right).\end{aligned}$$

Also, using the optimal solution, obtained in Step 7 without any additional constraint, the maximal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.3, is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = \left(\frac{11}{8}, 2, \frac{23}{8}\right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = \left(\frac{7}{8}, 1, \frac{11}{8}\right).$$

Step 10: Using optimal solution, obtained in Step 8, the minimal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.1

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = (2, 2, 2),$$

$$\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = (0, 1, 2).$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = \left(\frac{3}{2}, 2, \frac{5}{2}\right),$$

$$\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = \left(\frac{1}{2}, 1, \frac{3}{2}\right).$$

Also, using the optimal solution, obtained in Step 8 without any additional constraint, the minimal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.3, is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = (2, 2, 2), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = (0, 1, 2).$$

2.7.2. Maximal and minimal fuzzy solution of system of fuzzy linear equations chosen in Example 2.5

Using Mehar-Keerat method-I, proposed in Section 2.6, the maximal and minimal fuzzy solution of system of fuzzy linear equations, chosen in Example 2.5, can be obtained as follows.

Step 1: Assuming

$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1)$ and $\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2)$, the system of fuzzy linear equations, chosen in Example 2.5, can be transformed into the system of fuzzy linear equations (2.38).

$$\begin{aligned}
& (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') - 3(x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (-16, -3, 11, 15), \\
& 2(x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') + (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (-5, 4, 12, 17).
\end{aligned} \tag{2.38}$$

Step 2: Using Step 2 to Step 5 of the proposed Mehar-Keerat method-I, the system of fuzzy linear equations (2.38) can be transformed into the system of fuzzy linear equations (2.39).

$$\begin{aligned}
& (x_1^L - \sigma_1 - 3x_2^U - 3\sigma_2', x_1^L - 3x_2^U, x_1^U - 3x_2^L, x_1^U + \sigma_1' - 3x_2^L + 3\sigma_2') = (-16, -3, 11, 15), \\
& (2x_1^L - 2\sigma_1 + x_2^L - \sigma_2, 2x_1^L + x_2^L, 2x_1^U + x_2^U, 2x_1^U + 2\sigma_1' + x_2^U + \sigma_2') = (-5, 4, 12, 17).
\end{aligned} \tag{2.39}$$

Step 3: Using Step 6 of the proposed Mehar-Keerat method-I, the system of fuzzy linear equations (2.39) can be transformed into multi-objective mathematical programming problem (2.40).

$$\begin{aligned}
& \text{Maximize } \sigma_1, \\
& \text{Maximize } \sigma_1', \\
& \text{Maximize } \sigma_2, \\
& \text{Maximize } \sigma_2', \\
& \text{subject to} \\
& \sigma_1 + 3\sigma_2' \leq 13, \\
& x_1^L - 3x_2^U = -3, \\
& x_1^U - 3x_2^L = 11, \\
& \sigma_1' + 3\sigma_2 \leq 4, \\
& 2\sigma_1 + \sigma_2 \leq 9,
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
& 2x_1^L + x_2^L = 4, \\
& 2x_1^U + x_2^U = 12, \\
& 2\sigma_1' + \sigma_2' \leq 5, \\
& x_1^L - \sigma_1 \leq x_1^L \leq x_1^U \leq x_1^U + \sigma_1', x_2^L - \sigma_2 \leq x_2^L \leq x_2^U \leq x_2^U + \sigma_2', \sigma_1 \geq 0, \sigma_1' \geq 0, \sigma_2 \geq 0, \sigma_2' \geq 0.
\end{aligned}$$

Step 4: Using Step 7 of the proposed Mehar-Keerat method-I, the system of fuzzy linear equations (2.39) can be transformed into multi-objective mathematical programming problem (2.41).

$$\begin{aligned}
& \text{Minimize } \sigma_1, \\
& \text{Minimize } \sigma_1', \\
& \text{Minimize } \sigma_2, \\
& \text{Minimize } \sigma_2',
\end{aligned}$$

subject to

$$\begin{aligned}
&\sigma_1 + 3\sigma'_2 \geq 13, \\
&x_1^L - 3x_2^U = -3, \\
&x_1^U - 3x_2^L = 11, \\
&\sigma'_1 + 3\sigma_2 \geq 4, \\
&2\sigma_1 + \sigma_2 \geq 9, \\
&2x_1^L + x_2^L = 4, \\
&2x_1^U + x_2^U = 12, \\
&2\sigma'_1 + \sigma'_2 \geq 5,
\end{aligned} \tag{2.41}$$

$$x_1^L - \sigma_1 \leq x_1^L \leq x_1^U \leq x_1^U + \sigma'_1, x_2^L - \sigma_2 \leq x_2^L \leq x_2^U \leq x_2^U + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.$$

Step 5: Using Step 8 of the proposed Mehar-Keerat method-I, the multi-objective mathematical programming problem (2.40) can be transformed into single-objective mathematical programming problem (2.42).

$$\begin{aligned}
&\text{Maximize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
&\text{subject to} \\
&\text{Constraints of Problem (2.40)}.
\end{aligned} \tag{2.42}$$

Step 6: Using Step 9 of the proposed Mehar-Keerat method-I, the multi-objective mathematical programming problem (2.41) can be transformed into single-objective mathematical programming problem (2.43).

$$\begin{aligned}
&\text{Minimize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
&\text{subject to} \\
&\text{Constraints of Problem (2.41)}.
\end{aligned} \tag{2.43}$$

Step 7: The optimal solution $\{x_1^L, x_1^U, x_2^L, x_2^U, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (2.42)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{11}{5}, \sigma'_1 = \frac{11}{5}, \sigma_2 = \frac{3}{5}, \sigma'_2 = \frac{3}{5}, x_1^L = 3, x_1^U = 5, x_2^L = -2, x_2^U = 2.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = 1, \sigma'_1 = 1, \sigma_2 = 1, \sigma'_2 = 1, x_1^L = 3, x_1^U = 5, x_2^L = -2, x_2^U = 2.$$

(iii) without any additional constraint is

$$\sigma_1 = 4, \sigma'_1 = 1, \sigma_2 = 1, \sigma'_2 = 3, x_1^L = 3, x_1^U = 5, x_2^L = -2, x_2^U = 2.$$

Step 8: The optimal solution $\{x_1^L, x_1^U, x_2^L, x_2^U, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (2.43)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{14}{5}, \sigma'_1 = \frac{14}{5}, \sigma_2 = \frac{17}{5}, \sigma'_2 = \frac{17}{5}, x_1^L = 3, x_1^U = 5, x_2^L = -2, x_2^U = 2.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{13}{4}, \sigma'_1 = \frac{13}{4}, \sigma_2 = \frac{13}{4}, \sigma'_2 = \frac{13}{4}, x_1^L = 3, x_1^U = 5, x_2^L = -2, x_2^U = 2.$$

(iii) without any additional constraint is

$$\sigma_1 = 4, \sigma'_1 = 1, \sigma_2 = 1, \sigma'_2 = 3, x_1^L = 3, x_1^U = 5, x_2^L = -2, x_2^U = 2.$$

Step 9: Using optimal solution, obtained in Step 7, the maximal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.5

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\begin{aligned}\tilde{x}_1 &= (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = \left(\frac{4}{5}, 3, 5, \frac{36}{5}\right), \\ \tilde{x}_2 &= (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = \left(-\frac{13}{5}, -2, 2, \frac{13}{5}\right).\end{aligned}$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\begin{aligned}\tilde{x}_1 &= (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = (2, 3, 5, 6), \\ \tilde{x}_2 &= (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = (-3, -2, 2, 3).\end{aligned}$$

(iii) without any additional constraint is

$$\begin{aligned}\tilde{x}_1 &= (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = (-1, 3, 5, 6), \\ \tilde{x}_2 &= (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = (-3, -2, 2, 5).\end{aligned}$$

Step 10: Using optimal solution, obtained in Step 8, the minimal fuzzy solution of the system of fuzzy linear equations, chosen in Example 2.5

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\begin{aligned}\tilde{x}_1 &= (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = \left(\frac{1}{5}, 3, 5, \frac{39}{5}\right), \\ \tilde{x}_2 &= (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = \left(-\frac{27}{5}, -2, 2, \frac{27}{5}\right).\end{aligned}$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\begin{aligned}\tilde{x}_1 &= (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = \left(-\frac{1}{4}, 3, 5, \frac{33}{4}\right), \\ \tilde{x}_2 &= (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = \left(-\frac{21}{4}, -2, 2, \frac{21}{4}\right).\end{aligned}$$

(iii) without any additional constraint is

$$\begin{aligned}\tilde{x}_1 &= (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = (-1, 3, 5, 6), \\ \tilde{x}_2 &= (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = (-3, -2, 2, 5).\end{aligned}$$

2.8. Comparative study

The maximal and minimal fuzzy solution of the existing system of fuzzy linear equations and chosen system of fuzzy linear equations, obtained by the existing method [26] as well as proposed Mehar-Keerat method-I, are shown in Table 2.1.

Table 2.1. Maximal and minimal fuzzy symmetric/non-symmetric solution of the existing and chosen system of fuzzy linear equations.

Example	Maximal/Minimal fuzzy solution	Existing method [26]		Proposed Mehar-Keerat method-I	
		Symmetric solution	Non-symmetric solution	Symmetric solution	Non-symmetric solution
Example 2.1 [26, Section 4, Example 4.1, pp. 4550], Example 2.3	Maximal fuzzy solution	$\tilde{x}_1 = \left(\frac{7}{4}, 2, \frac{9}{4}\right), \tilde{x}_2 = \left(\frac{3}{4}, 1, \frac{5}{4}\right).$	N.A.	$\tilde{x}_1 = \left(\frac{7}{4}, 2, \frac{9}{4}\right), \tilde{x}_2 = \left(\frac{3}{4}, 1, \frac{5}{4}\right).$	$\tilde{x}_1 = \left(\frac{11}{8}, 2, \frac{23}{8}\right), \tilde{x}_2 = \left(\frac{7}{8}, 1, \frac{11}{8}\right).$
	Minimal fuzzy solution	$\tilde{x}_1 = \left(\frac{3}{2}, 2, \frac{5}{2}\right), \tilde{x}_2 = \left(\frac{1}{2}, 1, \frac{3}{2}\right).$	N.A.	$\tilde{x}_1 = \left(\frac{3}{2}, 2, \frac{5}{2}\right), \tilde{x}_2 = \left(\frac{1}{2}, 1, \frac{3}{2}\right).$	$\tilde{x}_1 = (2, 2, 2), \tilde{x}_2 = (0, 1, 2).$
Example 2.2 [26, Section 4, Example 4.2, pp. 4551], Example 2.4	Maximal fuzzy solution	$\tilde{x}_1 = (-0.0447, 0.0053, 0.0553),$ $\tilde{x}_2 = (0.6019, 0.6519, 0.7019),$ $\tilde{x}_3 = (0.3159, 0.3659, 0.4159),$ $\tilde{x}_4 = (0.2025, 0.2525, 0.3025),$ $\tilde{x}_5 = (0.0285, 0.0785, 0.1285).$	N.A.	$\tilde{x}_1 = (-0.0447, 0.0053, 0.0553),$ $\tilde{x}_2 = (0.6019, 0.6519, 0.7019),$ $\tilde{x}_3 = (0.3159, 0.3659, 0.4159),$ $\tilde{x}_4 = (0.2025, 0.2525, 0.3025),$ $\tilde{x}_5 = (0.0285, 0.0785, 0.1285).$	$\tilde{x}_1 = (-0.1143, 0.0053, 0.1250),$ $\tilde{x}_2 = (0.6328, 0.6519, 0.6710),$ $\tilde{x}_3 = (0.3067, 0.3659, 0.4251),$ $\tilde{x}_4 = (0.167, 0.2525, 0.338),$ $\tilde{x}_5 = (0.0785, 0.0785, 0.0785).$
	Minimal fuzzy Solution	$\tilde{x}_1 = (-0.0614, 0.0053, 0.072),$ $\tilde{x}_2 = (0.5852, 0.6519, 0.7186),$ $\tilde{x}_3 = (0.2992, 0.3659, 0.4326),$ $\tilde{x}_4 = (0.1858, 0.2525, 0.3192),$ $\tilde{x}_5 = (0.0118, 0.0785, 0.1452).$	N.A.	$\tilde{x}_1 = (-0.0614, 0.0053, 0.072),$ $\tilde{x}_2 = (0.5852, 0.6519, 0.7186),$ $\tilde{x}_3 = (0.2992, 0.3659, 0.4326),$ $\tilde{x}_4 = (0.1858, 0.2525, 0.3192),$ $\tilde{x}_5 = (0.0118, 0.0785, 0.1452).$	$\tilde{x}_1 = (0.0053, 0.0053, 0.0053),$ $\tilde{x}_2 = (0.6006, 0.6519, 0.7032),$ $\tilde{x}_3 = (0.3274, 0.3659, 0.4044),$ $\tilde{x}_4 = (0.1969, 0.2525, 0.3081),$ $\tilde{x}_5 = (-0.0511, 0.0785, 0.2081).$

Example 2.5	Maximal fuzzy solution	N.A.	N.A.	$\tilde{x}_1 = (2, 3, 5, 6),$ $\tilde{x}_2 = (-3, -2, 2, 3).$	$\tilde{x}_1 = (-1, 3, 5, 6),$ $\tilde{x}_2 = (-3, -2, 2, 5).$
	Minimal fuzzy Solution	N.A.	N.A.	$\tilde{x}_1 = \left(-\frac{1}{4}, 3, 5, \frac{33}{4}\right),$ $\tilde{x}_2 = \left(-\frac{21}{4}, -2, 2, \frac{21}{4}\right).$	$\tilde{x}_1 = (-1, 3, 5, 6),$ $\tilde{x}_2 = (-3, -2, 2, 5).$

2.9. Conclusion

On the basis of present study, it can be concluded that the system of fuzzy linear equations whose maximal and minimal fuzzy solution can be obtained by the existing method [26], can also be obtained by the proposed Mehar-Keerat method-I. However, there exist several system of fuzzy linear equations whose maximal and minimal fuzzy solution can be obtained by the proposed Mehar-Keerat method-I but cannot be obtained by the existing method [26]. Hence, it is better to use the proposed Mehar-Keerat method-I as compared to the existing method [26].

Chapter 3

Mehar-Keerat method-II to find maximal and minimal fuzzy solution of system of fully fuzzy linear equations*

Allahviranloo et al. [29] pointed out that the existing method [26], cannot be used to find maximal and minimal fuzzy solution of the system of fully fuzzy linear equations. Furthermore, to overcome this limitation of the existing method [26], Allahviranloo et al. [29] extended the existing method [26], to find maximal and minimal fuzzy solution of such system of fully fuzzy linear equations in which all the elements of the unknown vector are non-negative symmetric triangular fuzzy numbers, whereas, all the elements of the coefficient matrix as well as all the elements of the right hand side vector are non-negative/non-positive triangular fuzzy numbers. In this chapter, limitations of the existing method [29] are pointed out. Also, to overcome the limitations of the existing method [29], a new method (named as Mehar-Keerat method-II) is proposed by extending the Mehar-Keerat method-I, proposed in Chapter 2. The proposed Mehar-Keerat method-II is illustrated with the help of numerical examples.

3.1. Maximal and minimal fuzzy solution of system of fully fuzzy linear equations

Allahviranloo et al. [29] pointed out that the existing method [26] cannot be used to find the maximal and minimal fuzzy solution of any type of system of fully fuzzy linear equations. Furthermore, to overcome this limitation of the existing method [26], Allahviranloo et al. [29] extended the existing method [26] to find the maximal and minimal

* The contents of this chapter are communicated for possible publication in Journal of Computational and Applied Mathematics.

fuzzy solution of the system of fully fuzzy linear equations (3.1).

$$\begin{aligned}
&\tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \dots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1, \\
&\tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \dots + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2, \\
&\quad \vdots \\
&\tilde{a}_{m1}\tilde{x}_1 + \tilde{a}_{m2}\tilde{x}_2 + \dots + \tilde{a}_{mn}\tilde{x}_n = \tilde{b}_m,
\end{aligned}$$

i.e., $\tilde{A}\tilde{X} = \tilde{b}$ or $\sum_{j=1}^n \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i$ for $i=1,2,\dots,m$, (3.1)

where $\tilde{a}_{ij}, \tilde{b}_i$ are non-negative/non-positive triangular fuzzy numbers and $\tilde{x}_j$ is a non-negative symmetric triangular fuzzy number.

In this section, the existing definitions [29] of maximal and minimal fuzzy solution of the system of fully fuzzy linear equations (3.1) are presented in the modified form by considering the assumption that $\tilde{x}_j$ is a non-symmetric triangular/trapezoidal fuzzy number.

Definition 3.1. The set $\{\tilde{X} = (x_j - \sigma_j, x_j, x_j + \sigma'_j), j=1,2,\dots,n : \tilde{A}\tilde{X} \subseteq \tilde{b}\}$ is called the maximal fuzzy solution of the system of fully fuzzy linear equations (3.1), if for any arbitrary set $\{\tilde{Z} = (z_j - \alpha_j, z_j, z_j + \alpha'_j), j=1,2,\dots,n : \tilde{A}\tilde{Z} \subseteq \tilde{b}\}$, we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_j = x_j, \alpha_j \leq \sigma_j, \alpha'_j \leq \sigma'_j$.

Definition 3.2. The set $\{\tilde{X} = (x_j - \sigma_j, x_j, x_j + \sigma'_j), j=1,2,\dots,n : \tilde{A}\tilde{X} \supseteq \tilde{b}\}$ is called the minimal fuzzy solution of the system of fully fuzzy linear equations (3.1), if for any arbitrary set $\{\tilde{Z} = (z_j - \alpha_j, z_j, z_j + \alpha'_j), j=1,2,\dots,n : \tilde{A}\tilde{Z} \supseteq \tilde{b}\}$, we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_j = x_j, \alpha_j \geq \sigma_j, \alpha'_j \geq \sigma'_j$.

Definition 3.3. If in the system of fully fuzzy linear equations (3.1), the elements $\tilde{x}_j$ of the unknown vector $\tilde{X}$ are non-symmetric trapezoidal fuzzy numbers, then the set $\{\tilde{X} = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j), j=1,2,\dots,n : \tilde{A}\tilde{X} \subseteq \tilde{b}\}$ is called the maximal fuzzy solution of the system of fully fuzzy linear equations (3.1), if for any arbitrary set

$\{\tilde{Z} = (z_j^L - \alpha_j, z_j^L, z_j^U, z_j^U + \alpha'_j), j=1,2,\dots,n: \tilde{A}\tilde{Z} \subseteq \tilde{b}\}$, we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_j^L = x_j^L, z_j^U = x_j^U$, $\alpha_j \leq \sigma_j, \alpha'_j \leq \sigma'_j$.

Definition 3.4. If in the system of fully fuzzy linear equations (3.1), the elements $\tilde{x}_j$ of the unknown vector $\tilde{X}$ are non-symmetric trapezoidal fuzzy numbers, then the set $\{\tilde{X} = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j), j=1,2,\dots,n: \tilde{A}\tilde{X} \supseteq \tilde{b}\}$ is called the minimal fuzzy solution of the system of fully fuzzy linear equations (3.1), if for any arbitrary set $\{\tilde{Z} = (z_j^L - \alpha_j, z_j^L, z_j^U, z_j^U + \alpha'_j), j=1,2,\dots,n: \tilde{A}\tilde{Z} \supseteq \tilde{b}\}$, we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_j^L = x_j^L, z_j^U = x_j^U$, $\alpha_j \geq \sigma_j, \alpha'_j \geq \sigma'_j$.

3.2. Allahviranloo et al. method

Allahviranloo et al. [29] proposed the following method to find the maximal and minimal fuzzy solution of the system of fully fuzzy linear equations (3.1).

Step 1: Since, $\tilde{a}_{ij}$ is either a non-negative triangular fuzzy number or a non-positive triangular fuzzy number. So, assuming

$$S = \{\tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-negative fuzzy number}\} \text{ and}$$

$$T = \{\tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-positive fuzzy number}\}, \text{ the system of fully fuzzy linear equations (3.1)}$$

can be transformed into the system of fully fuzzy linear equations (3.2).

$$\sum_{\tilde{a}_{ij} \in S} \tilde{a}_{ij} \tilde{x}_j + \sum_{\tilde{a}_{ij} \in T} \tilde{a}_{ij} \tilde{x}_j = \tilde{b}_i, i=1,2,\dots,m. \quad (3.2)$$

Step 2: Assuming

$\tilde{a}_{ij} = (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}), \tilde{x}_j = (x_j - \sigma_j, x_j, x_j + \sigma'_j), \tilde{b}_i = (b_i - \beta_{i1}, b_i, b_i + \beta_{i2})$, the system of fully fuzzy linear equations (3.2) can be transformed into the system of fully fuzzy linear equations (3.3).

$$\sum_{a_{ij} \in S} (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}) (x_j - \sigma_j, x_j, x_j + \sigma_j) + \sum_{a_{ij} \in T} (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}) (x_j - \sigma_j, x_j, x_j + \sigma_j) \\ = (b_i - \beta_{i1}, b_i, b_i + \beta_{i2}), i = 1, 2, \dots, m. \quad (3.3)$$

Step 3: Replacing the triangular fuzzy numbers $(a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij})$, $(x_j - \sigma_j, x_j, x_j + \sigma_j)$ and $(b_i - \beta_{i1}, b_i, b_i + \beta_{i2})$ by the corresponding r -cuts $[a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}]$, $[x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j]$ and $[b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}]$, the system of fully fuzzy linear equations (3.3) can be transformed into the system of interval equations (3.4).

$$\sum_{a_{ij} \in S} [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}] [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] + \sum_{a_{ij} \in T} [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}] \\ [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] = [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], i = 1, 2, \dots, m. \quad (3.4)$$

Step 4: Using the property $[a_1, a_2][b_1, b_2] = \begin{cases} [a_1b_1, a_2b_2]; a_1 \geq 0, b_1 \geq 0, \\ [a_1b_2, a_2b_1]; a_2 \leq 0, b_1 \geq 0. \end{cases}$

the system of interval equations (3.4) can be transformed into the system of interval equations (3.5).

$$\sum_{a_{ij} \in S} [(a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j), (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j)] + \\ \sum_{a_{ij} \in T} [(a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j), (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j)] = \\ [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], i = 1, 2, \dots, m. \quad (3.5)$$

Step 5: Using the property, $\sum_{i=1}^m [a_i, b_i] = \left[\sum_{i=1}^m a_i, \sum_{i=1}^m b_i \right]$, the system of interval equations (3.5)

can be transformed into the system of interval equations (3.6).

$$\begin{aligned}
& \left[\sum_{a_{ij} \in S} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j), \sum_{a_{ij} \in S} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) \right] + \\
& \left[\sum_{a_{ij} \in T} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j), \sum_{a_{ij} \in T} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) \right] = \\
& [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], i = 1, 2, \dots, m.
\end{aligned} \tag{3.6}$$

Step 6: Using the property, $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, the system of interval equations (3.6) can be transformed into the system of interval equations (3.7).

$$\begin{aligned}
& \left[\sum_{a_{ij} \in S} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j) + \sum_{a_{ij} \in T} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j), \right. \\
& \left. \sum_{a_{ij} \in S} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) + \sum_{a_{ij} \in T} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) \right] \\
& = [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}], i = 1, 2, \dots, m.
\end{aligned} \tag{3.7}$$

Step 7: Using the property, $[a_1, a_2] = [b_1, b_2] \Rightarrow a_1 = b_1$ and $a_2 = b_2$, the system of interval equations (3.7) can be transformed into the system of crisp linear equations (3.8) and (3.9).

$$\sum_{a_{ij} \in S} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j) + \sum_{a_{ij} \in T} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j) = b_i - \beta_{i1} + r\beta_{i1}, i = 1, 2, \dots, m. \tag{3.8}$$

$$\sum_{a_{ij} \in S} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) + \sum_{a_{ij} \in T} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) = b_i + \beta_{i2} - r\beta_{i2}, i = 1, 2, \dots, m. \tag{3.9}$$

Step 8: Solve the system of crisp linear equations (3.10), obtained by putting $r = 1$ in the system of crisp linear equations (3.8) and (3.9), to find the values of $x_j, j = 1, 2, \dots, n$.

$$\sum_{a_{ij} \in S} a_{ij}x_j + \sum_{a_{ij} \in T} a_{ij}x_j = b_i, i = 1, 2, \dots, m. \tag{3.10}$$

Step 9: Replacing σ_j of the i^{th} equation of the system of crisp linear equations (3.8) by σ_{i1} and σ_j of the i^{th} equation of the system of crisp linear equations (3.9) by σ_{i2} , these system of crisp linear equations are transformed into the system of crisp linear equations (3.11) and (3.12).

$$\sum_{a_{ij} \in S} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_{i1} + r\sigma_{i1}) + \sum_{a_{ij} \in T} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_{i1} - r\sigma_{i1}) = b_i - \beta_{i1} + r\beta_{i1}, i = 1, 2, \dots, m. \quad (3.11)$$

$$\sum_{a_{ij} \in S} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_{i2} - r\sigma_{i2}) + \sum_{a_{ij} \in T} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_{i2} + r\sigma_{i2}) = b_i + \beta_{i2} - r\beta_{i2}, i = 1, 2, \dots, m. \quad (3.12)$$

Step 10: Put the values of $x_j, j = 1, 2, \dots, n$, obtained in Step 8, in the system of crisp linear equations (3.11) and (3.12) and then solve these to find the values of σ_{i1} and σ_{i2} .

Step 11: Find

$$\alpha_L(r) = \underset{r}{\text{minimum}} \{ (1-r)|\sigma_{i1}(r)|, (1-r)|\sigma_{i2}(r)| \}, i = 1, 2, \dots, m, 0 \leq r < 1,$$

$$\alpha_U(r) = \underset{r}{\text{maximum}} \{ (1-r)|\sigma_{i1}(r)|, (1-r)|\sigma_{i2}(r)| \}, i = 1, 2, \dots, m, 0 \leq r < 1.$$

Step 12: Using the values of $x_j, j = 1, 2, \dots, n$, obtained in Step 8 and the values of $\alpha_L(r)$ and $\alpha_U(r)$, obtained in Step 11, the r -cut of the maximal fuzzy symmetric solution $\tilde{X}_L$ and the minimal fuzzy symmetric solution $\tilde{X}_U$ of the system of fully fuzzy linear equations (3.1) are $[x_j - \alpha_L(r), x_j + \alpha_L(r)]$ and $[x_j - \alpha_U(r), x_j + \alpha_U(r)]$ respectively.

3.3. Maximal and minimal fuzzy solution of the existing problem

Allahviranloo et al. [29] solved the system of fully fuzzy linear equations, presented in Example 3.1, to illustrate their proposed method. In this section, the same example is solved to illustrate the existing method [29].

Example 3.1 [29, Section 4, Example 4.1, pp. 4659]. Find the maximal and minimal fuzzy solution of the following system of fully fuzzy linear equations.

$$(4, 5, 6)\tilde{x}_1 + (5, 6, 8)\tilde{x}_2 = (40, 50, 67),$$

$$(6, 7, 7)\tilde{x}_1 + (4, 4, 5)\tilde{x}_2 = (43, 48, 55),$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative symmetric triangular fuzzy numbers.

Using the existing method [29], the maximal and minimal fuzzy solution of the

system of fully fuzzy linear equations, presented in Example 3.1, can be obtained as follows:

Step 1: Assuming

$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1)$ and $\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2)$, the system of fully fuzzy linear equations, presented in Example 3.1, can be transformed into the system of fully fuzzy linear equations (3.13).

$$\begin{aligned} (4, 5, 6)(x_1 - \sigma_1, x_1, x_1 + \sigma_1) + (5, 6, 8)(x_2 - \sigma_2, x_2, x_2 + \sigma_2) &= (40, 50, 67), \\ (6, 7, 7)(x_1 - \sigma_1, x_1, x_1 + \sigma_1) + (4, 4, 5)(x_2 - \sigma_2, x_2, x_2 + \sigma_2) &= (43, 48, 55). \end{aligned} \quad (3.13)$$

Step 2: Replacing all the triangular fuzzy numbers of the system of fully fuzzy linear equations (3.13) by their r -cuts, it can be transformed into the system of interval equations (3.14).

$$\begin{aligned} [4+r, 6-r][x_1 - \sigma_1 + r\sigma_1, x_1 + \sigma_1 - r\sigma_1] + [5+r, 8-2r][x_2 - \sigma_2 + r\sigma_2, x_2 + \sigma_2 - r\sigma_2] &= [40+10r, 67-17r], \\ [6+r, 7][x_1 - \sigma_1 + r\sigma_1, x_1 + \sigma_1 - r\sigma_1] + [4, 5-r][x_2 - \sigma_2 + r\sigma_2, x_2 + \sigma_2 - r\sigma_2] &= [43+5r, 55-7r]. \end{aligned} \quad (3.14)$$

Step 3: Using the property $[a_1, a_2][b_1, b_2] = \begin{cases} [a_1b_1, a_2b_2]; a_1 \geq 0, b_1 \geq 0, \\ [a_1b_2, a_2b_1]; a_2 \leq 0, b_1 \geq 0. \end{cases}$

the system of interval equations (3.14) can be transformed into the system of interval equations (3.15).

$$\begin{aligned} [(4+r)(x_1 - \sigma_1 + r\sigma_1), (6-r)(x_1 + \sigma_1 - r\sigma_1)] + [(5+r)(x_2 - \sigma_2 + r\sigma_2), (8-2r)(x_2 + \sigma_2 - r\sigma_2)] &= [40+10r, 67-17r], \\ [(6+r)(x_1 - \sigma_1 + r\sigma_1), 7(x_1 + \sigma_1 - r\sigma_1)] + [4(x_2 - \sigma_2 + r\sigma_2), (5-r)(x_2 + \sigma_2 - r\sigma_2)] &= [43+5r, 55-7r]. \end{aligned} \quad (3.15)$$

Step 4: Using the property, $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, the system of interval equations (3.15) can be transformed into the system of interval equations (3.16).

$$[(4+r)(x_1 - \sigma_1 + r\sigma_1) + (5+r)(x_2 - \sigma_2 + r\sigma_2), (6-r)(x_1 + \sigma_1 - r\sigma_1) + (8-2r)(x_2 + \sigma_2 - r\sigma_2)] = [40+10r, 67-17r],$$

$$\left[(6+r)(x_1 - \sigma_1 + r\sigma_1) + 4(x_2 - \sigma_2 + r\sigma_2), 7(x_1 + \sigma_1 - r\sigma_1) + (5-r)(x_2 + \sigma_2 - r\sigma_2) \right] = [43+5r, 55-7r]. \quad (3.16)$$

Step 5: Using the property, $[a_1, a_2] = [b_1, b_2] \Rightarrow a_1 = b_1$ and $a_2 = b_2$, the system of interval equations (3.16) can be transformed into the system of crisp linear equations (3.17) and (3.18).

$$\begin{aligned} (4+r)(x_1 - \sigma_1 + r\sigma_1) + (5+r)(x_2 - \sigma_2 + r\sigma_2) &= 40+10r, \\ (6+r)(x_1 - \sigma_1 + r\sigma_1) + 4(x_2 - \sigma_2 + r\sigma_2) &= 43+5r. \end{aligned} \quad (3.17)$$

$$\begin{aligned} (6-r)(x_1 + \sigma_1 - r\sigma_1) + (8-2r)(x_2 + \sigma_2 - r\sigma_2) &= 67-17r, \\ 7(x_1 + \sigma_1 - r\sigma_1) + (5-r)(x_2 + \sigma_2 - r\sigma_2) &= 55-7r. \end{aligned} \quad (3.18)$$

Step 6: On solving the system of crisp linear equations (3.19), obtained by putting $r=1$ in the system of crisp linear equations (3.17) and (3.18), the obtained values of x_1 and x_2 are 4 and 5 respectively.

$$\begin{aligned} 5x_1 + 6x_2 &= 50, \\ 7x_1 + 4x_2 &= 48. \end{aligned} \quad (3.19)$$

Step 7: Replacing σ_1 and σ_2 of the i^{th} equation of the system of crisp linear equations (3.17) by σ_{i1} and replacing σ_1 and σ_2 of the i^{th} equation of the system of crisp linear equations (3.18) by σ_{i2} respectively, these system of crisp linear equations are transformed into the system of crisp linear equations (3.20) and (3.21).

$$\begin{aligned} (4+r)(x_1 - \sigma_{11} + r\sigma_{11}) + (5+r)(x_2 - \sigma_{11} + r\sigma_{11}) &= 40+10r, \\ (6+r)(x_1 - \sigma_{21} + r\sigma_{21}) + 4(x_2 - \sigma_{21} + r\sigma_{21}) &= 43+5r. \end{aligned} \quad (3.20)$$

$$\begin{aligned} (6-r)(x_1 + \sigma_{12} - r\sigma_{12}) + (8-2r)(x_2 + \sigma_{12} - r\sigma_{12}) &= 67-17r, \\ 7(x_1 + \sigma_{22} - r\sigma_{22}) + (5-r)(x_2 + \sigma_{22} - r\sigma_{22}) &= 55-7r. \end{aligned} \quad (3.21)$$

Step 8: On solving the system of crisp linear equations (3.20) and (3.21) after putting the values of x_1 and x_2 , obtained in Step 6, the following values of $\sigma_{11}, \sigma_{21}, \sigma_{12}$ and σ_{22} are obtained.

$$\sigma_{11} = \frac{1}{9+2r}, \sigma_{21} = \frac{1}{10+r}, \sigma_{12} = \frac{3}{14-3r} \text{ and } \sigma_{22} = \frac{2}{12-r}.$$

Step 9: The values of

$$\alpha_L(r) = \underset{r}{\text{minimum}} \left\{ (1-r)|\sigma_{11}(r)|, (1-r)|\sigma_{21}(r)|, (1-r)|\sigma_{12}(r)|, (1-r)|\sigma_{22}(r)| \right\}, 0 \leq r < 1 \text{ and}$$

$$\alpha_U(r) = \underset{r}{\text{maximum}} \left\{ (1-r)|\sigma_{11}(r)|, (1-r)|\sigma_{21}(r)|, (1-r)|\sigma_{12}(r)|, (1-r)|\sigma_{22}(r)| \right\}, 0 \leq r < 1, \text{ are}$$

$$\frac{1-r}{10} \text{ and } \frac{3-3r}{14} \text{ respectively.}$$

Step 10: Using the values of x_1 and x_2 , obtained in Step 6, and the values of $\alpha_L(r)$ and $\alpha_U(r)$, obtained in Step 9, the r -cuts of the obtained maximal fuzzy symmetric solution $\tilde{X}_L$ and minimal fuzzy symmetric solution $\tilde{X}_U$ of the system of fully fuzzy linear

equations, presented in Example 3.1, are $\left[4 - \frac{1-r}{10}, 4 + \frac{1-r}{10} \right], \left[5 - \frac{1-r}{10}, 5 + \frac{1-r}{10} \right]$ and

$\left[4 - \frac{3-3r}{14}, 4 + \frac{3-3r}{14} \right], \left[5 - \frac{3-3r}{14}, 5 + \frac{3-3r}{14} \right]$ respectively.

3.4. Limitations of Allahviranloo et al. method

The method, proposed by Allahviranloo et al. [29], can be used only to find the maximal and minimal fuzzy solution of such system of fully fuzzy linear equations $\tilde{A}\tilde{X} = \tilde{b}$, in which all the elements of the unknown vector $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers, whereas, all the elements of the coefficient matrix $\tilde{A}$ as well as the elements of the right hand side vector $\tilde{b}$ are non-negative/non-positive triangular fuzzy numbers. For example, the existing system of fully fuzzy linear equations, presented in Example 3.1 [29, Section 4, Example 4.1, pp. 4659], is same type of system of fully fuzzy linear equations and hence can be solved by the existing method [29].

However, if in the system of fully fuzzy linear equations (3.1), the variables $\tilde{x}_j, j=1,2,\dots,n$, are assumed as non-negative non-symmetric triangular/trapezoidal fuzzy numbers instead of non-negative symmetric triangular fuzzy numbers, then, the maximal and

minimal fuzzy solution of the transformed system of fully fuzzy linear equations cannot be obtained by the existing method [29]. For example, if in the existing system of fully fuzzy linear equations, presented in Example 3.1 [29, Section 4, Example 4.1, pp. 4659], the variables $\tilde{x}_j$, $j=1,2$, are assumed as non-symmetric triangular fuzzy numbers instead of symmetric triangular fuzzy numbers, then the maximal and minimal fuzzy solution of the transformed system of fully fuzzy linear equations, presented in Example 3.2, cannot be obtained by the existing method [29].

Moreover, the existing method [29] cannot be used to find the maximal and minimal fuzzy solution of such system of fully fuzzy linear equations in which the elements of the unknown vector are either symmetric trapezoidal fuzzy numbers or non-symmetric trapezoidal fuzzy numbers. For example, the maximal and minimal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.3, cannot be obtained by the existing method [29].

Example 3.2. Find the maximal and minimal fuzzy solution of the following system of fully fuzzy linear equations.

$$\begin{aligned}(4, 5, 6)\tilde{x}_1 + (5, 6, 8)\tilde{x}_2 &= (40, 50, 67), \\ (6, 7, 7)\tilde{x}_1 + (4, 4, 5)\tilde{x}_2 &= (43, 48, 55),\end{aligned}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative non-symmetric triangular fuzzy numbers.

Example 3.3. Find the maximal and minimal fuzzy solution of the following system of fully fuzzy linear equations.

$$\begin{aligned}(4, 5, 5, 6)\tilde{x}_1 + (1, 3, 4, 5)\tilde{x}_2 &= (6, 25, 39, 65), \\ (1, 2, 3, 4)\tilde{x}_1 + (3, 5, 8, 9)\tilde{x}_2 &= (7, 29, 57, 83),\end{aligned}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative symmetric/non-symmetric trapezoidal fuzzy numbers.

3.5. Proposed Mehar-Keerat method-II

In this section, to overcome the limitations of the existing method [29], discussed in

Section 3.4, a new method (named as Mehar-Keerat method-II) is proposed to find the maximal and minimal fuzzy solution of the system of fully fuzzy linear equations (3.22),

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j = \tilde{b}_i, i=1, 2, \dots, m, \quad (3.22)$$

in which the elements of the unknown vector $\tilde{x}_j$ are non-negative trapezoidal fuzzy numbers and $\tilde{a}_{ij}, \tilde{b}_i$ are non-negative/non-positive trapezoidal fuzzy numbers.

The steps of proposed Mehar-Keerat method-II are as follows:

Step 1: Since, $\tilde{a}_{ij}$ is either a non-negative trapezoidal fuzzy number or a non-positive trapezoidal fuzzy number. So, assuming

$$S = \{ \tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-negative fuzzy number} \} \text{ and}$$

$T = \{ \tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-positive fuzzy number} \}$, the system of fully fuzzy linear equations (3.22) can be transformed into the system of fully fuzzy linear equations (3.23).

$$\sum_{\tilde{a}_{ij} \in S} \tilde{a}_{ij} \tilde{x}_j + \sum_{\tilde{a}_{ij} \in T} \tilde{a}_{ij} \tilde{x}_j = \tilde{b}_i, i=1, 2, \dots, m, \quad (3.23)$$

where $\tilde{x}_j$ is a non-negative trapezoidal fuzzy number.

Step 2: Assuming

$$\tilde{a}_{ij} = (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}), \tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j), \tilde{b}_i = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}),$$

the system of fully fuzzy linear equations (3.23) can be transformed into the system of fully fuzzy linear equations (3.24).

$$\sum_{a_{ij} \in S} (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}) (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) + \sum_{a_{ij} \in T} (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}) (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}), i=1, 2, \dots, m, \quad (3.24)$$

where, $(x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j)$ is a non-negative trapezoidal fuzzy number.

Step 3: Using the property [92],

$$(a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1)(a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2) = \begin{cases} ((a_1^L - \alpha_1)(a_2^L - \alpha_2), a_1^L a_2^L, a_1^U a_2^U, (a_1^U + \alpha'_1)(a_2^U + \alpha'_2)) \\ \text{if } a_1^L - \alpha_1 \geq 0, a_2^L - \alpha_2 \geq 0, \\ ((a_1^L - \alpha_1)(a_2^U + \alpha'_2), a_1^L a_2^U, a_1^U a_2^L, (a_1^U + \alpha'_1)(a_2^L - \alpha_2)) \\ \text{if } a_1^U + \alpha'_1 \leq 0, a_2^L - \alpha_2 \geq 0. \end{cases},$$

the system of fully fuzzy linear equations (3.24) can be transformed into the system of fully fuzzy linear equations (3.25).

$$\begin{aligned} & \sum_{a_{ij} \in S} ((a_{ij}^L - \alpha_{ij})(x_j^L - \sigma_j), (a_{ij}^L x_j^L), (a_{ij}^U x_j^U), (a_{ij}^U + \alpha'_{ij})(x_j^U + \sigma'_j)) + \\ & \sum_{a_{ij} \in T} ((a_{ij}^L - \alpha_{ij})(x_j^U + \sigma'_j), (a_{ij}^L x_j^U), (a_{ij}^U x_j^L), (a_{ij}^U + \alpha'_{ij})(x_j^L - \sigma_j)) = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}), i = 1, 2, \dots, m, \end{aligned} \quad (3.25)$$

where, $(x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j)$ is a non-negative trapezoidal fuzzy number.

Step 4: Using the property,

$$\sum_{i=1}^n (a_i^L - \alpha_i, a_i^L, a_i^U, a_i^U + \alpha'_i) = \left(\sum_{i=1}^n a_i^L - \alpha_i, \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^U, \sum_{i=1}^n a_i^U + \alpha'_i \right), \text{ the system of fully fuzzy}$$

linear equations (3.25) can be transformed into the system of fully fuzzy linear equations (3.26).

$$\begin{aligned} & \left(\sum_{a_{ij} \in S} (a_{ij}^L - \alpha_{ij})(x_j^L - \sigma_j), \sum_{a_{ij} \in S} a_{ij}^L x_j^L, \sum_{a_{ij} \in S} a_{ij}^U x_j^U, \sum_{a_{ij} \in S} (a_{ij}^U + \alpha'_{ij})(x_j^U + \sigma'_j) \right) + \\ & \left(\sum_{a_{ij} \in T} (a_{ij}^L - \alpha_{ij})(x_j^U + \sigma'_j), \sum_{a_{ij} \in T} a_{ij}^L x_j^U, \sum_{a_{ij} \in T} a_{ij}^U x_j^L, \sum_{a_{ij} \in T} (a_{ij}^U + \alpha'_{ij})(x_j^L - \sigma_j) \right) \\ & = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}), i = 1, 2, \dots, m, \end{aligned} \quad (3.26)$$

where, $(x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j)$ is a non-negative trapezoidal fuzzy number.

Step 5: Using the property,

$$\begin{aligned} & (a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1) + (a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2) \\ & = (a_1^L - \alpha_1 + a_2^L - \alpha_2, a_1^L + a_2^L, a_1^U + a_2^U, a_1^U + \alpha'_1 + a_2^U + \alpha'_2), \end{aligned}$$

the system of fully fuzzy linear equations (3.26) can be transformed into the system of fully fuzzy linear equations (3.27).

$$\left(\begin{array}{l} \sum_{a_{ij} \in S} (a_{ij}^L - \alpha_{ij}) (x_j^L - \sigma_j) + \sum_{a_{ij} \in T} (a_{ij}^L - \alpha_{ij}) (x_j^U + \sigma'_j), \sum_{a_{ij} \in S} a_{ij}^L x_j^L + \sum_{a_{ij} \in T} a_{ij}^L x_j^U, \sum_{a_{ij} \in S} a_{ij}^U x_j^U + \sum_{a_{ij} \in T} a_{ij}^U x_j^L, \\ \sum_{a_{ij} \in S} (a_{ij}^U + \alpha'_{ij}) (x_j^U + \sigma'_j) + \sum_{a_{ij} \in T} (a_{ij}^U + \alpha'_{ij}) (x_j^L - \sigma_j) \end{array} \right)$$

$$= (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}), i = 1, 2, \dots, m, \quad (3.27)$$

where, $(x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j)$ is a non-negative trapezoidal fuzzy number.

Step 6: Using Definition 3.3 (definition of maximal fuzzy solution), the system of fully fuzzy linear equations (3.27) can be transformed into multi-objective mathematical programming problem (3.28).

Maximize $\sigma_j; j = 1, 2, \dots, n,$

Maximize $\sigma'_j; j = 1, 2, \dots, n,$

subject to

$$\left(\sum_{a_{ij} \in S} a_{ij}^L x_j^L + \sum_{a_{ij} \in T} a_{ij}^L x_j^U \right) - \left(\sum_{a_{ij} \in S} (a_{ij}^L - \alpha_{ij}) (x_j^L - \sigma_j) + \sum_{a_{ij} \in T} (a_{ij}^L - \alpha_{ij}) (x_j^U + \sigma'_j) \right) \leq \beta_{i1}; i = 1, 2, \dots, m,$$

$$\sum_{a_{ij} \in S} a_{ij}^L x_j^L + \sum_{a_{ij} \in T} a_{ij}^L x_j^U = b_i^L; i = 1, 2, \dots, m,$$

$$\sum_{a_{ij} \in S} a_{ij}^U x_j^U + \sum_{a_{ij} \in T} a_{ij}^U x_j^L = b_i^U; i = 1, 2, \dots, m, \quad (3.28)$$

$$\left(\sum_{a_{ij} \in S} (a_{ij}^U + \alpha'_{ij}) (x_j^U + \sigma'_j) + \sum_{a_{ij} \in T} (a_{ij}^U + \alpha'_{ij}) (x_j^L - \sigma_j) \right) - \left(\sum_{a_{ij} \in S} a_{ij}^U x_j^U + \sum_{a_{ij} \in T} a_{ij}^U x_j^L \right) \leq \beta_{i2}; i = 1, 2, \dots, m,$$

$$0 \leq x_j^L - \sigma_j \leq x_j^L \leq x_j^U \leq x_j^U + \sigma'_j \text{ and } \sigma_j \geq 0, \sigma'_j \geq 0.$$

Step 7: Using Definition 3.4 (definition of minimal fuzzy solution), the system of fully fuzzy linear equations (3.27) can be transformed into multi-objective mathematical programming problem (3.29).

Minimize $\sigma_j; j = 1, 2, \dots, n,$

Minimize $\sigma'_j; j = 1, 2, \dots, n,$

subject to

$$\left(\sum_{a_{ij} \in S} a_{ij}^L x_j^L + \sum_{a_{ij} \in T} a_{ij}^L x_j^U \right) - \left(\sum_{a_{ij} \in S} (a_{ij}^L - \alpha_{ij}) (x_j^L - \sigma_j) + \sum_{a_{ij} \in T} (a_{ij}^L - \alpha_{ij}) (x_j^U + \sigma'_j) \right) \geq \beta_{i1}; i = 1, 2, \dots, m,$$

$$\sum_{a_{ij} \in S} a_{ij}^L x_j^L + \sum_{a_{ij} \in T} a_{ij}^L x_j^U = b_i^L; i = 1, 2, \dots, m,$$

$$\sum_{a_{ij} \in S} a_{ij}^U x_j^U + \sum_{a_{ij} \in T} a_{ij}^U x_j^L = b_i^U; i = 1, 2, \dots, m, \quad (3.29)$$

$$\left(\sum_{a_{ij} \in S} (a_{ij}^U + \alpha'_{ij})(x_j^U + \sigma'_j) + \sum_{a_{ij} \in T} (a_{ij}^U + \alpha'_{ij})(x_j^L - \sigma_j) \right) - \left(\sum_{a_{ij} \in S} a_{ij}^U x_j^U + \sum_{a_{ij} \in T} a_{ij}^U x_j^L \right) \geq \beta_{i2}; i = 1, 2, \dots, m,$$

$$0 \leq x_j^L - \sigma_j \leq x_j^L \leq x_j^U \leq x_j^U + \sigma'_j \text{ and } \sigma_j \geq 0, \sigma'_j \geq 0.$$

Step 8: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (3.28) can be transformed into single-objective mathematical programming problem (3.30).

Maximize $(\sigma_j + \sigma'_j); j = 1, 2, \dots, n,$

subject to

Constraints of Problem (3.28). (3.30)

Step 9: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (3.29) can be transformed into single-objective mathematical programming problem (3.31).

Minimize $(\sigma_j + \sigma'_j); j = 1, 2, \dots, n,$

subject to

Constraints of Problem (3.29). (3.31)

Step 10: Find the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (3.30).

Step 11: Find the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (3.31).

Step 12: Using the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (3.30), obtained in Step 10, the maximal fuzzy solution of the system of fully fuzzy linear equations (3.22) is

$$\{\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j); j = 1, 2, \dots, n\}.$$

Step 13: Using the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical

programming problem (3.31), obtained in Step 11, the minimal fuzzy solution of the system of fully fuzzy linear equations (3.22) is

$$\{\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j); j = 1, 2, \dots, n\}.$$

Remark 3.1. The maximal and minimal fuzzy symmetric solution of the system of fully fuzzy linear equations (3.22) can be obtained by solving the mathematical programming problems (3.30) and (3.31) respectively, with the additional constraints $\sigma_j = \sigma'_j; j = 1, 2, \dots, n$.

3.6. Illustrative examples

In this section, to illustrate the proposed Mehar-Keerat method-II, the system of fully fuzzy linear equations, chosen in Example 3.2 and Example 3.3 which cannot be solved by the existing method [29], are solved by proposed Mehar-Keerat method-II. Also, the maximal and minimal fuzzy solution of system of fully fuzzy linear equations, presented in Example 3.1, is obtained.

3.6.1. Maximal and minimal fuzzy solution of system of fully fuzzy linear equations chosen in Example 3.1 and Example 3.2

Using Mehar-Keerat method-II, proposed in Section 3.5, the maximal and minimal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.1 and Example 3.2, can be obtained as follows.

Step 1: Assuming

$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1)$ and $\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2)$, the system of fully fuzzy linear equations, chosen in Example 3.2, can be transformed into the system of fully fuzzy linear equations (3.32).

$$\begin{aligned} (4, 5, 6)(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) + (5, 6, 8)(x_2 - \sigma_2, x_2, x_2 + \sigma'_2) &= (40, 50, 67), \\ (6, 7, 7)(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) + (4, 4, 5)(x_2 - \sigma_2, x_2, x_2 + \sigma'_2) &= (43, 48, 55), \end{aligned} \quad (3.32)$$

Step 2: Using Step 2 to Step 5 of the proposed Mehar-Keerat method-II, the system of fully fuzzy linear equations (3.32) can be transformed into the system of fully fuzzy linear equations (3.33).

$$\begin{aligned}(4x_1 - 4\sigma_1 + 5x_2 - 5\sigma_2, 5x_1 + 6x_2, 6x_1 + 6\sigma'_1 + 8x_2 + 8\sigma'_2) &= (40, 50, 67), \\ (6x_1 - 6\sigma_1 + 4x_2 - 4\sigma_2, 7x_1 + 4x_2, 7x_1 + 7\sigma'_1 + 5x_2 + 5\sigma'_2) &= (43, 48, 55),\end{aligned}\tag{3.33}$$

Step 3: Using Step 6 of the proposed Mehar-Keerat method-II, the system of fully fuzzy linear equations (3.33) can be transformed into multi-objective mathematical programming problem (3.34).

$$\begin{aligned}&\text{Maximize } \sigma_1, \\&\text{Maximize } \sigma'_1, \\&\text{Maximize } \sigma_2, \\&\text{Maximize } \sigma'_2, \\&\text{subject to} \\&x_1 + x_2 + 4\sigma_1 + 5\sigma_2 \leq 10, \\&5x_1 + 6x_2 = 50, \\&x_1 + 2x_2 + 6\sigma'_1 + 8\sigma'_2 \leq 17, \\&x_1 + 6\sigma_1 + 4\sigma_2 \leq 5, \\&7x_1 + 4x_2 = 48, \\&x_2 + 7\sigma'_1 + 5\sigma'_2 \leq 7, \\&0 \leq x_1 - \sigma_1 \leq x_1 \leq x_1 + \sigma'_1, 0 \leq x_2 - \sigma_2 \leq x_2 \leq x_2 + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.\end{aligned}\tag{3.34}$$

Step 4: Using Step 7 of the proposed Mehar-Keerat method-II, the system of fully fuzzy linear equations (3.33) can be transformed into multi-objective mathematical programming problem (3.35).

$$\begin{aligned}&\text{Minimize } \sigma_1, \\&\text{Minimize } \sigma'_1, \\&\text{Minimize } \sigma_2, \\&\text{Minimize } \sigma'_2, \\&\text{subject to} \\&x_1 + x_2 + 4\sigma_1 + 5\sigma_2 \geq 10, \\&5x_1 + 6x_2 = 50, \\&x_1 + 2x_2 + 6\sigma'_1 + 8\sigma'_2 \geq 17, \\&x_1 + 6\sigma_1 + 4\sigma_2 \geq 5,\end{aligned}\tag{3.35}$$

$$7x_1 + 4x_2 = 48,$$

$$x_2 + 7\sigma'_1 + 5\sigma'_2 \geq 7,$$

$$0 \leq x_1 - \sigma_1 \leq x_1 \leq x_1 + \sigma'_1, 0 \leq x_2 - \sigma_2 \leq x_2 \leq x_2 + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.$$

Step 5: Using Step 8 of the proposed Mehar-Keerat method-II, the multi-objective mathematical programming problem (3.34) can be transformed into single-objective mathematical programming problem (3.36).

$$\text{Maximize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2)$$

subject to

$$\text{Constraints of Problem (3.34).} \quad (3.36)$$

Step 6: Using Step 9 of the proposed Mehar-Keerat method-II, the multi-objective mathematical programming problem (3.35) can be transformed into single-objective mathematical programming problem (3.37).

$$\text{Minimize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2)$$

subject to

$$\text{Constraints of Problem (3.35).} \quad (3.37)$$

Step 7: The optimal solution $\{x_1, x_2, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (3.36)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{1}{14}, \sigma'_1 = \frac{1}{14}, \sigma_2 = \frac{1}{7}, \sigma'_2 = \frac{1}{7}, x_1 = 4, x_2 = 5.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{1}{10}, \sigma'_1 = \frac{1}{10}, \sigma_2 = \frac{1}{10}, \sigma'_2 = \frac{1}{10}, x_1 = 4, x_2 = 5.$$

(iii) without any additional constraint is

$$\sigma_1 = \frac{1}{14}, \sigma'_1 = \frac{1}{26}, \sigma_2 = \frac{1}{7}, \sigma'_2 = \frac{9}{26}, x_1 = 4, x_2 = 5.$$

Step 8: The optimal solution $\{x_1, x_2, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical

programming problem (3.37)

(i) with the additional constraints $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$ is

$$\sigma_1 = \frac{1}{26}, \sigma_1' = \frac{1}{26}, \sigma_2 = \frac{9}{26}, \sigma_2' = \frac{9}{26}, x_1 = 4, x_2 = 5.$$

(ii) with the additional constraints $\sigma_1 = \sigma_1' = \sigma_2 = \sigma_2'$ is

$$\sigma_1 = \frac{3}{14}, \sigma_1' = \frac{3}{14}, \sigma_2 = \frac{3}{14}, \sigma_2' = \frac{3}{14}, x_1 = 4, x_2 = 5.$$

(iii) without any additional constraint is

$$\sigma_1 = \frac{1}{14}, \sigma_1' = \frac{1}{26}, \sigma_2 = \frac{1}{7}, \sigma_2' = \frac{9}{26}, x_1 = 4, x_2 = 5.$$

Step 9: Using optimal solution, obtained in Step 7, the maximal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.1

(i) with the additional constraints $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$ is

$$\begin{aligned} \tilde{x}_1 &= (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{55}{14}, 4, \frac{57}{14} \right), \\ \tilde{x}_2 &= (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{34}{7}, 5, \frac{36}{7} \right). \end{aligned}$$

(ii) with the additional constraints $\sigma_1 = \sigma_1' = \sigma_2 = \sigma_2'$ is

$$\begin{aligned} \tilde{x}_1 &= (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{39}{10}, 4, \frac{41}{10} \right), \\ \tilde{x}_2 &= (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{49}{10}, 5, \frac{51}{10} \right). \end{aligned}$$

Also, using the optimal solution, obtained in Step 7 without any additional constraint, the maximal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.2,

$$\text{is } \tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{55}{14}, 4, \frac{105}{26} \right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{34}{7}, 5, \frac{139}{26} \right).$$

Step 10: Using optimal solution, obtained in Step 8, the minimal fully fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.1

(i) with the additional constraints $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$ is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{103}{26}, 4, \frac{105}{26} \right),$$

$$\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{121}{26}, 5, \frac{139}{26} \right).$$

(ii) with the additional constraints $\sigma_1 = \sigma_1' = \sigma_2 = \sigma_2'$ is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{53}{14}, 4, \frac{59}{14} \right),$$

$$\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{67}{14}, 5, \frac{73}{14} \right).$$

Also, using the optimal solution, obtained in Step 8 without any additional constraint, the minimal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.2,

$$\text{is } \tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{55}{14}, 4, \frac{105}{26} \right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{34}{7}, 5, \frac{139}{26} \right).$$

3.6.2. Maximal and minimal fuzzy solution of system of fully fuzzy linear equations chosen in Example 3.3

Using Mehar-Keerat method-II, proposed in Section 3.5, the maximal and minimal fuzzy solution of system of fully fuzzy linear equations, chosen in Example 3.3, can be obtained as follows.

Step 1: Assuming

$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1')$ and $\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2')$, the system of fully fuzzy linear equations, chosen in Example 3.3, can be transformed into the system of fully fuzzy linear equations (3.38).

$$\begin{aligned}
(4, 5, 5, 6) & (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') + (1, 3, 4, 5) (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (6, 25, 39, 65), \\
(1, 2, 3, 4) & (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') + (3, 5, 8, 9) (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (7, 29, 57, 83).
\end{aligned} \tag{3.38}$$

Step 2: Using Step 2 to Step 5 of the proposed Mehar-Keerat method-II, the system of fully fuzzy linear equations (3.38) can be transformed into the system of fully fuzzy linear equations (3.39).

$$\begin{aligned}
(4x_1^L - 4\sigma_1 + x_2^L - \sigma_2, 5x_1^L + 3x_2^L, 5x_1^U + 4x_2^U, 6x_1^U + 6\sigma_1' + 5x_2^U + 5\sigma_2') &= (6, 25, 39, 65), \\
(x_1^L - \sigma_1 + 3x_2^L - 3\sigma_2, 2x_1^L + 5x_2^L, 3x_1^U + 8x_2^U, 4x_1^U + 4\sigma_1' + 9x_2^U + 9\sigma_2') &= (7, 29, 57, 83).
\end{aligned} \tag{3.39}$$

Step 3: Using Step 6 of the proposed Mehar-Keerat method-II, the system of fully fuzzy linear equations (3.39) can be transformed into multi-objective mathematical programming problem (3.40).

$$\begin{aligned}
& \text{Maximize } \sigma_1, \\
& \text{Maximize } \sigma_1', \\
& \text{Maximize } \sigma_2, \\
& \text{Maximize } \sigma_2', \\
& \text{subject to} \\
& x_1^L + 2x_2^L + 4\sigma_1 + \sigma_2 \leq 19, \\
& 5x_1^L + 3x_2^L = 25, \\
& 5x_1^U + 4x_2^U = 39, \\
& x_1^U + x_2^U + 6\sigma_1' + 5\sigma_2' \leq 26, \\
& x_1^L + 2x_2^L + \sigma_1 + 3\sigma_2 \leq 22, \\
& 2x_1^L + 5x_2^L = 29, \\
& 3x_1^U + 8x_2^U = 57, \\
& x_1^U + x_2^U + 4\sigma_1' + 9\sigma_2' \leq 26, \\
& 0 \leq x_1^L - \sigma_1 \leq x_1^L \leq x_1^U \leq x_1^U + \sigma_1', 0 \leq x_2^L - \sigma_2 \leq x_2^L \leq x_2^U \leq x_2^U + \sigma_2', \sigma_1 \geq 0, \sigma_1' \geq 0, \sigma_2 \geq 0, \sigma_2' \geq 0.
\end{aligned} \tag{3.40}$$

Step 4: Using Step 7 of the proposed Mehar-Keerat method-I, the system of fully fuzzy linear equations (3.39) can be transformed into multi-objective mathematical programming problem (3.41).

$$\begin{aligned}
& \text{Minimize } \sigma_1, \\
& \text{Minimize } \sigma'_1, \\
& \text{Minimize } \sigma_2, \\
& \text{Minimize } \sigma'_2, \\
& \text{subject to} \\
& x_1^L + 2x_2^L + 4\sigma_1 + \sigma_2 \geq 19, \\
& 5x_1^L + 3x_2^L = 25, \\
& 5x_1^U + 4x_2^U = 39, \\
& x_1^U + x_2^U + 6\sigma'_1 + 5\sigma'_2 \geq 26, \\
& x_1^L + 2x_2^L + \sigma_1 + 3\sigma_2 \geq 22, \\
& 2x_1^L + 5x_2^L = 29, \\
& 3x_1^U + 8x_2^U = 57, \\
& x_1^U + x_2^U + 4\sigma'_1 + 9\sigma'_2 \geq 26, \\
& 0 \leq x_1^L - \sigma_1 \leq x_1^L \leq x_1^U \leq x_1^U + \sigma'_1, 0 \leq x_2^L - \sigma_2 \leq x_2^L \leq x_2^U \leq x_2^U + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.
\end{aligned} \tag{3.41}$$

Step 5: Using Step 8 of the proposed Mehar-Keerat method-II, the multi-objective mathematical programming problem (3.40) can be transformed into single-objective mathematical programming problem (3.42).

$$\begin{aligned}
& \text{Maximize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
& \text{subject to} \\
& \text{Constraints of Problem (3.40)}.
\end{aligned} \tag{3.42}$$

Step 6: Using Step 9 of the proposed Mehar-Keerat method-II, the multi-objective mathematical programming problem (3.41) can be transformed into single-objective mathematical programming problem (3.43).

$$\begin{aligned}
& \text{Minimize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
& \text{subject to} \\
& \text{Constraints of Problem (3.41)}.
\end{aligned} \tag{3.43}$$

Step 7: The optimal solution $\{x_1^L, x_1^U, x_2^L, x_2^U, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (3.42)

- (i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{23}{16}, \sigma'_1 = \frac{23}{16}, \sigma_2 = \frac{5}{4}, \sigma'_2 = \frac{5}{4}, x_1^L = 2, x_1^U = 3, x_2^L = 5, x_2^U = 6.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = \frac{17}{13}, \sigma'_1 = \frac{17}{13}, \sigma_2 = \frac{17}{13}, \sigma'_2 = \frac{17}{13}, x_1^L = 2, x_1^U = 3, x_2^L = 5, x_2^U = 6.$$

(iii) without any additional constraint is

$$\sigma_1 = 1, \sigma'_1 = 2, \sigma_2 = 3, \sigma'_2 = 1, x_1^L = 2, x_1^U = 3, x_2^L = 5, x_2^U = 6.$$

Step 8: The optimal solution $\{x_1^L, x_1^U, x_2^L, x_2^U, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2\}$, of the single-objective mathematical programming problem (3.43)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\sigma_1 = 0, \sigma'_1 = 0, \sigma_2 = 0, \sigma'_2 = 0, x_1^L = 2, x_1^U = 3, x_2^L = 5, x_2^U = 6.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\sigma_1 = 0, \sigma'_1 = 0, \sigma_2 = 0, \sigma'_2 = 0, x_1^L = 2, x_1^U = 3, x_2^L = 5, x_2^U = 6.$$

(iii) without any additional constraint is

$$\sigma_1 = 0, \sigma'_1 = 0, \sigma_2 = 0, \sigma'_2 = 0, x_1^L = 2, x_1^U = 3, x_2^L = 5, x_2^U = 6.$$

Step 9: Using optimal solution, obtained in Step 7, the maximal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.3

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma'_1) = \left(\frac{9}{16}, 2, 3, \frac{71}{16}\right),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma'_2) = \left(\frac{15}{4}, 5, 6, \frac{29}{4}\right).$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$ is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = \left(\frac{9}{13}, 2, 3, \frac{56}{13}\right),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = \left(\frac{48}{13}, 5, 6, \frac{95}{13}\right).$$

(iii) without any additional constraint is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = (1, 2, 3, 5),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (2, 5, 6, 7).$$

Step 10: Using optimal solution, obtained in Step 8, the minimal fuzzy solution of the system of fully fuzzy linear equations, chosen in Example 3.3

(i) with the additional constraints $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$ is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = (2, 2, 3, 3),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (5, 5, 6, 6).$$

(ii) with the additional constraints $\sigma_1 = \sigma_1' = \sigma_2 = \sigma_2'$ is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = (2, 2, 3, 3),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (5, 5, 6, 6).$$

(iii) without any additional constraint is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = (2, 2, 3, 3),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (5, 5, 6, 6).$$

3.7. Comparative study

The maximal and minimal fuzzy solution of the existing system of fully fuzzy linear equations and chosen system of fully fuzzy linear equations, obtained by the existing methods [26,29] as well as proposed Mehar-Keerat method-I and Mehar-Keerat method-II are shown in Table 3.1.

Table 3.1. Maximal and minimal fuzzy symmetric/non-symmetric solution of existing and chosen system of fully fuzzy linear equations.

Example	Maximal and minimal Fuzzy solution	Existing method [26]		Existing method [29]		Proposed Mehar-Keerat method-I		Proposed Mehar-Keerat method-II	
		Symmetric solution	Non-symmetric solution	Symmetric solution	Non-symmetric solution	Symmetric solution	Non-symmetric solution	Symmetric solution	Non-symmetric solution
Example 3.1 [29, Section 4, Example 4.1, pp. 4659], Example 3.2	Maximal fuzzy solution	N.A.	N.A.	$\tilde{x}_1 = \left(\frac{39}{10}, 4, \frac{41}{10}\right),$ $\tilde{x}_2 = \left(\frac{49}{10}, 5, \frac{51}{10}\right).$	N.A.	N.A.	N.A.	$\tilde{x}_1 = \left(\frac{39}{10}, 4, \frac{41}{10}\right),$ $\tilde{x}_2 = \left(\frac{49}{10}, 5, \frac{51}{10}\right).$	$\tilde{x}_1 = \left(\frac{55}{14}, 4, \frac{105}{26}\right),$ $\tilde{x}_2 = \left(\frac{34}{7}, 5, \frac{139}{26}\right).$
	Minimal fuzzy solution	N.A.	N.A.	$\tilde{x}_1 = \left(\frac{53}{14}, 4, \frac{59}{14}\right),$ $\tilde{x}_2 = \left(\frac{67}{14}, 5, \frac{73}{14}\right).$	N.A.	N.A.	N.A.	$\tilde{x}_1 = \left(\frac{53}{14}, 4, \frac{59}{14}\right),$ $\tilde{x}_2 = \left(\frac{67}{14}, 5, \frac{73}{14}\right).$	$\tilde{x}_1 = \left(\frac{55}{14}, 4, \frac{105}{26}\right),$ $\tilde{x}_2 = \left(\frac{34}{7}, 5, \frac{139}{26}\right).$
Example 3.3	Maximal fuzzy solution	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	$\tilde{x}_1 = \left(\frac{9}{13}, 2, 3, \frac{56}{13}\right),$ $\tilde{x}_2 = \left(\frac{48}{13}, 5, 6, \frac{95}{13}\right).$	$\tilde{x}_1 = (1, 2, 3, 5),$ $\tilde{x}_2 = (2, 5, 6, 7).$
	Minimal fuzzy solution	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	$\tilde{x}_1 = (2, 2, 3, 3),$ $\tilde{x}_2 = (5, 5, 6, 6).$	$\tilde{x}_1 = (2, 2, 3, 3),$ $\tilde{x}_2 = (5, 5, 6, 6).$

3.8. Conclusion

On the basis of present study, it can be concluded that the system of fully fuzzy linear equations whose maximal and minimal fuzzy solution can be obtained by the existing method [29], can also be obtained by the proposed Mehar-Keerat method-II. However, there exist several systems of fully fuzzy linear equations whose maximal and minimal fuzzy solution can be obtained by the proposed Mehar-Keerat method-II but cannot be obtained by the existing method [29]. Hence, it is better to use the proposed Mehar-Keerat method-II as compared to the existing method [29].

Chapter 4

Mehar-Keerat method-III to find maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations*

Ghomashi et al. [75] extended the existing method [29], to find maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \tilde{x}_j + \tilde{d}_i, i=1,2,\dots,m,$$
 in which $\tilde{x}_j$ are non-negative symmetric triangular fuzzy numbers and $\tilde{a}_{ij}, \tilde{c}_{ij}, \tilde{b}_i$ and $\tilde{d}_i$ are non-negative/non-positive triangular fuzzy numbers.

In this chapter, the flaws as well as limitations of the existing method [75] are pointed out. Also, to resolve the flaws as well as to overcome the limitations, a new method (named as Mehar-Keerat method-III) is proposed to find maximal and minimal fuzzy solution of such dual system of fully fuzzy linear equations in which all the parameters are trapezoidal fuzzy numbers. The proposed Mehar-Keerat method-III is illustrated with the help of numerical examples.

4.1. Maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations

Ghomashi et al. [75] pointed out that the existing method [29] cannot be used to find the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations. Furthermore, to overcome this limitation of the existing method [29], Ghomashi et al. [75] extended the existing method [29] to find the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.1).

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symmetric triangular fuzzy numbers and $\tilde{a}_{ij}, \tilde{c}_{ij}, \tilde{b}_i$ and $\tilde{d}_i$ are non-negative/non-positive triangular fuzzy numbers.

Definition 4.1. The set $\{\tilde{X} = (x_j - \sigma_j, x_j, x_j + \sigma'_j), j = 1, 2, \dots, n: \tilde{A}\tilde{X} + \tilde{b} \subseteq \tilde{C}\tilde{X} + \tilde{d}\}$ is called the maximal fuzzy solution of the dual system of fully fuzzy linear equations (4.1), if for any arbitrary set $\{\tilde{Z} = (z_j - \alpha_j, z_j, z_j + \alpha'_j), j = 1, 2, \dots, n: \tilde{A}\tilde{Z} + \tilde{b} \subseteq \tilde{C}\tilde{Z} + \tilde{d}\}$, we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_j = x_j, \alpha_j \leq \sigma_j, \alpha'_j \leq \sigma'_j$.

Definition 4.2. The set $\left\{\tilde{X}=\left(x_j-\sigma_j, x_j, x_j+\sigma'_j\right), j=1,2, \ldots, n: \tilde{A} \tilde{X}+\tilde{b} \supseteq \tilde{C} \tilde{X}+\tilde{d}\right\}$ is called the minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.1), if for any arbitrary set $\left\{\tilde{Z}=\left(z_j-\alpha_j, z_j, z_j+\alpha'_j\right), j=1,2, \ldots, n: \tilde{A} \tilde{Z}+\tilde{b} \supseteq \tilde{C} \tilde{Z}+\tilde{d}\right\}$, we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_j=x_j, \alpha_j \geq \sigma_j, \alpha'_j \geq \sigma'_j$.

Definition 4.3. If in the system of dual fully fuzzy linear equations (4.1), the elements $\tilde{x}_j$ of the unknown vector $\tilde{X}$ are trapezoidal fuzzy numbers, then the set $\{\tilde{X} = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma_j'), j = 1, 2, \dots, n : \tilde{A}\tilde{X} + \tilde{b} \subseteq \tilde{C}\tilde{X} + \tilde{d}\}$ is called the maximal fuzzy solution of the dual system of fully fuzzy linear equations (4.1), if for any arbitrary set

$\{\tilde{Z} = (z_j^L - \alpha_j, z_j^L, z_j^U, z_j^U + \alpha'_j), j=1, 2, \dots, n: \tilde{A}\tilde{Z} + \tilde{b} \subseteq \tilde{C}\tilde{Z} + \tilde{d}\}$, we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_j^L = x_j^L$, $z_j^U = x_j^U, \alpha_j \leq \sigma_j, \alpha'_j \leq \sigma'_j$.

Definition 4.4. If in the dual system of fully fuzzy linear equations (4.1), the elements $\tilde{x}_j$ of the unknown vector $\tilde{X}$ are trapezoidal fuzzy numbers, then the set $\{\tilde{X} = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j), j=1, 2, \dots, n: \tilde{A}\tilde{X} + \tilde{b} \supseteq \tilde{C}\tilde{X} + \tilde{d}\}$ is called the minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.1), if for any arbitrary set $\{\tilde{Z} = (z_j^L - \alpha_j, z_j^L, z_j^U, z_j^U + \alpha'_j), j=1, 2, \dots, n: \tilde{A}\tilde{Z} + \tilde{b} \supseteq \tilde{C}\tilde{Z} + \tilde{d}\}$, we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_j^L = x_j^L$, $z_j^U = x_j^U, \alpha_j \geq \sigma_j, \alpha'_j \geq \sigma'_j$.

4.2. Ghomashi et al. method

Ghomashi et al. [75] proposed the following method to find maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.1).

Step 1: Since, $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$ are either non-negative triangular fuzzy numbers or non-positive triangular fuzzy number. So, assuming

$$S_1 = \{\tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-negative fuzzy number}\}, S_2 = \{\tilde{c}_{ij} : \tilde{c}_{ij} \text{ is a non-negative fuzzy number}\},$$

$$T_1 = \{\tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-positive fuzzy number}\} \text{ and } T_2 = \{\tilde{c}_{ij} : \tilde{c}_{ij} \text{ is a non-positive fuzzy number}\},$$

the dual system of fully fuzzy linear equations (4.1) can be transformed into the dual system of fully fuzzy linear equations (4.2).

$$\sum_{\tilde{a}_{ij} \in S_1} \tilde{a}_{ij} \tilde{x}_j + \sum_{\tilde{a}_{ij} \in T_1} \tilde{a}_{ij} \tilde{x}_j + \tilde{b}_i = \sum_{\tilde{c}_{ij} \in S_2} \tilde{c}_{ij} \tilde{x}_j + \sum_{\tilde{c}_{ij} \in T_2} \tilde{c}_{ij} \tilde{x}_j + \tilde{d}_i, i=1, 2, \dots, m. \quad (4.2)$$

Step 2: Assuming

$$\tilde{a}_{ij} = (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}), \tilde{c}_{ij} = (c_{ij} - \gamma_{ij}, c_{ij}, c_{ij} + \gamma'_{ij}), \tilde{x}_j = (x_j - \sigma_j, x_j, x_j + \sigma'_j), \tilde{b}_i = (b_i - \beta_{i1}, b_i, b_i + \beta_{i2})$$

and $\tilde{d}_i = (d_i - \delta_{i1}, d_i, d_i + \delta_{i2})$, the dual system of fully fuzzy linear equations (4.2) can be transformed into the dual system of fully fuzzy linear equations (4.3).

$$\begin{aligned} & \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}) (x_j - \sigma_j, x_j, x_j + \sigma_j) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}) (x_j - \sigma_j, x_j, x_j + \sigma_j) \\ & + (b_i - \beta_{i1}, b_i, b_i + \beta_{i2}) = \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} - \gamma_{ij}, c_{ij}, c_{ij} + \gamma'_{ij}) (x_j - \sigma_j, x_j, x_j + \sigma_j) + \\ & \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} - \gamma_{ij}, c_{ij}, c_{ij} + \gamma'_{ij}) (x_j - \sigma_j, x_j, x_j + \sigma_j) + (d_i - \delta_{i1}, d_i, d_i + \delta_{i2}), i = 1, 2, \dots, m. \end{aligned} \quad (4.3)$$

Step 3: Replacing the triangular fuzzy numbers $(a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij})$, $(c_{ij} - \gamma_{ij}, c_{ij}, c_{ij} + \gamma'_{ij})$

$(x_j - \sigma_j, x_j, x_j + \sigma_j)$, $(b_i - \beta_{i1}, b_i, b_i + \beta_{i2})$ and $(d_i - \delta_{i1}, d_i, d_i + \delta_{i2})$ by the corresponding

r -cuts $[a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}]$, $[c_{ij} - \gamma_{ij} + r\gamma_{ij}, c_{ij} + \gamma'_{ij} - r\gamma'_{ij}]$, $[x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j]$,

$[b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}]$ and $[d_i - \delta_{i1} + r\delta_{i1}, d_i + \delta_{i2} - r\delta_{i2}]$, the dual system of fully fuzzy linear equations (4.3) can be transformed into the system of interval equations (4.4).

$$\begin{aligned} & \sum_{\tilde{a}_{ij} \in S_1} [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}] [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] + \sum_{\tilde{a}_{ij} \in T_1} [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}] \\ & [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] + [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}] = \sum_{\tilde{c}_{ij} \in S_2} [c_{ij} - \gamma_{ij} + r\gamma_{ij}, c_{ij} + \gamma'_{ij} - r\gamma'_{ij}] \\ & [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] + \sum_{\tilde{c}_{ij} \in T_2} [c_{ij} - \gamma_{ij} + r\gamma_{ij}, c_{ij} + \gamma'_{ij} - r\gamma'_{ij}] [x_j - \sigma_j + r\sigma_j, x_j + \sigma_j - r\sigma_j] + \\ & [d_i - \delta_{i1} + r\delta_{i1}, d_i + \delta_{i2} - r\delta_{i2}], i = 1, 2, \dots, m. \end{aligned} \quad (4.4)$$

Step 4: Using the property $[a_1, a_2][b_1, b_2] = \begin{cases} [a_1b_1, a_2b_2]; a_1 \geq 0, b_1 \geq 0, \\ [a_1b_2, a_2b_1]; a_2 \leq 0, b_1 \geq 0, \end{cases}$

the system of interval equations (4.4) can be transformed into the system of interval equations (4.5).

$$\begin{aligned}
& \sum_{\tilde{a}_{ij} \in S_1} \left[(a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j), (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) \right] + \\
& \sum_{\tilde{a}_{ij} \in T_1} \left[(a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j), (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) \right] + [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}] = \\
& \sum_{\tilde{c}_{ij} \in S_2} \left[(c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j - \sigma_j + r\sigma_j), (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j + \sigma_j - r\sigma_j) \right] + \\
& \sum_{\tilde{c}_{ij} \in T_2} \left[(c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j + \sigma_j - r\sigma_j), (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j - \sigma_j + r\sigma_j) \right] + [d_i - \delta_{i1} + r\delta_{i1}, d_i + \delta_{i2} - r\delta_{i2}], \\
& i = 1, 2, \dots, m.
\end{aligned} \tag{4.5}$$

Step 5: Using the property, $\sum_{i=1}^m [a_i, b_i] = \left[\sum_{i=1}^m a_i, \sum_{i=1}^m b_i \right]$, the system of interval equations (4.5)

can be transformed into the system of interval equations (4.6).

$$\begin{aligned}
& \left[\sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j), \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) \right] + \\
& \left[\sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j), \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) \right] + [b_i - \beta_{i1} + r\beta_{i1}, b_i + \beta_{i2} - r\beta_{i2}] = \\
& \left[\sum_{\tilde{c}_{ij} \in S_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j - \sigma_j + r\sigma_j), \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j + \sigma_j - r\sigma_j) \right] + \\
& \left[\sum_{\tilde{c}_{ij} \in T_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j + \sigma_j - r\sigma_j), \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j - \sigma_j + r\sigma_j) \right] + [d_i - \delta_{i1} + r\delta_{i1}, d_i + \delta_{i2} - r\delta_{i2}], \\
& i = 1, 2, \dots, m.
\end{aligned} \tag{4.6}$$

Step 6: Using the property, $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, the system of interval equations (4.6) can be transformed into the system of interval equations (4.7).

$$\begin{aligned}
& \left[\sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j) + b_i - \beta_{i1} + r\beta_{i1}, \right. \\
& \left. \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) + b_i + \beta_{i2} - r\beta_{i2} \right] = \\
& \left[\sum_{\tilde{c}_{ij} \in S_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j - \sigma_j + r\sigma_j) + \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j + \sigma_j - r\sigma_j) + d_i - \delta_{i1} + r\delta_{i1}, \right. \\
& \left. \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j + \sigma_j - r\sigma_j) + \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j - \sigma_j + r\sigma_j) + d_i + \delta_{i2} - r\delta_{i2} \right], \\
& i = 1, 2, \dots, m.
\end{aligned} \tag{4.7}$$

Step 7: Using the property, $[a_1, a_2] = [b_1, b_2] \Rightarrow a_1 = b_1$ and $a_2 = b_2$, the system of interval

equations (4.7) can be transformed into the system of crisp linear equations (4.8) and (4.9).

$$\begin{aligned} \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_j + r\sigma_j) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_j - r\sigma_j) + b_i - \beta_{i1} + r\beta_{i1} = \\ \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j - \sigma_j + r\sigma_j) + \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j + \sigma_j - r\sigma_j) + d_i - \delta_{i1} + r\delta_{i1}, i = 1, 2, \dots, m, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_j - r\sigma_j) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_j + r\sigma_j) + b_i + \beta_{i2} - r\beta_{i2} = \\ \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j + \sigma_j - r\sigma_j) + \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j - \sigma_j + r\sigma_j) + d_i + \delta_{i2} - r\delta_{i2}, i = 1, 2, \dots, m. \end{aligned} \quad (4.9)$$

Step 8: Solve the system of crisp linear equations (4.10), obtained by putting $r=1$ in the system of crisp linear equations (4.8) and (4.9), to find the values of $x_j, j = 1, 2, \dots, n$.

$$\sum_{\tilde{a}_{ij} \in S_1} a_{ij}x_j + \sum_{\tilde{a}_{ij} \in T_1} a_{ij}x_j + b_i = \sum_{\tilde{c}_{ij} \in S_2} c_{ij}x_j + \sum_{\tilde{c}_{ij} \in T_2} c_{ij}x_j + d_i, i = 1, 2, \dots, m. \quad (4.10)$$

Step 9: Replacing σ_j of the i^{th} equation of the system of crisp linear equations (4.8) by σ_{i1} and σ_j of the i^{th} equation of the system of crisp linear equations (4.9) by σ_{i2} , these system of crisp linear equations are transformed into the system of crisp linear equations (4.11) and (4.12).

$$\begin{aligned} \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j - \sigma_{i1} + r\sigma_{i1}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_j + \sigma_{i1} - r\sigma_{i1}) + b_i - \beta_{i1} + r\beta_{i1} = \\ \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j - \sigma_{i1} + r\sigma_{i1}) + \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} - \gamma_{ij} + r\gamma_{ij})(x_j + \sigma_{i1} - r\sigma_{i1}) + d_i - \delta_{i1} + r\delta_{i1}, i = 1, 2, \dots, m, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j + \sigma_{i2} - r\sigma_{i2}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_j - \sigma_{i2} + r\sigma_{i2}) + b_i + \beta_{i2} - r\beta_{i2} = \\ \sum_{\tilde{c}_{ij} \in S_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j + \sigma_{i2} - r\sigma_{i2}) + \sum_{\tilde{c}_{ij} \in T_2} (c_{ij} + \gamma'_{ij} - r\gamma'_{ij})(x_j - \sigma_{i2} + r\sigma_{i2}) + d_i + \delta_{i2} - r\delta_{i2}, i = 1, 2, \dots, m. \end{aligned} \quad (4.12)$$

Step 10: Put the values of $x_j, j = 1, 2, \dots, n$, obtained in Step 8, in the system of crisp linear equations (4.11) and (4.12) and then solve these to find the values of $\sigma_{i1}(r)$ and $\sigma_{i2}(r)$.

Step 11: Find

$$\alpha_s^{-l}(r) = (1-r) \text{minimum} \{|\sigma_{i1}(0)|, |\sigma_{i2}(0)|\}, i = 1, 2, \dots, m,$$

$$\alpha_s^{-u}(r) = (1-r) \text{maximum} \{|\sigma_{i1}(0)|, |\sigma_{i2}(0)|\}, i = 1, 2, \dots, m,$$

$$\alpha_s^{+l}(r) = (1-r) \text{minimum} \{|\sigma_{i1}(1)|, |\sigma_{i2}(1)|\}, i = 1, 2, \dots, m,$$

$$\alpha_s^{+u}(r) = (1-r) \text{maximum} \{|\sigma_{i1}(1)|, |\sigma_{i2}(1)|\}, i = 1, 2, \dots, m.$$

Step 12: Using the values of $x_j, j = 1, 2, \dots, n$, obtained in Step 8, and the values of

$\alpha_s^{-l}(r), \alpha_s^{-u}(r), \alpha_s^{+l}(r)$ and $\alpha_s^{+u}(r)$, obtained in Step 11, the r -cut of the maximal fuzzy

symmetric solution $\tilde{X}^{-u}, \tilde{X}^{+u}$ and the minimal fuzzy symmetric solution $\tilde{X}^{-l}, \tilde{X}^{+l}$ of the dual

system of fully fuzzy linear equations (4.1) are $[x_j - \alpha_s^{-u}(r), x_j + \alpha_s^{-u}(r)]$,

$[x_j - \alpha_s^{+u}(r), x_j + \alpha_s^{+u}(r)]$ and $[x_j - \alpha_s^{-l}(r), x_j + \alpha_s^{-l}(r)], [x_j - \alpha_s^{+l}(r), x_j + \alpha_s^{+l}(r)], j = 1, 2, \dots, n$

respectively.

4.3. Flaws of Ghomashi et al. method

Ghomashi et al. [75] solved the dual system of fully fuzzy linear equations, presented in Example 4.1, to illustrate their proposed method.

Example 4.1 [75, Section 5, pp. 375]. Find the maximal and minimal fuzzy solution of the following dual system of fully fuzzy linear equations.

$$\tilde{x}_1 = (-22, -20, -18)\tilde{x}_2 + (1200, 1230, 1250),$$

$$\tilde{x}_1 = (12, 13, 15)\tilde{x}_2 + (-250, -220, -200),$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative symmetric triangular fuzzy numbers.

Ghomashi et al. [75] claimed that on solving the dual system of fully fuzzy linear equations by their proposed method, the obtained values of $\sigma_{11}, \sigma_{21}, \sigma_{12}$ and σ_{22} are

$$\frac{-117.879}{21-2r}, \frac{-73.939}{11+r}, \frac{-107.879}{17+2r} \text{ and } \frac{-107.879}{14-2r} \text{ respectively. It is well known fact that the}$$

spreads σ and σ' of a triangular fuzzy number $(x - \sigma, x, x + \sigma')$ should always be non-

negative real numbers. However, it is obvious that the values of $\sigma_{11}, \sigma_{21}, \sigma_{12}$ and σ_{22} , obtained by Ghomashi et al. [75], are negative real numbers. To obtain non-negative spreads $\alpha_s^{-l}(r), \alpha_s^{-u}(r), \alpha_s^{+l}(r)$ and $\alpha_s^{+u}(r)$, Ghomashi et al. [75] have taken the modulus of the obtained values of $\sigma_{11}, \sigma_{21}, \sigma_{12}$ and σ_{22} . However, this transformation is not valid, e.g., let $x = 2, \sigma = -1$ and $\sigma' = -2$, then the actual number is $(x - \sigma, x, x + \sigma') = (3, 2, 0)$, which is not a triangular fuzzy number and the conclusion is that the problem has no feasible solution. However, according to Ghomashi et al. [75], the number is $(x - |\sigma|, x, x + |\sigma'|) = (1, 2, 4)$, which is a triangular fuzzy number but actually it is not the solution of the chosen problem. Although, this flaw of the existing method [75] can be easily resolved by the following modifications:

Use Phase-1 of Two-Phase method to solve the system of crisp linear equations (4.11) and (4.12) with the additional constraints $\sigma_{i1} \geq 0$ and $\sigma_{i2} \geq 0$. However, there are several limitations of Ghomashi et al. method [75] which are pointed out in next section. Therefore, in Section 4.5, a new method (named as Mehar-Keerat method III) is proposed to overcome the limitations as well as to resolve the flaws of the Ghomashi et al. method [75].

Remark 4.1. The same flaw may also be found in the existing method [29].

4.4. Limitations of Ghomashi et al. method

The method, proposed by Ghomashi et al. [75], can be used only to find the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.1), where, $\tilde{x}_j$ are non-negative symmetric triangular fuzzy numbers and $\tilde{a}_{ij}, \tilde{c}_{ij}, \tilde{b}_i$ and $\tilde{d}_i$ are non-negative/non-positive triangular fuzzy numbers.

For example, the existing dual system of fully fuzzy linear equations, presented in

Example 4.1 [75, Section 5, pp. 375], is same type of dual system of fully fuzzy linear equations and hence can be solved by the existing method [75].

However, if in the dual system of fully fuzzy linear equations (4.1), the variables $\tilde{x}_j, j=1,2,\dots,n$ are assumed as non-negative non-symmetric triangular/trapezoidal fuzzy numbers instead of non-negative symmetric triangular fuzzy numbers, then, the maximal and minimal fuzzy solution of the transformed dual system of fully fuzzy linear equations cannot be obtained by the existing method [75]. For example, if in the existing dual system of fully fuzzy linear equations, presented in Example 4.1 [75, Section 5, pp. 375], the variables $\tilde{x}_j, j=1,2$, are assumed as non-negative non-symmetric triangular fuzzy numbers instead of non-negative symmetric triangular fuzzy numbers, then the maximal and minimal fuzzy solution of the transformed dual system of fully fuzzy linear equations, presented in Example 4.2, cannot be obtained by the existing method [75].

Example 4.2. Find the maximal and minimal fuzzy solution of the following dual system of fully fuzzy linear equations.

$$\tilde{x}_1 = (-22, -20, -18)\tilde{x}_2 + (1200, 1230, 1250),$$

$$\tilde{x}_1 = (12, 13, 15)\tilde{x}_2 + (-250, -220, -200),$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are non-negative non-symmetric triangular fuzzy numbers.

Moreover, the existing method [75] cannot be used to find the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.13),

$$\tilde{A}\tilde{X} + \tilde{b} = \tilde{C}\tilde{X} + \tilde{d} \text{ or } \sum_{j=1}^n \tilde{a}_{ij}\tilde{x}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij}\tilde{x}_j + \tilde{d}_i, i=1,2,\dots,m, \quad (4.13)$$

where $\tilde{a}_{ij}, \tilde{c}_{ij}, \tilde{x}_j, \tilde{b}_i, \tilde{d}_i$ are unrestricted trapezoidal fuzzy numbers. For example, the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.3, cannot be obtained by the existing method [75].

Example 4.3. Find the maximal and minimal fuzzy solution of the following dual system of fully fuzzy linear equations.

$$\begin{aligned} (-4, -3, -2, 1)\tilde{x}_1 + (11, 11, 11, 21) &= (1, 2, 3, 5)\tilde{x}_2 + (0, 4, 8, 8), \\ (-7, -5, -4, -3)\tilde{x}_1 + (18, 18, 18, 24) &= (-2, 0, 1, 4)\tilde{x}_2 + (0, 13, 25, 25), \end{aligned}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are symmetric/non-symmetric unrestricted trapezoidal fuzzy numbers.

Remark 4.2. In the literature [94, Section 4, pp. 7145], the product of a known trapezoidal

fuzzy number $(a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1)$ with an unknown trapezoidal fuzzy number

$(a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2)$ is defined as

$$\begin{aligned} (a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1)(a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2) = & \begin{cases} \left(\min((a_1^L - \alpha_1)(a_2^L - \alpha_2), (a_2^L - \alpha_2)(a_1^U + \alpha'_1)), \min(a_1^L a_2^L, a_2^L a_1^U), \right. \\ \left. \max(a_1^U a_2^U, a_1^L a_2^U), \max((a_1^U + \alpha'_1)(a_2^U + \alpha'_2), (a_1^L - \alpha_1)(a_2^U + \alpha'_2)) \right), \\ a_1^L - \alpha_1 \geq 0, \\ \left(\min((a_1^L - \alpha_1)(a_2^U + \alpha'_2), (a_2^L - \alpha_2)(a_1^U + \alpha'_1)), \min(a_1^L a_2^L, a_2^L a_1^U), \right. \\ \left. \max(a_1^U a_2^U, a_1^L a_2^U), \max((a_1^U + \alpha'_1)(a_2^U + \alpha'_2), (a_1^L - \alpha_1)(a_2^L - \alpha_2)) \right), \\ a_1^L - \alpha_1 < 0, a_1^L \geq 0, \\ \left(\min((a_1^L - \alpha_1)(a_2^U + \alpha'_2), (a_2^L - \alpha_2)(a_1^U + \alpha'_1)), \min(a_1^L a_2^U, a_2^L a_1^U), \right. \\ \left. \max(a_1^U a_2^U, a_1^L a_2^L), \max((a_1^U + \alpha'_1)(a_2^U + \alpha'_2), (a_1^L - \alpha_1)(a_2^L - \alpha_2)) \right), \\ a_1^L - \alpha_1 < 0, a_1^L < 0, a_1^U \geq 0, \\ \left(\min((a_1^L - \alpha_1)(a_2^U + \alpha'_2), (a_2^L - \alpha_2)(a_1^U + \alpha'_1)), \min(a_1^L a_2^U, a_1^U a_2^U), \right. \\ \left. \max(a_1^U a_2^L, a_1^L a_2^L), \max((a_1^U + \alpha'_1)(a_2^U + \alpha'_2), (a_1^L - \alpha_1)(a_2^L - \alpha_2)) \right), \\ a_1^L - \alpha_1 < 0, a_1^L < 0, a_1^U < 0, a_1^U + \alpha'_1 \geq 0, \\ \left(\min((a_1^L - \alpha_1)(a_2^U + \alpha'_2), (a_1^U + \alpha'_1)(a_2^U + \alpha'_2)), \min(a_1^L a_2^U, a_1^U a_2^U), \right. \\ \left. \max(a_1^U a_2^L, a_1^L a_2^L), \max((a_1^U + \alpha'_1)(a_2^L - \alpha_2), (a_1^L - \alpha_1)(a_2^L - \alpha_2)) \right), \\ a_1^L - \alpha_1 < 0, a_1^L < 0, a_1^U < 0, a_1^U + \alpha'_1 < 0. \end{cases} \end{aligned}$$

$$\begin{aligned}
& \left(\left| \frac{(a_1^L - \alpha_1)(a_2^L - \alpha_2) + (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} - \left| \frac{(a_1^L - \alpha_1)(a_2^L - \alpha_2) - (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} \right|, \frac{a_1^L a_2^L + a_2^L a_1^U}{2} - \left| \frac{a_1^L a_2^L - a_2^L a_1^U}{2} \right|, \right. \\
& \left. \frac{a_1^U a_2^U + a_1^L a_2^U}{2} + \left| \frac{a_1^U a_2^U - a_1^L a_2^U}{2} \right|, \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) + (a_1^L - \alpha_1)(a_2^U + \alpha'_2)}{2} + \left| \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) - (a_1^L - \alpha_1)(a_2^U + \alpha'_2)}{2} \right| \right), \\
& a_1^L - \alpha_1 \geq 0, \\
& \left(\left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) + (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} - \left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) - (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} \right|, \frac{a_1^L a_2^L + a_2^L a_1^U}{2} - \left| \frac{a_1^L a_2^L - a_2^L a_1^U}{2} \right|, \right. \\
& \left. \frac{a_1^U a_2^U + a_1^L a_2^U}{2} + \left| \frac{a_1^U a_2^U - a_1^L a_2^U}{2} \right|, \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) + (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} + \left| \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) - (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} \right| \right), \\
& a_1^L - \alpha_1 < 0, a_1^L \geq 0, \\
& \left(\left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) + (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} - \left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) - (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} \right|, \frac{a_1^L a_2^U + a_2^L a_1^U}{2} - \left| \frac{a_1^L a_2^U - a_2^L a_1^U}{2} \right|, \right. \\
& \left. \frac{a_1^U a_2^U + a_1^L a_2^L}{2} + \left| \frac{a_1^U a_2^U - a_1^L a_2^L}{2} \right|, \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) + (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} + \left| \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) - (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} \right| \right), \\
& a_1^L - \alpha_1 < 0, a_1^L < 0, a_1^U \geq 0, \\
& \left(\left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) + (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} - \left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) - (a_2^L - \alpha_2)(a_1^U + \alpha'_1)}{2} \right|, \frac{a_1^L a_2^U + a_1^U a_2^U}{2} - \left| \frac{a_1^L a_2^U - a_1^U a_2^U}{2} \right|, \right. \\
& \left. \frac{a_1^U a_2^L + a_1^L a_2^L}{2} + \left| \frac{a_1^U a_2^L - a_1^L a_2^L}{2} \right|, \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) + (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} + \left| \frac{(a_1^U + \alpha'_1)(a_2^U + \alpha'_2) - (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} \right| \right), \\
& a_1^L - \alpha_1 < 0, a_1^L < 0, a_1^U < 0, a_1^U + \alpha'_1 \geq 0, \\
& \left(\left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) + (a_1^U + \alpha'_1)(a_2^U + \alpha'_2)}{2} - \left| \frac{(a_1^L - \alpha_1)(a_2^U + \alpha'_2) - (a_1^U + \alpha'_1)(a_2^U + \alpha'_2)}{2} \right|, \frac{a_1^L a_2^U + a_1^U a_2^U}{2} - \left| \frac{a_1^L a_2^U - a_1^U a_2^U}{2} \right|, \right. \\
& \left. \frac{a_1^U a_2^L + a_1^L a_2^L}{2} + \left| \frac{a_1^U a_2^L - a_1^L a_2^L}{2} \right|, \frac{(a_1^U + \alpha'_1)(a_2^L - \alpha_2) + (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} + \left| \frac{(a_1^U + \alpha'_1)(a_2^L - \alpha_2) - (a_1^L - \alpha_1)(a_2^L - \alpha_2)}{2} \right| \right), \\
& a_1^L - \alpha_1 < 0, a_1^L < 0, a_1^U < 0, a_1^U + \alpha'_1 < 0.
\end{aligned}$$

4.5. Proposed Mehar-Keerat method-III

In this section, to resolve the flaws of the existing method [75], pointed out in Section 4.3, as well as to overcome the limitations of the existing method [75], discussed in Section 4.4, a new method (named as Mehar-Keerat method-III) is proposed to find the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.13),

The steps of proposed Mehar-Keerat method-III are as follows:

Step 1: Assuming

$$\tilde{a}_{ij} = (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}), \tilde{c}_{ij} = (c_{ij}^L - \gamma_{ij}, c_{ij}^L, c_{ij}^U, c_{ij}^U + \gamma'_{ij}), \tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j),$$

$\tilde{b}_i = (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2})$ and $\tilde{d}_i = (d_i^L - \delta_{i1}, d_i^L, d_i^U, d_i^U + \delta_{i2})$, the dual system of fully fuzzy linear equations (4.13) can be transformed into the dual system of fully fuzzy linear equations (4.14).

$$\begin{aligned} \sum_{j=1}^n (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}) (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) + (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}) = \\ \sum_{j=1}^n (c_{ij}^L - \gamma_{ij}, c_{ij}^L, c_{ij}^U, c_{ij}^U + \gamma'_{ij}) (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) + (d_i^L - \delta_{i1}, d_i^L, d_i^U, d_i^U + \delta_{i2}), i = 1, 2, \dots, m. \end{aligned} \quad (4.14)$$

Step 2: Assuming

$$\begin{aligned} (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}) (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) &= (e_{ij}^L - \lambda_{ij}, e_{ij}^L, e_{ij}^U, e_{ij}^U + \lambda'_{ij}), \\ (c_{ij}^L - \gamma_{ij}, c_{ij}^L, c_{ij}^U, c_{ij}^U + \gamma'_{ij}) (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j) &= (f_{ij}^L - \mu_{ij}, f_{ij}^L, f_{ij}^U, f_{ij}^U + \mu'_{ij}), \end{aligned}$$

the dual system of fully fuzzy linear equations (4.14) can be transformed into the dual system of fully fuzzy linear equations (4.15).

$$\begin{aligned} \sum_{j=1}^n (e_{ij}^L - \lambda_{ij}, e_{ij}^L, e_{ij}^U, e_{ij}^U + \lambda'_{ij}) + (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}) = \sum_{j=1}^n (f_{ij}^L - \mu_{ij}, f_{ij}^L, f_{ij}^U, f_{ij}^U + \mu'_{ij}) + \\ (d_i^L - \delta_{i1}, d_i^L, d_i^U, d_i^U + \delta_{i2}), i = 1, 2, \dots, m. \end{aligned} \quad (4.15)$$

where, $(e_{ij}^L - \lambda_{ij}, e_{ij}^L, e_{ij}^U, e_{ij}^U + \lambda'_{ij}), (f_{ij}^L - \mu_{ij}, f_{ij}^L, f_{ij}^U, f_{ij}^U + \mu'_{ij})$ can be obtained by the existing product [94, Section 4, pp. 7145] of two trapezoidal fuzzy numbers $(a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1)$ and $(a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2)$, presented in Remark 4.2.

Step 3: Using the property,

$$\sum_{i=1}^n (a_i^L - \alpha_i, a_i^L, a_i^U, a_i^U + \alpha'_i) = \left(\sum_{i=1}^n a_i^L - \alpha_i, \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^U, \sum_{i=1}^n a_i^U + \alpha'_i \right),$$

the dual system of fully fuzzy linear equations (4.15) can be transformed into the dual system of fully fuzzy linear equations (4.16).

$$\left(\sum_{j=1}^n e_{ij}^L - \lambda_{ij}, \sum_{j=1}^n e_{ij}^L, \sum_{j=1}^n e_{ij}^U, \sum_{j=1}^n e_{ij}^U + \lambda'_{ij} \right) + (b_i^L - \beta_{i1}, b_i^L, b_i^U, b_i^U + \beta_{i2}) = \left(\sum_{j=1}^n f_{ij}^L - \mu_{ij}, \sum_{j=1}^n f_{ij}^L, \sum_{j=1}^n f_{ij}^U, \sum_{j=1}^n f_{ij}^U + \mu'_{ij} \right) + (d_i^L - \delta_{i1}, d_i^L, d_i^U, d_i^U + \delta_{i2}), i = 1, 2, \dots, m. \quad (4.16)$$

Step 4: Using the property

$$\begin{aligned} & (a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1) + (a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2) \\ & = (a_1^L - \alpha_1 + a_2^L - \alpha_2, a_1^L + a_2^L, a_1^U + a_2^U, a_1^U + \alpha'_1 + a_2^U + \alpha'_2), \end{aligned}$$

the dual system of fully fuzzy linear equations (4.16) can be transformed into the dual system of fully fuzzy linear equations (4.17).

$$\begin{aligned} & \left(\sum_{j=1}^n e_{ij}^L - \lambda_{ij} + b_i^L - \beta_{i1}, \sum_{j=1}^n e_{ij}^L + b_i^L, \sum_{j=1}^n e_{ij}^U + b_i^U, \sum_{j=1}^n e_{ij}^U + \lambda'_{ij} + b_i^U + \beta_{i2} \right) = \\ & \left(\sum_{j=1}^n f_{ij}^L - \mu_{ij} + d_i^L - \delta_{i1}, \sum_{j=1}^n f_{ij}^L + d_i^L, \sum_{j=1}^n f_{ij}^U + d_i^U, \sum_{j=1}^n f_{ij}^U + \mu'_{ij} + d_i^U + \delta_{i2} \right), i = 1, 2, \dots, m. \quad (4.17) \end{aligned}$$

Step 5: Using Definition 4.3 (definition of maximal fuzzy solution), the dual system of fully fuzzy linear equations (4.17) can be transformed into multi-objective mathematical programming problem (4.18).

Maximize $\sigma_j; j = 1, 2, \dots, n,$

Maximize $\sigma'_j; j = 1, 2, \dots, n,$

subject to

$$\begin{aligned} & \left(\sum_{j=1}^n e_{ij}^L + b_i^L \right) - \left(\sum_{j=1}^n e_{ij}^L - \lambda_{ij} + b_i^L - \beta_{i1} \right) \leq \left(\sum_{j=1}^n f_{ij}^L + d_i^L \right) - \left(\sum_{j=1}^n f_{ij}^L - \mu_{ij} + d_i^L - \delta_{i1} \right), i = 1, 2, \dots, m, \\ & \sum_{j=1}^n e_{ij}^L + b_i^L = \sum_{j=1}^n f_{ij}^L + d_i^L, i = 1, 2, \dots, m, \\ & \sum_{j=1}^n e_{ij}^U + b_i^U = \sum_{j=1}^n f_{ij}^U + d_i^U, i = 1, 2, \dots, m, \\ & \left(\sum_{j=1}^n e_{ij}^U + \lambda'_{ij} + b_i^U + \beta_{i2} \right) - \left(\sum_{j=1}^n e_{ij}^U + b_i^U \right) \leq \left(\sum_{j=1}^n f_{ij}^U + \mu'_{ij} + d_i^U + \delta_{i2} \right) - \left(\sum_{j=1}^n f_{ij}^U + d_i^U \right), i = 1, 2, \dots, m, \\ & x_j^L - \sigma_j \leq x_j^L \leq x_j^U \leq x_j^U + \sigma'_j \text{ and } \sigma_j \geq 0, \sigma'_j \geq 0. \end{aligned} \quad (4.18)$$

Step 6: Using Definition 4.4 (definition of minimal fuzzy solution), the system of fully fuzzy linear equations (4.17) can be transformed into multi-objective mathematical programming

problem (4.19).

Minimize $\sigma_j; j = 1, 2, \dots, n$,

Minimize $\sigma'_j; j = 1, 2, \dots, n$,

subject to

$$\begin{aligned}
& \left(\sum_{j=1}^n e_{ij}^L + b_i^L \right) - \left(\sum_{j=1}^n e_{ij}^L - \lambda_{ij} + b_i^L - \beta_{i1} \right) \geq \left(\sum_{j=1}^n f_{ij}^L + d_i^L \right) - \left(\sum_{j=1}^n f_{ij}^L - \mu_{ij} + d_i^L - \delta_{i1} \right), i = 1, 2, \dots, m, \\
& \sum_{j=1}^n e_{ij}^L + b_i^L = \sum_{j=1}^n f_{ij}^L + d_i^L, i = 1, 2, \dots, m, \\
& \sum_{j=1}^n e_{ij}^U + b_i^U = \sum_{j=1}^n f_{ij}^U + d_i^U, i = 1, 2, \dots, m, \\
& \left(\sum_{j=1}^n e_{ij}^U + \lambda'_{ij} + b_i^U + \beta_{i2} \right) - \left(\sum_{j=1}^n e_{ij}^U + b_i^U \right) \geq \left(\sum_{j=1}^n f_{ij}^U + \mu'_{ij} + d_i^U + \delta_{i2} \right) - \left(\sum_{j=1}^n f_{ij}^U + d_i^U \right), i = 1, 2, \dots, m, \\
& x_j^L - \sigma_j \leq x_j^L \leq x_j^U \leq x_j^U + \sigma'_j \text{ and } \sigma_j \geq 0, \sigma'_j \geq 0.
\end{aligned} \tag{4.19}$$

Step 7: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (4.18) can be transformed into single-objective mathematical programming problem (4.20).

Maximize $(\sigma_j + \sigma'_j); j = 1, 2, \dots, n$,

subject to

Constraints of Problem (4.18). (4.20)

Step 8: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (4.19) can be transformed into single-objective mathematical programming problem (4.21).

Minimize $(\sigma_j + \sigma'_j); j = 1, 2, \dots, n$,

subject to

Constraints of Problem (4.19). (4.21)

Step 9: Find the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (4.20).

Step 10: Find the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (4.21).

Step 11: Using the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (4.20), obtained in Step 9, the maximal fuzzy solution of the dual system of fully fuzzy linear equations (4.13) is

$$\{\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j); j = 1, 2, \dots, n\}.$$

Step 12: Using the optimal solution $\{x_j^L, x_j^U, \sigma_j, \sigma'_j\}$, of the single-objective mathematical programming problem (4.21), obtained in Step 10, the minimal fuzzy solution of the dual system of fully fuzzy linear equations (4.13) is

$$\{\tilde{x}_j = (x_j^L - \sigma_j, x_j^L, x_j^U, x_j^U + \sigma'_j); j = 1, 2, \dots, n\}.$$

Remark 4.3. The maximal and minimal fuzzy symmetric solution of the dual system of fully fuzzy linear equations (4.13) can be obtained by solving the mathematical programming problems (4.20) and (4.21) respectively, with the additional constraints $\sigma_j = \sigma'_j; j = 1, 2, \dots, n$.

4.6. Illustrative examples

In this section, to illustrate the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations, chosen in Example 4.2 and Example 4.3 which cannot be solved by Ghomashi et al. method [75], are solved by proposed Mehar-Keerat method-III. Also, the maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations, presented in Example 4.1, is obtained.

4.6.1. Maximal and minimal solution of dual system of fully fuzzy linear equations chosen in Example 4.1 and Example 4.2

Using Mehar-Keerat method-III, proposed in Section 4.5, the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.1 and Example 4.2, can be obtained as follows.

Step 1: Assuming $\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1)$ and $\tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2)$,

the dual system of fully fuzzy linear equations, chosen in Example 4.2, can be transformed into the dual system of fully fuzzy linear equations (4.22).

$$\begin{aligned}(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) &= (-22, -20, -18)(x_2 - \sigma_2, x_2, x_2 + \sigma'_2) + (1200, 1230, 1250), \\(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) &= (12, 13, 15)(x_2 - \sigma_2, x_2, x_2 + \sigma'_2) + (-250, -220, -200),\end{aligned}\tag{4.22}$$

Step 2: Using Step 2 to Step 4 of the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations (4.22) can be transformed into the dual system of fully fuzzy linear equations (4.23).

$$\begin{aligned}(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) &= (-22x_2 - 22\sigma'_2 + 1200, -20x_2 + 1230, -18x_2 + 18\sigma_2 + 1250), \\(x_1 - \sigma_1, x_1, x_1 + \sigma'_1) &= (12x_2 - 12\sigma_2 - 250, 13x_2 - 220, 15x_2 + 15\sigma'_2 - 200),\end{aligned}\tag{4.23}$$

Step 3: Using Step 5 of the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations (4.23) can be transformed into multi-objective mathematical programming problem (4.24).

$$\begin{aligned}&\text{Maximize } \sigma_1, \\&\text{Maximize } \sigma'_1, \\&\text{Maximize } \sigma_2, \\&\text{Maximize } \sigma'_2, \\&\text{subject to} \\&\sigma_1 \leq 2x_2 + 22\sigma'_2 + 30, \\&x_1 = -20x_2 + 1230, \\&\sigma'_1 \leq 2x_2 + 18\sigma_2 + 20, \\&\sigma_1 \leq x_2 + 12\sigma_2 + 30,\end{aligned}\tag{4.24}$$

$$\begin{aligned}x_1 &= 13x_2 - 220, \\ \sigma'_1 &\leq 2x_2 + 15\sigma'_2 + 20, \\ 0 \leq x_1 - \sigma_1 \leq x_1 \leq x_1 + \sigma'_1, 0 \leq x_2 - \sigma_2 \leq x_2 \leq x_2 + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.\end{aligned}$$

Step 4: Using Step 6 of the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations (4.23) can be transformed into multi-objective mathematical programming problem (4.25).

$$\begin{aligned}
&\text{Minimize } \sigma_1, \\
&\text{Minimize } \sigma'_1, \\
&\text{Minimize } \sigma_2, \\
&\text{Minimize } \sigma'_2, \\
&\text{subject to} \\
&\sigma_1 \geq 2x_2 + 22\sigma'_2 + 30, \\
&x_1 = -20x_2 + 1230, \\
&\sigma'_1 \geq 2x_2 + 18\sigma_2 + 20, \\
&\sigma_1 \geq x_2 + 12\sigma_2 + 30, \\
&x_1 = 13x_2 - 220, \\
&\sigma'_1 \geq 2x_2 + 15\sigma'_2 + 20, \\
&0 \leq x_1 - \sigma_1 \leq x_1 \leq x_1 + \sigma'_1, 0 \leq x_2 - \sigma_2 \leq x_2 \leq x_2 + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.
\end{aligned} \tag{4.25}$$

Step 5: Using Step 7 of the proposed Mehar-Keerat method-III, the multi-objective mathematical programming problem (4.24) can be transformed into single-objective mathematical programming problem (4.26).

$$\begin{aligned}
&\text{Maximize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
&\text{subject to} \\
&\text{Constraints of Problem (4.24)}.
\end{aligned} \tag{4.26}$$

Step 6: Using Step 8 of the proposed Mehar-Keerat method-III, the multi-objective mathematical programming problem (4.25) can be transformed into single-objective mathematical programming problem (4.27).

$$\begin{aligned}
&\text{Minimize } (\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2) \\
&\text{subject to} \\
&\text{Constraints of Problem (4.25)}.
\end{aligned} \tag{4.27}$$

Step 7: On solving the single-objective mathematical programming problem (4.26)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$, the obtained optimal solution is

$$\sigma_1 = \frac{11590}{33}, \sigma'_1 = \frac{11590}{33}, \sigma_2 = \frac{1450}{33}, \sigma'_2 = \frac{1450}{33}, x_1 = \frac{11590}{33}, x_2 = \frac{1450}{33}.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$, the obtained optimal solution is

$$\sigma_1 = \frac{1450}{33}, \sigma'_1 = \frac{1450}{33}, \sigma_2 = \frac{1450}{33}, \sigma'_2 = \frac{1450}{33}, x_1 = \frac{11590}{33}, x_2 = \frac{1450}{33}.$$

(iii) without any additional constraint, the obtained optimal solution is

$$\sigma_1 = 0, \sigma'_1 = 0, \sigma_2 = 0, \sigma'_2 = 0, x_1 = \frac{11590}{33}, x_2 = \frac{1450}{33}.$$

Step 8: On solving the single-objective mathematical programming problem (4.27)

(i) with the additional constraints $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$, the obtained optimal solution is

$$\sigma_1 = \frac{3890}{33}, \sigma'_1 = \frac{3890}{33}, \sigma_2 = 0, \sigma'_2 = 0, x_1 = \frac{11590}{33}, x_2 = \frac{1450}{33}.$$

(ii) with the additional constraints $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$, no feasible solution is obtained.

(iii) without any additional constraint, the obtained optimal solution is

$$\sigma_1 = \frac{3890}{33}, \sigma'_1 = \frac{3560}{33}, \sigma_2 = 0, \sigma'_2 = 0, x_1 = \frac{11590}{33}, x_2 = \frac{1450}{33}.$$

Step 9: Using the optimal solution, obtained in Step 7 with the additional constraints

$\sigma_1 = \sigma'_1, \sigma_2 = \sigma'_2$ and $\sigma_1 = \sigma'_1 = \sigma_2 = \sigma'_2$, the maximal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.1, are

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = \left(0, \frac{11590}{33}, \frac{23180}{33}\right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = \left(0, \frac{1450}{33}, \frac{2900}{33}\right) \text{ and}$$

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = \left(\frac{10140}{33}, \frac{11590}{33}, \frac{13040}{33}\right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = \left(0, \frac{1450}{33}, \frac{2900}{33}\right)$$

respectively.

Also, using the optimal solution, obtained in Step 7 without any additional constraint, the maximal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.2, is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma'_1) = \left(\frac{11590}{33}, \frac{11590}{33}, \frac{11590}{33}\right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma'_2) = \left(\frac{1450}{33}, \frac{1450}{33}, \frac{1450}{33}\right).$$

Step 10: Using the optimal solution, obtained in Step 8 with the additional constraints

$\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$, the minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.1, is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{7700}{33}, \frac{11590}{33}, \frac{15480}{33} \right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{1450}{33}, \frac{1450}{33}, \frac{1450}{33} \right).$$

Also, using the optimal solution, obtained in Step 8 without any additional constraint, the minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.2, is

$$\tilde{x}_1 = (x_1 - \sigma_1, x_1, x_1 + \sigma_1') = \left(\frac{7700}{33}, \frac{11590}{33}, \frac{15150}{33} \right), \tilde{x}_2 = (x_2 - \sigma_2, x_2, x_2 + \sigma_2') = \left(\frac{1450}{33}, \frac{1450}{33}, \frac{1450}{33} \right).$$

4.6.2. Maximal and minimal fuzzy solution of dual system of fully fuzzy linear equations chosen in Example 4.3

Using Mehar-Keerat method-III, proposed in Section 4.5, the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.3, can be obtained as follows.

Step 1: Assuming $\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1')$ and $\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2')$, the dual system of fully fuzzy linear equations, chosen in Example 4.3, can be transformed into the dual system of fully fuzzy linear equations (4.28).

$$\begin{aligned} (-4, -3, -2, 1)(x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') + (11, 11, 11, 21) &= (1, 2, 3, 5)(x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') + (0, 4, 8, 8), \\ (-7, -5, -4, -3)(x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') + (18, 18, 18, 24) &= (-2, 0, 1, 4)(x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') + (0, 13, 25, 25). \end{aligned} \quad (4.28)$$

Step 2: Using Step 2 of the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations (4.28) can be transformed into the dual system of fully fuzzy linear equations (4.29).

$$\begin{aligned}
& \left(\frac{-4x_1^U - 4\sigma_1' + x_1^L - \sigma_1}{2} - \left| \frac{-4x_1^U - 4\sigma_1' - x_1^L + \sigma_1}{2} \right|, -\frac{5}{2}x_1^U - \left| \frac{-x_1^U}{2} \right|, -\frac{5}{2}x_1^L + \left| \frac{x_1^L}{2} \right|, \frac{x_1^U + \sigma_1' - 4x_1^L + 4\sigma_1}{2} + \right. \\
& \left. \left| \frac{x_1^U + \sigma_1' + 4x_1^L - 4\sigma_1}{2} \right| \right) \\
& + (11, 11, 11, 21) = \left(\frac{6x_2^L - 6\sigma_2}{2} - \left| \frac{-4x_2^L + 4\sigma_2}{2} \right|, \frac{5}{2}x_2^L - \left| -\frac{1}{2}x_2^L \right|, \frac{5}{2}x_2^U + \left| \frac{1}{2}x_2^U \right|, \frac{6x_2^U + 6\sigma_2'}{2} + \left| \frac{4x_2^U + 4\sigma_2'}{2} \right| \right) \\
& + (0, 4, 8, 8), \\
& \left(\frac{-10x_1^U - 10\sigma_1'}{2} - \left| \frac{-4x_1^U - 4\sigma_1'}{2} \right|, -\frac{9}{2}x_1^U - \left| \frac{-x_1^U}{2} \right|, -\frac{9}{2}x_1^L + \left| \frac{x_1^L}{2} \right|, \frac{-10x_1^L + 10\sigma_1}{2} + \left| \frac{4x_1^L - 4\sigma_1}{2} \right| \right) + (18, 18, 18, 24) \\
& = \left(\frac{-2x_2^U - 2\sigma_2' + 4x_2^L - 4\sigma_2}{2} - \left| \frac{-2x_2^U - 2\sigma_2' - 4x_2^L + 4\sigma_2}{2} \right|, \frac{1}{2}x_2^L - \left| -\frac{1}{2}x_2^L \right|, \frac{1}{2}x_2^U + \left| \frac{1}{2}x_2^U \right|, \frac{4x_2^U + 4\sigma_2' - 2x_2^L + 2\sigma_2}{2} \right. \\
& \left. + \left| \frac{4x_2^U + 4\sigma_2' + 2x_2^L - 2\sigma_2}{2} \right| \right) \\
& + (0, 13, 25, 25). \tag{4.29}
\end{aligned}$$

Step 3: Using Step 3 and Step 4 of the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations (4.29) can be transformed into the dual system of fully fuzzy linear equations (4.30).

$$\begin{aligned}
& \left(\frac{-4x_1^U - 4\sigma_1' + x_1^L - \sigma_1}{2} - \left| \frac{-4x_1^U - 4\sigma_1' - x_1^L + \sigma_1}{2} \right| + 11, -\frac{5}{2}x_1^U - \left| \frac{-x_1^U}{2} \right| + 11, -\frac{5}{2}x_1^L + \left| \frac{x_1^L}{2} \right| + 11, \frac{x_1^U + \sigma_1' - 4x_1^L + 4\sigma_1}{2} \right. \\
& \left. + \left| \frac{x_1^U + \sigma_1' + 4x_1^L - 4\sigma_1}{2} \right| + 21 \right) \\
& = \left(\frac{6x_2^L - 6\sigma_2}{2} - \left| \frac{-4x_2^L + 4\sigma_2}{2} \right|, \frac{5}{2}x_2^L - \left| -\frac{1}{2}x_2^L \right| + 4, \frac{5}{2}x_2^U + \left| \frac{1}{2}x_2^U \right| + 8, \frac{6x_2^U + 6\sigma_2'}{2} + \left| \frac{4x_2^U + 4\sigma_2'}{2} \right| + 8 \right), \\
& \left(\frac{-10x_1^U - 10\sigma_1'}{2} - \left| \frac{-4x_1^U - 4\sigma_1'}{2} \right| + 18, -\frac{9}{2}x_1^U - \left| \frac{-x_1^U}{2} \right| + 18, -\frac{9}{2}x_1^L + \left| \frac{x_1^L}{2} \right| + 18, \frac{-10x_1^L + 10\sigma_1}{2} + \left| \frac{4x_1^L - 4\sigma_1}{2} \right| + 24 \right) \\
& = \left(\frac{-2x_2^U - 2\sigma_2' + 4x_2^L - 4\sigma_2}{2} - \left| \frac{-2x_2^U - 2\sigma_2' - 4x_2^L + 4\sigma_2}{2} \right|, \frac{1}{2}x_2^L - \left| -\frac{1}{2}x_2^L \right| + 13, \frac{1}{2}x_2^U + \left| \frac{1}{2}x_2^U \right| + 25, \right. \\
& \left. \frac{4x_2^U + 4\sigma_2' - 2x_2^L + 2\sigma_2}{2} + \left| \frac{4x_2^U + 4\sigma_2' + 2x_2^L - 2\sigma_2}{2} \right| + 25 \right) \tag{4.30}
\end{aligned}$$

Step 4: Using Step 5 of the proposed Mehar-Keerat method-III, the dual system of fully

fuzzy linear equations (4.30) can be transformed into multi-objective mathematical programming problem (4.31).

Maximize σ_1 ,

Maximize σ'_1 ,

Maximize σ_2 ,

Maximize σ'_2 ,

subject to

$$\begin{aligned}
& -\frac{5}{2}x_1^U - \left| \frac{-x_1^U}{2} \right| - \frac{-4x_1^U - 4\sigma'_1 + x_1^L - \sigma_1}{2} + \left| \frac{-4x_1^U - 4\sigma'_1 - x_1^L + \sigma_1}{2} \right| \leq \frac{5}{2}x_2^L - \left| \frac{1}{2}x_2^L \right| - 3x_2^L + 3\sigma_2 + \left| -2x_2^L + 2\sigma_2 \right| + 4, \\
& -\frac{5}{2}x_1^U - \frac{1}{2}|x_1^U| + 11 = \frac{5}{2}x_2^L - \left| \frac{1}{2}x_2^L \right| + 4, \\
& -\frac{5}{2}x_1^L + \frac{1}{2}|x_1^L| + 11 = \frac{5}{2}x_2^U + \frac{1}{2}|x_2^U| + 8, \\
& \frac{x_1^U + \sigma'_1 - 4x_1^L + 4\sigma_1}{2} + \left| \frac{x_1^U + \sigma'_1 + 4x_1^L - 4\sigma_1}{2} \right| + \frac{5}{2}x_1^L - \frac{1}{2}|x_1^L| + 10 \leq 3x_2^U + 3\sigma'_2 + \left| 2x_2^U + 2\sigma'_2 \right| - \frac{5}{2}x_2^U - \frac{1}{2}|x_2^U|, \\
& -\frac{9}{2}x_1^U - \frac{1}{2}|x_1^U| + 5x_1^U + 5\sigma'_1 + \left| 2x_1^U + 2\sigma'_1 \right| \leq \frac{1}{2}x_2^L - \frac{1}{2}|x_2^L| + x_2^U + \sigma'_2 - 2x_2^L + 2\sigma_2 + \left| -x_2^U - \sigma'_2 - 2x_2^L + 2\sigma_2 \right| + 13, \\
& -\frac{9}{2}x_1^U - \frac{1}{2}|x_1^U| + 18 = \frac{1}{2}x_2^L - \frac{1}{2}|x_2^L| + 13, \\
& -\frac{9}{2}x_1^L + \frac{1}{2}|x_1^L| + 18 = \frac{1}{2}x_2^U + \frac{1}{2}|x_2^U| + 25, \\
& -5x_1^L + 5\sigma_1 + \left| 2x_1^L - 2\sigma_1 \right| + \frac{9}{2}x_1^L - \frac{1}{2}|x_1^L| + 6 \leq 2x_2^U + 2\sigma'_2 - x_2^L + \sigma_2 + \left| 2x_2^U + 2\sigma'_2 + x_2^L - \sigma_2 \right| - \frac{1}{2}x_2^U - \frac{1}{2}|x_2^U|, \\
& x_1^L - \sigma_1 \leq x_1^L \leq x_1^U \leq x_1^U + \sigma'_1, x_2^L - \sigma_2 \leq x_2^L \leq x_2^U \leq x_2^U + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.
\end{aligned} \tag{4.31}$$

Step 5: Using Step 6 of the proposed Mehar-Keerat method-III, the dual system of fully fuzzy linear equations (4.30) can be transformed into multi-objective mathematical programming problem (4.32).

Minimize σ_1 ,

Minimize σ'_1 ,

Minimize σ_2 ,

Minimize σ'_2 ,

subject to

$$-\frac{5}{2}x_1^U - \left| \frac{-x_1^U}{2} \right| - \frac{-4x_1^U - 4\sigma'_1 + x_1^L - \sigma_1}{2} + \left| \frac{-4x_1^U - 4\sigma'_1 - x_1^L + \sigma_1}{2} \right| \leq \frac{5}{2}x_2^L - \left| \frac{1}{2}x_2^L \right| - 3x_2^L + 3\sigma_2 + \left| -2x_2^L + 2\sigma_2 \right| + 4,$$

$$\begin{aligned}
& -\frac{5}{2}x_1^U - \frac{1}{2}|x_1^U| + 11 = \frac{5}{2}x_2^L - \left|\frac{1}{2}x_2^L\right| + 4, \\
& -\frac{5}{2}x_1^L + \frac{1}{2}|x_1^L| + 11 = \frac{5}{2}x_2^U + \frac{1}{2}|x_2^U| + 8, \\
& -\frac{5}{2}x_1^U - \left|\frac{-x_1^U}{2}\right| - \frac{-4x_1^U - 4\sigma'_1 + x_1^L - \sigma_1}{2} + \left|\frac{-4x_1^U - 4\sigma'_1 - x_1^L + \sigma_1}{2}\right| \geq \frac{5}{2}x_2^L - \left|\frac{1}{2}x_2^L\right| - 3x_2^L + 3\sigma_2 + |-2x_2^L + 2\sigma_2| + 4, \\
& -\frac{5}{2}x_1^U - \frac{1}{2}|x_1^U| + 11 = \frac{5}{2}x_2^L - \left|\frac{1}{2}x_2^L\right| + 4, \\
& -\frac{5}{2}x_1^L + \frac{1}{2}|x_1^L| + 11 = \frac{5}{2}x_2^U + \frac{1}{2}|x_2^U| + 8, \\
& \frac{x_1^U + \sigma'_1 - 4x_1^L + 4\sigma_1}{2} + \left|\frac{x_1^U + \sigma'_1 + 4x_1^L - 4\sigma_1}{2}\right| + \frac{5}{2}x_1^L - \frac{1}{2}|x_1^L| + 10 \geq 3x_2^U + 3\sigma'_2 + |2x_2^U + 2\sigma'_2| - \frac{5}{2}x_2^U - \frac{1}{2}|x_2^U|, \\
& -\frac{9}{2}x_1^U - \frac{1}{2}|x_1^U| + 5x_1^U + 5\sigma'_1 + |2x_1^U + 2\sigma'_1| \geq \frac{1}{2}x_2^L - \frac{1}{2}|x_2^L| + x_2^U + \sigma'_2 - 2x_2^L + 2\sigma_2 + |-x_2^U - \sigma'_2 - 2x_2^L + 2\sigma_2| + 13, \\
& -\frac{9}{2}x_1^U - \frac{1}{2}|x_1^U| + 18 = \frac{1}{2}x_2^L - \frac{1}{2}|x_2^L| + 13, \\
& -\frac{9}{2}x_1^L + \frac{1}{2}|x_1^L| + 18 = \frac{1}{2}x_2^U + \frac{1}{2}|x_2^U| + 25, \\
& -5x_1^L + 5\sigma_1 + |2x_1^L - 2\sigma_1| + \frac{9}{2}x_1^L - \frac{1}{2}|x_1^L| + 6 \geq 2x_2^U + 2\sigma'_2 - x_2^L + \sigma_2 + |2x_2^U + 2\sigma'_2 + x_2^L - \sigma_2| - \frac{1}{2}x_2^U - \frac{1}{2}|x_2^U|, \\
& x_1^L - \sigma_1 \leq x_1^L \leq x_1^U \leq x_1^U + \sigma'_1, x_2^L - \sigma_2 \leq x_2^L \leq x_2^U \leq x_2^U + \sigma'_2, \sigma_1 \geq 0, \sigma'_1 \geq 0, \sigma_2 \geq 0, \sigma'_2 \geq 0.
\end{aligned} \tag{4.32}$$

Step 6: Using Step 7 of the proposed Mehar-Keerat method-III, the multi-objective mathematical programming problem (4.31) can be transformed into single-objective mathematical programming problem (4.33).

Maximize $(\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2)$

subject to

Constraints of Problem (4.31). (4.33)

Step 7: Using Step 8 of the proposed Mehar-Keerat method-III, the multi-objective mathematical programming problem (4.32) can be transformed into single-objective mathematical programming problem (4.34).

Minimize $(\sigma_1 + \sigma'_1 + \sigma_2 + \sigma'_2)$

subject to

Constraints of Problem (4.32). (4.34)

Step 8: On solving the single-objective mathematical programming problem (4.33)

(i) with the additional constraints $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$, the obtained optimal solution is

$$\sigma_1 = 0, \sigma_1' = 0, \sigma_2 = \frac{68327}{3665} \cong 18.6431, \sigma_2' = \frac{68327}{3665} \cong 18.6431, x_1^L = -2, x_1^U = 1, x_2^L = 2, \\ x_2^U = 3.$$

(ii) with the additional constraints $\sigma_1 = \sigma_1' = \sigma_2 = \sigma_2'$, no feasible solution is obtained.

(iii) without any additional constraint, the obtained optimal solution is

$$\sigma_1 = \frac{5187}{752} \cong 6.8976, \sigma_1' = \frac{8947}{1504} \cong 5.9488, \sigma_2 = \frac{2015}{376} \cong 5.3590, \sigma_2' = \frac{37061}{3008} \cong 12.3208, \\ x_1^L = -2, x_1^U = 1, x_2^L = 2, x_2^U = 3.$$

Step 9: On solving the single-objective mathematical programming problem (4.34)

(i) with the additional constraints $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$, the obtained optimal solution is

$$\sigma_1 = \frac{17}{7} \cong 2.4286, \sigma_1' = \frac{17}{7} \cong 2.4286, \sigma_2 = 0, \sigma_2' = 0, x_1^L = -2, x_1^U = 1, x_2^L = 2, x_2^U = 3.$$

(ii) with the additional constraints $\sigma_1 = \sigma_1' = \sigma_2 = \sigma_2'$, no feasible solution is obtained.

(iii) without any additional constraint, the obtained optimal solution is

$$\sigma_1 = 0, \sigma_1' = \frac{17}{7} \cong 2.4286, \sigma_2 = 0, \sigma_2' = 0, x_1^L = -2, x_1^U = 1, x_2^L = 2, x_2^U = 3.$$

Step 10: Using the optimal solution, obtained in Step 8 with the additional constraints

$\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$, the maximal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.3, is $\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = (-2, -2, 1, 1)$,

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = \left(\frac{-60997}{3665}, 2, 3, \frac{79322}{3665} \right) \cong (-16.6431, 2, 3, 21.6431).$$

Also, using the optimal solution, obtained in Step 8 without any additional constraint, the maximal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in

Example 4.3, is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = \left(\frac{-6691}{752}, -2, 1, \frac{10451}{1504} \right) \cong (-8.8976, -2, 1, 6.9488),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = \left(\frac{-1263}{376}, 2, 3, \frac{46085}{3008} \right) \cong (-3.3590, 2, 3, 15.3208).$$

Step 11: Using the optimal solution, obtained in Step 9 with the additional constraints

$\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$, the minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example 4.3, is

$$\tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = \left(\frac{-31}{7}, -2, 1, \frac{24}{7} \right) \cong (-4.4286, -2, 1, 3.4286),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (2, 2, 3, 3).$$

Also, using the optimal solution, obtained in Step 9 without any additional constraint, the minimal fuzzy solution of the dual system of fully fuzzy linear equations, chosen in Example

$$4.3, \text{ is } \tilde{x}_1 = (x_1^L - \sigma_1, x_1^L, x_1^U, x_1^U + \sigma_1') = \left(-2, -2, 1, \frac{24}{7} \right) \cong (-2, -2, 1, 3.4286),$$

$$\tilde{x}_2 = (x_2^L - \sigma_2, x_2^L, x_2^U, x_2^U + \sigma_2') = (2, 2, 3, 3).$$

4.7. Comparative study

The maximal and minimal fuzzy solution of the existing dual system of fully fuzzy linear equations and the chosen dual system of fully fuzzy linear equations, obtained by the existing methods [26,29,75] as well as proposed Mehar-Keerat method-I, Mehar-Keerat method-II and Mehar-Keerat method-III are shown in Table 4.1.

Table 4.1. Maximal and minimal fuzzy symmetric/non-symmetric solution of existing and chosen dual system of fully fuzzy linear equations.

Example	Maximal and minimal Fuzzy solution	Existing methods [26,29]	Existing method [75]		Proposed Mehar-Keerat method-I, Mehar-Keerat method II	Proposed Mehar-Keerat method-III	
		Symmetric/ Non-symmetric solution	Symmetric solution	Non-symmetric solution	Symmetric/Non-symmetric solution	Symmetric solution	Non-symmetric solution
Example 4.1 [75, Section 5, pp. 375], Example 4.2	Maximal fuzzy solution	N.A.	$\tilde{x}_1 = \left[\frac{11590}{33} - \frac{107.879(1-r)}{12}, \frac{11590}{33} + \frac{107.879(1-r)}{12} \right],$ $\tilde{x}_2 = \left[\frac{1450}{33} - \frac{107.879(1-r)}{12}, \frac{1450}{33} + \frac{107.879(1-r)}{12} \right].$	N.A.	N.A.	$\tilde{x}_1 = \left(\frac{10140}{33}, \frac{11590}{33}, \frac{13040}{33} \right),$ $\tilde{x}_2 = \left(0, \frac{1450}{33}, \frac{2900}{33} \right).$	$\tilde{x}_1 = \left(\frac{11590}{33}, \frac{11590}{33}, \frac{11590}{33} \right),$ $\tilde{x}_2 = \left(\frac{1450}{33}, \frac{1450}{33}, \frac{1450}{33} \right).$
	Minimal fuzzy solution	N.A.	$\tilde{x}_1 = \left[\frac{11590}{33} - \frac{117.879(1-r)}{21}, \frac{11590}{33} + \frac{117.879(1-r)}{21} \right],$ $\tilde{x}_2 = \left[\frac{1450}{33} - \frac{117.879(1-r)}{21}, \frac{1450}{33} + \frac{117.879(1-r)}{21} \right].$	N.A.	N.A.	No feasible solution	$\tilde{x}_1 = \left(\frac{7700}{33}, \frac{11590}{33}, \frac{15150}{33} \right),$ $\tilde{x}_2 = \left(\frac{1450}{33}, \frac{1450}{33}, \frac{1450}{33} \right).$
Example 4.3	Maximal fuzzy solution	N.A.	N.A.	N.A.	N.A.	No feasible solution	$\tilde{x}_1 = \left(\frac{-6691}{752}, -2, 1, \frac{10451}{1504} \right),$ $\tilde{x}_2 = \left(\frac{-1263}{4}, 2, 3, \frac{46085}{3008} \right).$
	Minimal fuzzy solution	N.A.	N.A.	N.A.	N.A.	No feasible solution	$\tilde{x}_1 = \left(-2, -2, 1, \frac{24}{7} \right),$ $\tilde{x}_2 = (2, 2, 3, 3).$

4.8. Conclusion

It is pointed out that there are flaws in the existing method [75] and hence, it is not genuine to use the existing method [75] for finding the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations. Also, a new method (named as Mehar-Keerat method-III) is proposed to find the maximal and minimal fuzzy solution of the dual system of fully fuzzy linear equations. Furthermore, it is shown that the dual system of fully fuzzy linear equations whose maximal and minimal fuzzy solution can be obtained by the existing method [75], can also be obtained by the proposed Mehar-Keerat method-III. However, there exist several dual system of fully fuzzy linear equations whose maximal and minimal fuzzy solution can be obtained by the proposed Mehar-Keerat method-III but cannot be obtained by the existing method [75]. Finally, it may be concluded that it is better to use the proposed Mehar-Keerat method-III as compared to the existing method [75].

Chapter 5

Mehar-Keerat method-IV to find maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation*

Dookhitram et al. [59] extended the existing method [29], to find maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$, where the elements of the matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are non-negative/non-positive triangular fuzzy numbers and the elements of unknown matrix $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers.

In this chapter, the limitations as well as flaws of the existing method [59] are pointed out. Also, to overcome the limitations as well as to resolve the flaws of the existing method [59], a new method (named as Mehar-Keerat method-IV) is proposed to find maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, where the elements of $\tilde{A}, \tilde{B}, \tilde{X}$ and $\tilde{C}$ are trapezoidal fuzzy numbers. The proposed Mehar-Keerat method-IV is illustrated with the help of numerical examples.

5.1. Maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation

Dookhitram et al. [59] pointed out that the existing method [29] cannot be used to find the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation. Furthermore, to overcome this limitation of the existing method [29], Dookhitram et al. [59]

* The contents of this chapter are communicated for possible publication in Fuzzy Sets and Systems.

extended the existing method [29] to find the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$,

$$\text{i.e., } \sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_{jk} - \sum_{l=1}^m \tilde{x}_{il} \tilde{b}_{lk} = \tilde{c}_{ik} \text{ for } i=1, 2, \dots, n, k=1, 2, \dots, m, \quad (5.1)$$

where the elements $\tilde{a}_{ij}$ and $\tilde{b}_{lk}$ are non-negative/non-positive triangular fuzzy numbers, whereas, the elements $\tilde{x}_{jk}$ and $\tilde{x}_{il}$ are non-negative symmetric triangular fuzzy numbers.

In this section, the existing definitions [59] of maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation (5.1) are presented in the modified form by considering the assumption that $\tilde{x}_{ik}$ is a non-symmetric triangular/trapezoidal fuzzy number.

Definition 5.1. A fuzzy matrix $\tilde{X} = [\tilde{x}_{ik}]_{n \times m}$, where $\tilde{x}_{ik} = (x_{ik} - \sigma_{ik}, x_{ik}, x_{ik} + \sigma'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} \subseteq \tilde{C}$ is called the maximal fuzzy solution matrix of fully fuzzy Sylvester matrix equation (5.1) if for any arbitrary fuzzy matrix $\tilde{Z} = [\tilde{z}_{ik}]_{n \times m}$, where $\tilde{z}_{ik} = (z_{ik} - \alpha_{ik}, z_{ik}, z_{ik} + \alpha'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{Z} - \tilde{Z}\tilde{B} \subseteq \tilde{C}$ we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_{ik} = x_{ik}, \alpha_{ik} \leq \sigma_{ik}, \alpha'_{ik} \leq \sigma'_{ik}$.

Definition 5.2. A fuzzy matrix $\tilde{X} = [\tilde{x}_{ik}]_{n \times m}$, where $\tilde{x}_{ik} = (x_{ik} - \sigma_{ik}, x_{ik}, x_{ik} + \sigma'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} \supseteq \tilde{C}$ is called the minimal fuzzy solution matrix of fully fuzzy Sylvester matrix equation (5.1) if for any arbitrary fuzzy matrix $\tilde{Z} = [\tilde{z}_{ik}]_{n \times m}$, where $\tilde{z}_{ik} = (z_{ik} - \alpha_{ik}, z_{ik}, z_{ik} + \alpha'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{Z} - \tilde{Z}\tilde{B} \supseteq \tilde{C}$ we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_{ik} = x_{ik}, \alpha_{ik} \geq \sigma_{ik}, \alpha'_{ik} \geq \sigma'_{ik}$.

Definition 5.3. If in the fully fuzzy Sylvester matrix equation (5.1), the elements $\tilde{x}_{ik}$ of the unknown matrix $\tilde{X}$ are trapezoidal fuzzy numbers, then the fuzzy matrix $\tilde{X} = [\tilde{x}_{ik}]_{n \times m}$, where $\tilde{x}_{ik} = (x_{ik}^L - \sigma_{ik}, x_{ik}^L, x_{ik}^U, x_{ik}^U + \sigma'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} \subseteq \tilde{C}$ is called the maximal fuzzy solution matrix of fully fuzzy Sylvester matrix equation (5.1) if for any

arbitrary fuzzy matrix $\tilde{Z} = [\tilde{z}_{ik}]_{n \times m}$, where $\tilde{z}_{ik} = (z_{ik}^L - \alpha_{ik}, z_{ik}^L, z_{ik}^U, z_{ik}^U + \alpha'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{Z} - \tilde{Z}\tilde{B} \subseteq \tilde{C}$ we have $\tilde{Z} \subseteq \tilde{X}$ i.e., $z_{ik}^L = x_{ik}^L, z_{ik}^U = x_{ik}^U, \alpha_{ik} \leq \sigma_{ik}, \alpha'_{ik} \leq \sigma'_{ik}$.

Definition 5.4. If in the fully fuzzy Sylvester matrix equation (5.1), the elements $\tilde{x}_{ik}$ of the unknown matrix $\tilde{X}$ are trapezoidal fuzzy numbers, then the fuzzy matrix $\tilde{X} = [\tilde{x}_{ik}]_{n \times m}$, where $\tilde{x}_{ik} = (x_{ik}^L - \sigma_{ik}, x_{ik}^L, x_{ik}^U, x_{ik}^U + \sigma'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} \supseteq \tilde{C}$ is called the minimal fuzzy solution matrix of fully fuzzy Sylvester matrix equation (5.1) if for any arbitrary fuzzy matrix $\tilde{Z} = [\tilde{z}_{ik}]_{n \times m}$, where $\tilde{z}_{ik} = (z_{ik}^L - \alpha_{ik}, z_{ik}^L, z_{ik}^U, z_{ik}^U + \alpha'_{ik})$ for $1 \leq i \leq n, 1 \leq k \leq m$; $\tilde{A}\tilde{Z} - \tilde{Z}\tilde{B} \supseteq \tilde{C}$ we have $\tilde{Z} \supseteq \tilde{X}$ i.e., $z_{ik}^L = x_{ik}^L, z_{ik}^U = x_{ik}^U, \alpha_{ik} \geq \sigma_{ik}, \alpha'_{ik} \geq \sigma'_{ik}$.

Remark 5.1. In general, the fully fuzzy Sylvester matrix equation of the form $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ is used instead of the fully fuzzy Sylvester matrix equation of the form $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$. Therefore, in this chapter, the form $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ is used instead of $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$.

5.2. Dookhitram et al. method

Dookhitram et al. [59] proposed the following method to find the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation (5.1).

Step 1: Since, $\tilde{a}_{ij}$ and $\tilde{b}_{lk}$ are non-negative/non-positive triangular fuzzy numbers. So, assuming

$$S_1 = \{\tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-negative fuzzy number}\}, S_2 = \{\tilde{b}_{lk} : \tilde{b}_{lk} \text{ is a non-negative fuzzy number}\},$$

$$T_1 = \{\tilde{a}_{ij} : \tilde{a}_{ij} \text{ is a non-positive fuzzy number}\} \text{ and } T_2 = \{\tilde{b}_{lk} : \tilde{b}_{lk} \text{ is a non-positive fuzzy number}\},$$

the fully fuzzy Sylvester matrix equation (5.1) can be transformed into the fully fuzzy Sylvester matrix equation (5.2).

$$\sum_{\tilde{a}_{ij} \in S_1} \tilde{a}_{ij} \tilde{x}_{jk} + \sum_{\tilde{a}_{ij} \in T_1} \tilde{a}_{ij} \tilde{x}_{jk} + \sum_{\tilde{b}_{lk} \in S_2} \tilde{x}_{il} \tilde{b}_{lk} + \sum_{\tilde{b}_{lk} \in T_2} \tilde{x}_{il} \tilde{b}_{lk} = \tilde{c}_{ik}, i=1,2,\dots,n, j=1,2,\dots,n, l=1,2,\dots,m, k=1,2,\dots,m. \quad (5.2)$$

Step 2: Assuming

$$\tilde{a}_{ij} = (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}), \tilde{x}_{jk} = (x_{jk} - \sigma_{jk}, x_{jk}, x_{jk} + \sigma_{jk}), \tilde{x}_{il} = (x_{il} - \sigma_{il}, x_{il}, x_{il} + \sigma_{il}), \\ \tilde{b}_{lk} = (b_{lk} - \beta_{lk}, b_{lk}, b_{lk} + \beta'_{lk}) \text{ and } \tilde{c}_{ik} = (c_{ik} - \gamma_{ik1}, c_{ik}, c_{ik} + \gamma_{ik2}), \text{ the fully fuzzy Sylvester matrix}$$

equation (5.2) can be transformed into the fully fuzzy Sylvester matrix equation (5.3).

$$\sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}) (x_{jk} - \sigma_{jk}, x_{jk}, x_{jk} + \sigma_{jk}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}) (x_{jk} - \sigma_{jk}, x_{jk}, x_{jk} + \sigma_{jk}) + \\ \sum_{\tilde{b}_{lk} \in S_2} (x_{il} - \sigma_{il}, x_{il}, x_{il} + \sigma_{il}) (b_{lk} - \beta_{lk}, b_{lk}, b_{lk} + \beta'_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} - \sigma_{il}, x_{il}, x_{il} + \sigma_{il}) (b_{lk} - \beta_{lk}, b_{lk}, b_{lk} + \beta'_{lk}) = \\ (c_{ik} - \gamma_{ik1}, c_{ik}, c_{ik} + \gamma_{ik2}), i=1,2,\dots,n, j=1,2,\dots,n, l=1,2,\dots,m, k=1,2,\dots,m. \quad (5.3)$$

Step 3: Replacing the triangular fuzzy numbers $(a_{ij} - \alpha_{ij}, a_{ij}, a_{ij} + \alpha'_{ij}),$

$$(x_{jk} - \sigma_{jk}, x_{jk}, x_{jk} + \sigma_{jk}), (x_{il} - \sigma_{il}, x_{il}, x_{il} + \sigma_{il}), (b_{lk} - \beta_{lk}, b_{lk}, b_{lk} + \beta'_{lk}) \text{ and } (c_{ik} - \gamma_{ik1}, c_{ik}, c_{ik} + \gamma_{ik2}),$$

$$\text{by the corresponding } r\text{-cuts } [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}], [x_{jk} - \sigma_{jk} + r\sigma_{jk}, x_{jk} + \sigma_{jk} - r\sigma_{jk}],$$

$$[x_{il} - \sigma_{il} + r\sigma_{il}, x_{il} + \sigma_{il} - r\sigma_{il}], [b_{lk} - \beta_{lk} + r\beta_{lk}, b_{lk} + \beta'_{lk} - r\beta'_{lk}] \text{ and } [c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, c_{ik} + \gamma_{ik2} - r\gamma_{ik2}],$$

the fully fuzzy Sylvester matrix equation (5.3) can be transformed into the system of interval equations (5.4).

$$\sum_{\tilde{a}_{ij} \in S_1} [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}] [x_{jk} - \sigma_{jk} + r\sigma_{jk}, x_{jk} + \sigma_{jk} - r\sigma_{jk}] + \sum_{\tilde{a}_{ij} \in T_1} [a_{ij} - \alpha_{ij} + r\alpha_{ij}, a_{ij} + \alpha'_{ij} - r\alpha'_{ij}] \\ [x_{jk} - \sigma_{jk} + r\sigma_{jk}, x_{jk} + \sigma_{jk} - r\sigma_{jk}] + \sum_{\tilde{b}_{lk} \in S_2} [x_{il} - \sigma_{il} + r\sigma_{il}, x_{il} + \sigma_{il} - r\sigma_{il}] [b_{lk} - \beta_{lk} + r\beta_{lk}, b_{lk} + \beta'_{lk} - r\beta'_{lk}] + \\ \sum_{\tilde{b}_{lk} \in T_2} [x_{il} - \sigma_{il} + r\sigma_{il}, x_{il} + \sigma_{il} - r\sigma_{il}] [b_{lk} - \beta_{lk} + r\beta_{lk}, b_{lk} + \beta'_{lk} - r\beta'_{lk}] = [c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, c_{ik} + \gamma_{ik2} - r\gamma_{ik2}], \\ i=1,2,\dots,n, j=1,2,\dots,n, l=1,2,\dots,m, k=1,2,\dots,m. \quad (5.4)$$

Step 4: Using the property, $[a_1, a_2][b_1, b_2] = \begin{cases} [a_1b_1, a_2b_2]; a_1 \geq 0, b_1 \geq 0, \\ [a_1b_2, a_2b_1]; a_2 \leq 0, b_1 \geq 0. \end{cases}$

the system of interval equations (5.4) can be transformed into the system of interval equations (5.5).

$$\begin{aligned}
& \sum_{\tilde{a}_{ij} \in S_1} \left[(a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}), (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}) \right] + \\
& \sum_{\tilde{a}_{ij} \in T_1} \left[(a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}), (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}) \right] + \\
& \sum_{\tilde{b}_{lk} \in S_2} \left[(x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}), (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) \right] + \\
& \sum_{\tilde{b}_{lk} \in T_2} \left[(x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}), (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}) \right] = [c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, c_{ik} + \gamma_{ik2} - r\gamma_{ik2}], \\
& i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m.
\end{aligned} \tag{5.5}$$

Step 5: Using the property, $\sum_{i=1}^m [a_i, b_i] = \left[\sum_{i=1}^m a_i, \sum_{i=1}^m b_i \right]$, the system of interval equations (5.5)

can be transformed into the system of interval equations (5.6).

$$\begin{aligned}
& \left[\sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}), \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}) \right] + \\
& \left[\sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}), \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}) \right] + \\
& \left[\sum_{\tilde{b}_{lk} \in S_2} (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}), \sum_{\tilde{b}_{lk} \in S_2} (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) \right] + \\
& \left[\sum_{\tilde{b}_{lk} \in T_2} (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}), \sum_{\tilde{b}_{lk} \in T_2} (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}) \right] = \\
& [c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, c_{ik} + \gamma_{ik2} - r\gamma_{ik2}], i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m.
\end{aligned} \tag{5.6}$$

Step 6: Using the property, $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, the system of interval equations (5.6) can be transformed into the system of interval equations (5.7).

$$\begin{aligned}
& \left[\sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}) + \right. \\
& \sum_{\tilde{b}_{lk} \in S_2} (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}), \\
& \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}) + \\
& \left. \sum_{\tilde{b}_{lk} \in S_2} (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}) \right] = \\
& [c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, c_{ik} + \gamma_{ik2} - r\gamma_{ik2}], i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m.
\end{aligned} \tag{5.7}$$

Step 7: Using the property, $[a_1, a_2] = [b_1, b_2] \Rightarrow a_1 = b_1$ and $a_2 = b_2$, the system of interval equations (5.7) can be transformed into the system of crisp matrix equations (5.8) and (5.9).

$$\begin{aligned} & \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}) + \\ & \sum_{\tilde{b}_{lk} \in S_2} (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) = \\ & c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m. \end{aligned} \quad (5.8)$$

$$\begin{aligned} & \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} + \sigma_{jk} - r\sigma_{jk}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} - \sigma_{jk} + r\sigma_{jk}) + \\ & \sum_{\tilde{b}_{lk} \in S_2} (x_{il} + \sigma_{il} - r\sigma_{il})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} - \sigma_{il} + r\sigma_{il})(b_{lk} - \beta_{lk} + r\beta_{lk}) = \\ & c_{ik} + \gamma_{ik2} - r\gamma_{ik2}, i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m. \end{aligned} \quad (5.9)$$

Step 8: Solve the system of crisp matrix equations (5.10), obtained by putting $r=1$ in the system of crisp matrix equations (5.8) and (5.9), to find the value of $x_{ik}, i = 1, 2, \dots, n, k = 1, 2, \dots, m$.

$$\sum_{\tilde{a}_{ij} \in S_1} a_{ij}x_{jk} + \sum_{\tilde{a}_{ij} \in T_1} a_{ij}x_{jk} + \sum_{\tilde{b}_{lk} \in S_2} x_{il}b_{lk} + \sum_{\tilde{b}_{lk} \in T_2} x_{il}b_{lk} = c_{ik}, i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m. \quad (5.10)$$

Step 9: Replacing σ_{jk} and σ_{lk} of the i^{th} equation of the system of crisp matrix equation (5.8) by σ_{ik1} and σ_{jk} and σ_{lk} of the i^{th} equation of the system of crisp matrix equation (5.9) by σ_{ik2} , these system of crisp matrix equation are transformed into the crisp matrix equations (5.11) and (5.12).

$$\begin{aligned} & \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} - \sigma_{ik1} + r\sigma_{ik1}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} - \alpha_{ij} + r\alpha_{ij})(x_{jk} + \sigma_{ik1} - r\sigma_{ik1}) + \\ & \sum_{\tilde{b}_{lk} \in S_2} (x_{il} - \sigma_{ik1} + r\sigma_{ik1})(b_{lk} - \beta_{lk} + r\beta_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} + \sigma_{ik1} - r\sigma_{ik1})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) = c_{ik} - \gamma_{ik1} + r\gamma_{ik1}, \\ & i = 1, 2, \dots, n, j = 1, 2, \dots, n, l = 1, 2, \dots, m, k = 1, 2, \dots, m. \end{aligned} \quad (5.11)$$

$$\begin{aligned}
& \sum_{\tilde{a}_{ij} \in S_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} + \sigma_{ik2} - r\sigma_{ik2}) + \sum_{\tilde{a}_{ij} \in T_1} (a_{ij} + \alpha'_{ij} - r\alpha'_{ij})(x_{jk} - \sigma_{ik2} + r\sigma_{ik2}) + \\
& \sum_{\tilde{b}_{lk} \in S_2} (x_{il} + \sigma_{ik2} - r\sigma_{ik2})(b_{lk} + \beta'_{lk} - r\beta'_{lk}) + \sum_{\tilde{b}_{lk} \in T_2} (x_{il} - \sigma_{ik2} + r\sigma_{ik2})(b_{lk} - \beta_{lk} + r\beta_{lk}) = \\
& c_{ik} + \gamma_{ik2} - r\gamma_{ik2}, i=1, 2, \dots, n, j=1, 2, \dots, n, l=1, 2, \dots, m, k=1, 2, \dots, m.
\end{aligned} \tag{5.12}$$

Step 10: Put the values of $x_{ik}, i=1, 2, \dots, n, k=1, 2, \dots, m$, obtained in Step 8, in the crisp matrix equations (5.11) and (5.12) and then solve these to find the values of $\sigma_{ik1}(r)$ and $\sigma_{ik2}(r)$.

Step 11: Find

$$\begin{aligned}
\alpha_{sk}^{-,l}(r) &= (1-r) \text{minimum} \{|\sigma_{ik1}(0)|, |\sigma_{ik2}(0)|\}, i=1, 2, \dots, n, k=1, 2, \dots, m, \\
\alpha_{sk}^{-,u}(r) &= (1-r) \text{maximum} \{|\sigma_{ik1}(0)|, |\sigma_{ik2}(0)|\}, i=1, 2, \dots, n, k=1, 2, \dots, m, \\
\alpha_{sk}^{+,l}(r) &= (1-r) \text{minimum} \{|\sigma_{ik1}(1)|, |\sigma_{ik2}(1)|\}, i=1, 2, \dots, n, k=1, 2, \dots, m, \\
\alpha_{sk}^{+,u}(r) &= (1-r) \text{maximum} \{|\sigma_{ik1}(1)|, |\sigma_{ik2}(1)|\}, i=1, 2, \dots, n, k=1, 2, \dots, m.
\end{aligned}$$

Step 12: Using the values of $x_{ik}, i=1, 2, \dots, n, k=1, 2, \dots, m$, obtained in Step 8, and the values of $\alpha_{sk}^{-,l}(r), \alpha_{sk}^{-,u}(r), \alpha_{sk}^{+,l}(r)$ and $\alpha_{sk}^{+,u}(r)$, obtained in Step 11, the r -cut of the maximal fuzzy symmetric solution $\tilde{X}^{-,u}, \tilde{X}^{+,u}$ and the minimal fuzzy symmetric solution $\tilde{X}^{-,l}, \tilde{X}^{+,l}$ of the fully fuzzy Sylvester matrix equation (5.1) are

$$\begin{aligned}
& [x_{ik} - \alpha_{sk}^{-,u}(r), x_{ik} + \alpha_{sk}^{-,u}(r)], [x_{ik} - \alpha_{sk}^{+,u}(r), x_{ik} + \alpha_{sk}^{+,u}(r)] \text{ and } [x_{ik} - \alpha_{sk}^{-,l}(r), x_{ik} + \alpha_{sk}^{-,l}(r)], \\
& [x_{ik} - \alpha_{sk}^{+,l}(r), x_{ik} + \alpha_{sk}^{+,l}(r)], i=1, 2, \dots, n, k=1, 2, \dots, m \text{ respectively.}
\end{aligned}$$

5.3. Flaws of Dookhitram et al. method

Dookhitram et al. [59] solved the fuzzy Sylvester matrix equation, presented in Example 5.1, to illustrate their proposed method.

Example 5.1 [59, Section 4, Example 4.1, pp. 2208]. Find the maximal and minimal fuzzy solution of the fuzzy Sylvester matrix equation $A\tilde{X} - \tilde{X}B = \tilde{C}$, with the matrices

$$\tilde{A} = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 5 \end{pmatrix}, \tilde{B} = \begin{pmatrix} -4 & 5 \\ 3 & -5 \end{pmatrix} \text{ and the right hand side fuzzy matrix}$$

$$\tilde{C} = \begin{pmatrix} (-25+16r, 10-19r) & (-28+26r, 14-16r) \\ (-15+22r, 26-19r) & (-18+20r, 31-29r) \\ (-10+17r, 24-17r) & (-4+20r, 35-19r) \end{pmatrix},$$

where all the elements of the unknown matrix $\tilde{X}$ are non-negative symmetric triangular fuzzy numbers.

Dookhitram et al. [59] claimed that on solving this fuzzy Sylvester matrix equation by their proposed method, the obtained maximal and minimal fuzzy solutions are

$$\tilde{X}^U = \begin{bmatrix} [2-19(r-1), 2+19(r-1)] & [3-10(r-1), 3+10(r-1)] \\ [3-19(r-1), 3+19(r-1)] & [3-10(r-1), 3+10(r-1)] \\ [3-19(r-1), 3+19(r-1)] & [4-10(r-1), 4+10(r-1)] \end{bmatrix}$$

and

$$\tilde{X}^L = \begin{bmatrix} \left[2 - \frac{19(r-1)}{2}, 2 + \frac{19(r-1)}{2}\right] & [3-4(r-1), 3+4(r-1)] \\ \left[3 - \frac{19(r-1)}{2}, 3 + \frac{19(r-1)}{2}\right] & [3-4(r-1), 3+4(r-1)] \\ \left[3 - \frac{19(r-1)}{2}, 3 + \frac{19(r-1)}{2}\right] & [4-4(r-1), 4+4(r-1)] \end{bmatrix}$$

respectively. It can be easily verified that on putting $r=0$ and $r=1$ in $\tilde{X}^U$ and $\tilde{X}^L$, these are transformed into

$$\tilde{X}^U = \begin{bmatrix} (21, 2, -17) & (13, 3, -7) \\ (22, 3, -16) & (13, 3, -7) \\ (22, 3, -16) & (14, 4, -6) \end{bmatrix} \text{ and } \tilde{X}^L = \begin{bmatrix} \left(\frac{23}{2}, 2, -\frac{15}{2}\right) & (7, 3, -1) \\ \left(\frac{25}{2}, 3, -\frac{13}{2}\right) & (7, 3, -1) \\ \left(\frac{25}{2}, 3, -\frac{13}{2}\right) & (8, 4, 0) \end{bmatrix}, \text{ respectively.}$$

It is obvious that the elements of $\tilde{X}^U$ and $\tilde{X}^L$ are not the fuzzy numbers. Although, this flaw of the existing method [59] can be easily resolved by the following modifications:

Use Phase-1 of Two-Phase method to solve the system of crisp matrix equations (5.11) and (5.12) with the additional constraints $\sigma_{ik1} \geq 0, \sigma_{ik2} \geq 0$ and $x_{ik} - \sigma_{ik} \leq x_{ik} \leq x_{ik} + \sigma_{ik}$. However, there are several limitations of the existing method [59] which are pointed out in next section. Therefore, in Section 5.6, a new method (named as Mehar-Keerat method-IV) is proposed to overcome the limitations as well as to resolve the flaws of the existing method [59].

5.4. Limitations of Dookhitram et al. method

The method, proposed by Dookhitram et al. [59], can be used only to find the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix (5.1), where the elements $\tilde{a}_{ij}, \tilde{b}_{lk}$ and $\tilde{c}_{ik}$ are non-negative/non-positive triangular fuzzy numbers and the elements $\tilde{x}_{jk}$ and $\tilde{x}_{il}$ are non-negative symmetric triangular fuzzy numbers.

For example, the existing fuzzy Sylvester matrix equation, presented in Example 5.1 [59, Section 4, Example 4.1, pp. 2208], is same type of fuzzy Sylvester matrix equation and hence can be solved by the existing method [59].

However, if in the fully fuzzy Sylvester matrix equation (5.1), the variables $\tilde{x}_{jk}$ and $\tilde{x}_{il}, j = 1, 2, \dots, n, k = 1, 2, \dots, m, i = 1, 2, \dots, n, l = 1, 2, \dots, m$, are assumed as non-negative non-symmetric triangular/trapezoidal fuzzy numbers instead of non-negative symmetric triangular fuzzy numbers, then, the maximal and minimal fuzzy solution of the transformed fully fuzzy Sylvester matrix equation cannot be obtained by the existing method [59]. For example, if in the existing fuzzy Sylvester matrix equation, presented in Example 5.1 [59, Section 4, Example 4.1, pp. 2208], the variables $\tilde{x}_{ik}, i = 1, 2, 3, k = 1, 2$, are assumed as non-negative non-symmetric triangular fuzzy numbers instead of non-negative symmetric triangular fuzzy numbers, then the maximal and minimal fuzzy solution of the transformed fuzzy Sylvester matrix equation, presented in Example 5.2, cannot be obtained by the

existing method [59].

Example 5.2. Find the maximal and minimal fuzzy solution of the fuzzy Sylvester matrix equation $A\tilde{X} - \tilde{X}B = \tilde{C}$, with the matrices

$$\tilde{A} = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 5 \end{pmatrix}, \tilde{B} = \begin{pmatrix} -4 & 5 \\ 3 & -5 \end{pmatrix} \text{ and the right hand side fuzzy matrix}$$

$$\tilde{C} = \begin{pmatrix} (-25+16r, 10-19r) & (-28+26r, 14-16r) \\ (-15+22r, 26-19r) & (-18+20r, 31-29r) \\ (-10+17r, 24-17r) & (-4+20r, 35-19r) \end{pmatrix},$$

where all the elements of the unknown solution matrix $\tilde{X}$ non-negative non-symmetric trapezoidal fuzzy numbers.

Furthermore, the existing method [59] cannot be used to find the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$, if

- (i) The elements of unknown matrix $\tilde{X}$ are unrestricted symmetric/non-symmetric triangular/trapezoidal fuzzy numbers instead of non-negative symmetric triangular fuzzy numbers.
- (ii) The elements of the coefficient matrices $\tilde{A}$ and $\tilde{B}$ are unrestricted triangular/trapezoidal fuzzy numbers instead of non-negative/non-positive triangular fuzzy numbers.
- (iii) The restrictions mentioned in (i) and (ii) hold simultaneously.

For example, the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation, chosen in Example 5.3, in which the restrictions mentioned in (i) and (ii) hold simultaneously, cannot be obtained by the existing method [59].

Example 5.3. Find the maximal and minimal fuzzy solution of the fully fuzzy Sylvester matrix equation.

$$\begin{pmatrix} (1, 2, 3, 4) & (-8, -5, -2, 1) \\ (-3, 2, 7, 12) & (-2, -1, 0, 1) \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} + \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} \begin{pmatrix} (-5, -4, -3, -2) & (4, 5, 6, 7) \\ (-1, 0, 1, 2) & (2, 3, 4, 5) \end{pmatrix} \\ = \begin{pmatrix} (-36, -6, 17, 48) & (-119, -66, -29, 1) \\ (-83, -35, 0, 40) & (-80, -24, 10, 46) \end{pmatrix},$$

where $\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}$ and $\tilde{x}_{22}$ are unrestricted symmetric/non-symmetric trapezoidal fuzzy numbers.

5.5. Proposed Mehar-Keerat method-IV

In this section, to overcome the limitations as well as to resolve the flaws of the existing method [59], a new method (named as Mehar-Keerat method-IV) is proposed to find the maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$,

$$\text{i.e., } \sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_{jk} + \sum_{l=1}^m \tilde{x}_{il} \tilde{b}_{lk} = \tilde{c}_{ik} \text{ for } i = 1, 2, \dots, n, k = 1, 2, \dots, m. \quad (5.13)$$

where $\tilde{a}_{ij}, \tilde{x}_{jk}, \tilde{b}_{lk}, \tilde{x}_{il}$ and $\tilde{c}_{ik}$ are unrestricted non-symmetric trapezoidal fuzzy numbers.

The same method can also be used to find the maximal and minimal fuzzy solution of all other types of system of fuzzy linear/matrix equations.

The steps of proposed Mehar-Keerat method-IV are as follows:

Step 1: Assuming

$\tilde{a}_{ij} = (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}), \tilde{x}_{jk} = (x_{jk}^L - \sigma_{jk}, x_{jk}^L, x_{jk}^U, x_{jk}^U + \sigma'_{jk}), \tilde{b}_{lk} = (b_{lk}^L - \beta_{lk}, b_{lk}^L, b_{lk}^U, b_{lk}^U + \beta'_{lk}),$
 $\tilde{x}_{il} = (x_{il}^L - \sigma_{il}, x_{il}^L, x_{il}^U, x_{il}^U + \sigma'_{il})$ and $\tilde{c}_{ik} = (c_{ik}^L - \gamma_{ik}, c_{ik}^L, c_{ik}^U, c_{ik}^U + \gamma'_{ik})$, the fully fuzzy Sylvester matrix equation (5.13) can be transformed into fully fuzzy Sylvester matrix equation (5.14).

$$\sum_{j=1}^n (a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij}) (x_{jk}^L - \sigma_{jk}, x_{jk}^L, x_{jk}^U, x_{jk}^U + \sigma'_{jk}) + \sum_{l=1}^m (x_{il}^L - \sigma_{il}, x_{il}^L, x_{il}^U, x_{il}^U + \sigma'_{il}) (b_{lk}^L - \beta_{lk}, b_{lk}^L, b_{lk}^U, b_{lk}^U + \beta'_{lk}) = (c_{ik}^L - \gamma_{ik}, c_{ik}^L, c_{ik}^U, c_{ik}^U + \gamma'_{ik}); i = 1, 2, \dots, n, k = 1, 2, \dots, m. \quad (5.14)$$

Step 2: Assuming

$(a_{ij}^L - \alpha_{ij}, a_{ij}^L, a_{ij}^U, a_{ij}^U + \alpha'_{ij})(x_{jk}^L - \sigma_{jk}, x_{jk}^L, x_{jk}^U, x_{jk}^U + \sigma'_{jk}) = (e_{ik}^L - \lambda_{ik}, e_{ik}^L, e_{ik}^U, e_{ik}^U + \lambda'_{ik})$ and
 $(x_{il}^L - \sigma_{il}, x_{il}^L, x_{il}^U, x_{il}^U + \sigma'_{il})(b_{lk}^L - \beta_{lk}, b_{lk}^L, b_{lk}^U, b_{lk}^U + \beta'_{lk}) = (f_{ik}^L - \mu_{ik}, f_{ik}^L, f_{ik}^U, f_{ik}^U + \mu'_{ik})$, the fully
fuzzy Sylvester matrix equation (5.14) can be transformed into fully fuzzy Sylvester matrix
equation (5.15).

$$\sum_{k=1}^m (e_{ik}^L - \lambda_{ik}, e_{ik}^L, e_{ik}^U, e_{ik}^U + \lambda'_{ik}) + \sum_{k=1}^m (f_{ik}^L - \mu_{ik}, f_{ik}^L, f_{ik}^U, f_{ik}^U + \mu'_{ik}) = (c_{ik}^L - \gamma_{ik}, c_{ik}^L, c_{ik}^U, c_{ik}^U + \gamma'_{ik}); i = 1, 2, \dots, n, \\ k = 1, 2, \dots, m. \quad (5.15)$$

where, $(e_{ik}^L - \lambda_{ik}, e_{ik}^L, e_{ik}^U, e_{ik}^U + \lambda'_{ik}), (f_{ik}^L - \mu_{ik}, f_{ik}^L, f_{ik}^U, f_{ik}^U + \mu'_{ik})$ can be obtained by the existing
product [94, Section 4, pp. 7145] of two trapezoidal fuzzy numbers $(a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1)$
and $(a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2)$, presented in Remark 4.2 of Chapter 4.

Step 3: Using the property,

$$\sum_{i=1}^n (a_i^L - \alpha_i, a_i^L, a_i^U, a_i^U + \alpha'_i) = \left(\sum_{i=1}^n a_i^L - \alpha_i, \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^U, \sum_{i=1}^n a_i^U + \alpha'_i \right), \text{ the fully fuzzy Sylvester}$$

matrix equation (5.15) can be transformed into fully fuzzy Sylvester matrix equation (5.16).

$$\left(\sum_{k=1}^m e_{ik}^L - \lambda_{ik}, \sum_{k=1}^m e_{ik}^L, \sum_{k=1}^m e_{ik}^U, \sum_{k=1}^m e_{ik}^U + \lambda'_{ik} \right) + \left(\sum_{k=1}^m f_{ik}^L - \mu_{ik}, \sum_{k=1}^m f_{ik}^L, \sum_{k=1}^m f_{ik}^U, \sum_{k=1}^m f_{ik}^U + \mu'_{ik} \right) = (c_{ik}^L - \gamma_{ik}, c_{ik}^L, c_{ik}^U, c_{ik}^U + \gamma'_{ik}); \\ i = 1, 2, \dots, n, k = 1, 2, \dots, m. \quad (5.16)$$

Step 4: Using the property

$$(a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha'_1) + (a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha'_2) \\ = (a_1^L - \alpha_1 + a_2^L - \alpha_2, a_1^L + a_2^L, a_1^U + a_2^U, a_1^U + \alpha'_1 + a_2^U + \alpha'_2),$$

the fully fuzzy Sylvester matrix equation (5.16) can be transformed into fully fuzzy Sylvester
matrix equation (5.17).

$$\left(\sum_{k=1}^m e_{ik}^L - \lambda_{ik} + \sum_{k=1}^m f_{ik}^L - \mu_{ik}, \sum_{k=1}^m e_{ik}^L + \sum_{k=1}^m f_{ik}^L, \sum_{k=1}^m e_{ik}^U + \sum_{k=1}^m f_{ik}^U, \sum_{k=1}^m e_{ik}^U + \lambda'_{ik} + \sum_{k=1}^m f_{ik}^U + \mu'_{ik} \right) = (c_{ik}^L - \gamma_{ik}, c_{ik}^L, c_{ik}^U, c_{ik}^U + \gamma'_{ik});$$

$$i = 1, 2, \dots, n, k = 1, 2, \dots, m. \quad (5.17)$$

Step 5: Using Definition 5.3 (definition of maximal fuzzy solution), the fully fuzzy Sylvester matrix equation (5.17) can be transformed into multi-objective mathematical programming problem (5.18).

Maximize $\sigma_{jk}; j = 1, 2, \dots, n, k = 1, 2, \dots, m,$

Maximize $\sigma'_{jk}; j = 1, 2, \dots, n, k = 1, 2, \dots, m,$

Maximize $\sigma_{il}; i = 1, 2, \dots, n, l = 1, 2, \dots, m,$

Maximize $\sigma'_{il}; i = 1, 2, \dots, n, l = 1, 2, \dots, m,$

subject to

$$\sum_{k=1}^m e_{ik}^L + \sum_{k=1}^m f_{ik}^L - \left(\sum_{k=1}^m e_{ik}^L - \lambda_{ik} + \sum_{k=1}^m f_{ik}^L - \mu_{ik} \right) \leq \gamma_{ik}; i = 1, 2, \dots, n, k = 1, 2, \dots, m,$$

$$\sum_{k=1}^m e_{ik}^L + \sum_{k=1}^m f_{ik}^L = c_{ik}^L; i = 1, 2, \dots, n, k = 1, 2, \dots, m, \quad (5.18)$$

$$\sum_{k=1}^m e_{ik}^U + \sum_{k=1}^m f_{ik}^U = c_{ik}^U; i = 1, 2, \dots, n, k = 1, 2, \dots, m,$$

$$\sum_{k=1}^m e_{ik}^U + \lambda'_{ik} + \sum_{k=1}^m f_{ik}^U + \mu'_{ik} - \left(\sum_{k=1}^m e_{ik}^U + \sum_{k=1}^m f_{ik}^U \right) \leq \gamma'_{ik}; i = 1, 2, \dots, n, k = 1, 2, \dots, m,$$

$$x_{jk}^L - \sigma_{jk} \leq x_{jk}^L \leq x_{jk}^U \leq x_{jk}^U + \sigma'_{jk}, x_{il}^L - \sigma_{il} \leq x_{il}^L \leq x_{il}^U \leq x_{il}^U + \sigma'_{il} \text{ and } \sigma_{jk} \geq 0, \sigma'_{jk} \geq 0, \sigma_{il} \geq 0, \sigma'_{il} \geq 0.$$

Step 6: Using Definition 5.4 (definition of minimal fuzzy solution), the fully fuzzy Sylvester matrix equation (5.17) can be transformed into multi-objective mathematical programming problem (5.19).

Minimize $\sigma_{jk}; j = 1, 2, \dots, n, k = 1, 2, \dots, m,$

Minimize $\sigma'_{jk}; j = 1, 2, \dots, n, k = 1, 2, \dots, m,$

Minimize $\sigma_{il}; i = 1, 2, \dots, n, l = 1, 2, \dots, m,$

Minimize $\sigma'_{il}; i = 1, 2, \dots, n, l = 1, 2, \dots, m,$

subject to

$$\sum_{k=1}^m e_{ik}^L + \sum_{k=1}^m f_{ik}^L - \left(\sum_{k=1}^m e_{ik}^L - \lambda_{ik} + \sum_{k=1}^m f_{ik}^L - \mu_{ik} \right) \geq \gamma_{ik}; i = 1, 2, \dots, n, k = 1, 2, \dots, m,$$

$$\sum_{k=1}^m e_{ik}^L + \sum_{k=1}^m f_{ik}^L = c_{ik}^L; i = 1, 2, \dots, n, k = 1, 2, \dots, m, \quad (5.19)$$

$$\sum_{k=1}^m e_{ik}^U + \sum_{k=1}^m f_{ik}^U = c_{ik}^U; i=1, 2, \dots, n, k=1, 2, \dots, m,$$

$$\sum_{k=1}^m e_{ik}^U + \lambda'_{ik} + \sum_{k=1}^m f_{ik}^U + \mu'_{ik} - \left(\sum_{k=1}^m e_{ik}^U + \sum_{k=1}^m f_{ik}^U \right) \geq \gamma'_{ik}; i=1, 2, \dots, n, k=1, 2, \dots, m,$$

$$x_{jk}^L - \sigma_{jk} \leq x_{jk}^L \leq x_{jk}^U \leq x_{jk}^U + \sigma'_{jk}, x_{il}^L - \sigma_{il} \leq x_{il}^L \leq x_{il}^U \leq x_{il}^U + \sigma'_{il} \text{ and } \sigma_{jk} \geq 0, \sigma'_{jk} \geq 0, \sigma_{il} \geq 0, \sigma'_{il} \geq 0.$$

Step 7: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (5.18) can be transformed into single-objective mathematical programming problem (5.20).

$$\begin{aligned} & \text{Maximize } (\sigma_{jk} + \sigma'_{jk} + \sigma_{il} + \sigma'_{il}); j=1, 2, \dots, n, k=1, 2, \dots, m, i=1, 2, \dots, n, l=1, 2, \dots, m, \\ & \text{subject to} \\ & \text{Constraints of Problem (5.18).} \end{aligned} \quad (5.20)$$

Step 8: By the equal-weighted summation method [69], the multi-objective mathematical programming problem (5.19) can be transformed into single-objective mathematical programming problem (5.21).

$$\begin{aligned} & \text{Minimize } (\sigma_{jk} + \sigma'_{jk} + \sigma_{il} + \sigma'_{il}); j=1, 2, \dots, n, k=1, 2, \dots, m, i=1, 2, \dots, n, l=1, 2, \dots, m, \\ & \text{subject to} \\ & \text{Constraints of Problem (5.19).} \end{aligned} \quad (5.21)$$

Step 9: Find the optimal solution $\{x_{jk}^L, x_{jk}^U, \sigma_{jk}, \sigma'_{jk}, x_{il}^L, x_{il}^U, \sigma_{il}, \sigma'_{il}\}$ of the single-objective mathematical programming problem (5.20).

Step 10: Find the optimal solution $\{x_{jk}^L, x_{jk}^U, \sigma_{jk}, \sigma'_{jk}, x_{il}^L, x_{il}^U, \sigma_{il}, \sigma'_{il}\}$ of the single-objective mathematical programming problem (5.21).

Step 11: Using the optimal solution $\{x_{jk}^L, x_{jk}^U, \sigma_{jk}, \sigma'_{jk}, x_{il}^L, x_{il}^U, \sigma_{il}, \sigma'_{il}\}$ of the single-objective mathematical programming problem (5.20), obtained in Step 9, the maximal fuzzy solution of the fully fuzzy Sylvester matrix equation (5.13) is

$$[\tilde{x}_{ik}]_{n \times m} = \left[\left(x_{ik}^L - \sigma_{ik}, x_{ik}^L, x_{ik}^U, x_{ik}^U + \sigma'_{ik} \right) \right]_{n \times m}.$$

Step 12: Using the optimal solution $\{x_{jk}^L, x_{jk}^U, \sigma_{jk}, \sigma'_{jk}, x_{il}^L, x_{il}^U, \sigma_{il}, \sigma'_{il}\}$ of the single-objective

mathematical programming problem (5.21), obtained in Step 10, the minimal fuzzy solution of the fully fuzzy Sylvester matrix equation (5.13) is

$$[\tilde{x}_{ik}]_{n \times m} = \left[\left(x_{ik}^L - \sigma_{ik}, x_{ik}^L, x_{ik}^U, x_{ik}^U + \sigma'_{ik} \right) \right]_{n \times m}.$$

Remark 5.2. The maximal and minimal fuzzy symmetric solution of fully fuzzy Sylvester matrix equation $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ can be obtained by solving the mathematical programming problems (5.20) and (5.21) respectively, with the additional constraints $\sigma_{jk} = \sigma'_{jk}, \sigma_{il} = \sigma'_{il}$.

5.6. Illustrative examples

In this section, to illustrate the proposed Mehar-Keerat method-IV, the fuzzy Sylvester matrix equation, chosen in Example 5.2 and the fully fuzzy Sylvester matrix equation, chosen in Example 5.3 which cannot be solved by the Dookhitram et al. method [59], are solved by proposed Mehar-Keerat method-IV. Also, the maximal and minimal fuzzy solution of the fuzzy Sylvester matrix equation, presented in Example 5.1, is obtained.

5.6.1. Maximal and minimal fuzzy solution of fuzzy Sylvester matrix equation chosen in Example 5.1 and Example 5.2

Using Mehar-Keerat method-IV, proposed in Section 5.5, the maximal and minimal fuzzy solution of the fuzzy Sylvester matrix equation, chosen in Example 5.1 and Example 5.2, can be obtained as follows.

Step 1: Assuming

$$\tilde{x}_{11} = (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}), \tilde{x}_{21} = (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}),$$

$$\tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}), \tilde{x}_{31} = (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) \text{ and } \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}),$$

the fuzzy Sylvester matrix equation, chosen in Example 5.2, can be transformed into the fuzzy Sylvester matrix equation (5.22).

$$\begin{aligned}
& 6(x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) - 3(x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) - (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) - 3(x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) \\
& = (-25, -9, 10), \\
& 7(x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) - 3(x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) - (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) - 5(x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) \\
& = (-28, -2, 14), \\
& -(x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) + 7(x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) - (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) - 3(x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) \\
& = (-15, 7, 26), \\
& -(x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) + 8(x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) - (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) - 5(x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) \\
& = (-18, 2, 31), \tag{5.22} \\
& -(x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) - 2(x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) + 9(x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) - 3(x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) \\
& = (-10, 7, 24), \\
& -(x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) - 2(x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) + 10(x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) - 5(x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) \\
& = (-4, 16, 35).
\end{aligned}$$

Step 2: Using Step 2 to Step 4 of the proposed Mehar-Keerat method-IV, the fuzzy Sylvester matrix equation (5.22) can be transformed into the fuzzy Sylvester matrix equation (5.23).

$$\begin{aligned}
& \begin{pmatrix} 6x_{11} - 6\sigma_{11} - 3x_{21} - 3\sigma'_{21} - x_{31} - \sigma'_{31} - 3x_{12} - 3\sigma'_{12}, 6x_{11} - 3x_{21} - x_{31} - 3x_{12}, 6x_{11} + 6\sigma'_{11} - 3x_{21} + 3\sigma_{21} - \\ x_{31} + \sigma_{31} - 3x_{12} + 3\sigma_{12} \end{pmatrix} \\
& = (-25, -9, 10), \\
& \begin{pmatrix} 7x_{12} - 7\sigma_{12} - 3x_{22} - 3\sigma'_{22} - x_{32} - \sigma'_{32} - 5x_{11} - 5\sigma'_{11}, 7x_{12} - 3x_{22} - x_{32} - 5x_{11}, 7x_{12} + 7\sigma'_{12} - 3x_{22} + 3\sigma_{22} - \\ x_{32} + \sigma_{32} - 5x_{11} + 5\sigma_{11} \end{pmatrix} \\
& = (-28, -2, 14), \\
& \begin{pmatrix} -x_{11} - \sigma'_{11} + 7x_{21} - 7\sigma_{21} - x_{31} - \sigma'_{31} - 3x_{22} - 3\sigma'_{22}, -x_{11} + 7x_{21} - x_{31} - 3x_{22}, -x_{11} + \sigma_{11} + 7x_{21} + 7\sigma'_{21} - \\ x_{31} + \sigma_{31} - 3x_{22} + 3\sigma_{22} \end{pmatrix} \\
& = (-15, 7, 26), \\
& \begin{pmatrix} -x_{12} - \sigma'_{12} + 8x_{22} - 8\sigma_{22} - x_{32} - \sigma'_{32} - 5x_{21} - 5\sigma'_{21}, -x_{12} + 8x_{22} - x_{32} - 5x_{21}, -x_{12} + \sigma_{12} + 8x_{22} + 8\sigma'_{22} - \\ x_{32} + \sigma_{32} - 5x_{21} + 5\sigma_{21} \end{pmatrix} \\
& = (-18, 2, 31), \tag{5.23} \\
& \begin{pmatrix} -x_{11} - \sigma'_{11} - 2x_{21} - 2\sigma'_{21} + 9x_{31} - 9\sigma_{31} - 3x_{32} - 3\sigma'_{32}, -x_{11} - 2x_{21} + 9x_{31} - 3x_{32}, -x_{11} + \sigma_{11} - 2x_{21} + 2\sigma_{21} + \\ 9x_{31} + 9\sigma'_{31} - 3x_{32} + 3\sigma_{32} \end{pmatrix} \\
& = (-10, 7, 24), \\
& \begin{pmatrix} -x_{12} - \sigma'_{12} - 2x_{22} - 2\sigma'_{22} + 10x_{32} - 10\sigma_{32} - 5x_{31} - 5\sigma'_{31}, -x_{12} - 2x_{22} + 10x_{32} - 5x_{31}, -x_{12} + \sigma_{12} - 2x_{22} + \\ 2\sigma_{22} + 10x_{32} + 10\sigma'_{32} - 5x_{31} + 5\sigma_{31} \end{pmatrix} \\
& = (-4, 16, 35).
\end{aligned}$$

Step 3: Using Step 5 of the proposed Mehar-Keerat method-IV, the fuzzy Sylvester matrix equation (5.23) can be transformed into multi-objective mathematical programming problem (5.24).

$$\begin{aligned}
& \text{Maximize } \sigma_{11}, \\
& \text{Maximize } \sigma'_{11}, \\
& \text{Maximize } \sigma_{12}, \\
& \text{Maximize } \sigma'_{12}, \\
& \text{Maximize } \sigma_{21}, \\
& \text{Maximize } \sigma'_{21}, \\
& \text{Maximize } \sigma_{22}, \\
& \text{Maximize } \sigma'_{22}, \\
& \text{Maximize } \sigma_{31}, \\
& \text{Maximize } \sigma'_{31}, \\
& \text{Maximize } \sigma_{32}, \\
& \text{Maximize } \sigma'_{32}, \\
& \text{subject to} \\
& 6\sigma_{11} + 3\sigma'_{21} + \sigma'_{31} + 3\sigma'_{12} \leq 16, \\
& 6x_{11} - 3x_{21} - x_{31} - 3x_{12} = -9, \\
& 6\sigma'_{11} + 3\sigma_{21} + \sigma_{31} + 3\sigma_{12} \leq 19, \\
& 7\sigma_{12} + 3\sigma'_{22} + \sigma'_{32} + 5\sigma'_{11} \leq 26, \\
& 7x_{12} - 3x_{22} - x_{32} - 5x_{11} = -2, \\
& 7\sigma'_{12} + 3\sigma_{22} + \sigma_{32} + 5\sigma_{11} \leq 16, \\
& \sigma'_{11} + 7\sigma_{21} + \sigma'_{31} + 3\sigma'_{22} \leq 22, \\
& -x_{11} + 7x_{21} - x_{31} - 3x_{22} = 7, \\
& \sigma_{11} + 7\sigma'_{21} + \sigma_{31} + 3\sigma_{22} \leq 19, \\
& \sigma'_{12} + 8\sigma_{22} + \sigma'_{32} + 5\sigma'_{21} \leq 20, \\
& -x_{12} + 8x_{22} - x_{32} - 5x_{21} = 2, \\
& \sigma_{12} + 8\sigma'_{22} + \sigma_{32} + 5\sigma_{21} \leq 29, \\
& \sigma'_{11} + 2\sigma'_{21} + 9\sigma_{31} + 3\sigma'_{32} \leq 17, \\
& -x_{11} - 2x_{21} + 9x_{31} - 3x_{32} = 7, \\
& \sigma_{11} + 2\sigma_{21} + 9\sigma'_{31} + 3\sigma_{32} \leq 17, \\
& \sigma'_{12} + 2\sigma'_{22} + 10\sigma_{32} + 5\sigma'_{31} \leq 20, \\
& -x_{12} - 2x_{22} + 10x_{32} - 5x_{31} = 16, \\
& \sigma_{12} + 2\sigma_{22} + 10\sigma'_{32} + 5\sigma_{31} \leq 19, \\
& 0 \leq x_{11} - \sigma_{11} \leq x_{11} \leq x_{11} + \sigma'_{11}, 0 \leq x_{12} - \sigma_{12} \leq x_{12} \leq x_{12} + \sigma'_{12}, 0 \leq x_{21} - \sigma_{21} \leq x_{21} \leq x_{21} + \sigma'_{21},
\end{aligned} \tag{5.24}$$

$$0 \leq x_{22} - \sigma_{22} \leq x_{22} \leq x_{22} + \sigma'_{22}, 0 \leq x_{31} - \sigma_{31} \leq x_{31} \leq x_{31} + \sigma'_{31}, 0 \leq x_{32} - \sigma_{32} \leq x_{32} \leq x_{32} + \sigma'_{32},$$

$$\sigma_{11} \geq 0, \sigma'_{11} \geq 0, \sigma_{12} \geq 0, \sigma'_{12} \geq 0, \sigma_{21} \geq 0, \sigma'_{21} \geq 0, \sigma_{22} \geq 0, \sigma'_{22} \geq 0, \sigma_{31} \geq 0, \sigma'_{31} \geq 0, \sigma_{32} \geq 0, \sigma'_{32} \geq 0.$$

Step 4: Using Step 6 of the proposed Mehar-Keerat method-IV, the fuzzy Sylvester matrix equation (5.23) can be transformed into multi-objective mathematical programming problem (5.25).

$$\begin{aligned} &\text{Minimize } \sigma_{11}, \\ &\text{Minimize } \sigma'_{11}, \\ &\text{Minimize } \sigma_{12}, \\ &\text{Minimize } \sigma'_{12}, \\ &\text{Minimize } \sigma_{21}, \\ &\text{Minimize } \sigma'_{21}, \\ &\text{Minimize } \sigma_{22}, \\ &\text{Minimize } \sigma'_{22}, \\ &\text{Minimize } \sigma_{31}, \\ &\text{Minimize } \sigma'_{31}, \\ &\text{Minimize } \sigma_{32}, \\ &\text{Minimize } \sigma'_{32}, \\ &\text{subject to} \\ &6\sigma_{11} + 3\sigma'_{21} + \sigma'_{31} + 3\sigma'_{12} \geq 16, \\ &6x_{11} - 3x_{21} - x_{31} - 3x_{12} = -9, \\ &6\sigma'_{11} + 3\sigma_{21} + \sigma_{31} + 3\sigma_{12} \geq 19, \\ &7\sigma_{12} + 3\sigma'_{22} + \sigma'_{32} + 5\sigma'_{11} \geq 26, \\ &7x_{12} - 3x_{22} - x_{32} - 5x_{11} = -2, \\ &7\sigma'_{12} + 3\sigma_{22} + \sigma_{32} + 5\sigma_{11} \geq 16, \\ &\sigma'_{11} + 7\sigma_{21} + \sigma'_{31} + 3\sigma'_{22} \geq 22, \\ &-x_{11} + 7x_{21} - x_{31} - 3x_{22} = 7, \\ &\sigma_{11} + 7\sigma'_{21} + \sigma_{31} + 3\sigma_{22} \geq 19, \\ &\sigma'_{12} + 8\sigma_{22} + \sigma'_{32} + 5\sigma'_{21} \geq 20, \\ &-x_{12} + 8x_{22} - x_{32} - 5x_{21} = 2, \\ &\sigma_{12} + 8\sigma'_{22} + \sigma_{32} + 5\sigma_{21} \geq 29, \\ &\sigma'_{11} + 2\sigma'_{21} + 9\sigma_{31} + 3\sigma'_{32} \geq 17, \\ &-x_{11} - 2x_{21} + 9x_{31} - 3x_{32} = 7, \\ &\sigma_{11} + 2\sigma_{21} + 9\sigma'_{31} + 3\sigma_{32} \geq 17, \\ &\sigma'_{12} + 2\sigma'_{22} + 10\sigma_{32} + 5\sigma'_{31} \geq 20, \end{aligned} \tag{5.25}$$

$$\begin{aligned}
& -x_{12} - 2x_{22} + 10x_{32} - 5x_{31} = 16, \\
& \sigma_{12} + 2\sigma_{22} + 10\sigma'_{32} + 5\sigma_{31} \geq 19, \\
& 0 \leq x_{11} - \sigma_{11} \leq x_{11} \leq x_{11} + \sigma'_{11}, 0 \leq x_{12} - \sigma_{12} \leq x_{12} \leq x_{12} + \sigma'_{12}, 0 \leq x_{21} - \sigma_{21} \leq x_{21} \leq x_{21} + \sigma'_{21}, \\
& 0 \leq x_{22} - \sigma_{22} \leq x_{22} \leq x_{22} + \sigma'_{22}, 0 \leq x_{31} - \sigma_{31} \leq x_{31} \leq x_{31} + \sigma'_{31}, 0 \leq x_{32} - \sigma_{32} \leq x_{32} \leq x_{32} + \sigma'_{32}, \\
& \sigma_{11} \geq 0, \sigma'_{11} \geq 0, \sigma_{12} \geq 0, \sigma'_{12} \geq 0, \sigma_{21} \geq 0, \sigma'_{21} \geq 0, \sigma_{22} \geq 0, \sigma'_{22} \geq 0, \sigma_{31} \geq 0, \sigma'_{31} \geq 0, \sigma_{32} \geq 0, \sigma'_{32} \geq 0.
\end{aligned}$$

Step 5: Using Step 7 of the proposed Mehar-Keerat method-IV, the multi-objective mathematical programming problem (5.24) can be transformed into single-objective mathematical programming problem (5.26).

$$\begin{aligned}
& \text{Maximize } (\sigma_{11} + \sigma'_{11} + \sigma_{12} + \sigma'_{12} + \sigma_{21} + \sigma'_{21} + \sigma_{22} + \sigma'_{22} + \sigma_{31} + \sigma'_{31} + \sigma_{32} + \sigma'_{32}) \\
& \text{subject to} \\
& \text{Constraints of Problem (5.24).}
\end{aligned} \tag{5.26}$$

Step 6: Using Step 8 of the proposed Mehar-Keerat method-IV, the multi-objective mathematical programming problem (5.25) can be transformed into single-objective mathematical programming problem (5.27).

$$\begin{aligned}
& \text{Minimize } (\sigma_{11} + \sigma'_{11} + \sigma_{12} + \sigma'_{12} + \sigma_{21} + \sigma'_{21} + \sigma_{22} + \sigma'_{22} + \sigma_{31} + \sigma'_{31} + \sigma_{32} + \sigma'_{32}) \\
& \text{subject to} \\
& \text{Constraints of Problem (5.25).}
\end{aligned} \tag{5.27}$$

Step 7: On solving the single-objective mathematical programming problem (5.26)

(i) with the additional constraints $\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}, \sigma_{31} = \sigma'_{31}$

and $\sigma_{32} = \sigma'_{32}$, the obtained optimal solution is

$$\sigma_{11} = \frac{16139}{16057}, \sigma'_{11} = \frac{16139}{16057}, \sigma_{12} = \frac{47626}{48171}, \sigma'_{12} = \frac{47626}{48171}, \sigma_{21} = \frac{32387}{16057}, \sigma'_{21} = \frac{32387}{16057},$$

$$\sigma_{22} = \frac{15648}{16057}, \sigma'_{22} = \frac{15648}{16057}, \sigma_{31} = \frac{15291}{16057}, \sigma'_{31} = \frac{15291}{16057}, \sigma_{32} = \frac{54437}{48171}, \sigma'_{32} = \frac{54437}{48171},$$

$$x_{11} = 2, x_{12} = 3, x_{21} = 3, x_{22} = 3, x_{31} = 3, x_{32} = 4.$$

(ii) with the additional constraints $\sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22} =$

$\sigma_{31} = \sigma'_{31} = \sigma_{32} = \sigma'_{32}$, the obtained optimal solution is

$$\sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22} = \sigma_{31} = \sigma'_{31} = \sigma_{32} = \sigma'_{32} = 1, x_{11} = 2, x_{12} = 3,$$

$$x_{21} = 3, x_{22} = 3, x_{31} = 3, x_{32} = 4.$$

(iii) without any additional constraint, the obtained optimal solution is

$$\sigma_{11} = 1, \sigma'_{11} = 1, \sigma_{12} = 2, \sigma'_{12} = 1, \sigma_{21} = 2, \sigma'_{21} = 2, \sigma_{22} = 1, \sigma'_{22} = 2, \sigma_{31} = 1, \sigma'_{31} = 1,$$

$$\sigma_{32} = 1, \sigma'_{32} = 1, x_{11} = 2, x_{12} = 3, x_{21} = 3, x_{22} = 3, x_{31} = 3, x_{32} = 4.$$

Step 8: On solving the single-objective mathematical programming problem (5.27)

(i) with the additional constraints $\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}, \sigma_{31} = \sigma'_{31}$

and $\sigma_{32} = \sigma'_{32}$, the obtained optimal solution is

$$\sigma_{11} = \frac{15975}{16057}, \sigma'_{11} = \frac{15975}{16057}, \sigma_{12} = \frac{96887}{48171}, \sigma'_{12} = \frac{96887}{48171}, \sigma_{21} = \frac{31841}{16057}, \sigma'_{21} = \frac{31841}{16057},$$

$$\sigma_{22} = \frac{32523}{16057}, \sigma'_{22} = \frac{32523}{16057}, \sigma_{31} = \frac{16823}{16057}, \sigma'_{31} = \frac{16823}{16057}, \sigma_{32} = \frac{41905}{48171}, \sigma'_{32} = \frac{41905}{48171},$$

$$x_{11} = 2, x_{12} = 3, x_{21} = 3, x_{22} = 3, x_{31} = 3, x_{32} = 4.$$

(ii) with the additional constraints $\sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22} =$

$\sigma_{31} = \sigma'_{31} = \sigma_{32} = \sigma'_{32}$, the obtained optimal solution is

$$\sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22} = \sigma_{31} = \sigma'_{31} = \sigma_{32} = \sigma'_{32} = \frac{29}{15}, x_{11} = 2, x_{12} = 3,$$

$$x_{21} = 3, x_{22} = 3, x_{31} = 3, x_{32} = 4.$$

(iii) without any additional constraint, the obtained optimal solution is

$$\sigma_{11} = 1, \sigma'_{11} = 1, \sigma_{12} = 2, \sigma'_{12} = 1, \sigma_{21} = 2, \sigma'_{21} = 2, \sigma_{22} = 1, \sigma'_{22} = 2, \sigma_{31} = 1, \sigma'_{31} = 1, \sigma_{32} = 1,$$

$$\sigma'_{32} = 1, x_{11} = 2, x_{12} = 3, x_{21} = 3, x_{22} = 3, x_{31} = 3, x_{32} = 4.$$

Step 9: Using the optimal solution, obtained in Step 7 with the additional constraints

$$\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}, \sigma_{31} = \sigma'_{31}, \sigma_{32} = \sigma'_{32} \text{ and } \sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} =$$

$$\sigma'_{21} = \sigma_{22} = \sigma'_{22} = \sigma_{31} = \sigma'_{31} = \sigma_{32} = \sigma'_{32}, \text{ the maximal fuzzy solutions of the fuzzy Sylvester}$$

matrix equation, chosen in Example 5.1, are

$$\begin{aligned}\tilde{x}_{11} &= (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) = \left(\frac{15975}{16057}, 2, \frac{48253}{16057}\right), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) = \left(\frac{96887}{48171}, 3, \frac{192139}{48171}\right), \\ \tilde{x}_{21} &= (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) = \left(\frac{15784}{16057}, 3, \frac{80558}{16057}\right), \tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) = \left(\frac{32523}{16057}, 3, \frac{63819}{16057}\right), \\ \tilde{x}_{31} &= (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) = \left(\frac{32880}{16057}, 3, \frac{63462}{16057}\right), \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) = \left(\frac{138247}{48171}, 4, \frac{247121}{48171}\right)\end{aligned}$$

and

$$\begin{aligned}\tilde{x}_{11} &= (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) = (1, 2, 3), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) = (2, 3, 4), \\ \tilde{x}_{21} &= (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) = (2, 3, 4), \tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) = (2, 3, 4), \\ \tilde{x}_{31} &= (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) = (2, 3, 4), \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) = (3, 4, 5) \quad \text{respectively.}\end{aligned}$$

Also, using the optimal solution, obtained in Step 7 without any additional constraint, the maximal fuzzy solution of the fuzzy Sylvester matrix equation, chosen in Example 5.2, is

$$\begin{aligned}\tilde{x}_{11} &= (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) = (1, 2, 3), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) = (1, 3, 4), \\ \tilde{x}_{21} &= (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) = (1, 3, 5), \tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) = (2, 3, 5), \\ \tilde{x}_{31} &= (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) = (2, 3, 4), \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) = (3, 4, 5).\end{aligned}$$

Step 10: Using the optimal solution, obtained in Step 8 with the additional constraints

$$\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}, \sigma_{31} = \sigma'_{31}, \sigma_{32} = \sigma'_{32} \quad \text{and} \quad \sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} =$$

$$\sigma_{22} = \sigma'_{22} = \sigma_{31} = \sigma'_{31} = \sigma_{32} = \sigma'_{32}, \quad \text{the minimal fuzzy solutions of the fuzzy Sylvester matrix}$$

equation, chosen in Example 5.1, are

$$\begin{aligned}\tilde{x}_{11} &= (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) = \left(\frac{16139}{16057}, 2, \frac{48089}{16057}\right), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) = \left(\frac{47626}{48171}, 3, \frac{241400}{48171}\right), \\ \tilde{x}_{21} &= (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) = \left(\frac{16330}{16057}, 3, \frac{80012}{16057}\right), \tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) = \left(\frac{15648}{16057}, 3, \frac{80694}{16057}\right), \\ \tilde{x}_{31} &= (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) = \left(\frac{31348}{16057}, 3, \frac{64994}{16057}\right), \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) = \left(\frac{150779}{48171}, 4, \frac{234589}{48171}\right)\end{aligned}$$

and

$$\begin{aligned}\tilde{x}_{11} &= (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) = \left(\frac{1}{15}, 2, \frac{59}{15}\right), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) = \left(\frac{16}{15}, 3, \frac{74}{15}\right), \\ \tilde{x}_{21} &= (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) = \left(\frac{16}{15}, 3, \frac{74}{15}\right), \tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) = \left(\frac{16}{15}, 3, \frac{74}{15}\right),\end{aligned}$$

$$\tilde{x}_{31} = (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) = \left(\frac{16}{15}, 3, \frac{74}{15}\right), \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) = \left(\frac{31}{15}, 4, \frac{89}{15}\right)$$

respectively.

Also, using the optimal solution, obtained in Step 8 without any additional constraint, the minimal fuzzy solution of the fuzzy Sylvester matrix equation, chosen in Example 5.2, is

$$\begin{aligned}\tilde{x}_{11} &= (x_{11} - \sigma_{11}, x_{11}, x_{11} + \sigma'_{11}) = (1, 2, 3), \tilde{x}_{12} = (x_{12} - \sigma_{12}, x_{12}, x_{12} + \sigma'_{12}) = (1, 3, 4), \\ \tilde{x}_{21} &= (x_{21} - \sigma_{21}, x_{21}, x_{21} + \sigma'_{21}) = (1, 3, 5), \tilde{x}_{22} = (x_{22} - \sigma_{22}, x_{22}, x_{22} + \sigma'_{22}) = (2, 3, 5), \\ \tilde{x}_{31} &= (x_{31} - \sigma_{31}, x_{31}, x_{31} + \sigma'_{31}) = (2, 3, 4), \tilde{x}_{32} = (x_{32} - \sigma_{32}, x_{32}, x_{32} + \sigma'_{32}) = (3, 4, 5).\end{aligned}$$

5.6.2. Maximal and minimal fuzzy solution of chosen fully fuzzy Sylvester matrix equation

Using Mehar-Keerat method-IV, proposed in Section 5.5, the maximal and minimal fuzzy solution of fully fuzzy Sylvester matrix equation, chosen in Example 5.3, can be obtained as follows.

Step 1: Assuming

$$\begin{aligned}\tilde{x}_{11} &= (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}), \tilde{x}_{21} = (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}), \\ \tilde{x}_{22} &= (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}),\end{aligned}$$

the fully fuzzy Sylvester matrix equation, chosen in Example 5.3, can be transformed into fully fuzzy Sylvester matrix equation (5.28).

$$\begin{aligned}&\begin{pmatrix} (1, 2, 3, 4) & (-8, -5, -2, 1) \\ (-3, 2, 7, 12) & (-2, -1, 0, 1) \end{pmatrix} \begin{pmatrix} (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) & (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) \\ (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) & (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) \end{pmatrix} \\ &+ \begin{pmatrix} (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) & (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) \\ (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) & (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) \end{pmatrix} \begin{pmatrix} (-5, -4, -3, -2) & (4, 5, 6, 7) \\ (-1, 0, 1, 2) & (2, 3, 4, 5) \end{pmatrix} \\ &= \begin{pmatrix} (-36, -6, 17, 48) & (-119, -66, -29, 1) \\ (-83, -35, 0, 40) & (-80, -24, 10, 46) \end{pmatrix}. \tag{5.28}\end{aligned}$$

Step 2: Using Step 2 of proposed Mehar-Keerat method-IV, the fully fuzzy Sylvester matrix

equation (5.28) can be transformed into fully fuzzy Sylvester matrix equation (5.29).

$$\begin{aligned}
& \left(\frac{5x_{11}^L - 5\sigma_{11}}{2} - \left| \frac{-3x_{11}^L + 3\sigma_{11}}{2} \right|, \frac{5}{2}x_{11}^L - \left| -\frac{1}{2}x_{11}^L \right|, \frac{5}{2}x_{11}^U + \left| \frac{1}{2}x_{11}^U \right|, \frac{5x_{11}^U + 5\sigma'_{11}}{2} + \left| \frac{3x_{11}^U + 3\sigma'_{11}}{2} \right| \right) + \\
& \left(\frac{-8x_{21}^U + x_{21}^L - 8\sigma'_{21} - \sigma_{21}}{2} - \left| \frac{-8x_{21}^U - x_{21}^L - 8\sigma'_{21} + \sigma_{21}}{2} \right|, -\frac{7}{2}x_{21}^U - \left| -\frac{3}{2}x_{21}^U \right|, -\frac{7}{2}x_{21}^L + \left| \frac{3}{2}x_{21}^L \right|, \right. \\
& \left. \frac{x_{21}^U - 8x_{21}^L + \sigma'_{21} + 8\sigma_{21}}{2} + \left| \frac{x_{21}^U + 8x_{21}^L + \sigma'_{21} - 8\sigma_{21}}{2} \right| \right) + \\
& \left(\frac{-7x_{11}^U - 7\sigma'_{11}}{2} - \left| \frac{-3x_{11}^U - 3\sigma'_{11}}{2} \right|, -\frac{7}{2}x_{11}^U - \left| -\frac{1}{2}x_{11}^U \right|, -\frac{7}{2}x_{11}^L + \left| \frac{1}{2}x_{11}^L \right|, \frac{-7x_{11}^L + 7\sigma_{11}}{2} + \left| \frac{3x_{11}^L - 3\sigma_{11}}{2} \right| \right) + \\
& \left(\frac{-x_{12}^U + 2x_{12}^L - \sigma'_{12} - 2\sigma_{12}}{2} - \left| \frac{-x_{12}^U - 2x_{12}^L - \sigma'_{12} + 2\sigma_{12}}{2} \right|, \frac{1}{2}x_{12}^L - \left| -\frac{1}{2}x_{12}^L \right|, \frac{1}{2}x_{12}^U + \left| \frac{1}{2}x_{12}^U \right|, \right. \\
& \left. \frac{2x_{12}^U - x_{12}^L + 2\sigma'_{12} + \sigma_{12}}{2} + \left| \frac{2x_{12}^U + x_{12}^L + 2\sigma'_{12} - \sigma_{12}}{2} \right| \right) \\
& = (-36, -6, 17, 48), \\
& \left(\frac{5x_{12}^L - 5\sigma_{12}}{2} - \left| \frac{-3x_{12}^L + 3\sigma_{12}}{2} \right|, \frac{5}{2}x_{12}^L - \left| -\frac{1}{2}x_{12}^L \right|, \frac{5}{2}x_{12}^U + \left| \frac{1}{2}x_{12}^U \right|, \frac{5x_{12}^U + 5\sigma'_{12}}{2} + \left| \frac{3x_{12}^U + 3\sigma'_{12}}{2} \right| \right) + \\
& \left(\frac{-8x_{22}^U + x_{22}^L - 8\sigma'_{22} - \sigma_{22}}{2} - \left| \frac{-8x_{22}^U - x_{22}^L - 8\sigma'_{22} + \sigma_{22}}{2} \right|, -\frac{7}{2}x_{22}^U - \left| -\frac{3}{2}x_{22}^U \right|, -\frac{7}{2}x_{22}^L + \left| \frac{3}{2}x_{22}^L \right|, \right. \\
& \left. \frac{x_{22}^U - 8x_{22}^L + \sigma'_{22} + 8\sigma_{22}}{2} + \left| \frac{x_{22}^U + 8x_{22}^L + \sigma'_{22} - 8\sigma_{22}}{2} \right| \right) + \\
& \left(\frac{11x_{11}^L - 11\sigma_{11}}{2} - \left| \frac{-3x_{11}^L + 3\sigma_{11}}{2} \right|, \frac{11}{2}x_{11}^L - \left| -\frac{1}{2}x_{11}^L \right|, \frac{11}{2}x_{11}^U + \left| \frac{1}{2}x_{11}^U \right|, \frac{11x_{11}^U + 11\sigma'_{11}}{2} + \left| \frac{3x_{11}^U + 3\sigma'_{11}}{2} \right| \right) + \\
& \left(\frac{7x_{12}^L - 7\sigma_{12}}{2} - \left| \frac{-3x_{12}^L + 3\sigma_{12}}{2} \right|, \frac{7}{2}x_{12}^L - \left| -\frac{1}{2}x_{12}^L \right|, \frac{7}{2}x_{12}^U + \left| \frac{1}{2}x_{12}^U \right|, \frac{7x_{12}^U + 7\sigma'_{12}}{2} + \left| \frac{3x_{12}^U + 3\sigma'_{12}}{2} \right| \right) \\
& = (-119, -66, -29, 1), \tag{5.29}
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{-3x_{11}^U + 12x_{11}^L - 3\sigma'_{11} - 12\sigma_{11}}{2} - \left| \frac{-3x_{11}^U - 12x_{11}^L - 3\sigma'_{11} + 12\sigma_{11}}{2} \right|, \frac{9}{2}x_{11}^L - \left| -\frac{5}{2}x_{11}^L \right|, \frac{9}{2}x_{11}^U + \left| \frac{5}{2}x_{11}^U \right|, \right. \\
& \left. \frac{12x_{11}^U - 3x_{11}^L + 12\sigma'_{11} + 3\sigma_{11}}{2} + \left| \frac{12x_{11}^U + 3x_{11}^L + 12\sigma'_{11} - 3\sigma_{11}}{2} \right| \right) + \\
& \left(\frac{-2x_{21}^U + x_{21}^L - 2\sigma'_{21} - \sigma_{21}}{2} - \left| \frac{-2x_{21}^U - x_{21}^L - 2\sigma'_{21} + \sigma_{21}}{2} \right|, -\frac{1}{2}x_{21}^U - \left| -\frac{1}{2}x_{21}^U \right|, -\frac{1}{2}x_{21}^L + \left| \frac{1}{2}x_{21}^L \right|, \right. \\
& \left. \frac{x_{21}^U - 2x_{21}^L + \sigma'_{21} + 2\sigma_{21}}{2} + \left| \frac{x_{21}^U + 2x_{21}^L + \sigma'_{21} - 2\sigma_{21}}{2} \right| \right) + \\
& \left(\frac{-7x_{21}^U - 7\sigma'_{21}}{2} - \left| \frac{-3x_{21}^U - 3\sigma'_{21}}{2} \right|, -\frac{7}{2}x_{21}^U - \left| -\frac{1}{2}x_{21}^U \right|, -\frac{7}{2}x_{21}^L + \left| \frac{1}{2}x_{21}^L \right|, \frac{-7x_{21}^L + 7\sigma_{21}}{2} + \left| \frac{3x_{21}^L - 3\sigma_{21}}{2} \right| \right) + \\
& \left(\frac{-x_{22}^U + 2x_{22}^L - \sigma'_{22} - 2\sigma_{22}}{2} - \left| \frac{-x_{22}^U - 2x_{22}^L - \sigma'_{22} + 2\sigma_{22}}{2} \right|, \frac{1}{2}x_{22}^L - \left| -\frac{1}{2}x_{22}^L \right|, \frac{1}{2}x_{22}^U + \left| \frac{1}{2}x_{22}^U \right|, \right. \\
& \left. \frac{2x_{22}^U - x_{22}^L + 2\sigma'_{22} + \sigma_{22}}{2} + \left| \frac{2x_{22}^U + x_{22}^L + 2\sigma'_{22} - \sigma_{22}}{2} \right| \right) \\
& = (-83, -35, 0, 40),
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{-3x_{12}^U + 12x_{12}^L - 3\sigma'_{12} - 12\sigma_{12}}{2} - \left| \frac{-3x_{12}^U - 12x_{12}^L - 3\sigma'_{12} + 12\sigma_{12}}{2} \right|, \frac{9}{2}x_{12}^L - \left| -\frac{5}{2}x_{12}^L \right|, \frac{9}{2}x_{12}^U + \left| \frac{5}{2}x_{12}^U \right|, \right. \\
& \left. \frac{12x_{12}^U - 3x_{12}^L + 12\sigma'_{12} + 3\sigma_{12}}{2} + \left| \frac{12x_{12}^U + 3x_{12}^L + 12\sigma'_{12} - 3\sigma_{12}}{2} \right| \right) + \\
& \left(\frac{-2x_{22}^U + x_{22}^L - 2\sigma'_{22} - \sigma_{22}}{2} - \left| \frac{-2x_{22}^U - x_{22}^L - 2\sigma'_{22} + \sigma_{22}}{2} \right|, -\frac{1}{2}x_{22}^U - \left| -\frac{1}{2}x_{22}^U \right|, -\frac{1}{2}x_{22}^L + \left| \frac{1}{2}x_{22}^L \right|, \right. \\
& \left. \frac{x_{22}^U - 2x_{22}^L + \sigma'_{22} + 2\sigma_{22}}{2} + \left| \frac{x_{22}^U + 2x_{22}^L + \sigma'_{22} - 2\sigma_{22}}{2} \right| \right) + \\
& \left(\frac{11x_{21}^L - 11\sigma_{21}}{2} - \left| \frac{-3x_{21}^L + 3\sigma_{21}}{2} \right|, \frac{11}{2}x_{21}^L - \left| -\frac{1}{2}x_{21}^L \right|, \frac{11}{2}x_{21}^U + \left| \frac{1}{2}x_{21}^U \right|, \frac{11x_{21}^U + 11\sigma'_{21}}{2} + \left| \frac{3x_{21}^U + 3\sigma'_{21}}{2} \right| \right) + \\
& \left(\frac{7x_{22}^L - 7\sigma_{22}}{2} - \left| \frac{-3x_{22}^L + 3\sigma_{22}}{2} \right|, \frac{7}{2}x_{22}^L - \left| -\frac{1}{2}x_{22}^L \right|, \frac{7}{2}x_{22}^U + \left| \frac{1}{2}x_{22}^U \right|, \frac{7x_{22}^U + 7\sigma'_{22}}{2} + \left| \frac{3x_{22}^U + 3\sigma'_{22}}{2} \right| \right) \\
& = (-80, -24, 10, 46).
\end{aligned}$$

Step 3: Using Step 3 and Step 4 of proposed Mehar-Keerat method-IV, the fully fuzzy Sylvester matrix equation (5.29) can be transformed into fully fuzzy Sylvester matrix equation (5.30).

$$\begin{aligned}
& \left(\frac{5x_{11}^L - 5\sigma_{11}}{2} - \left| \frac{-3x_{11}^L + 3\sigma_{11}}{2} \right| + \frac{-8x_{21}^U + x_{21}^L - 8\sigma'_{21} - \sigma_{21}}{2} - \left| \frac{-8x_{21}^U - x_{21}^L - 8\sigma'_{21} + \sigma_{21}}{2} \right| + \frac{-7x_{11}^U - 7\sigma'_{11}}{2} \right. \\
& \quad \left. - \left| \frac{-3x_{11}^U - 3\sigma'_{11}}{2} \right| + \frac{-x_{12}^U + 2x_{12}^L - \sigma'_{12} - 2\sigma_{12}}{2} - \left| \frac{-x_{12}^U - 2x_{12}^L - \sigma'_{12} + 2\sigma_{12}}{2} \right|, \right. \\
& \quad \frac{5}{2}x_{11}^L - \left| -\frac{1}{2}x_{11}^L \right| - \frac{7}{2}x_{21}^U - \left| -\frac{3}{2}x_{21}^U \right| - \frac{7}{2}x_{11}^U - \left| -\frac{1}{2}x_{11}^U \right| + \frac{1}{2}x_{12}^L - \left| -\frac{1}{2}x_{12}^L \right|, \\
& \quad \frac{5}{2}x_{11}^U + \left| \frac{1}{2}x_{11}^U \right| - \frac{7}{2}x_{21}^L + \left| \frac{3}{2}x_{21}^L \right| - \frac{7}{2}x_{11}^L + \left| \frac{1}{2}x_{11}^L \right| + \frac{1}{2}x_{12}^U + \left| \frac{1}{2}x_{12}^U \right|, \\
& \quad \frac{5x_{11}^U + 5\sigma'_{11}}{2} + \left| \frac{3x_{11}^U + 3\sigma'_{11}}{2} \right| + \frac{x_{21}^U - 8x_{21}^L + \sigma'_{21} + 8\sigma_{21}}{2} + \left| \frac{x_{21}^U + 8x_{21}^L + \sigma'_{21} - 8\sigma_{21}}{2} \right| + \frac{-7x_{11}^L + 7\sigma_{11}}{2} + \left| \frac{3x_{11}^L - 3\sigma_{11}}{2} \right| \\
& \quad \left. + \frac{2x_{12}^U - x_{12}^L + 2\sigma'_{12} + \sigma_{12}}{2} + \left| \frac{2x_{12}^U + x_{12}^L + 2\sigma'_{12} - \sigma_{12}}{2} \right| \right) \\
& = (-36, -6, 17, 48),
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{5x_{12}^L - 5\sigma_{12}}{2} - \left| \frac{-3x_{12}^L + 3\sigma_{12}}{2} \right| + \frac{-8x_{22}^U + x_{22}^L - 8\sigma'_{22} - \sigma_{22}}{2} - \left| \frac{-8x_{22}^U - x_{22}^L - 8\sigma'_{22} + \sigma_{22}}{2} \right| + \frac{11x_{11}^L - 11\sigma_{11}}{2} - \right. \\
& \quad \left| \frac{-3x_{11}^L + 3\sigma_{11}}{2} \right| + \frac{7x_{12}^L - 7\sigma_{12}}{2} - \left| \frac{-3x_{12}^L + 3\sigma_{12}}{2} \right|, \\
& \quad \frac{5}{2}x_{12}^L - \left| -\frac{1}{2}x_{12}^L \right| - \frac{7}{2}x_{22}^U - \left| -\frac{3}{2}x_{22}^U \right| + \frac{11}{2}x_{11}^L - \left| -\frac{1}{2}x_{11}^L \right| + \frac{7}{2}x_{12}^L - \left| -\frac{1}{2}x_{12}^L \right|, \\
& \quad \frac{5}{2}x_{12}^U + \left| \frac{1}{2}x_{12}^U \right| - \frac{7}{2}x_{22}^L + \left| \frac{3}{2}x_{22}^L \right| + \frac{11}{2}x_{11}^U + \left| \frac{1}{2}x_{11}^U \right| + \frac{7}{2}x_{12}^U + \left| \frac{1}{2}x_{12}^U \right|, \\
& \quad \frac{5x_{12}^U + 5\sigma'_{12}}{2} + \left| \frac{3x_{12}^U + 3\sigma'_{12}}{2} \right| + \frac{x_{22}^U - 8x_{22}^L + \sigma'_{22} + 8\sigma_{22}}{2} + \left| \frac{x_{22}^U + 8x_{22}^L + \sigma'_{22} - 8\sigma_{22}}{2} \right| + \frac{11x_{11}^U + 11\sigma'_{11}}{2} + \\
& \quad \left| \frac{3x_{11}^U + 3\sigma'_{11}}{2} \right| + \frac{7x_{12}^U + 7\sigma'_{12}}{2} + \left| \frac{3x_{12}^U + 3\sigma'_{12}}{2} \right| \right) + \\
& = (-119, -66, -29, 1), \tag{5.30}
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{-3x_{11}^U + 12x_{11}^L - 3\sigma'_{11} - 12\sigma_{11}}{2} - \left| \frac{-3x_{11}^U - 12x_{11}^L - 3\sigma'_{11} + 12\sigma_{11}}{2} \right| + \frac{-2x_{21}^U + x_{21}^L - 2\sigma'_{21} - \sigma_{21}}{2} \right. \\
& - \left| \frac{-2x_{21}^U - x_{21}^L - 2\sigma'_{21} + \sigma_{21}}{2} \right| + \frac{-7x_{21}^U - 7\sigma'_{21}}{2} - \left| \frac{-3x_{21}^U - 3\sigma'_{21}}{2} \right| + \frac{-x_{22}^U + 2x_{22}^L - \sigma'_{22} - 2\sigma_{22}}{2} \\
& - \left| \frac{-x_{22}^U - 2x_{22}^L - \sigma'_{22} + 2\sigma_{22}}{2} \right|, \\
& \frac{9}{2}x_{11}^L - \left| -\frac{5}{2}x_{11}^L \right| - \frac{1}{2}x_{21}^U - \left| -\frac{1}{2}x_{21}^U \right| - \frac{7}{2}x_{21}^U - \left| -\frac{1}{2}x_{21}^U \right| + \frac{1}{2}x_{22}^L - \left| -\frac{1}{2}x_{22}^L \right|, \\
& \frac{9}{2}x_{11}^U + \left| \frac{5}{2}x_{11}^U \right| - \frac{1}{2}x_{21}^L + \left| \frac{1}{2}x_{21}^L \right| - \frac{7}{2}x_{21}^L + \left| \frac{1}{2}x_{21}^L \right| + \frac{1}{2}x_{22}^U + \left| \frac{1}{2}x_{22}^U \right|, \\
& \frac{12x_{11}^U - 3x_{11}^L + 12\sigma'_{11} + 3\sigma_{11}}{2} + \left| \frac{12x_{11}^U + 3x_{11}^L + 12\sigma'_{11} - 3\sigma_{11}}{2} \right| + \frac{x_{21}^U - 2x_{21}^L + \sigma'_{21} + 2\sigma_{21}}{2} \\
& + \left| \frac{x_{21}^U + 2x_{21}^L + \sigma'_{21} - 2\sigma_{21}}{2} \right| + \frac{-7x_{21}^L + 7\sigma_{21}}{2} + \left| \frac{3x_{21}^L - 3\sigma_{21}}{2} \right| + \frac{2x_{22}^U - x_{22}^L + 2\sigma'_{22} + \sigma_{22}}{2} \\
& + \left| \frac{2x_{22}^U + x_{22}^L + 2\sigma'_{22} - \sigma_{22}}{2} \right| \\
& \left. \right) \\
& = (-83, -35, 0, 40),
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{-3x_{12}^U + 12x_{12}^L - 3\sigma'_{12} - 12\sigma_{12}}{2} - \left| \frac{-3x_{12}^U - 12x_{12}^L - 3\sigma'_{12} + 12\sigma_{12}}{2} \right| + \frac{-2x_{22}^U + x_{22}^L - 2\sigma'_{22} - \sigma_{22}}{2} \right. \\
& - \left| \frac{-2x_{22}^U - x_{22}^L - 2\sigma'_{22} + \sigma_{22}}{2} \right| + \frac{11x_{21}^L - 11\sigma_{21}}{2} - \left| \frac{-3x_{21}^L + 3\sigma_{21}}{2} \right| + \frac{7x_{22}^L - 7\sigma_{22}}{2} - \left| \frac{-3x_{22}^L + 3\sigma_{22}}{2} \right|, \\
& \frac{9}{2}x_{12}^L - \left| -\frac{5}{2}x_{12}^L \right| - \frac{1}{2}x_{22}^U - \left| -\frac{1}{2}x_{22}^U \right| + \frac{11}{2}x_{21}^L - \left| -\frac{1}{2}x_{21}^L \right| + \frac{7}{2}x_{22}^L - \left| -\frac{1}{2}x_{22}^L \right|, \\
& \frac{9}{2}x_{12}^U + \left| \frac{5}{2}x_{12}^U \right| - \frac{1}{2}x_{22}^L + \left| \frac{1}{2}x_{22}^L \right| + \frac{11}{2}x_{21}^U + \left| \frac{1}{2}x_{21}^U \right| + \frac{7}{2}x_{22}^U + \left| \frac{1}{2}x_{22}^U \right|, \\
& \frac{12x_{12}^U - 3x_{12}^L + 12\sigma'_{12} + 3\sigma_{12}}{2} + \left| \frac{12x_{12}^U + 3x_{12}^L + 12\sigma'_{12} - 3\sigma_{12}}{2} \right| + \frac{x_{22}^U - 2x_{22}^L + \sigma'_{22} + 2\sigma_{22}}{2} + \\
& \left| \frac{x_{22}^U + 2x_{22}^L + \sigma'_{22} - 2\sigma_{22}}{2} \right| + \frac{11x_{21}^U + 11\sigma'_{21}}{2} + \left| \frac{3x_{21}^U + 3\sigma'_{21}}{2} \right| + \frac{7x_{22}^U + 7\sigma'_{22}}{2} + \left| \frac{3x_{22}^U + 3\sigma'_{22}}{2} \right| \\
& \left. \right) \\
& = (-80, -24, 10, 46).
\end{aligned}$$

Step 4: Using Step 5 of the proposed Mehar-Keerat method-IV, the fully fuzzy Sylvester matrix equation (5.30) can be transformed into multi-objective mathematical programming problem (5.31).

Maximize σ_{11} ,

Maximize σ'_{11} ,

Maximize σ_{12} ,

Maximize σ'_{12} ,

Maximize σ_{21} ,

Maximize σ'_{21} ,

Maximize σ_{22} ,

Maximize σ'_{22} ,

subject to

$$\begin{aligned} & \frac{5}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| - \frac{7}{2}x_{21}^U - \frac{3}{2}|x_{21}^U| - \frac{7}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| + \frac{1}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{5}{2}(x_{11}^L - \sigma_{11}) + \frac{3}{2}\left|\frac{x_{11}^L - \sigma_{11}}{2}\right| - \\ & \frac{-8x_{21}^U + x_{21}^L - 8\sigma'_{21} - \sigma_{21}}{2} + \left|\frac{-8x_{21}^U - x_{21}^L - 8\sigma'_{21} + \sigma_{21}}{2}\right| + \frac{7}{2}(x_{11}^U + \sigma'_{11}) + \frac{3}{2}\left|\frac{x_{11}^U + \sigma'_{11}}{2}\right| - \\ & \frac{-x_{12}^U + 2x_{12}^L - \sigma'_{12} - 2\sigma_{12}}{2} + \left|\frac{-x_{12}^U - 2x_{12}^L - \sigma'_{12} + 2\sigma_{12}}{2}\right| \leq 30, \end{aligned}$$

$$\frac{5}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| - \frac{7}{2}x_{21}^U - \frac{3}{2}|x_{21}^U| - \frac{7}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| + \frac{1}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| = -6,$$

$$\frac{5}{2}x_{11}^U + \frac{1}{2}|x_{11}^U| - \frac{7}{2}x_{21}^L + \frac{3}{2}|x_{21}^L| - \frac{7}{2}x_{11}^L + \frac{1}{2}|x_{11}^L| + \frac{1}{2}x_{12}^U + \frac{1}{2}|x_{12}^U| = 17,$$

$$\begin{aligned} & \frac{5}{2}(x_{11}^U + \sigma'_{11}) + \frac{3}{2}\left|\frac{x_{11}^U + \sigma'_{11}}{2}\right| + \frac{x_{21}^U - 8x_{21}^L + \sigma'_{21} + 8\sigma_{21}}{2} + \left|\frac{x_{21}^U + 8x_{21}^L + \sigma'_{21} - 8\sigma_{21}}{2}\right| + \frac{7}{2}(-x_{11}^L + \sigma_{11}) + \\ & \frac{3}{2}\left|\frac{x_{11}^L - \sigma_{11}}{2}\right| + \frac{2x_{12}^U - x_{12}^L + 2\sigma'_{12} + \sigma_{12}}{2} + \left|\frac{2x_{12}^U + x_{12}^L + 2\sigma'_{12} - \sigma_{12}}{2}\right| - \frac{5}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| + \frac{7}{2}x_{21}^L - \frac{3}{2}|x_{21}^L| + \end{aligned}$$

$$\frac{7}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| - \frac{1}{2}x_{12}^U - \frac{1}{2}|x_{12}^U| \leq 31,$$

$$\begin{aligned} & \frac{5}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{7}{2}x_{22}^U - \frac{3}{2}|x_{22}^U| + \frac{11}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| + \frac{7}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{5}{2}(x_{12}^L - \sigma_{12}) + \frac{3}{2}|x_{12}^L - \sigma_{12}| - \\ & \frac{-8x_{22}^U + x_{22}^L - 8\sigma'_{22} - \sigma_{22}}{2} + \left|\frac{-8x_{22}^U - x_{22}^L - 8\sigma'_{22} + \sigma_{22}}{2}\right| - \frac{11}{2}(x_{11}^L - \sigma_{11}) + \frac{3}{2}|x_{11}^L - \sigma_{11}| - \end{aligned}$$

$$\frac{7}{2}(x_{12}^L - \sigma_{12}) + \frac{3}{2}|x_{12}^L - \sigma_{12}| \leq 53,$$

$$\frac{5}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{7}{2}x_{22}^U - \frac{3}{2}|x_{22}^U| + \frac{11}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| + \frac{7}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| = -66,$$

$$\frac{5}{2}x_{12}^U + \frac{1}{2}|x_{12}^U| - \frac{7}{2}x_{22}^L + \frac{3}{2}|x_{22}^L| + \frac{11}{2}x_{11}^U + \frac{1}{2}|x_{11}^U| + \frac{7}{2}x_{12}^U + \frac{1}{2}|x_{12}^U| = -29,$$

$$\begin{aligned} & \frac{5}{2}(x_{12}^U + \sigma'_{12}) + \frac{3}{2}|x_{12}^U + \sigma'_{12}| + \frac{x_{22}^U - 8x_{22}^L + \sigma'_{22} + 8\sigma_{22}}{2} + \left| \frac{x_{22}^U + 8x_{22}^L + \sigma'_{22} - 8\sigma_{22}}{2} \right| + \frac{11}{2}(x_{11}^U + \sigma'_{11}) + \\ & \frac{3}{2}|x_{11}^U + \sigma'_{11}| + \frac{7}{2}(x_{12}^U + \sigma'_{12}) + \frac{3}{2}|x_{12}^U + \sigma'_{12}| - \frac{5}{2}x_{12}^U - \frac{1}{2}|x_{12}^U| + \frac{7}{2}x_{22}^L - \frac{3}{2}|x_{22}^L| - \frac{11}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| - \frac{7}{2}x_{12}^U - \\ & \frac{1}{2}|x_{12}^U| \leq 30, \end{aligned} \quad (5.31)$$

$$\begin{aligned} & \frac{9}{2}x_{11}^L - \frac{5}{2}|x_{11}^L| - \frac{1}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| - \frac{7}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| + \frac{1}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| - \frac{-3x_{11}^U + 12x_{11}^L - 3\sigma'_{11} - 12\sigma_{11}}{2} + \\ & \left| \frac{-3x_{11}^U - 12x_{11}^L - 3\sigma'_{11} + 12\sigma_{11}}{2} \right| - \frac{-2x_{21}^U + x_{21}^L - 2\sigma'_{21} - \sigma_{21}}{2} + \left| \frac{-2x_{21}^U - x_{21}^L - 2\sigma'_{21} + \sigma_{21}}{2} \right| + \frac{7}{2}(x_{21}^U + \sigma'_{21}) \\ & + \frac{3}{2}|x_{21}^U + \sigma'_{21}| - \frac{-x_{22}^U + 2x_{22}^L - \sigma'_{22} - 2\sigma_{22}}{2} + \left| \frac{-x_{22}^U - 2x_{22}^L - \sigma'_{22} + 2\sigma_{22}}{2} \right| \leq 48, \end{aligned}$$

$$\frac{9}{2}x_{11}^L - \frac{5}{2}|x_{11}^L| - \frac{1}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| - \frac{7}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| + \frac{1}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| = -35,$$

$$\frac{9}{2}x_{11}^U + \frac{5}{2}|x_{11}^U| - \frac{1}{2}x_{21}^L + \frac{1}{2}|x_{21}^L| - \frac{7}{2}x_{21}^L + \frac{1}{2}|x_{21}^L| + \frac{1}{2}x_{22}^U + \frac{1}{2}|x_{22}^U| = 0,$$

$$\begin{aligned} & \frac{12x_{11}^U - 3x_{11}^L + 12\sigma'_{11} + 3\sigma_{11}}{2} + \left| \frac{12x_{11}^U + 3x_{11}^L + 12\sigma'_{11} - 3\sigma_{11}}{2} \right| + \frac{x_{21}^U - 2x_{21}^L + \sigma'_{21} + 2\sigma_{21}}{2} \\ & + \left| \frac{x_{21}^U + 2x_{21}^L + \sigma'_{21} - 2\sigma_{21}}{2} \right| + \frac{7}{2}(-x_{21}^L + \sigma_{21}) + \frac{3}{2}|x_{21}^L - \sigma_{21}| + \frac{2x_{22}^U - x_{22}^L + 2\sigma'_{22} + \sigma_{22}}{2} \\ & + \left| \frac{2x_{22}^U + x_{22}^L + 2\sigma'_{22} - \sigma_{22}}{2} \right| - \frac{9}{2}x_{11}^U - \frac{5}{2}|x_{11}^U| + \frac{1}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| + \frac{7}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| - \frac{1}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| \leq 40, \end{aligned}$$

$$\begin{aligned} & \frac{9}{2}x_{12}^L - \frac{5}{2}|x_{12}^L| - \frac{1}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| + \frac{11}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| + \frac{7}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| - \frac{-3x_{12}^U + 12x_{12}^L - 3\sigma'_{12} - 12\sigma_{12}}{2} + \\ & \left| \frac{-3x_{12}^U - 12x_{12}^L - 3\sigma'_{12} + 12\sigma_{12}}{2} \right| - \frac{-2x_{22}^U + x_{22}^L - 2\sigma'_{22} - \sigma_{22}}{2} + \left| \frac{-2x_{22}^U - x_{22}^L - 2\sigma'_{22} + \sigma_{22}}{2} \right| - \\ & \frac{11}{2}(x_{21}^L - \sigma_{21}) + \frac{3}{2}|-x_{21}^L + \sigma_{21}| - \frac{7}{2}(x_{22}^L - \sigma_{22}) + \frac{3}{2}|x_{22}^L - \sigma_{22}| \leq 56, \end{aligned}$$

$$\frac{9}{2}x_{12}^L - \frac{5}{2}|x_{12}^L| - \frac{1}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| + \frac{11}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| + \frac{7}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| = -24,$$

$$\frac{9}{2}x_{12}^U + \frac{5}{2}|x_{12}^U| - \frac{1}{2}x_{22}^L + \frac{1}{2}|x_{22}^L| + \frac{11}{2}x_{21}^U + \frac{1}{2}|x_{21}^U| + \frac{7}{2}x_{22}^U + \frac{1}{2}|x_{22}^U| = 10,$$

$$\begin{aligned} & \frac{12x_{12}^U - 3x_{12}^L + 12\sigma'_{12} + 3\sigma_{12}}{2} + \left| \frac{12x_{12}^U + 3x_{12}^L + 12\sigma'_{12} - 3\sigma_{12}}{2} \right| + \frac{x_{22}^U - 2x_{22}^L + \sigma'_{22} + 2\sigma_{22}}{2} + \\ & \left| \frac{x_{22}^U + 2x_{22}^L + \sigma'_{22} - 2\sigma_{22}}{2} \right| + \frac{11}{2}(x_{21}^U + \sigma'_{21}) + \frac{3}{2}|x_{21}^U + \sigma'_{21}| + \frac{7}{2}(x_{22}^U + \sigma'_{22}) + \frac{3}{2}|x_{22}^U + \sigma'_{22}| - \\ & \frac{9}{2}x_{12}^U - \frac{5}{2}|x_{12}^U| + \frac{1}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| - \frac{11}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| - \frac{7}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| \leq 36. \end{aligned}$$

$$\begin{aligned} & x_{11}^L - \sigma_{11} \leq x_{11}^L \leq x_{11}^U \leq x_{11}^U + \sigma'_{11}, x_{12}^L - \sigma_{12} \leq x_{12}^L \leq x_{12}^U \leq x_{12}^U + \sigma'_{12}, x_{21}^L - \sigma_{21} \leq x_{21}^L \leq x_{21}^U \leq x_{21}^U + \sigma'_{21}, \\ & x_{22}^L - \sigma_{22} \leq x_{22}^L \leq x_{22}^U \leq x_{22}^U + \sigma'_{22} \text{ and } \sigma_{11} \geq 0, \sigma'_{11} \geq 0, \sigma_{12} \geq 0, \sigma'_{12} \geq 0, \sigma_{21} \geq 0, \sigma'_{21} \geq 0, \sigma_{22} \geq 0, \sigma'_{22} \geq 0. \end{aligned}$$

Step 5: Using Step 6 of the proposed Mehar-Keerat method-IV, the fully fuzzy Sylvester matrix equation (5.30) can be transformed into multi-objective mathematical programming problem (5.32).

Minimize σ_{11} ,

Minimize σ'_{11} ,

Minimize σ_{12} ,

Minimize σ'_{12} ,

Minimize σ_{21} ,

Minimize σ'_{21} ,

Minimize σ_{22} ,

Minimize σ'_{22} ,

subject to

$$\begin{aligned} & \frac{5}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| - \frac{7}{2}x_{21}^U - \frac{3}{2}|x_{21}^U| - \frac{7}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| + \frac{1}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{5}{2}(x_{11}^L - \sigma_{11}) + \frac{3}{2}\left|\frac{x_{11}^L - \sigma_{11}}{2}\right| - \\ & \frac{-8x_{21}^U + x_{21}^L - 8\sigma'_{21} - \sigma_{21}}{2} + \left| \frac{-8x_{21}^U - x_{21}^L - 8\sigma'_{21} + \sigma_{21}}{2} \right| + \frac{7}{2}(x_{11}^U + \sigma'_{11}) + \frac{3}{2}\left|\frac{x_{11}^U + \sigma'_{11}}{2}\right| - \\ & \frac{-x_{12}^U + 2x_{12}^L - \sigma'_{12} - 2\sigma_{12}}{2} + \left| \frac{-x_{12}^U - 2x_{12}^L - \sigma'_{12} + 2\sigma_{12}}{2} \right| \geq 30, \end{aligned}$$

$$\frac{5}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| - \frac{7}{2}x_{21}^U - \frac{3}{2}|x_{21}^U| - \frac{7}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| + \frac{1}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| = -6,$$

$$\frac{5}{2}x_{11}^U + \frac{1}{2}|x_{11}^U| - \frac{7}{2}x_{21}^L + \frac{3}{2}|x_{21}^L| - \frac{7}{2}x_{11}^L + \frac{1}{2}|x_{11}^L| + \frac{1}{2}x_{12}^U + \frac{1}{2}|x_{12}^U| = 17,$$

$$\begin{aligned} & \frac{5}{2}(x_{11}^U + \sigma'_{11}) + \frac{3}{2} \left| \frac{x_{11}^U + \sigma'_{11}}{2} \right| + \frac{x_{21}^U - 8x_{21}^L + \sigma'_{21} + 8\sigma_{21}}{2} + \left| \frac{x_{21}^U + 8x_{21}^L + \sigma'_{21} - 8\sigma_{21}}{2} \right| + \frac{7}{2}(-x_{11}^L + \sigma_{11}) + \\ & \frac{3}{2} \left| \frac{x_{11}^L - \sigma_{11}}{2} \right| + \frac{2x_{12}^U - x_{12}^L + 2\sigma'_{12} + \sigma_{12}}{2} + \left| \frac{2x_{12}^U + x_{12}^L + 2\sigma'_{12} - \sigma_{12}}{2} \right| - \frac{5}{2}x_{11}^U - \left| \frac{1}{2}x_{11}^U \right| + \frac{7}{2}x_{21}^L - \frac{3}{2}|x_{21}^L| + \\ & \frac{7}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| - \frac{1}{2}x_{12}^U - \frac{1}{2}|x_{12}^U| \geq 31, \end{aligned}$$

$$\begin{aligned} & \frac{5}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{7}{2}x_{22}^U - \frac{3}{2}|x_{22}^U| + \frac{11}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| + \frac{7}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{5}{2}(x_{12}^L - \sigma_{12}) + \frac{3}{2}|x_{12}^L - \sigma_{12}| - \\ & \frac{-8x_{22}^U + x_{22}^L - 8\sigma'_{22} - \sigma_{22}}{2} + \left| \frac{-8x_{22}^U - x_{22}^L - 8\sigma'_{22} + \sigma_{22}}{2} \right| - \frac{11}{2}(x_{11}^L - \sigma_{11}) + \frac{3}{2}|x_{11}^L - \sigma_{11}| - \\ & \frac{7}{2}(x_{12}^L - \sigma_{12}) + \frac{3}{2}|x_{12}^L - \sigma_{12}| \geq 53, \end{aligned}$$

$$\frac{5}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| - \frac{7}{2}x_{22}^U - \frac{3}{2}|x_{22}^U| + \frac{11}{2}x_{11}^L - \frac{1}{2}|x_{11}^L| + \frac{7}{2}x_{12}^L - \frac{1}{2}|x_{12}^L| = -66,$$

$$\frac{5}{2}x_{12}^U + \frac{1}{2}|x_{12}^U| - \frac{7}{2}x_{22}^L + \frac{3}{2}|x_{22}^L| + \frac{11}{2}x_{11}^U + \frac{1}{2}|x_{11}^U| + \frac{7}{2}x_{12}^U + \frac{1}{2}|x_{12}^U| = -29,$$

$$\begin{aligned} & \frac{5}{2}(x_{12}^U + \sigma'_{12}) + \frac{3}{2} \left| \frac{x_{12}^U + \sigma'_{12}}{2} \right| + \frac{x_{22}^U - 8x_{22}^L + \sigma'_{22} + 8\sigma_{22}}{2} + \left| \frac{x_{22}^U + 8x_{22}^L + \sigma'_{22} - 8\sigma_{22}}{2} \right| + \frac{11}{2}(x_{11}^U + \sigma'_{11}) + \\ & \frac{3}{2} \left| \frac{x_{11}^U + \sigma'_{11}}{2} \right| + \frac{7}{2}(x_{12}^U + \sigma'_{12}) + \frac{3}{2} \left| \frac{x_{12}^U + \sigma'_{12}}{2} \right| - \frac{5}{2}x_{12}^U - \frac{1}{2}|x_{12}^U| + \frac{7}{2}x_{22}^L - \frac{3}{2}|x_{22}^L| - \frac{11}{2}x_{11}^U - \frac{1}{2}|x_{11}^U| - \\ & \frac{7}{2}x_{12}^U - \frac{1}{2}|x_{12}^U| \geq 30, \end{aligned} \tag{5.32}$$

$$\begin{aligned} & \frac{9}{2}x_{11}^L - \frac{5}{2}|x_{11}^L| - \frac{1}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| - \frac{7}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| + \frac{1}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| - \frac{-3x_{11}^U + 12x_{11}^L - 3\sigma'_{11} - 12\sigma_{11}}{2} + \\ & \left| \frac{-3x_{11}^U - 12x_{11}^L - 3\sigma'_{11} + 12\sigma_{11}}{2} \right| - \frac{-2x_{21}^U + x_{21}^L - 2\sigma'_{21} - \sigma_{21}}{2} + \left| \frac{-2x_{21}^U - x_{21}^L - 2\sigma'_{21} + \sigma_{21}}{2} \right| + \frac{7}{2}(x_{21}^U + \sigma'_{21}) \\ & + \frac{3}{2} \left| \frac{x_{21}^U + \sigma'_{21}}{2} \right| - \frac{-x_{22}^U + 2x_{22}^L - \sigma'_{22} - 2\sigma_{22}}{2} + \left| \frac{-x_{22}^U - 2x_{22}^L - \sigma'_{22} + 2\sigma_{22}}{2} \right| \geq 48, \end{aligned}$$

$$\frac{9}{2}x_{11}^L - \frac{5}{2}|x_{11}^L| - \frac{1}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| - \frac{7}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| + \frac{1}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| = -35,$$

$$\frac{9}{2}x_{11}^U + \frac{5}{2}|x_{11}^U| - \frac{1}{2}x_{21}^L + \frac{1}{2}|x_{21}^L| - \frac{7}{2}x_{21}^L + \frac{1}{2}|x_{21}^L| + \frac{1}{2}x_{22}^U + \frac{1}{2}|x_{22}^U| = 0,$$

$$\begin{aligned}
& \frac{12x_{11}^U - 3x_{11}^L + 12\sigma'_{11} + 3\sigma_{11}}{2} + \left| \frac{12x_{11}^U + 3x_{11}^L + 12\sigma'_{11} - 3\sigma_{11}}{2} \right| + \frac{x_{21}^U - 2x_{21}^L + \sigma'_{21} + 2\sigma_{21}}{2} \\
& + \left| \frac{x_{21}^U + 2x_{21}^L + \sigma'_{21} - 2\sigma_{21}}{2} \right| + \frac{7}{2}(-x_{21}^L + \sigma_{21}) + \frac{3}{2}|x_{21}^L - \sigma_{21}| + \frac{2x_{22}^U - x_{22}^L + 2\sigma'_{22} + \sigma_{22}}{2} \\
& + \left| \frac{2x_{22}^U + x_{22}^L + 2\sigma'_{22} - \sigma_{22}}{2} \right| - \frac{9}{2}x_{11}^U - \frac{5}{2}|x_{11}^U| + \frac{1}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| + \frac{7}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| - \frac{1}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| \geq 40,
\end{aligned}$$

$$\begin{aligned}
& \frac{9}{2}x_{12}^L - \frac{5}{2}|x_{12}^L| - \frac{1}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| + \frac{11}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| + \frac{7}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| - \frac{-3x_{12}^U + 12x_{12}^L - 3\sigma'_{12} - 12\sigma_{12}}{2} + \\
& \left| \frac{-3x_{12}^U - 12x_{12}^L - 3\sigma'_{12} + 12\sigma_{12}}{2} \right| - \frac{-2x_{22}^U + x_{22}^L - 2\sigma'_{22} - \sigma_{22}}{2} + \left| \frac{-2x_{22}^U - x_{22}^L - 2\sigma'_{22} + \sigma_{22}}{2} \right| - \\
& \frac{11}{2}(x_{21}^L - \sigma_{21}) + \frac{3}{2}|-x_{21}^L + \sigma_{21}| - \frac{7}{2}(x_{22}^L - \sigma_{22}) + \frac{3}{2}|x_{22}^L - \sigma_{22}| \geq 56,
\end{aligned}$$

$$\frac{9}{2}x_{12}^L - \frac{5}{2}|x_{12}^L| - \frac{1}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| + \frac{11}{2}x_{21}^L - \frac{1}{2}|x_{21}^L| + \frac{7}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| = -24,$$

$$\frac{9}{2}x_{12}^U + \frac{5}{2}|x_{12}^U| - \frac{1}{2}x_{22}^L + \frac{1}{2}|x_{22}^L| + \frac{11}{2}x_{21}^U + \frac{1}{2}|x_{21}^U| + \frac{7}{2}x_{22}^U + \frac{1}{2}|x_{22}^U| = 10,$$

$$\begin{aligned}
& \frac{12x_{12}^U - 3x_{12}^L + 12\sigma'_{12} + 3\sigma_{12}}{2} + \left| \frac{12x_{12}^U + 3x_{12}^L + 12\sigma'_{12} - 3\sigma_{12}}{2} \right| + \frac{x_{22}^U - 2x_{22}^L + \sigma'_{22} + 2\sigma_{22}}{2} + \\
& \left| \frac{x_{22}^U + 2x_{22}^L + \sigma'_{22} - 2\sigma_{22}}{2} \right| + \frac{11}{2}(x_{21}^U + \sigma'_{21}) + \frac{3}{2}|x_{21}^U + \sigma'_{21}| + \frac{7}{2}(x_{22}^U + \sigma'_{22}) + \frac{3}{2}|x_{22}^U + \sigma'_{22}| - \\
& \frac{9}{2}x_{12}^U - \frac{5}{2}|x_{12}^U| + \frac{1}{2}x_{22}^L - \frac{1}{2}|x_{22}^L| - \frac{11}{2}x_{21}^U - \frac{1}{2}|x_{21}^U| - \frac{7}{2}x_{22}^U - \frac{1}{2}|x_{22}^U| \geq 36.
\end{aligned}$$

$$\begin{aligned}
& x_{11}^L - \sigma_{11} \leq x_{11}^L \leq x_{11}^U \leq x_{11}^U + \sigma'_{11}, x_{12}^L - \sigma_{12} \leq x_{12}^L \leq x_{12}^U \leq x_{12}^U + \sigma'_{12}, x_{21}^L - \sigma_{21} \leq x_{21}^L \leq x_{21}^U \leq x_{21}^U + \sigma'_{21}, \\
& x_{22}^L - \sigma_{22} \leq x_{22}^L \leq x_{22}^U \leq x_{22}^U + \sigma'_{22} \text{ and } \sigma_{11} \geq 0, \sigma'_{11} \geq 0, \sigma_{12} \geq 0, \sigma'_{12} \geq 0, \sigma_{21} \geq 0, \sigma'_{21} \geq 0, \sigma_{22} \geq 0, \sigma'_{22} \geq 0.
\end{aligned}$$

Step 6: Using Step 7 of the proposed Mehar-Keerat method-IV, the multi-objective mathematical programming problem (5.31) can be transformed into single-objective mathematical programming problem (5.33).

Maximize $(\sigma_{11} + \sigma'_{11} + \sigma_{12} + \sigma'_{12} + \sigma_{21} + \sigma'_{21} + \sigma_{22} + \sigma'_{22})$,

subject to

Constraints of Problem (5.31). (5.33)

Step 7: Using Step 8 of the proposed Mehar-Keerat method-IV, the multi-objective

mathematical programming problem (5.32) can be transformed into single-objective mathematical programming problem (5.34).

$$\begin{aligned} &\text{Minimize } (\sigma_{11} + \sigma'_{11} + \sigma_{12} + \sigma'_{12} + \sigma_{21} + \sigma'_{21} + \sigma_{22} + \sigma'_{22}), \\ &\text{subject to} \\ &\text{Constraints of Problem (5.32)}. \end{aligned} \tag{5.34}$$

Step 8: On solving the single-objective mathematical programming problem (5.33)

(i) with the additional constraints $\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}$, the obtained optimal solution is

$$\begin{aligned} \sigma_{11} = 1, \sigma'_{11} = 1, \sigma_{12} = 2, \sigma'_{12} = 2, \sigma_{21} = 1, \sigma'_{21} = 1, \sigma_{22} = 1, \sigma'_{22} = 1, x_{11}^L = -5, x_{11}^U = -4, x_{12}^L = -3, \\ x_{12}^U = -1, x_{21}^L = -1, x_{21}^U = 0, x_{22}^L = 2, x_{22}^U = 3. \end{aligned}$$

(ii) with the additional constraints $\sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22}$, the obtained optimal solution is

$$\begin{aligned} \sigma_{11} = 1, \sigma'_{11} = 1, \sigma_{12} = 1, \sigma'_{12} = 1, \sigma_{21} = 1, \sigma'_{21} = 1, \sigma_{22} = 1, \sigma'_{22} = 1, x_{11}^L = -5, x_{11}^U = -4, x_{12}^L = -3, \\ x_{12}^U = -1, x_{21}^L = -1, x_{21}^U = 0, x_{22}^L = 2, x_{22}^U = 3. \end{aligned}$$

(iii) without any additional constraint, the obtained optimal solution is

$$\begin{aligned} \sigma_{11} = 0.8587, \sigma'_{11} = 3.0948, \sigma_{12} = 1.5583, \sigma'_{12} = 1, \sigma_{21} = 0.8817, \sigma'_{21} = 0.6574, \sigma_{22} = 2.5776, \\ \sigma'_{22} = 1.6206, x_{11}^L = -5, x_{11}^U = -4, x_{12}^L = -3, x_{12}^U = -1, x_{21}^L = -1, x_{21}^U = 0, x_{22}^L = 2, x_{22}^U = 3. \end{aligned}$$

Step 9: On solving the single-objective mathematical programming problem (5.34)

(i) with the additional constraints $\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}$, the obtained optimal solution is

$$\begin{aligned} \sigma_{11} = 7, \sigma'_{11} = 7, \sigma_{12} = 2.3333, \sigma'_{12} = 2.3333, \sigma_{21} = 1, \sigma'_{21} = 1, \sigma_{22} = 2, \sigma'_{22} = 2, x_{11}^L = -5, \\ x_{11}^U = -4, x_{12}^L = -3, x_{12}^U = -1, x_{21}^L = -1, x_{21}^U = 0, x_{22}^L = 2, x_{22}^U = 3. \end{aligned}$$

(ii) with the additional constraints $\sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22}$, the obtained the optimal solution is

$$\sigma_{11} = 7, \sigma'_{11} = 7, \sigma_{12} = 7, \sigma'_{12} = 7, \sigma_{21} = 7, \sigma'_{21} = 7, \sigma_{22} = 7, \sigma'_{22} = 7, x_{11}^L = -5, x_{11}^U = -4,$$

$$x_{12}^L = -3, x_{12}^U = -1, x_{21}^L = -1, x_{21}^U = 0, x_{22}^L = 2, x_{22}^U = 3.$$

(iii) without any additional constraint, the obtained the optimal solution is

$$\sigma_{11} = 1.373, \sigma'_{11} = 6.2465, \sigma_{12} = 2.5798, \sigma'_{12} = 3.7899, \sigma_{21} = 0, \sigma'_{21} = 0.5, \sigma_{22} = 2,$$

$$\sigma'_{22} = 0.0211, x_{11}^L = -5, x_{11}^U = -4, x_{12}^L = -3, x_{12}^U = -1, x_{21}^L = -1, x_{21}^U = 0, x_{22}^L = 2, x_{22}^U = 3.$$

Step 10: Using the optimal solution, obtained in Step 8 with the additional constraints

$$\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}, \text{ and } \sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22}, \quad \text{the}$$

maximal fuzzy solutions of the fully fuzzy Sylvester matrix equation, chosen in Example 5.3,

are

$$\tilde{x}_{11} = (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) = (-6, -5, -4, -3), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) = (-5, -3, -1, 1),$$

$$\tilde{x}_{21} = (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) = (-2, -1, 0, 1), \tilde{x}_{22} = (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) = (1, 2, 3, 4)$$

and

$$\tilde{x}_{11} = (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) = (-6, -5, -4, -3), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) = (-4, -3, -1, 0),$$

$$\tilde{x}_{21} = (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) = (-2, -1, 0, 1), \tilde{x}_{22} = (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) = (1, 2, 3, 4)$$

respectively.

Also, using the optimal solution, obtained in Step 8 without any additional constraint, the

maximal fuzzy solution of the fully fuzzy Sylvester matrix equation, chosen in Example 5.3,

is

$$\tilde{x}_{11} = (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) = (-5.8587, -5, -4, -0.9052), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12})$$

$$= (-4.5583, -3, -1, 0), \tilde{x}_{21} = (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) = (-1.8817, -1, 0, 0.6574),$$

$$\tilde{x}_{22} = (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) = (-0.5776, 2, 3, 4.6206).$$

Step 11: Using the optimal solution, obtained in Step 9 with the additional constraints

$$\sigma_{11} = \sigma'_{11}, \sigma_{12} = \sigma'_{12}, \sigma_{21} = \sigma'_{21}, \sigma_{22} = \sigma'_{22}, \text{ and } \sigma_{11} = \sigma'_{11} = \sigma_{12} = \sigma'_{12} = \sigma_{21} = \sigma'_{21} = \sigma_{22} = \sigma'_{22}, \quad \text{the}$$

minimal fuzzy solutions of the fully fuzzy Sylvester matrix equation, chosen in Example 5.3, are

$$\begin{aligned}\tilde{x}_{11} &= (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) = (-12, -5, -4, 3), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) = (-5.3333, -3, -1, 1.3333), \\ \tilde{x}_{21} &= (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) = (-2, -1, 0, 1), \tilde{x}_{22} = (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) = (0, 2, 3, 5).\end{aligned}$$

and

$$\begin{aligned}\tilde{x}_{11} &= (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) = (-12, -5, -4, 3), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) = (-10, -3, -1, 6), \\ \tilde{x}_{21} &= (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) = (-8, -1, 0, 7), \tilde{x}_{22} = (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) = (-5, 2, 3, 10)\end{aligned}$$

respectively.

Also, using the optimal solution, obtained in Step 9 without any additional constraint, the minimal fuzzy solution of the fully fuzzy Sylvester matrix equation, chosen in Example 5.3, is

$$\begin{aligned}\tilde{x}_{11} &= (x_{11}^L - \sigma_{11}, x_{11}^L, x_{11}^U, x_{11}^U + \sigma'_{11}) = (-6.373, -5, -4, 2.2465), \tilde{x}_{12} = (x_{12}^L - \sigma_{12}, x_{12}^L, x_{12}^U, x_{12}^U + \sigma'_{12}) \\ &= (-5.5798, -3, -1, 2.7899), \tilde{x}_{21} = (x_{21}^L - \sigma_{21}, x_{21}^L, x_{21}^U, x_{21}^U + \sigma'_{21}) = (-1, -1, 0, 0.5), \\ \tilde{x}_{22} &= (x_{22}^L - \sigma_{22}, x_{22}^L, x_{22}^U, x_{22}^U + \sigma'_{22}) = (0, 2, 3, 3.0211).\end{aligned}$$

5.7. Comparative study

The maximal and minimal fuzzy solution of the existing fuzzy Sylvester matrix equation and chosen fully fuzzy Sylvester matrix equation, obtained by the existing methods [26,29,59,75] as well as proposed Mehar-Keerat method-I, Mehar-Keerat method-II, Mehar-Keerat method-III and Mehar-Keerat method-IV are shown in Table 5.1.

Table 5.1. Maximal and minimal fuzzy symmetric/non-symmetric solution of existing and chosen system of fuzzy/fully fuzzy Sylvester matrix equations.

Example	Maximal and minimal Fuzzy solution	Existing methods [26,29,75]	Existing method [59]		Proposed Mehar-Keerat method-I, Mehar-Keerat method II, Mehar-Keerat method III	Proposed Mehar-Keerat method-IV	
		Symmetric/ Non-symmetric solution	Symmetric solution	Non-symmetric solution	Symmetric/Non-symmetric solution	Symmetric solution	Non-symmetric solution
Example 5.1 [59, Section 4, Example 4.1, pp. 2208], Example 5.2	Maximal fuzzy solution	N.A.	$\tilde{x}_{11} = [2 - 19(r - 1), 2 + 19(r - 1)],$ $\tilde{x}_{12} = [3 - 10(r - 1), 3 + 10(r - 1)],$ $\tilde{x}_{21} = [3 - 19(r - 1), 3 + 19(r - 1)],$ $\tilde{x}_{22} = [3 - 10(r - 1), 3 + 10(r - 1)],$ $\tilde{x}_{31} = [3 - 19(r - 1), 3 + 19(r - 1)],$ $\tilde{x}_{32} = [4 - 10(r - 1), 4 + 10(r - 1)].$	N.A.	N.A.	$\tilde{x}_{11} = (1, 2, 3), \tilde{x}_{12} = (2, 3, 4),$ $\tilde{x}_{21} = (2, 3, 4), \tilde{x}_{22} = (2, 3, 4),$ $\tilde{x}_{31} = (2, 3, 4), \tilde{x}_{32} = (3, 4, 5).$	$\tilde{x}_{11} = (1, 2, 3), \tilde{x}_{12} = (1, 3, 4),$ $\tilde{x}_{21} = (1, 3, 5), \tilde{x}_{22} = (2, 3, 5),$ $\tilde{x}_{31} = (2, 3, 4), \tilde{x}_{32} = (3, 4, 5).$
	Minimal fuzzy solution	N.A.	$\tilde{x}_{11} = \left[2 - \frac{19(r - 1)}{2}, 2 + \frac{19(r - 1)}{2} \right],$ $\tilde{x}_{12} = [3 - 4(r - 1), 3 + 4(r - 1)],$ $\tilde{x}_{21} = \left[3 - \frac{19(r - 1)}{2}, 3 + \frac{19(r - 1)}{2} \right],$ $\tilde{x}_{22} = [3 - 4(r - 1), 3 + 4(r - 1)],$ $\tilde{x}_{31} = \left[3 - \frac{19(r - 1)}{2}, 3 + \frac{19(r - 1)}{2} \right],$ $\tilde{x}_{32} = [4 - 4(r - 1), 4 + 4(r - 1)].$	N.A.	N.A.	$\tilde{x}_{11} = \left(\frac{1}{15}, 2, \frac{59}{15} \right), \tilde{x}_{12} = \left(\frac{16}{15}, 3, \frac{74}{15} \right),$ $\tilde{x}_{21} = \left(\frac{16}{15}, 3, \frac{74}{15} \right), \tilde{x}_{22} = \left(\frac{16}{15}, 3, \frac{74}{15} \right),$ $\tilde{x}_{31} = \left(\frac{16}{15}, 3, \frac{74}{15} \right), \tilde{x}_{32} = \left(\frac{31}{15}, 4, \frac{89}{15} \right).$	$\tilde{x}_{11} = (1, 2, 3), \tilde{x}_{12} = (1, 3, 4),$ $\tilde{x}_{21} = (1, 3, 5), \tilde{x}_{22} = (2, 3, 5),$ $\tilde{x}_{31} = (2, 3, 4), \tilde{x}_{32} = (3, 4, 5).$

Example 5.3	Maximal fuzzy solution	N.A.	N.A.	N.A.	N.A.	$\tilde{x}_{11} = (-6, -5, -4, -3),$ $\tilde{x}_{12} = (-4, -3, -1, 0),$ $\tilde{x}_{21} = (-2, -1, 0, 1),$ $\tilde{x}_{22} = (1, 2, 3, 4).$	$\tilde{x}_{11} = (-5.8587, -5, -4, -0.9052),$ $\tilde{x}_{12} = (-4.5583, -3, -1, 0),$ $\tilde{x}_{21} = (-1.8817, -1, 0, 0.6574),$ $\tilde{x}_{22} = (-0.5776, 2, 3, 4.6206).$
	Minimal fuzzy solution	N.A.	N.A.	N.A.	N.A.	$\tilde{x}_{11} = (-12, -5, -4, 3),$ $\tilde{x}_{12} = (-10, -3, -1, 6),$ $\tilde{x}_{21} = (-8, -1, 0, 7),$ $\tilde{x}_{22} = (-5, 2, 3, 10).$	$\tilde{x}_{11} = (-6.373, -5, -4, 2.2465),$ $\tilde{x}_{12} = (-5.5798, -3, -1, 2.7899),$ $\tilde{x}_{21} = (-1, -1, 0, 0.5),$ $\tilde{x}_{22} = (0, 2, 3, 3.0211).$

5.8. Conclusion

On the basis of present study, it can be concluded that due to the flaws of the existing method [59], it is not genuine to use the existing method [59] to find the maximal and minimal fuzzy solution the fully fuzzy Sylvester matrix equation. Also, the fully fuzzy Sylvester matrix equation whose maximal and minimal fuzzy solution can be obtained by the existing method [59], can also be obtained by the proposed Mehar-Keerat method-IV. However, there exist several fully fuzzy Sylvester matrix equations whose maximal and minimal fuzzy solution can be obtained by the proposed Mehar-Keerat method-IV but cannot be obtained by the existing method [59]. Hence, it is better to use the proposed Mehar-Keerat method-IV as compared to the existing method [59].

Chapter 6

Future scope*

After reviewing literature, it is noticed that there are few methods for solving the system of fuzzy complex linear equations. However, there is no method in the literature to find the maximal and minimal fuzzy solution of any type of system of fuzzy complex linear equations. Also, after a deep study of the existing methods for solving the system of fuzzy complex linear equations, it is noticed that in all the existing methods for solving the system of fuzzy complex linear equations, some mathematical incorrect assumptions are used. Therefore, it is not valid to use the existing methods. Furthermore, there exist several system of fuzzy complex linear equations and its variants which cannot be solved by any of the existing methods.

Keeping the same in mind, in this chapter, new methods (named Mehar method, Keerat method and Mehar-Keerat method-V) are proposed to resolve the shortcomings as well as to overcome the limitations of the existing methods for solving system of fuzzy complex linear equations. To propose methods to find the maximal and minimal fuzzy solution of system of fuzzy complex linear equations and its variants may be considered as an open challenging research problem.

6.1. Preliminaries

In this section, some basic definitions are presented [45].

Definition 6.1. If $\tilde{X}$ and $\tilde{Y}$ are real fuzzy numbers with membership functions $\mu(x|\tilde{X})$ and $\mu(y|\tilde{Y})$ respectively, then $\tilde{Z} = \tilde{X} + i\tilde{Y}$ will be a fuzzy complex number with membership

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function $\mu(z|\tilde{Z}) = \min(\mu(x|\tilde{X}), \mu(y|\tilde{Y}))$, where $z = x + iy$.

Definition 6.2. A triangular complex fuzzy number $(a_1, b_1, c_1) + i(a_2, b_2, c_2)$ represents the fuzzy set,

$$\{(x + iy, \mu(x + iy)) : x \in [a_1, c_1], y \in [a_2, c_2], \mu(x + iy) = \min\{\mu(x), \mu(y)\}\}.$$

Definition 6.3. A trapezoidal complex fuzzy number $(a_1, b_1, c_1, d_1) + i(a_2, b_2, c_2, d_2)$ represents the fuzzy set,

$$\{(x + iy, \mu(x + iy)) : x \in [a_1, d_1], y \in [a_2, d_2], \mu(x + iy) = \min\{\mu(x), \mu(y)\}\}.$$

Remark 6.1. In this chapter, triangular fuzzy number and trapezoidal fuzzy number are represented by (a, b, c) and (a, b, c, d) instead of $(a - \alpha, a, a + \alpha')$ and $(a^L - \alpha, a^L, a^U, a^U + \alpha')$ respectively.

6.2. Proposed membership functions and their graphical representation

In this section, with the help of Definition 6.2 and Definition 6.3, the membership functions of triangular and trapezoidal fuzzy complex numbers as well as graphical representation of these fuzzy complex numbers are introduced.

According to Buckley [45], a triangular complex fuzzy number $(a_1, b_1, c_1) + i(a_2, b_2, c_2)$ represents the fuzzy set,

$$\{(x + iy, \mu(x + iy)) : x \in [a_1, c_1], y \in [a_2, c_2], \mu(x + iy) = \min\{\mu(x), \mu(y)\}\}.$$

Therefore, the membership function of a triangular complex fuzzy number $(a_1, b_1, c_1) + i(a_2, b_2, c_2)$ may be defined as follows:

$$\mu(x+iy) = \begin{cases} \min \left\{ \frac{x-a_1}{b_1-a_1}, \frac{y-a_2}{b_2-a_2} \right\}, & a_1 \leq x < b_1, a_2 \leq y < b_2 \\ \min \left\{ \frac{x-a_1}{b_1-a_1}, \frac{c_2-y}{c_2-b_2} \right\}, & a_1 \leq x < b_1, b_2 \leq y < c_2 \\ \min \left\{ \frac{c_1-x}{c_1-b_1}, \frac{y-a_2}{b_2-a_2} \right\}, & b_1 \leq x < c_1, a_2 < y \leq b_2 \\ \min \left\{ \frac{c_1-x}{c_1-b_1}, \frac{c_2-y}{c_2-b_2} \right\}, & b_1 \leq x \leq c_1, b_2 < y \leq c_2 \\ 1, & x = b_1, y = b_2. \end{cases}$$

and its graphical representation will be a surface in three dimensions e.g., Fig 1 and Fig 2 (obtained by rotating Fig 1) represent the graphical representation of the triangular complex fuzzy number $(2,3,5)+i(6,7,9)$.



Fig 1. Triangular complex fuzzy number $(2,3,5)+i(6,7,9)$.

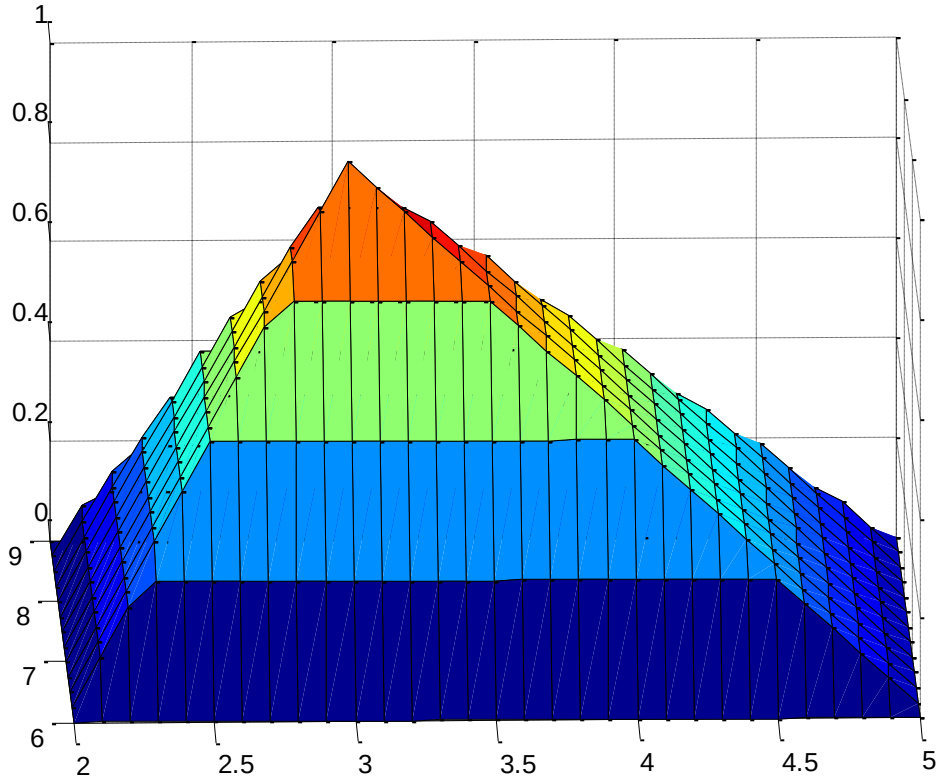


Fig 2. Triangular complex fuzzy number $(2,3,5)+i(6,7,9)$.

Similarly, a trapezoidal complex fuzzy number $(a_1, b_1, c_1, d_1) + i(a_2, b_2, c_2, d_2)$ represents the fuzzy set,

$$\{((x+iy, \mu(x+iy)) : x \in [a_1, d_1], y \in [a_2, d_2], \mu(x+iy) = \min\{\mu(x), \mu(y)\}\}.$$

Therefore, the membership function of a trapezoidal complex fuzzy number $(a_1, b_1, c_1, d_1) + i(a_2, b_2, c_2, d_2)$ may be defined as follows:

$$\mu(x+iy) = \begin{cases} \min \left\{ \frac{x-a_1}{b_1-a_1}, \frac{y-a_2}{b_2-a_2} \right\}, & a_1 \leq x < b_1, a_2 \leq y < b_2 \\ \min \left\{ \frac{x-a_1}{b_1-a_1}, \frac{d_2-y}{d_2-c_2} \right\}, & a_1 \leq x < b_1, c_2 \leq y < d_2 \\ \min \left\{ \frac{d_1-x}{d_1-c_1}, \frac{y-a_2}{b_2-a_2} \right\}, & c_1 \leq x < d_1, a_2 \leq y \leq b_2 \\ \min \left\{ \frac{d_1-x}{d_1-c_1}, \frac{d_2-y}{d_2-c_2} \right\}, & c_1 \leq x \leq d_1, c_2 \leq y \leq d_2 \\ 1, & x \in [b_1, c_1], y \in [b_2, c_2]. \end{cases}$$

and its graphical representation will be a surface in three dimensions e.g., Fig 3 and Fig 4 (obtained by rotating Fig 3) represent the graphical representation of the trapezoidal complex fuzzy number $(2, 3, 4, 5) + i(6, 7, 8, 9)$.

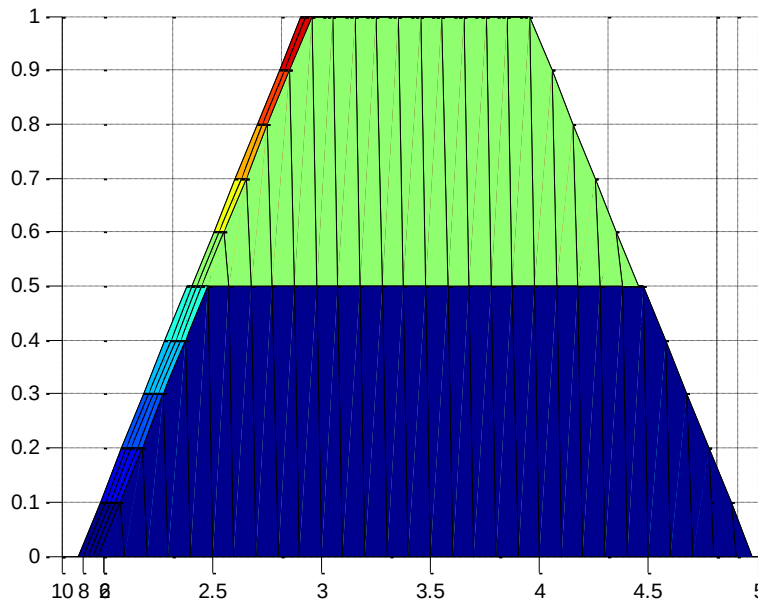


Fig 3. Trapezoidal complex fuzzy number $(2, 3, 4, 5) + i(6, 7, 8, 9)$.

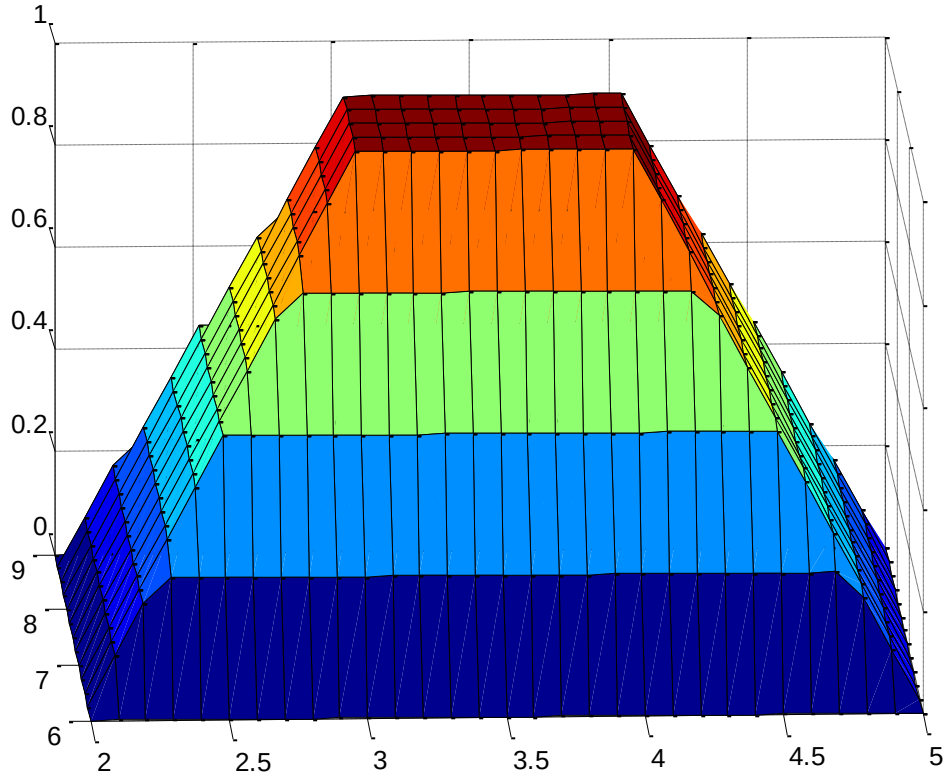


Fig 4. Trapezoidal complex fuzzy number $(2,3,4,5)+i(6,7,8,9)$.

6.3. Flaws of the existing methods

Rahgooy et al. [134] pointed out that there is no method in the literature to find the solution of such system of linear equations in which some or all the parameters are represented by fuzzy complex numbers [45] and proposed a method to find the solution of the system of fuzzy complex linear equations $\sum_{j=1}^n c_{ij} \tilde{z}_j = \tilde{w}_i, i=1,2,\dots,m$, where, $\tilde{z}_j$ and $\tilde{w}_i$ are fuzzy complex numbers and c_{ij} are crisp complex numbers.

Later, Jahantigh et al. [88], Behera and Chakraverty [40], Behera and Chakraverty [41] and Behera and Chakraverty [42], proposed different methods for solving the same system of fuzzy complex linear equations. Majumdar [102] proposed numerical methods for

solving the system of fuzzy complex linear equations $\sum_{j=1}^n c_{ij} \tilde{z}_j = \tilde{w}_i, i=1,2,\dots,m$, where,

$\tilde{z}_j$ and $\tilde{w}_i$ are fuzzy complex numbers and c_{ij} are crisp complex numbers as well as for

solving the system of fuzzy complex linear equations $\sum_{j=1}^n \tilde{c}_{ij} z_j = \tilde{w}_i, i=1,2,\dots,m$, where,

$\tilde{c}_{ij}$ and $\tilde{w}_i$ are fuzzy complex numbers and z_j are crisp complex numbers.

In this section, flaws of the existing numerical methods [102] as well as analytical methods [40-42,88,134] are pointed out.

6.3.1. Flaws of the numerical methods

In this section, flaws of the existing numerical methods [102] are pointed out.

It is well known fact that if $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are three fuzzy numbers, then $\tilde{a} + \tilde{b} = \tilde{c}$ does not imply $\tilde{a} = \tilde{c} - \tilde{b}$. However, Majumdar [102] has used this mathematical incorrect assumption to transform the system of fuzzy complex linear equations

$\sum_{j=1}^n \tilde{c}_{ij} x_j = \tilde{w}_i, i=1,2,\dots,n$ into system of fuzzy complex linear equations

$$x_i = \frac{\tilde{w}_i - \sum_{\substack{j=1 \\ j \neq i}}^n \tilde{c}_{ij} x_j}{\tilde{c}_{ii}}, i=1,2,\dots,n, \text{ as well as to transform the system of fuzzy complex linear}$$

equations $\sum_{j=1}^n c_{ij} \tilde{x}_j = \tilde{w}_i, i=1,2,\dots,n$ into system of fuzzy complex linear equations

$$\tilde{x}_i = \frac{\tilde{w}_i - \sum_{\substack{j=1 \\ j \neq i}}^n c_{ij} \tilde{x}_j}{c_{ii}}, i=1,2,\dots,n.$$

Since, the system of fuzzy complex linear equations $\sum_{j=1}^n \tilde{c}_{ij} x_j = \tilde{w}_i, i=1,2,\dots,n$, cannot

be transformed into system of fuzzy complex linear equations $x_i = \frac{\tilde{w}_i - \sum_{\substack{j=1 \\ j \neq i}}^n \tilde{c}_{ij} x_j}{\tilde{c}_{ii}}, i=1,2,\dots,n$

and the system of fuzzy complex linear equations $\sum_{j=1}^n c_{ij} \tilde{x}_j = \tilde{w}_i, i=1,2,\dots,n$, cannot be

transformed into system of fuzzy complex linear equations $\tilde{x}_i = \frac{\tilde{w}_i - \sum_{\substack{j=1 \\ j \neq i}}^n c_{ij} \tilde{x}_j}{c_{ii}}, i=1,2,\dots,n$.

Therefore, the numerical methods, proposed by Majumdar [102], are not valid.

6.3.2. Flaws of the analytical methods

In this section, flaws of the existing analytical methods [40-42,88,134], are pointed out.

It is well known fact that $\lambda[a,b] = \begin{cases} [\lambda a, \lambda b], \lambda \geq 0, \\ [\lambda b, \lambda a], \lambda < 0. \end{cases}$

Since, i is a purely imaginary number i.e., neither $i \geq 0$ nor $i < 0$. Therefore, neither $i[a,b] = [ia, ib]$ nor $i[a,b] = [ib, ia]$.

Also, it is well known that if z_1 and z_2 are two distinct complex numbers then it is not possible to check that $z_1 > z_2$ or $z_1 < z_2$. Due to the same reason, the interval of complex numbers $[z_1, z_2]$ does not exist. However, the authors [40-42,88,134] have used the invalid property $i[a,b] = [ia, ib]$ and/or assumed that the interval of complex numbers $[z_1, z_2]$ exist at the following places.

1. Rahgooy et al. [134, Section A, Sub-section IV, pp. 537] have assumed that

$\underline{z} = \underline{a}(r) + i\underline{b}(r)$ and $\bar{z} = \bar{a}(r) + i\bar{b}(r)$ represents the lower bound and upper bound of

an interval $[\underline{z}, \bar{z}]$ i.e., Rahgooy et al. [134] have assumed that interval of fuzzy complex numbers exist.

2. Jahantigh et al. [88, Theorem 3.1, pp. 25] have transformed the equation 3.20 [88, pp. 25], $z' = (b_1^-, \dots, b_n^-, -b_1^+, \dots, -b_n^+)^t + i(c_1^-, \dots, c_n^-, -c_1^+, \dots, -c_n^+)^t$ into $z' = (b_1^- + ic_1^-, \dots, b_n^- + ic_n^-, -(b_1^+ + ic_1^+), \dots, -(b_n^+ + ic_n^+))^t$ by considering the wrong assumption $i(a_1, a_2, \dots, a_n) = (ia_1, ia_2, \dots, ia_n)$.
3. Behera and Chakraverty [40, Section 2.4, pp. 509] have assumed that the interval of complex numbers exist and in Section 4.1 [40, pp. 511], for solving the system of fuzzy complex linear equation (18), Behera and Chakraverty [40] have transformed $z_j = p_j + iq_j = [\underline{p}_j(\alpha), \bar{p}_j(\alpha)] + i[\underline{q}_j(\alpha), \bar{q}_j(\alpha)]$ into $[\underline{p}_j(\alpha) + i\underline{q}_j(\alpha), \bar{p}_j(\alpha) + i\bar{q}_j(\alpha)]$ by considering the mathematically incorrect assumption $i[a, b] = [ia, ib]$. Also, in equation 9 [40, Section 4.1, pp. 511], Behera and Chakraverty [40] have considered the mathematically incorrect assumption that the interval of complex numbers $[\underline{z}_j, \bar{z}_j]$ exist.
4. Behera and Chakraverty [41, Definition 5, pp. 632] have assumed that an arbitrary fuzzy complex number may be represented as $\tilde{X} = [\underline{p}(\alpha), \bar{p}(\alpha)] + i[\underline{q}(\alpha), \bar{q}(\alpha)] = [\underline{p}(\alpha) + i\underline{q}(\alpha), \bar{p}(\alpha) + i\bar{q}(\alpha)]$, by considering the mathematically incorrect assumption that the interval of complex numbers exist as well as by considering the mathematically incorrect assumption $i[a, b] = [ia, ib]$. Also, Behera and Chakraverty [41, Section 3, pp. 633] have transformed $\tilde{z}_j = p_j + iq_j = [\underline{p}_j(\alpha), \bar{p}_j(\alpha)] + i[\underline{q}_j(\alpha), \bar{q}_j(\alpha)]$ into $[\underline{p}_j(\alpha) + i\underline{q}_j(\alpha), \bar{p}_j(\alpha) + i\bar{q}_j(\alpha)]$ and $\tilde{w}_k = u_k + iv_k = [\underline{u}_k(\alpha), \bar{u}_k(\alpha)] + i[\underline{v}_k(\alpha), \bar{v}_k(\alpha)]$ into

$\tilde{w}_k = [\underline{u}_k(\alpha) + i\underline{v}_k(\alpha), \bar{u}_k(\alpha) + i\bar{v}_k(\alpha)]$ by considering the mathematically incorrect assumption $i[a, b] = [ia, ib]$.

5. Behera and Chakraverty [42, Definition 2.4, pp. 156] have assumed that an arbitrary fuzzy complex number may be represented as $\tilde{X} = [\underline{p}(\alpha), \bar{p}(\alpha)] + i[\underline{q}(\alpha), \bar{q}(\alpha)] = [\underline{p}(\alpha) + i\underline{q}(\alpha), \bar{p}(\alpha) + i\bar{q}(\alpha)]$, by considering the mathematically incorrect assumption that the interval of complex numbers exist as well as by considering the mathematically incorrect assumption $i[a, b] = [ia, ib]$. Also, Behera and Chakraverty [42, Section 3, pp. 156] have transformed $\tilde{z}_j = p_j + iq_j = [\underline{p}_j(\alpha), \bar{p}_j(\alpha)] + i[\underline{q}_j(\alpha), \bar{q}_j(\alpha)]$ into $[\underline{p}_j(\alpha) + i\underline{q}_j(\alpha), \bar{p}_j(\alpha) + i\bar{q}_j(\alpha)]$ and $\tilde{w}_k = u_k + iv_k = [\underline{u}_k(\alpha), \bar{u}_k(\alpha)] + i[\underline{v}_k(\alpha), \bar{v}_k(\alpha)]$ into $\tilde{w}_k = [\underline{u}_k(\alpha) + i\underline{v}_k(\alpha), \bar{u}_k(\alpha) + i\bar{v}_k(\alpha)]$ by considering the mathematically incorrect assumption $i[a, b] = [ia, ib]$.

6.4. Limitations of the numerical and analytical methods

In this section, limitations of the existing numerical as well as analytical methods are discussed.

The existing methods [40-42, 88, 102, 134] cannot be used for solving the following system of fuzzy complex equations:

1. The system of fuzzy complex linear equations $\sum_{j=1}^n \tilde{a}_{ij} \tilde{z}_j = \tilde{b}_i, i = 1, 2, \dots, m$, where,

$\tilde{a}_{ij}, \tilde{z}_j$ and $\tilde{b}_i$ are triangular/trapezoidal fuzzy complex numbers.

For example, the system of fuzzy complex linear equations, chosen in Example 6.1, cannot be solved by any of the existing methods [40-42, 88, 102, 134].

Example 6.1

$$[(1, 2, 3, 4) + i(-1, 0, 1, 2)]\tilde{z}_1 - [(2, 4, 6, 8) + i(-4, -3, -2, -1)]\tilde{z}_2 = (-49, -12, 33, 68) + i(-2, 23, 66, 112),$$

$$[(4, 5, 6, 7) + i(-2, 0, 2, 4)]\tilde{z}_1 + [(1, 3, 5, 7) + i(-5, -2, 1, 4)]\tilde{z}_2 = (-88, -19, 39, 100) + i(-61, -11, 33, 85).$$

2. The dual system of fuzzy complex linear equations

$$\sum_{j=1}^n a_{ij}\tilde{z}_j + \tilde{b}_i = \sum_{j=1}^n c_{ij}\tilde{z}_j + \tilde{d}_i, i = 1, 2, \dots, m, \text{ where, } a_{ij}, c_{ij} \text{ are crisp complex numbers and}$$

$\tilde{z}_j, \tilde{b}_i, \tilde{d}_i$ are triangular/trapezoidal fuzzy complex numbers.

For example, the dual system of fuzzy complex linear equations, chosen in Example 6.2, cannot be solved by any of the existing methods [40-42,88,102,134].

Example 6.2

$$(4 + 3i)\tilde{z}_1 + (-1 - 5i)\tilde{z}_2 + [(0, 25, 41, 46) + i(13, 20, 27, 32)] = (2 + 3i)\tilde{z}_1 + (10 - 5i)\tilde{z}_2 + [(9, 9, 9, 9) + i(0, 0, 0, 0)],$$

$$(-6 + 4i)\tilde{z}_1 - (7 - 2i)\tilde{z}_2 + [(75, 75, 75, 75) + i(0, 0, 0, 0)] = (1 + 2i)\tilde{z}_1 + (2 - 4i)\tilde{z}_2 + [(0, 7, 22, 52) + i(12, 34, 42, 43)].$$

3. The dual system of fuzzy complex linear equations

$$\sum_{j=1}^n \tilde{a}_{ij}\tilde{z}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij}\tilde{z}_j + \tilde{d}_i, i = 1, 2, \dots, m, \quad \text{where, } \tilde{a}_{ij}, \tilde{z}_j, \tilde{b}_i, \tilde{c}_{ij}, \text{ and } \tilde{d}_i \text{ are}$$

triangular/trapezoidal fuzzy complex numbers.

For example, the dual system of fuzzy complex linear equations, chosen in Example 6.3, cannot be solved by any of the existing methods [40-42,88,102,134].

Example 6.3

$$[(0, 1, 2, 3) + i(-3, -1, 1, 3)]\tilde{z}_1 + [(2, 4, 6, 8) + i(1, 2, 3, 4)]\tilde{z}_2 + [(0, 0, 0, 0) + i(74, 74, 74, 82)] = [(1, 2, 3, 4) + i(-1, 0, 1, 2)]\tilde{z}_1 - [(2, 4, 6, 8) + i(-4, -3, -2, -1)]\tilde{z}_2 + [(9, 10, 15, 28) + i(0, 14, 16, 16)],$$

$$[(-4, -3, -2, -1) + i(1, 3, 5, 6)]\tilde{z}_1 + [(1, 2, 3, 5) + i(-5, -3, -2, -1)]\tilde{z}_2 + [(21, 41, 57, 83) + i(32, 32, 37, 62)] = [(4, 5, 6, 7) + i(-2, 0, 2, 4)]\tilde{z}_1 + [(1, 3, 5, 7) + i(-5, -2, 1, 4)]\tilde{z}_2 + [(0, 0, 0, 0) + i(0, 4, 4, 4)].$$

4. The dual system of fuzzy complex matrix equations

$$\sum_{j=1}^n a_{ij} \tilde{z}_{jk} + \tilde{b}_{ik} = \sum_{j=1}^n c_{ij} \tilde{z}_{jk} + \tilde{d}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p, \text{ where, } a_{ij}, c_{ij} \text{ are crisp complex}$$

numbers and $\tilde{b}_{ik}, \tilde{d}_{ik}, \tilde{z}_{jk}$ are triangular/trapezoidal fuzzy complex numbers.

For example, the dual system of fuzzy complex matrix equations, chosen in Example 6.4, cannot be solved by any of the existing methods [40-42,88,102,134].

Example 6.4

$$\begin{aligned} (4 + 3i)\tilde{z}_{11} + (-1 - 5i)\tilde{z}_{21} + [(0, 25, 41, 46) + i(13, 20, 27, 32)] &= (2 + 3i)\tilde{z}_{11} + (10 - 5i)\tilde{z}_{21} + [(9, 9, 9, \\ &9) + i(0, 0, 0, 0)], \\ (-6 + 4i)\tilde{z}_{11} - (7 - 2i)\tilde{z}_{21} + [(75, 75, 75, 75) + i(0, 0, 0, 0)] &= (1 + 2i)\tilde{z}_{11} + (2 - 4i)\tilde{z}_{21} + [(0, 7, 22, 52) \\ &+ i(12, 34, 42, 43)], \\ (4 + 3i)\tilde{z}_{12} + (-1 - 5i)\tilde{z}_{22} + [(0, 7, 12, 19) + i(0, 16, 41, 57)] &= (2 + 3i)\tilde{z}_{12} + (10 - 5i)\tilde{z}_{22} + [(76, 76, \\ &76, 76) + i(22, 22, 22, 22)], \\ (-6 + 4i)\tilde{z}_{12} - (7 - 2i)\tilde{z}_{22} + [(0, 0, 0, 0) + i(54, 54, 54, 54)] &= (1 + 2i)\tilde{z}_{12} + (2 - 4i)\tilde{z}_{22} + [(13, 21, 32, \\ &40) + i(0, 15, 37, 52)], \\ (4 + 3i)\tilde{z}_{13} + (-1 - 5i)\tilde{z}_{23} + [(0, 6, 13, 30) + i(0, 24, 58, 93)] &= (2 + 3i)\tilde{z}_{13} + (10 - 5i)\tilde{z}_{23} + [(56, 56, \\ &56, 56) + i(102, 102, 102, 102)], \\ (-6 + 4i)\tilde{z}_{13} - (7 - 2i)\tilde{z}_{23} + [(0, 0, 0, 0) + i(89, 89, 89, 89)] &= (1 + 2i)\tilde{z}_{13} + (2 - 4i)\tilde{z}_{23} + [(41, 56, 60, \\ &60) + i(0, 23, 48, 68)]. \end{aligned}$$

5. The dual system of fuzzy complex matrix equations

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{z}_{jk} + \tilde{b}_{ik} = \sum_{j=1}^n \tilde{c}_{ij} \tilde{z}_{jk} + \tilde{d}_{ik}, i = 1, 2, \dots, m; k = 1, 2, \dots, p, \text{ where, } \tilde{a}_{ij}, \tilde{z}_{jk}, \tilde{b}_{ik}, \tilde{d}_{ik} \text{ are}$$

triangular/trapezoidal fuzzy complex numbers.

For example, the dual system of fuzzy complex matrix equations, chosen in Example 6.5, cannot be solved by any of the existing methods [40-42,88,102,134].

Example 6.5

$$[(2, 3, 5, 6) + i(1, 2, 3, 5)]\tilde{z}_{11} + [(-1, 1, 2, 3) + i(2, 3, 4, 5)]\tilde{z}_{21} - [(-4, -3, -2, -1) + i(4, 5, 6, 7)]\tilde{z}_{31} + [(0, 0, 15, 38) + i(0, 35, 55, 55)] = [(-9, -7, -5, -4) + i(3, 4, 6, 8)]\tilde{z}_{11} + [(0, 1, 2, 3) + i(-5, -3, -1, 1)]\tilde{z}_{21} + [(1, 2, 3, 4) + i(5, 6, 7, 8)]\tilde{z}_{31} + [(53, 59, 59, 59) + i(26, 26, 26, 35)],$$

$$[(1, 3, 5, 6) + i(2, 3, 6, 7)]\tilde{z}_{11} + [(-5, -4, -3, -1) + i(2, 4, 6, 8)]\tilde{z}_{21} + [(-1, 0, 1, 2) + i(-6, -5, 0, 3)]\tilde{z}_{31} + [(0, 52, 58, 82) + i(0, 43, 43, 43)] = [(2, 3, 4, 5) + i(0, 1, 2, 3)]\tilde{z}_{11} - [(5, 9, 10, 11) + i(-2, -1, 0, 1)]\tilde{z}_{21} + [(7, 9, 10, 12) + i(-11, -10, -9, -8)]\tilde{z}_{31} + [(29, 29, 29, 29) + i(73, 73, 97, 97)],$$

$$[(2, 3, 5, 6) + i(1, 2, 3, 5)]\tilde{z}_{12} + [(-1, 1, 2, 3) + i(2, 3, 4, 5)]\tilde{z}_{22} - [(-4, -3, -2, -1) + i(4, 5, 6, 7)]\tilde{z}_{32} + [(0, 20, 36, 36) + i(0, 37, 58, 58)] = [(-9, -7, -5, -4) + i(3, 4, 6, 8)]\tilde{z}_{12} + [(0, 1, 2, 3) + i(-5, -3, -1, 1)]\tilde{z}_{22} + [(1, 2, 3, 4) + i(5, 6, 7, 8)]\tilde{z}_{32} + [(111, 111, 111, 111) + i(48, 48, 48, 65)],$$

$$[(1, 3, 5, 6) + i(2, 3, 6, 7)]\tilde{z}_{12} + [(-5, -4, -3, -1) + i(2, 4, 6, 8)]\tilde{z}_{22} + [(-1, 0, 1, 2) + i(-6, -5, 0, 3)]\tilde{z}_{32} + [(0, 36, 36, 44) + i(4, 27, 27, 50)] = [(2, 3, 4, 5) + i(0, 1, 2, 3)]\tilde{z}_{12} - [(5, 9, 10, 11) + i(-2, -1, 0, 1)]\tilde{z}_{22} + [(7, 9, 10, 12) + i(-11, -10, -9, -8)]\tilde{z}_{32} + [(0, 0, 13, 13) + i(0, 0, 20, 20)],$$

$$[(2, 3, 5, 6) + i(1, 2, 3, 5)]\tilde{z}_{13} + [(-1, 1, 2, 3) + i(2, 3, 4, 5)]\tilde{z}_{23} - [(-4, -3, -2, -1) + i(4, 5, 6, 7)]\tilde{z}_{33} + [(0, 13, 23, 26) + i(121, 121, 132, 164)] = [(-9, -7, -5, -4) + i(3, 4, 6, 8)]\tilde{z}_{13} + [(0, 1, 2, 3) + i(-5, -3, -1, 1)]\tilde{z}_{23} + [(1, 2, 3, 4) + i(5, 6, 7, 8)]\tilde{z}_{33} + [(72, 72, 72, 72) + i(0, 13, 13, 13)],$$

$$[(1, 3, 5, 6) + i(2, 3, 6, 7)]\tilde{z}_{13} + [(-5, -4, -3, -1) + i(2, 4, 6, 8)]\tilde{z}_{23} + [(-1, 0, 1, 2) + i(-6, -5, 0, 3)]\tilde{z}_{33} + [(93, 93, 97, 128) + i(0, 16, 16, 27)] = [(2, 3, 4, 5) + i(0, 1, 2, 3)]\tilde{z}_{13} - [(5, 9, 10, 11) + i(-2, -1, 0, 1)]\tilde{z}_{23} + [(7, 9, 10, 12) + i(-11, -10, -9, -8)]\tilde{z}_{33} + [(0, 1, 1, 1) + i(18, 18, 19, 19)].$$

6.5. Proposed methods

In this section, to resolve the flaws of the existing numerical methods, new methods (named as Mehar method and Keerat method) for solving system of fuzzy complex linear equations are proposed. Also, to resolve the shortcomings as well as to overcome the limitations of the analytical methods, a new method (named as Mehar-Keerat method-V) is proposed to find the solution of dual system of fuzzy complex linear equations

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{z}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \tilde{z}_j + \tilde{d}_i, i = 1, 2, \dots, m, \quad \text{where, } \tilde{a}_{ij}, \tilde{z}_j, \tilde{b}_i, \tilde{c}_{ij}, \text{ and } \tilde{d}_i \text{ are trapezoidal fuzzy}$$

complex numbers.

6.5.1. Mehar method for solving system of fuzzy complex linear equations with fuzzy complex coefficients

In this section, a new method (named as Mehar method) for solving system of fuzzy complex linear equations with fuzzy complex coefficients (6.1) is proposed.

$$\begin{aligned}\tilde{c}_{11}z_1 + \tilde{c}_{12}z_2 + \dots + \tilde{c}_{1n}z_n &= \tilde{w}_1, \\ \tilde{c}_{21}z_1 + \tilde{c}_{22}z_2 + \dots + \tilde{c}_{2n}z_n &= \tilde{w}_2, \\ &\vdots \\ \tilde{c}_{n1}z_1 + \tilde{c}_{n2}z_2 + \dots + \tilde{c}_{nn}z_n &= \tilde{w}_n.\end{aligned}$$

$$\text{i.e. } \sum_{j=1}^n \tilde{c}_{ij}z_j = \tilde{w}_i, \quad i = 1, 2, \dots, n. \quad (6.1)$$

where $\tilde{c}_{ij}, \tilde{w}_i$ are trapezoidal fuzzy complex numbers and z_j is a crisp complex number.

The steps of the proposed Mehar method are as follows:

Step 1: Assuming

$$\tilde{c}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) + i(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4}), z_j = (x_j + iy_j), \tilde{w}_i = (u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4}),$$

the system of fuzzy complex linear equations (6.1) can be transformed into system of fuzzy complex linear equations (6.2).

$$\sum_{j=1}^n \{[(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) + i(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4})](x_j + iy_j)\} = [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n. \quad (6.2)$$

Step 2: Using the existing product [134, Section IV, pp. 537]

$$\begin{aligned}&[(x + iy)][(a_1, a_2, a_3, a_4) + i(b_1, b_2, b_3, b_4)] \\ &= [x(a_1, a_2, a_3, a_4) - y(b_1, b_2, b_3, b_4)] + i[x(b_1, b_2, b_3, b_4) + y(a_1, a_2, a_3, a_4)],\end{aligned}$$

the system of fuzzy complex linear equations (6.2) can be transformed into system of fuzzy complex linear equations (6.3).

$$\begin{aligned}&\sum_{j=1}^n \{[x_j(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) - y_j(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4})] + i[x_j(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4}) + y_j(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})]\} \\ &= [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n.\end{aligned} \quad (6.3)$$

Step 3: Using the scalar multiplication [92, Section 3.4, pp. 33]

$$x_j(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) = (e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) = \begin{cases} (x_j a_{ij1}, x_j a_{ij2}, x_j a_{ij3}, x_j a_{ij4}) & \text{if } x_j \geq 0, \\ (x_j a_{ij4}, x_j a_{ij3}, x_j a_{ij2}, x_j a_{ij1}) & \text{if } x_j < 0, \end{cases}$$

$$y_j(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4}) = (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}) = \begin{cases} (y_j b_{ij1}, y_j b_{ij2}, y_j b_{ij3}, y_j b_{ij4}) & \text{if } y_j \geq 0, \\ (y_j b_{ij4}, y_j b_{ij3}, y_j b_{ij2}, y_j b_{ij1}) & \text{if } y_j < 0, \end{cases}$$

$$x_j(b_{ij1}, b_{ij2}, b_{ij3}, b_{ij4}) = (e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) = \begin{cases} (x_j b_{ij1}, x_j b_{ij2}, x_j b_{ij3}, x_j b_{ij4}) & \text{if } x_j \geq 0, \\ (x_j b_{ij4}, x_j b_{ij3}, x_j b_{ij2}, x_j b_{ij1}) & \text{if } x_j < 0, \end{cases}$$

$$y_j(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) = (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}) = \begin{cases} (y_j a_{ij1}, y_j a_{ij2}, y_j a_{ij3}, y_j a_{ij4}) & \text{if } y_j \geq 0, \\ (y_j a_{ij4}, y_j a_{ij3}, y_j a_{ij2}, y_j a_{ij1}) & \text{if } y_j < 0, \end{cases}$$

the system of fuzzy complex linear equations (6.3) can be transformed into system of fuzzy complex linear equations (6.4).

$$\begin{aligned} & \sum_{j=1}^n \{[(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})] + i[(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) + (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4})]\} \\ & = [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n. \end{aligned} \quad (6.4)$$

Step 4: Using the property

$$\sum_{i=1}^n [(a_{i1}, a_{i2}, a_{i3}, a_{i4}) + i(a_{i5}, a_{i6}, a_{i7}, a_{i8})] = \sum_{i=1}^n (a_{i1}, a_{i2}, a_{i3}, a_{i4}) + i \sum_{i=1}^n (a_{i5}, a_{i6}, a_{i7}, a_{i8}),$$

the system of fuzzy complex linear equations (6.4) can be transformed into system of fuzzy complex linear equations (6.5).

$$\begin{aligned} & \sum_{j=1}^n [(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})] + i \sum_{j=1}^n [(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) + (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4})] \\ & = [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n. \end{aligned} \quad (6.5)$$

Step 5: Equating real and imaginary parts of the system of fuzzy complex linear equations (6.5), it can be transformed into system of fuzzy linear equations (6.6).

$$\begin{aligned} & \sum_{j=1}^n [(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})] = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), i = 1, 2, \dots, n; \\ & \sum_{j=1}^n [(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) + (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4})] = (v_{i1}, v_{i2}, v_{i3}, v_{i4}), i = 1, 2, \dots, n. \end{aligned} \quad (6.6)$$

Step 6: Using the property

$$\sum_{i=1}^n (a_{i1}, a_{i2}, a_{i3}, a_{i4}) = \left(\sum_{j=1}^n a_{j1}, \sum_{j=1}^n a_{j2}, \sum_{j=1}^n a_{j3}, \sum_{j=1}^n a_{j4} \right),$$

the system of fuzzy linear equations (6.6) can be transformed into system of fuzzy linear equations (6.7).

$$\begin{aligned} & \left(\sum_{j=1}^n e_{ij1}, \sum_{j=1}^n e_{ij2}, \sum_{j=1}^n e_{ij3}, \sum_{j=1}^n e_{ij4} \right) - \left(\sum_{j=1}^n f_{ij1}, \sum_{j=1}^n f_{ij2}, \sum_{j=1}^n f_{ij3}, \sum_{j=1}^n f_{ij4} \right) = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), i = 1, 2, \dots, n; \\ & \left(\sum_{j=1}^n e'_{ij1}, \sum_{j=1}^n e'_{ij2}, \sum_{j=1}^n e'_{ij3}, \sum_{j=1}^n e'_{ij4} \right) + \left(\sum_{j=1}^n f'_{ij1}, \sum_{j=1}^n f'_{ij2}, \sum_{j=1}^n f'_{ij3}, \sum_{j=1}^n f'_{ij4} \right) = (v_{i1}, v_{i2}, v_{i3}, v_{i4}), i = 1, 2, \dots, n. \end{aligned} \quad (6.7)$$

Step 7: Using the properties

$$\begin{aligned} & \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right) + \left(\sum_{i=1}^n b_{i1}, \sum_{i=1}^n b_{i2}, \sum_{i=1}^n b_{i3}, \sum_{i=1}^n b_{i4} \right) \\ & = \left(\sum_{i=1}^n a_{i1} + \sum_{i=1}^n b_{i1}, \sum_{i=1}^n a_{i2} + \sum_{i=1}^n b_{i2}, \sum_{i=1}^n a_{i3} + \sum_{i=1}^n b_{i3}, \sum_{i=1}^n a_{i4} + \sum_{i=1}^n b_{i4} \right) \end{aligned}$$

and

$$\begin{aligned} & \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right) - \left(\sum_{i=1}^n b_{i1}, \sum_{i=1}^n b_{i2}, \sum_{i=1}^n b_{i3}, \sum_{i=1}^n b_{i4} \right) \\ & = \left(\sum_{i=1}^n a_{i1} - \sum_{i=1}^n b_{i4}, \sum_{i=1}^n a_{i2} - \sum_{i=1}^n b_{i3}, \sum_{i=1}^n a_{i3} - \sum_{i=1}^n b_{i2}, \sum_{i=1}^n a_{i4} - \sum_{i=1}^n b_{i1} \right), \end{aligned}$$

the system of fuzzy linear equations (6.7) can be transformed into system of fuzzy linear equations (6.8).

$$\begin{aligned} & \left(\sum_{j=1}^n e_{ij1} - \sum_{j=1}^n f_{ij4}, \sum_{j=1}^n e_{ij2} - \sum_{j=1}^n f_{ij3}, \sum_{j=1}^n e_{ij3} - \sum_{j=1}^n f_{ij2}, \sum_{j=1}^n e_{ij4} - \sum_{j=1}^n f_{ij1} \right) = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), i = 1, 2, \dots, n; \\ & \left(\sum_{j=1}^n e'_{ij1} + \sum_{j=1}^n f'_{ij4}, \sum_{j=1}^n e'_{ij2} + \sum_{j=1}^n f'_{ij3}, \sum_{j=1}^n e'_{ij3} + \sum_{j=1}^n f'_{ij2}, \sum_{j=1}^n e'_{ij4} + \sum_{j=1}^n f'_{ij1} \right) = (v_{i1}, v_{i2}, v_{i3}, v_{i4}), i = 1, 2, \dots, n. \end{aligned} \quad (6.8)$$

Step 8: Using the property

$$(a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4) \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4,$$

the system of fuzzy linear equations (6.8) can be transformed into system of crisp linear equations (6.9).

$$\begin{aligned}
\sum_{j=1}^n e_{ij1} - \sum_{j=1}^n f_{ij4} &= u_{i1}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e_{ij2} - \sum_{j=1}^n f_{ij3} &= u_{i2}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e_{ij3} - \sum_{j=1}^n f_{ij2} &= u_{i3}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e_{ij4} - \sum_{j=1}^n f_{ij1} &= u_{i4}, i = 1, 2, \dots, n;
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
\sum_{j=1}^n e'_{ij1} + \sum_{j=1}^n f'_{ij1} &= v_{i1}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij2} + \sum_{j=1}^n f'_{ij2} &= v_{i2}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij3} + \sum_{j=1}^n f'_{ij3} &= v_{i3}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij4} + \sum_{j=1}^n f'_{ij4} &= v_{i4}, i = 1, 2, \dots, n.
\end{aligned}$$

Step 9: Find the solution $\{x_j, y_j\}$ of system of crisp linear equations (6.9).

Step 10: Using the solution, obtained in Step 9, the solution of system of fuzzy complex linear equations (6.1) is $\{z_j = x_j + iy_j\}$.

6.5.2. Keerat method for solving system of fuzzy complex linear equations with fuzzy complex variables

In this section, a new method (named as Keerat method) for solving system of fuzzy complex linear equations with fuzzy complex variables (6.10) is proposed.

$$\begin{aligned}
c_{11}\tilde{z}_1 + c_{12}\tilde{z}_2 + \dots + c_{1n}\tilde{z}_n &= \tilde{w}_1 \\
c_{21}\tilde{z}_1 + c_{22}\tilde{z}_2 + \dots + c_{2n}\tilde{z}_n &= \tilde{w}_2 \\
&\vdots \\
c_{n1}\tilde{z}_1 + c_{n2}\tilde{z}_2 + \dots + c_{nn}\tilde{z}_n &= \tilde{w}_n
\end{aligned}$$

i.e. $\sum_{j=1}^n c_{ij}\tilde{z}_j = \tilde{w}_i, i = 1, 2, \dots, n.$ (6.10)

where $\tilde{z}_j, \tilde{w}_i$ are trapezoidal fuzzy complex numbers and c_{ij} is a crisp complex number.

The steps of the proposed Keerat method are as follows:

Step 1: Assuming

$$\begin{aligned} c_{ij} &= (a_{ij} + ib_{ij}), \\ \tilde{z}_j &= (x_{j1}, x_{j2}, x_{j3}, x_{j4}) + i(y_{j1}, y_{j2}, y_{j3}, y_{j4}), \\ \tilde{w}_i &= (u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4}), \end{aligned}$$

the system of fuzzy complex linear equations (6.10) can be transformed into system of fuzzy complex linear equations (6.11).

$$\sum_{j=1}^n \{(a_{ij} + ib_{ij})[(x_{j1}, x_{j2}, x_{j3}, x_{j4}) + i(y_{j1}, y_{j2}, y_{j3}, y_{j4})]\} = [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n. \quad (6.11)$$

Step 2: Using the existing product [134, Section IV, pp. 537]

$$\begin{aligned} &[(x + iy)][(a_1, a_2, a_3, a_4) + i(b_1, b_2, b_3, b_4)] \\ &= [x(a_1, a_2, a_3, a_4) - y(b_1, b_2, b_3, b_4)] + i[x(b_1, b_2, b_3, b_4) + y(a_1, a_2, a_3, a_4)], \end{aligned}$$

the system of fuzzy complex linear equations (6.11) can be transformed into system of fuzzy complex linear equations (6.12).

$$\begin{aligned} &\sum_{j=1}^n \{[a_{ij}(x_{j1}, x_{j2}, x_{j3}, x_{j4}) - b_{ij}(y_{j1}, y_{j2}, y_{j3}, y_{j4})] + i[a_{ij}(y_{j1}, y_{j2}, y_{j3}, y_{j4}) + b_{ij}(x_{j1}, x_{j2}, x_{j3}, x_{j4})]\} \\ &= [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n. \end{aligned} \quad (6.12)$$

Step 3: Using the scalar multiplication [92, Section 3.4, pp. 33]

$$\begin{aligned} a_{ij}(x_{j1}, x_{j2}, x_{j3}, x_{j4}) &= (e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) = \begin{cases} (a_{ij}x_{j1}, a_{ij}x_{j2}, a_{ij}x_{j3}, a_{ij}x_{j4}) & \text{if } a_{ij} \geq 0, \\ (a_{ij}x_{j4}, a_{ij}x_{j3}, a_{ij}x_{j2}, a_{ij}x_{j1}) & \text{if } a_{ij} < 0, \end{cases} \\ b_{ij}(y_{j1}, y_{j2}, y_{j3}, y_{j4}) &= (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}) = \begin{cases} (b_{ij}y_{j1}, b_{ij}y_{j2}, b_{ij}y_{j3}, b_{ij}y_{j4}) & \text{if } b_{ij} \geq 0, \\ (b_{ij}y_{j4}, b_{ij}y_{j3}, b_{ij}y_{j2}, b_{ij}y_{j1}) & \text{if } b_{ij} < 0, \end{cases} \\ a_{ij}(y_{j1}, y_{j2}, y_{j3}, y_{j4}) &= (e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) = \begin{cases} (a_{ij}y_{j1}, a_{ij}y_{j2}, a_{ij}y_{j3}, a_{ij}y_{j4}) & \text{if } a_{ij} \geq 0, \\ (a_{ij}y_{j4}, a_{ij}y_{j3}, a_{ij}y_{j2}, a_{ij}y_{j1}) & \text{if } a_{ij} < 0, \end{cases} \end{aligned}$$

$$b_{ij}(x_{j1}, x_{j2}, x_{j3}, x_{j4}) = (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}) = \begin{cases} (b_{ij}x_{j1}, b_{ij}x_{j2}, b_{ij}x_{j3}, b_{ij}x_{j4}) & \text{if } b_{ij} \geq 0, \\ (b_{ij}x_{j4}, b_{ij}x_{j3}, b_{ij}x_{j2}, b_{ij}x_{j1}) & \text{if } b_{ij} < 0, \end{cases}$$

the system of fuzzy complex linear equations (6.12) can be transformed into system of fuzzy complex linear equations (6.13).

$$\begin{aligned} & \sum_{j=1}^n \{[(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})] + i[(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) + (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4})]\} \\ & = \{(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})\}; i = 1, 2, \dots, n. \end{aligned} \quad (6.13)$$

Step 4: Using the property

$$\sum_{i=1}^n [(a_{i1}, a_{i2}, a_{i3}, a_{i4}) + i(a_{i5}, a_{i6}, a_{i7}, a_{i8})] = \sum_{i=1}^n (a_{i1}, a_{i2}, a_{i3}, a_{i4}) + i \sum_{i=1}^n (a_{i5}, a_{i6}, a_{i7}, a_{i8}),$$

the system of fuzzy complex linear equations (6.13) can be transformed into system of fuzzy complex linear equations (6.14).

$$\begin{aligned} & \sum_{j=1}^n [(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})] + i \sum_{j=1}^n [(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) + (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4})] \\ & = [(u_{i1}, u_{i2}, u_{i3}, u_{i4}) + i(v_{i1}, v_{i2}, v_{i3}, v_{i4})]; i = 1, 2, \dots, n. \end{aligned} \quad (6.14)$$

Step 5: Equating real and imaginary parts of the system of fuzzy complex linear equations (6.14), it can be transformed into system of fuzzy linear equations (6.15).

$$\begin{aligned} & \sum_{j=1}^n [(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4})] = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), i = 1, 2, \dots, n; \\ & \sum_{j=1}^n [(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) + (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4})] = (v_{i1}, v_{i2}, v_{i3}, v_{i4}), i = 1, 2, \dots, n. \end{aligned} \quad (6.15)$$

Step 6: Using the property

$$\sum_{i=1}^n (a_{i1}, a_{i2}, a_{i3}, a_{i4}) = \left(\sum_{j=1}^n a_{j1}, \sum_{j=1}^n a_{j2}, \sum_{j=1}^n a_{j3}, \sum_{j=1}^n a_{j4} \right),$$

the system of fuzzy linear equations (6.15) can be transformed into system of fuzzy linear equations (6.16).

$$\begin{aligned}
& \left(\sum_{j=1}^n e_{ij1}, \sum_{j=1}^n e_{ij2}, \sum_{j=1}^n e_{ij3}, \sum_{j=1}^n e_{ij4} \right) - \left(\sum_{j=1}^n f_{ij1}, \sum_{j=1}^n f_{ij2}, \sum_{j=1}^n f_{ij3}, \sum_{j=1}^n f_{ij4} \right) = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), i = 1, 2, \dots, n; \\
& \left(\sum_{j=1}^n e'_{ij1}, \sum_{j=1}^n e'_{ij2}, \sum_{j=1}^n e'_{ij3}, \sum_{j=1}^n e'_{ij4} \right) + \left(\sum_{j=1}^n f'_{ij1}, \sum_{j=1}^n f'_{ij2}, \sum_{j=1}^n f'_{ij3}, \sum_{j=1}^n f'_{ij4} \right) = (v_{i1}, v_{i2}, v_{i3}, v_{i4}), i = 1, 2, \dots, n.
\end{aligned} \tag{6.16}$$

Step 7: Using the properties

$$\begin{aligned}
& \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right) + \left(\sum_{i=1}^n b_{i1}, \sum_{i=1}^n b_{i2}, \sum_{i=1}^n b_{i3}, \sum_{i=1}^n b_{i4} \right) \\
& = \left(\sum_{i=1}^n a_{i1} + \sum_{i=1}^n b_{i1}, \sum_{i=1}^n a_{i2} + \sum_{i=1}^n b_{i2}, \sum_{i=1}^n a_{i3} + \sum_{i=1}^n b_{i3}, \sum_{i=1}^n a_{i4} + \sum_{i=1}^n b_{i4} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right) - \left(\sum_{i=1}^n b_{i1}, \sum_{i=1}^n b_{i2}, \sum_{i=1}^n b_{i3}, \sum_{i=1}^n b_{i4} \right) \\
& = \left(\sum_{i=1}^n a_{i1} - \sum_{i=1}^n b_{i4}, \sum_{i=1}^n a_{i2} - \sum_{i=1}^n b_{i3}, \sum_{i=1}^n a_{i3} - \sum_{i=1}^n b_{i2}, \sum_{i=1}^n a_{i4} - \sum_{i=1}^n b_{i1} \right),
\end{aligned}$$

the system of fuzzy linear equations (6.16) can be transformed into system of fuzzy linear equations (6.17).

$$\begin{aligned}
& \left(\sum_{j=1}^n e_{ij1} - \sum_{j=1}^n f_{ij4}, \sum_{j=1}^n e_{ij2} - \sum_{j=1}^n f_{ij3}, \sum_{j=1}^n e_{ij3} - \sum_{j=1}^n f_{ij2}, \sum_{j=1}^n e_{ij4} - \sum_{j=1}^n f_{ij1} \right) = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), i = 1, 2, \dots, n; \\
& \left(\sum_{j=1}^n e'_{ij1} + \sum_{j=1}^n f'_{ij1}, \sum_{j=1}^n e'_{ij2} + \sum_{j=1}^n f'_{ij2}, \sum_{j=1}^n e'_{ij3} + \sum_{j=1}^n f'_{ij3}, \sum_{j=1}^n e'_{ij4} + \sum_{j=1}^n f'_{ij4} \right) = (v_{i1}, v_{i2}, v_{i3}, v_{i4}), i = 1, 2, \dots, n.
\end{aligned} \tag{6.17}$$

Step 8: Using the property

$$(a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4) \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4,$$

the system of fuzzy linear equations (6.17) can be transformed into system of crisp linear equations (6.18).

$$\begin{aligned}
\sum_{j=1}^n e_{ij1} - \sum_{j=1}^n f_{ij4} &= u_{i1}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e_{ij2} - \sum_{j=1}^n f_{ij3} &= u_{i2}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e_{ij3} - \sum_{j=1}^n f_{ij2} &= u_{i3}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e_{ij4} - \sum_{j=1}^n f_{ij1} &= u_{i4}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij1} + \sum_{j=1}^n f'_{ij1} &= v_{i1}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij2} + \sum_{j=1}^n f'_{ij2} &= v_{i2}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij3} + \sum_{j=1}^n f'_{ij3} &= v_{i3}, i = 1, 2, \dots, n; \\
\sum_{j=1}^n e'_{ij4} + \sum_{j=1}^n f'_{ij4} &= v_{i4}, i = 1, 2, \dots, n.
\end{aligned} \tag{6.18}$$

Step 9: Find the solution $\{x_{j1}, x_{j2}, x_{j3}, x_{j4}, y_{j1}, y_{j2}, y_{j3}, y_{j4}\}$ of system of crisp linear equations (6.18) with the restrictions $x_{j1} \leq x_{j2} \leq x_{j3} \leq x_{j4}$ and $y_{j1} \leq y_{j2} \leq y_{j3} \leq y_{j4}$.

Step 10: Using the solution $\{x_{j1}, x_{j2}, x_{j3}, x_{j4}, y_{j1}, y_{j2}, y_{j3}, y_{j4}\}$, obtained in Step 9, the fuzzy complex solution of system of fuzzy complex linear equations (6.10) is

$$\{\tilde{z}_j = (x_{j1}, x_{j2}, x_{j3}, x_{j4}) + i(y_{j1}, y_{j2}, y_{j3}, y_{j4})\}.$$

Remark 6.1. The solution, obtained by proposed Mehar method, will be a real complex number and the solution, obtained by proposed Keerat method, will always be a fuzzy complex number.

Remark 6.2. The restrictions $x_{j1} \leq x_{j2} \leq x_{j3} \leq x_{j4}$ and $y_{j1} \leq y_{j2} \leq y_{j3} \leq y_{j4}$ in Step 9 in Section 6.4.2, guarantee that the solutions are always fuzzy number solutions.

Remark 6.3. The system of fuzzy complex linear equations (6.1) will be consistent if the equivalent system of crisp linear equations (6.9) is consistent.

Remark 6.4. The system of fuzzy complex linear equations (6.10) will be consistent if the equivalent system of crisp linear equations (6.18) is consistent.

6.5.3. Mehar-Keerat method-V

In this section, a new method (named as Mehar-Keerat method-V) is proposed for solving the dual system of fuzzy complex linear equations (6.19).

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{z}_j + \tilde{b}_i = \sum_{j=1}^n \tilde{c}_{ij} \tilde{z}_j + \tilde{d}_i, i = 1, 2, \dots, m, \quad (6.19)$$

where, $\tilde{a}_{ij}, \tilde{z}_j, \tilde{b}_i, \tilde{c}_{ij}$ and $\tilde{d}_i$ are trapezoidal fuzzy complex numbers.

The steps of the proposed Mehar-Keerat method-V are as follows:

Step 1: Assuming

$$\begin{aligned} \tilde{a}_{ij} &= (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) + i(a_{ij5}, a_{ij6}, a_{ij7}, a_{ij8}), \\ \tilde{b}_i &= (b_{i1}, b_{i2}, b_{i3}, b_{i4}) + i(b_{i5}, b_{i6}, b_{i7}, b_{i8}), \\ \tilde{c}_{ij} &= (c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4}) + i(c_{ij5}, c_{ij6}, c_{ij7}, c_{ij8}), \\ \tilde{d}_i &= (d_{i1}, d_{i2}, d_{i3}, d_{i4}) + i(d_{i5}, d_{i6}, d_{i7}, d_{i8}), \\ \tilde{z}_j &= (x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) + i(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}), \end{aligned}$$

the dual system of fuzzy complex linear equations (6.19) can be transformed into the dual system of fuzzy complex linear equations (6.20).

$$\begin{aligned} &\sum_{j=1}^n [(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}) + i(a_{ij5}, a_{ij6}, a_{ij7}, a_{ij8})][(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) + i(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j})] \\ &+ [(b_{i1}, b_{i2}, b_{i3}, b_{i4}) + i(b_{i5}, b_{i6}, b_{i7}, b_{i8})] \\ &= \sum_{j=1}^n [(c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4}) + i(c_{ij5}, c_{ij6}, c_{ij7}, c_{ij8})][(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) + i(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j})] \\ &+ [(d_{i1}, d_{i2}, d_{i3}, d_{i4}) + i(d_{i5}, d_{i6}, d_{i7}, d_{i8})], i = 1, 2, \dots, m. \end{aligned} \quad (6.20)$$

Step 2: Using the existing product of fuzzy complex numbers [102, pp. 22],

$$\begin{aligned}
& [(a_{11}, a_{12}, a_{13}, a_{14}) + i(a_{15}, a_{16}, a_{17}, a_{18})][(b_{11}, b_{12}, b_{13}, b_{14}) + i(b_{15}, b_{16}, b_{17}, b_{18})] \\
& = [(a_{11}, a_{12}, a_{13}, a_{14})(b_{11}, b_{12}, b_{13}, b_{14}) - (a_{15}, a_{16}, a_{17}, a_{18})(b_{15}, b_{16}, b_{17}, b_{18})] \\
& + i[(a_{11}, a_{12}, a_{13}, a_{14})(b_{15}, b_{16}, b_{17}, b_{18}) + (a_{15}, a_{16}, a_{17}, a_{18})(b_{11}, b_{12}, b_{13}, b_{14})],
\end{aligned}$$

the dual system of fuzzy complex linear equations (6.20) can be transformed into the dual system of fuzzy complex linear equations (6.21).

$$\begin{aligned}
& \sum_{j=1}^n \{[(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) - (a_{ij5}, a_{ij6}, a_{ij7}, a_{ij8})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j})] \\
& + i[(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}) + (a_{ij5}, a_{ij6}, a_{ij7}, a_{ij8})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j})]\} \\
& + [(b_{i1}, b_{i2}, b_{i3}, b_{i4}) + i(b_{i5}, b_{i6}, b_{i7}, b_{i8})] \\
& = \sum_{j=1}^n \{[(c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) - (c_{ij5}, c_{ij6}, c_{ij7}, c_{ij8})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j})] \\
& + i[(c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}) + (c_{ij5}, c_{ij6}, c_{ij7}, c_{ij8})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j})]\} \\
& + [(d_{i1}, d_{i2}, d_{i3}, d_{i4}) + i(d_{i5}, d_{i6}, d_{i7}, d_{i8})], i = 1, 2, \dots, m.
\end{aligned} \tag{6.21}$$

Step 3: Assuming

$$\begin{aligned}
& (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) = (e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}), \\
& (a_{ij5}, a_{ij6}, a_{ij7}, a_{ij8})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}) = (e_{ij5}, e_{ij6}, e_{ij7}, e_{ij8}), \\
& (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}) = (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}), \\
& (a_{ij5}, a_{ij6}, a_{ij7}, a_{ij8})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) = (f_{ij5}, f_{ij6}, f_{ij7}, f_{ij8}), \\
& (c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) = (e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}), \\
& (c_{ij5}, c_{ij6}, c_{ij7}, c_{ij8})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}) = (e'_{ij5}, e'_{ij6}, e'_{ij7}, e'_{ij8}), \\
& (c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4})(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}) = (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}), \\
& (c_{ij5}, c_{ij6}, c_{ij7}, c_{ij8})(x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) = (f'_{ij5}, f'_{ij6}, f'_{ij7}, f'_{ij8}),
\end{aligned}$$

the dual system of fuzzy complex linear equations (6.21) can be transformed into the dual system of fuzzy complex equations (6.22).

$$\begin{aligned}
& \sum_{j=1}^n \{[(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (e_{ij5}, e_{ij6}, e_{ij7}, e_{ij8})] + i[(f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}) + (f_{ij5}, f_{ij6}, f_{ij7}, f_{ij8})]\} \\
& + [(b_{i1}, b_{i2}, b_{i3}, b_{i4}) + i(b_{i5}, b_{i6}, b_{i7}, b_{i8})] \\
& = \sum_{j=1}^n \{[(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) - (e'_{ij5}, e'_{ij6}, e'_{ij7}, e'_{ij8})] + i[(f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}) + (f'_{ij5}, f'_{ij6}, f'_{ij7}, f'_{ij8})]\} \\
& + [(d_{i1}, d_{i2}, d_{i3}, d_{i4}) + i(d_{i5}, d_{i6}, d_{i7}, d_{i8})], i = 1, 2, \dots, m.
\end{aligned} \tag{6.22}$$

where, $(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}), (e_{ij5}, e_{ij6}, e_{ij7}, e_{ij8}), (f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}), (f_{ij5}, f_{ij6}, f_{ij7}, f_{ij8}), (e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}),$
 $(e'_{ij5}, e'_{ij6}, e'_{ij7}, e'_{ij8}), (f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}), (f'_{ij5}, f'_{ij6}, f'_{ij7}, f'_{ij8}),$ can be obtained by the existing product
[94, Section 4, pp. 7145] of two trapezoidal fuzzy numbers $(a_1^L - \alpha_1, a_1^L, a_1^U, a_1^U + \alpha_1')$ and
 $(a_2^L - \alpha_2, a_2^L, a_2^U, a_2^U + \alpha_2')$, presented in Remark 4.2 of Chapter 4.

Step 4: Using the property

$$\sum_{i=1}^n [(a_{i1}, a_{i2}, a_{i3}, a_{i4}) + i(a_{i5}, a_{i6}, a_{i7}, a_{i8})] = \sum_{i=1}^n (a_{i1}, a_{i2}, a_{i3}, a_{i4}) + i \sum_{i=1}^n (a_{i5}, a_{i6}, a_{i7}, a_{i8}),$$

the dual system of fuzzy complex equations (6.22) can be transformed into the dual system of fuzzy complex equations (6.23).

$$\begin{aligned} & \sum_{j=1}^n [(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (e_{ij5}, e_{ij6}, e_{ij7}, e_{ij8})] + i \sum_{j=1}^n [(f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}) + (f_{ij5}, f_{ij6}, f_{ij7}, f_{ij8})] \\ & + [(b_{i1}, b_{i2}, b_{i3}, b_{i4}) + i(b_{i5}, b_{i6}, b_{i7}, b_{i8})] \\ & = \sum_{j=1}^n [(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) - (e'_{ij5}, e'_{ij6}, e'_{ij7}, e'_{ij8})] + i \sum_{j=1}^n [(f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}) + (f'_{ij5}, f'_{ij6}, f'_{ij7}, f'_{ij8})] \\ & + [(d_{i1}, d_{i2}, d_{i3}, d_{i4}) + i(d_{i5}, d_{i6}, d_{i7}, d_{i8})], i = 1, 2, \dots, m. \end{aligned} \quad (6.23)$$

Step 5: Equating the real and imaginary parts of the dual system of fuzzy complex equations (6.23), it can be transformed into the dual system of fuzzy equations (6.24).

$$\begin{aligned} & \sum_{j=1}^n [(e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}) - (e_{ij5}, e_{ij6}, e_{ij7}, e_{ij8})] + (b_{i1}, b_{i2}, b_{i3}, b_{i4}) \\ & = \sum_{j=1}^n [(e'_{ij1}, e'_{ij2}, e'_{ij3}, e'_{ij4}) - (e'_{ij5}, e'_{ij6}, e'_{ij7}, e'_{ij8})] + (d_{i1}, d_{i2}, d_{i3}, d_{i4}), i = 1, 2, \dots, m; \\ & \sum_{j=1}^n [(f_{ij1}, f_{ij2}, f_{ij3}, f_{ij4}) + (f_{ij5}, f_{ij6}, f_{ij7}, f_{ij8})] + (b_{i5}, b_{i6}, b_{i7}, b_{i8}) \\ & = \sum_{j=1}^n [(f'_{ij1}, f'_{ij2}, f'_{ij3}, f'_{ij4}) + (f'_{ij5}, f'_{ij6}, f'_{ij7}, f'_{ij8})] + (d_{i5}, d_{i6}, d_{i7}, d_{i8}), i = 1, 2, \dots, m. \end{aligned} \quad (6.24)$$

Step 6: Using the property

$$\sum_{i=1}^n (a_{i1}, a_{i2}, a_{i3}, a_{i4}) = \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right),$$

the dual system of fuzzy equations (6.24) can be transformed into the dual system of fuzzy equations (6.25).

$$\begin{aligned}
& \left(\sum_{j=1}^n e_{ij1}, \sum_{j=1}^n e_{ij2}, \sum_{j=1}^n e_{ij3}, \sum_{j=1}^n e_{ij4} \right) - \left(\sum_{j=1}^n e_{ij5}, \sum_{j=1}^n e_{ij6}, \sum_{j=1}^n e_{ij7}, \sum_{j=1}^n e_{ij8} \right) + (b_{i1}, b_{i2}, b_{i3}, b_{i4}) \\
&= \left(\sum_{j=1}^n e'_{ij1}, \sum_{j=1}^n e'_{ij2}, \sum_{j=1}^n e'_{ij3}, \sum_{j=1}^n e'_{ij4} \right) - \left(\sum_{j=1}^n e'_{ij5}, \sum_{j=1}^n e'_{ij6}, \sum_{j=1}^n e'_{ij7}, \sum_{j=1}^n e'_{ij8} \right) \\
&+ (d_{i1}, d_{i2}, d_{i3}, d_{i4}), i = 1, 2, \dots, m; \\
& \left(\sum_{j=1}^n f_{ij1}, \sum_{j=1}^n f_{ij2}, \sum_{j=1}^n f_{ij3}, \sum_{j=1}^n f_{ij4} \right) + \left(\sum_{j=1}^n f_{ij5}, \sum_{j=1}^n f_{ij6}, \sum_{j=1}^n f_{ij7}, \sum_{j=1}^n f_{ij8} \right) + (b_{i5}, b_{i6}, b_{i7}, b_{i8}) \\
&= \left(\sum_{j=1}^n f'_{ij1}, \sum_{j=1}^n f'_{ij2}, \sum_{j=1}^n f'_{ij3}, \sum_{j=1}^n f'_{ij4} \right) + \left(\sum_{j=1}^n f'_{ij5}, \sum_{j=1}^n f'_{ij6}, \sum_{j=1}^n f'_{ij7}, \sum_{j=1}^n f'_{ij8} \right) \\
&+ (d_{i5}, d_{i6}, d_{i7}, d_{i8}), i = 1, 2, \dots, m. \tag{6.25}
\end{aligned}$$

Step 7: Using the property

$$\begin{aligned}
& \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right) + \left(\sum_{i=1}^n b_{i1}, \sum_{i=1}^n b_{i2}, \sum_{i=1}^n b_{i3}, \sum_{i=1}^n b_{i4} \right) \\
&= \left(\sum_{i=1}^n a_{i1} + \sum_{i=1}^n b_{i1}, \sum_{i=1}^n a_{i2} + \sum_{i=1}^n b_{i2}, \sum_{i=1}^n a_{i3} + \sum_{i=1}^n b_{i3}, \sum_{i=1}^n a_{i4} + \sum_{i=1}^n b_{i4} \right) \text{ and} \\
& \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \sum_{i=1}^n a_{i3}, \sum_{i=1}^n a_{i4} \right) - \left(\sum_{i=1}^n b_{i1}, \sum_{i=1}^n b_{i2}, \sum_{i=1}^n b_{i3}, \sum_{i=1}^n b_{i4} \right) \\
&= \left(\sum_{i=1}^n a_{i1} - \sum_{i=1}^n b_{i4}, \sum_{i=1}^n a_{i2} - \sum_{i=1}^n b_{i3}, \sum_{i=1}^n a_{i3} - \sum_{i=1}^n b_{i2}, \sum_{i=1}^n a_{i4} - \sum_{i=1}^n b_{i1} \right),
\end{aligned}$$

the dual system of fuzzy equations (6.25) can be transformed into the dual system of fuzzy equations (6.26).

$$\begin{aligned}
& \left(\sum_{j=1}^n e_{ij1} - \sum_{j=1}^n e_{ij8} + b_{i1}, \sum_{j=1}^n e_{ij2} - \sum_{j=1}^n e_{ij7} + b_{i2}, \sum_{j=1}^n e_{ij3} - \sum_{j=1}^n e_{ij6} + b_{i3}, \sum_{j=1}^n e_{ij4} - \sum_{j=1}^n e_{ij5} + b_{i4} \right) = \\
& \left(\sum_{j=1}^n e'_{ij1} - \sum_{j=1}^n e'_{ij8} + d_{i1}, \sum_{j=1}^n e'_{ij2} - \sum_{j=1}^n e'_{ij7} + d_{i2}, \sum_{j=1}^n e'_{ij3} - \sum_{j=1}^n e'_{ij6} + d_{i3}, \sum_{j=1}^n e'_{ij4} - \sum_{j=1}^n e'_{ij5} + d_{i4} \right), \\
& i = 1, 2, \dots, m; \\
& \left(\sum_{j=1}^n f_{ij1} + \sum_{j=1}^n f_{ij5} + b_{i5}, \sum_{j=1}^n f_{ij2} + \sum_{j=1}^n f_{ij6} + b_{i6}, \sum_{j=1}^n f_{ij3} + \sum_{j=1}^n f_{ij7} + b_{i7}, \sum_{j=1}^n f_{ij4} + \sum_{j=1}^n f_{ij8} + b_{i8} \right) = \\
& \left(\sum_{j=1}^n f'_{ij1} + \sum_{j=1}^n f'_{ij5} + d_{i5}, \sum_{j=1}^n f'_{ij2} + \sum_{j=1}^n f'_{ij6} + d_{i6}, \sum_{j=1}^n f'_{ij3} + \sum_{j=1}^n f'_{ij7} + d_{i7}, \sum_{j=1}^n f'_{ij4} + \sum_{j=1}^n f'_{ij8} + d_{i8} \right), \\
& i = 1, 2, \dots, m.
\end{aligned} \tag{6.26}$$

Step 8: Using the property

$$(a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4) \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4,$$

the dual system of fuzzy equations (6.26) can be transformed into the dual system of crisp equations (6.27).

$$\begin{aligned}
& \sum_{j=1}^n e_{ij1} - \sum_{j=1}^n e_{ij8} + b_{i1} = \sum_{j=1}^n e'_{ij1} - \sum_{j=1}^n e'_{ij8} + d_{i1}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n e_{ij2} - \sum_{j=1}^n e_{ij7} + b_{i2} = \sum_{j=1}^n e'_{ij2} - \sum_{j=1}^n e'_{ij7} + d_{i2}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n e_{ij3} - \sum_{j=1}^n e_{ij6} + b_{i3} = \sum_{j=1}^n e'_{ij3} - \sum_{j=1}^n e'_{ij6} + d_{i3}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n e_{ij4} - \sum_{j=1}^n e_{ij5} + b_{i4} = \sum_{j=1}^n e'_{ij4} - \sum_{j=1}^n e'_{ij5} + d_{i4}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n f_{ij1} + \sum_{j=1}^n f_{ij5} + b_{i5} = \sum_{j=1}^n f'_{ij1} + \sum_{j=1}^n f'_{ij5} + d_{i5}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n f_{ij2} + \sum_{j=1}^n f_{ij6} + b_{i6} = \sum_{j=1}^n f'_{ij2} + \sum_{j=1}^n f'_{ij6} + d_{i6}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n f_{ij3} + \sum_{j=1}^n f_{ij7} + b_{i7} = \sum_{j=1}^n f'_{ij3} + \sum_{j=1}^n f'_{ij7} + d_{i7}, \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n f_{ij4} + \sum_{j=1}^n f_{ij8} + b_{i8} = \sum_{j=1}^n f'_{ij4} + \sum_{j=1}^n f'_{ij8} + d_{i8}, \quad i = 1, 2, \dots, m.
\end{aligned} \tag{6.27}$$

Step 9: Find the solution $\{x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}, y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}\}$ of the dual system of crisp equations (6.27) with the restrictions $x_{4j-3} \leq x_{4j-2} \leq x_{4j-1} \leq x_{4j}$ and $y_{4j-3} \leq y_{4j-2} \leq y_{4j-1} \leq y_{4j}$.

Step 10: Using the solution $\{x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}, y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}\}$ of the dual system of equations, obtained in Step 9, the fuzzy complex solution $\tilde{z}_j$ of the dual system of fuzzy complex linear equations (6.19) is

$$\{\tilde{z}_j = (x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}) + i(y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}), j = 1, 2, \dots, n\}.$$

6.6. Illustrative examples

In this section, to illustrate the proposed Mehar method, Keerat method and Mehar-Keerat method-V, system of fuzzy complex linear equations with fuzzy complex coefficients, system of fuzzy complex linear equations with fuzzy complex variables and the dual system of fuzzy complex linear equations are solved.

Example 6.6. Solve the following system of fuzzy complex linear equations by the Mehar method proposed in Section 6.5.1.

$$\begin{aligned} [(1, 2, 3, 4) + i(2, 3, 4, 5)]z_1 + [(0, 1, 2, 3) + i(6, 7, 8, 9)]z_2 &= [(-23, -15, -7, 1) + i(23, 31, 39, 47)], \\ [(-2, -1, 0, 1) + i(3, 4, 5, 6)]z_1 + [(0, 2, 4, 6) + i(-3, -2, -1, 0)]z_2 &= [(-20, -9, 2, 13) + i(-12, -3, 6, 15)]. \end{aligned} \quad (6.28)$$

where, z_1 and z_2 are crisp complex numbers.

Solution: Using the Mehar method, proposed in Section 6.5.1, the solution of system of fuzzy complex linear equations (6.28) can be obtained as follows:

Step 1: Assuming $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the system of fuzzy complex linear equations (6.28) can be transformed into system of fuzzy complex linear equations (6.29).

$$\begin{aligned} [(1, 2, 3, 4) + i(2, 3, 4, 5)](x_1 + iy_1) + [(0, 1, 2, 3) + i(6, 7, 8, 9)](x_2 + iy_2) &= [(-23, -15, -7, 1) + i(23, 31, 39, 47)], \\ [(-2, -1, 0, 1) + i(3, 4, 5, 6)](x_1 + iy_1) + [(0, 2, 4, 6) + i(-3, -2, -1, 0)](x_2 + iy_2) &= [(-20, -9, 2, 13) + i(-12, -3, 6, 15)]. \end{aligned} \quad (6.29)$$

Step 2: Using Step 2 of the Mehar method, proposed in Section 6.5.1, the system of fuzzy complex linear equations (6.29) can be transformed into system of fuzzy complex linear equations (6.30).

$$\begin{aligned} [(1, 2, 3, 4)x_1 - (2, 3, 4, 5)y_1 + (0, 1, 2, 3)x_2 - (6, 7, 8, 9)y_2] + i[(1, 2, 3, 4)y_1 + (2, 3, 4, 5)x_1 + (0, 1, 2, 3)y_2 + (6, 7, 8, 9)x_2] \\ = [(-23, -15, -7, 1) + i(23, 31, 39, 47)], \\ [(-2, -1, 0, 1)x_1 - (3, 4, 5, 6)y_1 + (0, 2, 4, 6)x_2 - (-3, -2, -1, 0)y_2] + i[(-2, -1, 0, 1)y_1 + (3, 4, 5, 6)x_1 + (0, 2, 4, 6)y_2 \\ + (-3, -2, -1, 0)x_2] = [(-20, -9, 2, 13) + i(-12, -3, 6, 15)]. \end{aligned} \quad (6.30)$$

Step 3: Using Step 3 of the Mehar method, proposed in Section 6.5.1, the system of fuzzy complex linear equations (6.30) can be transformed into system of fuzzy complex linear equations (6.31).

$$\begin{aligned} [(1x_1, 2x_1, 3x_1, 4x_1) + (-5y_1, -4y_1, -3y_1, -2y_1) + (0, 1x_2, 2x_2, 3x_2) + (-9y_2, -8y_2, -7y_2, -6y_2)] + i[(1y_1, 2y_1, 3y_1, 4y_1) \\ + (2x_1, 3x_1, 4x_1, 5x_1) + (0, 1y_2, 2y_2, 3y_2) + (6x_2, 7x_2, 8x_2, 9x_2)] = [(-23, -15, -7, 1) + i(23, 31, 39, 47)], \\ [(-2x_1, -1x_1, 0, 1x_1) + (-6y_1, -5y_1, -4y_1, -3y_1) + (0, 2x_2, 4x_2, 6x_2) + (0, 1y_2, 2y_2, 3y_2)] + i[(-2y_1, -1y_1, 0, 1y_1) \\ + (3x_1, 4x_1, 5x_1, 6x_1) + (0, 2y_2, 4y_2, 6y_2) + (-3x_2, -2x_2, -1x_2, 0)] = [(-20, -9, 2, 13) + i(-12, -3, 6, 15)]. \end{aligned} \quad (6.31)$$

Step 4: Using Step 4 to Step 8 of the Mehar method, proposed in Section 6.5.1, the system of fuzzy complex linear equations (6.31) can be transformed into system of crisp linear equations (6.32).

$$\begin{aligned} x_1 - 5y_1 - 9y_2 &= -23, \\ 2x_1 + x_2 - 4y_1 - 8y_2 &= -15, \\ 3x_1 + 2x_2 - 3y_1 - 7y_2 &= -7, \\ 4x_1 + 3x_2 - 2y_1 - 6y_2 &= 1, \\ -2x_1 - 6y_1 &= -20, \\ -x_1 + 2x_2 - 5y_1 + y_2 &= -9, \\ 4x_2 - 4y_1 + 2y_2 &= 2, \\ x_1 + 6x_2 - 3y_1 + 3y_2 &= 13, \\ 2x_1 + 6x_2 + y_1 &= 23, \\ 3x_1 + 7x_2 + 2y_1 + y_2 &= 31, \end{aligned} \quad (6.32)$$

$$\begin{aligned}
4x_1 + 8x_2 + 3y_1 + 2y_2 &= 39, \\
5x_1 + 9x_2 + 4y_1 + 3y_2 &= 47, \\
3x_1 - 3x_2 - 2y_1 &= -12, \\
4x_1 - 2x_2 - y_1 + 2y_2 &= -3, \\
5x_1 - x_2 + 4y_2 &= 6, \\
6x_1 + y_1 + 6y_2 &= 15.
\end{aligned}$$

Step 5: The solution of the system of crisp linear equations (6.32) is $x_1 = 1, x_2 = 3, y_1 = 3$ and $y_2 = 1$.

Step 6: Using Step 10 of the Mehar method, proposed in Section 6.5.1, the solution of system of fuzzy complex linear equations (6.28) is

$$\begin{aligned}
z_1 &= x_1 + iy_1 = 1 + 3i, \\
z_2 &= x_2 + iy_2 = 3 + i.
\end{aligned}$$

Example 6.7. Solve the following system of fuzzy complex linear equations by the Keerat method proposed in Section 6.5.2.

$$\begin{aligned}
(10 - 7.5i)\tilde{z}_1 - (6 - 5i)\tilde{z}_2 &= [(4, 5, 5, 6) + i(-1, 0, 0, 1)], \\
-(6 - 5i)\tilde{z}_1 + (16 + 3i)\tilde{z}_2 &= (-2, -1, -1, 0) + i(-3, -2, -2, -1).
\end{aligned} \tag{6.33}$$

where, $\tilde{z}_1$ and $\tilde{z}_2$ are trapezoidal fuzzy complex numbers.

Solution: Using the Keerat method, proposed in Section 6.5.2, the solution of system of fuzzy complex linear equations (6.33) can be obtained as follows:

Step 1: Assuming

$$\begin{aligned}
\tilde{z}_1 &= (x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4), \\
\tilde{z}_2 &= (x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8),
\end{aligned}$$

the system of fuzzy complex linear equations (6.33) can be transformed into system of fuzzy complex linear equations (6.34).

$$\begin{aligned}
[(10 - 7.5i)(x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4)] - (6 - 5i)[(x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8)] &= [(4, 5, 5, 6) + i(-1, 0, 0, 1)], \\
-(6 - 5i)[(x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4)] + (16 + 3i)[(x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8)] &= [(-2, -1, -1, 0) + i(-3, -2, -2, -1)].
\end{aligned} \tag{6.34}$$

Step 2: Using Step 2 of the Keerat method, proposed in Section 6.5.2, the system of fuzzy complex linear equations (6.34) can be transformed into system of fuzzy complex linear equations (6.35).

$$\begin{aligned}
& [10(x_1, x_2, x_3, x_4) + 7.5(y_1, y_2, y_3, y_4) - 6(x_5, x_6, x_7, x_8) - 5(y_5, y_6, y_7, y_8)] + i[10(y_1, y_2, y_3, y_4) - 7.5(x_1, x_2, x_3, x_4) \\
& - 6(y_5, y_6, y_7, y_8) + 5(x_5, x_6, x_7, x_8)] = [(4, 5, 5, 6) + i(-1, 0, 0, 1)], \\
& [-6(x_1, x_2, x_3, x_4) - 5(y_1, y_2, y_3, y_4) + 16(x_5, x_6, x_7, x_8) - 3(y_5, y_6, y_7, y_8)] + i[-6(y_1, y_2, y_3, y_4) + 5(x_1, x_2, x_3, x_4) \\
& + 16(y_5, y_6, y_7, y_8) + 3(x_5, x_6, x_7, x_8)] = [(-2, -1, -1, 0) + i(-3, -2, -2, -1)]. \quad (6.35)
\end{aligned}$$

Step 3: Using Step 3 of the Keerat method, proposed in Section 6.5.2, the system of fuzzy complex linear equations (6.35) can be transformed into system of fuzzy complex linear equations (6.36).

$$\begin{aligned}
& [(10x_1, 10x_2, 10x_3, 10x_4) + (7.5y_1, 7.5y_2, 7.5y_3, 7.5y_4) + (-6x_8, -6x_7, -6x_6, -6x_5) + (-5y_8, -5y_7, -5y_6, -5y_5)] \\
& + i[(10y_1, 10y_2, 10y_3, 10y_4) + (-7.5x_4, -7.5x_3, -7.5x_2, -7.5x_1) + (-6y_8, -6y_7, -6y_6, -6y_5) + (5x_5, 5x_6, 5x_7, 5x_8)] \\
& = [(4, 5, 5, 6) + i(-1, 0, 0, 1)], \\
& [(-6x_4, -6x_3, -6x_2, -6x_1) + (-5y_4, -5y_3, -5y_2, -5y_1) + (16x_5, 16x_6, 16x_7, 16x_8) + (-3y_8, -3y_7, -3y_6, -3y_5)] \\
& + i[(-6y_4, -6y_3, -6y_2, -6y_1) + (5x_1, 5x_2, 5x_3, 5x_4) + (16y_5, 16y_6, 16y_7, 16y_8) + (3x_5, 3x_6, 3x_7, 3x_8)] \\
& = [(-2, -1, -1, 0) + i(-3, -2, -2, -1)]. \quad (6.36)
\end{aligned}$$

Step 4: Using Step 4 to Step 8 of the Keerat method, proposed in Section 6.5.2, the system of fuzzy complex linear equations (6.36) can be transformed into system of crisp linear equations (6.37).

$$\begin{aligned}
& 10x_1 - 6x_8 + 7.5y_1 - 5y_8 = 4, \\
& 10x_2 - 6x_7 + 7.5y_2 - 5y_7 = 5, \\
& 10x_3 - 6x_6 + 7.5y_3 - 5y_6 = 5, \\
& 10x_4 - 6x_5 + 7.5y_4 - 5y_5 = 6, \\
& -6x_4 + 16x_5 - 5y_4 - 3y_8 = -2, \\
& -6x_3 + 16x_6 - 5y_3 - 3y_7 = -1, \\
& -6x_2 + 16x_7 - 5y_2 - 3y_6 = -1, \\
& -6x_1 + 16x_8 - 5y_1 - 3y_5 = 0, \\
& -7.5x_4 + 5x_5 + 10y_1 - 6y_8 = -1, \\
& -7.5x_3 + 5x_6 + 10y_2 - 6y_7 = 0, \\
& -7.5x_2 + 5x_7 + 10y_3 - 6y_6 = 0,
\end{aligned} \quad (6.37)$$

$$\begin{aligned}
-7.5x_1 + 5x_8 + 10y_4 - 6y_5 &= 1, \\
5x_1 + 3x_5 - 6y_4 + 16y_5 &= -3, \\
5x_2 + 3x_6 - 6y_3 + 16y_6 &= -2, \\
5x_3 + 3x_7 - 6y_2 + 16y_7 &= -2, \\
5x_4 + 3x_8 - 6y_1 + 16y_8 &= -1.
\end{aligned}$$

Step 5: The solution $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$ of the system of crisp linear equations (6.37) with the restrictions $x_1 \leq x_2 \leq x_3 \leq x_4, x_5 \leq x_6 \leq x_7 \leq x_8$ and

$y_1 \leq y_2 \leq y_3 \leq y_4, y_5 \leq y_6 \leq y_7 \leq y_8$, is

$$\begin{aligned}
x_1 &= \frac{16225328}{51288327}, x_2 = \frac{42944}{121249}, x_3 = \frac{42944}{121249}, x_4 = \frac{20105296}{51288327}, \\
x_5 &= \frac{1778153}{51288327}, x_6 = \frac{7930}{121249}, x_7 = \frac{7930}{121249}, x_8 = \frac{4930627}{51288327}, \\
y_1 &= \frac{3630080}{51288327}, y_2 = \frac{13168}{121249}, y_3 = \frac{13168}{121249}, y_4 = \frac{7510048}{51288327}, \\
y_5 &= -\frac{12204112}{51288327}, y_6 = -\frac{25125}{121249}, y_7 = -\frac{25125}{121249}, y_8 = -\frac{9051638}{51288327}.
\end{aligned}$$

Step 6: Using Step 10 of the Keerat method, proposed in Section 6.5.2, the solution of system of fuzzy complex system of linear equations (6.33) is

$$\begin{aligned}
\tilde{z}_1 &= (x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4) \\
&= \left(\frac{16225328}{51288327}, \frac{42944}{121249}, \frac{42944}{121249}, \frac{20105296}{51288327} \right) + i \left(\frac{3630080}{51288327}, \frac{13168}{121249}, \frac{13168}{121249}, \frac{7510048}{51288327} \right), \\
\tilde{z}_2 &= (x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8) \\
&= \left(\frac{1778153}{51288327}, \frac{7930}{121249}, \frac{7930}{121249}, \frac{4930627}{51288327} \right) + i \left(-\frac{12204112}{51288327}, -\frac{25125}{121249}, -\frac{25125}{121249}, -\frac{9051638}{51288327} \right).
\end{aligned}$$

Example 6.8. Solve the dual system of fuzzy complex linear equations, chosen in Example 6.3, by the Mehar-Keerat method-V proposed in Section 6.5.3.

Solution: Using the Mehar-Keerat method-V, proposed in Section 6.5.3, the solution of the dual system of fuzzy complex linear equations, chosen in Example 6.3, can be obtained as

follows:

Step 1: Assuming

$$\tilde{z}_1 = (x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4) \text{ and } \tilde{z}_2 = (x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8),$$

the dual system of fuzzy complex linear equations, chosen in Example 6.3, can be transformed into the dual system of fuzzy complex linear equations (6.38).

$$\begin{aligned} & [(0, 1, 2, 3) + i(-3, -1, 1, 3)][(x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4)] + [(2, 4, 6, 8) + i(1, 2, 3, 4)][(x_5, x_6, x_7, \\ & x_8) + i(y_5, y_6, y_7, y_8)] + [(0, 0, 0, 0) + i(74, 74, 74, 82)] = [(1, 2, 3, 4) + i(-1, 0, 1, 2)][(x_1, x_2, x_3, x_4) + \\ & i(y_1, y_2, y_3, y_4)] - [(2, 4, 6, 8) + i(-4, -3, -2, -1)][(x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8)] + [(9, 10, 15, 28) \\ & + i(0, 14, 16, 16)], \\ & [(-4, -3, -2, -1) + i(1, 3, 5, 6)][(x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4)] + [(1, 2, 3, 5) + i(-5, -3, -2, \\ & -1)][(x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8)] + [(21, 41, 57, 83) + i(32, 32, 37, 62)] = [(4, 5, 6, 7) + \\ & i(-2, 0, 2, 4)][(x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4)] + [(1, 3, 5, 7) + i(-5, -2, 1, 4)][(x_5, x_6, x_7, x_8) \\ & + i(y_5, y_6, y_7, y_8)] + [(0, 0, 0, 0) + i(0, 4, 4, 4)]. \end{aligned} \quad (6.38)$$

Step 2: Using Step 2 of the proposed Mehar-Keerat method-V, the dual system of fuzzy complex linear equations (6.38) can be transformed into the dual system of fuzzy complex linear equations (6.39).

$$\begin{aligned} & [(0, 1, 2, 3)(x_1, x_2, x_3, x_4) - (-3, -1, 1, 3)(y_1, y_2, y_3, y_4) + (2, 4, 6, 8)(x_5, x_6, x_7, x_8) - (1, 2, 3, 4) \\ & (y_5, y_6, y_7, y_8)] + i[(0, 1, 2, 3)(y_1, y_2, y_3, y_4) + (-3, -1, 1, 3)(x_1, x_2, x_3, x_4) + (2, 4, 6, 8)(y_5, y_6, \\ & y_7, y_8) + (1, 2, 3, 4)(x_5, x_6, x_7, x_8)] + [(0, 0, 0, 0) + i(74, 74, 74, 82)] = [(1, 2, 3, 4)(x_1, x_2, x_3, x_4) \\ & - (-1, 0, 1, 2)(y_1, y_2, y_3, y_4) - (2, 4, 6, 8)(x_5, x_6, x_7, x_8) + (-4, -3, -2, -1)(y_5, y_6, y_7, y_8)] + \\ & i[(1, 2, 3, 4)(y_1, y_2, y_3, y_4) + (-1, 0, 1, 2)(x_1, x_2, x_3, x_4) - (2, 4, 6, 8)(y_5, y_6, y_7, y_8) - (-4, -3, \\ & -2, -1)(x_5, x_6, x_7, x_8)] + [(9, 10, 15, 28) + i(0, 14, 16, 16)], \\ & [(-4, -3, -2, -1)(x_1, x_2, x_3, x_4) - (1, 3, 5, 6)(y_1, y_2, y_3, y_4) + (1, 2, 3, 5)(x_5, x_6, x_7, x_8) - (-5, -3, -2, \\ & -1)(y_5, y_6, y_7, y_8)] + i[(-4, -3, -2, -1)(y_1, y_2, y_3, y_4) + (1, 3, 5, 6)(x_1, x_2, x_3, x_4) + (1, 2, 3, 5)(y_5, y_6, \\ & y_7, y_8) + (-5, -3, -2, -1)(x_5, x_6, x_7, x_8)] + [(21, 41, 57, 83) + i(32, 32, 37, 62)] = [(4, 5, 6, 7)(x_1, x_2, \\ & x_3, x_4) - (-2, 0, 2, 4)(y_1, y_2, y_3, y_4) + (1, 3, 5, 7)(x_5, x_6, x_7, x_8) - (-5, -2, 1, 4)(y_5, y_6, y_7, y_8)] + \\ & i[(4, 5, 6, 7)(y_1, y_2, y_3, y_4) + (-2, 0, 2, 4)(x_1, x_2, x_3, x_4) + (1, 3, 5, 7)(y_5, y_6, y_7, y_8) + (-5, -2, 1, 4) \\ & (x_5, x_6, x_7, x_8)] + [(0, 0, 0, 0) + i(0, 4, 4, 4)]. \end{aligned} \quad (6.39)$$

Step 3: Using Step 3 of proposed Mehar-Keerat method-V, the dual system of fuzzy complex linear equations (6.39) can be transformed into the dual system of fuzzy complex equations (6.40).

$$\begin{aligned}
& \left\{ \begin{aligned} & \left(\frac{3}{2}x_1 - \left| \frac{3}{2}x_1 \right|, \frac{3}{2}x_2 - \left| \frac{1}{2}x_2 \right|, \frac{3}{2}x_3 + \left| -\frac{1}{2}x_3 \right|, \frac{3}{2}x_4 + \left| -\frac{3}{2}x_4 \right| \right) \\ & + \left(\frac{-3y_4 + 3y_1}{2} - \left| \frac{3y_1 + 3y_4}{2} \right|, \frac{-y_3 + y_2}{2} - \left| \frac{y_2 + y_3}{2} \right|, \frac{y_3 - y_2}{2} + \left| \frac{-y_2 - y_3}{2} \right|, \frac{3y_4 - 3y_1}{2} + \left| \frac{-3y_1 - 3y_4}{2} \right| \right) \\ & + (5x_5 - |3x_5|, 5x_6 - |x_6|, 5x_7 + |-x_7|, 5x_8 + |-3x_8|) \\ & + \left(-\frac{5}{2}y_8 - \left| \frac{3}{2}y_8 \right|, -\frac{5}{2}y_7 - \left| \frac{1}{2}y_7 \right|, -\frac{5}{2}y_6 + \left| -\frac{1}{2}y_6 \right|, -\frac{5}{2}y_5 + \left| -\frac{3}{2}y_5 \right| \right) + (0, 0, 0, 0) \end{aligned} \right\} \\
& + i \left\{ \begin{aligned} & \left(\frac{3}{2}y_1 - \left| \frac{3}{2}y_1 \right|, \frac{3}{2}y_2 - \left| \frac{1}{2}y_2 \right|, \frac{3}{2}y_3 + \left| -\frac{1}{2}y_3 \right|, \frac{3}{2}y_4 + \left| -\frac{3}{2}y_4 \right| \right) \\ & + \left(\frac{-3x_4 + 3x_1}{2} - \left| \frac{3x_1 + 3x_4}{2} \right|, \frac{-x_3 + x_2}{2} - \left| \frac{x_2 + x_3}{2} \right|, \frac{x_3 - x_2}{2} + \left| \frac{-x_2 - x_3}{2} \right|, \frac{3x_4 - 3x_1}{2} + \left| \frac{-3x_1 - 3x_4}{2} \right| \right) \\ & + (5y_5 - |3y_5|, 5y_6 - |y_6|, 5y_7 + |-y_7|, 5y_8 + |-3y_8|) \\ & + \left(\frac{5}{2}x_5 - \left| \frac{3}{2}x_5 \right|, \frac{5}{2}x_6 - \left| \frac{1}{2}x_6 \right|, \frac{5}{2}x_7 + \left| -\frac{1}{2}x_7 \right|, \frac{5}{2}x_8 + \left| -\frac{3}{2}x_8 \right| \right) + (74, 74, 74, 82) \end{aligned} \right\} \\
& = \left\{ \begin{aligned} & \left(\frac{5}{2}x_1 - \left| \frac{3}{2}x_1 \right|, \frac{5}{2}x_2 - \left| \frac{1}{2}x_2 \right|, \frac{5}{2}x_3 + \left| -\frac{1}{2}x_3 \right|, \frac{5}{2}x_4 + \left| -\frac{3}{2}x_4 \right| \right) \\ & + \left(\frac{-2y_4 + y_1}{2} - \left| \frac{y_1 + 2y_4}{2} \right|, \frac{-y_3}{2} - \left| \frac{y_3}{2} \right|, \frac{-y_2}{2} + \left| \frac{-y_2}{2} \right|, \frac{y_4 - 2y_1}{2} + \left| \frac{-2y_1 - y_4}{2} \right| \right) \\ & + (-5x_8 - |3x_8|, -5x_7 - |x_7|, -5x_6 + |-x_6|, -5x_5 + |-3x_5|) \\ & + \left(-\frac{5}{2}y_8 - \left| \frac{3}{2}y_8 \right|, -\frac{5}{2}y_7 - \left| \frac{1}{2}y_7 \right|, -\frac{5}{2}y_6 + \left| -\frac{1}{2}y_6 \right|, -\frac{5}{2}y_5 + \left| -\frac{3}{2}y_5 \right| \right) + (9, 10, 15, 28) \end{aligned} \right\} \\
& + i \left\{ \begin{aligned} & \left(\frac{5}{2}y_1 - \left| \frac{3}{2}y_1 \right|, \frac{5}{2}y_2 - \left| \frac{1}{2}y_2 \right|, \frac{5}{2}y_3 + \left| -\frac{1}{2}y_3 \right|, \frac{5}{2}y_4 + \left| -\frac{3}{2}y_4 \right| \right) \\ & + \left(\frac{-x_4 + 2x_1}{2} - \left| \frac{2x_1 + x_4}{2} \right|, \frac{x_2}{2} - \left| \frac{x_2}{2} \right|, \frac{x_3}{2} + \left| -\frac{x_3}{2} \right|, \frac{2x_4 - x_1}{2} + \left| \frac{-x_1 - 2x_4}{2} \right| \right) \\ & + (-5y_8 - |3y_8|, -5y_7 - |y_7|, -5y_6 + |-y_6|, -5y_5 + |-3y_5|) \\ & + \left(\frac{5}{2}x_5 - \left| \frac{3}{2}x_5 \right|, \frac{5}{2}x_6 - \left| \frac{1}{2}x_6 \right|, \frac{5}{2}x_7 + \left| -\frac{1}{2}x_7 \right|, \frac{5}{2}x_8 + \left| -\frac{3}{2}x_8 \right| \right) + (0, 14, 16, 16) \end{aligned} \right\}, \tag{6.40}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{aligned} & \left(-\frac{5}{2}x_4 - \left| \frac{3}{2}x_4 \right|, -\frac{5}{2}x_3 - \left| \frac{1}{2}x_3 \right|, -\frac{5}{2}x_2 + \left| -\frac{1}{2}x_2 \right|, -\frac{5}{2}x_1 + \left| -\frac{3}{2}x_1 \right| \right) \\ & + \left(-\frac{7}{2}y_4 - \left| \frac{5}{2}y_4 \right|, -4y_3 - |y_3|, -4y_2 + |-y_2|, -\frac{7}{2}y_1 + \left| -\frac{5}{2}y_1 \right| \right) \\ & + \left(3x_5 - |2x_5|, \frac{5}{2}x_6 - \left| \frac{1}{2}x_6 \right|, \frac{5}{2}x_7 + \left| -\frac{1}{2}x_7 \right|, 3x_8 + |-2x_8| \right) \\ & + \left(3y_5 - |2y_5|, \frac{5}{2}y_6 - \left| \frac{1}{2}y_6 \right|, \frac{5}{2}y_7 + \left| -\frac{1}{2}y_7 \right|, 3y_8 + |-2y_8| \right) + (21, 41, 57, 83) \end{aligned} \right\} \\
& + i \left\{ \begin{aligned} & \left(-\frac{5}{2}y_4 - \left| \frac{3}{2}y_4 \right|, -\frac{5}{2}y_3 - \left| \frac{1}{2}y_3 \right|, -\frac{5}{2}y_2 + \left| -\frac{1}{2}y_2 \right|, -\frac{5}{2}y_1 + \left| -\frac{3}{2}y_1 \right| \right) \\ & + \left(\frac{7}{2}x_1 - \left| \frac{5}{2}x_1 \right|, 4x_2 - |x_2|, 4x_3 + |-x_3|, \frac{7}{2}x_4 + \left| -\frac{5}{2}x_4 \right| \right) \\ & + \left(3y_5 - |2y_5|, \frac{5}{2}y_6 - \left| \frac{1}{2}y_6 \right|, \frac{5}{2}y_7 + \left| -\frac{1}{2}y_7 \right|, 3y_8 + |-2y_8| \right) \\ & + \left(-3x_8 - |2x_8|, -\frac{5}{2}x_7 - \left| \frac{1}{2}x_7 \right|, -\frac{5}{2}x_6 + \left| -\frac{1}{2}x_6 \right|, -3x_5 + |-2x_5| \right) + (32, 32, 37, 62) \end{aligned} \right\}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{aligned} & \left(\frac{11}{2}x_1 - \left| \frac{3}{2}x_1 \right|, \frac{11}{2}x_2 - \left| \frac{1}{2}x_2 \right|, \frac{11}{2}x_3 + \left| -\frac{1}{2}x_3 \right|, \frac{11}{2}x_4 + \left| -\frac{3}{2}x_4 \right| \right) \\ & + \left(\frac{-4y_4 + 2y_1}{2} - \left| \frac{2y_1 + 4y_4}{2} \right|, -y_3 - |y_3|, -y_2 + |-y_2|, \frac{2y_4 - 4y_1}{2} + \left| \frac{-4y_1 - 2y_4}{2} \right| \right) \\ & + (4x_5 - |3x_5|, 4x_6 - |x_6|, 4x_7 + |-x_7|, 4x_8 + |-3x_8|) \\ & + \left(\frac{-4y_8 + 5y_5}{2} - \left| \frac{5y_5 + 4y_8}{2} \right|, \frac{-y_7 + 2y_6}{2} - \left| \frac{2y_6 + y_7}{2} \right|, \frac{2y_7 - y_6}{2} + \left| \frac{-y_6 - 2y_7}{2} \right|, \frac{5y_8 - 4y_5}{2} + \left| \frac{-4y_5 - 5y_8}{2} \right| \right) \\ & + (0, 0, 0, 0) \end{aligned} \right\} \\
& + i \left\{ \begin{aligned} & \left(\frac{11}{2}y_1 - \left| \frac{3}{2}y_1 \right|, \frac{11}{2}y_2 - \left| \frac{1}{2}y_2 \right|, \frac{11}{2}y_3 + \left| -\frac{1}{2}y_3 \right|, \frac{11}{2}y_4 + \left| -\frac{3}{2}y_4 \right| \right) \\ & + \left(\frac{-2x_4 + 4x_1}{2} - \left| \frac{4x_1 + 2x_4}{2} \right|, x_2 - |x_2|, x_3 + |-x_3|, \frac{4x_4 - 2x_1}{2} + \left| \frac{-2x_1 - 4x_4}{2} \right| \right) \\ & + (4y_5 - |3y_5|, 4y_6 - |y_6|, 4y_7 + |-y_7|, 4y_8 + |-3y_8|) \\ & + \left(\frac{-5x_8 + 4x_5}{2} - \left| \frac{4x_5 + 5x_8}{2} \right|, \frac{-2x_7 + x_6}{2} - \left| \frac{x_6 + 2x_7}{2} \right|, \frac{x_7 - 2x_6}{2} + \left| \frac{-2x_6 - x_7}{2} \right|, \frac{4x_8 - 5x_5}{2} + \left| \frac{-5x_5 - 4x_8}{2} \right| \right) \\ & + (0, 4, 4, 4) \end{aligned} \right\}.
\end{aligned}$$

Step 4: Using Step 4 to Step 8 of proposed Mehar-Keerat method-V, the dual system of fuzzy complex equations (6.40) can be transformed into the dual system of crisp equations (6.41).

$$\frac{3}{2}x_1 - \left| \frac{3}{2}x_1 \right| - \frac{3}{2}y_4 + \frac{3}{2}y_1 - \left| \frac{3y_1 + 3y_4}{2} \right| + 5x_5 - |3x_5| - \frac{5}{2}y_8 - \left| \frac{3}{2}y_8 \right| = \frac{5}{2}x_1 - \left| \frac{3}{2}x_1 \right| - y_4 + \frac{1}{2}y_1 - \left| \frac{y_1 + 2y_4}{2} \right| - 5x_8 - |3x_8| - \frac{5}{2}y_8 - \left| \frac{3}{2}y_8 \right| + 9,$$

$$\frac{3}{2}x_2 - \left| \frac{1}{2}x_2 \right| - \frac{1}{2}y_3 + \frac{1}{2}y_2 - \left| \frac{y_2 + y_3}{2} \right| + 5x_6 - |x_6| - \frac{5}{2}y_7 - \left| \frac{1}{2}y_7 \right| = \frac{5}{2}x_2 - \left| \frac{1}{2}x_2 \right| - \frac{1}{2}y_3 - \left| \frac{y_3}{2} \right| - 5x_7 - |x_7| - \frac{5}{2}y_7 - \left| \frac{1}{2}y_7 \right| + 10,$$

$$\frac{3}{2}x_3 + \left| -\frac{1}{2}x_3 \right| + \frac{1}{2}y_3 - \frac{1}{2}y_2 + \left| \frac{-y_2 - y_3}{2} \right| + 5x_7 + |-x_7| - \frac{5}{2}y_6 + \left| -\frac{1}{2}y_6 \right| = \frac{5}{2}x_3 + \left| -\frac{1}{2}x_3 \right| - \frac{1}{2}y_2 + \left| \frac{-y_2}{2} \right| - 5x_6 + |-x_6| - \frac{5}{2}y_6 + \left| -\frac{1}{2}y_6 \right| + 15,$$

$$\frac{3}{2}x_4 + \left| -\frac{3}{2}x_4 \right| + \frac{3}{2}y_4 - \frac{3}{2}y_1 + \left| \frac{-3y_1 - 3y_4}{2} \right| + 5x_8 + |-3x_8| - \frac{5}{2}y_5 + \left| -\frac{3}{2}y_5 \right| = \frac{5}{2}x_4 + \left| -\frac{3}{2}x_4 \right| + \frac{1}{2}y_4 - y_1 + \left| \frac{-2y_1 - y_4}{2} \right| - 5x_5 + |-3x_5| - \frac{5}{2}y_5 + \left| -\frac{3}{2}y_5 \right| + 28,$$

$$\frac{3}{2}y_1 - \left| \frac{3}{2}y_1 \right| - \frac{3}{2}x_4 + \frac{3}{2}x_1 - \left| \frac{3x_1 + 3x_4}{2} \right| + 5y_5 - |3y_5| + \frac{5}{2}x_5 - \left| \frac{3}{2}x_5 \right| + 74 = \frac{5}{2}y_1 - \left| \frac{3}{2}y_1 \right| - \frac{1}{2}x_4 + x_1 - \left| \frac{2x_1 + x_4}{2} \right| - 5y_8 - |3y_8| + \frac{5}{2}x_5 - \left| \frac{3}{2}x_5 \right|,$$

$$\frac{3}{2}y_2 - \left| \frac{1}{2}y_2 \right| - \frac{1}{2}x_3 + \frac{1}{2}x_2 - \left| \frac{x_2 + x_3}{2} \right| + 5y_6 - |y_6| + \frac{5}{2}x_6 - \left| \frac{1}{2}x_6 \right| + 74 = \frac{5}{2}y_2 - \left| \frac{1}{2}y_2 \right| + \frac{1}{2}x_2 - \left| \frac{x_2}{2} \right| - 5y_7 - |y_7| + \frac{5}{2}x_6 - \left| \frac{1}{2}x_6 \right| + 14,$$

$$\frac{3}{2}y_3 + \left| -\frac{1}{2}y_3 \right| + \frac{1}{2}x_3 - \frac{1}{2}x_2 + \left| \frac{-x_2 - x_3}{2} \right| + 5y_7 + |-y_7| + \frac{5}{2}x_7 + \left| -\frac{1}{2}x_7 \right| + 74 = \frac{5}{2}y_3 + \left| -\frac{1}{2}y_3 \right| + \frac{1}{2}x_3 + \left| -\frac{x_3}{2} \right| - 5y_6 + |-y_6| + \frac{5}{2}x_7 + \left| -\frac{1}{2}x_7 \right| + 16,$$

$$\begin{aligned}
& \frac{3}{2}y_4 + \left| -\frac{3}{2}y_4 \right| + \frac{3}{2}x_4 - \frac{3}{2}x_1 + \left| \frac{-3x_1 - 3x_4}{2} \right| + 5y_8 + |-3y_8| + \frac{5}{2}x_8 + \left| -\frac{3}{2}x_8 \right| + 82 = \frac{5}{2}y_4 + \left| -\frac{3}{2}y_4 \right| \\
& + x_4 - \frac{1}{2}x_1 + \left| \frac{-x_1 - 2x_4}{2} \right| - 5y_5 + |-3y_5| + \frac{5}{2}x_8 + \left| -\frac{3}{2}x_8 \right| + 16, \\
& -\frac{5}{2}x_4 - \left| \frac{3}{2}x_4 \right| - \frac{7}{2}y_4 - \left| \frac{5}{2}y_4 \right| + 3x_5 - |2x_5| + 3y_5 - |2y_5| + 21 = \frac{11}{2}x_1 - \left| \frac{3}{2}x_1 \right| - 2y_4 + y_1 - \left| \frac{2y_1 + 4y_4}{2} \right| \\
& + 4x_5 - |3x_5| - 2y_8 + \frac{5}{2}y_5 - \left| \frac{5y_5 + 4y_8}{2} \right|, \tag{6.41}
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{2}x_3 - \left| \frac{1}{2}x_3 \right| - 4y_3 - |y_3| + \frac{5}{2}x_6 - \left| \frac{1}{2}x_6 \right| + \frac{5}{2}y_6 - \left| \frac{1}{2}y_6 \right| + 41 = \frac{11}{2}x_2 - \left| \frac{1}{2}x_2 \right| - y_3 - |y_3| + 4x_6 - |x_6| \\
& - \frac{y_7}{2} + y_6 - \left| \frac{2y_6 + y_7}{2} \right|,
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{2}x_2 + \left| -\frac{1}{2}x_2 \right| - 4y_2 + |-y_2| + \frac{5}{2}x_7 + \left| -\frac{1}{2}x_7 \right| + \frac{5}{2}y_7 + \left| -\frac{1}{2}y_7 \right| + 57 = \frac{11}{2}x_3 + \left| -\frac{1}{2}x_3 \right| - y_2 + |-y_2| \\
& + 4x_7 + |-x_7| + y_7 - \frac{1}{2}y_6 + \left| \frac{-y_6 - 2y_7}{2} \right|,
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{2}x_1 + \left| -\frac{3}{2}x_1 \right| - \frac{7}{2}y_1 + \left| -\frac{5}{2}y_1 \right| + 3x_8 + |-2x_8| + 3y_8 + |-2y_8| + 83 = \frac{11}{2}x_4 + \left| -\frac{3}{2}x_4 \right| + y_4 - 2y_1 \\
& + \left| \frac{-4y_1 - 2y_4}{2} \right| + 4x_8 + |-3x_8| + \frac{5}{2}y_8 - 2y_5 + \left| \frac{-4y_5 - 5y_8}{2} \right|,
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{2}y_4 - \left| \frac{3}{2}y_4 \right| + \frac{7}{2}x_1 - \left| \frac{5}{2}x_1 \right| + 3y_5 - |2y_5| - 3x_8 - |2x_8| + 32 = \frac{11}{2}y_1 - \left| \frac{3}{2}y_1 \right| - x_4 + 2x_1 - \left| \frac{4x_1 + 2x_4}{2} \right| \\
& + 4y_5 - |3y_5| - \frac{5}{2}x_8 + 2x_5 - \left| \frac{4x_5 + 5x_8}{2} \right|,
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{2}y_3 - \left| \frac{1}{2}y_3 \right| + 4x_2 - |x_2| + \frac{5}{2}y_6 - \left| \frac{1}{2}y_6 \right| - \frac{5}{2}x_7 - \left| \frac{1}{2}x_7 \right| + 32 = \frac{11}{2}y_2 - \left| \frac{1}{2}y_2 \right| + x_2 - |x_2| + 4y_6 - |y_6| \\
& - x_7 + \frac{1}{2}x_6 - \left| \frac{x_6 + 2x_7}{2} \right| + 4,
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{2}y_2 + \left| -\frac{1}{2}y_2 \right| + 4x_3 + |-x_3| + \frac{5}{2}y_7 + \left| -\frac{1}{2}y_7 \right| - \frac{5}{2}x_6 + \left| -\frac{1}{2}x_6 \right| + 37 = \frac{11}{2}y_3 + \left| -\frac{1}{2}y_3 \right| + x_3 + |-x_3| \\
& + 4y_7 + |-y_7| + \frac{1}{2}x_7 - x_6 + \left| \frac{-2x_6 - x_7}{2} \right| + 4,
\end{aligned}$$

$$-\frac{5}{2}y_1 + \left| -\frac{3}{2}y_1 \right| + \frac{7}{2}x_4 + \left| -\frac{5}{2}x_4 \right| + 3y_8 + \left| -2y_8 \right| - 3x_5 + \left| -2x_5 \right| + 62 = \frac{11}{2}y_4 + \left| -\frac{3}{2}y_4 \right| + 2x_4 - x_1 \\ + \left| \frac{-2x_1 - 4x_4}{2} \right| + 4y_8 + \left| -3y_8 \right| + 2x_8 - \frac{5}{2}x_5 + \left| \frac{-5x_5 - 4x_8}{2} \right| + 4.$$

Step 5: Solving the system of crisp equations (6.41) with the restrictions

$$x_1 \leq x_2, x_2 \leq x_3, x_3 \leq x_4, x_5 \leq x_6, x_6 \leq x_7, x_7 \leq x_8 \text{ and}$$

$y_1 \leq y_2, y_2 \leq y_3, y_3 \leq y_4, y_5 \leq y_6, y_6 \leq y_7, y_7 \leq y_8$, the obtained solution is

$$x_1 = -1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = -2, x_6 = -1, x_7 = 3, x_8 = 4 \text{ and}$$

$$y_1 = 4, y_2 = 5, y_3 = 6, y_4 = 8, y_5 = -7, y_6 = -6, y_7 = -4, y_8 = -3.$$

Step 6: Using Step 10 of the proposed Mehar-Keerat method-V, the fuzzy complex solution of the system of fuzzy complex linear equations, chosen in Example 6.3, is

$$\begin{aligned} \tilde{z}_1 &= (x_1, x_2, x_3, x_4) + i(y_1, y_2, y_3, y_4) \\ &= (-1, 2, 3, 4) + i(4, 5, 6, 8), \\ \tilde{z}_2 &= (x_5, x_6, x_7, x_8) + i(y_5, y_6, y_7, y_8) \\ &= (-2, -1, 3, 4) + i(-7, -6, -4, -3). \end{aligned}$$

6.7. Comparative study

The fuzzy complex solution of some existing system of fuzzy complex linear equations and chosen system of fuzzy complex linear equations, obtained by the existing methods [40-42,88,134] as well as the proposed Mehar method, Keerat method and Mehar-Keerat method-V are shown in Table 6.1.

Table 6.1. Fuzzy complex solution of the existing and chosen system of fuzzy complex linear equations.

Example	Existing methods [40-42,88,134]	Proposed Mehar method/Keerat method	Proposed Mehar-Keerat method-V
Example 6.1	N.A.	N.A.	$\tilde{z}_1 = (-1, 2, 3, 4) + i(4, 5, 6, 8),$ $\tilde{z}_2 = (-2, -1, 3, 4) + i(-7, -6, -4, -3).$
Example 6.2	N.A.	N.A.	$\tilde{z}_1 = (2, 4, 5, 6) + i(-5, -4, -3, -1),$ $\tilde{z}_2 = (-1, 2, 4, 5) + i(0, 1, 2, 3).$
Example 6.3	N.A.	N.A.	$\tilde{z}_1 = (-1, 2, 3, 4) + i(4, 5, 6, 8),$ $\tilde{z}_2 = (-2, -1, 3, 4) + i(-7, -6, -4, -3).$
Example 6.4	N.A.	N.A.	$\tilde{z}_{11} = (2, 4, 5, 6) + i(-5, -4, -3, -1),$ $\tilde{z}_{21} = (-1, 2, 4, 5) + i(0, 1, 2, 3),$ $\tilde{z}_{12} = (1, 2, 4, 5) + i(-2, -1, 0, 1),$ $\tilde{z}_{22} = (-7, -6, -5, -4) + i(-3, -1, 2, 4),$ $\tilde{z}_{13} = (-3, -2, -1, 0) + i(7, 8, 9, 10),$ $\tilde{z}_{23} = (-6, -5, -4, -2) + i(-9, -6, -2, 2).$

Example 6.5	N.A.	N.A.	$\tilde{z}_{11} = (-2, -1, 3, 4) + i(1, 2, 3, 4),$ $\tilde{z}_{21} = (4, 5, 6, 8) + i(-4, -3, -2, -1),$ $\tilde{z}_{31} = (2, 5, 7, 8) + i(-3, 1, 2, 3),$ $\tilde{z}_{12} = (-3, -2, 4, 5) + i(0, 1, 2, 3),$ $\tilde{z}_{22} = (1, 2, 4, 6) + i(-5, -3, -1, 0),$ $\tilde{z}_{32} = (-1, 2, 3, 4) + i(2, 4, 5, 6),$ $\tilde{z}_{13} = (1, 2, 3, 4) + i(-4, -3, -2, -1),$ $\tilde{z}_{23} = (-5, -4, -3, -2) + i(1, 2, 3, 5),$ $\tilde{z}_{33} = (2, 4, 6, 8) + i(2, 3, 5, 6).$
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Appendix A

A note on “Approximate solution of dual fuzzy matrix equations”*

Gong et al. [77] proposed a method to solve the dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$ and claimed that their method is valid for all real matrices A and C of appropriate order. In this appendix, it is pointed out that this claim is not valid and the method, proposed by Gong et al. [77], is valid only if all the elements of A and C are non-negative real numbers. Also, it is pointed out that the dual fuzzy matrix equations, solved by Gong et al. [77] to illustrate their proposed method, have no significance as some of the elements of the matrices A and/or C are negative real numbers in these dual fuzzy matrix equations. Furthermore, it is pointed out that such a dual fuzzy matrix equation in which some/all the elements of A and/or C are negative real numbers can never exist and hence, an attempt to overcome this limitation of Gong et al. method [77] is meaningless.

A.1. Introduction

Gong et al. [77, Section 3, pp. 115] proposed a method to solve the dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$, where A, C are $m \times n$ matrices, whose elements are real numbers and $\tilde{B}, \tilde{D}$ are $m \times p$ matrices, whose elements are LR fuzzy numbers. Gong et al. [77, Section 3, pp. 117] transformed the dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$ into equivalent dual fuzzy matrix equation (1).

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$$\sum_{k=1}^n a_{ik} \tilde{x}_{kj} + \tilde{b}_{ij} = \sum_{k=1}^n c_{ik} \tilde{x}_{kj} + \tilde{d}_{ij}, i=1,2,\dots,m; j=1,2,\dots,p. \quad (1)$$

Furthermore, Gong et al. [77, Section 3, pp. 117] transformed the dual fuzzy matrix equation (1) into dual fuzzy matrix equation (2) by assuming $U_i^+ = \{k : a_{ik} \geq 0\}, U_i^- = \{k : a_{ik} < 0\}, V_i^+ = \{k : c_{ik} \geq 0\}$ and $V_i^- = \{k : c_{ik} < 0\}$.

$$\sum_{k \in U_i^+} a_{ik} \tilde{x}_{kj} + \sum_{k \in U_i^-} a_{ik} \tilde{x}_{kj} + \tilde{b}_{ij} = \sum_{k \in V_i^+} c_{ik} \tilde{x}_{kj} + \sum_{k \in V_i^-} c_{ik} \tilde{x}_{kj} + \tilde{d}_{ij}, i=1,2,\dots,m; k=1,2,\dots,n; j=1,2,\dots,p. \quad (2)$$

Finally, Gong et al. [77, Section 3, pp. 117-119] obtained the solution of dual fuzzy matrix equation (2) by transforming it into system of crisp linear equations.

A.2. Flaws of the existing method

It is well known fact that [63, Section e, Subsection γ , pp. 55; 77, Section 2, Scalar multiplication of Remark 2.1, pp. 114], if $\tilde{x}$ is an *LR* fuzzy number and λ is a negative real number, then $\lambda\tilde{x}$ will be a *RL* fuzzy number as well as a *RL* fuzzy number cannot be added in an *LR* fuzzy number. However, it is obvious from Section 1 that Gong et al. [77] have assumed the mathematical incorrect assumption that the *RL* fuzzy number $\sum_{k \in U_i^-} a_{ik} \tilde{x}_{kj}$ (due to multiplication of negative real number a_{ik} with an *LR* fuzzy number $\tilde{x}_{kj}$) can be added in the *LR* fuzzy number $\sum_{k \in U_i^+} a_{ik} \tilde{x}_{kj}$ as well as the *RL* fuzzy number $\sum_{k \in V_i^-} c_{ik} \tilde{x}_{kj}$ (due to multiplication of negative real number c_{ik} with an *LR* fuzzy number $\tilde{x}_{kj}$) can be added in the *LR* fuzzy number $\sum_{k \in V_i^+} c_{ik} \tilde{x}_{kj}$, to re-write the dual fuzzy matrix equation (1) into the dual fuzzy matrix equation (2).

Furthermore, Gong et al. [77] solved the dual fuzzy matrix equations [77, Section 4, Example 4.1, Example 4.2, Example 4.3, Example 4.4, pp. 123-128; Section 5, Example 5.1, Example 5.2, Example 5.3, Example 5.4, pp. 128-132], to illustrate their proposed method.

However, none of these dual fuzzy matrix equations is valid, as in these dual fuzzy matrix equations, some of the elements of the coefficient matrices A and/or C are negative real numbers. For example, in the dual fuzzy matrix equation [77, Section 4, Example 4.1, pp. 124], the elements a_{12} as well as c_{21} are negative real numbers.

A.3. Validity of the existing method

It can be easily verified that to find the solution of Example 5.4 [77, Section 5, pp. 130], Gong et al. [77, pp. 132], have assumed that $L(x) = \max\left\{0, \frac{1}{e-1}(e^{1-x^2} - 1)\right\}$ and $R(x) = \max\left\{0, \frac{2}{1+x^2} - 1\right\}$. This clearly indicates that for the LR fuzzy numbers, considered by Gong et al. [77], the condition $L(x) = R(x)$ is not satisfying. Therefore, to assume that Gong et al. [77] have proposed a method by assuming $L(x) = R(x)$, is also meaningless. Hence, the existing method [77] is valid only if all the elements of A and C are non-negative real numbers. It is also pertinent to mention that only the existing dual fuzzy matrix equation [77, Section 4, Example 4.3, pp. 127] is valid, as all the elements of A and C of this dual fuzzy matrix equation are non-negative real numbers.

A.4. Conclusion

It is pointed out that the claim of Gong et al. [77] that their proposed method is valid for all the dual fuzzy matrix equations $A\tilde{X} + \tilde{B} = C\tilde{X} + \tilde{D}$, is false and the existing method [77] is valid only if all the elements of A and C are non-negative real numbers. Furthermore, it is pointed out that out of the eight dual fuzzy matrix equations, considered by Gong et al. [77], only one is valid and the remaining seven have no significance.

It is pertinent to mention that such a dual fuzzy matrix equation, in which some/all the elements of A, C are negative real numbers and some/all the elements of $\tilde{X}, \tilde{B}, \tilde{D}$ are

LR fuzzy numbers with $L(x) \neq R(x)$, will never exist and hence an attempt to propose a method for solving such type of dual fuzzy matrix equations is meaningless.

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